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THE WINNER-TAKE-ALL DILEMMA

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Abstract

This paper considers collective decision-making when individuals are partitioned into groups (e.g., states or parties) endowed with voting weights. We study a game in which each group chooses an internal rule that specifies the allocation of its weight to the alternatives as a function of its members' preferences. We show that under quite general conditions, the game is a Prisoner's Dilemma: while the winner-take-all rule is a dominant strategy, the equilibrium is Pareto dominated. We also show asymptotic Pareto dominance of the proportional rule. Our numerical computation for the US Electoral College verifies the sensibility of the asymptotic results.

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1 Introduction

A fundamental question about representative democracy is how social decisions should reflect the opinions of individuals belonging to distinct groups, such as states or parties. Typically, each group has a voting weight, in the form of a number of representatives or a weighted vote assigned to a unique representative. The groups allocate the weights to decision alternatives, and the one that receives the most weight becomes the social decision. In such cases, the quality of social decision-making depends not only on the apportionment of weights among the groups, but also on the rules that allocate the groups' weights to alternatives, based on the preferences of their individual members. The present paper is concerned with how the weight allocation rules affect individuals' welfare.

Existing institutions use different weight allocation rules. On the one hand, the *winner-take-all rule* devotes all the weight of a group to the alternative preferred by the majority of its members. Most states in the Untied States use this rule to allocate presidential electoral votes. A council of national ministers, each with a weighted vote (e.g., the Council of the European Union), is another example, provided the ministers can be thought of as representing their countries' majority interests. Party discipline frequently observed in legislative voting may also be understood as the winner-take-all rule used by parties.

On the other hand, the *proportional rule* allocates a group's weight in proportion to how many members prefer the respective alternatives. In many parliamentary institutions at the national or international level, each constituency (e.g., state or prefecture) elects a set of representatives whose composition more or less proportionally reflects its residents' preferences. Alternatively, when the representatives are viewed as standing for parties rather than states or prefectures, the proportional rule corresponds to a party's rule that allows its representatives to vote for or against proposals based on their own preferences, provided the composition of the party's representatives proportionally reflects the opinions of all party members.

The weight allocation rules are often exogenously given to all groups, but there are also cases where each group chooses its own rule. For instance, in national parliaments, how the representatives are elected from the respective constituencies is stipulated by national law. By contrast, parties often have control over how their representatives vote, by punishing those who violate the party lines. As another example, the US Constitution stipulates that it is up to each state to decide the way in which the presidential electoral votes are allocated (Article II, Section 1, Clause 2).

If groups are allowed to choose their rules, it is possible that each group has an incentive to allocate the weight so as to increase the influence of its members' opinions on social decisions, at the cost of the other groups' influence. It is not clear whether such an incentive at the group level is compatible with desirable properties of the overall preference aggregation, such as efficiency. To address this issue, we need to model the choice of rules as a non-cooperative game.

In this paper, we consider a model of social decision-making where individuals are partitioned into groups endowed with voting weights. The society makes a binary decision through two stages: first, all individuals vote; then each group allocates its weight to the alternatives, based on the number of votes they received from the group's individual members. The winner is the alternative with the most weight. A *rule* for a group is a function that maps each possible vote result in the group to an allocation of its weight to the alternatives. Examples are the winner-take-all and proportional rules. A *profile* is a specification of rules for all groups. We measure an individual's expected welfare under a profile by the *probability of success*, i.e., the probability that the profile produces the social decision preferred by the individual, where the probability is defined at the ex ante stage in which individuals' preferences between the alternatives are unknown. We study the game in which the groups independently choose their rules, so as to maximize their members' expected welfare.

The main result of this paper is that the game is a *Prisoner's Dilemma*. On the one hand, the winner-take-all rule is a *dominant strategy*, i.e., it is an optimal strategy for each group regardless of the rules chosen by the other groups. On the other hand, the winner-take-all profile is *Pareto dominated*, i.e., some other profile makes *every* group better off. In brief, no group has an incentive to deviate from the winner-take-all rule, but every group will be better off if all groups jointly move to other rules. The dilemma structure exists for any number of groups (> 2) and with little restriction on the joint distribution of preferences. Individuals' preferences may be biased, and also correlated within and across groups, which would be true when the groups are parties with different but overlapping political goals, or states that tend to support specific alternatives, e.g., blue, red or swing states in the US.

We then turn to an asymptotic and normative analysis of the model. We consider situations where the number of groups is sufficiently large, and the preferences are independent across groups and distributed symmetrically with respect to the alternatives. In this case, we show that the proportional profile *Pareto dom-*

inates every other symmetric profile (i.e., one in which all groups use the same rule), including the winner-take-all one. The assumptions on the preference distribution abstract from the fact that in reality, some groups tend to prefer specific alternatives. Such an abstraction would be reasonable on the grounds that normative judgment about rules should not favor particular groups because of their characteristic preference biases. To see how many groups are typically sufficient for the asymptotic result, we perform numerical computations in a model based on the US Electoral College, using the current apportionment of electoral votes. The numerical comparisons indicate that the proportional profile does Pareto dominate the winner-take-all profile in the model with fifty states and a federal district.

The observation that the winner-take-all rule is a dominant strategy is consistent with the fact that all but two states in the US currently use it to allocate presidential electoral votes,[1] and also with the widely observed party discipline in assemblies. Despite the various problems or limitations that have been pointed out concerning the winner-take-all rule,[2] it is still used prevalently.

While the above result suggests that the proportional profile asymptotically performs well in terms of efficiency, it is silent about the equality of individuals' welfare. In fact, our model also provides some insight into how rules affect the distribution of welfare. We examine an asymmetric profile called the *congressional district profile*. This profile is inspired by the Congressional District Method currently used by Maine and Nebraska, in which two electoral votes are allocated by the winner-take-all rule and the remaining ones are awarded to the winner of each district-wide popular vote.[3] We show that the congressional district profile achieves a more equal distribution of welfare than any symmetric profile by making

[1] One of the most recent attempts of reform by a state took place in 2004, when a ballot initiative for an amendment to the state constitution was raised in Colorado. The suggested procedure is the proportional rule, in which the state electoral votes are allocated proportionally to the state popular votes. The amendment did not pass, garnering only 34.1% approval.

[2] Some of the major arguments against the winner-take-all rule are the following. First, the winner of the election may be inconsistent with that of the popular votes. Such a discrepancy has happened five times in the history of the US presidential elections, including recently in 2000 and 2016. Second, it may cause reduced dimensionality: (i) the parties have an incentive to concentrate campaign resources only in the battleground states, and (ii) voters' incentive to turn out or to invest in information may be small and/or uneven across states, since the probability of each voter to be pivotal is so small under the winner-take-all rule, and even smaller in the non-swing states.

[3] The idea of allocating a part of the votes by the winner-take-all rule and allowing the rest to be awarded to potentially distinct candidates can be seen as a compromise between the winner-take-all and the proportional rules. Symbolically, the two votes allocated by the winner-take-all rule is the same number as the Senators in each state, while the rest is equal to the number of the House representatives. The idea behind such a mixture is in line with the logic supporting bicameralism, which is supposed to provide checks and balances between the states and the federal governance.

individuals in smaller groups better off.

A technical contribution of this paper is to develop an asymptotic method for analyzing players' expected welfare in weighted voting games. One of the major challenges in analyzing such games is their discreteness. By the nature of combinatorial problems, obtaining an analytical result often requires a large number of classifications by cases, which may include prohibitively tedious and complex tasks in order to obtain general insights. We overcome this difficulty by considering asymptotic properties of games in which there are a sufficiently large number of groups. This technique allows us to obtain an explicit formula that captures the asymptotic behavior of the probability of success for each individual, which holds for a wide class of distributions of weights among groups (the correlation lemma: Lemma 3).

1.1 Literature Review

The incentives for groups to use the winner-take-all rule have been studied by several papers. Hummel (2011) and Beisbart and Bovens (2008) analyze models of the US presidential elections. Gelman (2003) and Eguia (2011a,b) give theoretical explanations as to why voters in an assembly form parties or voting blocs to coordinate their votes. Their findings are coherent with our observation that the winner-take-all rule is a dominant strategy.

Beisbart and Bovens (2008) and Gelman (2003) also contain comparisons of the winner-take-all and proportional profiles. Under the current apportionment of electoral votes in the US, Beisbart and Bovens (2008) numerically compares these profiles, in terms of inequality indices on citizens' voting power and the mean majority deficit, on the basis of a priori and a posteriori voting power measures. Gelman (2003) compares the case with coalitions of equal sizes in which voters coordinate their votes to the case without such coordination. Our analysis is based on Pareto dominance between profiles, and provides results which hold under general distribution of groups' weights or sizes. In that sense, Beisbart and Bovens's positive analysis is complementary to our normative analysis of properties of the proportional profile.

De Mouzon et al. (2019) provides a welfare analysis of popular vote interstate compacts, and shows that, for the regional compact, welfare of the member states is single-peaked as a function of the number of the participating states, while it is monotonically decreasing for the non-member states. The second effect dominates in terms of the social welfare, unless a large majority (approximately more than

$2/\pi \simeq 64\%$) of the states join the compact, implying that a small- or middle-sized regional compact is welfare detrimental. For the national compact, the total welfare is increasing, as it turns out that even the non-members would mostly benefit from the compact, implying that the social optimum is attained when a majority joins the compact, i.e. the winner is determined by the national popular vote. Their findings are coherent with ours: if the winner-take-all rule is applied only to a subset of the groups, then the member states enjoy the benefit at the expense of the welfare loss of the non-member states, and the total welfare decreases. The social optimum is attained when the entire nation uses the popular vote.

The winner-take-all rule has been a regular focus of the literature. The history, objectives, problems, and reforms of the US Electoral College are summarized, for example, in Edwards (2004) and Bugh (2010). One of the most scrutinized problems of the Electoral College is its reduced dimensionality. The incentive of the candidates to concentrate their campaign resources in the swing and decisive states is modeled in Strömberg (2008), which is coherent with the findings of the seminal paper in probabilistic voting by Lindbeck and Weibull (1987). Strömberg (2008) also finds that uneven resource allocation and unfavorable treatment of minority states would be mitigated by implementing a national popular vote, which is coherent with the classical findings by Brams and Davis (1974). Voters' incentive to turn out is investigated by Kartal (2015), which finds that the winner-take-all rule discourages turnout when the voting cost is heterogeneous.

Constitutional design of weighted voting is studied extensively in the literature. Seminal contributions are found in the context of power measurement: Penrose (1946), Shapley and Shubik (1954), Banzhaf (1968) and Rae (1946). Excellent summaries of theory and applications of power measurement are given by, above all, Felsenthal and Machover (1998) and Laruelle and Valenciano (2008). The tools and insights obtained in the power measurement literature are often used in the apportionment problem: e.g., Barberà and Jackson (2006), Koriyama et al. (2013), and Kurz et al. (2017).

2 The Model

Let us consider a society which consists of n disjoint *groups*: $i \in \{1, 2, \cdots, n\}$. Let M_i be the set of members of group i. The society makes a collective decision between two *alternatives*, denoted -1 and $+1$. Each group i is endowed with a *weight* $w_i > 0$. To exclude trivial cases, we assume that each group's weight is less than half the total weight: $w_i < \frac{1}{2}\sum_{j=1}^{n} w_j$ for all $i = 1, \cdots, n$.

Let

$$X_{im} \in \{-1, 1\}$$

be the random variable which represents the alternative preferred by member $m(\in M_i)$ of group i. Since the model is concerned with the weight allocation by each group which aggregates the preferences of its members, it is most appropriate to suppose that the groups' aggregation rules are fixed prior to the realization of the preferences. The following is the assumption on the preference distribution.

Assumption 1. There exists a latent random vector $(\Theta_i)_{i=1}^n$ that is absolutely continuous and has support $[-1,1]^n$, such that conditional on $\Theta = \theta$, the individuals' preferences X_{im} are independent and distributed with:

$$\begin{aligned}
\mathbb{P}\{X_{im} = +1 | \Theta_1 = \theta_1, \cdots, \Theta_n = \theta_n\} &= (1 + \theta_i)/2, \\
\mathbb{P}\{X_{im} = -1 | \Theta_1 = \theta_1, \cdots, \Theta_n = \theta_n\} &= (1 - \theta_i)/2,
\end{aligned}$$

for each group $i = 1, \cdots, n$ and member $m \in M_i$.[4]

In this paper, we focus on the situation in which the group size $|M_i|$ is sufficiently large for all $i = 1, \cdots, n$. By the Law of Large Numbers,

$$\frac{1}{|M_i|} \sum_{m \in M_i} X_{im} \to \Theta_i \text{ for all } i = 1, \cdots, n, \text{ almost surely.}$$

The left-hand side is the *group-wide margin* for alternative $+1$ in group i, i.e., the fraction of members of i preferring $+1$ minus the fraction of those preferring -1. Throughout the paper, we therefore regard the random variable Θ_i itself as representing the group-wide margin for alternative $+1$.

The latent variables Θ_i allow us to capture intra-group correlations of the preferences. For any two members m, m' of group i, the correlation is given by $\mathrm{Corr}(X_{im}, X_{im'}) = \mathrm{Var}(\Theta_i)/(1 - \mathbb{E}(\Theta_i^2))$. Since Θ_i has full support, the preferences of members of group i are positively correlated. In addition, since Assumption 1 does not exclude correlation of $(\Theta_i)_{i=1}^n$ across groups, individuals' preferences may also be correlated across groups. See Remark 1 for more details.

The society decides between the alternatives through two stages: (i) each individual votes for his preferred alternative; (ii) each group allocates its weight to the two alternatives, based on the group-wide margin. The winner is the alternative which receives a majority of the weight.

[4] Throughout the paper, we use capital Θ_i for representation of a random variable, and small θ_i for the realization.

At the second stage, each group's allocation of weight is determined as a function of the group-wide margin. A *rule* for group i is defined as a measurable function

$$\phi_i : [-1, 1] \to [-1, 1].$$

The value $\phi_i(\theta_i)$ means the fraction of w_i allocated to alternative $+1$ minus that allocated to -1, given that the group-wide margin is θ_i. That is, the rule allocates $w_i \phi_i(\theta_i)$ more weight to alternative $+1$ than alternative -1. The following rules deserve particular attention:

Winner-take-all rule: $\phi_i^{\mathrm{WTA}}(\theta_i) = \operatorname{sgn} \theta_i$,
Proportional rule: $\phi_i^{\mathrm{PR}}(\theta_i) = \theta_i$,
Mixed rules: $\phi_i^a(\theta_i) = a\phi_i^{\mathrm{WTA}}(\theta_i) + (1-a)\phi_i^{\mathrm{PR}}(\theta_i)$, $0 \le a \le 1$.

The winner-take-all rule devotes all the weight of a group to the winning alternative in the group. The proportional rule allocates the weight in proportion to the vote shares of the respective alternatives in the group. The mixed rule ϕ^a allocates the fixed ratio a of the weight by the winner-take-all rule and the remaining $1-a$ part by the proportional rule.

A profile $\phi = (\phi_i)_{i=1}^n$ consists of rules specified for all groups. By *symmetric* profile, we mean that the same rule is used by all groups. For instance, the above three rules naturally define the following symmetric profiles: the *winner-take-all profile* $\phi^{\mathrm{WTA}} = (\phi_i^{\mathrm{WTA}})_{i=1}^n$, the *proportional profile* $\phi^{\mathrm{PR}} = (\phi_i^{\mathrm{PR}})_{i=1}^n$, and *mixed profiles* $\phi^a = (\phi_i^a)_{i=1}^n$, $a \in [0, 1]$.

The winning alternative is the one which obtains more weight from the groups. In the case of a tie, we assume that both alternatives are chosen with equal probability. To define it formally, let

$$S_\phi = \sum_{i=1}^n w_i \phi_i(\Theta_i)$$

be the difference between the total weight cast for alternatives $+1$ and -1. The *social decision* D_ϕ is

$$D_\phi = \begin{cases} \operatorname{sgn} S_\phi & \text{if } S_\phi \neq 0 \\ \pm 1 \text{ with equal probabilities} & \text{if } S_\phi = 0. \end{cases} \tag{1}$$

We say that the social decision is a *success* for member m of group i, if it coincides with his preferred alternative: $X_{im} = D_\phi$. Each group's objective is to maximize the ex ante probability of success for its members. Since Assumption 1 implies that the ex ante probability is the same for all members of the same group, we can define

$$\pi_i(\phi) = \mathbb{P}\{X_{im} = D_\phi\}$$

to be the probability of success for the members of group i, given the profile $\phi = (\phi_i)_{i=1}^n$. Since each group chooses a rule as a function of the group-wide margin, maximizing $\pi_i(\phi)$ with respect to its own rule ϕ_i is equivalent to maximizing the conditional probability of success given the group-wide margin $\Theta_i = \theta_i$,

$$\pi_i(\phi|\theta_i) = \mathbb{P}\{X_{im} = D_\phi | \Theta_i = \theta_i\},$$

for almost every $\theta_i \in [-1, 1]$.

The probabilistic models that have been extensively studied in the literature also assume existence of the latent variables.

Remark 1. The *Impartial Culture (IC)* assumes that all members' preferences are independently distributed and they are equally likely to prefer the two alternatives. In our model, this corresponds to the case where $\Theta_i = 0$ for all i. In particular, $\pi_i(\phi^{\mathrm{WTA}})$ coincides with an affine transformation of the Banzhaf-Penrose index when the coalitions are formed according to the IC distribution (Straffin (1988)). The *Impartial Anonymous Culture (IAC)* assumes that the fraction of members in the society who prefer a specific alternative is uniformly distributed. IAC corresponds to the case in which all the latent variables are perfectly correlated, $\Theta_1 = \cdots = \Theta_n$, so that the preferences of the members in the society are equally correlated both across and within groups, and are uniformly distributed on $[-1, 1]$. Similar to IC, $\pi_i(\phi^{\mathrm{WTA}})$ coincides with an affine transformation of the Shapley-Shubik index when the coalitions are formed according to the IAC distribution. A variant of IAC, called the *Impartial Anonymous Culture* (IAC*)*, assumes that the fraction of members who prefer one alternative to the other is uniformly distributed in each group, and is independent across groups. This corresponds to the case where $(\Theta_i)_{i=1}^n$ are independent and uniformly distributed on $[-1, 1]$. Under IAC*, the members in a group are allowed to share common interests, and thus their preferences are positively correlated *within* the group, but not *across* groups. By contrast, no correlation among the members is allowed under IC, whereas the correlation is allowed but in exactly the same way within and across groups under IAC. IAC* fits best to the analysis if we consider the situation in which a group is not merely a collection of members with independent preferences, but rather they may share common interests and/or values, although the correlation is not required to be perfect.[5]

3 The Dilemma

We consider a non-cooperative game Γ in which each group chooses a rule to allocate its weight to the alternatives. Each group's objective is to maximize the ex ante probability of success for its members. Formally, the *game* Γ is defined as follows. The set of players is the set of groups, i.e., $\{1, \cdots, n\}$. The strategy space for group i is the set of all rules, i.e., {all measurable functions $\phi_i : [-1,1] \to [-1,1]$}. The payoff of group i is the probability of success, i.e., $\pi_i(\phi)$.

Two rules ϕ_i and ψ_i are called *equivalent* if $\phi_i(\Theta_i) = \psi_i(\Theta_i)$ almost surely. Two profiles ϕ and ψ are called *equivalent* if $D_\phi = D_\psi$ almost surely.

A rule (or strategy) ϕ_i for group i *dominates* another rule ψ_i if $\pi_i(\phi_i, \phi_{-i}) \geq \phi_i(\psi_i, \phi_{-i})$ for any ϕ_{-i}, with strict inequality for at least one ϕ_{-i}. A rule ϕ_i is a *dominant strategy* for group i if it dominates every rule not equivalent to ϕ_i. A profile ϕ *Pareto dominates* another profile ψ if $\pi_i(\phi) \geq \pi_i(\psi)$ for all i, with strict inequality for at least one i. If ϕ is not Pareto dominated by any profile, it is called *Pareto efficient.*

The main result of this paper is the following.

Theorem 1. *Under Assumption 1, game Γ is a Prisoner's Dilemma:*

(i) the winner-take-all rule ϕ_i^{WTA} is a dominant strategy for each group i;

(ii) the winner-take-all profile ϕ^{WTA} is Pareto dominated.

We use the following lemmata to prove the theorem. The proofs of the lemmata are relegated to the Appendix.

Lemma 1. *(Formula of the probability of success) Under Assumption 1,*

$$\begin{aligned}&2\pi_i(\phi|\theta_i) - 1\\&= \theta_i \left(\mathbb{P}\{w_i\phi_i(\theta_i) + S_{\phi_{-i}} > 0|\Theta_i = \theta_i\} - \mathbb{P}\{w_i\phi_i(\theta_i) + S_{\phi_{-i}} < 0|\Theta_i = \theta_i\}\right),\end{aligned}$$

where $S_{\phi_{-i}} = \sum_{j \neq i} w_j \phi_j(\Theta_j)$.

[5] De Mouzon et al. (2019) provide a detailed comparison of IC, IAC, and IAC* and find, in particular, a peculiarity of IAC in their numerical computations.

In Lemma 2 below, a *generalized proportional profile* refers to a profile in which $\phi_i(\theta_i) = \lambda_i\theta_i$, $i = 1, \cdots, n$, for some vector $\lambda \in [0,1]^n \setminus \{0\}$.

Lemma 2. *(Characterization of the Pareto set) Under Assumption 1, a profile ϕ is Pareto efficient if and only if it is equivalent to some generalized proportional profile.*

Remark 2. Lemma 2 characterizes the entire Pareto set of game Γ. In the Appendix, we prove the lemma by showing: (i) a profile is Pareto efficient if and only if it maximizes a weighted sum of probabilities of success of the groups; (ii) the weighted sum $\sum q_i\pi_i(\phi)$ is maximized if and only if ϕ is equivalent to the generalized proportional profile with coefficients proportional to q_i/w_i. As a simplest example, the profile that maximizes the (unweighted) sum of probabilities of success for all individuals is the generalized proportional profile with coefficients proportional to n_i/w_i, where n_i is the population share of group i.

Proof of Theorem 1. Part (i). By Lemma 1, if $\theta_i > 0$ (resp. $\theta_i < 0$), then $\pi_i(\phi|\theta_i)$ is non-decreasing (resp. non-increasing) in $\phi_i(\theta_i) \in [-1,1]$. We thus have $\pi_i(\phi_i^{\mathrm{WTA}}, \phi_{-i}|\theta_i) \geq \pi_i(\phi_i, \phi_{-i}|\theta_i)$ for any (ϕ_i, ϕ_{-i}) and θ_i. Therefore

$$\pi_i(\phi_i^{\mathrm{WTA}}, \phi_{-i}) \geq \pi_i(\phi_i, \phi_{-i})$$

for any (ϕ_i, ϕ_{-i}). Now we show that for any subprofile ϕ_{-i} in which each $\phi_j : [-1,1] \to [-1,1]$ $(j \neq i)$ is onto (e.g., ϕ_j^{PR}), the strict inequality

$$\pi_i(\phi_i^{\mathrm{WTA}}, \phi_{-i}) > \pi_i(\phi_i, \phi_{-i}) \tag{2}$$

holds for any rule ϕ_i that is not equivalent to ϕ_i^{WTA}. To see this, note that for such ϕ_{-i}, the full-support assumption on $(\Theta_j)_{j=1}^n$ implies that the conditional distribution of $S_{\phi_{-i}}$ given $\Theta_i = \theta_i$ has support $[-\sum_{j\neq i} w_j, \sum_{j\neq i} w_j]$. Since $w_i < \sum_{j\neq i} w_j$, the formula in Lemma 1 implies that if $\theta_i > 0$ (resp. $\theta_i < 0$), then $\pi_i(\phi|\theta_i)$ is strictly increasing (resp. decreasing) in $\phi_i(\theta_i) \in [-1,1]$. Thus $\pi_i(\phi_i^{\mathrm{WTA}}, \phi_{-i}|\theta_i) > \pi_i(\phi_i, \phi_{-i}|\theta_i)$ holds at any θ_i for which $\phi^{\mathrm{WTA}}(\theta_i) \neq \phi_i(\theta_i)$. Since Θ_i has full support, this implies that (2) holds for any ϕ_i that is not equivalent to ϕ_i^{WTA}.

Part (ii). By the characterization of the Pareto set (Lemma 2), it suffices to check that ϕ^{WTA} is not equivalent to any generalized proportional profile. Suppose, on the contrary, that ϕ^{WTA} is equivalent to a generalized proportional profile with coefficients $\lambda \in [0,1]^n \setminus \{0\}$. Then, since $(\Theta_i)_{i=1}^n$ has full support,

$$D_{\phi^{\mathrm{WTA}}}(\theta) = \operatorname{sgn} \sum_{i=1}^{n} w_i \lambda_i \theta_i \text{ at almost every } \theta \in [-1,1]^n. \tag{3}$$

Since no group dictates the social decision, the coefficients λ_i are positive for at least two groups. Without loss of generality, assume $\lambda_1 > 0$ and $\lambda_2 > 0$. Now, fix θ_i for $i \neq 1, 2$ so that they are sufficiently small in absolute value. Then, according to (3), for (almost any) sufficiently small $\varepsilon > 0$, $D_{\phi^{\mathrm{WTA}}}(\theta) = +1$ if $\theta_1 = 1 - \varepsilon$ and $\theta_2 = -\varepsilon$, while $D_{\phi^{\mathrm{WTA}}}(\theta) = -1$ if $\theta_1 = \varepsilon$ and $\theta_2 = -1 + \varepsilon$. This contradicts the fact that $D_{\phi^{\mathrm{WTA}}}(\theta)$ depends only on the signs of $(\theta_i)_{i=1}^n$. $\square$

In contrast with the winner-take-all profile, the proportional profile has the following property.

Proposition 1. *Under Assumption 1, the proportional profile ϕ^{PR} is Pareto efficient.*

Proof. This follows from the characterization of the Pareto set (Lemma 2). $\square$

However, the proportional profile does not necessarily Pareto dominate the winner-take-all profile. This is illustrated by the following example.

Example 1. Let us consider three groups with weights $(w_1, w_2, w_3) = (49, 49, 2)$. The group-wide margins Θ_i are independent and uniformly distributed on $[-1, 1]$. On the one hand, under the winner-take-all profile ϕ^{WTA}, all groups are perfectly symmetric, and a simple calculation shows that the probability of success is $\pi_i(\phi^{\mathrm{WTA}}) = 0.625$ for all $i = 1, 2, 3$. On the other hand, under the proportional profile ϕ^{PR}, group 3 is extremely unlikely to affect the social decision, and $\pi_3(\phi^{\mathrm{PR}})$ is close to 0.5 (approximately 0.507). Group 3 is better off under ϕ^{WTA} than ϕ^{PR}, and so ϕ^{PR} does not Pareto dominate ϕ^{WTA}. By what profile is ϕ^{WTA} Pareto dominated? The characterization lemma provides an answer. Consider the generalized proportional profile $\hat{\phi}$ with coefficients $\lambda_i = 1/w_i$. Then, the distribution of the weight assigned to the alternative is exactly the same across groups, and thus $\pi_i(\hat{\phi})$ is the same for all i. By Pareto efficiency of the generalized proportional profile, $\pi_i(\hat{\phi}) > 0.625$ for all i. $\square$

4 Asymptotic and Computational Results

4.1 Asymptotic analysis

We saw above that the game is a Prisoner's Dilemma. In this section, we provide further insights on the welfare properties, by focusing on the following situations in which: (i) the number of groups is sufficiently large, and (ii) the preferences of the members are distributed symmetrically. These properties allows us to provide an asymptotic and normative analysis.

Often the difficulty of analysis arises from the discrete nature of the problem. Since the social decision D_ϕ is determined as a function of the sum of the weights allocated to the alternatives across the groups, computing the success probability may require classification of a large number of success configurations which increases exponentially as the number of groups increases, rendering the analysis prohibitively costly. We overcome this difficulty by studying asymptotic properties. In order to check the sensibility of our analysis, we provide Monte Carlo simulation results later in the section, using the example of the US Electoral College.

In order to study asymptotic properties, let us consider a sequence of weights $(w_i)_{i=1}^{\infty}$, exogenously given as a fixed parameter.

Assumption 2. The sequence of weights $(w_i)_{i=1}^{\infty}$ satisfies the following properties.

(i) There exists $\bar{w}$ such that $w_i \in [0, \bar{w}]$ for all i.

(ii) As $n \to \infty$, the statistical distribution G_n induced by $(w_i)_{i=1}^{n}$ weakly converges to a distribution G whose support contains an open interval.[6]

Assumption 2 guarantees that for large n, the statistical distribution of weights G_n is sufficiently close to some well-behaved distribution G, on which our asymptotic analysis is based.

Additionally, we impose an impartiality assumption for our normative analysis:

Assumption 3. The variables $(\Theta_i)_{i=1}^{\infty}$ are drawn independently from a common symmetric distribution F.

[6] The statistical distribution function G_n induced by $(w_i)_{i=1}^{n}$ is defined by $G_n(x) = \#\{i \leq m : w_i \leq x\}/n$ for each x. G_n weakly converges to G if $G_n(x) \to G(x)$ at every point x of continuity of G.

As in Felsenthal and Machover (1998), a normative analysis requires impartiality, and a study of fundamental rules in the society, such as a constitution, should be free from any dependence on the ex post realization of the group characteristics. Assumption 3 allows our normative analysis to abstract away the distributional details. Of course, a normative analysis is best complemented by a positive analysis which takes into account the actual characteristics of the distributions (as in Beisbart and Bovens (2008)).

Following the symmetry of the preferences, our analysis also focuses on *symmetric* profiles, in which all groups use the same rule: $\phi_i = \phi$ for all i. With a slight abuse of notation, we write ϕ both for a *single rule* ϕ and for the *symmetric profile* $(\phi, \phi, \cdots)$, which should not create any confusion as long as we refer to symmetric profiles. As for the alternatives, it is natural to consider that the label should not matter when the group-wide vote margin is translated into the weight allocation, given the symmetry of the preferences.

Assumption 4. We assume that the rule is monotone and neutral, that is, ϕ is a non-decreasing, odd function: $\phi(\theta_i) = -\phi(-\theta_i)$.

Let $\pi_i(\phi; n)$ denote the probability of success for group $i(\leq n)$ under profile ϕ when the set of groups is $\{1, \cdots, n\}$ and each group j's weight is w_j, the jth component of the sequence of weights. The definition of $\pi_i(\phi; n)$ is the same as $\pi_i(\phi)$ in the preceding sections; the new notation just clarifies its dependence on the number of groups n.

The main welfare criterion employed in this section is the asymptotic Pareto dominance.

Definition 1. For two symmetric profiles ϕ and ψ, we say that ϕ *asymptotically Pareto dominates* ψ if there exists N such that for all $n > N$ and all $i = 1, \cdots, n$,

$$\pi_i(\phi; n) > \pi_i(\psi; n).$$

4.2 Pareto Dominance

The following is the main result in our asymptotic analysis.

Theorem 2. *Under Assumptions 1-4, the proportional profile asymptotically Pareto dominates all other symmetric profiles. In particular, it Pareto dominates the Nash equilibrium of the game, i.e., the symmetric winner-take-all profile.*

We use the following lemma to prove Theorem 2. The proof of Lemma 3 is relegated to the Appendix. The proof of part (ii) uses a more general result, Lemma 4, stated in the next subsection, whose proof also appears in the Appendix.

Lemma 3. *Under Assumptions 1-4, the following statements hold.*

(i) For any symmetric profile ϕ, as $n \to \infty$,

$$\pi_i(\phi; n) - \frac{1}{2} = \int_0^1 \theta_i \mathbb{P}\left\{ -w_i\phi(\theta_i) < \sum_{j \le n, j \ne i} w_j \phi(\Theta_j) \le w_i \phi(\theta_i) \right\} dF(\theta_i).$$

(ii) For any symmetric profile ϕ, as $n \to \infty$,

$$\sqrt{2\pi n}\left(\pi_i(\phi; n) - \frac{1}{2}\right) \to w_i \sqrt{\frac{\mathbb{E}[\Theta^2]}{\int_0^{\bar{w}} w^2 dG(w)}} \cdot \mathrm{Corr}[\Theta, \phi(\Theta)],$$ [7][8]

uniformly in $w_i \in [0, \bar{w}]$. The limit depends on ϕ only through the factor $\mathrm{Corr}[\Theta, \phi(\Theta)]$.

Proof of Theorem 2.
The heart of the proof is in the correlation result shown in part (ii) of Lemma 3. It follows that if $\phi(\Theta)$ is more correlated with Θ than $\psi(\Theta)$ is, then for each group i, there exists N_i such that if the number of groups (n) is greater than N_i, group i ($\le n$) will be better off under ϕ than ψ.

Note that the convergence in part (ii) of Lemma 3 is uniform in $w_i \in [0, \bar{w}]$. This implies that the convergence is uniform in $i = 1, 2, \cdots$.[9] Thus there is N with the above property, without subscript i, which applies to all groups $i = 1, 2, \cdots$. Therefore, if $\phi(\Theta)$ is more correlated with Θ than $\psi(\Theta)$ is, then ϕ asymptotically Pareto dominates ψ.

[7] We write Θ for a random variable having the same distribution F as Θ_i.

[8] Since Θ and $\phi(\Theta)$ are symmetric, the correlation is given by $\mathrm{Corr}[\Theta, \phi(\Theta)] = \mathbb{E}[\Theta\phi(\Theta)]/\sqrt{\mathbb{E}[\Theta^2]\mathbb{E}[\phi(\Theta)^2]}$ unless $\phi(\Theta)$ is almost surely zero. If $\phi(\Theta)$ is almost surely zero, then the correlation is zero.

[9] A more detailed explanation of this step is the following. By Lemma 3 (i), $\sqrt{2\pi n}(\pi_i(\phi; n) - 1/2)$ asymptotically behaves as $\sqrt{2\pi n}\int_0^1 \theta \mathbb{P}\{-w_i\phi(\theta) < \sum_{j \le n} w_j\phi(\Theta_j) \le w_i\phi(\theta)\} dF(\theta)$, where whether or not to include the ith term $w_i\phi(\Theta_i)$ in the sum does not matter in the limit. The estimate of $\sqrt{2\pi n}(\pi_i(\phi; n) - 1/2)$ therefore has the form $f_n(w_i)$, where $f_n(x) := \sqrt{2\pi n}\int_0^1 \theta \mathbb{P}\{-x\phi(\theta) < \sum_{j \le n} w_j\phi(\Theta_j) \le x\phi(\theta)\} dF(\theta)$. Lemma 3 (ii) implies that $f_n(x)$ converges uniformly in $x \in [0, \bar{w}]$, which in turn implies that the convergence of $\sqrt{2\pi n}(\pi_i(\phi; n) - 1/2) \approx f_n(w_i)$ is uniform in $i = 1, 2, \cdots$.

Since the perfect correlation $\text{Corr}[\Theta, \phi^{\text{PR}}(\Theta)] = 1$ is attained by the proportional rule, Theorem 2 follows. $\square$

Theorem 2 shows the Pareto dominance of the symmetric proportional profile over any other symmetric profile. Intuitively, when there are sufficiently many groups, the members' preferences are most efficiently aggregated to the social decision if the weights are allocated proportionally to the alternatives *by all groups*. However, such a profile cannot be sustained as a Nash equilibrium of the game, since each group has an incentive to deviate to a dominant strategy, i.e., the winner-take-all rule. This typical Prisoner's Dilemma situation suggests to us that a Pareto efficient outcome is not expected to be achieved under decentralized decision making, and a coordination device is necessary in order to attain a Pareto improvement.

The above results show that the winner-take-all rule is characterized by its strategic dominance, while the proportional rule is characterized by its asymptotic Pareto dominance. The following proposition provides a complete Pareto order among all the linear combinations of the two rules.

Remember that we defined the mixed rules in Section 2 above. For $0 \le a \le 1$, a fraction a of the weight is assigned to the winner of the popular vote, while the rest, $1 - a$, is distributed proportionally to each alternative:

$$\phi^a(\theta_i) = a\phi^{\text{WTA}}(\theta_i) + (1 - a)\phi^{\text{PR}}(\theta_i).$$

Proposition 2. *Under Assumptions 1-3, mixed profile ϕ^a asymptotically Pareto dominates mixed profile $\phi^{a'}$ for any $0 \le a < a' \le 1$. In particular, the proportional profile asymptotically Pareto dominates any mixed profile ϕ^a for $0 < a < 1$, which in turn asymptotically Pareto dominates the winner-take-all profile. In other words, all mixed profiles can be ordered by asymptotic Pareto dominance, from the proportional rule as the best, to the winner-take-all rule as the worst.*

Proof. In Appendix. $\square$

The winner-take-all rule is not only asymptotically Pareto inefficient, but the worst among the symmetric mixed profiles. Is it worse than *any* other symmetric profile? We provide an answer in Remark 3 below.

Remark 3. Theorem 2 leaves the natural question of whether the winner-take-all profile is the worst among all symmetric profiles, in terms of asymptotic Pareto dominance. The answer is negative. To see this, note first that, for the winner-take-all profile, the correlation in Lemma 3 is strictly positive: $\text{Corr}[\Theta, \phi^{\text{WTA}}(\Theta)] =$

$\mathbb{E}(|\Theta|)/\sqrt{\mathbb{E}(\Theta^2)} > 0$. On the other hand, for the symmetric profile ϕ^0 in which the rule is defined by $\phi^0(\theta) = 0$ for any θ, the correlation is obviously zero. This rule assigns exactly half of the weight to each alternative, regardless of the group-wide vote. The profile ϕ^0 thus ignores the group-wide vote results and is the worst among all symmetric profiles.

4.3 Congressional District Method

The analysis in the preceding subsection suggests that the proportional profile is optimal in terms of Pareto efficiency. However, our model also implies that this profile produces an unequal distribution of welfare among individuals; in fact, this unequal nature pertains to all symmetric profiles. The Correlation Lemma 3 (ii) shows that for these profiles, normalized probability of success for a group is asymptotically proportional to its weight, providing a high success probability to the members in a group with a large weight.

In this subsection, we examine whether such inequality can be alleviated without impairing efficiency by using an asymmetric profile, based on the Congressional District Method, currently used in Maine and Nebraska. This profile allocates a fixed amount c of each group's weight by the winner-take-all rule and the rest by the proportional rule:

$$w_i\phi^{\mathrm{CD}}(\theta_i, w_i) = c\phi^{\mathrm{WTA}}(\theta_i) + (w_i - c)\phi^{\mathrm{PR}}(\theta_i).$$

We consider the profile in which the rule is used by all groups. To ensure that the profile is well-defined, we impose that the weight sequence $(w_i)_{i=1}^{\infty}$ has a positive lower bound.

Assumption 5. There exists $\underline{w} > 0$ such that $w_i > \underline{w}$ for all i.

Theorem 3. *Under Assumptions 1-5, let us consider the congressional district profile with parameter $c \leq \underline{w}$. For any symmetric profile ϕ, there exists $w^* \in [0, \bar{w}]$ with the following property: for any $\varepsilon > 0$, there is N such that for all $n > N$ and $i = 1, \cdots, n$,*

$$\begin{aligned} w_i < w^* - \varepsilon &\Rightarrow \pi_i(\phi^{\mathrm{CD}}; n) > \pi_i(\phi; n), \\ w_i > w^* + \varepsilon &\Rightarrow \pi_i(\phi^{\mathrm{CD}}; n) < \pi_i(\phi; n). \end{aligned}$$

The proof of Theorem 3 uses the following lemma. Its proof and the Local Limit Theorem used in the proof are stated in the Appendix.

Lemma 4. *Under Assumptions 1-4, let ϕ be a symmetric profile, or the congressional district profile (in which case Assumption 5 is also assumed), so that the rule for group i is denoted $\phi(\cdot, w_i)$. Then, as $n \to \infty$,*

$$\sqrt{2\pi n}\left(\pi_i(\phi; n) - \frac{1}{2}\right) \to \frac{w_i \mathbb{E}[\Theta\phi(\Theta, w_i)]}{\sqrt{\int_0^{\bar{w}} w^2 \mathbb{E}[\phi(\Theta, w)^2] dG(w)}},[10]$$

uniformly in $w_i \in [0, \bar{w}]$.

Proof of Theorem 3. By Lemma 4, the normalized success probability for group i under a symmetric profile ϕ tends to a linear function of w_i. Let A^ϕ be the coefficient:

$$\begin{aligned}\lim_{n\to\infty} \sqrt{2\pi n}\left(\pi_i(\phi; n) - \frac{1}{2}\right) &= \frac{w_i \mathbb{E}[\Theta\phi(\Theta)]}{\sqrt{\mathbb{E}[\phi(\Theta)^2] \int_0^{\bar{w}} w^2 dG(w)}} \\ &=: A^\phi w_i.\end{aligned} \tag{4}$$

For the congressional district profile, remember the definition:

$$\begin{aligned} w_j \phi^{\mathrm{CD}}(\theta_j, w_j) &= c\phi^{\mathrm{WTA}}(\theta_j) + (w_j - c)\,\phi^{\mathrm{PR}}(\theta_j) \\ &= c \operatorname{sgn}(\theta_j) + (w_j - c)\,\theta_j. \end{aligned}$$

We claim that the limit function is affine in w_i:

$$\lim_{n\to\infty} \sqrt{2\pi n}\left(\pi_i(\phi^{\mathrm{CD}}; n) - \frac{1}{2}\right) = Bw_i + C. \tag{5}$$

To see that, let us apply Lemma 4 again:

$$\begin{aligned}\lim_{n\to\infty} \sqrt{2\pi n}\left(\pi_i(\phi^{\mathrm{CD}}; n) - \frac{1}{2}\right) &= \frac{w_i \mathbb{E}\left[\Theta\phi^{\mathrm{CD}}(\Theta, w_i)\right]}{\sqrt{\int_0^{\bar{w}} w^2 \mathbb{E}\left[\phi^{\mathrm{CD}}(\Theta, w)^2\right] dG(w)}} \\ &= \frac{c\mathbb{E}\left[|\Theta|\right] + (w_i - c)\,\mathbb{E}\left[\Theta^2\right]}{\sqrt{\int_0^{\bar{w}} w^2 \mathbb{E}\left[\phi^{\mathrm{CD}}(\Theta, w)^2\right] dG(w)}}.\end{aligned}$$

Since $|\theta| \geq \theta^2$ with a strict inequality for $0 < |\theta| < 1$, the full support condition

[10] This formula implicitly excludes the case where the denominator is zero, but this causes no problem. First suppose ϕ is a symmetric profile in which the rule takes the zero value almost surely, i.e., $\mathbb{E}[\phi(\Theta)^2] = 0$. Then the social decision is almost always determined by coin tossing, and so the probability of success is $1/2$ for all groups. Therefore the limit in the lemma is zero. For the congressional district profile, $\mathbb{E}[\phi^{\mathrm{CD}}(\Theta, x)^2] > 0$ for each $x > \underline{w}$, so the formula in the lemma is well-defined.

for Θ implies $\mathbb{E}[|\Theta|] > \mathbb{E}[\Theta^2]$, which induces that the intercept C is positive. The coefficient of w_i is:

$$B = \frac{\mathbb{E}[\Theta^2]}{\sqrt{\int_0^{\bar{w}} w^2 \mathbb{E}\left[\phi^{\mathrm{CD}}(\Theta, w)^2\right] dG(w)}}.$$

If $A^\phi < B$, combined with $C > 0$, the right-hand side of (5) is above that of (4). Then, set $w^* = \bar{w}$. If $A^\phi > B$, again combined with $C > 0$, the two limit functions (4) and (5) intersect only once at a positive value $\hat{w}$. Let $w^* = \min\{\hat{w}, \bar{w}\}$.

Since the convergences (4) and (5) are uniform in w_i, for any $\varepsilon > 0$ there is N with the property stated in Theorem 3. □

If the weight is an increasing function of the group size, the theorem implies that the congressional district profile makes members of small groups better off, compared with any symmetric profile.

The intuitive reason why the congressional district profile is advantageous for small groups is as follows. Under this profile, the ratio of weights cast by the winner-take-all rule (i.e. c/w_i) is higher for small groups than large groups. The congressional district profile therefore resembles the situation where the rules used by the smaller groups are relatively close to the winner-take-all rule, whereas those by the larger groups are close to the proportional rule. The strategic dominance of the winner-take-all rule suggests that this deviation is profitable for the small groups.

In addition to Theorem 3, we can also show that the congressional district profile allocates success probabilities to individuals more equally than any symmetric profile does, in the sense of Lorenz dominance. A distribution of success probabilities among individuals is said to *Lorenz dominate* another distribution if the share of success probabilities acquired by any bottom fraction of individuals is larger in the former distribution than in the latter.[11] Lorenz dominance, whenever it occurs, agrees with equality comparisons by various inequality indices including coefficient of variation, Gini coefficient, Atkinson index, and Theil index (see Fields and Fei (1978) and Atkinson (1970)). To see why the congressional district profile is more equal than any symmetric profile, recall equations (4) and (5) in the proof of Theorem 3, which assert that when the number of groups is large, the

[11]Formally, if H is a distribution of success probabilities among individuals, the *Lorenz curve* of H is the graph of the function $\int_0^{H^{-1}(p)} \pi dH(\pi) / \int_0^1 \pi dH(\pi)$, $0 \leq p \leq 1$, where we define $H^{-1}(p) = \sup\{\pi : H(\pi) \leq p\}$. A distribution *Lorenz dominates* another if the Lorenz curve of the former lies above that of the latter.

normalized success probability for each member of group i is approximately $A^{\phi}w_i$ for the symmetric profile, and it is approximately $Bw_i + C$ for the congressional district profile. The constant term $C > 0$ for the congressional district profile means equal additions to all individuals' probabilities of success, which results in a more equal distribution than when there is no such term. More precisely, we can prove the following statement. The proof is relegated to the Appendix.

Theorem 4. *Fix n as the number of groups. Under Assumptions 1-5, let us consider the distributions of success probabilities among individuals under the congressional district profile and any symmetric profile ϕ, in which each member of group i receives $\pi_i(\phi^{\mathrm{CD}}; n)$ and $\pi_i(\phi; n)$, respectively. For sufficiently large n, the distribution under the congressional district profile Lorenz dominates the distribution under the symmetric profile.*

4.4 Computational Results

The results in the previous subsection concern cases with a large number of groups. The question remains as to whether the conclusions obtained there are also valid for a finite number of groups. In this section, we study this question by numerically analyzing a model of the US presidential election.

There are 50 states and one federal district. The weight w_i for state i is the number of electoral votes currently assigned to it. This number equals the state's total number of seats in the Senate and House of Representatives. Thus, w_i is two plus a number that is roughly proportional to the state's population. The first and second columns of Table 1 describe the distribution of weights among the states.

We assume IAC* (Impartial Anonymous Culture*): the statewide popular vote margins Θ_i are independent and uniformly distributed on $[-1, 1]$. For any profile ϕ, we can compute the probability of success for state i via the formula:

$$\tilde{\pi}_i(\phi) := \pi_i(\phi) - \frac{1}{2} = 0.5^{51} \int_{-1}^{1} \cdots \int_{-1}^{1} \theta_i 1_A(\theta_1, \cdots, \theta_{51}) d\theta_1 \cdots d\theta_{51} \qquad (6)$$

where $A = \left\{ (\theta_1, \cdots, \theta_{51}) \middle| \sum_{j=1}^{51} w_j \phi_j(\theta_j) > 0 \right\}$.[12]

We consider four distinct profiles: ϕ^{WTA}, ϕ^{PR}, ϕ^{a} with $a = 102/538$, and ϕ^{CD} with coefficient $c = 2$. As before, these are respectively the winner-take-all profile, the proportional profile, a mixed profile, and a congressional district profile. The

[12] It is easy to check that under the uniform distribution assumption, (6) is equivalent to the expression in Lemma 3 (i).

parameter $c = 2$ of the congressional district profile is the number currently used in Maine and Nebraska, namely, it corresponds to two seats assigned to each state in the Senate. The parameter $a = 102/538$ of the mixed profile is chosen so that the proportion of electoral votes allocated on the winner-take-all basis is the same for all states, and the total number of electoral votes allocated in this way is the same as in the congressional district profile.

We compute (6) under these four profiles by a Monte Carlo simulation with 10^{10} iterations. The results are summarized in Tables 1 and 2. Table 1 shows the probabilities of success ($\pi_i(\phi)$) under the respective profiles. Table 2 shows the ratios of normalized success probabilities between different profiles:

$$\frac{\tilde{\pi}_i(\phi)}{\tilde{\pi}_i(\psi)} = \frac{\pi_i(\phi) - 1/2}{\pi_i(\psi) - 1/2}.$$

If the ratio is below 1, state i prefers ψ to ϕ.

It follows from Lemma 3 (ii) that as the number n of states increases, the ratios $\tilde{\pi}_i\left(\phi^{\mathrm{WTA}}\right)/\tilde{\pi}_i\left(\phi^{\mathrm{PR}}\right)$ and $\tilde{\pi}_i\left(\phi^{a}\right)/\tilde{\pi}_i\left(\phi^{\mathrm{PR}}\right)$ converges to the respective correlations $\mathrm{Corr}[\Theta, \phi^{\mathrm{WTA}}(\Theta)] \approx 0.866$ and $\mathrm{Corr}[\Theta, \phi^{a}(\Theta)] \approx 0.989$, where the values are computed for Θ uniformly distributed on $[-1, 1]$. Table 2 indicates that for the present example with 50 states plus DC, these ratios are indeed close to the respective correlations, which suggests that convergence of the $\tilde{\pi}$-ratios is fairly quick. In particular, as expected by Theorem 2, the proportional profile Pareto dominates the winner-take-all profile in the present case. As suggested by Proposition 2, all states prefer the mixed profile ϕ^a to the winner-take-all profile, and all states except California prefer the proportional profile to ϕ^a.

The ratios $\tilde{\pi}_i(\phi^{\mathrm{CD}})/\tilde{\pi}_i(\phi^{\mathrm{PR}})$ in Table 2 are consistent with the result in Theorem 3. Small states prefer the congressional district profile to the proportional one.

In addition, the values of $\tilde{\pi}_i(\phi^{\mathrm{CD}})/\tilde{\pi}_i(\phi^{\mathrm{WTA}})$ in the table show that the winner-take-all profile is Pareto dominated by the congressional district profile, and the welfare improvement by switching to the congressional district profile is greater for small states than for large states.

Table 1: Estimated probabilities of success in the US presidential election, based on the apportionment in 2016, via Monte Carlo simulation with 10^{10} iterations. The estimated standard errors are in the range between 3.9 and 4.1×10^{-6}.

electoral votes	number of states	$\pi(\phi^{\mathrm{WTA}})$	$\pi(\phi^{\mathrm{PR}})$	$\pi(\phi^{a})$	$\pi(\phi^{\mathrm{CD}})$
3	8	0.5057	0.5066	0.5065	0.5084
4	5	0.5075	0.5089	0.5087	0.5105
5	3	0.5094	0.5111	0.5109	0.5126
6	6	0.5113	0.5133	0.5130	0.5146
7	3	0.5132	0.5155	0.5152	0.5168
8	2	0.5151	0.5177	0.5174	0.5189
9	3	0.5170	0.5199	0.5196	0.5210
10	4	0.5189	0.5222	0.5218	0.5231
11	4	0.5208	0.5244	0.5240	0.5252
12	1	0.5227	0.5266	0.5262	0.5273
13	1	0.5246	0.5288	0.5283	0.5294
14	1	0.5265	0.5311	0.5306	0.5315
15	1	0.5284	0.5333	0.5328	0.5336
16	2	0.5304	0.5356	0.5350	0.5357
18	1	0.5342	0.5401	0.5394	0.5400
20	2	0.5381	0.5446	0.5438	0.5443
29	2	0.5560	0.5652	0.5642	0.5637
38	1	0.5747	0.5864	0.5853	0.5838
55	1	0.6178	0.6307	0.6307	0.6253

Table 2: Ratios between normalized success probabilities.

electoral votes	number of states	$\frac{\tilde{\pi}(\phi^{\mathrm{WTA}})}{\tilde{\pi}(\phi^{\mathrm{PR}})}$	$\frac{\tilde{\pi}(\phi^{a})}{\tilde{\pi}(\phi^{\mathrm{PR}})}$	$\frac{\tilde{\pi}(\phi^{\mathrm{CD}})}{\tilde{\pi}(\phi^{\mathrm{PR}})}$	$\frac{\tilde{\pi}(\phi^{\mathrm{CD}})}{\tilde{\pi}(\phi^{\mathrm{WTA}})}$
3	8	0.852	0.982	1.260	1.479
4	5	0.852	0.982	1.182	1.387
5	3	0.852	0.982	1.134	1.331
6	6	0.852	0.982	1.103	1.294
7	3	0.852	0.982	1.080	1.268
8	2	0.852	0.982	1.064	1.248
9	3	0.852	0.982	1.050	1.232
10	4	0.853	0.983	1.040	1.220
11	4	0.853	0.983	1.031	1.210
12	1	0.853	0.983	1.024	1.201
13	1	0.853	0.983	1.018	1.194
14	1	0.853	0.983	1.013	1.187
15	1	0.854	0.983	1.009	1.181
16	2	0.854	0.983	1.005	1.177
18	1	0.854	0.983	0.998	1.168
20	2	0.855	0.983	0.993	1.161
29	2	0.859	0.985	0.978	1.138
38	1	0.864	0.987	0.970	1.122
55	1	0.901	1.000	0.959	1.064

5 Concluding Remarks

This paper shows that the decentralized choice of the weight allocation rule in representative voting constitutes a Prisoner's Dilemma: the winner-take-all rule is a dominant strategy for each group, whereas the Nash equilibrium is Pareto dominated. We also show that the proportional rule Pareto dominates every other symmetric profile, when the number of the groups is sufficiently large. Each group has an incentive to put its entire weight on the alternative supported by the majority of its members in order to reflect their preferences in the social decision, although it fails to efficiently aggregate the preferences of all members in the society, if the winner-take-all rule is employed by all groups.

Our model may provide explanations for the phenomena that we observe in existing collective decision making. In the United States Electoral College, the rule used by the states varied in early elections until it converged by 1832 to the winner-take-all rule, which remains dominantly employed by nearly all states since then. In many parliamentary voting situations, we often observe parties and/or factions forcing their members to align their votes in order to maximally reflect their preferences in the social decision, although some members may disagree with the party's alignment. The voting outcome obtained by the winner-take-all rule may fail to efficiently aggregate preferences, as observed in the discrepancy between the electoral result and the national popular vote winner in the US presidential elections in 2000 and 2016. Party discipline or factional voting may also cause welfare loss when each group pushes their votes maximally toward their ideological goals, failing to reflect all members' preferences in the social decision.

The Winner-Take-All Dilemma tells us that the society should call for some device different from each group's unilateral effort, in order to obtain a more socially preferable outcome. As we see in the failure of various attempts to modify or abolish the winner-take-all rule, such as the ballot initiative for an amendment to the State Constitution in Colorado in 2004, each state has no incentive to unilaterally deviate from the equilibrium. The National Popular Vote Interstate Compact is a well-suited example of a coordination device. As it comes into effect only when the number of electoral votes attains the majority, each state does not suffer from the payoff loss by unilateral (or coalitional) deviation until sufficient coordination is attained. The emergence of such an attempt is coherent with the insights obtained in this paper that the game is a Prisoner's Dilemma, and a coordination device is necessary for a Pareto improvement.

Our analysis is abstract in that we do not impose assumptions on the preferences distribution based on the observed characteristics in the real representative voting problems. Additionally, we impose an impartiality assumption in our asymptotic analysis. Obviously, our normative analysis would be best complemented by a positive analysis, which we leave for future research.

Appendix

A1 Proof of Lemma 1

We prove the lemma for group 1. Consider a fixed member m of group 1. Suppose m obtains utility 1 or -1, depending on whether the social decision is a success or failure for him. Suppose also that the group-wide margin has realized: $\Theta_1 = \theta_1$. Then $2\pi_1(\phi|\theta_1) - 1 = 1 \cdot \pi_1(\phi|\theta_1) + (-1) \cdot (1 - \pi_1(\phi|\theta_1))$ is the conditional expected utility for member m. If, in addition, member m prefers $+1$, then his conditional expected utility will be $\mathbb{P}\{S_\phi > 0|\Theta_1 = \theta_1\} - \mathbb{P}\{S_\phi < 0|\Theta_1 = \theta_1\}$; if he prefers -1 then it will be $\mathbb{P}\{S_\phi < 0|\Theta_1 = \theta_1\} - \mathbb{P}\{S_\phi > 0|\Theta_1 = \theta_1\}$.[13] So, taking the average by weighting the two cases by their probabilities (i.e., $(1 + \theta_1)/2$ and $(1 - \theta_1)/2$), the conditional expected utility for member m before his preference is realized $2\pi_1(\phi|\theta_1) - 1 = \theta_1(\mathbb{P}\{S_\phi > 0|\Theta_1 = \theta_1\} - \mathbb{P}\{S_\phi < 0|\Theta_1 = \theta_1\})$, which is the formula stated in the lemma. □

A2 Proof of Lemma 2

Preliminaries. In this proof, we denote by ϕ^λ the generalized proportional profile with coefficients $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$:

$$\phi_i^\lambda(\theta_i) = \lambda_i \theta_i,\ i = 1, \cdots, n,$$

which should not be confused with the notation ϕ^a for mixed profiles.

We write $\tilde{\pi}_i(\phi) = 2\pi_i(\phi) - 1$ for the normalized payoff (i.e., probability of success) for group i, and $\tilde{\pi}(\phi) = (\tilde{\pi}_i(\phi))_{i=1}^n$ for the vector of normalized payoffs.

[13]Note that by the tie-breaking rule, the case with $S_\phi = 0$ has zero expected utility. Note also that the probabilities appearing in the formulas are conditional only on the group-wide margin Θ_1, and not on member m's preference, since Assumption 1 implies that once the group-wide margin (i.e., *how many* members of group 1 prefer $+1$) is known, any additional information about the preference profile of group 1 (i.e., *which* members of group 1 prefer $+1$) is irrelevant to the distribution of S_ϕ.

Let Π be the set of all possible normalized payoff vectors in game Γ:

$$\Pi = \{\tilde{\pi}(\phi) : \phi \text{ is a profile}\}.$$

For any $X \subset \mathbb{R}^n$, let Pareto$\,(X)$ be the Pareto frontier of X, i.e., the set of points $x \in X$ for which there exists no $y \in X$ such that $y_i \geq x_i$ for all i, with strict inequality for at least one i. Let $\mathrm{co}\, X$ denote the closed convex hull of X.

We will refer to the following maximization problem (M_q) parametrized by vector $q \in \mathbb{R}_+ \setminus \{0\}$:

Problem $\mathbf{M}_q$: $\max\limits_{x \in \mathrm{co}\,\Pi} q \cdot x.$

Note that the maximization is *not* directly with respect to profile ϕ. Moreover, Π may be non-closed or non-convex. Thus, there may be a solution $x \in \mathrm{co}\,\Pi$ that is not the normalized payoff vector of any profile ϕ, although we will later disprove this possibility.

We divide the proof of Lemma 2 into several claims. Claims 2.1-2.4 concern properties of the solutions to Problem M_q. Claim 2.5 describes the relation between Problem M_q and Pareto efficiency. Finally, Claim 2.6 completes the proof.

Claim 2.1. *A solution of Problem M_q is $x = \tilde{\pi}(\phi^{\lambda^q})$, where $\lambda_i^q = cq_i/w_i$, $i = 1, \cdots, n$.*[14]

Proof of Claim 2.1. Rewrite the formula in Lemma 1 as

$$\tilde{\pi}_i(\phi|\theta_i) = \theta_i \mathbb{E}(\operatorname{sgn} S_\phi | \Theta_i = \theta_i).$$

Integrating this with respect to θ_i gives

$$\tilde{\pi}_i(\phi) = \mathbb{E}\left(\Theta_i \operatorname{sgn} S_\phi\right). \tag{7}$$

Thus, for any profile ϕ,

$$q \cdot \tilde{\pi}(\phi) = \mathbb{E}\left[(q \cdot \Theta)\left(\operatorname{sgn} S_\phi\right)\right] \leq \mathbb{E}(|q \cdot \Theta|). \tag{8}$$

That is, $q \cdot x \leq \mathbb{E}(|q \cdot \Theta|)$ for any $x \in \Pi$. The linearity of the objective function $q \cdot x$ implies that $q \cdot x \leq \mathbb{E}(|q \cdot \Theta|)$ for all $x \in \mathrm{co}\,\Pi$. If $\phi = \phi^{\lambda^q}$, then S_ϕ has the same sign as $q \cdot \Theta$. Thus for $x = \tilde{\pi}(\phi^{\lambda^q})$, we have $q \cdot x = \mathbb{E}(|q \cdot \Theta|)$. $\square$

[14] $c > 0$ is a constant such that $cq_i/w_i \leq 1$ for all i.

Claim 2.2. *Let* $v(q) := \max_{x \in \operatorname{co} \Pi} q \cdot x$ *be the maximum value of Problem* M_q*. Then*

$$v(q) = \mathbb{E}(|q \cdot \Theta|).$$

Proof of Claim 2.2. This follows from the proof of Claim 2.1, in which we showed that $q \cdot x \leq \mathbb{E}(|q \cdot \Theta|)$ for all $x \in \operatorname{co} \Pi$ and the upper bound is attained by $x = \tilde{\pi}(\phi^{\lambda^q})$. □

Claim 2.3. *A profile* ϕ *satisfies* $q \cdot \tilde{\pi}(\phi) = v(q)$ *if and only if* ϕ *is equivalent to* ϕ^{λ^q}*.*

Proof of Claim 2.3. Since $q \neq 0$ and Θ is absolutely continuous, we have $q \cdot \Theta \neq 0$ almost surely. Thus, (8) holds with equality if and only if

$$\operatorname{sgn} S_\phi = \operatorname{sgn}(q \cdot \Theta) \text{ almost surely.}$$

Since $cq \cdot \Theta = S_{\phi^{\lambda^q}}$, this holds if and only if ϕ is equivalent to ϕ^{λ^q}. □

Claim 2.4. $x = \tilde{\pi}(\phi^{\lambda^q})$ *is the unique solution of Problem* M_q*.*

Proof of Claim 2.4. We use the absolute continuity of Θ to show that the value function $v(q) = \mathbb{E}(|q \cdot \Theta|)$ is differentiable, with gradient $\nabla v(q) = \tilde{\pi}(\phi^{\lambda^q})$. Then the uniqueness follows by the Duality Theorem (Mas-Colell et al. (1995, Proposition 3.F.1)).

To show that $v(q)$ is differentiable, it suffices to show that as vector $\varepsilon \in \mathbb{R}^n$ approaches 0,

$$v(q+\varepsilon) - v(q) - \tilde{\pi}(\phi^{\lambda^q}) \cdot \varepsilon = o(\|\varepsilon\|). \tag{9}$$

Using (7) and Claim 2.2, we can rewrite the left-hand side of (9) as:

$$\begin{aligned}
&v(q+\varepsilon) - v(q) - \tilde{\pi}(\phi^{\lambda^q}) \cdot \varepsilon \\
&= \mathbb{E}\left[\{(q+\varepsilon) \cdot \Theta\} \times \operatorname{sgn}\{(q+\varepsilon) \cdot \Theta\}\right] \\
&\quad - \mathbb{E}\left[(q \cdot \Theta) \times \operatorname{sgn}(q \cdot \Theta)\right] \\
&\quad - \mathbb{E}\left[(\varepsilon \cdot \Theta) \times \operatorname{sgn}(q \cdot \Theta)\right] \\
&= \mathbb{E}[\{(q+\varepsilon) \cdot \Theta\} \times \{\operatorname{sgn}((q+\varepsilon) \cdot \Theta) - \operatorname{sgn}(q \cdot \Theta)\}].
\end{aligned}$$

This expression has the following bound:

$$|v(q+\varepsilon) - v(q) - \tilde{\pi}(\phi^{\lambda^q}) \cdot \varepsilon| \leq 2\mathbb{E}\left(|(q+\varepsilon) \cdot \Theta| \, 1_{\{\operatorname{sgn}((q+\varepsilon) \cdot \Theta) \neq \operatorname{sgn}(q \cdot \Theta)\}}\right).$$

The expectation on the right-hand side is

$$\int_{A^+_{q,\varepsilon}} \{(q+\varepsilon)\cdot\theta\}h(\theta)d\theta - \int_{A^-_{q,\varepsilon}} \{(q+\varepsilon)\cdot\theta\}h(\theta)d\theta, \tag{10}$$

where h is the joint density of Θ and

$$A^+_{q,\varepsilon} = \{\theta \in [-1,1]^n : (q+\varepsilon)\cdot\theta \geq 0 \geq q\cdot\theta\},$$
$$A^-_{q,\varepsilon} = \{\theta \in [-1,1]^n : (q+\varepsilon)\cdot\theta \leq 0 \leq q\cdot\theta\}.$$

We show that for ε sufficiently close to 0, $(q+\varepsilon)\cdot\theta \leq \sqrt{n}\|\varepsilon\|$ for all $\theta \in A^+_{q,\varepsilon}$. To do this, we fix a sufficiently small ε so that for each $e \in \{-1,1\}^n$ (i.e., each vertex of the hypercube $[-1,1]^n$), either both $q\cdot e$ and $(q+\varepsilon)\cdot e$ are non-negative or both are non-positive.[15] Now, consider the following linear-programming problem ($\mathrm{L}_{q,\varepsilon}$):

Problem $\mathbf{L}_{q,\varepsilon}$: $\max\limits_{\theta\in A^+_{q,\varepsilon}} (q+\varepsilon)\cdot\theta.$

Let θ^* be a solution of Problem $\mathrm{L}_{q,\varepsilon}$ that is a vertex of $A^+_{q,\varepsilon}$. Then θ^* belongs to at least one of the following sets:

$$H_{q+\varepsilon} = \{\theta : (q+\varepsilon)\cdot\theta = 0\},$$
$$H_q = \{\theta : q\cdot\theta = 0\},$$
$$\{-1,1\}^n.$$

We claim that $\theta^* \in H_q$. First, we have $\theta^* \notin H_{q+\varepsilon}$, since otherwise θ^* minimizes the objective function $(q+\varepsilon)\cdot\theta$ subject to $\theta \in A^+_{q,\varepsilon}$, while the n-dimensional polytope $A^+_{q,\varepsilon}$ contains points that attain larger values of the function. Now, suppose $\theta^* \in \{-1,1\}^n \setminus H_q$. The fact that $\theta^* \in \{-1,1\}^n \cap A^+_{q,\varepsilon} \cap H^c_q \cap H^c_{q+\varepsilon}$ implies that θ^* is a vertex of the hypercube $[-1,1]^n$ such that $q\cdot\theta^* < 0 < (q+\varepsilon)\cdot\theta^*$. This contradicts the fact that for any vertex e of the hypercube, either both $(q+\varepsilon)\cdot e$ and $q\cdot e$ are non-negative or both are non-positive. Therefore $\theta^* \in H_q$.

We have shown that $q\cdot\theta^* = 0$. This implies that for any $\theta \in A^+_{q,\varepsilon}$, $(q+\varepsilon)\cdot\theta \leq (q+\varepsilon)\cdot\theta^* = \varepsilon\cdot\theta^* \leq \|\theta^*\|\|\varepsilon\| \leq \sqrt{n}\|\varepsilon\|$. It similarly follows that $-(q+\varepsilon)\cdot\theta \leq \sqrt{n}\|\varepsilon\|$ for any $\theta \in A^-_{q,\varepsilon}$. Therefore, (10) is bounded by $\sqrt{n}\|\varepsilon\| \int_{A^+_{q,\varepsilon}\cup A^-_{q,\varepsilon}} h(\theta)d\theta$. Noting that the integral $\int_{A^+_{q,\varepsilon}\cup A^-_{q,\varepsilon}} h(\theta)d\theta$ vanishes as $\varepsilon \to 0$, we obtain (9). $\square$

[15] For each vertex $e \in \{-1,1\}^n$ there is $\delta_e > 0$ such that if $\|\varepsilon\| < \delta_e$ then either both $q\cdot e$ and $(q+\varepsilon)\cdot e$ are non-negative or both are non-positive. Thus, it suffices to choose ε so that $\|\varepsilon\| < \min\{\delta_e : e \in \{-1,1\}^n\}$.

Claim 2.5. *Let $x \in \mathrm{co}\,\Pi$. Then, $x \in \mathrm{Pareto}\,(\mathrm{co}\,\Pi)$ if and only if there exists $q \in \mathbb{R}^n_+ \setminus \{0\}$ such that x is the unique solution of Problem M_q, i.e., $x = \tilde{\pi}(\phi^{\lambda^q})$.*

Proof Claim 2.5. For any $x \in \mathbb{R}^n$, let $D(x) = \{x + a : a \in \mathbb{R}^n_+ \setminus \{0\}\}$ be the (convex) set of all points that dominate x. Note that $x \in \mathrm{Pareto}\,(\mathrm{co}\,\Pi)$ if and only if $D(x) \cap \mathrm{co}\,\Pi = \emptyset$. To prove Claim 2.5, suppose $x \in \mathrm{Pareto}\,(\mathrm{co}\,\Pi)$. Then there exists a hyperplane with some normal vector $q \in \mathbb{R}^n_+ \setminus \{0\}$ that separates $\mathrm{co}\,\Pi$ and $D(x)$.[16] Clearly this hyperplane contains x, which means that x is the solution of Problem M_q. Conversely, suppose x is the unique solution of Problem M_q. Then the supporting hyperplane of $\mathrm{co}\,\Pi$ with normal vector q separates $\mathrm{co}\,\Pi$ and $D(x)$. The uniqueness of the solution implies that the hyperplane intersects $\mathrm{co}\,\Pi$ only at x. This implies that $D(x) \cap \mathrm{co}\,\Pi = \emptyset$. □

Claim 2.6. *A profile ϕ satisfies $\tilde{\pi}(\phi) \in \mathrm{Pareto}\,(\Pi)$ if and only if there exists $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ such that ϕ is equivalent to ϕ^λ. That is, Lemma 2 holds.*

Proof of Claim 2.6. By Claim 2.5,

$$\mathrm{Pareto}\,(\mathrm{co}\,\Pi) = \mathrm{Pareto}\,(\Pi) = \{\tilde{\pi}(\phi^{\lambda^q}) : q \in \mathbb{R}^n_+ \setminus \{0\}\}.$$

By Claim 2.3, $\tilde{\pi}(\phi)$ belongs to this set if and only if ϕ is equivalent to ϕ^{λ^q} for some $q \in \mathbb{R}^n_+ \setminus \{0\}$. This condition is the same as saying that ϕ is equivalent to ϕ^λ for some $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$. □

A3 Proof of Part (i) of Lemma 3

We prove the statement for group 1. Let $\pi_1(\phi; n|\theta)$ be the conditional probability of success for group 1 given that the group-wide margin is $\Theta_1 = \theta_1$. We may apply the formula appeared in Lemma 1. By independence, the formula becomes

$$\pi_1(\phi; n|\theta_1) - \frac{1}{2} = \frac{\theta_1}{2}(\mathbb{P}\{w_1\phi(\theta_1) + S_{\phi_{-1}} > 0\} - \mathbb{P}\{w_1\phi(\theta_1) + S_{\phi_{-1}} < 0\}).$$

Since $S_{\phi_{-1}}$ is symmetrically distributed, the second probability can be written as $\mathbb{P}\{-w_1\phi(\theta_1) + S_{\phi_{-1}} > 0\}$. Thus, for $\theta_1 \in [0, 1]$, the above expression equals

$$\pi_1(\phi; n|\theta_1) - \frac{1}{2} = \frac{\theta_1}{2}\mathbb{P}\{-w_1\phi(\theta_1) < S_{\phi_{-1}} \leq w_1\phi(\theta_1)\}.$$

[16] Here, separation is in the weak sense that the hyperplane may contain boundary points of the two sets.

By symmetry, twice the integral of this expression over $\theta_1 \in [0,1]$ (instead of $[-1,1]$) equals the unconditional probability $\pi_1(\phi;n) - 1/2$, which proves part (i) of Lemma 3. $\square$

A4 Local Limit Theorem

We quote a version of the Local Limit Theorem shown in Mineka and Silverman (1970). We will use it in the proof of part (ii) of Lemma 3.

LLT. *(Mineka and Silverman (1970, Theorem 1)) Let (X_i) be a sequence of independent random variables with mean 0 and variances $0 < \sigma_i^2 < \infty$. Write F_i for the distribution of X_i. Write also $S_n = \sum_{i=1}^n X_i$ and $s_n^2 = \sum_{i=1}^n \sigma_i^2$. Suppose the sequence (X_i) satisfies the following conditions:*

(α) There exists $\bar{x} > 0$ and $c > 0$ such that for all i,

$$\frac{1}{\sigma_i^2} \int_{|x|<\bar{x}} x^2 dF_i(x) > c.$$

(β) Let (a_i) be any bounded sequence of numbers such that $\inf_i \mathbb{P}\{|X_i - a_i| < \delta\} > 0$ for all $\delta > 0$.[17] *Define the set*

$$A(t,\varepsilon) = \{x : |x| < \bar{x} \text{ and } |xt - \pi m| > \varepsilon \text{ for all integer } m \text{ with } |m| < \bar{x}\}.$$

Then for each $t \neq 0$, there exists $\varepsilon > 0$ such that

$$\frac{1}{\log s_n} \sum_{i=1}^n \mathbb{P}\{X_i - a_i \in A(t,\varepsilon)\} \to \infty.$$

(γ) (Lindeberg's condition.) For any $\varepsilon > 0$,

$$\frac{1}{s_n^2} \sum_{i=1}^n \int_{|x|/s_n > \varepsilon} x^2 dF_i(x) \to 0.$$

Under conditions (α)-(γ), if $s_n^2 \to \infty$, we have

$$\sqrt{2\pi s_n^2}\,\mathbb{P}\{S_n \in (a,b]\} \to b - a.^{18} \tag{11}$$

[17] Such a sequence (a_i) exists under assumption (α).

[18] The original conclusion of Theorem 1 in Mineka and Silverman (1970) is stated in terms of the open interval (a,b). Applying the theorem to $(a, b+c)$ and $(b, b+c)$ and then taking the difference gives the result for $(a,b]$. In addition, the original statement allows for cases where

A5 Proof of Lemma 4

Preliminaries. We prove the lemma for group 1. In the proof, we use the notation of LLT. Let

$$X_i := w_i\phi(\Theta_i, w_i),\ i = 1, 2, \cdots ,$$

and $S_n := \sum_{i=1}^n X_i$.[19] Then X_i has mean 0 and variance $\sigma_i^2 := w_i^2\mathbb{E}[\phi(\Theta, w_i)^2]$, and so the partial sum of variances is $s_n^2 := \sum_{i=1}^n w_i^2\mathbb{E}[\phi(\Theta, w_i)^2]$.[20]

Define the event

$$\Omega_n(\theta_1, w_1) = \left\{ -w_1\phi(\theta_1, w_1) < \sum_{i=2}^n X_i \leq w_1\phi(\theta_1, w_1) \right\}.$$

We divide the proof into several claims. Claims 5.1-5.3 show that the sequence (X_i) defined above satisfies the conditions of the Local Limit Theorem (LLT) in Section A4. Claim 5.4 applies LLT to complete the proof of Lemma 4.

Claim 5.1. $\frac{s_n^2}{n} \to \int_0^{\bar{w}} w^2\mathbb{E}[\phi(\Theta, w)^2]dG(w)$.

Proof of Claim 5.1. This holds since sequence (σ_i^2) is bounded and the statistical distribution G_n induced by $(w_i)_{i=1}^n$ converges weakly to G. □

Claim 5.2. *Conditions (α) and (γ) in LLT hold.*

Proof of Claim 5.2. This immediately follows from the fact that sequence (X_i) is bounded and $s_n^2 \to \infty$. □

Claim 5.3. *Condition (β) in LLT holds.*

Proof of Claim 5.3. Fix $t \neq 0$. Recall that the support of G contains an open interval O. Hence G is strictly increasing on O. Recall also that set $A(t, \varepsilon)$ is the union of open intervals of the form

$$A_m(t, \varepsilon) = \left(\frac{\pi m + \varepsilon}{|t|}, \frac{\pi(m+1) - \varepsilon}{|t|} \right),$$

s_n^2 does not go to infinity, and also mentions uniform convergence. These considerations are not necessary for our purpose, so we omit them.

[19] Whether the sum includes the first term X_1 or not does not matter in the limit.

[20] Recall that in Lemma 4, Θ represents a random variable that has the same distribution (F) as each Θ_i.

for integers $m = \pm 1, \pm 2, \cdots$.[21]

First suppose ϕ is a symmetric profile. Let $r > 0$ be such that $\pm r$ are in the support of $\phi(\Theta)$.[22] Define $a_i = -rw_i$ for each i. Then

$$\mathbb{P}\{|X_i - a_i| < \delta\} = \mathbb{P}\{w_i|\phi(\Theta) - (-r)| < \delta\} \geq \mathbb{P}\{|\phi(\Theta) - (-r)| < \delta/\bar{w}\} > 0$$

for all i and $\delta > 0$, and sequence (a_i) is bounded since sequence (w_i) is bounded. Thus (a_i) satisfies the requirement in condition (β).

For any sufficiently small $\varepsilon > 0$ and an appropriate subinterval $(\underline{v}, \bar{v}) \subset O$, there is an integer m such that $((2r - \varepsilon)\underline{v}, (2r + \varepsilon)\bar{v}) \subset A_m(t, \varepsilon)$, and so

$$((2r - \varepsilon)\underline{v}, (2r + \varepsilon)\bar{v}) \subset A(t, \varepsilon).$$

Fix such $\varepsilon > 0$ and $(\underline{v}, \bar{v}) \subset O$, so that $\underline{v}$ and $\bar{v}$ are points of continuity of G. Define

$$I := \{i : w_i \in (\underline{v}, \bar{v})\}.$$

Since r belongs to the support of $\phi(\Theta)$, we have

$$p := \mathbb{P}\{\phi(\Theta) \in (r - \varepsilon, r + \varepsilon)\} > 0.$$

Note that if $w_i \in (\underline{v}, \bar{v})$ and $\phi(\theta_i) \in (r - \varepsilon, r + \varepsilon)$, then $(\phi(\theta_i) + r)w_i \in ((2r - \varepsilon)\underline{v}, (2r + \varepsilon)\bar{v}) \subset A(t, \varepsilon)$. Thus, for all $i \in I$,

$$\mathbb{P}\{X_i - a_i \in A(t, \varepsilon)\} = \mathbb{P}\{(\phi(\Theta) + r)w_i \in A(t, \varepsilon)\} \geq p > 0.$$

Therefore,

$$\frac{1}{\log s_n} \sum_{i \leq n} \mathbb{P}\{X_i - a_i \in A(t, \varepsilon)\} \geq \frac{n}{\log s_n} \cdot \frac{1}{n} \cdot \#\{i \leq n : i \in I\} \cdot p.$$

The right-hand side goes to infinity as $n \to \infty$, since $\frac{1}{n}\#\{i \leq n : i \in I\} \to G(\bar{v}) - G(\underline{v}) > 0$ and s_n has asymptotic order of $\sqrt{n}$. Thus condition (β) is satisfied.

Next suppose ϕ is the congressional district profile ϕ^{CD}. In this case, $X_i =$

[21] In the present case, the upper bound $\bar{x}$ appearing in the definition of $A(t, \varepsilon)$ does not matter: one can take it to be arbitrarily large without violating conditions (α) and (γ).

[22] This is possible since $\phi(\Theta)$ is symmetrically distributed, and since we may exclude the trivial case in which $\phi(\Theta) = 0$ almost surely.

$c \operatorname{sgn} \Theta_i + (w_i - c)\Theta_i$. Define sequence (a_i) by letting $a_i = c$ for all i. Then

$$\mathbb{P}\{|X_i - a_i| < \delta\} \geq \mathbb{P}\{(w_i - c)\Theta \in (0, \delta)\} \geq \mathbb{P}\{\Theta \in (0, \delta/(\bar{w} - c))\} > 0$$

for all i and $\delta > 0$, where the lower bound is positive since Θ has full support. Thus sequence (a_i) passes the requirement in condition (β).

Let $\varepsilon > 0$ be sufficiently small. Then there is an interval $(\underline{\theta}, \bar{\theta}) \subset [0, 1]$ such that for all $w \in O$,

$$((w - c)\underline{\theta}, (w - c)\bar{\theta}) \subset A_0(t, \varepsilon) \subset A(t, \varepsilon).$$

Let $p := \mathbb{P}\{\Theta \in (\underline{\theta}, \bar{\theta})\} > 0$ and $I := \{i : w_i \in O\}$. Note that if $w_i \in O$ and $\theta_i \in (\underline{\theta}, \bar{\theta})$, then $(w_i - c)\theta_i \in A(t, \varepsilon)$. Thus, for any $i \in I$,

$$\mathbb{P}\{X_i - a_i \in A(t, \varepsilon)\} = \mathbb{P}\{(w_i - c)\Theta \in A(t, \varepsilon)\} \geq p > 0.$$

The rest of the argument is the same as in the previous paragraph, and so omitted. Thus (β) holds when $\phi = \phi^{\mathrm{CD}}$ as well. $\square$

Claim 5.4. *As $n \to \infty$, uniformly in $w_1 \in [0, \bar{w}]$,*

$$\int_0^1 \theta_1 \sqrt{2\pi n}\, \mathbb{P}\{\Omega_n(\theta_1, w_1)\} dF(\theta_1) \to \frac{w_1 \mathbb{E}[\Theta \phi(\Theta, w_1)]}{\sqrt{\int_0^{\bar{w}} w^2 \mathbb{E}[\phi(\Theta, w)^2] dG(w)}}. \tag{12}$$

By part (i) of Lemma 3,[23] *the left-hand side of (12) is $\sqrt{2\pi n}\left(\pi_i(\phi; n) - \frac{1}{2}\right)$, and therefore Lemma 4 holds.*

Proof of Claim 5.4. By Claims 5.2 and 5.3, we may apply LLT to obtain

$$\sqrt{2\pi s_n^2}\, \mathbb{P}\{\Omega_n(\theta_1, w_1)\} \to 2w_1 \phi(\theta_1, w_1).$$

By Claim 5.1, this means that

$$\sqrt{2\pi n}\, \theta_1 \mathbb{P}\{\Omega_n(\theta_1, w_1)\} \to \frac{2w_1 \theta_1 \phi(\theta_1, w_1)}{\sqrt{\int_0^{\bar{w}} w^2 \mathbb{E}[\phi(\Theta, w)^2] dG(w)}}. \tag{13}$$

Letting $\theta_1 = 1$ maximizes the left-hand side of (13) with the maximum value $\sqrt{2\pi n}\, \mathbb{P}\{\Omega_n(1, w_1)\}$. This maximum itself converges to a finite limit. Hence the

[23] It is easy to check that part (i) of Lemma 3 holds for rules $\phi(\cdot, w_i)$ that depend on weight w_i as well.

expression $\sqrt{2\pi n}\theta_1\mathbb{P}\{\Omega_n(\theta_1, w_1)\}$ is uniformly bounded for all n and $\theta_1 \in [0,1]$. By the Bounded Convergence Theorem,

$$\int_0^1 \theta_1\sqrt{2\pi n}\mathbb{P}\{\Omega_n(\theta_1, w_1)\}dF(\theta_1) \to \frac{2w_1\int_0^1 \theta_1\phi(\theta_1, w_1)dF(\theta_1)}{\sqrt{\int_0^{\bar{w}} w^2\mathbb{E}[\phi(\Theta, w)^2]dG(w)}}.$$

Since F is symmetric and ϕ is odd, this limit is exactly the one in (12).

To check the uniform convergence, note that for each n, the integral on the left-hand side of (12) is non-decreasing in w_1, since event $\Omega_n(\theta_1, w_1)$ weakly expands as w_1 increases.[24] We have shown that this integral converges pointwise to a limit that is proportional to the factor $w_1\mathbb{E}[\Theta\phi(\Theta, w_1)]$, which is continuous in w_1.[25] Therefore, the convergence in (12) is uniform in $w_1 \in [0, \bar{w}]$.[26] □

A6 Proof of Part (ii) of Lemma 3

This follows immediately from Lemma 4, by noting that if ϕ is a symmetric profile, each group's rule can be written as $\phi(\theta_j, w_j) = \phi(\theta_j)$. □

A7 Proof of Proposition 2

By part (ii) of Lemma 3, we must show that $\mathrm{Corr}[\Theta, \phi^a(\Theta)]$ is decreasing in $a \in [0,1]$. By simple calculation,

$$\mathbb{E}(\Theta^2)\cdot\mathrm{Corr}[\Theta, \phi^a(\Theta)]^2 = \frac{a\mathbb{E}(|\Theta|) + (1-a)\mathbb{E}(\Theta^2)}{a^2 + 2a(1-a)\mathbb{E}(|\Theta|) + (1-a)^2\mathbb{E}(\Theta^2)}.$$

The derivative of this expression with respect to a has the same sign as

$$\begin{aligned}
&\left\{\tfrac{d}{da}(a\mathbb{E}(|\Theta|) + (1-a)\mathbb{E}(\Theta^2))^2\right\}\Big(a^2 + 2a(1-a)\mathbb{E}(|\Theta|) + (1-a)^2\mathbb{E}(\Theta^2)\Big)\\
&- \Big(a\mathbb{E}(|\Theta|) + (1-a)\mathbb{E}(\Theta^2)\Big)^2\left\{\tfrac{d}{da}(a^2 + 2a(1-a)\mathbb{E}(|\Theta|) + (1-a)^2\mathbb{E}(\Theta^2))\right\}\\
&= a(a\mathbb{E}(|\Theta|) + (1-a)\mathbb{E}(\Theta^2))(\mathbb{E}(|\Theta|)^2 - \mathbb{E}(\Theta^2)).
\end{aligned}$$

[24] Let $\theta_1 \in [0,1]$. If ϕ is a symmetric profile, i.e. if $\phi(\theta_1, w_1) = \phi(\theta_1)$, then $w_1\phi(\theta_1)$ is non-decreasing in w_1. If $\phi = \phi^{\mathrm{CD}}$, then $w_1\phi^{\mathrm{CD}}(\theta_1, w_1) = c\,\mathrm{sgn}(\theta_1) + (w_1 - c)\theta_1$, which is non-decreasing in w_1 again. Thus event $\Omega_n(\theta_1, w_1)$ weakly expands as w_1 increases.

[25] If ϕ is a symmetric profile, this factor is linear in w_i. If $\phi = \phi^{\mathrm{CD}}$, the factor equals $c\mathbb{E}(|\Theta|) + (w_i - c)\mathbb{E}(\Theta^2)$, which is affine in w_i.

[26] It is known that if (f_n) is a sequence of non-decreasing functions on a fixed finite interval and f_n converges pointwise to a continuous function, then the convergence is uniform. See Buchanan and Hildebrandt (1908).

This is negative for any $a \in (0,1]$, since $\mathbb{E}(|\Theta|)^2 \leq \mathbb{E}(\Theta^2)$ in general, and the full-support assumption implies that this holds with strict inequality. □

A8 Proof of Theorem 4

Clearly, Lorenz dominance is invariant to affine transformations of success probabilities (see Moyes (1989)). Thus, it suffices to prove that for large enough n, the distribution in which each member of group i receives the amount $\sqrt{2\pi n}(\pi_i(\phi^{\mathrm{CD}}; n) - 1/2)$ Lorenz dominates the distribution in which the corresponding amount is $\sqrt{2\pi n}(\pi_i(\phi; n) - 1/2)$. By equations (4) and (5) in the proof of Theorem 3, as $n \to \infty$ these amounts converge to $Bw_i + C$ and $A^\phi w_i$, respectively, where $C > 0$ is a constant. A result by Moyes (1994, Proposition 2.3) implies that if f and g are continuous, nondecreasing, and positive-valued functions such that $f(w_i)/g(w_i)$ is decreasing in w_i, then the distribution of $f(w_i)$ Lorenz dominates that of $g(w_i)$. The ratio $(Bw_i + C)/(A^\phi w_i)$ is decreasing in w_i, and so the claimed Lorenz dominance holds in the limit as $n \to \infty$. Recalling that the convergences are uniform, the dominance holds for sufficiently large n. □

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