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THE WINNER-TAKE-ALL DILEMMA

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Abstract

This paper considers collective decision-making when individuals are partitioned into groups (e.g., states or parties) endowed with voting weights. We study a game in which each group chooses an internal rule that specifies the allocation of its weight to the alternatives as a function of its members' preferences. We show that under quite general conditions, the game is a Prisoner's Dilemma: while the winner-take-all rule is a dominant strategy, the equilibrium is Pareto dominated. We also show asymptotic Pareto dominance of the proportional rule. Our numerical computation for the US Electoral College verifies the sensibility of the asymptotic results.

JEL classification: C72, D70, D72

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1 Introduction

A fundamental question about representative democracy is how social decisions should reflect the opinions of individuals belonging to distinct groups, such as states or parties. Typically, each group has a voting weight, in the form of a number of representatives or a weighted vote assigned to a unique representative. The groups allocate the weights to decision alternatives, and the one that receives the most weight becomes the social decision. In such cases, the quality of social decision-making depends not only on the apportionment of weights among the groups, but also on the rules that allocate the groups' weights to alternatives, based on the preferences of their individual members. The present paper is concerned with how the weight allocation rules affect individuals' welfare.

Existing institutions use different weight allocation rules. On the one hand, the *winner-take-all rule* devotes all the weight of a group to the alternative preferred by the majority of its members. Most states in the Untied States use this rule to allocate presidential electoral votes. A council of national ministers, each with a weighted vote (e.g., the Council of the European Union), is another example, provided the ministers can be thought of as representing their countries' majority interests. Party discipline frequently observed in legislative voting may also be understood as the winner-take-all rule used by parties.

On the other hand, the *proportional rule* allocates a group's weight in proportion to the number of members who prefer the respective alternatives. In many parliamentary institutions at the national or international level, each constituency (e.g., state or prefecture) elects a set of representatives whose composition more or less proportionally reflects its residents' preferences. Alternatively, when the representatives are viewed as standing for parties rather than states or prefectures, the proportional rule corresponds to a party's rule that allows its representatives to vote for or against proposals based on their own preferences, provided the composition of the party's representatives proportionally reflects the opinions of all party members.

The weight allocation rules are often exogenously given to all groups, but there are also cases where each group chooses its own rule. For instance, in national parliaments, how the representatives are elected from the respective constituencies is stipulated by national law. By contrast, parties often have control over how their representatives vote, by punishing those who violate the party lines. As another example, the US Constitution stipulates that it is up to each state to decide the way in which the presidential electoral votes are allocated (Article II, Section 1, Clause 2).

If groups are allowed to choose their rules, it is possible that each group has an incentive to allocate the weight so as to increase the influence of its members' opinions on social decisions, at the cost of the other groups' influence. It is not clear whether such an incentive at the group level is compatible with desirable properties of the overall preference aggregation, such as efficiency. To address this issue, we need to model the choice of rules as a non-cooperative game.

In this paper, we consider a model of social decision-making where individuals are partitioned into groups endowed with voting weights. The society makes a binary decision through two stages: first, all individuals vote; then each group allocates its weight to the alternatives, based on the number of votes they received from the group's individual members. The winner is the alternative with the most weight. A *rule* for a group is a function that maps each possible vote result in the group to an allocation of its weight to the alternatives. Examples are the winner-take-all and proportional rules. A *profile* is a specification of rules for all groups. We study the game in which the groups independently choose their rules, so as to maximize their members' expected welfare.

The main result of this paper is that the game is a n-player *Prisoner's Dilemma* (Theorem 1). On the one hand, the winner-take-all rule is a *dominant strategy*, i.e., it is an optimal strategy for each group regardless of the rules chosen by the other groups. On the other hand, if each group has less than a half of the total weight, then the winner-take-all profile is *Pareto dominated*, i.e., some other profile makes *every* group better off. In brief, no group has an incentive to deviate from the winner-take-all rule, but every group will be better off if all groups jointly move to another profile.

The dilemma structure exists for any number of groups (> 2) and with little restriction on the joint distribution of preferences. Individuals' preferences may be biased, and also correlated within and across groups, or not, which would be true when the groups are parties with different but overlapping political goals, or states that tend to support specific alternatives, e.g., blue, red or swing states in the US elections.

The observation that the winner-take-all rule is a dominant strategy is consistent with the fact that it has been dominantly employed by the states in the US Electoral College since 1830s in order to allocate presidential electoral votes,[1] and also with the widely observed party discipline in assemblies. Despite the various problems or limitations that have been pointed out concerning the winner-take-all rule,[2] it is still used prevalently.

Our conventional knowledge that direct majority voting by all individuals maximizes the *utilitarian* welfare of the society is not sufficient to see whether *every* group is better off under the proportional profile than the winner-take-all profile. We provide a counterexample later (Example 1): a small group may be strictly better off under the winner-take-all profile than the proportional profile. Indeed, this is an oft-used argument by the small states in the US, on which their support for the winner-take-all rule is based. The welfare criterion used in Theorem 1 (ii) is Pareto dominance, which is obviously stronger than the utilitarian welfare evaluation: there exists a profile under which *every* group is better off than the winner-take-all profile. Example 1 shows that it is not necessarily the proportional profile. Then, what profile Pareto dominates the winner-take-all profile?

[1] One of the recent attempts of reform by a state took place in 2004, when a ballot initiative for an amendment to the state constitution was raised in Colorado. The suggested procedure is the proportional rule, in which the state electoral votes are allocated proportionally to the state popular votes. The amendment did not pass, garnering only 34.1% approval.

[2] Some of the major arguments against the winner-take-all rule are the following. First, the winner of the election may be inconsistent with that of the popular votes (May (1948), Feix et al. (2004)). Such a discrepancy has happened five times in the history of the US presidential elections, including recently in 2000 and 2016. Second, it may cause reduced dimensionality: (i) the parties have an incentive to concentrate campaign resources only in the battleground states, and (ii) voters' incentive to turn out or to invest in information may be small and/or uneven across states, since the probability of each voter to be pivotal is so small under the winner-take-all rule, and even smaller in the non-swing states. Although campaign resource allocation and voter turnout are important issues, they are beyond the scope of this paper.

A full characterization of the Pareto set is provided in Lemma 1.

To further study welfare properties, we turn to an asymptotic and normative analysis of the model. We consider situations where the number of groups is sufficiently large, and the preferences are independent across groups and distributed symmetrically with respect to the alternatives. In this case, we show that the proportional profile *Pareto dominates* every other symmetric profile (i.e., one in which all groups use the same rule), including the winner-take-all one. The assumptions on the preference distribution abstract from the fact that in reality, some groups tend to prefer specific alternatives. Such an abstraction would be reasonable on the grounds that normative judgment about rules should not favor particular groups because of their characteristic preference biases. To see how many groups are typically sufficient for the asymptotic result, we provide numerical computations in a model based on the US Electoral College, using the current apportionment of electoral votes. The numerical comparisons indicate that the proportional profile does Pareto dominate the winner-take-all profile in the model with fifty states and a federal district.

While the above result suggests that the proportional profile asymptotically performs well in terms of efficiency, it is silent about the equality of individuals' welfare. In fact, our model also provides some insight into how rules affect the distribution of welfare. We examine an asymmetric profile called the *congressional district profile*. This profile is inspired by the Congressional District Method currently used by Maine and Nebraska, in which two electoral votes are allocated by the winner-take-all rule and the remaining ones are awarded to the winner of each district-wide popular vote.[3] We show that the congressional district profile achieves a more equal distribution of welfare than any symmetric profile by making individuals in smaller groups better off.

A technical contribution of this paper is to develop an asymptotic method for analyzing players' expected welfare in weighted voting games.

[3]The idea of allocating a part of the votes by the winner-take-all rule and allowing the rest to be awarded to potentially distinct candidates can be seen as a compromise between the winner-take-all and the proportional rules. Symbolically, the two votes allocated by the winner-take-all rule is the same number as the Senators in each state, while the rest is equal to the number of the House representatives. The idea behind such a mixture is in line with the logic supporting bicameralism, which is supposed to provide checks and balances between the states and the federal governance.

One of the major challenges in analyzing such games is their discreteness. By the nature of combinatorial problems, obtaining an analytical result often requires a large number of classifications by cases, which may include prohibitively tedious and complex tasks in order to obtain general insights. We overcome this difficulty by considering asymptotic properties of games in which there are a sufficiently large number of groups. This technique allows us to obtain an explicit formula that captures the asymptotic behavior of the probability of success for each individual, which holds for a wide class of distributions of weights among groups (the correlation lemma: Lemma 2).

1.1 Literature Review

The incentives for groups to use the winner-take-all rule have been studied by several papers. Hummel (2011) and Beisbart and Bovens (2008) analyze models of the US presidential elections. Gelman (2003) and Eguia (2011a,b) give theoretical explanations as to why voters in an assembly form parties or voting blocs to coordinate their votes. Their findings are coherent with our observation that the winner-take-all rule is a dominant strategy.

Beisbart and Bovens (2008) and Gelman (2003) also contain comparisons of the winner-take-all and proportional profiles. Under the current apportionment of electoral votes in the US, Beisbart and Bovens (2008) numerically compares these profiles, in terms of inequality indices on citizens' voting power and the mean majority deficit, on the basis of a priori and a posteriori voting power measures. Gelman (2003) compares the case with coalitions of equal sizes in which voters coordinate their votes to the case without such coordination. Our analysis is based on Pareto dominance between profiles, and provides results which hold under general distribution of groups' weights or sizes. In that sense, Beisbart and Bovens's positive analysis is complementary to our normative analysis of properties of the proportional profile.

De Mouzon et al. (2019) provides a welfare analysis of popular vote interstate compacts, and shows that, for the regional compact, welfare of the member states is single-peaked as a function of the number of the participating states, while it is monotonically decreasing for the non-member states. The second effect dominates in terms of the social welfare, unless a large majority (approximately more than $2/\pi \simeq 64\%$) of the states join the compact, implying that a small- or middle-sized regional compact is welfare detrimental. For the national compact, the total welfare is increasing, as it turns out that even the non-members would mostly benefit from the compact, implying that the social optimum is attained when a majority joins the compact, i.e. the winner is determined by the national popular vote. Their findings are coherent with ours: if the winner-take-all rule is applied only to a subset of the groups, then the member states enjoy the benefit at the expense of the welfare loss of the non-member states, and the total welfare decreases. The social optimum is attained when the entire nation uses the popular vote.

The winner-take-all rule has been a regular focus of the literature. The history, objectives, problems, and reforms of the US Electoral College are summarized, for example, in Edwards (2004) and Bugh (2010). One of the most scrutinized problems of the Electoral College is its reduced dimensionality. The incentive of the candidates to concentrate their campaign resources in the swing and decisive states is modeled in Strömberg (2008), which is coherent with the findings of the seminal paper in probabilistic voting by Lindbeck and Weibull (1987). Strömberg (2008) also finds that uneven resource allocation and unfavorable treatment of minority states would be mitigated by implementing a national popular vote, which is coherent with the classical findings by Brams and Davis (1974). Voters' incentive to turn out is investigated by Kartal (2015), which finds that the winner-take-all rule discourages turnout when the voting cost is heterogeneous.

Constitutional design of weighted voting is studied extensively in the literature. Seminal contributions are found in the context of power measurement: Penrose (1946), Shapley and Shubik (1954), Banzhaf (1968) and Rae (1946). Excellent summaries of theory and applications of power measurement are given by, above all, Felsenthal and Machover (1998) and Laruelle and Valenciano (2008). The tools and insights obtained in the power measurement literature are often used in the apportionment problem: e.g., Barberà and Jackson (2006), Koriyama et al. (2013), and Kurz et al. (2017).

2 The Model

Let us consider a society which consists of n disjoint *groups*: $i \in \{1, 2, \cdots, n\}$. The society makes a collective decision between two *alternatives*, denoted -1 and $+1$. For instance, the alternatives may represent presidential candidates in a two-party system, or the status quo and a proposal in a legislature. Each group i is endowed with a *weight* $w_i > 0$.

Let $\Theta_i \in [-1, 1]$ be the random variable which represents the *group-wide margin*, i.e., the fraction of members of i preferring alternative $+1$ minus the fraction of those preferring -1.[4] For instance, $\Theta_i = -0.2$ means that 60% of members of group i prefer alternative -1 and the remaining 40% prefer $+1$. Since the model is concerned with the weight allocation by each group which aggregates the preferences of its members, it is most appropriate to suppose that the groups' aggregation rules are fixed prior to the realization of the preferences, and hence of the group-wide margins. The following is the assumption on the joint distribution of the group-wide margins.

Assumption 1. The joint distribution of group-wide margins $(\Theta_i)_{i=1}^n$ is absolutely continuous and has full support $[-1, 1]^n$.

Assumption 1 permits a wide variety of joint distributions of individuals' preferences, in which intra- and inter-group correlations and biases are possible. First, the assumption imposes no restriction on preference correlations within each group. Second, individuals' preferences may also be correlated across groups, since the group-wide margins $(\Theta_i)_{i=1}^n$ can be correlated. This allows us to capture situations where, for instance, residents of different states or members of different parties have common interest on some issues. Third, preferences may be biased toward a particular alternative, since Θ_i can be asymmetrically distributed. For instance, blue (resp. red) states in the US might be described as groups whose group-wide margins have a distribution biased to the left (resp. right). In contrast, swing states might be described as groups whose distributions are concentrated around zero.

[4]Throughout the paper, we use capital Θ_i for representation of a random variable, and small θ_i for the realization.

The society decides between the alternatives through two stages: (i) each individual votes for his preferred alternative; (ii) each group allocates its weight to the two alternatives, based on the group-wide margin. The winner is the alternative which receives a majority of the weight.

At the second stage, each group's allocation of weight is determined as a function of the group-wide margin.

Definition 1. A *rule* for group i is defined as a Borel-measurable[5] function:

$$\phi_i : [-1, 1] \to [-1, 1].$$

The value $\phi_i(\theta_i)$ is the fraction of the weight w_i allocated to alternative $+1$ minus that allocated to -1, given that the group-wide margin is θ_i. That is, the rule allocates $w_i \phi_i(\theta_i)$ more weight to alternative $+1$ than alternative -1. For example, if $w_i = 50$ and $\phi_i(\theta_i) = -0.2$, it means that the rule allocates 20 (resp. 30) units of weight to the alternative $+1$ (resp. -1).

Let

$$\Phi = \{\phi_i | \text{Borel-measurable}\}$$

be the set of all admissible rules.

Examples of rules. Among all admissible rules, the following ones deserve particular attention:

(i) *Winner-take-all rule*: $\phi_i^{\mathrm{WTA}}(\theta_i) = \operatorname{sgn} \theta_i$.

(ii) *Proportional rule*: $\phi_i^{\mathrm{PR}}(\theta_i) = \theta_i$.

(iii) *Mixed rules*: $\phi_i^a(\theta_i) = a\phi_i^{\mathrm{WTA}}(\theta_i) + (1-a)\phi_i^{\mathrm{PR}}(\theta_i)$, $0 \le a \le 1$.

The winner-take-all rule devotes all the weight of a group to the winning alternative in the group. The proportional rule allocates the weight in proportion to the vote shares of the respective alternatives in the group. The mixed rule ϕ^a allocates the fixed ratio a of the weight by the winner-take-all rule and the remaining $1-a$ part by the proportional rule.

[5] Borel-measurability is needed to ensure that each $\phi_i(\Theta_i)$ is a well-defined random variable.

A profile $\phi = (\phi_i)_{i=1}^n$ consists of rules specified for all groups. By *symmetric* profile, we mean that the same rule is used by all groups. For instance, the above three rules naturally define the following symmetric profiles: the *winner-take-all profile* $\phi^{\text{WTA}} = (\phi_i^{\text{WTA}})_{i=1}^n$, the *proportional profile* $\phi^{\text{PR}} = (\phi_i^{\text{PR}})_{i=1}^n$, and *mixed profiles* $\phi^a = (\phi_i^a)_{i=1}^n$, $a \in [0, 1]$.

The winning alternative is the one which obtains more weight from the groups. In the case of a tie, we assume that both alternatives are chosen with equal probability. To define it formally, let

$$S_\phi = \sum_{i=1}^n w_i \phi_i(\Theta_i)$$

be the difference between the total weight allocated for alternatives $+1$ and -1. The *social decision* D_ϕ is

$$D_\phi = \begin{cases} \operatorname{sgn} S_\phi & \text{if } S_\phi \neq 0 \\ \pm 1 \text{ with equal probabilities} & \text{if } S_\phi = 0. \end{cases} \tag{1}$$

The payoff of each individual is 1 or -1, depending on whether she prefers the social decision or not. We define *group i's (expected) payoff* as the average expected payoff of its members. Since the average payoff of group-i members is Θ_i or $-\Theta_i$ depending on whether the social decision is $+1$ or -1, the group's expected payoff is

$$\pi_i(\phi) = \mathbb{E}(\Theta_i D_\phi).$$

Since each group chooses a rule as a function of the group-wide margin, maximizing $\pi_i(\phi)$ with respect to its own rule ϕ_i is equivalent to maximizing, for almost every $\theta_i \in [-1, 1]$, the conditional expected payoff given the group-wide margin $\Theta_i = \theta_i$:

$$\begin{aligned} &\pi_i(\phi|\theta_i) \\ &= \theta_i \mathbb{E}(D_\phi|\Theta_i = \theta_i) \\ &= \theta_i \left(\mathbb{P}\left\{D_\phi = +1|\Theta_i = \theta_i\right\} - \mathbb{P}\left\{D_\phi = -1|\Theta_i = \theta_i\right\}\right) \\ &= \theta_i(\mathbb{P}\{w_i\phi_i(\theta_i) + S_{\phi_{-i}} > 0|\Theta_i = \theta_i\} - \mathbb{P}\{w_i\phi_i(\theta_i) + S_{\phi_{-i}} < 0|\Theta_i = \theta_i\}), \end{aligned} \tag{2}$$

where $S_{\phi_{-i}} = \sum_{j \neq i} w_j \phi_j(\Theta_j)$.[6]

Remark 1. Our definition of group payoffs has the following interpretation based on the members' preferences. Let M_i be the set of individuals in group i, and $X_{im} \in \{-1, 1\}$ be the preferred alternative of member $m \in M_i$ in group i. Let us here redefine Θ_i as a *latent variable* that parametrizes the distribution of the random preferences in group i. Specifically, suppose X_{im} are independently and identically distributed conditional on the realization $(\theta_i)_{i=1}^n$ with the following probabilities for all $i = 1, \cdots, n$ and $m \in M_i$:

$$\begin{cases} \mathbb{P}\{X_{im} = +1 | \Theta_1 = \theta_1, \cdots, \Theta_n = \theta_n\} = (1 + \theta_i)/2, \\ \mathbb{P}\{X_{im} = -1 | \Theta_1 = \theta_1, \cdots, \Theta_n = \theta_n\} = (1 - \theta_i)/2. \end{cases} \tag{3}$$

Then, as the group size becomes large ($|M_i| \to \infty$), the Law of Large Numbers implies that the group-wide margin $\frac{1}{M_i} \sum_{m \in M_i} X_{im}$ indeed converges to Θ_i almost surely, which is consistent with our original definition of Θ_i as the group-wide margin. Moreover,

$$\begin{aligned} \mathbb{P}\{X_{im} = D_\phi\} &= \mathbb{E}\left[\mathbb{P}\{X_{im} = D_\phi | \Theta\}\right] \\ &= \mathbb{E}\left[\mathbb{P}\{X_{im} = 1, D_\phi = 1 | \Theta\} + \mathbb{P}\{X_{im} = -1, D_\phi = -1 | \Theta\}\right] \\ &= \mathbb{E}\left[\mathbb{P}\{D_\phi = 1 | \Theta\} \frac{1 + \Theta_i}{2} + \mathbb{P}\{D_\phi = -1 | \Theta\} \frac{1 - \Theta_i}{2}\right] \\ &= \frac{1}{2}\left(1 + \mathbb{E}\left[\mathbb{P}\{D_\phi = 1 | \Theta\} \Theta_i + \mathbb{P}\{D_\phi = -1 | \Theta\} (-\Theta_i)\right]\right) \\ &= \frac{1}{2}\left(1 + \mathbb{E}\left[\Theta_i D_\phi\right]\right). \end{aligned}$$

Therefore, $\pi_i(\phi) = \mathbb{E}(\Theta_i D_\phi)$ is an affine transformation of the probability that the preferred alternative of a member m in group i coincides with the social decision ($X_{im} = D_\phi$), which is called success in the literature of voting power measurement (Laruelle and Valenciano (2008)). The objective of the group, formulated as the maximization of π_i, is thus equivalent to maximization of the probability of success.

Under the winner-take-all profile ϕ^{WTA}, π_i is closely related to the classical voting power indices studied in the literature. If $(\Theta_i)_{i=1}^n$ are independently, identically and symmetrically distributed (thus each group's pre-

[6] The last expression in (2) uses the assumption that ties are broken by tossing a fair coin.

ferred alternative is independently and equally distributed over $\{-1, +1\}$, called Impartial Culture), then π_i corresponds to the Banzhaf-Penrose index (Banzhaf (1965), Penrose (1946)) and $\mathbb{P}\{X_{im} = D_\phi\}$ to the Rae index (Rae (1946)), up to a multiplication by the constant $\mathbb{E}[|\Theta_i|]$. If $(\Theta_i)_{i=1}^n$ are perfectly correlated and symmetrically distributed (called Impartial Anonymous Culture. See, for example, Le Breton et al. (2016)), then π_i corresponds to the Shapley-Shubik index (Shapley and Shubik (1954)).

Remark 2. Our specification of group's payoff may sound at first as if it excludes the case where individuals have different preference intensities. However, even for the cases in which each group is allowed to use not only the ordinal, but also the cardinal information of its members' preferences (as in Barberà and Jackson (2006), Beisbart et al. (2005), Beisbart and Bovens (2008), Beisbart and Hartmann (2010)), our assumption comes along with no loss of generality: it suffices to redefine Θ_i as the average payoff difference between alternatives $+1$ and -1 over group i's members.

3 The Dilemma

We consider a non-cooperative game Γ in which each group chooses a rule to allocate its weight to the alternatives. Each group's objective is to maximize the average expected payoff of its members. Formally, the *game* Γ is defined as follows. The set of players is the set of groups: $\{1, \cdots, n\}$. The strategy space for group i is the set of all rules: {all measurable functions $\phi_i : [-1, 1] \to [-1, 1]$}. The payoff of group i is its per capita expected payoff: $\pi_i(\phi)$.

Two rules ϕ_i and ψ_i are called *equivalent* if $\phi_i(\Theta_i) = \psi_i(\Theta_i)$ almost surely. Two profiles ϕ and ψ are called *equivalent* if $D_\phi = D_\psi$ almost surely.

A rule (or strategy) ϕ_i for group i *dominates* another rule ψ_i if $\pi_i(\phi_i, \phi_{-i}) \geq \pi_i(\psi_i, \phi_{-i})$ for any ϕ_{-i}, with strict inequality for at least one ϕ_{-i}. A rule ϕ_i is a *dominant strategy* for group i if it dominates every rule not equivalent to ϕ_i. A profile ϕ *Pareto dominates* another profile ψ if $\pi_i(\phi) \geq \pi_i(\psi)$ for all i, with strict inequality for at least one i. If ϕ is not Pareto dominated by any profile, it is called *Pareto efficient.*

To show the main theorem, we need the following assumption on the allocation of weights among the groups.

Assumption 2. Each group has less than half the total weight: $w_i < \frac{1}{2}\sum_{j=1}^{n} w_j$ for all $i = 1, \cdots, n$.

Theorem 1. *Under Assumptions 1 and 2, game* Γ *is a Prisoner's Dilemma:*

(i) the winner-take-all rule ϕ_i^{WTA} *is a dominant strategy for each group* i*;*

(ii) the winner-take-all profile ϕ^{WTA} *is Pareto dominated.*

We use the following lemma to prove the theorem. A *generalized proportional profile* refers to a profile in which $\phi_i(\theta_i) = \lambda_i\theta_i$, $i = 1, \cdots, n$, for some vector $\lambda \in [0,1]^n \setminus \{0\}$.

Lemma 1. *(Characterization of the Pareto set) Under Assumption 1, a profile* ϕ *is Pareto efficient if and only if it is equivalent to some generalized proportional profile.*

Proof of Theorem 1. Part (i). By (2), if $\theta_i > 0$ (resp. $\theta_i < 0$), then the conditional expected payoff $\pi_i(\phi|\theta_i)$ is non-decreasing (resp. non-increasing) in $\phi_i(\theta_i) \in [-1,1]$. We thus have $\pi_i(\phi_i^{\text{WTA}}, \phi_{-i}|\theta_i) \geq \pi_i(\phi_i, \phi_{-i}|\theta_i)$ for any (ϕ_i, ϕ_{-i}) and θ_i. Therefore

$$\pi_i(\phi_i^{\text{WTA}}, \phi_{-i}) \geq \pi_i(\phi_i, \phi_{-i})$$

for any (ϕ_i, ϕ_{-i}). Now we show that for any subprofile ϕ_{-i} in which each $\phi_j : [-1,1] \to [-1,1]$ $(j \neq i)$ is onto (e.g., ϕ_j^{PR}), the strict inequality

$$\pi_i(\phi_i^{\text{WTA}}, \phi_{-i}) > \pi_i(\phi_i, \phi_{-i}) \tag{4}$$

holds for any rule ϕ_i that is not equivalent to ϕ_i^{WTA}. To see this, note that for such ϕ_{-i}, the full-support assumption on $(\Theta_j)_{j=1}^n$ implies that the conditional distribution of $S_{\phi_{-i}}$ given $\Theta_i = \theta_i$ has support $[-\sum_{j\neq i} w_j, \sum_{j\neq i} w_j]$. Since $w_i < \sum_{j\neq i} w_j$ by Assumption 2, formula (2) implies that if $\theta_i > 0$ (resp. $\theta_i < 0$), then $\pi_i(\phi|\theta_i)$ is strictly increasing (resp. decreasing) in $\phi_i(\theta_i) \in [-1,1]$. Thus $\pi_i(\phi_i^{\text{WTA}}, \phi_{-i}|\theta_i) > \pi_i(\phi_i, \phi_{-i}|\theta_i)$ holds at any θ_i for which $\phi_i^{\mathrm{WTA}}(\theta_i) \neq \phi_i(\theta_i)$. Since Θ_i has full support, this implies that (4) holds for any ϕ_i that is not equivalent to ϕ_i^{WTA}.

Part (ii). By the characterization of the Pareto set (Lemma 1), it suffices to check that ϕ^{WTA} is not equivalent to any generalized proportional profile. Suppose, on the contrary, that ϕ^{WTA} is equivalent to a generalized proportional profile with coefficients $\lambda \in [0,1]^n \setminus \{0\}$. Then, since $(\Theta_i)_{i=1}^n$ has full support,

$$D_{\phi^{\mathrm{WTA}}}(\theta) = \operatorname{sgn} \sum_{i=1}^{n} w_i \lambda_i \theta_i \text{ at almost every } \theta \in [-1,1]^n. \tag{5}$$

Since no group dictates the social decision, the coefficients λ_i are positive for at least two groups. Without loss of generality, assume $\lambda_1 > 0$ and $\lambda_2 > 0$. Now, fix θ_i for $i \neq 1, 2$ so that they are sufficiently small in absolute value. Then, according to (5), for (almost any) sufficiently small $\varepsilon > 0$, $D_{\phi^{\mathrm{WTA}}}(\theta) = +1$ if $\theta_1 = 1 - \varepsilon$ and $\theta_2 = -\varepsilon$, while $D_{\phi^{\mathrm{WTA}}}(\theta) = -1$ if $\theta_1 = \varepsilon$ and $\theta_2 = -1 + \varepsilon$. This contradicts the fact that $D_{\phi^{\mathrm{WTA}}}(\theta)$ depends only on the signs of $(\theta_i)_{i=1}^n$. □

Together with Lemma 1, Theorem 1 shows that while the dominant strategy for each group is the winner-take-all rule, the dominant-strategy equilibrium is Pareto dominated by some generalized proportional profile. This typical Prisoner's Dilemma situation suggests to us that a Pareto efficient outcome is not expected to be achieved under decentralized decision making, and a coordination device is necessary in order to attain a Pareto improvement.

If Assumption 2 fails and some group has more than half the total weight, the winner-take-all profile is Pareto efficient.

Proposition 1. *Under Assumption 1, if there exists a group i^* such that $w_{i^*} > \frac{1}{2}\sum_{j=1}^n w_j$, then the winner-take-all profile ϕ^{WTA} is Pareto efficient.*

Proof. Under ϕ^{WTA}, the social decision always coincides with group i^*'s majority preference. Thus ϕ^{WTA} is equivalent to the generalized proportional profile with coefficients $\lambda_{i^*} > 0$ and $\lambda_i = 0$ for all $i \neq i^*$. The proposition follows from the characterization of the Pareto set (Lemma 1). □

Intuitively, the winner-take-all profile gives dictatorial power to the group with more than half the total weight, while any non-equivalent profile creates a positive probability of social decision against that group's will. Hence Pareto improvement is impossible. One might further speculate that Pareto efficiency of the winner-take-all profile will still hold as long as there are groups with sufficiently large weights, even if no group's weight exceeds half the total weight. Theorem 1 disproves this possibility.

In contrast with the winner-take-all profile, the proportional profile is Pareto efficient, regardless of the allocation of weights across the groups.

Proposition 2. *Under Assumption 1, the proportional profile ϕ^{PR} is Pareto efficient.*

Proof. This follows from the characterization of the Pareto set (Lemma 1). □

However, the proportional profile does not necessarily Pareto dominate the winner-take-all profile, even when Assumption 2 holds. This is illustrated by the following example.

Example 1. Let us consider three groups with weights $(w_1, w_2, w_3) = (49, 49, 2)$. The group-wide margins Θ_i are independent and uniformly distributed on $[-1, 1]$. On the one hand, under the winner-take-all profile ϕ^{WTA}, all groups are perfectly symmetric, and a simple calculation shows that the expected payoff is $\pi_i(\phi^{\mathrm{WTA}}) = 0.25$ for all $i = 1, 2, 3$. On the other hand, under the proportional profile ϕ^{PR}, group 3 is extremely unlikely to affect the social decision, and $\pi_3(\phi^{\mathrm{PR}})$ is close to 0 (approximately 0.014). Group 3 is better off under ϕ^{WTA} than ϕ^{PR}, and so ϕ^{PR} does not Pareto dominate ϕ^{WTA}. By what profile is ϕ^{WTA} Pareto dominated? The characterization lemma provides an answer. Consider the generalized proportional profile $\hat{\phi}$ with coefficients $\lambda_i = 1/w_i$. Then, the distribution of the weight assigned to the alternative is exactly the same across groups, and thus $\pi_i(\hat{\phi})$ is the same for all i. By Pareto efficiency of the generalized proportional profile, $\pi_i(\hat{\phi}) > 0.25$ for all i. □

Remark 3. An interesting extension of the model would be to assume that each group chooses a rule through voting by its members. Does this extension lead to an equilibrium different from the winner-take-all profile? The answer is relatively clear in the case of group-wide majority voting between the winner-take-all rule and some other rule (e.g., the proportional rule), as when a state in the US holds a referendum to switch from the current winner-take-all rule to some proposed rule (see, e.g., Beisbart and Bovens (2008)). In that case, a group's choice of a rule depends on the prior joint distribution of its members' preferences, which we have not specified so far. If the group is ex ante sufficiently homogeneous,[7] the choice by majority voting will coincide with the choice that maximizes the per capita expected payoff, i.e., the winner-take-all rule. However, if the group is ex ante sufficiently heterogeneous, group-wide majority voting may select the other rule. The following example illustrates this point.

Suppose there are two groups with weights $w_1 = 4, w_2 = 3$.[8] Each group consists of two types of members, L and R. Type-L members are more likely to prefer alternative -1 than $+1$, and type-R members are more likely to prefer $+1$ than -1. The fraction of type-L members is 51% in group 1, and 80% in group 2. In that sense, ex ante heterogeneity is high in group 1 and low in group 2.

To define the types more precisely, suppose there are four random variables $(\Theta_{1L}, \Theta_{1R}, \Theta_{2L}, \Theta_{2R})$ in which Θ_{it} is the latent variable for the preferences of type-t members in group i, in the same sense as in Remark 1.[9] The latent variable Θ_{iL} is uniformly distributed on $[-1, 0]$, and Θ_{iR} is uniformly distributed on $[0, 1]$. In particular, each type-L (resp. type-R) member is always more likely to prefer -1 than $+1$ (resp. $+1$ than -1). We also

[7] Ex ante homogeneity of a group does not necessarily mean that most of its members tend to support the same alternative. What it means is that the members' preferences follow similar probability laws. For instance, conditionally independent and identically distributed preferences in Remark 1 provide an example of perfect ex ante homogeneity.

[8] Here we use the case with $n = 2$ for ease of exposition. A similar example can be constructed with a larger number of groups. We also note that while Assumption 2 excludes the case with $n = 2$, strategic dominance of the winner-take-all rule (i.e., part (i) of Theorem 1) does not depend on that assumption.

[9] That is, conditional on the realization $(\theta_{1L}, \theta_{1R}, \theta_{2L}, \theta_{2R})$, all individual members' preferences are independent and identically distributed, where the alternative X_{im} preferred by a type-t member m of group i has the following conditional distribution:

$$\begin{cases} \mathbb{P}\{X_{im} = +1 | \Theta_{1L} = \theta_{1L}, \cdots, \Theta_{2R} = \theta_{2R}\} = (1 + \theta_{it})/2, \\ \mathbb{P}\{X_{im} = -1 | \Theta_{1L} = \theta_{1L}, \cdots, \Theta_{2R} = \theta_{2R}\} = (1 - \theta_{it})/2. \end{cases}$$

assume that the four latent variables are independent. Given any profile $\phi = (\phi_1, \phi_2)$, the (expected) payoff for each type-t member of group i is

$$\pi_{it}(\phi) = \mathbb{E}(\Theta_{it} D_\phi).^{10}$$

In other words, the probability of success for this member (i.e., the probability that the social decision will be his preferred alternative) is $\frac{1+\pi_{it}(\phi)}{2}$.

Let us assume that the only rules available for each group are the winner-take-all rule ϕ_i^{WTA} and the proportional rule ϕ_i^{PR}. Each group chooses a rule by majority voting, where each member votes for the rule that gives him a higher payoff. Since the type-L members are the majority in both groups, group i's majority preference over profiles is represented by the payoff function $\pi_{iL}(\phi)$. Thus the situation can be formally represented as the following 2×2 game played by groups 1 and 2, where group i's payoff is the payoff $\pi_{iL}(\phi)$ for a type-L member[11] (not the per capita expected payoff $\pi_i(\phi)$):

	ϕ_2^{WTA}	ϕ_2^{PR}
ϕ_1^{WTA}	0.192, 0.020	0.192, 0.020
ϕ_1^{PR}	0.375, 0.479	0.386, 0.402

The equilibrium is $(\phi_1^{\mathrm{PR}}, \phi_2^{\mathrm{WTA}})$. In contrast with Theorem 1, *the proportional rule is the dominant strategy for group 1 (or more precisely, voting for the proportional rule is the dominant strategy for each type-L member in group 1).* The intuitive reason is as follows. If group 1 uses the winner-take-all rule, then it will dictate the social decision, and hence the decision will be alternative -1 or $+1$ with almost equal probabilities, since group 1 is almost evenly split into the two types. Alternatively, if group 1 adopts the proportional rule, then the social decision will be more likely to be alternative -1 (i.e., the alternative which the majority of group 1's members are more likely to support), since there are now chances that the decision reflects the will of group 2 in which 80% are of type L. This explains why

[10] Assuming that the number of members of each type is sufficiently large in each group, the group-wide margins in groups 1 and 2 are $\Theta_1 = 0.51\Theta_{1L} + 0.49\Theta_{1R}$ and $\Theta_2 = 0.8\Theta_{2L} + 0.2\Theta_{2R}$, respectively. The definition of the social decision D_ϕ is then the same as before (equation (1)).

[11] We obtained the payoffs in the table by numerical computation.

the winner-take-all rule is the dominant strategy for group 1. For the relatively homogeneous group 2, on the other hand, the dominant strategy is the winner-take-all rule.

This example highlights the importance of the internal decision procedure employed within each group. In order to fully explain the incentives of the group, it would be interesting to build a full-fledge model which includes a detailed description of internal heterogeneity of the preference distributions, but it is beyond the scope of the current paper.

4 Asymptotic and Computational Results

4.1 Asymptotic Analysis

We saw above that the game is a Prisoner's Dilemma. In this section, we provide further insights on the welfare properties, by focusing on the following situations in which: (i) the number of groups is sufficiently large, and (ii) the preferences of the members are distributed symmetrically. These properties allows us to provide an asymptotic and normative analysis.

Often the difficulty of analysis arises from the discrete nature of the problem. Since the social decision D_ϕ is determined as a function of the sum of the weights allocated to the alternatives across the groups, computing the expected payoffs may require classification of a large number of success configurations which increases exponentially as the number of groups increases, rendering the analysis prohibitively costly. We overcome this difficulty by studying asymptotic properties. In order to check the sensibility of our analysis, we provide Monte Carlo simulation results later in the section, using the example of the US Electoral College.

In order to study asymptotic properties, let us consider a sequence of weights $(w_i)_{i=1}^{\infty}$, exogenously given as a fixed parameter.

Assumption 3. The sequence of weights $(w_i)_{i=1}^{\infty}$ satisfies the following properties.

(i) $w_1, w_2, \cdots$ are in a finite interval $[\underline{w}, \bar{w}]$ for some $0 \leq \underline{w} < \bar{w}$.

(ii) As $n \to \infty$, the statistical distribution G_n induced by $(w_i)_{i=1}^n$ weakly converges to a distribution G with support $[\underline{w}, \bar{w}]$.[12]

Assumption 3 guarantees that for large n, the statistical distribution of weights G_n is sufficiently close to some well-behaved distribution G, on which our asymptotic analysis is based.

Additionally, we impose an impartiality assumption for our normative analysis:

Assumption 4. The variables $(\Theta_i)_{i=1}^{\infty}$ are drawn independently from a common symmetric distribution F.

As in Felsenthal and Machover (1998), a normative analysis requires impartiality, and a study of fundamental rules in the society, such as a constitution, should be free from any dependence on the ex post realization of the group characteristics. Assumption 4 allows our normative analysis to abstract away the distributional details. Of course, a normative analysis is best complemented by a positive analysis which takes into account the actual characteristics of the distributions (as in Beisbart and Bovens (2008)).

Following the symmetry of the preferences, our analysis also focuses on *symmetric* profiles, in which all groups use the same rule: $\phi_i = \phi$ for all i. With a slight abuse of notation, we write ϕ both for a *single rule* ϕ and for the *symmetric profile* $(\phi, \phi, \cdots)$, which should not create any confusion as long as we refer to symmetric profiles. As for the alternatives, it is natural to consider that the label should not matter when the group-wide vote margin is translated into the weight allocation, given the symmetry of the preferences.

Assumption 5. We assume that the rule is monotone and neutral, that is, ϕ is a non-decreasing, odd function: $\phi(\theta_i) = -\phi(-\theta_i)$.

Let $\pi_i(\phi; n)$ denote the expected payoff for group $i (\leq n)$ under profile ϕ when the set of groups is $\{1, \cdots, n\}$ and each group j's weight is w_j, the

[12] The statistical distribution function G_n induced by $(w_i)_{i=1}^n$ is defined by $G_n(x) = \#\{i \leq n : w_i \leq x\}/n$ for each x. G_n weakly converges to G if $G_n(x) \to G(x)$ at every point x of continuity of G.

jth component of the sequence of weights. The definition of $\pi_i(\phi; n)$ is the same as $\pi_i(\phi)$ in the preceding sections; the new notation just clarifies its dependence on the number of groups n.

The main welfare criterion employed in this section is the asymptotic Pareto dominance.

Definition 2. For two symmetric profiles ϕ and ψ, we say that ϕ *asymptotically Pareto dominates* ψ if there exists N such that for all $n > N$ and all $i = 1, \cdots, n$,

$$\pi_i(\phi; n) > \pi_i(\psi; n).$$

4.2 Pareto Dominance

The following is the main result in our asymptotic analysis.

Theorem 2. *Under Assumptions 1-5, the proportional profile asymptotically Pareto dominates all other symmetric profiles. In particular, it asymptotically Pareto dominates the dominant-strategy equilibrium of the game, i.e., the symmetric winner-take-all profile.*

We use the following lemma to prove Theorem 2. The proof of Lemma 2 is relegated to the Appendix. The proof of part (ii) uses a more general result, Lemma 3, stated in the next subsection, whose proof also appears in the Appendix.

Lemma 2. *Under Assumptions 1-5, the following statements hold.*

(i) For any symmetric profile ϕ,

$$\begin{aligned} &\pi_i(\phi; n) \\ &= 2\int_0^1 \theta_i \mathbb{P}\left\{ -w_i\phi(\theta_i) < \sum_{j\leq n, j\neq i} w_j\phi(\Theta_j) \leq w_i\phi(\theta_i) \right\} dF(\theta_i). \end{aligned}$$

(ii) For any symmetric profile ϕ, as $n \to \infty$,

$$\sqrt{2\pi n}\pi_i(\phi; n) \to 2w_i\sqrt{\frac{\mathbb{E}[\Theta^2]}{\int_{\underline{w}}^{\bar{w}} w^2 dG(w)}}\,\mathrm{Corr}\,[\Theta, \phi(\Theta)],^{13}$$

[13]Since Θ and $\phi(\Theta)$ are symmetrically distributed, the correlation is given by

uniformly in $w_i \in [\underline{w}, \bar{w}]$, where Θ is a random variable having the same distribution F as Θ_i. The limit depends on the profile ϕ only through the factor $\mathrm{Corr}[\Theta, \phi(\Theta)]$.

Proof of Theorem 2.
The heart of the proof is in the correlation result shown in part (ii) of Lemma 2. It follows that if $\phi(\Theta)$ is more correlated with Θ than $\psi(\Theta)$ is, then for each group i, there exists N_i such that if the number of groups (n) is greater than N_i, group i ($\leq n$) will be better off under ϕ than ψ.

Note that the convergence in part (ii) of Lemma 2 is uniform in $w_i \in [\underline{w}, \bar{w}]$. This implies that the convergence is uniform in $i = 1, 2, \cdots$.[14] Thus there is N with the above property, without subscript i, which applies to all groups $i = 1, 2, \cdots$. Therefore, if $\phi(\Theta)$ is more correlated with Θ than $\psi(\Theta)$ is, then ϕ asymptotically Pareto dominates ψ.

Since the perfect correlation $\mathrm{Corr}[\Theta, \phi^{\mathrm{PR}}(\Theta)] = 1$ is attained by the proportional rule, Theorem 2 follows. □

The above results show that the winner-take-all rule is characterized by its strategic dominance, while the proportional rule is characterized by its asymptotic Pareto dominance. The following proposition provides a complete Pareto order among all the linear combinations of the two rules.

Remember that we defined the mixed rules in Section 2 above. For $0 \leq a \leq 1$, a fraction a of the weight is assigned to the winner of the group-wide vote, while the rest, $1 - a$, is distributed proportionally to each alternative:

$$\phi^a(\theta_i) = a\phi^{\mathrm{WTA}}(\theta_i) + (1 - a)\phi^{\mathrm{PR}}(\theta_i).$$

Proposition 3. *Under Assumptions 1-4, mixed profile ϕ^a asymptotically Pareto dominates mixed profile $\phi^{a'}$ for any $0 \leq a < a' \leq 1$. In particular,*

$\mathrm{Corr}[\Theta, \phi(\Theta)] = \mathbb{E}[\Theta\phi(\Theta)]/\sqrt{\mathbb{E}[\Theta^2]\mathbb{E}[\phi(\Theta)^2]}$ unless $\phi(\Theta)$ is almost surely zero. If $\phi(\Theta)$ is almost surely zero, then the correlation is zero.

[14] A more detailed explanation of this step is the following. By Lemma 2 (i), $\sqrt{2\pi n}\pi_i(\phi; n))$ asymptotically behaves as $2\sqrt{2\pi n}\int_0^1 \theta\mathbb{P}\{-w_i\phi(\theta) < \sum_{j\leq n} w_j\phi(\Theta_j) \leq w_i\phi(\theta)\}dF(\theta)$, where whether the sum $\sum_{j\leq n} w_j\phi(\Theta_j)$ includes the ith term or not is immaterial in the limit. The estimate of $\sqrt{2\pi n}\pi_i(\phi; n)$ therefore has the form $f_n(w_i)$, where $f_n(x) := 2\sqrt{2\pi n}\int_0^1 \theta\mathbb{P}\{-x\phi(\theta) < \sum_{j\leq n} w_j\phi(\Theta_j) \leq x\phi(\theta)\}dF(\theta)$. Lemma 2 (ii) implies that $f_n(x)$ converges uniformly in $x \in [\underline{w}, \bar{w}]$, which in turn implies that the convergence of $\sqrt{2\pi n}\pi_i(\phi; n) \approx f_n(w_i)$ is uniform in $i = 1, 2, \cdots$.

the proportional profile asymptotically Pareto dominates any mixed profile ϕ^a for $0 < a < 1$, which in turn asymptotically Pareto dominates the winner-take-all profile. In other words, all mixed profiles can be ordered by asymptotic Pareto dominance, from the proportional profile as the best, to the winner-take-all profile as the worst.

Proof. In Appendix. □

The winner-take-all rule is not only asymptotically Pareto inefficient, but the worst among the symmetric mixed profiles. Is it worse than *any* other symmetric profile? We provide an answer in Remark 4 below.

Remark 4. Theorem 2 leaves the natural question of whether the winner-take-all profile is the worst among all symmetric profiles, in terms of asymptotic Pareto dominance. The answer is negative. To see this, note first that, for the winner-take-all profile, the correlation in Lemma 2 is strictly positive: $\text{Corr}[\Theta, \phi^{\text{WTA}}(\Theta)] = \mathbb{E}(|\Theta|)/\sqrt{\mathbb{E}(\Theta^2)} > 0$. On the other hand, for the symmetric profile ϕ^0 in which the rule is defined by $\phi^0(\theta) = 0$ for almost all θ, the correlation is obviously zero. This rule assigns exactly half of the weight to each alternative, regardless of the group-wide vote. Thus the profile ϕ^0 is the worst among all symmetric profiles, as the social decision is made by a coin toss almost surely, yielding expected payoff 0 to all groups. In the following, we exclude such a trivial profile from our consideration.

4.3 Congressional District Method

The analysis in the preceding subsection suggests that the proportional profile is optimal in terms of Pareto efficiency. However, our model also implies that this profile produces an unequal distribution of welfare; in fact, this unequal nature pertains to all symmetric profiles. The Correlation Lemma 2 (ii) shows that for these profiles, the expected payoff for a group is asymptotically proportional to its weight, providing high expected payoffs to the members in a group with a large weight.

In this subsection, we examine whether such inequality can be alleviated without impairing efficiency by using an asymmetric profile, based on the Congressional District Method, currently used in Maine and Nebraska. This profile allocates a fixed amount c of each group's weight by the winner-take-all rule and the rest by the proportional rule:

$$w_i \phi^{\mathrm{CD}}(\theta_i, w_i) = c\phi^{\mathrm{WTA}}(\theta_i) + (w_i - c)\phi^{\mathrm{PR}}(\theta_i).$$

We consider the *congressional district profile* ϕ^{CD} in which the rule is used by all groups. Note that this profile is not symmetric as it depends on w_i, but the way ϕ^{CD} depends on w_i is the same for all groups. To ensure that the profile is well-defined, we impose that its parameter c is below the lower bound of weights: $c \in [0, \underline{w}]$.

Theorem 3. *Under Assumptions 1-5, let us consider the congressional district profile with parameter $c \leq \underline{w}$. For any symmetric profile ϕ, there exists $w^* \in [\underline{w}, \bar{w}]$ with the following property: for any $\varepsilon > 0$, there is N such that for all $n > N$ and $i = 1, \cdots, n$,*

$$w_i < w^* - \varepsilon \Rightarrow \pi_i(\phi^{\mathrm{CD}}; n) > \pi_i(\phi; n),$$
$$w_i > w^* + \varepsilon \Rightarrow \pi_i(\phi^{\mathrm{CD}}; n) < \pi_i(\phi; n).$$

The proof of Theorem 3 uses the following lemma, which shows that the correlation lemma holds for a class of profiles such that the weight allocation rules have the following specific form of separability. Its proof and the Local Limit Theorem used in the proof are relegated to the Appendix.

Assumption 6. Let $\phi = (\phi_i)_{i=1}^{\infty}$ be a profile. There exist functions h_1, h_2, h_3 such that

$$w_i \phi_i(\theta_i, w_i) = h_1(w_i) h_2(\theta_i) + h_3(w_i) \operatorname{sgn} \theta_i, \text{ for all } i$$

where (i) h_1 is bounded, (ii) h_2 is an odd function such that the support of the distribution of $h_2(\Theta_i)$ contains 0, and (iii) h_3 is continuous but not constant.[15]

It is straightforward to show that Assumption 6 is satisfied for any symmetric profile as well as the congressional district profile. For a symmetric profile ϕ, let $h_1(w_i) = w_i$, $h_2(\theta_i) = \phi(\theta_i) - r \operatorname{sgn} \theta_i$, and $h_3(w_i) = w_i r$

[15]Under this form, $\phi_i(\cdot, \cdot)$ is the same for all i so that we can omit subscript i whenever there is no confusion.

where $r > 0$ is any positive number in the support of the distribution of $\phi(\Theta)$.[16] For the congressional district profile ϕ^{CD}, let $h_1(w_i) = w_i - c$, $h_2(\theta_i) = \theta_i - \operatorname{sgn}\theta_i$, and $h_3(w_i) = w_i$.

Lemma 3. *Under Assumptions 1-5, let ϕ be a profile which satisfies Assumption 6. Then, as $n \to \infty$,*

$$\sqrt{2\pi n}\pi_i(\phi; n) \to \frac{2w_i \mathbb{E}[\Theta\phi(\Theta, w_i)]}{\sqrt{\int_{\underline{w}}^{\bar{w}} w^2 \mathbb{E}[\phi(\Theta, w)^2] dG(w)}},$$

uniformly in $w_i \in [\underline{w}, \bar{w}]$, where Θ is a random variable having the same distribution F as Θ_i.

Proof of Theorem 3. By Lemma 3, the expected payoff for group i under a symmetric profile ϕ tends to a linear function of w_i. Let A^ϕ be the coefficient:

$$\begin{aligned} \lim_{n\to\infty} \sqrt{2\pi n}\pi_i(\phi; n) &= \frac{2w_i \mathbb{E}[\Theta\phi(\Theta)]}{\sqrt{\mathbb{E}[\phi(\Theta)^2] \int_{\underline{w}}^{\bar{w}} w^2 dG(w)}} \\ &=: A^\phi w_i. \end{aligned} \tag{6}$$

For the congressional district profile, remember the definition:

$$\begin{aligned} w_j \phi^{\mathrm{CD}}(\theta_j, w_j) &= c\phi^{\mathrm{WTA}}(\theta_j) + (w_j - c)\phi^{\mathrm{PR}}(\theta_j) \\ &= c \operatorname{sgn}(\theta_j) + (w_j - c)\theta_j. \end{aligned}$$

We claim that the limit function is affine in w_i:

$$\lim_{n\to\infty} \sqrt{2\pi n}\pi_i(\phi^{\mathrm{CD}}; n) = Bw_i + C. \tag{7}$$

To see that, let us apply Lemma 3 again:

$$\begin{aligned} \lim_{n\to\infty} \sqrt{2\pi n}\pi_i(\phi^{\mathrm{CD}}; n) &= 2 \cdot \frac{w_i \mathbb{E}\left[\Theta\phi^{\mathrm{CD}}(\Theta, w_i)\right]}{\sqrt{\int_{\underline{w}}^{\bar{w}} w^2 \mathbb{E}\left[\phi^{\mathrm{CD}}(\Theta, w)^2\right] dG(w)}} \\ &= 2 \cdot \frac{c\mathbb{E}\left[|\Theta|\right] + (w_i - c)\mathbb{E}\left[\Theta^2\right]}{\sqrt{\int_{\underline{w}}^{\bar{w}} w^2 \mathbb{E}\left[\phi^{\mathrm{CD}}(\Theta, w)^2\right] dG(w)}}. \end{aligned}$$

[16] This is possible since $\phi(\Theta)$ is symmetrically distributed, and since we exclude the trivial case in which $\phi(\Theta) = 0$ almost surely.

Since $|\theta| \geq \theta^2$ with a strict inequality for $0 < |\theta| < 1$, the full support condition for Θ implies $\mathbb{E}[|\Theta|] > \mathbb{E}[\Theta^2]$, which induces that the intercept C is positive. The coefficient of w_i is:

$$B = \frac{2\mathbb{E}[\Theta^2]}{\sqrt{\int_{\underline{w}}^{\bar{w}} w^2 \mathbb{E}\left[\phi^{\mathrm{CD}}(\Theta, w)^2\right] dG(w)}}.$$

If $A^\phi < B$, combined with $C > 0$, the right-hand side of (7) is above that of (6). Then, set $w^* = \bar{w}$. If $A^\phi > B$, again combined with $C > 0$, the two limit functions (6) and (7) intersect only once at a positive value $\hat{w}$. Let $w^* = \max\{\underline{w}, \min\{\hat{w}, \bar{w}\}\}$.

Since the convergences (6) and (7) are uniform in w_i, for any $\varepsilon > 0$ there is N with the property stated in Theorem 3. □

Theorem 3 implies that the congressional district profile makes the members of groups with small weights better off, compared with *any* symmetric profile. If the weight is an increasing function of the group size, it means that the congressional district profile is favorable for the members of small groups.

The intuitive reason why the congressional district profile is advantageous for small groups is as follows. Under this profile, the ratio of weights cast by the winner-take-all rule (i.e. c/w_i) is higher for small groups than large groups. The congressional district profile therefore resembles the situation where the rules used by the smaller groups are relatively close to the winner-take-all rule, whereas those by the larger groups are close to the proportional rule. The strategic dominance of the winner-take-all rule suggests that this deviation is profitable for the small groups. We provide a numerical result in the following subsection using an example of the US Electoral College.

In addition to Theorem 3, we can also show that the congressional district profile distributes payoffs more equally than any symmetric profile does, in the sense of Lorenz dominance. A profile of per capita payoffs for the groups, $\pi = (\pi_1, \cdots, \pi_n)$, is said to *Lorenz dominate* another profile $\pi' = (\pi'_1, \cdots, \pi'_n)$ if the share of payoffs acquired by any bottom fraction of groups is larger in the former profile than in the latter.[17] Lorenz dominance, whenever it occurs, agrees with equality comparisons by various inequality indices including coefficient of variation, Gini coefficient, Atkinson index, and Theil index (see Fields and Fei (1978) and Atkinson (1970)). To see why the congressional district profile is more equal than any symmetric profile, recall equations (6) and (7) in the proof of Theorem 3, which assert that when the number of groups is large, the per capita payoff for group i is approximately $A^{\phi} w_i$ for the symmetric profile, and it is approximately $Bw_i + C$ for the congressional district profile. The constant term $C > 0$ for the congressional district profile means equal additions to all groups' payoffs, which result in a more equal distribution than when there is no such term. More precisely, we can prove the following statement. The proof is relegated to the Appendix.

Theorem 4. *Under Assumptions 1-5, let us consider the payoff profile under the congressional district profile:* $\pi\left(\phi^{\mathrm{CD}}; n\right) = \left(\pi_i\left(\phi^{\mathrm{CD}}; n\right)\right)_{i=1}^{n}$. *Let* ϕ *be any symmetric profile and* $\pi(\phi; n) = (\pi_i(\phi; n))_{i=1}^{n}$ *the payoff profile under* ϕ. *For sufficiently large* n, $\pi\left(\phi^{\mathrm{CD}}; n\right)$ *Lorenz dominates* $\pi(\phi; n)$.

4.4 Computational Results

The results in the previous subsection concern cases with a large number of groups. The question remains as to whether the conclusions obtained there are also valid for a finite number of groups. In this section, we study this question by numerically analyzing a model of the US presidential election.

There are 50 states and one federal district. The weight w_i for state i is the number of electoral votes currently assigned to it. This number equals the state's total number of seats in the Senate and House of Representatives. Thus, w_i is two plus a number that is roughly proportional to the state's population. The first and second columns of Table 1 describe the distribution of weights among the states.

[17] Formally, for each $x \in [-1, 1]$, let $H_\pi(x)$ be the total population share of those groups whose per capita welfare is not greater than x under the payoff profile π. Then H_π is a distribution function. The *Lorenz curve* of H_π is the graph of the function $\int_0^{H_\pi^{-1}(p)} x dH_\pi(x) / \int_0^1 x dH_\pi(x)$, $0 \le p \le 1$, where we define $H_\pi^{-1}(p) = \sup\{x : H_\pi(x) \le p\}$. A payoff profile π *Lorenz dominates* another profile π' if the Lorenz curve of H_π lies above that of $H_{\pi'}$.

We assume IAC* (Impartial Anonymous Culture*): the statewide popular vote margins Θ_i are independent and uniformly distributed on $[-1, 1]$, first introduced by May (1948) and studied thoroughly by, for example, De Mouzon et al. (2019). For any profile ϕ, we can compute the per capita payoff for state i via the formula:

$$\pi_i(\phi) = 0.5^{50} \int_{-1}^{1} \cdots \int_{-1}^{1} \theta_i 1_A(\theta_1, \cdots, \theta_{51}) d\theta_1 \cdots d\theta_{51} \tag{8}$$

where $A = \left\{ (\theta_1, \cdots, \theta_{51}) \middle| \sum_{j=1}^{51} w_j \phi_j(\theta_j) > 0 \right\}$.[18]

We consider four distinct profiles: ϕ^{WTA}, ϕ^{PR}, ϕ^a with $a = 102/538$, and ϕ^{CD} with coefficient $c = 2$. As before, these are respectively the winner-take-all profile, the proportional profile, a mixed profile, and a congressional district profile. The parameter $c = 2$ of the congressional district profile is the number currently used in Maine and Nebraska, namely, it corresponds to two seats assigned to each state in the Senate. The parameter $a = 102/538$ of the mixed profile is chosen so that the proportion of electoral votes allocated on the winner-take-all basis is the same for all states, and the total number of electoral votes allocated in this way is the same as in the congressional district profile.

We compute (8) under these four profiles by a Monte Carlo simulation with 10^{10} iterations. The results are summarized in Tables 1 and 2. Table 1 shows the per capita payoff $(\pi_i(\phi))$ under the respective profiles. Table 2 shows the ratios of per capita payoff between different profiles $(\pi_i(\phi)/\pi_i(\psi))$. If the ratio is below 1, state i prefers ψ to ϕ.

It follows from Lemma 2 (ii) that as the number n of states increases, the ratios $\pi_i\left(\phi^{\mathrm{WTA}}\right)/\pi_i\left(\phi^{\mathrm{PR}}\right)$ and $\pi_i\left(\phi^a\right)/\pi_i\left(\phi^{\mathrm{PR}}\right)$ converge to the respective correlations $\mathrm{Corr}[\Theta, \phi^{\mathrm{WTA}}(\Theta)] \approx 0.866$ and $\mathrm{Corr}[\Theta, \phi^a(\Theta)] \approx 0.989$, where the values are computed for Θ uniformly distributed on $[-1, 1]$. Table 2 indicates that for the present example with 50 states plus DC, these ratios are indeed close to the respective correlations, which suggests that convergence of the π-ratios is fairly quick. In particular, as expected by Theorem 2, the proportional profile Pareto dominates the winner-take-all profile in the present case. As suggested by Proposition 3, all states prefer

[18] It is easy to check that under the uniform distribution assumption, (8) is equivalent to the expression in Lemma 2 (i).

the mixed profile ϕ^a to the winner-take-all profile, and the proportional profile to ϕ^a.

The ratios $\pi_i(\phi^{\mathrm{CD}})/\pi_i(\phi^{\mathrm{PR}})$ in Table 2 are consistent with the result in Theorem 3. Small states prefer the congressional district profile to the proportional one.

In addition, the values of $\pi_i(\phi^{\mathrm{CD}})/\pi_i(\phi^{\mathrm{WTA}})$ in the table show that the winner-take-all profile is Pareto dominated by the congressional district profile, and the welfare improvement by switching to the congressional district profile is greater for small states than for large states in terms of the ratio.

Table 1: Estimated payoffs in the US presidential election, based on the apportionment in 2016, via Monte Carlo simulation with 10^{10} iterations. The estimated standard errors are in the range between 3.9 and 4.1×10^{-6}.

electoral votes	number of states	$\pi(\phi^{\mathrm{WTA}})$	$\pi(\phi^{\mathrm{PR}})$	$\pi(\phi^{a})$	$\pi(\phi^{\mathrm{CD}})$
3	8	0.0113	0.0133	0.0130	0.0167
4	5	0.0151	0.0177	0.0174	0.0209
5	3	0.0189	0.0221	0.0217	0.0251
6	6	0.0226	0.0266	0.0261	0.0293
7	3	0.0264	0.0310	0.0305	0.0335
8	2	0.0302	0.0354	0.0348	0.0377
9	3	0.0340	0.0399	0.0392	0.0419
10	4	0.0378	0.0443	0.0436	0.0461
11	4	0.0416	0.0488	0.0479	0.0503
12	1	0.0454	0.0532	0.0523	0.0545
13	1	0.0492	0.0577	0.0567	0.0587
14	1	0.0531	0.0622	0.0611	0.0630
15	1	0.0569	0.0666	0.0655	0.0672
16	2	0.0607	0.0711	0.0699	0.0715
18	1	0.0684	0.0801	0.0788	0.0800
20	2	0.0762	0.0891	0.0877	0.0885
29	2	0.1120	0.1303	0.1284	0.1275
38	1	0.1494	0.1729	0.1706	0.1677
55	1	0.2356	0.2614	0.2615	0.2507

Table 2: Ratios between payoffs.

electoral votes	number of states	$\frac{\pi(\phi^{\mathrm{WTA}})}{\pi(\phi^{\mathrm{PR}})}$	$\frac{\pi(\phi^{a})}{\pi(\phi^{\mathrm{PR}})}$	$\frac{\pi(\phi^{\mathrm{CD}})}{\pi(\phi^{\mathrm{PR}})}$	$\frac{\pi(\phi^{\mathrm{CD}})}{\pi(\phi^{\mathrm{WTA}})}$
3	8	0.852	0.982	1.260	1.479
4	5	0.852	0.982	1.182	1.387
5	3	0.852	0.982	1.134	1.331
6	6	0.852	0.982	1.103	1.294
7	3	0.852	0.982	1.080	1.268
8	2	0.852	0.982	1.064	1.248
9	3	0.852	0.982	1.050	1.232
10	4	0.853	0.983	1.040	1.220
11	4	0.853	0.983	1.031	1.210
12	1	0.853	0.983	1.024	1.201
13	1	0.853	0.983	1.018	1.194
14	1	0.853	0.983	1.013	1.187
15	1	0.854	0.983	1.009	1.181
16	2	0.854	0.983	1.005	1.177
18	1	0.854	0.983	0.998	1.168
20	2	0.855	0.983	0.993	1.161
29	2	0.859	0.985	0.978	1.138
38	1	0.864	0.987	0.970	1.122
55	1	0.901	1.000	0.959	1.064

5 Concluding Remarks

This paper shows that the decentralized choice of the weight allocation rule in representative voting constitutes a Prisoner's Dilemma: the winner-take-all rule is a dominant strategy for each group, whereas the Nash equilibrium is Pareto dominated. We also show that the proportional rule Pareto dominates every other symmetric profile, when the number of the groups is sufficiently large. Each group has an incentive to put its entire weight on the alternative supported by the majority of its members in order to reflect their preferences in the social decision, although it fails to efficiently aggregate the preferences of all members in the society, if the winner-take-all rule is employed by all groups.

Our model may provide explanations for the phenomena that we observe in existing collective decision making. In the United States Electoral College, the rule used by the states varied in early elections until it converged by 1832 to the winner-take-all rule, which remains dominantly employed by nearly all states since then. In many parliamentary voting situations, we often observe parties and/or factions forcing their members to align their votes in order to maximally reflect their preferences in the social decision, although some members may disagree with the party's alignment. The voting outcome obtained by the winner-take-all rule may fail to efficiently aggregate preferences, as observed in the discrepancy between the electoral result and the national popular vote winner in the US presidential elections in 2000 and 2016. Party discipline or factional voting may also cause welfare loss when each group pushes their votes maximally toward their ideological goals, failing to reflect all members' preferences in the social decision.

The Winner-Take-All Dilemma tells us that the society should call for some device different from each group's unilateral effort, in order to obtain a more socially preferable outcome. As we see in the failure of various attempts to modify or abolish the winner-take-all rule, such as the ballot initiative for an amendment to the State Constitution in Colorado in 2004, each state has no incentive to unilaterally deviate from the equilibrium. The National Popular Vote Interstate Compact is a well-suited example of a coordination device (Koza et al. (2013)). As it comes into effect only when the number of electoral votes attains the majority, each state does not suffer from the payoff loss by unilateral (or coalitional) deviation until sufficient coordination is attained. The emergence of such an attempt is coherent with the insights obtained in this paper that the game is a Prisoner's Dilemma, and a coordination device is necessary for a Pareto improvement.

Our analysis is abstract in that we do not impose assumptions on the preferences distribution based on the observed characteristics in the real representative voting problems. Additionally, we impose an impartiality assumption in our asymptotic analysis. Obviously, our normative analysis would be best complemented by a positive analysis, which we leave for future research.

We have assumed that social decisions are binary. There are situations where this assumption may not fit. In the US presidential elections, third-party or independent candidates can, and do, have a non-negligible impact on the election outcome. It is not clear how the presence of such candidates alters the comparison of rules to allocate electoral votes. When the model is applied to legislative voting, the assumption of binary decision might be justified on the grounds that choices are ultimately made between the status quo and a proposal. However, such an argument abstracts away the process that gives rise to the particular pair of alternatives (e.g., what becomes the status quo, how much proposal power each party has, and so on). Cases with more than two alternatives require further investigation.

References

Atkinson, A. B. (1970): "On the measurement of inequality," *Journal of Economic Theory*, 2, 244–263.

Banzhaf, J. F. (1965): "Weighted Voting Doesn't Work: A Mathematical Analysis," *Rutgers Law Review*, 19, 317–343.

——— (1968): "One man, 3.312 votes: A mathematical analysis of the Electoral College," *Villanova Law Review*, 13, 304–332.

Barberà, S. and M. O. Jackson (2006): "On the weights of nations:

Assigning voting weights in a heterogeneous union," *Journal of Political Economy*, 114, 317–339.

Beisbart, C. and L. Bovens (2008): "A power analysis of Amendment 36 in Colorado," *Public Choice*, 134, 231–246.

Beisbart, C., L. Bovens, and S. Hartmann (2005): "A Utilitarian Assessment of Alternative Decision Rules in the Council of Ministers," *European Union Politics*, 6, 395–418.

Beisbart, C. and S. Hartmann (2010): "Welfarist evaluations of decision rules under interstate utility dependencies," *Social Choice and Welfare*, 34, 315–344.

Brams, S. J. and M. D. Davis (1974): "The 3/2's Rule in Presidential Campaigning," *Americal Political Science Review*, 68, 113–134.

Buchanan, H. and T. Hildebrandt (1908): "Note on Convergence of a Sequence of Functions of a Certain Type," *Annals of Mathematics*, 9, 123–126.

Bugh, G., ed. (2010): *Electoral College Reform: Challenges and Possibilities*, Ashgate Publishing.

De Mouzon, O., T. Laurent, M. Le Breton, and D. Lepelley (2019): "Exploring the effects of national and regional popular vote interstate compact on a toy symmetric version of the Electoral College: an electoral engineering perspective," *Public Choice*, 179, 51–95.

Edwards, G. C. (2004): *Why the Electoral College is Bad for America*, Yale University Press.

Eguia, J. X. (2011a): "Voting blocs, party discipline and party formation," *Games and Economic Behavior*, 73, 111–135.

——— (2011b): "Endogenous parties in an assembly," *American Journal of Political Science*, 55, 16–26.

Feix, M. R., D. Lepelley, V. R. Merlin, and J.-L. Rouet (2004): "The probability of conflicts in a U.S. presidential type election," *Economic Theory*, 23, 227–257.

Felsenthal, D. S. and M. Machover (1998): *The measurement of voting power*, Northampton: Edward Edgar.

Fields, G. S. and J. C. H. Fei (1978): "On inequality comparisons," *Econometrica*, 46, 303–316.

Gelman, A. (2003): "Forming Voting Blocs and Coalitions as a Prisoner's Dilemma: A Possible Theoretical Explanation for Political Instability," *The B.E. Journal of Economic Analysis and Policy*, 2, 1–16.

Hummel, P. (2011): "Proportional versus winner-take-all electoral vote allocations," *Public Choice*, 148, 381–393.

Kartal, M. (2015): "A Comparative Welfare Analysis of Electoral Systems with Endogenous Turnout," *Economic Journal*, 125, 1369–1392.

Koriyama, Y., J.-F. Laslier, A. Macé, and R. Treibich (2013): "Optimal Apportionment," *Journal of Political Economy*, 121, 584–608.

Koza, J. R., B. F. Fadem, M. Grueskin, M. S. Mandell, R. Richie, and J. F. Zimmerman (2013): *Every Vote Equal: A State-Based Plan for Electing the President by National Popular Vote*, National Popular Vote Press, 4th ed.

Kurz, S., N. Maaser, and S. Napel (2017): "On the Democratic Weights of Nations," *Journal of Political Economy*, 125, 1599–1634.

Laruelle, A. and F. Valenciano (2008): *Voting and Collective Decision-Making*, Cambridge University Press.

Le Breton, M., D. Lepelley, and H. Smaoui (2016): "Correlation, partitioning and the probability of casting a decisive vote under the majority rule," *Journal of Mathematical Economics*, 64, 11–22.

Lindbeck, A. and J. W. Weibull (1987): "Balanced-budget redistribution as the outcome of political competition," *Public Choice*, 52, 273–297.

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995): *Microeconomic Theory*, Oxford University Press.

May, K. (1948): "Probabilities of Certain Election Results," *American Mathematical Monthly*, 55, 203–209.

Mineka, J. and S. Silverman (1970): "A Local Limit Theorem and Recurrence Conditions for Sums of Independent and Non-Lattice Random Variables," *Annals of Probability*, 41, 592–600.

Moyes, P. (1994): "Inequality reducing and inequality preserving transformations of incomes: symmetric and individualistic transformations," *Journal of Economic Theory*, 63, 271–298.

Penrose, L. S. (1946): "The elementary statistics of majority voting," *Journal of the Royal Statistical Society*, 109, 53–57.

Rae, D. W. (1946): "Decision-Rules and Individual Values in Constitutional Choice," *American Political Science Review*, 63, 40–56.

Shapley, L. S. and M. Shubik (1954): "A method for evaluating the distribution of power in a committee system," *American Political Science Review*, 48, 787–792.

Strömberg, D. (2008): "How the Electoral College Influences Campaigns and Policy: The Probability of Being Florida," *American Economic Review*, 98, 769–807.

Appendix
"The Winner-Take-All Dilemma"
Kazuya Kikuchi and Yukio Koriyama

A1 Proof of Lemma 1

Preliminaries. In this proof, we denote by ϕ^λ the generalized proportional profile with coefficients $\lambda \in [-1,1]^n \setminus \{0\}$:

$$\phi_i^\lambda(\theta_i) = \lambda_i \theta_i,\ i = 1, \cdots, n,$$

which should not be confused with the notation ϕ^a for mixed profiles.

We write $\pi(\phi) = (\pi_i(\phi))_{i=1}^n$ for the vector of payoffs. Let Π be the set of all possible payoff vectors in game Γ:

$$\Pi = \{\pi(\phi) : \phi \text{ is a profile}\}.$$

For any $X \subset \mathbb{R}^n$, let Pareto$\,(X)$ be the Pareto frontier of X, i.e., the set of points $x \in X$ for which there exists no $y \in X$ such that $y_i \geq x_i$ for all i, with strict inequality for at least one i. Let $\operatorname{co} X$ denote the closed convex hull of X.

We will refer to the following maximization problem parametrized by vector $q \in \mathbb{R}_+ \setminus \{0\}$ as Problem M_q:

Problem $\mathbf{M}_q$: $\max_{x \in \operatorname{co}\Pi} q \cdot x.$

Note that the maximization is *not* directly with respect to profile ϕ. Moreover, Π may be non-closed or non-convex. Thus, there may be a solution $x \in \operatorname{co}\Pi$ that is not the payoff vector of any profile ϕ, although we will later disprove this possibility.

We divide the proof of Lemma 1 into several claims. Claims 2.1-2.4 concern properties of the solutions to Problem M_q. Claim 2.5 describes the relation between Problem M_q and Pareto efficiency. Finally, Claim 2.6 completes the proof.

Claim 2.1. *A solution of Problem M_q is $x = \pi(\phi^{\lambda^q})$, where $\lambda_i^q = cq_i/w_i$,*

$i = 1, \cdots, n$.[19]

Proof of Claim 2.1. Recall that the payoff is given by

$$\pi_i(\phi) = \mathbb{E}\left(\Theta_i \operatorname{sgn} S_\phi\right). \tag{9}$$

Thus, for any profile ϕ,

$$q \cdot \pi(\phi) = \mathbb{E}\left[(q \cdot \Theta)\left(\operatorname{sgn} S_\phi\right)\right] \leq \mathbb{E}(|q \cdot \Theta|). \tag{10}$$

That is, $q \cdot x \leq \mathbb{E}(|q \cdot \Theta|)$ for any $x \in \Pi$. The linearity of the objective function $q \cdot x$ implies that $q \cdot x \leq \mathbb{E}(|q \cdot \Theta|)$ for all $x \in \operatorname{co} \Pi$. If $\phi = \phi^{\lambda^q}$, then S_ϕ has the same sign as $q \cdot \Theta$. Thus for $x = \pi(\phi^{\lambda^q})$, we have $q \cdot x = \mathbb{E}(|q \cdot \Theta|)$. □

Claim 2.2. *Let* $v(q) := \max_{x \in \operatorname{co} \Pi} q \cdot x$ *be the maximum value of Problem* M_q. *Then*

$$v(q) = \mathbb{E}(|q \cdot \Theta|).$$

Proof of Claim 2.2. This follows from the proof of Claim 2.1, in which we showed that $q \cdot x \leq \mathbb{E}(|q \cdot \Theta|)$ for all $x \in \operatorname{co} \Pi$ and the upper bound is attained by $x = \pi(\phi^{\lambda^q})$. □

Claim 2.3. *A profile* ϕ *satisfies* $q \cdot \pi(\phi) = v(q)$ *if and only if* ϕ *is equivalent to* ϕ^{λ^q}.

Proof of Claim 2.3. Since $q \neq 0$ and Θ is absolutely continuous, we have $q \cdot \Theta \neq 0$ almost surely. Thus, (10) holds with equality if and only if

$$\operatorname{sgn} S_\phi = \operatorname{sgn}(q \cdot \Theta) \text{ almost surely.}$$

Since $cq \cdot \Theta = S_{\phi^{\lambda^q}}$, this holds if and only if ϕ is equivalent to ϕ^{λ^q}. □

Claim 2.4. $x = \pi(\phi^{\lambda^q})$ *is the unique solution of Problem* M_q.

Proof of Claim 2.4. We use the absolute continuity of Θ to show that the value function $v(q) = \mathbb{E}(|q \cdot \Theta|)$ is differentiable, with gradient $\nabla v(q) =$

[19] $c > 0$ is a constant such that $cq_i / w_i \leq 1$ for all i.

$\pi(\phi^{\lambda^q})$. Then the uniqueness follows by the Duality Theorem (Mas-Colell et al. (1995, Proposition 3.F.1)).

To show that $v(q)$ is differentiable, it suffices to show that as vector $\varepsilon \in \mathbb{R}^n$ approaches 0,

$$v(q+\varepsilon) - v(q) - \pi(\phi^{\lambda^q}) \cdot \varepsilon = o(\|\varepsilon\|). \tag{11}$$

Using (9) and Claim 2.2, we can rewrite the left-hand side of (11) as:

$$\begin{aligned}
& v(q+\varepsilon) - v(q) - \pi(\phi^{\lambda^q}) \cdot \varepsilon \\
&= \mathbb{E}\left[\{(q+\varepsilon)\cdot\Theta\} \times \operatorname{sgn}\{(q+\varepsilon)\cdot\Theta\}\right] \\
&\quad - \mathbb{E}\left[(q\cdot\Theta) \times \operatorname{sgn}(q\cdot\Theta)\right] \\
&\quad - \mathbb{E}\left[(\varepsilon\cdot\Theta) \times \operatorname{sgn}(q\cdot\Theta)\right] \\
&= \mathbb{E}[\{(q+\varepsilon)\cdot\Theta\} \times \{\operatorname{sgn}((q+\varepsilon)\cdot\Theta) - \operatorname{sgn}(q\cdot\Theta)\}].
\end{aligned}$$

This expression has the following bound:

$$|v(q+\varepsilon) - v(q) - \pi(\phi^{\lambda^q}) \cdot \varepsilon| \leq 2\mathbb{E}\left(|(q+\varepsilon)\cdot\Theta|\, 1_{\{\operatorname{sgn}((q+\varepsilon)\cdot\Theta) \neq \operatorname{sgn}(q\cdot\Theta)\}}\right).$$

The expectation on the right-hand side is

$$\int_{A^+_{q,\varepsilon}} \{(q+\varepsilon)\cdot\theta\} h(\theta) d\theta - \int_{A^-_{q,\varepsilon}} \{(q+\varepsilon)\cdot\theta\} h(\theta) d\theta, \tag{12}$$

where h is the joint density of Θ and

$$\begin{aligned}
A^+_{q,\varepsilon} &= \{\theta \in [-1,1]^n : (q+\varepsilon)\cdot\theta \geq 0 \geq q\cdot\theta\}, \\
A^-_{q,\varepsilon} &= \{\theta \in [-1,1]^n : (q+\varepsilon)\cdot\theta \leq 0 \leq q\cdot\theta\}.
\end{aligned}$$

We show that for ε sufficiently close to 0, $(q+\varepsilon)\cdot\theta \leq \sqrt{n}\|\varepsilon\|$ for all $\theta \in A^+_{q,\varepsilon}$. To do this, we fix a sufficiently small ε so that for each $e \in \{-1,1\}^n$ (i.e., each vertex of the hypercube $[-1,1]^n$), either both $q\cdot e$ and $(q+\varepsilon)\cdot e$ are non-negative or both are non-positive.[20] Now, consider the following linear-programming problem ($\mathrm{L}_{q,\varepsilon}$):

[20] For each vertex $e \in \{-1,1\}^n$ there is $\delta_e > 0$ such that if $\|\varepsilon\| < \delta_e$ then either both $q\cdot e$ and $(q+\varepsilon)\cdot e$ are non-negative or both are non-positive. Thus, it suffices to choose ε so that $\|\varepsilon\| < \min\{\delta_e : e \in \{-1,1\}^n\}$.

Problem $\mathbf{L}_{q,\varepsilon}$: $\max_{\theta \in A^+_{q,\varepsilon}} (q+\varepsilon)\cdot\theta.$

Let θ^* be a solution of Problem $\mathrm{L}_{q,\varepsilon}$ that is a vertex of $A^+_{q,\varepsilon}$. Then θ^* belongs to at least one of the following sets:

$$H_{q+\varepsilon} = \{\theta : (q+\varepsilon)\cdot\theta = 0\},$$
$$H_q = \{\theta : q\cdot\theta = 0\},$$
$$\{-1,1\}^n.$$

We claim that $\theta^* \in H_q$. First, we have $\theta^* \notin H_{q+\varepsilon}$, since otherwise θ^* minimizes the objective function $(q+\varepsilon)\cdot\theta$ subject to $\theta \in A^+_{q,\varepsilon}$, while the n-dimensional polytope $A^+_{q,\varepsilon}$ contains points that attain larger values of the function. Now, suppose $\theta^* \in \{-1,1\}^n \setminus H_q$. The fact that $\theta^* \in \{-1,1\}^n \cap A^+_{q,\varepsilon} \cap H^c_q \cap H^c_{q+\varepsilon}$ implies that θ^* is a vertex of the hypercube $[-1,1]^n$ such that $q\cdot\theta^* < 0 < (q+\varepsilon)\cdot\theta^*$. This contradicts the fact that for any vertex e of the hypercube, either both $(q+\varepsilon)\cdot e$ and $q\cdot e$ are non-negative or both are non-positive. Therefore $\theta^* \in H_q$.

We have shown that $q\cdot\theta^* = 0$. This implies that for any $\theta \in A^+_{q,\varepsilon}$, $(q+\varepsilon)\cdot\theta \le (q+\varepsilon)\cdot\theta^* = \varepsilon\cdot\theta^* \le \|\theta^*\|\|\varepsilon\| \le \sqrt{n}\|\varepsilon\|$. It similarly follows that $-(q+\varepsilon)\cdot\theta \le \sqrt{n}\|\varepsilon\|$ for any $\theta \in A^-_{q,\varepsilon}$. Therefore, (12) is bounded by $\sqrt{n}\|\varepsilon\| \int_{A^+_{q,\varepsilon}\cup A^-_{q,\varepsilon}} h(\theta)d\theta$. Noting that the integral $\int_{A^+_{q,\varepsilon}\cup A^-_{q,\varepsilon}} h(\theta)d\theta$ vanishes as $\varepsilon \to 0$, we obtain (11). □

Claim 2.5. *Let $x \in \operatorname{co}\Pi$. Then, $x \in \operatorname{Pareto}(\operatorname{co}\Pi)$ if and only if there exists $q \in \mathbb{R}^n_+ \setminus \{0\}$ such that x is the unique solution of Problem M_q, i.e., $x = \pi(\phi^{\lambda^q})$.*

Proof Claim 2.5. For any $x \in \mathbb{R}^n$, let $D(x) = \{x + a : a \in \mathbb{R}^n_+ \setminus \{0\}\}$ be the (convex) set of all points that dominate x. Note that $x \in \operatorname{Pareto}(\operatorname{co}\Pi)$ if and only if $D(x) \cap \operatorname{co}\Pi = \emptyset$. To prove Claim 2.5, suppose $x \in \operatorname{Pareto}(\operatorname{co}\Pi)$. Then there exists a hyperplane with some normal vector $q \in \mathbb{R}^n_+ \setminus \{0\}$ that separates $\operatorname{co}\Pi$ and $D(x)$.[21] Clearly this hyperplane contains x, which means that x is the solution of Problem M_q. Conversely, suppose x is the unique solution of Problem M_q. Then the supporting hyperplane of

[21] Here, separation is in the weak sense that the hyperplane may contain boundary points of the two sets.

$\operatorname{co}\Pi$ with normal vector q separates $\operatorname{co}\Pi$ and $D(x)$. The uniqueness of the solution implies that the hyperplane intersects $\operatorname{co}\Pi$ only at x. This implies that $D(x) \cap \operatorname{co}\Pi = \emptyset$. □

Claim 2.6. *A profile ϕ satisfies $\pi(\phi) \in \text{Pareto}(\Pi)$ if and only if there exists $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ such that ϕ is equivalent to ϕ^λ. That is, Lemma 1 holds.*

Proof of Claim 2.6. By Claim 2.5,

$$\text{Pareto}(\operatorname{co}\Pi) = \text{Pareto}(\Pi) = \{\pi(\phi^{\lambda^q}) : q \in \mathbb{R}^n_+ \setminus \{0\}\}.$$

By Claim 2.3, $\pi(\phi)$ belongs to this set if and only if ϕ is equivalent to ϕ^{λ^q} for some $q \in \mathbb{R}^n_+ \setminus \{0\}$. This condition is the same as saying that ϕ is equivalent to ϕ^λ for some $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$. □

A2 Proof of Part (i) of Lemma 2

We prove the statement for group 1. Let $\pi_1(\phi; n|\theta_1)$ be the conditional expected payoff for group 1 given that the group-wide margin is $\Theta_1 = \theta_1$, which by (2) is:

$$\pi_1(\phi; n|\theta_1) = \theta_1(\mathbb{P}\{w_1\phi(\theta_1) + S_{\phi_{-1}} > 0\} - \mathbb{P}\{w_1\phi(\theta_1) + S_{\phi_{-1}} < 0\}).$$

Since $S_{\phi_{-1}}$ is symmetrically distributed, the second probability can be written as $\mathbb{P}\{-w_1\phi(\theta_1) + S_{\phi_{-1}} > 0\}$. Thus, for $\theta_1 \in [0, 1]$, the above expression equals

$$\pi_1(\phi; n|\theta_1) = \theta_1\mathbb{P}\{-w_1\phi(\theta_1) < S_{\phi_{-1}} \leq w_1\phi(\theta_1)\}.$$

By symmetry, twice the integral of this expression over $\theta_1 \in [0, 1]$ (instead of $[-1, 1]$) equals the unconditional expected payoff $\pi_1(\phi; n)$, which proves part (i) of Lemma 2. □

A3 Local Limit Theorem

We quote a version of the Local Limit Theorem shown in Mineka and Silverman (1970). We will use it in the proof of part (ii) of Lemma 2.

LLT. *(Mineka and Silverman (1970, Theorem 1)) Let* (X_i) *be a sequence of independent random variables with mean 0 and variances* $0 < \sigma_i^2 < \infty$*. Write* F_i *for the distribution of* X_i*. Write also* $S_n = \sum_{i=1}^n X_i$ *and* $s_n^2 = \sum_{i=1}^n \sigma_i^2$*. Suppose the sequence* (X_i) *satisfies the following conditions:*

(α) *There exists* $\bar{x} > 0$ *and* $c > 0$ *such that for all* i*,*

$$\frac{1}{\sigma_i^2}\int_{|x|<\bar{x}} x^2 dF_i(x) > c.$$

(β) *Define the set*

$$A(t,\varepsilon) = \{x : |x| < \bar{x} \text{ and } |xt - \pi m| > \varepsilon \text{ for all integer } m \text{ with } |m| < \bar{x}\}.$$

Then, for some bounded sequence (a_i) *such that* $\inf_i \mathbb{P}\{|X_i - a_i| < \delta\} > 0$ *for all* $\delta > 0$*, and for any* $t \neq 0$*, there exists* $\varepsilon > 0$ *such that*

$$\frac{1}{\log s_n}\sum_{i=1}^n \mathbb{P}\{X_i - a_i \in A(t,\varepsilon)\} \to \infty.$$

(γ) *(Lindeberg's condition.) For any* $\varepsilon > 0$*,*

$$\frac{1}{s_n^2}\sum_{i=1}^n \int_{|x|/s_n > \varepsilon} x^2 dF_i(x) \to 0.$$

Under conditions (α)*-*(γ)*, if* $s_n^2 \to \infty$*, we have*

$$\sqrt{2\pi s_n^2}\,\mathbb{P}\{S_n \in (a,b]\} \to b - a.\text{[22]} \tag{13}$$

[22] The original conclusion of Theorem 1 in Mineka and Silverman (1970) is stated in terms of the open interval (a, b). Applying the theorem to $(a, b + c)$ and $(b, b + c)$ and then taking the difference gives the result for $(a, b]$. In addition, the original statement allows for cases where s_n^2 does not go to infinity, and also mentions uniform convergence. These considerations are not necessary for our purpose, so we omit them.

A4 Proof of Lemma 3

Preliminaries. We prove the lemma for group 1. In the proof, we use the notation of LLT. Let

$$X_i := w_i\phi(\Theta_i, w_i),\ i = 1, 2, \cdots,$$

and $S_n := \sum_{i=1}^n X_i$. Then X_i has mean 0 and variance $\sigma_i^2 := w_i^2\mathbb{E}[\phi(\Theta, w_i)^2]$, and so the partial sum of variances is $s_n^2 := \sum_{i=1}^n w_i^2\mathbb{E}[\phi(\Theta, w_i)^2]$, where Θ represents a random variable that has the same distribution F as Θ_i.

Define the event

$$\Omega_n(\theta_1, w_1) = \left\{-w_1\phi(\theta_1, w_1) < \sum_{i=2}^n X_i \le w_1\phi(\theta_1, w_1)\right\}.$$

We divide the proof into several claims. Claims 5.1-5.3 show that the sequence (X_i) defined above satisfies the conditions of the Local Limit Theorem (LLT) in Section A4. Claim 5.4 applies LLT to complete the proof of Lemma 3.

Claim 5.1. $\frac{s_n^2}{n} \to \int_{\underline{w}}^{\bar{w}} w^2\mathbb{E}[\phi(\Theta, w)^2]dG(w)$.

Proof of Claim 5.1. This holds since sequence (σ_i^2) is bounded and the statistical distribution G_n induced by $(w_i)_{i=1}^n$ converges weakly to G. □

Claim 5.2. *Conditions (α) and (γ) in LLT hold.*

Proof of Claim 5.2. This immediately follows from the fact that sequence (X_i) is bounded and $s_n^2 \to \infty$. In particular, it is enough to define $\bar{x}$ to be any finite number greater than $\bar{w}$. □

Claim 5.3. *Condition (β) in LLT holds.*

Proof of Claim 5.3. Recall that ϕ has the form

$$w_i\phi(\theta_i, w_i) = h_1(w_i)h_2(\theta_i) + h_3(w_i)\operatorname{sgn}\theta_i.$$

Let $a_i = h_3(w_i)$. We first check that the sequence (a_i) satisfies the requirements in condition (β). First, (a_i) is bounded since h_3 is bounded.

Now, for any i and any $\delta > 0$,

$$\begin{aligned}\mathbb{P}\{|X_i - a_i| < \delta\} &\geq \mathbb{P}\{|X_i - a_i| < \delta \text{ and } \Theta_i > 0\} \\ &\geq \mathbb{P}\left\{|w_i\phi(\Theta_i, w_i) - h_3(w_i)\operatorname{sgn}\Theta_i| < \delta \text{ and } \Theta_i > 0\right\} \\ &= \mathbb{P}\{|h_1(w_i)h_2(\Theta_i)| < \delta \text{ and } \Theta_i > 0\}.\end{aligned}$$

Letting $\bar{h}_1 > 0$ be an upper bound of $|h_1|$ and Θ a random variable distributed as Θ_i, the last expression has the following lower bound independent of i:

$$\mathbb{P}\{|h_2(\Theta)| < \delta/\bar{h}_1 \text{ and } \Theta > 0\} > 0,$$

which is positive by the assumptions on h_2 and on the distribution of Θ.

Next we check the limit condition in (β). Recall that $A(t, \varepsilon)$ is the union of intervals

$$\left(\frac{\pi m + \varepsilon}{|t|}, \frac{\pi(m+1) - \varepsilon}{|t|}\right), \ m = 0, \pm 1, \pm 2, \cdots,$$

restricted to $(-\bar{x}, \bar{x})$, where we can choose $\bar{x}$ to be any number greater than $\bar{w}$. To prove the limit condition in (β), it therefore suffices to verify that one such interval contains $X_i - a_i$ with probability bounded away from zero, for all groups i in some sufficiently large subset of groups. To do this, note that if $\Theta_i < 0$, then $X_i - a_i = h_1(w_i)h_2(\Theta_i) - 2h_3(w_i)$. The assumptions on h_2 and on the distribution of Θ imply that for any $\eta > 0$, there exists a set $O_\eta \subset [-1, 0]$ with $\mathbb{P}\{\Theta \in O_\eta\} > 0$ such that if $\Theta \in O_\eta$ then $|h_2(\Theta)| \leq \eta$. Therefore,

$$\Theta_i \in O_\eta \Rightarrow X_i - a_i \in T_{w_i,\eta},$$

where

$$T_{w_i,\eta} := [-2h_3(w_i) - \eta h_1(w_i), \ -2h_3(w_i) + \eta h_1(w_i)].$$

Since h_1 is bounded, we can make $T_{w_i,\eta}$ an arbitrarily small interval around $-2h_3(w_i)$ by letting $\eta > 0$ sufficiently small. Moreover, since h_3 is continuous and not a constant, we can find a sufficiently small interval $[\underline{v}, \bar{v}] \subset [\underline{w}, \bar{w}]$ with $\underline{v} < \bar{v}$ such that if $w_i \in [\underline{v}, \bar{v}]$, then $-2h_3(w_i)$ is between, and bounded away from, $\frac{\pi m}{|t|}$ and $\frac{\pi(m+1)}{|t|}$ for some integer m. Fix such an interval $[\underline{v}, \bar{v}]$ and define

$$I := \{i : w_i \in [\underline{v}, \bar{v}]\}.$$

Then, for sufficiently small $\eta > 0$ and $\varepsilon > 0$, we have $T_{w_i,\eta} \subset A(t,\varepsilon)$ for all $i \in I$. Fixing such $\eta > 0$ and $\varepsilon > 0$, it follows that

$$\Theta_i \in O_\eta \text{ and } i \in I \Rightarrow X_i - a_i \in A(t,\varepsilon).$$

This implies that

$$\mathbb{P}\{X_i - a_i \in A(t,\varepsilon)\} \geq \mathbb{P}\{\Theta \in O_\eta\} =: p > 0 \text{ for all } i \in I,$$

and hence

$$\frac{1}{\log s_n} \sum_{i=1}^{n} \mathbb{P}\{X_i - a_i \in A(t,\varepsilon)\} \geq \frac{n}{\log s_n} \cdot \frac{\sharp\{i \in I : i \leq n\}}{n} \cdot p.$$

As $n \to \infty$, the first factor on the right-hand side tends to ∞ since s_n has an asymptotic order of $\sqrt{n}$. The second factor tends to $G(\bar{v}) - G(\underline{v}) > 0$, which is positive since G has full support on $[\underline{w}, \bar{w}]$. Therefore the left-hand side tends to ∞. □

Claim 5.4. *As $n \to \infty$, uniformly in $w_1 \in [\underline{w}, \bar{w}]$,*

$$2\int_0^1 \theta_1 \sqrt{2\pi n}\mathbb{P}\{\Omega_n(\theta_1, w_1)\} dF(\theta_1) \to \frac{2w_1 \mathbb{E}[\Theta\phi(\Theta, w_1)]}{\sqrt{\int_{\underline{w}}^{\bar{w}} w^2 \mathbb{E}[\phi(\Theta, w)^2] dG(w)}}. \quad (14)$$

By part (i) of Lemma 2,[23] *the left-hand side of (14) is $\sqrt{2\pi n}\pi_i(\phi; n)$, and therefore Lemma 3 holds.*

Proof of Claim 5.4. By Claims 5.2 and 5.3, we may apply LLT to obtain

$$\sqrt{2\pi s_n^2}\mathbb{P}\{\Omega_n(\theta_1, w_1)\} \to 2w_1\phi(\theta_1, w_1).$$

By Claim 5.1, this means that

$$\sqrt{2\pi n}\theta_1 \mathbb{P}\{\Omega_n(\theta_1, w_1)\} \to \frac{2w_1\theta_1\phi(\theta_1, w_1)}{\sqrt{\int_{\underline{w}}^{\bar{w}} w^2 \mathbb{E}[\phi(\Theta, w)^2] dG(w)}}. \quad (15)$$

Letting $\theta_1 = 1$ maximizes the left-hand side of (15) with the maximum

[23] It is easy to check that part (i) of Lemma 2 holds for rules $\phi(\cdot, w_i)$ that depend on weight w_i as well.

value $\sqrt{2\pi n}\mathbb{P}\{\Omega_n(1, w_1)\}$. This maximum value itself converges to a finite limit. Hence the expression $\sqrt{2\pi n}\theta_1\mathbb{P}\{\Omega_n(\theta_1, w_1)\}$ is uniformly bounded for all n and $\theta_1 \in [0,1]$. By the Bounded Convergence Theorem,

$$2\int_0^1 \theta_1\sqrt{2\pi n}\mathbb{P}\{\Omega_n(\theta_1, w_1)\}dF(\theta_1) \to 2\cdot\frac{2w_1\int_0^1 \theta_1\phi(\theta_1, w_1)dF(\theta_1)}{\sqrt{\int_{\underline{w}}^{\bar{w}} w^2\mathbb{E}[\phi(\Theta, w)^2]dG(w)}}.$$

Since F is symmetric and ϕ is odd, this limit is exactly the one in (14).

To check the uniform convergence, note that for each n, the integral on the left-hand side of (14) is non-decreasing in w_1, since event $\Omega_n(\theta_1, w_1)$ weakly expands as w_1 increases.[24] We have shown that this integral converges pointwise to a limit that is proportional to the factor $w_1\mathbb{E}[\Theta\phi(\Theta, w_1)]$, which is continuous in w_1.[25] Therefore, the convergence in (14) is uniform in $w_1 \in [\underline{w}, \bar{w}]$.[26] □

A5 Proof of Part (ii) of Lemma 2

This follows immediately from Lemma 3, by noting that if ϕ is a symmetric profile, each group's rule can be written as $\phi(\theta_j, w_j) = \phi(\theta_j)$. □

A6 Proof of Proposition 2

By part (ii) of Lemma 2, we must show that $\mathrm{Corr}\,[\Theta, \phi^a(\Theta)]$ is decreasing in $a \in [0,1]$. By simple calculation,

$$\mathbb{E}(\Theta^2)\cdot\mathrm{Corr}\,[\Theta, \phi^a(\Theta)]^2 = \frac{a\mathbb{E}(|\Theta|) + (1-a)\mathbb{E}(\Theta^2)}{a^2 + 2a(1-a)\mathbb{E}(|\Theta|) + (1-a)^2\mathbb{E}(\Theta^2)}.$$

[24] Let $\theta_1 \in [0,1]$. If ϕ is a symmetric profile, i.e. if $\phi(\theta_1, w_1) = \phi(\theta_1)$, then $w_1\phi(\theta_1)$ is non-decreasing in w_1. If $\phi = \phi^{\mathrm{CD}}$, then $w_1\phi^{\mathrm{CD}}(\theta_1, w_1) = c\,\mathrm{sgn}(\theta_1) + (w_1 - c)\theta_1$, which is non-decreasing in w_1 again. Thus event $\Omega_n(\theta_1, w_1)$ weakly expands as w_1 increases.

[25] If ϕ is a symmetric profile, this factor is linear in w_i. If $\phi = \phi^{\mathrm{CD}}$, the factor equals $c\mathbb{E}(|\Theta|) + (w_i - c)\mathbb{E}(\Theta^2)$, which is affine in w_i.

[26] It is known that if (f_n) is a sequence of non-decreasing functions on a fixed finite interval and f_n converges pointwise to a continuous function, then the convergence is uniform. See Buchanan and Hildebrandt (1908).

The derivative of this expression with respect to a has the same sign as

$$\left\{\tfrac{d}{da}(a\mathbb{E}(|\Theta|)+(1-a)\mathbb{E}(\Theta^2))^2\right\}\Big(a^2+2a(1-a)\mathbb{E}(|\Theta|)+(1-a)^2\mathbb{E}(\Theta^2)\Big)$$
$$-\Big(a\mathbb{E}(|\Theta|)+(1-a)\mathbb{E}(\Theta^2)\Big)^2\left\{\tfrac{d}{da}(a^2+2a(1-a)\mathbb{E}(|\Theta|)+(1-a)^2\mathbb{E}(\Theta^2))\right\}$$
$$=a(a\mathbb{E}(|\Theta|)+(1-a)\mathbb{E}(\Theta^2))(\mathbb{E}(|\Theta|)^2-\mathbb{E}(\Theta^2)).$$

This is negative for any $a \in (0,1]$, since $\mathbb{E}(|\Theta|)^2 \leq \mathbb{E}(\Theta^2)$ in general, and the full-support assumption implies that this holds with strict inequality. □

A7 Proof of Theorem 4

Clearly, Lorenz dominance is invariant to linear transformations of payoffs. Thus, it suffices to prove that for large enough n, the payoff profile $\sqrt{2\pi n}\pi(\phi^{\mathrm{CD}};n)$ Lorenz dominates the payoff profile $\sqrt{2\pi n}\pi(\phi;n)$. By equations (6) and (7) in the proof of Theorem 3, as $n \to \infty$ these amounts converge to $Bw_i + C$ and $A^\phi w_i$, respectively. A result by Moyes (1994, Proposition 2.3) implies that if f and g are continuous, nondecreasing, and positive-valued functions such that $f(w_i)/g(w_i)$ is decreasing in w_i, then the distribution of $f(w_i)$ Lorenz dominates that of $g(w_i)$. The ratio $(Bw_i+C)/(A^\phi w_i)$ is decreasing in w_i, and so the claimed Lorenz dominance holds in the limit as $n \to \infty$. Recalling that the convergences are uniform, the dominance holds for sufficiently large n. □