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BEHAVIOR-BASED PRICE DISCRIMINATION AND PRODUCT CHOICE

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Behavior-Based Price Discrimination and Product Choice*

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Abstract

We study a two-period model of behavior-based price discrimination in Fudenberg and Tirole (2000) but allow firms to make product choice in the first period. We show that the only possible equilibrium involves maximal differentiation. This is in contrast to Choe et al. (2018) where equilibrium features less than maximal differentiation when competition is in personalized pricing. Thus, our result highlights an important interplay between the type of price competition and product choice.

Keywords: Behavior-based price discrimination, spatial competition
JEL Classification Number: D43, L13

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1 Introduction

Behavior-base price discrimination (BBPD) refers to the practice whereby firms condition their price offers on customers' purchase histories. The advances in technologies have lowered the costs of firms' investment in customer information, leading to proliferation of various types of BBPD.[1] In the simplest two-period model of BBPD, firms segment a market into two based on the first-period purchase, existing customers and new customers, and exercise third-degree price discrimination in the second period.

Existing studies on BBPD typically focus only on the pricing game assuming product differentiation is exogenously fixed. For example, Fudenberg and Tirole (2000) and many subsequent studies consider a two-period Hotelling model taking maximal differentiation as given. While maximal differentiation is an equilibrium outcome in the static Hotelling model with quadratic transportation cost, it is not obvious whether it continues to be so in the dynamic context. Suppose a firm chooses an interior location in the first period while its rival chooses the opposite end. In the second period, the first firm is in a more strategic position than its rival, which it may be able to leverage and poach the rival's customers more effectively.

This line of argument suggests that maximal differentiation may not be an equilibrium outcome when locations are chosen in the first period. Indeed this is shown by Choe et al. (2018): when firms choose locations in the first period and compete in *personalized pricing* in the second period, BBPD results in equilibrium where one firm chooses an interior location. But we are not aware of any study that endogenizes location choice when the second-period competition is in third-degree price discrimination. The purpose of our study is to fill this gap.

Our main result is that, unlike Choe et al. (2018), maximal differentiation is the only possible equilibrium outcome. The intuition is as follows. When the second-period competition is in third-degree price discrimination, the firm with a larger market share loses more customers to its rival. The reason is that, with third-degree price discrimination, a firm has to charge the same price to all its loyal customers, some of whom are inevitably closer to the rival when the firm has a larger market share. Thus the second period does not matter much to the firm's decision in the first period. As a result, firms choose maximal differentiation to soften competition in the first period.

Given that the result in Choe et al. (2018) is based on competition in personalized pricing, our result is driven mainly by third-degree price discrimination. This highlights an important interplay between the type of price competition and product choice. We proceed below by providing a brief literature review, followed by the model and analysis, and some discussions and conclusion.

[1] See Ezrachi and Stucke (2016) for various examples of BBPD.

2 Related Literature

The first strand of literature on BBPD shows that BBPD generally lowers firm profitability by intensifying competition, unless there are sufficient asymmetries at firm- or consumer-level.[2] This is true whether firms compete in third-degree price discrimination (Chen, 1997; Villas-Boas, 1999; Fudenberg and Tirole, 2000; Pazgal and Soberman, 2008; Esteves, 2010) or in personalized pricing (Zhang, 2011; Choe et al., 2018).

The second strand of literature introduces various asymmetries and shows how BBPD can improve profitability. Such asymmetries include enhanced services that firms can offer only to loyal customers (Acquisti and Varian, 2005; Pazgal and Soberman, 2008), quality difference between firms (Jing, 2017; Rhee and Thomadsen, 2017), asymmetry in consumer preferences (Chen and Zhang, 2009; Shin and Sudhir, 2010), or consumers' fairness concern (Li and Jain, 2016).

Our paper belongs to the first strand. The general intuition behind the competition-intensifying effect of BBPD is that customer information made available from past purchases makes firms more aggressive in pricing. Choe et al. (2018) further shows that firms also make more aggressive product choice when competition is in personalized pricing.[3] But all the above studies that assume third-degree price discrimination take exogenously fixed product choice as given. Thus it remains unanswered if the result in Choe et al. (2018) continues to hold when competition is in third-degree price discrimination. Our paper addresses this question.

3 The Model

Our model is an extension of Fudenberg and Tirole (2000) where we incorporate location choice by firms. There are two periods, $\tau = 1, 2$. Consumers are located uniformly over a Hotelling linear city $[0, 1]$, and each consumer's location stays the same over the two periods. Each consumer buys one unit of good in each period and derives utility v from each unit. We assume v is sufficiently large so that the entire market is covered in equilibrium. Two firms, $i = A, B$, have the same constant marginal cost of production, which is normalized to zero. Consumers have quadratic transportation costs: if firm i is located at a and sets a price p_i, then consumer x gets a surplus of $v - p_i - t(x - a)^2$ by purchasing from firm i.

In $\tau = 1$, competition follows the standard Hotelling model. Firms simultaneously

[2] See Chen (2005) or Fudenberg and Villas-Boas (2006, 2012) for more comprehensive reviews.

[3] Zhang (2011) considers a two-period model with personalized pricing in the second period, but allows costless personalization of products as well as prices. Thus she departs from the standard BBPD assumption that price is the only choice variable in the second period. Her assumption of product personalization leads to substantially different results from ours. For example, there is no customer poaching in Zhang (2011) in contrast to ours.

choose locations which are fixed over two periods, after which they compete in price. Let $\mathcal{A}$ ($\mathcal{B}$) be the set of consumers that choose firm A (B) in $\tau = 1$. In $\tau = 2$, firms compete using third-degree price discrimination where each firm chooses two prices, one for its $\tau = 1$ customers and the other for its rival's $\tau = 1$ customers, the latter we call the poaching price.

In making $\tau = 1$ decisions, firms discount $\tau = 2$ profits by $\delta_f \in [0, 1]$ and consumers discount $\tau = 2$ surplus by $\delta_c \in [0, 1]$. As shown in Choe et al. (2018), however, solving the game for general discount factors is not possible even in the simpler case where the $\tau = 2$ competition is in personalized pricing. Accordingly, for our main result (Proposition 1), we follow Fudenberg and Tirole (2000) and assume the common discount factor $\delta_c = \delta_f = \delta \in [0, 1]$. For additional results (Propositions 2, 3), we follow Choe et al. (2018) and assume consumers are myopic in that $\delta_c = 0$.

4 Analysis

Fix firm A's location at a and firm B's location at b and, without loss of generality, assume $0 \leq a \leq b \leq 1$. Let z be the marginal consumer in $\tau = 1$ who is indifferent between choosing either firm in $\tau = 1$. For $i = A, B$, denote firm i's $\tau = 1$ price by p_i, its $\tau = 2$ price for its own $\tau = 1$ customers by p_{io}, and its $\tau = 2$ poaching price by p_{in}. Thus in $\tau = 2$, firm A chooses p_{Ao} for consumers in $\mathcal{A}$ and p_{An} for consumers in $\mathcal{B}$. Similarly firm B chooses p_{Bo} for consumers in $\mathcal{B}$ and p_{Bn} for consumers in $\mathcal{A}$. We solve the game backwards.

4.1 Second period

Given the $\tau = 1$ marginal consumer z, we have $\mathcal{A} = [0, z]$ and $\mathcal{B} = [z, 1]$. In each set, there may be consumers who want to switch to a new firm in $\tau = 2$. Let $z_A \in \mathcal{A}$ be a marginal consumer such that consumers in $[0, z_A]$ continue to choose firm A while those in $[z_A, z]$ switch to firm B. Then z_A satisfies

$$p_{Ao} + t(z_A - a)^2 = p_{Bn} + t(z_A - b)^2,$$

which leads to

$$z_A = \frac{(b^2 - a^2)t - (p_{Ao} - p_{Bn})}{2(b - a)t}.$$

Similarly let z_B be the marginal consumer in $\mathcal{B}$, i.e., $p_{An} + t(z_B - a)^2 = p_{Bo} + t(z_B - b)^2$. Then we have

$$z_B = \frac{(b^2 - a^2)t - (p_{An} - p_{Bo})}{2(b - a)t}.$$

Firms' $\tau = 2$ profits are

$$\begin{aligned} \pi_{A2} &= p_{Ao} z_A + p_{An}(z_B - z), \\ \pi_{B2} &= p_{Bn}(z - z_A) + p_{Bo}(1 - z_B). \end{aligned}$$

Firm i chooses p_{io} and p_{in} to maximize π_{i2}. Solving the first-order conditions simultaneously, we have

$$p_{Ao} = \frac{(b-a)(2z+a+b)t}{3}, \quad p_{An} = \frac{(b-a)(2+a+b-4z)t}{3}, \tag{1}$$

$$p_{Bo} = \frac{(b-a)(4-2z-a-b)t}{3}, \; p_{Bn} = \frac{(b-a)(4z-a-b)t}{3}. \tag{2}$$

Plugging these prices back into z_A and z_B above, we obtain

$$z_A = \frac{a+b+2z}{6}, \quad z_B = \frac{2+a+b+2z}{6}. \tag{3}$$

Note that $z_A \leq z$ if and only if $z \geq (a+b)/4$; if $z \leq (a+b)/4$, we need to consider a corner solution $z_A = z$. Similarly, $z_B \geq z$ if and only if $z \leq (2+a+b)/4$; otherwise, we need to consider the case $z_B = z$. Thus there are three possibilities in $\tau = 2$: (i) when $z = z_A$, firm B cannot poach any of firm A's customers and the only possibility is one-way poaching by firm A; (ii) when $z_A < z < z_B$, there is two-way poaching; (iii) when $z_B = z$, the only possibility is one-way poaching by firm B. In the proof of Proposition 1, we show that only case (ii) is possible in equilibrium. Thus we will focus our discussion on this case, of which condition can be stated as

$$\frac{a+b}{4} < z < \frac{2+a+b}{4}.$$

Substituting the above prices, z_A and z_B into each firm's $\tau = 2$ profit, we have

$$\pi_{A2} = \frac{t(b-a)(2+2a+a^2+2b+2ab+b^2-8z-2az-2bz+10z^2)}{9},$$

$$\pi_{B2} = \frac{t(b-a)(8-4a+a^2-4b+2ab+b^2-8z-2az-2bz+10z^2)}{9}.$$

To see how market shares change in $\tau = 2$, let us denote firm i's market share in $\tau = 2$ by S_i, the fraction of consumers switching from firm A to firm B by $S_{A\to B}$, and the fraction switching the other way by $S_{B\to A}$. Consumers in $[z_A, z]$ switch from firm A to firm B, hence $S_{A\to B} = z - z_A = (4z-a-b)/6$. On the other hand, firm A serves new consumers in $[z, z_B]$ who switch from firm B to firm A, leading to $S_{B\to A} = z_B - z = (2-4z+a+b)/6$. As a result, the $\tau = 2$ market shares become $S_A = z + S_{B\to A} - S_{A\to B} = (1+a+b-z)/3$ and $S_B = (1-z) - S_{B\to A} + S_{A\to B} = (2-a-b+z)/3$.

Fudenberg and Tirole (2000) consider the case with $a = 0$ and $b = 1$. In this case, it is easy to see $S_{A\to B} \geq S_{B\to A}$ and $S_A \leq S_B$ if and only if $z \geq 1/2$. Thus the firm with a larger market share in $\tau = 1$ loses more customers and ends up having a smaller market share in $\tau = 2$. Moreover, one can check $\pi_{A2} = \pi_{B2}$ for all z when $a = 0$ and $b = 1$. In sum, there is no benefit in having a larger market share in $\tau = 1$ due to the two-way customer switching in $\tau = 2$. This is why the equilibrium in Fudenberg and Tirole (2000) is symmetric with $z = 1/2$.

Similar reasoning applies to our case. Suppose firm B chooses $b = 1$ but firm A chooses an interior location $a > 0$ and secures a larger market share in $\tau = 1$. But this increases poaching by firm B, which has a negative effect on firm A's $\tau = 2$ market share. Moreover, less than maximal differentiation intensifies competition in both periods. These discussions imply that firms do not have strong incentives to secure a larger market share in $\tau = 1$, which in turn reduces their incentives to choose aggressive locations. Thus we may conjecture that maximal differentiation becomes an equilibrium outcome, which we confirm below.

4.2 First period

Since consumers are forward-looking, the marginal consumer's location in $\tau = 1$ depends on the outcome consumers anticipate in $\tau = 2$. From the previous analysis, we know there are three possibilities: (i) $z = z_A$, (ii) $z_A < z < z_B$, and (iii) $z_B = z$.

As before, we focus on the case with two-way poaching in $\tau = 2$: $z_A \leq z \leq z_B$. If consumers anticipate this outcome, then the $\tau = 1$ marginal consumer z is indifferent between choosing firm A in $\tau = 1$ but switching to firm B in $\tau = 2$, and choosing firm B in $\tau = 1$ but switching to firm A in $\tau = 2$. Given consumers' discount factor δ_c, z is then given by

$$p_A + t(z-a)^2 + \delta_c \left(p_{Bn} + t(z-b)^2\right) = p_B + t(z-b)^2 + \delta_c \left(p_{An} + t(z-a)^2\right).$$

Thus the marginal consumer's location in this case is

$$z = \frac{(b-a)t\left((b+a)(3-\delta_c)+2\delta_c\right) - 3(p_A - p_B)}{2(3+\delta_c)(b-a)t}. \tag{4}$$

Given firms' discount factor δ_f, each firm's total discounted profit is

$$\begin{aligned}\Pi_A &= p_A z + \delta_f \pi_{A2}, \\ \Pi_B &= p_B(1-z) + \delta_f \pi_{B2}.\end{aligned}$$

Of course, the $\tau = 1$ marginal consumer's location will change if consumers anticipate different outcomes in $\tau = 2$. This will in turn lead to different total discounted profit for each firm. This complicates the analysis of the $\tau = 1$ game. We will first outline how to solve for the equilibrium of the $\tau = 1$ game. We will then discuss the complications in solving for the equilibrium.

The $\tau = 1$ game consists of two stages: firms choose a and b first, which is followed by the choice of p_A and p_B. We solve the game backwards. Given (a, b), we first solve for the equilibrium of the pricing game denoted by $p_A^*(a, b)$, $p_B^*(a, b)$, hence the marginal consumer's location denoted by $z^*(a, b)$. We then substitute these back into total discounted profit functions, denoted by $\Pi_A^*(a, b)$ and $\Pi_B^*(a, b)$, and find equilibrium locations (a^*, b^*) by solving simultaneously $\partial\Pi_A^*(a^*, b^*)/\partial a \leq 0$ with equality if $a^* > 0$, and $\partial\Pi_B^*(a^*, b^*)/\partial b \geq 0$ with equality if $b^* < 1$. Finally we substitute (a^*, b^*) into $p_A^*(a, b)$, $p_B^*(a, b)$, (1), (2), (3), and (4) to find the equilibrium of the whole game.

The main difficulty in solving for the $\tau = 1$ equilibrium arises from the fact that there are three possible outcomes in $\tau = 2$. Corresponding to each outcome, we have a different profit function for each firm. Thus each firm's reaction function in the pricing stage consists of three possibly discontinuous pieces. This means that, following location choice by one firm, we need to consider all possible deviations by the other firm including those that lead to points on the different pieces of the reaction function.

When the $\tau = 2$ competition is in personalized pricing as in Choe et al. (2018), there are only two possible outcomes with one-way poaching in $\tau = 2$. This is because a firm can use personalized pricing to protect its turf effectively unless its market share is too large. In contrast, third-degree price discrimination is a blunt tool to protect one's turf. It is because, with third-degree price discrimination, a firm has to charge the same price to all its loyal customers. Thus competition 'around the middle' cannot be too tough, leading to two-way poaching as a possible outcome. With three possible outcomes, the analysis becomes significantly more complicated.

Moreover, as shown in Choe et al. (2018), finding the $\tau = 1$ equilibrium for general discount factors is not possible even when competition is in personalized pricing in $\tau = 2$. In addition, pure-strategy equilibria do not exist when both firms are located sufficiently close to one end in that $a + b \approx 2$ or $a + b \approx 0$. Since our case with third-degree price discrimination involves more complicated analysis than with personalized pricing, we also expect non-existence of pure-strategy equilibria in some cases.

We present two results below. First, maximal differentiation is a unique equilibrium under certain conditions on discount factors and $a+b$. Second, if firms' discount factor is $\delta_f = \delta \in [0, 1]$ while consumers are myopic in that $\delta_c = 0$, then maximal differentiation is a unique equilibrium for all δ.[4] The full analysis is long and tedious. So we only provide a sketch of the proof in the appendix while referring to the technical appendix for the complete proof.

[4]Choe et al. (2018, Proposition 5) considered the case where $\delta_f = 1$ and $\delta_c = 0$. In contrast to our result, they obtained two asymmetric equilibria with less-than-maximal differentiation.

Proposition 1. *Suppose $\delta_c = \delta_f = \delta \in [0, 1]$. Then there is a unique equilibrium with maximal differentiation (i) if $\delta < 0.826$ or (ii) for all $\delta \in [0, 1]$ if $a + b$ is restricted to the range $[0.415, 1.585]$. The equilibrium is given by $a = 0,\ b = 1$, $p_A = p_B = (3 + \delta)t/3$, $p_{Ao} = p_{Bo} = (2t)/3$, $p_{An} = p_{Bn} = t/3$, and $z = 1/2, z_A = 1/3, z_B = 2/3$.*

Proof. See the appendix. □

The reason for the sufficient conditions stated in Proposition 1 is as follows. Suppose firms choose locations close enough to each other at one end of the market in that $a+b$ is close to 0 or 2. Then the $\tau = 2$ outcome is more likely to involve one-way poaching. For example, if $a+b$ is close to 2, then in $\tau = 2$, firm B is likely to poach firm A's customers. If firm B values its $\tau = 2$ profit sufficiently, i.e., large δ, then it has incentives to cut its $\tau = 1$ price further to secure a more advantageous location in $\tau = 2$. This destabilizes the pricing equilibrium in $\tau = 1$.

Our next result shows that if consumers are myopic in that $\delta_c = 0$, then the above incentives disappear, leading to the equilibrium with maximal differentiation for all values of firms' discount factor. The intuition is that, as consumers become myopic, their $\tau = 1$ decisions do not depend much on favorable poaching offers in $\tau = 2$. This makes their $\tau = 1$ demands more price elastic, which can intensify price competition in $\tau = 1$. Thus firms benefit from choosing differentiation to soften the price competition in $\tau = 1$.

Proposition 2. *Suppose $\delta_c = 0$. Then for all $\delta_f = \delta \in [0, 1]$, the game has a unique equilibrium described in Proposition 1 with the only difference $p_A = p_B = t$.*

Proof. From the proof of Proposition 1, one can show that, if $\delta_c = 0$, then only the first-period pricing equilibrium corresponding to the two-way poaching outcome exists for all $\delta_f = \delta \in [0, 1]$ and $a, b \in [0, 1]$. The details are in the technical appendix. □

5 Discussions

5.1 Comparison with Choe et al. (2018)

Our result of maximal differentiation differs from that in Choe et al. (2018). The difference stems from the difference in the way firms compete in the second period. Competition is in personalized pricing in Choe et al. (2018) whereas it is in third degree price discrimination in our case. We discuss below why firms' product choice hinges on the type of price competition that follows.

When competition is in personalized pricing, firms compete under asymmetric information: a firm knows more about its own past customers than about its rival's past customers. Since personalized pricing allows a firm to extract surplus from its own customers more effectively than uniform price, the ability to exercise personalized pricing creates incentives for firms to secure a large market share. This intensifies competition in the first period, leading to less than maximal differentiation.

When competition is in third-degree price discrimination, however, firms compete under minimal, symmetric information: two firms share identical information about which market segment made purchase from each firm in the first period. As discussed previously, uniform pricing based on symmetric information undermines the benefits of having a large market share because of customer switching in the second period. Given quadratic transportation costs, firms then have more incentives to soften competition in the first period than compete for larger profit in the second period. Thus firms choose maximal differentiation.

5.2 Welfare

In this section, we discuss welfare implications of BBPD. Given that the market is fully covered, social optimum depends only on the average distance traveled by a consumer. Then it follows that the optimal location choice involves $a = 1/4,\ b = 3/4$ with average distance traveled equal to 1/8. Thus in $\tau = 1$, there is too much differentiation as in the standard Hotelling equilibrium where the average distance traveled is 1/4. The $\tau = 2$ equilibrium is the same as that in Fudenberg and Tirole (2000): firm A continues to serve customers in $[0, 1/3]$ while those in $[1/3, 1/2]$ switch to firm B; firm B continues to serve customers in $[2/3, 1]$ while those in $[1/2, 2/3]$ switch to firm A. Due to such inefficient customer switching, welfare is even lower than the Hotelling equilibrium. The average distance traveled in $\tau = 2$ is equal to $11/36 > 1/4$.[5]

Firms also have lower profits due to BBPD compared to the Hotelling equilibrium. For simplicity, suppose $\delta_c = \delta_f = \delta$ as in Fudenberg and Tirole (2000). In the Hotelling equilibrium adapted to our setting, each firm earns profit equal to $t/2$ each period, hence the total discounted profit is $(1+\delta)t/2$. With BBPD, each firm earns profit $(3+\delta)t/6$ in $\tau = 1$ and $5t/18$ in $\tau = 2$. Thus the total discounted profit is $(9 + 8\delta)t/18 < (1 + \delta)t/2$. This is consistent with the general thrust of the literature that shows BBPD lowers firm profitability unless there are sufficient asymmetries.

5.3 Location constraint

In our model, we assumed that location choice is restricted to $[0, 1]$. We now consider the case where firms may locate outside $[0, 1]$. The main purpose of this exercise is to see whether location choice continues to be identical to what is obtained in the absence

[5] Since firm A serves customers in $[0, 1/3] \cup [1/2, 2/3]$ and firm B serves the rest, the average distance traveled is $\int_0^{1/3} x dx + \int_{1/3}^{1/2} (1 - x)dx + \int_{1/2}^{2/3} x dx + \int_{2/3}^{1} (1 - x)dx = 11/36$.

of dynamic consideration.[6] When adapted to our model, one can verify that the static location equilibrium without location constraint in Tabuchi and Thisse (1995) is given by $a = -1/4,\ b = 5/4$.

As in Proposition 2, we simplify analysis by assuming consumers are myopic. We show that firms' discount factor ($\delta_f = \delta$) matters in that the static outcome in Tabuchi and Thisse (1995) obtains only when firms are also myopic ($\delta = 0$). Otherwise, firms choose locations closer to each other with location choice converging to that in Tabuchi and Thisse (1995) as δ decreases to 0. This suggests that, given location constraint and quadratic transportation costs, the incentives to soften competition dominate the dynamic consideration. When the location constraint is relaxed, however, the dynamic consideration has bite, as we formalize below.

Proposition 3. *Suppose* $\delta_c = 0$ *and that firms can choose locations outside* $[0, 1]$*. Then for all* $\delta_f = \delta \in [0, 1]$*, the game has a unique equilibrium with locations given by*

$$a = -\frac{81 - 99\delta + 20\delta^2}{12(27 + 9\delta - 20\delta^2)}, \quad b = 1 - a.$$

As δ *decreases to 0, a decreases monotonically and the location equilibrium converges to the static equilibrium in Tabuchi and Thisse (1995):* $a = -1/4,\ b = 5/4$.

Proof. See the technical appendix. □

6 Conclusion

We have studied spatial competition in a model of behavior-based price discrimination in Fudenberg and Tirole (2000). We find that the static Hotelling outcome of maximal differentiation continues to emerge in equilibrium when the second-period competition is in third-degree price discrimination. Thus endogenous product choice does not intensify competition further compared to the case where product choice is fixed exogenously. The result is driven primarily by third-degree price discrimination, which no longer holds when firms can avail themselves of more sophisticated pricing tools such as personalized pricing, as shown in Choe et al. (2018). Thus our result identifies an important interplay between the type of price competition and product choice.

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[6]For studies of Hotelling model without the location constraint, see, for example, Tabuchi and Thisse (1995), Matsumura and Matsushima (2012) and Li and Shuai (2018).

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Appendix

Proof of Proposition 1 (Sketch). Recall that there are three possible outcomes in $\tau = 2$: (1) $z = z_A$, (2) $z_A < z < z_B$, (3) $z_B = z$. We use subscript $k = 1, 2, 3$ to denote each of these outcomes. For each k, denote the location of $\tau = 1$ marginal consumer by z_k and each firm's total discounted profit by Π_{ik}, $i = A, B$.

First, consider the $\tau = 1$ pricing game given (a, b). For each k, firms simultaneously choose p_A, p_B to maximize Π_{Ak}, Π_{Bk}, leading to reaction functions $p_{Ak}(p_B; a, b)$ and $p_{Bk}(p_A; a, b)$. These reaction functions represent locally optimal prices given k. For firm i, the 'true' reaction function is derived from comparing $\Pi_{ik}, k = 1, 2, 3$ to find p_{ik} that leads to a global optimum. We then solve the two true reaction functions simultaneously for equilibrium prices $p_A^*(a, b)$, $p_B^*(a, b)$, and the marginal consumer's location $z^*(a, b)$. In the technical appendix, we show that the solution exists only when $k = 2$ with sufficient conditions as given in Proposition 1; in other cases $k = 1, 3$, the two reaction functions do not intersect.

Next, consider the $\tau = 1$ location game. Substitute $p_A^*(a, b)$, $p_B^*(a, b)$, and $z^*(a, b)$ into each firm's profit function and denote them by $\Pi_{A2}^*(a, b), \Pi_{B2}^*(a, b)$ where the second subscript indicates that the pricing equilibrium is possible only when $k = 2$. Differentiating these profit functions, one can show $\partial\Pi_{A2}^*/\partial a < 0$ for all $a, b \in [0, 1]$ and $\partial\Pi_{B2}^*/\partial b > 0$ for all $a, b \in [0, 1]$. Thus $a = 0$, $b = 1$ is a candidate equilibrium, which leads to the two-way poaching outcome in $\tau = 2$. Substituting $a = 0, b = 1$ into $p_A^*(a, b)$, $p_B^*(a, b)$, (1), (2), (3), and (4) gives us the equilibrium stated in the proposition. □

Technical Appendix

In this appendix, we provide the detailed proof of the results in our paper.We have used Mathematica for the calculations and figures included in this appendix.

Proof of Proposition 1

1. Second period

We start with the second-period prices.

We define four prices:
p_{Ao} is the price of Firm A for its old customers,
p_{An} is the price of Firm A for its new customers,
p_{Bo} is the price of Firm B for its old customers,
p_{Bn} is the price of Firm B for its new customers.

The location of the indifferent consumer in Firm A's turf, z_A, is derived by solving the following equation with respect to z_A:

```
Solve[-p_Ao - t (z_A - a)^2 == -p_Bn - t (z_A - b)^2, z_A]
```

$$\left\{\left\{z_A \to \frac{a^2\, t - b^2\, t + p_{Ao} - p_{Bn}}{2\,(a-b)\,t}\right\}\right\}$$

We set the location of the indifferent consumer in Firm A's turf, z_A:

$$z_A = \frac{a^2\, t - b^2\, t + p_{Ao} - p_{Bn}}{2\,(a-b)\,t}$$

$$\frac{a^2\, t - b^2\, t + p_{Ao} - p_{Bn}}{2\,(a-b)\,t}$$

The location of the indifferent consumer in Firm B's turf, z_B, is derived by solving the following equation with respect to z_B:

```
Solve[-p_Bo - t (z_B - b)^2 == -p_An - t (z_B - a)^2, z_B]
```

$$\left\{\left\{z_B \to \frac{a^2\, t - b^2\, t + p_{An} - p_{Bo}}{2\,(a-b)\,t}\right\}\right\}$$

We set the location of the indifferent consumer in Firm B's turf, z_B:

$$z_B = \frac{a^2\, t - b^2\, t + p_{An} - p_{Bo}}{2\,(a-b)\,t}$$

$$\frac{a^2\, t - b^2\, t + p_{An} - p_{Bo}}{2\,(a-b)\,t}$$

The first-order differentials of Firm A's profit with respect to p_{Ao} and p_{An} are

```
Factor[D[p_Ao z_A + p_An (z_B - z), p_Ao]]
```

$$\frac{a^2\, t - b^2\, t + 2\, p_{Ao} - p_{Bn}}{2\,(a-b)\,t}$$

```
Factor[D[p_Ao z_A + p_An (z_B - z), p_An]]
```

$$-\frac{-a^2\, t + b^2\, t + 2\, a\, t\, z - 2\, b\, t\, z - 2\, p_{An} + p_{Bo}}{2\,(a-b)\,t}$$

Similarly, the first-order differentials of Firm B's profit with respect to p_{Bo} and p_{Bn} are

```
Factor[D[p_Bn (z - z_A) + p_Bo (1 - z_B), p_Bo]]
```

$$-\frac{-2\,a\,t + a^2\,t + 2\,b\,t - b^2\,t + p_{An} - 2\,p_{Bo}}{2\,(a-b)\,t}$$

```
Factor[D[p_Bn (z - z_A) + p_Bo (1 - z_B), p_Bn]]
```

$$\frac{-a^2\,t + b^2\,t + 2\,a\,t\,z - 2\,b\,t\,z - p_{Ao} + 2\,p_{Bn}}{2\,(a-b)\,t}$$

The four first-order differentials give us the following simultaneous equations, which we solve for the second-period prices.

$$\text{Simplify}\Big[\text{Solve}\Big[\Big\{\frac{a^2\,t - b^2\,t + 2\,p_{Ao} - p_{Bn}}{2\,(a-b)\,t} == 0,$$
$$-\frac{-a^2\,t + b^2\,t + 2\,a\,t\,z - 2\,b\,t\,z - 2\,p_{An} + p_{Bo}}{2\,(a-b)\,t} == 0,\ -\frac{-2\,a\,t + a^2\,t + 2\,b\,t - b^2\,t + p_{An} - 2\,p_{Bo}}{2\,(a-b)\,t} == 0,$$
$$\frac{-a^2\,t + b^2\,t + 2\,a\,t\,z - 2\,b\,t\,z - p_{Ao} + 2\,p_{Bn}}{2\,(a-b)\,t} == 0\Big\},\ \{p_{Ao}, p_{An}, p_{Bn}, p_{Bo}\}\Big]\Big]$$

$$\Big\{\Big\{p_{Ao} \to -\frac{1}{3}\,(a-b)\,t\,(a+b+2\,z),\ p_{An} \to -\frac{1}{3}\,(a-b)\,t\,(2+a+b-4\,z),$$
$$p_{Bn} \to \frac{1}{3}\,(a-b)\,t\,(a+b-4\,z),\ p_{Bo} \to \frac{1}{3}\,(a-b)\,t\,(-4+a+b+2\,z)\Big\}\Big\}$$

Substituting the above prices into z_A, we have the equilibrium z_A in period 2:

$$\text{Factor}\Big[z_A\ /.\ \Big\{p_{Ao} \to -\frac{1}{3}\,(a-b)\,t\,(a+b+2\,z),\ p_{An} \to -\frac{1}{3}\,(a-b)\,t\,(2+a+b-4\,z),$$
$$p_{Bn} \to \frac{1}{3}\,(a-b)\,t\,(a+b-4\,z),\ p_{Bo} \to \frac{1}{3}\,(a-b)\,t\,(-4+a+b+2\,z)\Big\}\Big]$$

$$\frac{1}{6}\,(a+b+2\,z)$$

This z_A is in the range [0, z] if and only if z ≥ (a + b)/4. If z≤(a+b)/4, we need to consider a corner solution (z_A=z), which is discussed later.

Similarly, substituting the above prices into z_B, we have the equilibrium z_B in period 2:

$$\text{Factor}\Big[z_B\ /.\ \Big\{p_{Ao} \to -\frac{1}{3}\,(a-b)\,t\,(a+b+2\,z),\ p_{An} \to -\frac{1}{3}\,(a-b)\,t\,(2+a+b-4\,z),$$
$$p_{Bn} \to \frac{1}{3}\,(a-b)\,t\,(a+b-4\,z),\ p_{Bo} \to \frac{1}{3}\,(a-b)\,t\,(-4+a+b+2\,z)\Big\}\Big]$$

$$\frac{1}{6}\,(2+a+b+2\,z)$$

This z_B is in the range [z, 1] if and only if z ≤ (2+a+b)/4. If z≥(2+a+b)/4, we need to consider a corner solution (z_B=z), which is discussed later.

Based on the above discussions, we have three cases: (i) 0≤z≤(a+b)/4, (ii) (a+b)/4<z<(2+a+b)/4, (iii) (2+a+b)/4≤z≤1.

(Case i) 0≤z≤ (a + b)/4.

In this case, we have $z_A = z$, hence Firm B cannot poach any customer in Firm A's turf. As a result, p_{Bn}=0. Anticipating this, Firm A sets the highest p_{Ao} that leads to $z_A = z$. z_A, just equals to z. This is found below.

```
Solve[{z_A == z, p_Bn == 0}, {p_Ao, p_Bn}]
```

$$\{\{p_{Ao} \to -(a-b)\,t\,(a+b-2\,z),\ p_{Bn} \to 0\}\}$$

For the optimal pricing in Firm B's turf, we can use the first-order differentials we have already derived:

$$\text{The first - order differential of Firm A } (p_{An}) : -\frac{-a^2 t + b^2 t + 2 a t z - 2 b t z - 2 p_{An} + p_{Bo}}{2 (a - b) t}$$

$$\text{The first - order differential of Firm B } (p_{Bo}) : -\frac{-2 a t + a^2 t + 2 b t - b^2 t + p_{An} - 2 p_{Bo}}{2 (a - b) t}$$

This leads to the following second-period prices:

$$\text{Simplify}\left[\text{Solve}\left[\left\{-\frac{-a^2 t + b^2 t + 2 a t z - 2 b t z - 2 p_{An} + p_{Bo}}{2 (a - b) t} == 0, -\frac{-2 a t + a^2 t + 2 b t - b^2 t + p_{An} - 2 p_{Bo}}{2 (a - b) t} == 0\right\}, \{p_{An}, p_{Bo}\}\right]\right]$$

$$\left\{\left\{p_{An} \to -\frac{1}{3} (a - b) t (2 + a + b - 4 z), p_{Bo} \to \frac{1}{3} (a - b) t (-4 + a + b + 2 z)\right\}\right\}$$

Substituting the above prices into z_B, we have the equilibrium z_B in period 2 when 0≤z≤(a+b)/4:

$$\text{Factor}\left[z_B \text{ /. } \left\{p_{An} \to -\frac{1}{3} (a - b) t (2 + a + b - 4 z), p_{Bo} \to \frac{1}{3} (a - b) t (-4 + a + b + 2 z)\right\}\right]$$

$$\frac{1}{6} (2 + a + b + 2 z)$$

Using the above outcomes, we derive the second period profit of Firm A in case (i) given the first-period z:

$$\text{Factor}\left[p_{Ao} z_A + p_{An} (z_B - z) \text{ /. } \left\{p_{Ao} \to -(a - b) t (a + b - 2 z), p_{An} \to -\frac{1}{3} (a - b) t (2 + a + b - 4 z), p_{Bo} \to \frac{1}{3} (a - b) t (-4 + a + b + 2 z), p_{Bn} \to 0\right\}\right]$$

$$-\frac{1}{18} (a - b) t \left(4 + 4 a + a^2 + 4 b + 2 a b + b^2 - 16 z + 10 a z + 10 b z - 20 z^2\right)$$

Similarly, using the above outcomes, we derive the second period profit of Firm B in case (i) given the first-period z:s

$$\text{Factor}\left[p_{Bn} (z - z_A) + p_{Bo} (1 - z_B) \text{ /. } \left\{p_{Ao} \to -(a - b) t (a + b - 2 z), p_{An} \to -\frac{1}{3} (a - b) t (2 + a + b - 4 z), p_{Bo} \to \frac{1}{3} (a - b) t (-4 + a + b + 2 z), p_{Bn} \to 0\right\}\right]$$

$$-\frac{1}{18} (a - b) t (-4 + a + b + 2 z)^2$$

(Case ii) (a+b)/4<z<(2+a+b)/4.

In this case, we have an interior solution with two-way poaching. Therefore, we can use the second-period prices we have already obtained previously, reproduced below:

$$\left\{p_{Ao} \to -\frac{1}{3} (a - b) t (a + b + 2 z), p_{An} \to -\frac{1}{3} (a - b) t (2 + a + b - 4 z), p_{Bn} \to \frac{1}{3} (a - b) t (a + b - 4 z), p_{Bo} \to \frac{1}{3} (a - b) t (-4 + a + b + 2 z)\right\}$$

Thus firm A's second-period profit given *z* can be derived as

$$\text{Factor}\left[p_{Ao}\,z_A + p_{An}\,(z_B - z)\ /.\ \left\{p_{Ao} \to -\frac{1}{3}\,(a-b)\,t\,(a+b+2\,z),\ p_{An} \to -\frac{1}{3}\,(a-b)\,t\,(2+a+b-4\,z),\ p_{Bn} \to \frac{1}{3}\,(a-b)\,t\,(a+b-4\,z),\ p_{Bo} \to \frac{1}{3}\,(a-b)\,t\,(-4+a+b+2\,z)\right\}\right]$$

$$-\frac{1}{9}\,(a-b)\,t\,\left(2+2\,a+a^2+2\,b+2\,a\,b+b^2-8\,z-2\,a\,z-2\,b\,z+10\,z^2\right)$$

Similarly, firm B's profit given z is

$$\text{Factor}\left[p_{Bn}\,(z - z_A) + p_{Bo}\,(1 - z_B)\ /.\ \left\{p_{Ao} \to -\frac{1}{3}\,(a-b)\,t\,(a+b+2\,z),\ p_{An} \to -\frac{1}{3}\,(a-b)\,t\,(2+a+b-4\,z),\ p_{Bn} \to \frac{1}{3}\,(a-b)\,t\,(a+b-4\,z),\ p_{Bo} \to \frac{1}{3}\,(a-b)\,t\,(-4+a+b+2\,z)\right\}\right]$$

$$-\frac{1}{9}\,(a-b)\,t\,\left(8-4\,a+a^2-4\,b+2\,a\,b+b^2-8\,z-2\,a\,z-2\,b\,z+10\,z^2\right)$$

(Case iii) (2 + a + b)/4≤z≤1.

In this case, $z_B = z$, hence Firm A cannot poach any customer in Firm B's turf. As a result, p_{An}=0. Given p_{An}=0, Firm B chooses the highest p_{Bo} that leads to $z_B = z$:

$$\text{Solve}[\{z_B == z,\ p_{An} == 0\},\ \{p_{Bo},\ p_{An}\}]$$

$$\{\{p_{Bo} \to (a-b)\,t\,(a+b-2\,z),\ p_{An} \to 0\}\}$$

For the optimal pricing in Firm A's turf, we can use the first-order differentials we have already derived:

$$\text{The first - order differential of Firm A }(p_{Ao}) : \frac{a^2\,t - b^2\,t + 2\,p_{Ao} - p_{Bn}}{2\,(a-b)\,t}$$

$$\text{The first - order differential of Firm B }(p_{Bn}) : \frac{-a^2\,t + b^2\,t + 2\,a\,t\,z - 2\,b\,t\,z - p_{Ao} + 2\,p_{Bn}}{2\,(a-b)\,t}$$

Solving the following simultaneous equations gives us the second-period prices.

$$\text{Simplify}\left[\text{Solve}\left[\left\{\frac{a^2\,t - b^2\,t + 2\,p_{Ao} - p_{Bn}}{2\,(a-b)\,t} == 0,\ \frac{-a^2\,t + b^2\,t + 2\,a\,t\,z - 2\,b\,t\,z - p_{Ao} + 2\,p_{Bn}}{2\,(a-b)\,t} == 0\right\},\ \{p_{Ao},\ p_{Bn}\}\right]\right]$$

$$\left\{\left\{p_{Ao} \to -\frac{1}{3}\,(a-b)\,t\,(a+b+2\,z),\ p_{Bn} \to \frac{1}{3}\,(a-b)\,t\,(a+b-4\,z)\right\}\right\}$$

Substituting the second-period prices into z_A, we have the equilibrium z_A in period 2 when z ≥ (2+a + b)/4:

$$\text{Factor}\left[z_A\ /.\ \left\{p_{Ao} \to -\frac{1}{3}\,(a-b)\,t\,(a+b+2\,z),\ p_{An} \to 0,\ p_{Bo} \to (a-b)\,t\,(a+b-2\,z),\ p_{Bn} \to \frac{1}{3}\,(a-b)\,t\,(a+b-4\,z)\right\}\right]$$

$$\frac{1}{6}\,(a+b+2\,z)$$

Using the above prices, z_A and $z_B = z$, we derive firm A's second-period profit given z as follows:

$$\text{Factor}\left[p_{Ao}\,z_A + p_{An}\,(z_B - z)\ /.\ \left\{p_{Ao} \to -\frac{1}{3}\,(a-b)\,t\,(a+b+2\,z),\ p_{An} \to 0,\ p_{Bo} \to (a-b)\,t\,(a+b-2\,z),\ p_{Bn} \to \frac{1}{3}\,(a-b)\,t\,(a+b-4\,z)\right\}\right]$$

$$-\frac{1}{18}\,(a-b)\,t\,(a+b+2\,z)^2$$

Similarly, firm B's second-period profit given z is

$$\text{Factor}\left[p_{Bn}\,(z-z_A)+p_{Bo}\,(1-z_B)\;/.\;\left\{p_{Ao}\to -\frac{1}{3}(a-b)\,t\,(a+b+2z),\;p_{An}\to 0,\;p_{Bo}\to (a-b)\,t\,(a+b-2z),\;p_{Bn}\to \frac{1}{3}(a-b)\,t\,(a+b-4z)\right\}\right]$$

$$-\frac{1}{18}(a-b)\,t\,\left(-18a+a^2-18b+2ab+b^2+36z+10az+10bz-20z^2\right)$$

2. First period - Prices

In our calculation, we denote firms' discount factor by δ_f and consumers' discount factor by δ_c. In Proposition 1, we focus on the case, δ_f=δ_f=δ. In Propositions 2 and 3, we focus on the case, δ_f=δ and δ_c=0.

We need to consider three cases: (i) 0≤z ≤ (a + b)/4, (ii) (a + b)/4<z<(2+a+b)/4, (iii) (2+a+b)/4≤ z≤1.

(Case i) 0≤z≤(a + b)/4.

From the previous analysis, we have the second-period prices given as follow.

$$\left\{p_{Ao}\to -(a-b)\,t\,(a+b-2z),\;p_{An}\to -\frac{1}{3}(a-b)\,t\,(2+a+b-4z),\;p_{Bo}\to \frac{1}{3}(a-b)\,t\,(-4+a+b+2z),\;p_{Bn}\to 0\right\}$$

Anticipating the second period prices, consumers choose one of the first-period prices p_A or p_B (p_A is the first-period price of firm A and p_B is the first-period price of firm B)

The location of the indifferent consumer, z, is derived from the following equation.:

$$\text{Solve}\left[-p_A-t\,(z-a)^2-\delta c\left((-(a-b)\,t\,(a+b-2z))+t\,(z-a)^2\right)==-p_B-t\,(z-b)^2-\delta c\left(\left(\frac{1}{3}(b-a)\,t\,(2+a+b-4z)\right)+t\,(z-a)^2\right),\;z\right]$$

$$\left\{\left\{z\to \left(-3a^2t+3b^2t-2at\,\delta c+2a^2t\,\delta c+2bt\,\delta c-2b^2t\,\delta c-3p_A+3p_B\right)/\left(2\,(a-b)\,t\,(-3+\delta c)\right)\right\}\right\}$$

We set the location of the indifferent consumers z:

$$z=\left(-3a^2t+3b^2t-2at\,\delta c+2a^2t\,\delta c+2bt\,\delta c-2b^2t\,\delta c-3p_A+3p_B\right)/\left(2\,(a-b)\,t\,(-3+\delta c)\right)$$

$$\frac{-3a^2t+3b^2t-2at\,\delta c+2a^2t\,\delta c+2bt\,\delta c-2b^2t\,\delta c-3p_A+3p_B}{2\,(a-b)\,t\,(-3+\delta c)}$$

Next, we derive the condition for z to be in the range [0, (a+b)/4] by solving the following equations.

```
Factor[Solve[(a + b) / 4 - z == 0, pA]]
Factor[Solve[0 - z == 0, pA]]
```

$$\left\{\left\{p_A\to \frac{1}{6}\left(-3a^2t+3b^2t-4at\,\delta c+3a^2t\,\delta c+4bt\,\delta c-3b^2t\,\delta c+6p_B\right)\right\}\right\}$$

$$\left\{\left\{p_A\to \frac{1}{3}\left(-3a^2t+3b^2t-2at\,\delta c+2a^2t\,\delta c+2bt\,\delta c-2b^2t\,\delta c+3p_B\right)\right\}\right\}$$

Simplifying the above, the condition can be stated as follows.

$$p_B+\frac{(b-a)\,t\,((a+b)\,(3-2\delta c)+2\delta c)}{3}\geq p_A\geq p_B+\frac{(b-a)\,t\,(3\,(a+b)\,(1-\delta c)+4\delta c)}{6}$$

Note that if p_A is larger than $p_B+\frac{(b-a)\,t\,((a+b)\,(3-2\delta c)+2\delta c)}{3}$, z becomes zero.

Next, we solve for the pricing equilibrium in the first period.
From the previous analysis, we have each firm's second-period profit given as follows.

$\pi_{A2}: -\frac{1}{18}(a-b)\,t\,\left(4+4a+a^2+4b+2ab+b^2-16z+10az+10bz-20z^2\right)$

$\pi_{B2}: -\frac{1}{18}(a-b)\,t\,(-4+a+b+2z)^2$

First, the derivative of firm A's total discounted profit with respect to p_A is

$$\text{Factor}\left[D\left[p_A z+\delta f\left(\frac{1}{18}(b-a)\,t\,\left(4+4a+a^2+4b+2ab+b^2-16z+10az+10bz-20z^2\right)\right),\ p_A\right]\right]$$

$$\frac{1}{2(a-b)\,t\,(-3+\delta c)^2}\left(9a^2t-9b^2t+6at\delta c-9a^2t\delta c-6bt\delta c+9b^2t\delta c-2at\delta c^2+2a^2t\delta c^2+2bt\delta c^2-2b^2t\delta c^2+8at\delta f+5a^2t\delta f-8bt\delta f-5b^2t\delta f+4at\delta c\,\delta f-5a^2t\delta c\,\delta f-4bt\delta c\,\delta f+5b^2t\delta c\,\delta f+18p_A-6\delta c\,p_A+10\delta f\,p_A-9p_B+3\delta c\,p_B-10\delta f\,p_B\right)$$

Using the derivative, we obtain the reaction function of Firm A in the range, 0≤z≤(a + b)/4:

$$\text{Simplify}\left[\text{Solve}\left[\frac{1}{2(a-b)\,t\,(-3+\delta c)^2}\left(9a^2t-9b^2t+6at\delta c-9a^2t\delta c-6bt\delta c+9b^2t\delta c-2at\delta c^2+2a^2t\delta c^2+2bt\delta c^2-2b^2t\delta c^2+8at\delta f+5a^2t\delta f-8bt\delta f-5b^2t\delta f+4at\delta c\,\delta f-5a^2t\delta c\,\delta f-4bt\delta c\,\delta f+5b^2t\delta c\,\delta f+18p_A-6\delta c\,p_A+10\delta f\,p_A-9p_B+3\delta c\,p_B-10\delta f\,p_B\right)==0,\ p_A\right]\right]$$

$$\left\{\left\{p_A\to\frac{1}{2(-9+3\delta c-5\delta f)}\left((a-b)\,t\,\left(6\delta c-2\delta c^2+8\delta f+4\delta c\,\delta f+b\left(9-9\delta c+2\delta c^2+5\delta f-5\delta c\,\delta f\right)+a\left(9+2\delta c^2+5\delta f-\delta c\,(9+5\delta f)\right)\right)+(-9+3\delta c-10\delta f)\,p_B\right)\right\}\right\}$$

Since the above reaction function may prescribe z outside the required range, we need the condition that indeed guarantees,0≤z≤(a + b)/4.

We have already obtained the condition that z is between 0 and (a + b)/4 as follows:

$$p_B+\frac{(b-a)\,t\,((a+b)(3-2\delta c)+2\delta c)}{3}\geq p_A\geq p_B+\frac{(b-a)\,t\,(3(a+b)(1-\delta c)+4\delta c)}{6}$$

If the following outcomes are positive, the reaction function satisfies the above inequalities:

$$\text{Simplify}\left[\text{Factor}\left[\frac{1}{2(-9+3\delta c-5\delta f)}\left((a-b)\,t\,\left(6\delta c-2\delta c^2+8\delta f+4\delta c\,\delta f+b\left(9-9\delta c+2\delta c^2+5\delta f-5\delta c\,\delta f\right)+a\left(9+2\delta c^2+5\delta f-\delta c\,(9+5\delta f)\right)\right)+(-9+3\delta c-10\delta f)\,p_B\right)-\left(p_B+\frac{(b-a)\,t\,(3(a+b)(1-\delta c)+4\delta c)}{6}\right)\right]\right]$$

$$\text{Simplify}\left[\text{Factor}\left[p_B+\frac{(b-a)\,t\,((a+b)(3-2\delta c)+2\delta c)}{3}-\frac{1}{2(-9+3\delta c-5\delta f)}\left((a-b)\,t\,\left(6\delta c-2\delta c^2+8\delta f+4\delta c\,\delta f+b\left(9-9\delta c+2\delta c^2+5\delta f-5\delta c\,\delta f\right)+a\left(9+2\delta c^2+5\delta f-\delta c\,(9+5\delta f)\right)\right)+(-9+3\delta c-10\delta f)\,p_B\right)\right]\right]$$

$$-\frac{(-3+\delta c)\,((a-b)\,t\,(3(-2+a+b)\,\delta c+8\delta f)+9p_B)}{6(-9+3\delta c-5\delta f)}$$

$$((-3+\delta c)\,((a-b)\,t\,(-6\delta c+a(-9+6\delta c-5\delta f)+b(-9+6\delta c-5\delta f)+8\delta f)+9p_B))\,/\,(6(-9+3\delta c-5\delta f))$$

We derive the threshold values of p_B such that each of the outcomes equals zero:

$$\text{Simplify}\left[\text{Solve}\left[-\frac{(-3+\delta c)\ ((a-b)\ t\ (3\ (-2+a+b)\ \delta c+8\ \delta f)+9\ p_B)}{6\ (-9+3\ \delta c-5\ \delta f)}==0,\ p_B\right]\right]$$

$$\text{Simplify}[\text{Solve}[((-3+\delta c)\ ((a-b)\ t\ (-6\ \delta c+a\ (-9+6\ \delta c-5\ \delta f)+b\ (-9+6\ \delta c-5\ \delta f)+8\ \delta f)+9\ p_B))\ /\ (6\ (-9+3\ \delta c-5\ \delta f))==0,\ p_B]]$$

$$\left\{\left\{p_B\to-\frac{1}{9}\ (a-b)\ t\ (3\ (-2+a+b)\ \delta c+8\ \delta f)\right\}\right\}$$

$$\left\{\left\{p_B\to-\frac{1}{9}\ (a-b)\ t\ (-6\ \delta c+a\ (-9+6\ \delta c-5\ \delta f)+b\ (-9+6\ \delta c-5\ \delta f)+8\ \delta f)\right\}\right\}$$

Therefore, if p_B satisfies the following inequalities, the reaction function of Firm A is in the range, 0≤z≤(a + b)/4:

$$-\frac{1}{9}\ (b-a)\ t\ ((a+b)\ (9-6\ \delta c+5\ \delta f)+6\ \delta c-8\ \delta f)\le p_B\le\frac{(b-a)\ t\ (3\ (-2+a+b)\ \delta c+8\ \delta f)}{9}$$

Note that if p_B is smaller than the left-hand side value of the inequality, Firm A abandons to supply in period 1.

Similarly, using the above outcomes, we derive the first-order derivative of Firm B's profit with respect to p_B

$$\text{Factor}\left[D\left[p_B\ (1-z)+\delta f\ \left(\frac{1}{18}\ (b-a)\ t\ (-4+a+b+2\ z)^2\right),\ p_B\right]\right]$$

$$-\frac{1}{2\ (a-b)\ t\ (-3+\delta c)^2}\Big(-18\ a\ t+9\ a^2\ t+18\ b\ t-9\ b^2\ t+18\ a\ t\ \delta c-9\ a^2\ t\ \delta c-18\ b\ t\ \delta c+9\ b^2\ t\ \delta c-4\ a\ t\ \delta c^2+2\ a^2\ t\ \delta c^2+4\ b\ t\ \delta c^2-2\ b^2\ t\ \delta c^2+8\ a\ t\ \delta f-4\ a^2\ t\ \delta f-8\ b\ t\ \delta f+4\ b^2\ t\ \delta f-4\ a\ t\ \delta c\ \delta f+2\ a^2\ t\ \delta c\ \delta f+4\ b\ t\ \delta c\ \delta f-2\ b^2\ t\ \delta c\ \delta f+9\ p_A-3\ \delta c\ p_A-2\ \delta f\ p_A-18\ p_B+6\ \delta c\ p_B+2\ \delta f\ p_B\Big)$$

Using the derivative, we obtain the reaction function of Firm B in the range, 0≤z ≤ (a + b)/4:

$$\text{Simplify}\Big[\text{Solve}\Big[-\frac{1}{2\ (a-b)\ t\ (-3+\delta c)^2}\Big(-18\ a\ t+9\ a^2\ t+18\ b\ t-9\ b^2\ t+18\ a\ t\ \delta c-9\ a^2\ t\ \delta c-18\ b\ t\ \delta c+9\ b^2\ t\ \delta c-4\ a\ t\ \delta c^2+2\ a^2\ t\ \delta c^2+4\ b\ t\ \delta c^2-2\ b^2\ t\ \delta c^2+8\ a\ t\ \delta f-4\ a^2\ t\ \delta f-8\ b\ t\ \delta f+4\ b^2\ t\ \delta f-4\ a\ t\ \delta c\ \delta f+2\ a^2\ t\ \delta c\ \delta f+4\ b\ t\ \delta c\ \delta f-2\ b^2\ t\ \delta c\ \delta f+9\ p_A-3\ \delta c\ p_A-2\ \delta f\ p_A-18\ p_B+6\ \delta c\ p_B+2\ \delta f\ p_B\Big)==0,\ p_B\Big]\Big]$$

$$\left\{\left\{p_B\to\left(-\left(-2\ a+a^2-(-2+b)\ b\right)\ t\ \left(9+2\ \delta c^2-4\ \delta f+\delta c\ (-9+2\ \delta f)\right)+(-9+3\ \delta c+2\ \delta f)\ p_A\right)/\ (2\ (-9+3\ \delta c+\delta f))\right\}\right\}$$

Since the above reaction function may prescribe z outside the required range, we need the condition that indeed guarantees 0≤z ≤ (a + b)/4.

We have already obtained the condition that z is between 0 and (a + b)/4 as follows:

$$p_B+\frac{(b-a)\ t\ ((a+b)\ (3-2\ \delta c)+2\ \delta c)}{3}\ge p_A\ge p_B+\frac{(b-a)\ t\ (3\ (a+b)\ (1-\delta c)+4\ \delta c)}{6}.$$

If p_A is larger than $p_B+\frac{(b-a)\ t\ ((a+b)\ (3-2\ \delta c)+2\ \delta c)}{3}$, z becomes zero. In this case, Firm B chooses the following p_B which just leads to z=0.

$$p_B\to\ p_A-\frac{(b-a)\ t\ ((a+b)\ (3-2\ \delta c)+2\ \delta c)}{3}.$$

If the following outcomes are positive, the reaction function satisfies the above inequalities, 0≤z ≤ (a + b)/4:

$$\text{Simplify}\Big[\text{Factor}\Big[\big(-\big(-2a+a^2-(-2+b)\,b\big)\,t\,\big(9+2\,\delta c^2-4\,\delta f+\delta c\,(-9+2\,\delta f)\big)+(-9+3\,\delta c+2\,\delta f)\,p_A\big)\big/\,(2\,(-9+3\,\delta c+\delta f))+\frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)+2\,\delta c)}{3}-p_A\Big]\Big]$$

$$\text{Simplify}\Big[\text{Factor}\Big[p_A-\Big(\big(-\big(-2a+a^2-(-2+b)\,b\big)\,t\,\big(9+2\,\delta c^2-4\,\delta f+\delta c\,(-9+2\,\delta f)\big)+(-9+3\,\delta c+2\,\delta f)\,p_A\big)\big/\,(2\,(-9+3\,\delta c+\delta f))+\frac{(b-a)\,t\,(3\,(a+b)\,(1-\delta c)+4\,\delta c)}{6}\Big)\Big]\Big]$$

$$((-3+\delta c)\,((a-b)\,t\,(-18+a\,(-9+6\,\delta c-2\,\delta f)+b\,(-9+6\,\delta c-2\,\delta f)+8\,\delta f)-9\,p_A))\,/\,(6\,(-9+3\,\delta c+\delta f))$$

$$-(((-3+\delta c)\,((a-b)\,t\,(-18+3\,b\,\delta c+3\,a\,(\delta c-\delta f)+8\,\delta f-3\,b\,\delta f)-9\,p_A))\,/\,(6\,(-9+3\,\delta c+\delta f)))$$

We derive the threshold values of p_A such that each of the outcomes equals zero:

$$\text{Simplify}[\text{Solve}[((-3+\delta c)\,((a-b)\,t\,(-18+a\,(-9+6\,\delta c-2\,\delta f)+b\,(-9+6\,\delta c-2\,\delta f)+8\,\delta f)-9\,p_A))\,/\,(6\,(-9+3\,\delta c+\delta f)) == 0,\ p_A]]$$

$$\text{Simplify}[\text{Solve}[-(((-3+\delta c)\,((a-b)\,t\,(-18+3\,b\,\delta c+3\,a\,(\delta c-\delta f)+8\,\delta f-3\,b\,\delta f)-9\,p_A))\,/\,(6\,(-9+3\,\delta c+\delta f))) == 0,\ p_A]]$$

$$\Big\{\Big\{p_A\to\frac{1}{9}\,(a-b)\,t\,(-18+a\,(-9+6\,\delta c-2\,\delta f)+b\,(-9+6\,\delta c-2\,\delta f)+8\,\delta f)\Big\}\Big\}$$

$$\Big\{\Big\{p_A\to\frac{1}{9}\,(a-b)\,t\,(-18+3\,a\,(\delta c-\delta f)+3\,b\,(\delta c-\delta f)+8\,\delta f)\Big\}\Big\}$$

Therefore, if p_A satisfies the following inequalities, the reaction function of Firm B is in the range, $0 \le z \le (a+b)/4$:

$$\frac{(b-a)\,t\,(18+(a+b)\,(9-6\,\delta c+2\,\delta f)-8\,\delta f)}{9}\ \ge p_A \ge\ \frac{(b-a)\,t\,(18-3\,(a+b)\,(\delta c-\delta f)-8\,\delta f)}{9}$$

Note that if p_A is larger than the left-hand side value of the inequality, Firm B chooses the following p_B, which leads to z=0.

$$p_B\to p_A-\frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)+2\,\delta c)}{3}$$

(Case ii) (a + b)/4 < z < (2 + a + b)/4

From the previous analysis, we have the second-period prices given as follow.

$$\Big\{p_{Ao}\to-\frac{1}{3}\,(a-b)\,t\,(a+b+2\,z),\ p_{An}\to-\frac{1}{3}\,(a-b)\,t\,(2+a+b-4\,z),\ p_{Bn}\to\frac{1}{3}\,(a-b)\,t\,(a+b-4\,z),\ p_{Bo}\to\frac{1}{3}\,(a-b)\,t\,(-4+a+b+2\,z)\Big\}$$

Anticipating the second period prices, consumers choose one of the first-period prices p_A or p_B (p_A is the first-period price of firm A and p_B is the first-period price of firm B)

The location of the indifferent consumer, *z*, is derived by the following equation:

$$\text{Solve}\Big[-p_A-t\,(z-a)^2-\delta c\,\Big(\frac{1}{3}\,(a-b)\,t\,(a+b-4\,z)+t\,(z-b)^2\Big) == -p_B-t\,(z-b)^2-\delta c\,\Big(\Big(\frac{1}{3}\,(b-a)\,t\,(2+a+b-4\,z)\Big)+t\,(z-a)^2\Big),\ z\Big]$$

$$\Big\{\Big\{z\to\frac{3\,a^2\,t-3\,b^2\,t+2\,a\,t\,\delta c-a^2\,t\,\delta c-2\,b\,t\,\delta c+b^2\,t\,\delta c+3\,p_A-3\,p_B}{2\,(a-b)\,t\,(3+\delta c)}\Big\}\Big\}$$

We set the location of the indifferent consumers z:

$$z = \frac{3\ (p_B - p_A)}{2\ (b - a)\ t\ (3 + \delta c)} + \frac{((a + b)\ (3 - \delta c) + 2\ \delta c)}{2\ (3 + \delta c)}$$

$$\frac{(a + b)\ (3 - \delta c) + 2\ \delta c}{2\ (3 + \delta c)} + \frac{3\ (-p_A + p_B)}{2\ (-a + b)\ t\ (3 + \delta c)}$$

We derive the condition that the location of the indifferent consumers, *z*, is between (a+b)/4 and (2+a+b)/4, by solving the following simultaneous equations

```
Simplify[Solve[z - (a + b) / 4 == 0, pA]]
Simplify[Solve[(2 + a + b) / 4 - z == 0, pA]]
```

$$\left\{\left\{p_A \to \frac{1}{6}\ ((a - b)\ t\ (3\ a\ (-1 + \delta c) + 3\ b\ (-1 + \delta c) - 4\ \delta c) + 6\ p_B)\right\}\right\}$$

$$\left\{\left\{p_A \to \frac{1}{6}\ ((a - b)\ t\ (6 + 3\ a\ (-1 + \delta c) + 3\ b\ (-1 + \delta c) - 2\ \delta c) + 6\ p_B)\right\}\right\}$$

By simplifying the above values of p_A, we have the condition that *z* is between (a+b)/4 and (2+a+b)/4 as follows:

$$p_B + \frac{(b - a)\ t\ (3\ (a + b)\ (1 - \delta c) - 2\ (3 - \delta c))}{6} < p_A < p_B + \frac{(b - a)\ t\ (3\ (a + b)\ (1 - \delta c) + 4\ \delta c)}{6}$$

We have already derived the 2nd period profits of Firms A and B as follows:

$$\pi_{A2} : -\frac{1}{9}\ (a - b)\ t\ \left(2 + 2\ a + a^2 + 2\ b + 2\ a\ b + b^2 - 8\ z - 2\ a\ z - 2\ b\ z + 10\ z^2\right)$$

$$\pi_{B2} : -\frac{1}{9}\ (a - b)\ t\ \left(8 - 4\ a + a^2 - 4\ b + 2\ a\ b + b^2 - 8\ z - 2\ a\ z - 2\ b\ z + 10\ z^2\right)$$

Using the above outcomes, we derive the first-order derivative of Firm A's total discounted profit with respect to p_A

$$\text{Factor}\left[D\left[p_A\ z + \delta f\ \frac{1}{9}\ (b - a)\ t\ \left(2 + 2\ a + a^2 + 2\ b + 2\ a\ b + b^2 - 8\ z - 2\ a\ z - 2\ b\ z + 10\ z^2\right),\ p_A\right]\right]$$

$$-\frac{1}{2\ (a - b)\ t\ (3 + \delta c)^2}\ \left(-9\ a^2\ t + 9\ b^2\ t - 6\ a\ t\ \delta c + 6\ b\ t\ \delta c - 2\ a\ t\ \delta c^2 + a^2\ t\ \delta c^2 + 2\ b\ t\ \delta c^2 - b^2\ t\ \delta c^2 - 8\ a\ t\ \delta f + 8\ a^2\ t\ \delta f + 8\ b\ t\ \delta f - 8\ b^2\ t\ \delta f + 4\ a\ t\ \delta c\ \delta f - 4\ a^2\ t\ \delta c\ \delta f - 4\ b\ t\ \delta c\ \delta f + 4\ b^2\ t\ \delta c\ \delta f - 18\ p_A - 6\ \delta c\ p_A + 10\ \delta f\ p_A + 9\ p_B + 3\ \delta c\ p_B - 10\ \delta f\ p_B\right)$$

Using the derivative, we obtain the reaction function of Firm A in the range, (a+b)/4<z<(2+a+b)/4:

$$\text{Simplify}\left[\text{Solve}\left[-\frac{1}{2\ (a - b)\ t\ (3 + \delta c)^2}\ \left(-9\ a^2\ t + 9\ b^2\ t - 6\ a\ t\ \delta c + 6\ b\ t\ \delta c - 2\ a\ t\ \delta c^2 + a^2\ t\ \delta c^2 + 2\ b\ t\ \delta c^2 - b^2\ t\ \delta c^2 - 8\ a\ t\ \delta f + 8\ a^2\ t\ \delta f + 8\ b\ t\ \delta f - 8\ b^2\ t\ \delta f + 4\ a\ t\ \delta c\ \delta f - 4\ a^2\ t\ \delta c\ \delta f - 4\ b\ t\ \delta c\ \delta f + 4\ b^2\ t\ \delta c\ \delta f - 18\ p_A - 6\ \delta c\ p_A + 10\ \delta f\ p_A + 9\ p_B + 3\ \delta c\ p_B - 10\ \delta f\ p_B\right) == 0,\ p_A\right]\right]$$

$$\left\{\left\{p_A \to \frac{1}{2\ (9 + 3\ \delta c - 5\ \delta f)}\ \left((a - b)\ t\ \left(a\ \left(-9 + \delta c^2 + 8\ \delta f - 4\ \delta c\ \delta f\right) + b\ \left(-9 + \delta c^2 + 8\ \delta f - 4\ \delta c\ \delta f\right) - 2\ \left(3\ \delta c + \delta c^2 + 4\ \delta f - 2\ \delta c\ \delta f\right)\right) + (9 + 3\ \delta c - 10\ \delta f)\ p_B\right)\right\}\right\}$$

The function might be outside the range, (a+b)/4<z<(2+a+b)/4.
We derive the condition that the reaction function is indeed in the range, (a+b)/4<z<(2+a+b)/4.

We have already obtained the condition that z is between (a+b)/4<z<(2+a+b)/4 as follows:

$$p_B + \frac{(b - a)\ t\ (3\ (a + b)\ (1 - \delta c) - 2\ (3 - \delta c))}{6} < p_A < p_B + \frac{(b - a)\ t\ (3\ (a + b)\ (1 - \delta c) + 4\ \delta c)}{6}$$

If the following outcomes are positive, the reaction function satisfies the above inequalities:

```
Simplify[pB + (b - a) t (3 (a + b) (1 - δc) + 4 δc)/6 - 1/(2 (9 + 3 δc - 5 δf))
  ((a - b) t (a (-9 + δc^2 + 8 δf - 4 δc δf) + b (-9 + δc^2 + 8 δf - 4 δc δf) - 2 (3 δc + δc^2 + 4 δf - 2 δc δf)) +
   (9 + 3 δc - 10 δf) pB)]
Simplify[1/(2 (9 + 3 δc - 5 δf)) ((a - b) t (a (-9 + δc^2 + 8 δf - 4 δc δf) +
     b (-9 + δc^2 + 8 δf - 4 δc δf) - 2 (3 δc + δc^2 + 4 δf - 2 δc δf)) +
   (9 + 3 δc - 10 δf) pB) - (pB + (b - a) t (3 (a + b) (1 - δc) - 2 (3 - δc))/6)]
```

$$\frac{(3+\delta c)\,((a-b)\,t\,(6\,(-1+a+b)\,\delta c+(8-3\,a-3\,b)\,\delta f)+9\,p_B)}{6\,(9+3\,\delta c-5\,\delta f)}$$

$$-(((3+\delta c)\,((a-b)\,t\,(18+6\,a\,\delta c+6\,b\,\delta c-2\,\delta f-3\,a\,\delta f-3\,b\,\delta f)+9\,p_B))\,/\,(6\,(9+3\,\delta c-5\,\delta f)))$$

We derive the threshold values of p_A such that each of the outcomes equals zero:

```
Simplify[Solve[(3 + δc) ((a - b) t (6 (-1 + a + b) δc + (8 - 3 a - 3 b) δf) + 9 pB)/(6 (9 + 3 δc - 5 δf)) == 0, pB]]
Simplify[Solve[
  -(((3 + δc) ((a - b) t (18 + 6 a δc + 6 b δc - 2 δf - 3 a δf - 3 b δf) + 9 pB)) / (6 (9 + 3 δc - 5 δf))) == 0,
  pB]]
```

$$\left\{\left\{p_B\to-\frac{1}{9}\,(a-b)\,t\,(6\,(-1+a+b)\,\delta c+(8-3\,a-3\,b)\,\delta f)\right\}\right\}$$

$$\left\{\left\{p_B\to-\frac{1}{9}\,(a-b)\,t\,(18+6\,a\,\delta c+6\,b\,\delta c-2\,\delta f-3\,a\,\delta f-3\,b\,\delta f)\right\}\right\}$$

Therefore, if p_B satisfies the following inequalities, the reaction function of Firm B is in the range, (a+b)/4<z<(2+a+b)/4.

$$\frac{(b-a)\,t\,(6\,(-1+a+b)\,\delta c+(8-3\,a-3\,b)\,\delta f)}{9}<p_B<\frac{(b-a)\,t\,(3\,(a+b)\,(2\,\delta c-\delta f)+2\,(9-\delta f))}{9}$$

Similarly, using the above outcomes, we derive the first-order derivative of Firm B's profit with respect to p_B

```
Factor[D[pB (1 - z) + δf 1/9 (b - a) t (8 - 4 a + a^2 - 4 b + 2 a b + b^2 - 8 z - 2 a z - 2 b z + 10 z^2), pB]]
```

$$\frac{1}{2\,(a-b)\,t\,(3+\delta c)^2}\left(18\,a\,t-9\,a^2\,t-18\,b\,t+9\,b^2\,t+6\,a\,t\,\delta c-6\,b\,t\,\delta c+a^2\,t\,\delta c^2-b^2\,t\,\delta c^2-8\,a\,t\,\delta f+8\,a^2\,t\,\delta f+8\,b\,t\,\delta f-8\,b^2\,t\,\delta f+4\,a\,t\,\delta c\,\delta f-4\,a^2\,t\,\delta c\,\delta f-4\,b\,t\,\delta c\,\delta f+4\,b^2\,t\,\delta c\,\delta f-9\,p_A-3\,\delta c\,p_A+10\,\delta f\,p_A+18\,p_B+6\,\delta c\,p_B-10\,\delta f\,p_B\right)$$

Using the derivative, we obtain the reaction function of Firm B in the range, (a+b)/4<z<(2+a+b)/4.

```
Simplify[
 Solve[1/(2 (a - b) t (3 + δc)^2) (18 a t - 9 a^2 t - 18 b t + 9 b^2 t + 6 a t δc - 6 b t δc + a^2 t δc^2 - b^2 t δc^2 -
     8 a t δf + 8 a^2 t δf + 8 b t δf - 8 b^2 t δf + 4 a t δc δf - 4 a^2 t δc δf - 4 b t δc δf +
     4 b^2 t δc δf - 9 pA - 3 δc pA + 10 δf pA + 18 pB + 6 δc pB - 10 δf pB) == 0, pB]]
```

$$\left\{\left\{p_B\to\frac{1}{2\,(9+3\,\delta c-5\,\delta f)}\left(-(a-b)\,t\,\left(18+6\,\delta c-8\,\delta f+4\,\delta c\,\delta f+a\,(-9+\delta c^2+8\,\delta f-4\,\delta c\,\delta f)+b\,(-9+\delta c^2+8\,\delta f-4\,\delta c\,\delta f)\right)+(9+3\,\delta c-10\,\delta f)\,p_A\right)\right\}\right\}$$

We have already obtained the condition that *z* is between (a+b)/4 and (2+a+b)/4 as follows:

$$p_B + \frac{(b-a)\ t\ (3\ (a+b)\ (1-\delta c) - 2\ (3-\delta c))}{6} < p_A < p_B + \frac{(b-a)\ t\ (3\ (a+b)\ (1-\delta c) + 4\ \delta c)}{6}$$

which can be rewritten as

$$p_A - \frac{(b-a)\ t\ (3\ (a+b)\ (1-\delta c) + 4\ \delta c)}{6} < p_B < p_A - \frac{(b-a)\ t\ (3\ (a+b)\ (1-\delta c) - 2\ (3-\delta c))}{6}$$

If the following outcomes are positive, the reaction function satisfies the above inequalities:

$$\text{Factor}\Big[p_A - \frac{(b-a)\ t\ (3\ (a+b)\ (1-\delta c) - 2\ (3-\delta c))}{6} - \frac{1}{2\ (9+3\ \delta c - 5\ \delta f)} \left(-(a-b)\ t\ \left(18+6\ \delta c - 8\ \delta f + 4\ \delta c\ \delta f + a\left(-9+\delta c^2 + 8\ \delta f - 4\ \delta c\ \delta f\right) + b\left(-9+\delta c^2+8\ \delta f - 4\ \delta c\ \delta f\right)\right) + (9+3\ \delta c - 10\ \delta f)\ p_A\right)\Big]$$

$$\text{Factor}\Big[\frac{1}{2\ (9+3\ \delta c - 5\ \delta f)} \left(-(a-b)\ t\ \left(18+6\ \delta c - 8\ \delta f + 4\ \delta c\ \delta f + a\left(-9+\delta c^2+8\ \delta f - 4\ \delta c\ \delta f\right) + b\left(-9+\delta c^2+8\ \delta f-4\ \delta c\ \delta f\right)\right) + (9+3\ \delta c - 10\ \delta f)\ p_A\right) - \left(p_A - \frac{(b-a)\ t\ (3\ (a+b)\ (1-\delta c) + 4\ \delta c)}{6}\right)\Big]$$

$$-\left(\left((3+\delta c)\ \left(-6\ a\ t\ \delta c + 6\ a^2\ t\ \delta c + 6\ b\ t\ \delta c - 6\ b^2\ t\ \delta c - 2\ a\ t\ \delta f - 3\ a^2\ t\ \delta f + 2\ b\ t\ \delta f + 3\ b^2\ t\ \delta f - 9\ p_A\right)\right) \Big/ (6\ (9+3\ \delta c - 5\ \delta f))\right)$$

$$\frac{1}{6\ (9+3\ \delta c - 5\ \delta f)} (3+\delta c)\ \left(-18\ a\ t + 18\ b\ t - 12\ a\ t\ \delta c + 6\ a^2\ t\ \delta c + 12\ b\ t\ \delta c - 6\ b^2\ t\ \delta c + 8\ a\ t\ \delta f - 3\ a^2\ t\ \delta f - 8\ b\ t\ \delta f + 3\ b^2\ t\ \delta f - 9\ p_A\right)$$

We derive the threshold values of p_A such that each of the outcomes equals zero:

$$\text{Simplify}\Big[\text{Solve}\Big[-\left(\left((3+\delta c)\ \left(-6\ a\ t\ \delta c + 6\ a^2\ t\ \delta c + 6\ b\ t\ \delta c - 6\ b^2\ t\ \delta c - 2\ a\ t\ \delta f - 3\ a^2\ t\ \delta f + 2\ b\ t\ \delta f + 3\ b^2\ t\ \delta f - 9\ p_A\right)\right) \Big/ (6\ (9+3\ \delta c - 5\ \delta f))\right) == 0,\ p_A\Big]\Big]$$

$$\text{Simplify}\Big[\text{Solve}\Big[\frac{1}{6\ (9+3\ \delta c - 5\ \delta f)} (3+\delta c)\ \left(-18\ a\ t + 18\ b\ t - 12\ a\ t\ \delta c + 6\ a^2\ t\ \delta c + 12\ b\ t\ \delta c - 6\ b^2\ t\ \delta c + 8\ a\ t\ \delta f - 3\ a^2\ t\ \delta f - 8\ b\ t\ \delta f + 3\ b^2\ t\ \delta f - 9\ p_A\right) == 0,\ p_A\Big]\Big]$$

$$\left\{\left\{p_A \to \frac{1}{9}\ (a-b)\ t\ (6\ (-1+a+b)\ \delta c - (2+3\ a+3\ b)\ \delta f)\right\}\right\}$$

$$\left\{\left\{p_A \to \frac{1}{9}\ (a-b)\ t\ (-18+6\ (-2+a+b)\ \delta c + (8-3\ a-3\ b)\ \delta f)\right\}\right\}$$

Therefore, if p_A satisfies the following inequalities, the reaction function of Firm B is in the range, (a+b)/4<z<(2+a+b)/4

$$\frac{(b-a)\ t\ (6\ (1-a-b)\ \delta c + (2+3\ a+3\ b)\ \delta f)}{9} < p_A < \frac{(b-a)\ t\ (18+6\ (2-a-b)\ \delta c - (8-3\ a-3\ b)\ \delta f)}{9}$$

(Case iii) (2 + a + b)/4 ≤ z ≤1

The following prices are the second period prices.

$$\left\{p_{Ao} \to -\frac{1}{3}\ (a-b)\ t\ (a+b+2\ z),\ p_{An} \to 0,\ p_{Bo} \to (a-b)\ t\ (a+b-2\ z),\ p_{Bn} \to \frac{1}{3}\ (a-b)\ t\ (a+b-4\ z)\right\}$$

Anticipating the second period prices, consumers choose one of the first-period prices p_A or p_B (p_A is the first-period price of firm A and p_B is the first-period price of firm B)

The location of the indifferent consumer, *z*, is derived by the following equation:

$$\text{Solve}\left[-p_A - t\,(z-a)^2 - \delta c\left(\frac{1}{3}(a-b)\,t\,(a+b-4z) + t\,(z-b)^2\right) == -p_B - t\,(z-b)^2 - \delta c\left(((a-b)\,t\,(a+b-2z)) + t\,(z-b)^2\right),\ z\right]$$

$$\left\{\left\{z \to \frac{-3a^2 t + 3b^2 t + 2a^2 t\,\delta c - 2b^2 t\,\delta c - 3p_A + 3p_B}{2(a-b)\,t\,(-3+\delta c)}\right\}\right\}$$

We set the location of the indifferent consumer z:

$$z = \frac{-3a^2 t + 3b^2 t + 2a^2 t\,\delta c - 2b^2 t\,\delta c - 3p_A + 3p_B}{2(a-b)\,t\,(-3+\delta c)}$$

$$\frac{-3a^2 t + 3b^2 t + 2a^2 t\,\delta c - 2b^2 t\,\delta c - 3p_A + 3p_B}{2(a-b)\,t\,(-3+\delta c)}$$

We derive the condition that the location of the indifferent consumer, z, is between (2+a+b)/4 and 1, by solving the following equations

```
Factor[Solve[z - (2 + a + b) / 4 == 0, pA]]
Factor[Solve[1 - z == 0, pA]]
```

$$\left\{\left\{p_A \to \frac{1}{6}\left(6at - 3a^2 t - 6bt + 3b^2 t - 2at\,\delta c + 3a^2 t\,\delta c + 2bt\,\delta c - 3b^2 t\,\delta c + 6p_B\right)\right\}\right\}$$

$$\left\{\left\{p_A \to \frac{1}{3}\left(6at - 3a^2 t - 6bt + 3b^2 t - 2at\,\delta c + 2a^2 t\,\delta c + 2bt\,\delta c - 2b^2 t\,\delta c + 3p_B\right)\right\}\right\}$$

By simplifying the above values of p_A, we have the condition that z is between (2+a+b)/4 and 1 as follows:

$$p_B + \frac{(b-a)\,t\,((a+b)(3-2\delta c) - 2(3-\delta c))}{3} \le p_A \le p_B + \frac{(b-a)\,t\,(3(a+b)(1-\delta c) - 2(3-\delta c))}{6}$$

We have already derived the 2nd period profits of Firms A and B as follows:

$$\pi_{A2} : -\frac{1}{18}(a-b)\,t\,(a+b+2z)^2$$

$$\pi_{B2} : -\frac{1}{18}(a-b)\,t\left(-18a + a^2 - 18b + 2ab + b^2 + 36z + 10az + 10bz - 20z^2\right)$$

The first-order condition of Firm A with respect to p_A is

$$\text{Factor}\left[D\left[p_A z + \delta f\left(\frac{1}{18}(b-a)\,t\,(a+b+2z)^2\right),\ p_A\right]\right]$$

$$\frac{1}{2(a-b)\,t\,(-3+\delta c)^2}\left(9a^2 t - 9b^2 t - 9a^2 t\,\delta c + 9b^2 t\,\delta c + 2a^2 t\,\delta c^2 - 2b^2 t\,\delta c^2 - 4a^2 t\,\delta f + 4b^2 t\,\delta f + 2a^2 t\,\delta c\,\delta f - 2b^2 t\,\delta c\,\delta f + 18p_A - 6\,\delta c\,p_A - 2\,\delta f\,p_A - 9p_B + 3\,\delta c\,p_B + 2\,\delta f\,p_B\right)$$

The reaction function of Firm A within the range in which (2 + a + b)/4≤z≤1 is

$$\text{Simplify}\left[\text{Solve}\left[\frac{1}{2(a-b)\,t\,(-3+\delta c)^2}\left(9a^2 t - 9b^2 t - 9a^2 t\,\delta c + 9b^2 t\,\delta c + 2a^2 t\,\delta c^2 - 2b^2 t\,\delta c^2 - 4a^2 t\,\delta f + 4b^2 t\,\delta f + 2a^2 t\,\delta c\,\delta f - 2b^2 t\,\delta c\,\delta f + 18p_A - 6\,\delta c\,p_A - 2\,\delta f\,p_A - 9p_B + 3\,\delta c\,p_B + 2\,\delta f\,p_B\right) == 0,\ p_A\right]\right]$$

$$\left\{\left\{p_A \to \left(\left(a^2 - b^2\right) t\left(9 + 2\,\delta c^2 - 4\,\delta f + \delta c\,(-9 + 2\,\delta f)\right) + (-9 + 3\,\delta c + 2\,\delta f)\,p_B\right) / (2\,(-9 + 3\,\delta c + \delta f))\right\}\right\}$$

$$p_A \to \frac{(9 - 3\,\delta c - 2\,\delta f)\,p_B}{2(9 - 3\,\delta c - \delta f)} + \frac{(b-a)(a+b)\,t\,((3-\delta c)(3-2\,\delta c) - 2(2-\delta c)\,\delta f)}{2(9 - 3\,\delta c - \delta f)}$$

The function might be outside the range, (2 + a + b)/4≤z≤1.
We derive the condition that the reaction function is indeed in the range, (2 + a + b)/4≤z≤1.

We have already obtained the condition that z is between (2+a + b)/4 and 1 as follows:

$$p_B + \frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)-2\,(3-\delta c))}{3} \le p_A \le p_B + \frac{(b-a)\,t\,(3\,(a+b)\,(1-\delta c)-2\,(3-\delta c))}{6}$$

If p_B is larger than $p_A - \frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)-2\,(3-\delta c))}{3}$, z becomes 1.

In this case, Firm A chooses the following p_A which just leads to z=1.

$$p_A \to p_B + \frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)-2\,(3-\delta c))}{3}.$$

The reaction function is within the range (2 + a + b)/4≤z≤1 if the following are positive:

$$\text{Simplify}\Big[\text{Factor}\Big[\Big(\frac{(9-3\,\delta c-2\,\delta f)\,p_B}{2\,(9-3\,\delta c-\delta f)} + \frac{(b-a)\,(a+b)\,t\,((3-\delta c)\,(3-2\,\delta c)-2\,(2-\delta c)\,\delta f)}{2\,(9-3\,\delta c-\delta f)}\Big) - \Big(p_B + \frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)-2\,(3-\delta c))}{3}\Big)\Big]\Big]$$

$$\text{Simplify}\Big[\text{Factor}\Big[p_B + \frac{(b-a)\,t\,(3\,(a+b)\,(1-\delta c)-2\,(3-\delta c))}{6} - \Big(\frac{(9-3\,\delta c-2\,\delta f)\,p_B}{2\,(9-3\,\delta c-\delta f)} + \frac{(b-a)\,(a+b)\,t\,((3-\delta c)\,(3-2\,\delta c)-2\,(2-\delta c)\,\delta f)}{2\,(9-3\,\delta c-\delta f)}\Big)\Big]\Big]$$

$$-(((-3+\delta c)\,((a-b)\,t\,(a\,(-9+6\,\delta c-2\,\delta f)+b\,(-9+6\,\delta c-2\,\delta f)-4\,(-9+3\,\delta c+\delta f))+9\,p_B))\,/\,(6\,(-9+3\,\delta c+\delta f)))$$

$$((-3+\delta c)\,((a-b)\,t\,(18+3\,(-2+a+b)\,\delta c-(2+3\,a+3\,b)\,\delta f)+9\,p_B))\,/\,(6\,(-9+3\,\delta c+\delta f))$$

Therefore, the reaction function of Firm A is within the range (2 + a + b)/4≤z≤1 if

$$\frac{(b-a)\,t\,(4\,(9-3\,\delta c-\delta f)-(a+b)\,(9-6\,\delta c+2\,\delta f))}{9} \ge p_B \ge \frac{(b-a)\,t\,(2\,(9-3\,\delta c-\delta f)+3\,(a+b)\,(\delta c-\delta f))}{9}$$

Note that if p_B is larger than the left-hand side value of the inequality, Firm A chooses the following p_A, which leads to z=1.

$$p_A \to p_B + \frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)-2\,(3-\delta c))}{3}.$$

Similarly, using the above outcomes, we derive the first-order condition of Firm B with respect to p_B

$$\text{Factor}\Big[D\Big[p_B\,(1-z) + \delta f\,\Big(\frac{1}{18}\,(b-a)\,t\,\big(-18\,a+a^2-18\,b+2\,a\,b+b^2+36\,z+10\,a\,z+10\,b\,z-20\,z^2\big)\Big),\ p_B\Big]\Big]$$

$$-\frac{1}{2\,(a-b)\,t\,(-3+\delta c)^2}\big(-18\,a\,t+9\,a^2\,t+18\,b\,t-9\,b^2\,t+12\,a\,t\,\delta c-9\,a^2\,t\,\delta c-12\,b\,t\,\delta c+9\,b^2\,t\,\delta c-2\,a\,t\,\delta c^2+2\,a^2\,t\,\delta c^2+2\,b\,t\,\delta c^2-2\,b^2\,t\,\delta c^2-18\,a\,t\,\delta f+5\,a^2\,t\,\delta f+18\,b\,t\,\delta f-5\,b^2\,t\,\delta f+6\,a\,t\,\delta c\,\delta f-5\,a^2\,t\,\delta c\,\delta f-6\,b\,t\,\delta c\,\delta f+5\,b^2\,t\,\delta c\,\delta f+9\,p_A-3\,\delta c\,p_A+10\,\delta f\,p_A-18\,p_B+6\,\delta c\,p_B-10\,\delta f\,p_B\big)$$

The reaction function of Firm B is

$$\text{Simplify}\Big[\text{Solve}\Big[-\frac{1}{2\,(a-b)\,t\,(-3+\delta c)^2}\big(-18\,a\,t+9\,a^2\,t+18\,b\,t-9\,b^2\,t+12\,a\,t\,\delta c-9\,a^2\,t\,\delta c-12\,b\,t\,\delta c+9\,b^2\,t\,\delta c-2\,a\,t\,\delta c^2+2\,a^2\,t\,\delta c^2+2\,b\,t\,\delta c^2-2\,b^2\,t\,\delta c^2-18\,a\,t\,\delta f+5\,a^2\,t\,\delta f+18\,b\,t\,\delta f-5\,b^2\,t\,\delta f+6\,a\,t\,\delta c\,\delta f-5\,a^2\,t\,\delta c\,\delta f-6\,b\,t\,\delta c\,\delta f+5\,b^2\,t\,\delta c\,\delta f+9\,p_A-3\,\delta c\,p_A+10\,\delta f\,p_A-18\,p_B+6\,\delta c\,p_B-10\,\delta f\,p_B\big)==0,\ p_B\Big]\Big]$$

$$\Big\{\Big\{p_B\to\frac{1}{2\,(-9+3\,\delta c-5\,\delta f)}\Big(-(a-b)\,t\,\big(-2\,(-3+\delta c)\,(-3+\delta c-3\,\delta f)+b\,(9-9\,\delta c+2\,\delta c^2+5\,\delta f-5\,\delta c\,\delta f)+a\,(9+2\,\delta c^2+5\,\delta f-\delta c\,(9+5\,\delta f))\big)+(-9+3\,\delta c-10\,\delta f)\,p_A\Big)\Big\}\Big\}$$

$$p_B\to\frac{(9-3\,\delta c+10\,\delta f)\,p_A}{2\,(9-3\,\delta c+5\,\delta f)}+((b-a)\,t\,(2\,(3-\delta c)\,(3-\delta c+3\,\delta f)-(a+b)\,((3-\delta c)\,(3-2\,\delta c)+5\,(1-\delta c)\,\delta f)))\,/\,(2\,(9-3\,\delta c+5\,\delta f))$$

We check the condition that the reaction function of Firm B is within the range (2 + a + b)/4≤z≤ 1.

$$p_B+\frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)-2\,(3-\delta c))}{3}<p_A<p_B+\frac{(b-a)\,t\,(3\,(a+b)\,(1-\delta c)-2\,(3-\delta c))}{6}$$

Note that if p_B is larger than $p_A-\frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)-2\,(3-\delta c))}{3}$, z becomes 1.

The reaction function of Firm B is within the range in which (2 + a + b)/4≤z≤1 if the following are positive:

$$\text{Simplify}\Big[\text{Factor}\Big[p_A-\Big(\frac{(9-3\,\delta c+10\,\delta f)\,p_A}{2\,(9-3\,\delta c+5\,\delta f)}+((b-a)\,t\,(2\,(3-\delta c)\,(3-\delta c+3\,\delta f)-(a+b)\,((3-\delta c)\,(3-2\,\delta c)+5\,(1-\delta c)\,\delta f)))\,/\,(2\,(9-3\,\delta c+5\,\delta f))+\frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)-2\,(3-\delta c))}{3}\Big)\Big]\Big]$$

$$\text{Simplify}\Big[\text{Factor}\Big[\frac{(9-3\,\delta c+10\,\delta f)\,p_A}{2\,(9-3\,\delta c+5\,\delta f)}+((b-a)\,t\,(2\,(3-\delta c)\,(3-\delta c+3\,\delta f)-(a+b)\,((3-\delta c)\,(3-2\,\delta c)+5\,(1-\delta c)\,\delta f)))\,/\,(2\,(9-3\,\delta c+5\,\delta f))+\frac{(b-a)\,t\,(3\,(a+b)\,(1-\delta c)-2\,(3-\delta c))}{6}-p_A\Big]\Big]$$

$$-(((-3+\delta c)\,((a-b)\,t\,(a\,(-9+6\,\delta c-5\,\delta f)+b\,(-9+6\,\delta c-5\,\delta f)+2\,(9-3\,\delta c+\delta f))-9\,p_A))\,/\,(6\,(-9+3\,\delta c-5\,\delta f)))$$

$$\frac{(-3+\delta c)\,((a-b)\,t\,(3\,a\,\delta c+3\,b\,\delta c-8\,\delta f)-9\,p_A)}{6\,(-9+3\,\delta c-5\,\delta f)}$$

$$\text{Simplify}[\text{Solve}[-(((-3+\delta c)\,((a-b)\,t\,(a\,(-9+6\,\delta c-5\,\delta f)+b\,(-9+6\,\delta c-5\,\delta f)+2\,(9-3\,\delta c+\delta f))-9\,p_A))\,/\,(6\,(-9+3\,\delta c-5\,\delta f)))==0,\ p_A]]$$

$$\text{Simplify}\Big[\text{Solve}\Big[\frac{(-3+\delta c)\,((a-b)\,t\,(3\,a\,\delta c+3\,b\,\delta c-8\,\delta f)-9\,p_A)}{6\,(-9+3\,\delta c-5\,\delta f)}==0,\ p_A\Big]\Big]$$

$$\Big\{\Big\{p_A\to\frac{1}{9}\,(a-b)\,t\,(a\,(-9+6\,\delta c-5\,\delta f)+b\,(-9+6\,\delta c-5\,\delta f)+2\,(9-3\,\delta c+\delta f))\Big\}\Big\}$$

$$\Big\{\Big\{p_A\to\frac{1}{9}\,(a-b)\,t\,(3\,a\,\delta c+3\,b\,\delta c-8\,\delta f)\Big\}\Big\}$$

Thus, the reaction function of Firm B is within the range (2 + a + b)/4≤z≤1 if

$$\frac{(b-a)\,t\,((a+b)\,(9-6\,\delta c+5\,\delta f)-2\,(9-3\,\delta c+\delta f))}{9}\le p_A\le\frac{(b-a)\,t\,(8\,\delta f-3\,(a+b)\,\delta c)}{9}$$

We summarize below the results from our analysis of the reaction functions in the three cases.

(Case i) 0≤z ≤ (a + b)/4.

Firm A's reaction function in this range is

$$p_A \to \left((b-a)\ t\ \left(6\ \delta c - 2\ \delta c^2 + 8\ \delta f + 4\ \delta c\ \delta f + (a+b)\ \left(9 - 9\ \delta c + 2\ \delta c^2 + 5\ \delta f - 5\ \delta c\ \delta f\right)\right) + (9 - 3\ \delta c + 10\ \delta f)\ p_B\right) \Big/ (2\ (9 - 3\ \delta c + 5\ \delta f))$$

This is applicable for the following range of p_B

$$-\frac{1}{9}\ (b-a)\ t\ ((a+b)\ (9 - 6\ \delta c + 5\ \delta f) + 6\ \delta c - 8\ \delta f) < p_B < \frac{(b-a)\ t\ (3\ (-2+a+b)\ \delta c + 8\ \delta f)}{9}$$

We check the endpoints of Firm A's reaction function in the range of p_B (see the inequalities above). Substituting the minimum and maximum values of p_B into Firm A's reaction function, we obtain the endpoints (vectors) of Firm A's reaction function in the $(p_A,\ p_B)$ coordinate system:

$$\text{FullSimplify}\left[\left((b-a)\ t\ \left(6\ \delta c - 2\ \delta c^2 + 8\ \delta f + 4\ \delta c\ \delta f + (a+b)\ \left(9 - 9\ \delta c + 2\ \delta c^2 + 5\ \delta f - 5\ \delta c\ \delta f\right)\right) + (9 - 3\ \delta c + 10\ \delta f)\ p_B\right) \Big/ (2\ (9 - 3\ \delta c + 5\ \delta f))\ /.\ p_B \to -\frac{1}{9}\ (b-a)\ t\ ((a+b)\ (9 - 6\ \delta c + 5\ \delta f) + 6\ \delta c - 8\ \delta f)\right]$$

$$\text{FullSimplify}\left[\left((b-a)\ t\ \left(6\ \delta c - 2\ \delta c^2 + 8\ \delta f + 4\ \delta c\ \delta f + (a+b)\ \left(9 - 9\ \delta c + 2\ \delta c^2 + 5\ \delta f - 5\ \delta c\ \delta f\right)\right) + (9 - 3\ \delta c + 10\ \delta f)\ p_B\right) \Big/ (2\ (9 - 3\ \delta c + 5\ \delta f))\ /.\ p_B \to \frac{(b-a)\ t\ (3\ (-2+a+b)\ \delta c + 8\ \delta f)}{9}\right]$$

$$\frac{1}{9}\ (a-b)\ (-8 + 5\ a + 5\ b)\ t\ \delta f$$

$$\frac{1}{18}\ (a-b)\ t\ (3\ (a+b)\ (-3 + \delta c) - 16\ \delta f)$$

Firm A's reaction function in this range consists of the line segment between the following two points:

$$\left(\frac{1}{9}\ (a-b)\ (-8 + 5\ a + 5\ b)\ t\ \delta f,\ -\frac{1}{9}\ (b-a)\ t\ ((a+b)\ (9 - 6\ \delta c + 5\ \delta f) + 6\ \delta c - 8\ \delta f)\right),$$

$$\left(\frac{1}{18}\ (a-b)\ t\ (3\ (a+b)\ (-3 + \delta c) - 16\ \delta f),\ \frac{(b-a)\ t\ (3\ (-2+a+b)\ \delta c + 8\ \delta f)}{9}\right)$$

Firm B's reaction function in this range is

$$p_B \to \frac{1}{2\ (9 - 3\ \delta c - \delta f)}\left(((b-a)\ (2-a-b))\ t\ \left(9 + 2\ \delta c^2 - 4\ \delta f + \delta c\ (-9 + 2\ \delta f)\right) + (9 - 3\ \delta c - 2\ \delta f)\ p_A\right)$$

This is applicable for the following range of p_A

$$\frac{(b-a)\ t\ (18 + (a+b)\ (9 - 6\ \delta c + 2\ \delta f) - 8\ \delta f)}{9} \geq p_A \geq \frac{(b-a)\ t\ (18 - 3\ (a+b)\ (\delta c - \delta f) - 8\ \delta f)}{9}$$

Note that if p_A is larger than the left-hand side value of the inequality, Firm B chooses the following p_B, which just leads to z=0.

$$p_B \to p_A - \frac{(b-a)\ t\ ((a+b)\ (3 - 2\ \delta c) + 2\ \delta c)}{3}$$

We check the endpoints of Firm B's reaction function in the range of p_A, based on the above inequalities. Substituting the minimum and maximum values of p_A into Firm B's reaction function, we obtain the endpoints (vectors) of Firm B's reaction function in the (p_A, p_B) coordinate system:

$$\text{FullSimplify}\Big[\frac{1}{2\,(9-3\,\delta c-\delta f)}\Big(((b-a)\,(2-a-b))\,t\,\big(9+2\,\delta c^2-4\,\delta f+\delta c\,(-9+2\,\delta f)\big)+(9-3\,\delta c-2\,\delta f)\,p_A\Big)\ /.\ p_A\to\frac{(b-a)\,t\,(18-3\,(a+b)\,(\delta c-\delta f)-8\,\delta f)}{9}\Big]$$

$$\text{FullSimplify}\Big[\frac{1}{2\,(9-3\,\delta c-\delta f)}\Big(((b-a)\,(2-a-b))\,t\,\big(9+2\,\delta c^2-4\,\delta f+\delta c\,(-9+2\,\delta f)\big)+(9-3\,\delta c-2\,\delta f)\,p_A\Big)\ /.\ p_A\to\frac{(b-a)\,t\,(18+(a+b)\,(9-6\,\delta c+2\,\delta f)-8\,\delta f)}{9}\Big]$$

$$-\frac{1}{18}\,(a-b)\,t\,(3\,(-4+a+b)\,(-3+\delta c)+2\,(-8+3\,a+3\,b)\,\delta f)$$

$$-\frac{2}{9}\,(a-b)\,t\,(9-3\,\delta c+(-4+a+b)\,\delta f)$$

Firm B's reaction function in this range consists of the line segment between the following two points:

$$\Big(\frac{(b-a)\,t\,(18-3\,(a+b)\,(\delta c-\delta f)-8\,\delta f)}{9},\ -\frac{1}{18}\,(a-b)\,t\,(3\,(-4+a+b)\,(-3+\delta c)+2\,(-8+3\,a+3\,b)\,\delta f)\Big),$$

$$\Big(\frac{(b-a)\,t\,(18+(a+b)\,(9-6\,\delta c+2\,\delta f)-8\,\delta f)}{9},\ -\frac{2}{9}\,(a-b)\,t\,(9-3\,\delta c+(-4+a+b)\,\delta f)\Big)$$

Note that if the following inequality,

$$\frac{(b-a)\,t\,(18+(a+b)\,(9-6\,\delta c+2\,\delta f)-8\,\delta f)}{9}\le p_A,$$

holds, then Firm B chooses the following p_B, which just leads to z=0.

$$p_B\to p_A-\frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)+2\,\delta c)}{3}.$$

The reaction function of firm B leading to z=0 consists of the line segment connecting the following two points.

$$\Big(\frac{(b-a)\,t\,(18+(a+b)\,(9-6\,\delta c+2\,\delta f)-8\,\delta f)}{9},\ -\frac{2}{9}\,(a-b)\,t\,(9-3\,\delta c+(-4+a+b)\,\delta f)\Big),$$

$$\Big(\frac{(b-a)\,t\,(18+(a+b)\,(9-6\,\delta c+2\,\delta f)-8\,\delta f)}{9}+k,\ -\frac{2}{9}\,(a-b)\,t\,(9-3\,\delta c+(-4+a+b)\,\delta f)+k\Big),$$

where *k* is a (sufficiently large) positive constant.

(Case ii) (a + b)/4 < z < (2 + a + b)/4

Firm A's reaction function in this range is

$$p_A\to\Big((9+3\,\delta c-10\,\delta f)\,p_B+(b-a)\,t\,\Big((a+b)\,\big(9-\delta c^2-8\,\delta f+4\,\delta c\,\delta f\big)+2\,\big(3\,\delta c+\delta c^2+4\,\delta f-2\,\delta c\,\delta f\big)\Big)\Big)\Big/(2\,(9+3\,\delta c-5\,\delta f))$$

This is applicable for the following range of p_B

$$\frac{(b-a)\,t\,(3\,(a+b)\,(2\,\delta c-\delta f)-2\,(3\,\delta c-4\,\delta f))}{9}<p_B<\frac{(b-a)\,t\,(3\,(a+b)\,(2\,\delta c-\delta f)+2\,(9-\delta f))}{9}$$

We check the endpoints of Firm A's reaction function in the range of p_B (see the inequalities above). Substituting the minimum and maximum values of p_B into Firm A's reaction function, we obtain the endpoints (vectors) of Firm A's reaction function in the (p_A, p_B) coordinate system:

$$\text{FullSimplify}\left[\left((9+3\,\delta c-10\,\delta f)\,p_B+(b-a)\,t\,\left((a+b)\,\left(9-\delta c^2-8\,\delta f+4\,\delta c\,\delta f\right)+2\,\left(3\,\delta c+\delta c^2+4\,\delta f-2\,\delta c\,\delta f\right)\right)\right)\Big/\left(2\,(9+3\,\delta c-5\,\delta f)\right)\,/.\,p_B\to\frac{(b-a)\,t\,(3\,(a+b)\,(2\,\delta c-\delta f)-2\,(3\,\delta c-4\,\delta f))}{9}\right]$$

$$\text{FullSimplify}\left[\left((9+3\,\delta c-10\,\delta f)\,p_B+(b-a)\,t\,\left((a+b)\,\left(9-\delta c^2-8\,\delta f+4\,\delta c\,\delta f\right)+2\,\left(3\,\delta c+\delta c^2+4\,\delta f-2\,\delta c\,\delta f\right)\right)\right)\Big/\left(2\,(9+3\,\delta c-5\,\delta f)\right)\,/.\,p_B\to\frac{(b-a)\,t\,(3\,(a+b)\,(2\,\delta c-\delta f)+2\,(9-\delta f))}{9}\right]$$

$$-\frac{1}{18}\,(a-b)\,t\,(3\,(a+b)\,(3+\delta c)-2\,(-8+3\,a+3\,b)\,\delta f)$$

$$-\frac{1}{18}\,(a-b)\,t\,(3\,(2+a+b)\,(3+\delta c)-2\,(2+3\,a+3\,b)\,\delta f)$$

Firm A's reaction function in this range consists of the line segment between the following two points:

$$\left(-\frac{1}{18}\,(a-b)\,t\,(3\,(a+b)\,(3+\delta c)-2\,(-8+3\,a+3\,b)\,\delta f),\ \frac{(b-a)\,t\,(3\,(a+b)\,(2\,\delta c-\delta f)-2\,(3\,\delta c-4\,\delta f))}{9}\right),$$

$$\left(-\frac{1}{18}\,(a-b)\,t\,(3\,(2+a+b)\,(3+\delta c)-2\,(2+3\,a+3\,b)\,\delta f),\ \frac{(b-a)\,t\,(3\,(a+b)\,(2\,\delta c-\delta f)+2\,(9-\delta f))}{9}\right)$$

Firm B's reaction function in this range is

$$p_B\to\left((b-a)\,t\,\left(2\,(9+3\,\delta c-4\,\delta f+2\,\delta c\,\delta f)+(a+b)\,\left(-9+\delta c^2+8\,\delta f-4\,\delta c\,\delta f\right)\right)+(9+3\,\delta c-10\,\delta f)\,p_A\right)\Big/\left(2\,(9+3\,\delta c-5\,\delta f)\right)$$

This is applicable for the following range of p_A

$$\frac{(b-a)\,t\,(6\,(1-a-b)\,\delta c+(2+3\,a+3\,b)\,\delta f)}{9}<p_A<\frac{(b-a)\,t\,(18+6\,(2-a-b)\,\delta c-(8-3\,a-3\,b)\,\delta f)}{9}$$

We check the endpoints of Firm B's reaction function in the range of p_A (see the inequalities above). Substituting the minimum and maximum values of p_A into Firm B's reaction function, we obtain the endpoints (vectors) of Firm B's reaction function in the (p_A, p_B) coordinate system:

$$\text{FullSimplify}\left[\left((b-a)\,t\,\left(2\,(9+3\,\delta c-4\,\delta f+2\,\delta c\,\delta f)+(a+b)\,\left(-9+\delta c^2+8\,\delta f-4\,\delta c\,\delta f\right)\right)+(9+3\,\delta c-10\,\delta f)\,p_A\right)\Big/\left(2\,(9+3\,\delta c-5\,\delta f)\right)\,/.\,p_A\to\frac{(b-a)\,t\,(6\,(1-a-b)\,\delta c+(2+3\,a+3\,b)\,\delta f)}{9}\right]$$

$$\text{FullSimplify}\left[\left((b-a)\,t\,\left(2\,(9+3\,\delta c-4\,\delta f+2\,\delta c\,\delta f)+(a+b)\,\left(-9+\delta c^2+8\,\delta f-4\,\delta c\,\delta f\right)\right)+(9+3\,\delta c-10\,\delta f)\,p_A\right)\Big/\left(2\,(9+3\,\delta c-5\,\delta f)\right)\,/.\,p_A\to\frac{(b-a)\,t\,(18+6\,(2-a-b)\,\delta c-(8-3\,a-3\,b)\,\delta f)}{9}\right]$$

$$\frac{1}{18}\,(a-b)\,t\,(3\,(-2+a+b)\,(3+\delta c)-2\,(2+3\,a+3\,b)\,\delta f)$$

$$\frac{1}{18}\,(a-b)\,t\,(3\,(-4+a+b)\,(3+\delta c)-2\,(-8+3\,a+3\,b)\,\delta f)$$

Firm B's reaction function in this range consists of the line segment between the following two points:

$$\left(\frac{(b-a)\,t\,(6\,(1-a-b)\,\delta c+(2+3\,a+3\,b)\,\delta f)}{9},\right.$$
$$\left.\frac{1}{18}\,(a-b)\,t\,(3\,(-2+a+b)\,(3+\delta c)-2\,(2+3\,a+3\,b)\,\delta f)\right),$$
$$\left(\frac{(b-a)\,t\,(18+6\,(2-a-b)\,\delta c-(8-3\,a-3\,b)\,\delta f)}{9},\right.$$
$$\left.\frac{1}{18}\,(a-b)\,t\,(3\,(-4+a+b)\,(3+\delta c)-2\,(-8+3\,a+3\,b)\,\delta f)\right)$$

(Case iii) (2 + a + b)/4≤1≤z

Firm A's reaction function in this range is

$$p_A \to \frac{(9-3\,\delta c-2\,\delta f)\,p_B}{2\,(9-3\,\delta c-\delta f)}+\frac{(b-a)\,(a+b)\,t\,((3-\delta c)\,(3-2\,\delta c)-2\,(2-\delta c)\,\delta f)}{2\,(9-3\,\delta c-\delta f)}$$

This is applicable for the following range of p_B

$$\frac{(b-a)\,t\,(4\,(9-3\,\delta c-\delta f)-(a+b)\,(9-6\,\delta c+2\,\delta f))}{9} \geq$$
$$p_B \geq \frac{(b-a)\,t\,(2\,(9-3\,\delta c-\delta f)+3\,(a+b)\,(\delta c-\delta f))}{9}$$

Note that if p_B is larger than the left-hand side value of the inequality, Firm A chooses the following p_A, which leads to z=1.

$$p_A \to p_B+\frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)-2\,(3-\delta c))}{3}.$$

We check the endpoints of Firm A's reaction function in the range of p_B (see the inequalities above). Substituting the minimum and maximum values of p_B into Firm A's reaction function, we obtain the endpoints (vectors) of Firm A's reaction function in the (p_A, p_B) coordinate system:

```
FullSimplify[(9 - 3 δc - 2 δf) pB/(2 (9 - 3 δc - δf)) + (b - a) (a + b) t ((3 - δc) (3 - 2 δc) - 2 (2 - δc) δf)/(2 (9 - 3 δc - δf)) /.
  pB -> (b - a) t (2 (9 - 3 δc - δf) + 3 (a + b) (δc - δf))/9]
FullSimplify[(9 - 3 δc - 2 δf) pB/(2 (9 - 3 δc - δf)) + (b - a) (a + b) t ((3 - δc) (3 - 2 δc) - 2 (2 - δc) δf)/(2 (9 - 3 δc - δf)) /.
  pB -> (b - a) t (4 (9 - 3 δc - δf) - (a + b) (9 - 6 δc + 2 δf))/9]
```

$$\frac{1}{18}\,(a-b)\,t\,(3\,(2+a+b)\,(-3+\delta c)+2\,(2+3\,a+3\,b)\,\delta f)$$

$$\frac{2}{9}\,(a-b)\,t\,(-9+3\,\delta c+(2+a+b)\,\delta f)$$

Firm A's reaction function in this range consists of the line segment between the following two points:

$$\left(\frac{1}{18}\,(a-b)\,t\,(3\,(2+a+b)\,(-3+\delta c)+2\,(2+3\,a+3\,b)\,\delta f),\right.$$
$$\left.\frac{(b-a)\,t\,(2\,(9-3\,\delta c-\delta f)+3\,(a+b)\,(\delta c-\delta f))}{9}\right),$$
$$\left(\frac{2}{9}\,(a-b)\,t\,(-9+3\,\delta c+(2+a+b)\,\delta f),\ \frac{(b-a)\,t\,(4\,(9-3\,\delta c-\delta f)-(a+b)\,(9-6\,\delta c+2\,\delta f))}{9}\right)$$

Note that if the following inequality,

$$\frac{(b-a)\, t\, (4\, (9-3\,\delta c-\delta f)-(a+b)\, (9-6\,\delta c+2\,\delta f))}{9} \le p_B,$$

holds, then Firm A chooses the following p_A, which leads to z=1.

$$p_A \to p_B + \frac{(b-a)\, t\, ((a+b)\, (3-2\,\delta c)-2\, (3-\delta c))}{3}.$$

The reaction function of firm A leading to z=1 consists of the line segment connecting the following two points.

$$\left(\frac{2}{9}\, (a-b)\, t\, (-9+3\,\delta c+(2+a+b)\,\delta f),\ \frac{(b-a)\, t\, (4\, (9-3\,\delta c-\delta f)-(a+b)\, (9-6\,\delta c+2\,\delta f))}{9}\right),$$

$$\left(\frac{2}{9}\, (a-b)\, t\, (-9+3\,\delta c+(2+a+b)\,\delta f)+k,\ \frac{(b-a)\, t\, (4\, (9-3\,\delta c-\delta f)-(a+b)\, (9-6\,\delta c+2\,\delta f))}{9}+k\right),$$

where k is a (sufficiently large) positive constant.

Firm B's reaction function in this range is

$$p_B \to \frac{(9-3\,\delta c+10\,\delta f)\, p_A}{2\, (9-3\,\delta c+5\,\delta f)} + ((b-a)\, t\, (2\, (3-\delta c)\, (3-\delta c+3\,\delta f)-(a+b)\, ((3-\delta c)\, (3-2\,\delta c)+5\, (1-\delta c)\,\delta f)))\, /\, (2\, (9-3\,\delta c+5\,\delta f))$$

This is applicable for the following range of p_A

$$\frac{(b-a)\, t\, ((a+b)\, (9-6\,\delta c+5\,\delta f)-2\, (9-3\,\delta c+\delta f))}{9} < p_A < \frac{(b-a)\, t\, (8\,\delta f-3\, (a+b)\,\delta c)}{9}$$

We check the endpoints of Firm B's reaction function in the range of p_A (see the inequalities above). Substituting the minimum and maximum values of p_A into Firm B's reaction function, we obtain the endpoints (vectors) of Firm B's reaction function in the (p_A, p_B) coordinate system:

$$\text{FullSimplify}\left[\frac{(9-3\,\delta c+10\,\delta f)\, p_A}{2\, (9-3\,\delta c+5\,\delta f)} + ((b-a)\, t\, (2\, (3-\delta c)\, (3-\delta c+3\,\delta f)-(a+b)\, ((3-\delta c)\, (3-2\,\delta c)+5\, (1-\delta c)\,\delta f)))\, /\, (2\, (9-3\,\delta c+5\,\delta f))\ /.\ p_A \to \frac{(b-a)\, t\, ((a+b)\, (9-6\,\delta c+5\,\delta f)-2\, (9-3\,\delta c+\delta f))}{9}\right]$$

$$\text{FullSimplify}\left[\frac{(9-3\,\delta c+10\,\delta f)\, p_A}{2\, (9-3\,\delta c+5\,\delta f)} + ((b-a)\, t\, (2\, (3-\delta c)\, (3-\delta c+3\,\delta f)-(a+b)\, ((3-\delta c)\, (3-2\,\delta c)+5\, (1-\delta c)\,\delta f)))\, /\, (2\, (9-3\,\delta c+5\,\delta f))\ /.\ p_A \to \frac{(b-a)\, t\, (8\,\delta f-3\, (a+b)\,\delta c)}{9}\right]$$

$$-\frac{1}{9}\, (a-b)\, (-2+5\,a+5\,b)\, t\,\delta f$$

$$-\frac{1}{18}\, (a-b)\, t\, (3\, (-2+a+b)\, (-3+\delta c)+16\,\delta f)$$

Firm B's reaction function in this range consists of the line segment between the following two points:

$$\left(\frac{(b-a)\, t\, ((a+b)\, (9-6\,\delta c+5\,\delta f)-2\, (9-3\,\delta c+\delta f))}{9},\ -\frac{1}{9}\, (a-b)\, (-2+5\,a+5\,b)\, t\,\delta f\right),$$

$$\left(\frac{(b-a)\, t\, (8\,\delta f-3\, (a+b)\,\delta c)}{9},\ -\frac{1}{18}\, (a-b)\, t\, (3\, (-2+a+b)\, (-3+\delta c)+16\,\delta f)\right)$$

Because each firm's reaction function consists of three different pieces, we need to derive the 'true' reaction function by checking when the firm's profit obtains a global, rather than, local maximum.

From here on, we assume that the discount factors are common δ. We set the following :

```
δf = δ
```

δ

```
δc = δ
```

δ

From Firm A's reaction function derived above, we check the endpoints of each of the three line segments corresponding to the three cases: (i)0≤z≤(a+b)/4, (ii) (a+b)/4<z<(2+a+b)/4, (iii) (2+a+b)/4≤z≤1.

(Case i): The first endpoint below locates the left-hand side of Firm A's reaction function, and the second endpoint below locates the right-hand side of Firm A's reaction function in period 1.

$$\text{c1aL} = \text{Simplify}\left[\left\{\frac{1}{9}(a-b)(-8+5a+5b)\,t\,\delta f,\ -\frac{1}{9}(b-a)\,t\,((a+b)(9-6\,\delta c+5\,\delta f)+6\,\delta c-8\,\delta f)\right\}\right]$$

$$\text{c1aR} = \text{FullSimplify}\left[\text{Factor}\left[\left\{\frac{1}{18}(a-b)\,t\,(3(a+b)(-3+\delta c)-16\,\delta f),\ \frac{(b-a)\,t\,(3(-2+a+b)\,\delta c+8\,\delta f)}{9}\right\}\right]\right]$$

$$\left\{\frac{1}{9}(a-b)(-8+5a+5b)\,t\,\delta,\ -\frac{1}{9}(-a+b)\,t\,(-(a+b)(-9+\delta)-2\,\delta)\right\}$$

$$\left\{\frac{1}{18}(a-b)\,t\,(-9(a+b)+(-16+3a+3b)\,\delta),\ -\frac{1}{9}(a-b)(2+3a+3b)\,t\,\delta\right\}$$

Case (ii): The first endpoint below locates the left-hand side of Firm A's reaction function, and the second endpoint below locates the right-hand side of Firm A's reaction function in period 1.

$$\text{c2aL} = \text{FullSimplify}\left[\text{Factor}\left[\left\{-\frac{1}{18}(a-b)\,t\,(3(a+b)(3+\delta c)-2(-8+3a+3b)\,\delta f),\ \frac{(b-a)\,t\,(3(a+b)(2\,\delta c-\delta f)-2(3\,\delta c-4\,\delta f))}{9}\right\}\right]\right]$$

$$\text{c2aR} = \text{Simplify}\left[\left\{-\frac{1}{18}(a-b)\,t\,(3(2+a+b)(3+\delta c)-2(2+3a+3b)\,\delta f),\ \frac{(b-a)\,t\,(3(a+b)(2\,\delta c-\delta f)+2(9-\delta f))}{9}\right\}\right]$$

$$\left\{\frac{1}{18}(a-b)\,t\,(-9(a+b)+(-16+3a+3b)\,\delta),\ -\frac{1}{9}(a-b)(2+3a+3b)\,t\,\delta\right\}$$

$$\left\{\frac{1}{18}(a-b)\,t\,(3a(-3+\delta)+3b(-3+\delta)-2(9+\delta)),\ \frac{1}{9}(-a+b)\,t\,(18+(-2+3a+3b)\,\delta)\right\}$$

As shown above, the left endpoint of Firm A's reaction function in (Case ii) coincides with the right endpoint of Firm A's reaction function in (Case i). That is, Firm A's reaction function is continuous in (Case i) and (Case ii).

(Case iii): The first endpoint below locates the left-hand side of Firm A's reaction function, and the second endpoint below locates the right-hand side of Firm A's reaction function in period 1.

```
c3aL = Simplify[{1/18 (a - b) t (3 (2 + a + b) (-3 + δc) + 2 (2 + 3 a + 3 b) δf),
    ((b - a) t (2 (9 - 3 δc - δf) + 3 (a + b) (δc - δf)))/9}]
c3aR = Simplify[{2/9 (a - b) t (-9 + 3 δc + (2 + a + b) δf),
    ((b - a) t (4 (9 - 3 δc - δf) - (a + b) (9 - 6 δc + 2 δf)))/9}]
```

$$\left\{\frac{1}{18}(a-b)\,t\,(3\,(2+a+b)\,(-3+\delta)+2\,(2+3a+3b)\,\delta),\ \frac{2}{9}(a-b)\,t\,(-9+4\delta)\right\}$$

$$\left\{\frac{2}{9}(a-b)\,t\,(-9+(5+a+b)\,\delta),\ -\frac{1}{9}(a-b)\,(-4+a+b)\,t\,(-9+4\delta)\right\}$$

At this stage, we cannot say if Firm A's reaction function is also continuous in (Case ii) and (Case iii). We will come back to this shortly. Note that the left endpoint in (Case iii) corresponds to (2+a+b)/4=z.

(Case iii)': When z=1, the reaction function of Firm A consists of the segment connecting the following two points.

```
c3daL = Simplify[
  {2/9 (a - b) t (-9 + 3 δc + (2 + a + b) δf), ((b - a) t (4 (9 - 3 δc - δf) - (a + b) (9 - 6 δc + 2 δf)))/9}]
c3daR = Simplify[{2/9 (a - b) t (-9 + 3 δc + (2 + a + b) δf) + k,
    ((b - a) t (4 (9 - 3 δc - δf) - (a + b) (9 - 6 δc + 2 δf)))/9 + k}]
```

$$\left\{\frac{2}{9}(a-b)\,t\,(-9+(5+a+b)\,\delta),\ -\frac{1}{9}(a-b)\,(-4+a+b)\,t\,(-9+4\delta)\right\}$$

$$\left\{k+\frac{2}{9}(a-b)\,t\,(-9+(5+a+b)\,\delta),\ k-\frac{1}{9}(a-b)\,(-4+a+b)\,t\,(-9+4\delta)\right\}$$

where k is a sufficient large positive number (to keep p_A at the monopoly price leading to $z=1$).

Next, from Firm B's reaction function derived above, we check the endpoints of each of the three line segments corresponding to the three cases: (i)0≤z≤(a+b)/4, (ii) (a+b)/4<z<(2+a+b)/4, (iii) (2+a+b)/4≤z≤1.

(Case i): The first endpoint below locates the left-hand side of Firm B's reaction function, and the second endpoint below locates the right-hand side of Firm B's reaction function in period 1.

$$c1bL = Simplify\left[\left\{\frac{(b-a)\ t\ (18-3\ (a+b)\ (\delta c-\delta f)-8\ \delta f)}{9},\ -\frac{1}{18}\ (a-b)\ t\ (3\ (-4+a+b)\ (-3+\delta c)+2\ (-8+3\ a+3\ b)\ \delta f)\right\}\right]$$

$$c1bR = Simplify\left[\left\{\frac{(b-a)\ t\ (18+(a+b)\ (9-6\ \delta c+2\ \delta f)-8\ \delta f)}{9},\ -\frac{2}{9}\ (a-b)\ t\ (9-3\ \delta c+(-4+a+b)\ \delta f)\right\}\right]$$

$$\left\{\frac{2}{9}\ (a-b)\ t\ (-9+4\ \delta),\ -\frac{1}{18}\ (a-b)\ t\ (36+9\ a\ (-1+\delta)+9\ b\ (-1+\delta)-28\ \delta)\right\}$$

$$\left\{\frac{1}{9}\ (a-b)\ (2+a+b)\ t\ (-9+4\ \delta),\ -\frac{2}{9}\ (a-b)\ t\ (9+(-7+a+b)\ \delta)\right\}$$

(Case i)’: When z=0, The reaction function of Firm B consists of the segment connecting the following two points.

$$c1dbL = Simplify\left[\left\{\frac{(b-a)\ t\ (18+(a+b)\ (9-6\ \delta c+2\ \delta f)-8\ \delta f)}{9},\ -\frac{2}{9}\ (a-b)\ t\ (9-3\ \delta c+(-4+a+b)\ \delta f)\right\}\right]$$

$$c1dbR = Simplify\left[\left\{\frac{(b-a)\ t\ (18+(a+b)\ (9-6\ \delta c+2\ \delta f)-8\ \delta f)}{9}+k,\ -\frac{2}{9}\ (a-b)\ t\ (9-3\ \delta c+(-4+a+b)\ \delta f)+k\right\}\right]$$

$$\left\{\frac{1}{9}\ (a-b)\ (2+a+b)\ t\ (-9+4\ \delta),\ -\frac{2}{9}\ (a-b)\ t\ (9+(-7+a+b)\ \delta)\right\}$$

$$\left\{k+\frac{1}{9}\ (a-b)\ (2+a+b)\ t\ (-9+4\ \delta),\ k-\frac{2}{9}\ (a-b)\ t\ (9+(-7+a+b)\ \delta)\right\}$$

where k is a sufficient large positive number (to keep p_B at the monopoly price leading to $z=0$).

Case (ii): The first endpoint below locates the left-hand side of Firm B’s reaction function, and the second endpoint below locates the right-hand side of Firm B’s reaction function in period 1.

$$c2bL = FullSimplify\left[\left\{\frac{(b-a)\ t\ (6\ (1-a-b)\ \delta c+(2+3\ a+3\ b)\ \delta f)}{9},\ \frac{1}{18}\ (a-b)\ t\ (3\ (-2+a+b)\ (3+\delta c)-2\ (2+3\ a+3\ b)\ \delta f)\right\}\right]$$

$$c2bR = Simplify\left[\left\{\frac{(b-a)\ t\ (18+6\ (2-a-b)\ \delta c-(8-3\ a-3\ b)\ \delta f)}{9},\ \frac{1}{18}\ (a-b)\ t\ (3\ (-4+a+b)\ (3+\delta c)-2\ (-8+3\ a+3\ b)\ \delta f)\right\}\right]$$

$$\left\{\frac{1}{9}\ (a-b)\ (-8+3\ a+3\ b)\ t\ \delta,\ -\frac{1}{18}\ (a-b)\ t\ (-9\ (-2+a+b)+(10+3\ a+3\ b)\ \delta)\right\}$$

$$\left\{\frac{1}{9}\ (a-b)\ t\ (-18+(-4+3\ a+3\ b)\ \delta),\ -\frac{1}{18}\ (a-b)\ t\ (-4\ (-9+\delta)+3\ a\ (-3+\delta)+3\ b\ (-3+\delta))\right\}$$

At this stage, we cannot say if Firm B’s reaction function is continuous in (Case i) and (Case ii). We will come back to this shortly. Note that the left-hand endpoint in (Case i) corresponds to (a+b)/4=z.

(Case iii): The first endpoint below locates the left-hand side of Firm B’s reaction function, and the second endpoint below locates the right-hand side of Firm B’s reaction function in period 1.

```
c3bL =
  Simplify[{((b - a) t ((a + b) (9 - 6 δc + 5 δf) - 2 (9 - 3 δc + δf)))/9, -1/9 (a - b) (-2 + 5 a + 5 b) t δf}]
c3bR = FullSimplify[
   Factor[{((b - a) t (8 δf - 3 (a + b) δc))/9, -1/18 (a - b) t (3 (-2 + a + b) (-3 + δc) + 16 δf)}]]
```

$\left\{\frac{1}{9}(a-b)\,t\,(18+a(-9+\delta)+b(-9+\delta)-4\delta),\ -\frac{1}{9}(a-b)(-2+5a+5b)\,t\,\delta\right\}$

$\left\{\frac{1}{9}(a-b)(-8+3a+3b)\,t\,\delta,\ -\frac{1}{18}(a-b)\,t\,(-9(-2+a+b)+(10+3a+3b)\,\delta)\right\}$

We find that the left endpoint of Firm B's reaction function in (Case ii) coincides with the right endpoint of Firm B's reaction function in (Case iii). That is, Firm B's reaction function is continuous in (Case ii) and (Case iii).

The exact shape of the above reaction functions depends on various parameters of the model and the values of (a, b) that are given at this stage. In what follows, we show first that, for any values of (a, b), each firm's reaction function has one discontinuity point. Second, we show that the two reaction functions intersect only in (Case ii) given the restrictions on a + b and δ stated in Proposition 1. This shows that the equilibrium is possible only in (Case ii). We then show that the unique location equilibrium leading to the pricing equilibrium corresponding to (Case ii) is given by a = 0, b = 1.

We start by deriving Firm A's 'true' reaction function illustrating the discontinuity using various examples. We repeat the same for Firm B's 'true' reaction function. Calculations in the two parts are quite messy. If necessary, readers can refer to various figures provided, and skip the calculations to jump directly to the third part, where we show the pricing equilibrium in the first period is possible only in (Case ii).

For illustrative purposes, we start with an example where we set a = 0, b = 1, t = 1, and δ = 1/2.

```
a = 0
b = 1
δ = 1 / 2
t = 1
k = 1
```

0

1

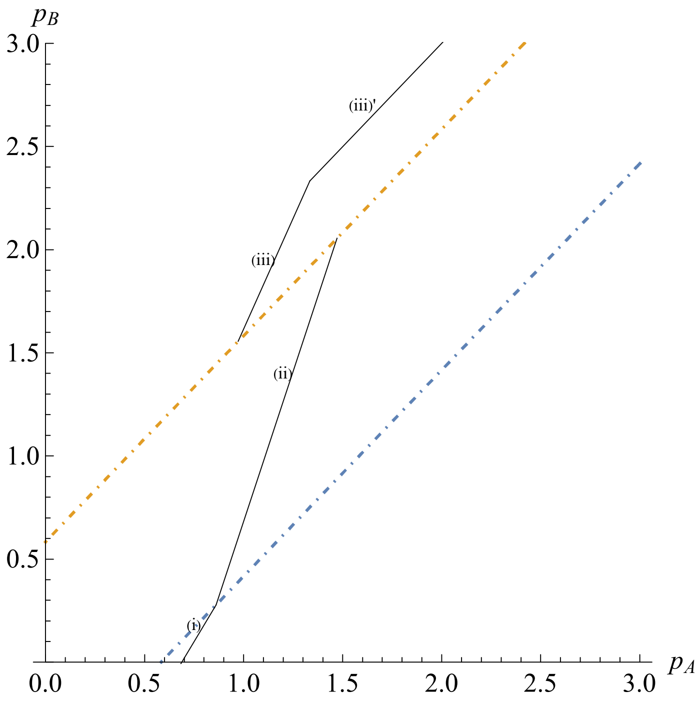
$\frac{1}{2}$

1

1

First, we plot Firm A's reaction function corresponding to the three cases.

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
   x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3},
  Epilog → {Line[{c1aL, c1aR}], Line[{c2aL, c2aR}], , Line[{c3aL, c3aR}],
    Line[{c3daL, c3daR}], Text["(iii)'", {1.6, 2.7}], Text["(iii)", {1.1, 1.95}],
    Text["(ii)", {1.2, 1.4}], Text["(i)", {0.75, 0.18}]}, PlotRange → {0, 3},
  LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"}, AspectRatio → 1, PlotStyle → DotDashed]
```



As shown above, for some p_B, there are two local optimal prices for Firm A.
We can show that the multiplicity of local optimal prices always appears.

```
Clear[a, b, δ, t, k]
```

To show the multiplicity, we check the locations of the three endpoints: The right-hand endpoint in (ii) (c2aR), the left-hand and right-hand endpoints (c3aL and c3aR) in (iii) (see below)

```
c2aR
c3aL
c3aR
```

$$\left\{\frac{1}{18}\,(a-b)\,t\,(3\,a\,(-3+\delta)+3\,b\,(-3+\delta)-2\,(9+\delta)),\ \frac{1}{9}\,(-a+b)\,t\,(18+(-2+3\,a+3\,b)\,\delta)\right\}$$

$$\left\{\frac{1}{18}\,(a-b)\,t\,(3\,(2+a+b)\,(-3+\delta)+2\,(2+3\,a+3\,b)\,\delta),\ \frac{2}{9}\,(a-b)\,t\,(-9+4\,\delta)\right\}$$

$$\left\{\frac{2}{9}\,(a-b)\,t\,(-9+(5+a+b)\,\delta),\ -\frac{1}{9}\,(a-b)\,(-4+a+b)\,t\,(-9+4\,\delta)\right\}$$

First, we compare the elements of the right-hand endpoint in (ii) and the left-hand endpoint in (iii):

```
Factor[1/18 (a - b) t (3 a (-3 + δ) + 3 b (-3 + δ) - 2 (9 + δ)) -
  1/18 (a - b) t (3 (2 + a + b) (-3 + δ) + 2 (2 + 3 a + 3 b) δ)]
Factor[1/9 (-a + b) t (18 + (-2 + 3 a + 3 b) δ) - 2/9 (a - b) t (-9 + 4 δ)]
```

$-\frac{1}{3}(a-b)(2+a+b)\,t\,\delta$

$-\frac{1}{3}(a-b)(2+a+b)\,t\,\delta$

The outcome means that for any a∈[0,1], b∈[0,1] (a≥b), t, and δ, the right-hand endpoint in (ii) is located above the left-hand endpoint in (iii) as in the above Figure.

Second, we compare the p_B-elements of the right-hand endpoint in (ii) and the right-hand end-point in (iii):

```
Factor[1/9 (-a + b) t (18 + (-2 + 3 a + 3 b) δ) - (-1/9 (a - b) (-4 + a + b) t (-9 + 4 δ))]
```

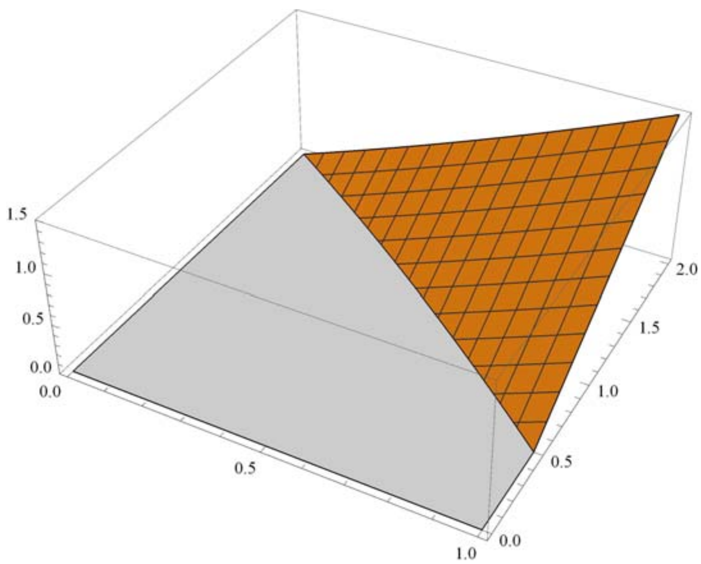
$\frac{1}{9}(a-b)\,t\,(18-9a-9b-14\delta+a\delta+b\delta)$

The p_B-element of the right-hand endpoint in (ii) larger than that of the right-hand endpoint in (iii) if and only if

$\frac{2(9-7\delta)}{9-\delta} < a+b$

If $\frac{2(9-7\delta)}{9-\delta} \geq a+b$, we simply compare the reaction function in (ii) and the reaction function in (iii); If $\frac{2(9-7\delta)}{9-\delta} < a+b$, in addition to the previous comparison, we also compare the reaction function in (ii) and the reaction function in (iii)'.

```
Plot3D[g - 2 (9 - 7 δ)/(9 - δ), {δ, 0, 1}, {g, 0, 2}, PlotRange → {0, 1.5}]
```



We need to find the global optimal price of Firm A, p_A, when there are two local optima for a given p_B. There is a price p_B such that choosing the reaction function in (ii) and choosing the reaction function in (iii) or the reaction function in (iii)' are indifferent for Firm A. This p_B is the threshold for which choosing the reaction function in (ii) is preferred by Firm A if p_B is smaller than this threshold; otherwise, choosing the reaction function in (iii) is preferred by Firm A. We need to find the threshold value of p_B.

To check the threshold value of p_B for Firm A's reaction function, we derive the profits under cases (ii), (iii), and (iii)'.
The interior profit of firm A under case (ii) for p_B is

$$\text{Factor}\Big[p_A\, z + \delta f \frac{1}{9} (b - a)\, t \left(2 + 2 a + a^2 + 2 b + 2 a b + b^2 - 8 z - 2 a z - 2 b z + 10 z^2\right) /. \; z \to \frac{3 (p_B - p_A)}{2 (b - a)\, t\, (3 + \delta c)} + \frac{((a + b)(3 - \delta c) + 2 \delta c)}{2 (3 + \delta c)} /. \{p_A \to ((9 + 3 \delta c - 10 \delta f)\, p_B + (b - a)\, t\, ((a + b)(9 - \delta c^2 - 8 \delta f + 4 \delta c\, \delta f) + 2 (3 \delta c + \delta c^2 + 4 \delta f - 2 \delta c\, \delta f))) / (2 (9 + 3 \delta c - 5 \delta f))\}\Big]$$

$$\frac{1}{72 (a - b)\, t\, (-9 + 2 \delta)} \left(81 a^4 t^2 - 162 a^2 b^2 t^2 + 81 b^4 t^2 + 144 a^2 t^2 \delta + 108 a^3 t^2 \delta - 18 a^4 t^2 \delta - 288 a b t^2 \delta - 108 a^2 b t^2 \delta + 144 b^2 t^2 \delta - 108 a b^2 t^2 \delta + 36 a^2 b^2 t^2 \delta + 108 b^3 t^2 \delta - 18 b^4 t^2 \delta - 28 a^2 t^2 \delta^2 - 12 a^3 t^2 \delta^2 + 9 a^4 t^2 \delta^2 + 56 a b t^2 \delta^2 + 12 a^2 b t^2 \delta^2 - 28 b^2 t^2 \delta^2 + 12 a b^2 t^2 \delta^2 - 18 a^2 b^2 t^2 \delta^2 - 12 b^3 t^2 \delta^2 + 9 b^4 t^2 \delta^2 - 162 a^2 t\, p_B + 162 b^2 t\, p_B + 36 a t \delta p_B + 90 a^2 t \delta p_B - 36 b t \delta p_B - 90 b^2 t \delta p_B + 81 p_B^2\right)$$

The interior profit of firm A under case (iii) for p_B

$$\text{Factor}\Big[p_A\, z + \delta f \left(\frac{1}{18} (b - a)\, t\, (a + b + 2 z)^2\right) /. \; z \to \frac{-3 p_A + 3 p_B - 3 a^2 t + 3 b^2 t + 2 a^2 t\, \delta c - 2 b^2 t\, \delta c}{2 (a - b)\, t\, (-3 + \delta c)} /. \; p_A \to \frac{(9 - 3 \delta c - 2 \delta f)\, p_B}{2 (9 - 3 \delta c - \delta f)} + \frac{(b - a)(a + b)\, t\, ((3 - \delta c)(3 - 2 \delta c) - 2 (2 - \delta c)\, \delta f)}{2 (9 - 3 \delta c - \delta f)}\Big]$$

$$-\left(\left(-9 a^4 t^2 + 18 a^2 b^2 t^2 - 9 b^4 t^2 + 4 a^4 t^2 \delta - 8 a^2 b^2 t^2 \delta + 4 b^4 t^2 \delta + 18 a^2 t\, p_B - 18 b^2 t\, p_B - 8 a^2 t \delta p_B + 8 b^2 t \delta p_B - 9 p_B^2\right) / (8 (a - b)\, t\, (-9 + 4 \delta))\right)$$

We derive the threshold value of p_B by finding p_B that equalizes the above two profits:

$$\text{FullSimplify}\Big[\text{Solve}\Big[\Big\{\frac{1}{72 (a - b)\, t\, (-9 + 2 \delta)} \left(81 a^4 t^2 - 162 a^2 b^2 t^2 + 81 b^4 t^2 + 144 a^2 t^2 \delta + 108 a^3 t^2 \delta - 18 a^4 t^2 \delta - 288 a b t^2 \delta - 108 a^2 b t^2 \delta + 144 b^2 t^2 \delta - 108 a b^2 t^2 \delta + 36 a^2 b^2 t^2 \delta + 108 b^3 t^2 \delta - 18 b^4 t^2 \delta - 28 a^2 t^2 \delta^2 - 12 a^3 t^2 \delta^2 + 9 a^4 t^2 \delta^2 + 56 a b t^2 \delta^2 + 12 a^2 b t^2 \delta^2 - 28 b^2 t^2 \delta^2 + 12 a b^2 t^2 \delta^2 - 18 a^2 b^2 t^2 \delta^2 - 12 b^3 t^2 \delta^2 + 9 b^4 t^2 \delta^2 - 162 a^2 t\, p_B + 162 b^2 t\, p_B + 36 a t \delta p_B + 90 a^2 t \delta p_B - 36 b t \delta p_B - 90 b^2 t \delta p_B + 81 p_B^2\right) == -\left(\left(-9 a^4 t^2 + 18 a^2 b^2 t^2 - 9 b^4 t^2 + 4 a^4 t^2 \delta - 8 a^2 b^2 t^2 \delta + 4 b^4 t^2 \delta + 18 a^2 t\, p_B - 18 b^2 t\, p_B - 8 a^2 t \delta p_B + 8 b^2 t \delta p_B - 9 p_B^2\right) / (8 (a - b)\, t\, (-9 + 4 \delta))\right)\Big\}, p_B\Big]\Big]$$

$$\Big\{\Big\{p_B \to -\frac{1}{18 \delta} t\, (-9 + 4 \delta) \left(2 a \delta + 3 a^2 \delta - 2 b \delta - 3 b^2 \delta + 3 (-9 + 2 \delta) \sqrt{\frac{(a - b)^2 (2 + a + b)^2 \delta^2}{(-9 + 2 \delta)(-9 + 4 \delta)}}\right)\Big\}, \Big\{p_B \to \frac{1}{18 \delta} t\, (-9 + 4 \delta) \left(-2 a \delta - 3 a^2 \delta + 2 b \delta + 3 b^2 \delta + 3 (-9 + 2 \delta) \sqrt{\frac{(a - b)^2 (2 + a + b)^2 \delta^2}{(-9 + 2 \delta)(-9 + 4 \delta)}}\right)\Big\}\Big\}$$

We can easily show that the former outcome is negative. So, we use the latter one.
We simplify the expression of the latter outcome, and obtain the following p_B:

$$p_B \rightarrow \frac{t\,(9-4\,\delta)\,(b-a)\left(3\,(2+a+b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}-(2+3\,a+3\,b)\right)}{18}\quad \text{(pb1)}$$

We rewrite the locations of the two endpoints: The right-hand endpoint in (ii), the right-hand endpoints in (iii) (see below)

```
c2aR
c3aR
```

$$\left\{\frac{1}{18}\,(a-b)\,t\,(3\,a\,(-3+\delta)+3\,b\,(-3+\delta)-2\,(9+\delta)),\ \frac{1}{9}\,(-a+b)\,t\,(18+(-2+3\,a+3\,b)\,\delta)\right\}$$

$$\left\{\frac{2}{9}\,(a-b)\,t\,(-9+(5+a+b)\,\delta),\ -\frac{1}{9}\,(a-b)\,(-4+a+b)\,t\,(-9+4\,\delta)\right\}$$

We check the condition that the derived p_B (pb1) is below the p_B-element of the right-hand endpoints in (iii).

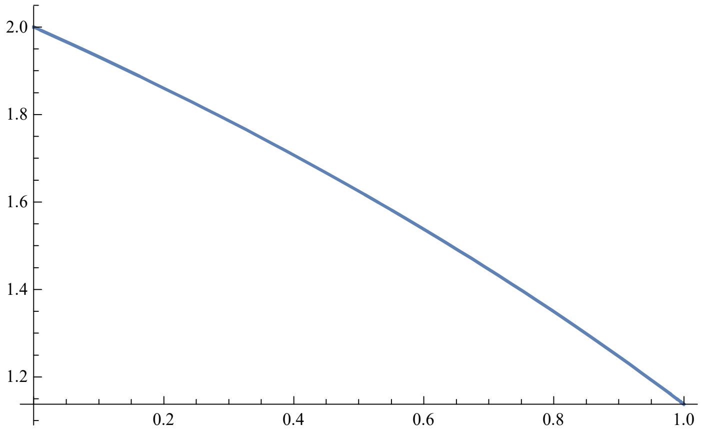
$$\text{Simplify}\left[\text{Factor}\left[-\frac{1}{9}\,(a-b)\,(-4+a+b)\,t\,(-9+4\,\delta)-\frac{t\,(9-4\,\delta)\,(b-a)\left(3\,(2+a+b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}-(2+3\,a+3\,b)\right)}{18}\right]\right]$$

$$-\frac{1}{18}\,(a-b)\,t\left(-10+a\left(-1+3\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)+b\left(-1+3\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)+6\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)(-9+4\,\delta)$$

This is positive if and only if the following inequality holds

$$(a+b)\left(-1+3\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)<10-6\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}$$

$$\text{Plot}\left[\left(10-6\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)\Big/\left(-1+3\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right),\ \{\delta,\,0,\,1\}\right]$$



If $(a+b)\left(-1+3\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)<10-6\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}$ holds, the derived p_B (pb1) is on the interval between the left-hand and right-hand endpoints in (iii). Otherwise, the reaction function in (ii) is always better than the interval between the left-hand and right-hand endpoints in (iii).

Now we derive the profit of firm A under case (iii)' for p_B

$$\text{Factor}\left[p_A z + \delta f\left(\frac{1}{18}(b-a)\,t\,(a+b+2z)^2\right) /.\ z \to \frac{-3p_A + 3p_B - 3a^2 t + 3b^2 t + 2a^2 t\,\delta c - 2b^2 t\,\delta c}{2(a-b)\,t\,(-3+\delta c)} /.\ p_A \to p_B + \frac{(b-a)\,t\,((a+b)(3-2\delta c) - 2(3-\delta c))}{3}\right]$$

$$\frac{1}{18}\left(36\,a\,t - 18\,a^2\,t - 36\,b\,t + 18\,b^2\,t - 16\,a\,t\,\delta + 8\,a^2\,t\,\delta - a^3\,t\,\delta + 16\,b\,t\,\delta - a^2\,b\,t\,\delta - 8\,b^2\,t\,\delta + a\,b^2\,t\,\delta + b^3\,t\,\delta + 18\,p_B\right)$$

We derive the threshold value of p_B by finding p_B that equalizes the two profits in cases (ii) and (iii)’:

$$\text{FullSimplify}\Big[\text{Solve}\Big[\Big\{\frac{1}{72(a-b)\,t\,(-9+2\delta)}\big(81a^4t^2 - 162a^2b^2t^2 + 81b^4t^2 + 144a^2t^2\delta + 108a^3t^2\delta - 18a^4t^2\delta - 288\,a\,b\,t^2\delta - 108a^2b\,t^2\delta + 144b^2t^2\delta - 108\,a\,b^2t^2\delta + 36a^2b^2t^2\delta + 108b^3t^2\delta - 18b^4t^2\delta - 28a^2t^2\delta^2 - 12a^3t^2\delta^2 + 9a^4t^2\delta^2 + 56\,a\,b\,t^2\delta^2 + 12a^2b\,t^2\delta^2 - 28b^2t^2\delta^2 + 12\,a\,b^2t^2\delta^2 - 18a^2b^2t^2\delta^2 - 12b^3t^2\delta^2 + 9b^4t^2\delta^2 - 162a^2t\,p_B + 162b^2t\,p_B + 36\,a\,t\,\delta\,p_B + 90a^2t\,\delta\,p_B - 36\,b\,t\,\delta\,p_B - 90b^2t\,\delta\,p_B + 81p_B^2\big) == \frac{1}{18}\big(36\,a\,t - 18a^2t - 36\,b\,t + 18b^2t - 16\,a\,t\,\delta + 8a^2t\,\delta - a^3t\,\delta + 16\,b\,t\,\delta - a^2b\,t\,\delta - 8b^2t\,\delta + a\,b^2t\,\delta + b^3t\,\delta + 18p_B\big)\Big\}, p_B\Big]\Big]$$

$$\left\{\left\{p_B \to -\frac{1}{9}(a-b)\,t\left(-9(-4+a+b) + (-6+5a+5b)\,\delta - 18\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}} + 4\delta\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}}\right)\right\}, \left\{p_B \to -\frac{1}{9}(a-b)\,t\left(36 - 9a - 9b - 6\delta + 5(a+b)\,\delta + 18\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}} - 4\delta\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}}\right)\right\}\right\}$$

We pick up the first outcome as the threshold p_B.

$$p_B \to -\frac{1}{9}(a-b)\,t\left(-9(-4+a+b) + (-6+5a+5b)\,\delta - 18\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}} + 4\delta\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}}\right) \quad (\text{pb2})$$

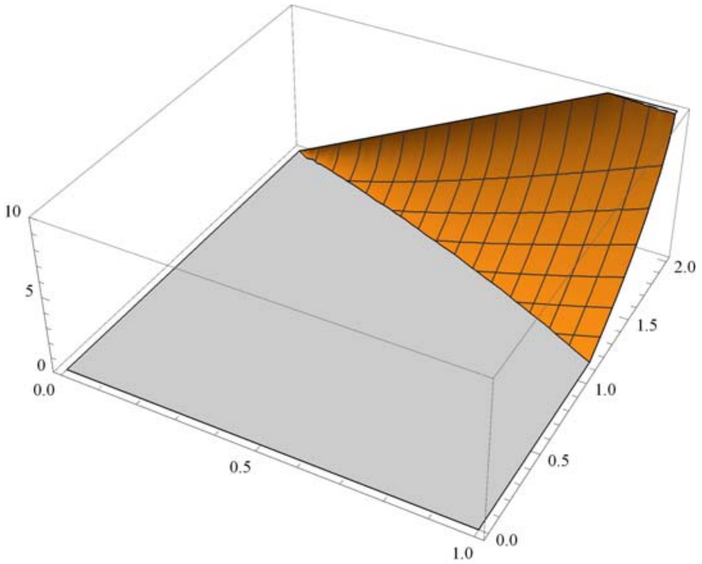
We check the condition that the derived p_B (pb2) is above the p_B-element of the right-hand endpoints in (iii).

$$\text{Simplify}\left[\text{Factor}\left[-\frac{1}{9}(a-b)\,t\left(-9(-4+a+b) + (-6+5a+5b)\,\delta - 18\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}} + 4\delta\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}}\right) - \left(-\frac{1}{9}(a-b)(-4+a+b)\,t\,(-9+4\delta)\right)\right]\right]$$

$$-\frac{1}{9}(a-b)\,t\left(10\,\delta + a\,\delta + b\,\delta - 18\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}} + 4\delta\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}}\right)$$

Denoting a+b by g, we check the value between the largest parentheses:

```
Plot3D[(10 δ + g δ - 18 Sqrt[((-2 + g) (4 + g) δ)/(-9 + 2 δ)] + 4 δ Sqrt[((-2 + g) (4 + g) δ)/(-9 + 2 δ)]),
 {δ, 0, 1}, {g, 0, 2}, PlotRange → {0, 10}]
```



We check the condition that the value between the large parentheses is positive:

```
Solve[(10 δ + g δ - 18 Sqrt[((-2 + g) (4 + g) δ)/(-9 + 2 δ)] + 4 δ Sqrt[((-2 + g) (4 + g) δ)/(-9 + 2 δ)]) == 0, g]
```

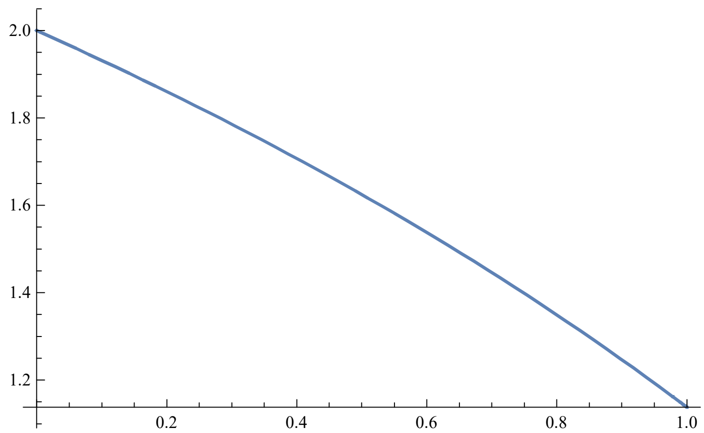
$$\left\{\left\{g \to \frac{2\left(18+\delta-6\sqrt{81-54\,\delta+8\,\delta^2}\right)}{-36+7\,\delta}\right\}, \left\{g \to \frac{2\left(18+\delta+6\sqrt{81-54\,\delta+8\,\delta^2}\right)}{-36+7\,\delta}\right\}\right\}$$

If $g = a + b > \frac{2\left(18+\delta-6\sqrt{81-54\,\delta+8\,\delta^2}\right)}{-36+7\,\delta}$,

the derived p_B (pb2) is above the p_B-element of the right-hand endpoints in (iii).

This coincides with the following condition that the derived p_B (pb1) is above the p_B-element of the right-hand endpoints in (iii).

```
Plot[(2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]))/(36 - 7 δ), {δ, 0, 1}]
```



If $(a + b) < \frac{2\left(-18-\delta+6\sqrt{81-54\,\delta+8\,\delta^2}\right)}{36-7\,\delta}$ holds, the threshold p_B is on the line segment between the left-hand and right-hand endpoints in (iii). The threshold is given as

$$p_B \to \frac{t\ (9-4\,\delta)\ (b-a)\ \left(3\ (2+a+b)\ \sqrt{\frac{9-2\,\delta}{9-4\,\delta}} - (2+3\,a+3\,b)\right)}{18}\ \text{(pb1)}$$

For the threshold value of p_B (pb1), the point of p_A-element in Firm A's reaction function in (ii) is

$$\text{Simplify}\Big[\text{Expand}\Big[p_A \to \big((9+3\,\delta c-10\,\delta f)\ p_B + (b-a)\ t\ \big((a+b)\ (9-\delta c^2-8\,\delta f+4\,\delta c\,\delta f) + 2\ (3\,\delta c+\delta c^2+4\,\delta f-2\,\delta c\,\delta f)\big)\big)\,/\,(2\ (9+3\,\delta c-5\,\delta f))\ /.\ p_B \to \frac{t\ (9-4\,\delta)\ (b-a)\ \left(3\ (2+a+b)\ \sqrt{\frac{9-2\,\delta}{9-4\,\delta}} - (2+3\,a+3\,b)\right)}{18}\Big]\Big]$$

$$p_A \to \frac{1}{36\ (-9+2\,\delta)}\ (a-b)\ t\ \Bigg(3\,a\left(-27+81\sqrt{\frac{9-2\,\delta}{9-4\,\delta}} + \left(51-99\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)\delta + 2\left(-5+14\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)\delta^2\right) + 3\,b\left(-27+81\sqrt{\frac{9-2\,\delta}{9-4\,\delta}} + \left(51-99\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)\delta + 2\left(-5+14\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)\delta^2\right) + 2\left(81\left(-1+3\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right) - 9\left(-25+33\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)\delta + \left(-46+84\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)\delta^2\right)\Bigg)$$

We simplify the above outcome, and obtain the following:

$$p_A \to \frac{1}{36\ (9-2\,\delta)}\ (b-a)\ t\ \left((9-2\,\delta)\ (-18+46\,\delta+3\ (a+b)\ (-3+5\,\delta)) + 3\ (2+a+b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\ \left(81-99\,\delta+28\,\delta^2\right)\right)$$

We can define the jumping point of Firm A's reaction function in (ii) as c2ja1

$$\text{c2ja1} = \Big\{\frac{1}{36\ (9-2\,\delta)}\ (b-a)\ t\ \left((9-2\,\delta)\ (-18+46\,\delta+3\ (a+b)\ (-3+5\,\delta)) + 3\ (2+a+b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\ \left(81-99\,\delta+28\,\delta^2\right)\right),\ \frac{t\ (9-4\,\delta)\ (b-a)\ \left(3\ (2+a+b)\ \sqrt{\frac{9-2\,\delta}{9-4\,\delta}} - (2+3\,a+3\,b)\right)}{18}\Big\}$$

$$\Big\{\frac{1}{36\ (9-2\,\delta)}\ (-a+b)\ t\ \left(3\ (2+a+b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\ \left(81-99\,\delta+28\,\delta^2\right) + (9-2\,\delta)\ (-18+46\,\delta+3\ (a+b)\ (-3+5\,\delta))\right),\ \frac{1}{18}\ (-a+b)\ t\ \left(-2-3\,a-3\,b+3\ (2+a+b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)\ (9-4\,\delta)\Big\}$$

Also, for the threshold value of p_B (pb1), the point of p_A-element in Firm A's reaction function in (iii) is

$$\text{Simplify}\Big[\text{Expand}\Big[p_A \to \frac{(9 - 3\,\delta c - 2\,\delta f)\,p_B}{2\,(9 - 3\,\delta c - \delta f)} + \frac{(b-a)\,(a+b)\,t\,((3-\delta c)\,(3 - 2\,\delta c) - 2\,(2-\delta c)\,\delta f)}{2\,(9 - 3\,\delta c - \delta f)}\ /.$$

$$p_B \to \frac{t\,(9 - 4\,\delta)\,(b-a)\left(3\,(2+a+b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}} - (2 + 3\,a + 3\,b)\right)}{18}\Big]\Big]$$

$$p_A \to \frac{1}{36}\,(a-b)\,t\left(2\left(-1 + 3\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)(-9 + 5\,\delta) + 3\,a\left(3 - 9\sqrt{\frac{9-2\,\delta}{9-4\,\delta}} + \delta + 5\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\,\delta\right) + 3\,b\left(3 - 9\sqrt{\frac{9-2\,\delta}{9-4\,\delta}} + \delta + 5\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\,\delta\right)\right)$$

We simplify the above outcome, and obtain the following:

$$p_A \to \frac{1}{36}\,(b-a)\,t\left(-2\,(9 - 5\,\delta) - 3\,(a+b)\,(3+\delta) + 3\,(9 - 5\,\delta)\,(2+a+b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)$$

We can define the jumping point of Firm A's reaction function in (iii) as c3ja1

$$\text{c3ja1} = \left\{\frac{1}{36}\,(b-a)\,t\left(-2\,(9 - 5\,\delta) - 3\,(a+b)\,(3+\delta) + 3\,(9 - 5\,\delta)\,(2+a+b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right),\ \frac{t\,(9 - 4\,\delta)\,(b-a)\left(3\,(2+a+b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}} - (2 + 3\,a + 3\,b)\right)}{18}\right\}$$

$$\left\{\frac{1}{36}\,(-a+b)\,t\left(-2\,(9 - 5\,\delta) + 3\,(2+a+b)\,(9 - 5\,\delta)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}} - 3\,(a+b)\,(3+\delta)\right),\ \frac{1}{18}\,(-a+b)\,t\left(-2 - 3\,a - 3\,b + 3\,(2+a+b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)(9 - 4\,\delta)\right\}$$

If $(a+b) \geq \frac{2\left(-18-\delta+6\sqrt{81-54\,\delta+8\,\delta^2}\right)}{36-7\,\delta}$ holds, the threshold p_B is on the half-line starting from the right-hand endpoints in (iii), that is, case (iii)'. The threshold is given as

$$p_B \to \frac{1}{9}\,(b-a)\,t\left(9\,(4 - a - b) - (6 - 5\,a - 5\,b)\,\delta - 2\sqrt{(9 - 2\,\delta)\,(2 - a - b)\,(4 + a + b)\,\delta}\right)\ \text{(pb2)}$$

For the threshold value of p_B (pb2), the point of p_A-element in Firm A's reaction function in (ii) is

$$\text{Simplify}\Big[\text{Expand}\Big[p_A \to \left((9 + 3\,\delta c - 10\,\delta f)\,p_B + (b-a)\,t\left((a+b)\left(9 - \delta c^2 - 8\,\delta f + 4\,\delta c\,\delta f\right) + 2\left(3\,\delta c + \delta c^2 + 4\,\delta f - 2\,\delta c\,\delta f\right)\right)\right) / (2\,(9 + 3\,\delta c - 5\,\delta f))\ /.$$

$$p_B \to \frac{1}{9}\,(b-a)\,t\left(9\,(4 - a - b) - (6 - 5\,a - 5\,b)\,\delta - 2\sqrt{(9 - 2\,\delta)\,(2 - a - b)\,(4 + a + b)\,\delta}\right)\Big]\Big]$$

$$p_A \to -\frac{1}{9\,(-9 + 2\,\delta)}\,(a-b)\,t\left(4\,(-3 + a + b)\,\delta^2 + 9\left(-18 + \sqrt{\left(-8 + a^2 + 2\,b + b^2 + 2\,a\,(1+b)\right)\delta\,(-9 + 2\,\delta)}\right) - \delta\left(-90 + 18\,a + 18\,b + 7\sqrt{\left(-8 + a^2 + 2\,b + b^2 + 2\,a\,(1+b)\right)\delta\,(-9 + 2\,\delta)}\right)\right)$$

We can define the jumping point of Firm A's reaction function in (ii) as c2ja2

```
c2ja2 =
```

$$\left\{-\frac{1}{9(-9+2\delta)}(a-b)\,t\left(4(-3+a+b)\delta^2+9\left(-18+\sqrt{(-8+a^2+2b+b^2+2a(1+b))\,\delta\,(-9+2\delta)}\right)-\delta\left(-90+18a+18b+7\sqrt{(-8+a^2+2b+b^2+2a(1+b))\,\delta\,(-9+2\delta)}\right)\right),\ \frac{1}{9}(b-a)\,t\left(9(4-a-b)-(6-5a-5b)\delta-2\sqrt{(9-2\delta)(2-a-b)(4+a+b)\delta}\right)\right\}$$

$$\left\{-\frac{1}{9(-9+2\delta)}(a-b)\,t\left(4(-3+a+b)\delta^2+9\left(-18+\sqrt{(-8+a^2+2b+b^2+2a(1+b))\,\delta\,(-9+2\delta)}\right)-\delta\left(-90+18a+18b+7\sqrt{(-8+a^2+2b+b^2+2a(1+b))\,\delta\,(-9+2\delta)}\right)\right),\ \frac{1}{9}(-a+b)\,t\left(9(4-a-b)-(6-5a-5b)\delta-2\sqrt{(2-a-b)(4+a+b)(9-2\delta)\delta}\right)\right\}$$

Also, for the threshold value of p_B (pb2), the point of p_A-element in Firm A's reaction function in (iii)' is

```
Simplify[pA → pB + ((b - a) t ((a + b) (3 - 2 δc) - 2 (3 - δc)))/3 /.
  pB → 1/9 (b - a) t (9 (4 - a - b) - (6 - 5 a - 5 b) δ - 2 √((9 - 2 δ) (2 - a - b) (4 + a + b) δ))]
```

$$p_A\to\frac{1}{9}(a-b)\,t\left(-18+a\delta+b\delta+2\sqrt{(-8+a^2+2b+b^2+2a(1+b))\,\delta\,(-9+2\delta)}\right)$$

We can define the jumping point of Firm A's reaction function in (iii) as c3ja1

```
c3ja2 = {1/9 (a - b) t (-18 + a δ + b δ + 2 √((-8 + a^2 + 2 b + b^2 + 2 a (1 + b)) δ (-9 + 2 δ))),
  1/9 (b - a) t (9 (4 - a - b) - (6 - 5 a - 5 b) δ - 2 √((9 - 2 δ) (2 - a - b) (4 + a + b) δ))}
```

$$\left\{\frac{1}{9}(a-b)\,t\left(-18+a\delta+b\delta+2\sqrt{(-8+a^2+2b+b^2+2a(1+b))\,\delta\,(-9+2\delta)}\right),\ \frac{1}{9}(-a+b)\,t\left(9(4-a-b)-(6-5a-5b)\delta-2\sqrt{(2-a-b)(4+a+b)(9-2\delta)\delta}\right)\right\}$$

Next we show various examples of Firm A's true reaction function for different values of a, b, δ, t, and k.

```
a = 0
b = 1
δ = 1/2
t = 1
k = 2
```

0

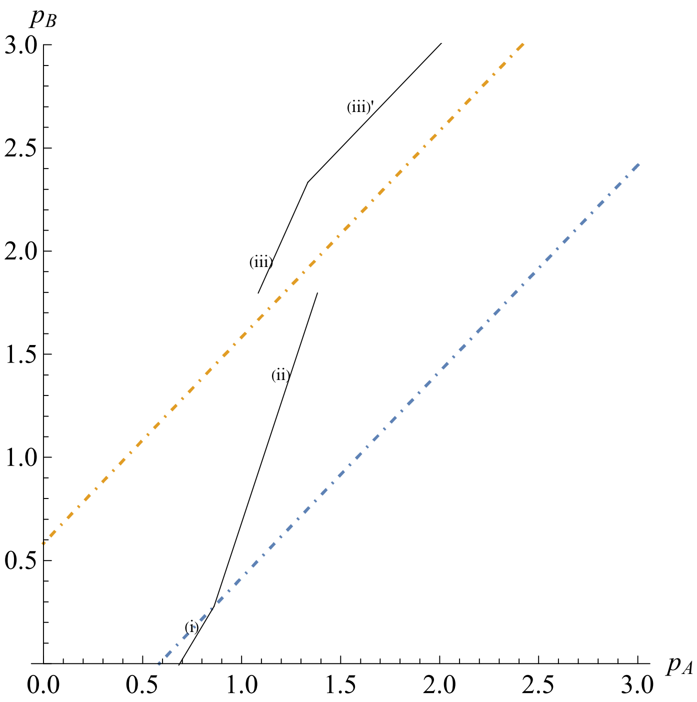
1

$\frac{1}{2}$

1

2

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
   x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3}, Epilog → {Line[{c1aL, c1aR}],
    If[a + b < 2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]) / (36 - 7 δ), Line[{c2aL, c2ja1}], Line[{c2aL, c2ja2}]],
    If[a + b < 2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]) / (36 - 7 δ), Line[{c3ja1, c3aR}], Line[{{0, 0}, {0, 0}}]],
    If[a + b < 2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]) / (36 - 7 δ), Line[{c3daL, c3daR}], Line[{c3ja2, c3daR}]],
    Text["(iii)'", {1.6, 2.7}], Text["(iii)", {1.1, 1.95}],
    Text["(ii)", {1.2, 1.4}], Text["(i)", {0.75, 0.18}]},
  PlotRange → {0, 3}, LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"},
  AspectRatio → 1, PlotStyle → DotDashed]
```




```
a = 0.1376881861101862`
b = 1
δ = 1
t = 1
k = 2
```

0.137688

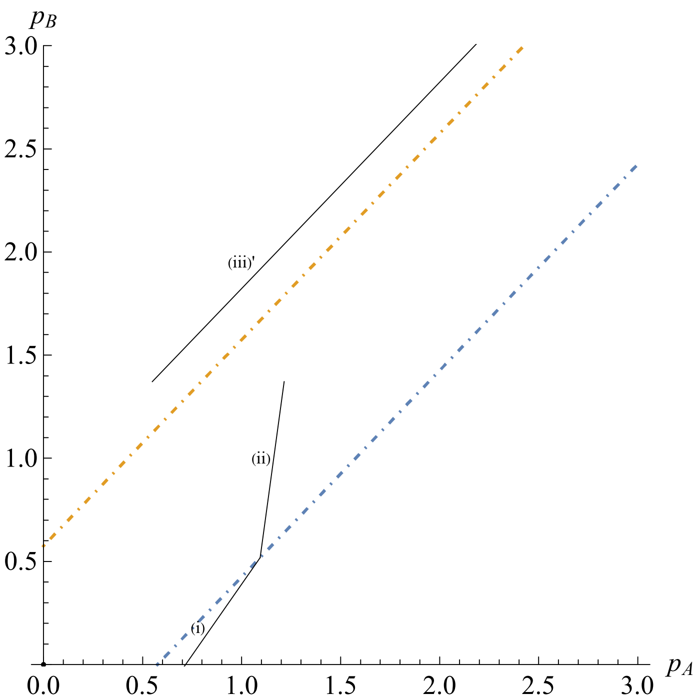
1

1

1

2

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
  x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3}, Epilog → {Line[{c1aL, c1aR}],
   If[a + b < (2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]))/(36 - 7 δ), Line[{c2aL, c2ja1}], Line[{c2aL, c2ja2}]],
   If[a + b < (2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]))/(36 - 7 δ), Line[{c3ja1, c3aR}], Line[{{0, 0}, {0, 0}}]],
   If[a + b < (2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]))/(36 - 7 δ), Line[{c3daL, c3daR}], Line[{c3ja2, c3daR}]],
   Text["(iii)'", {1, 1.95}], Text["(ii)", {1.1, 1}], Text["(i)", {0.78, 0.18}]},
 PlotRange → {0, 3}, LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"},
 AspectRatio → 1, PlotStyle → DotDashed]
```




```
a = 0
b = 1
δ = 1
t = 1
k = 2
```

0

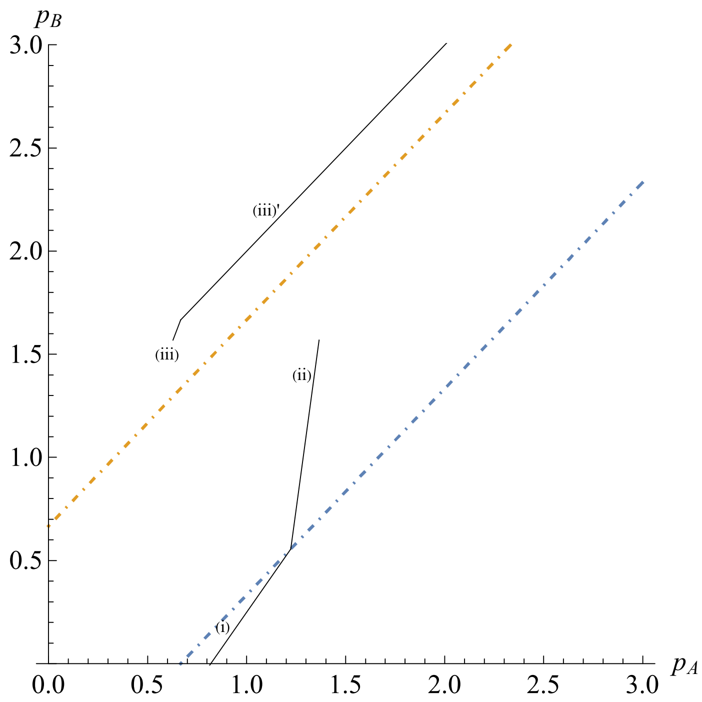
1

1

1

2

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
  x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3}, Epilog → {Line[{c1aL, c1aR}],
   If[a + b < 2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]) / (36 - 7 δ), Line[{c2aL, c2ja1}], Line[{c2aL, c2ja2}]],
   If[a + b < 2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]) / (36 - 7 δ), Line[{c3ja1, c3aR}], Line[{{0, 0}, {0, 0}}]],
   If[a + b < 2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]) / (36 - 7 δ), Line[{c3daL, c3daR}], Line[{c3ja2, c3daR}]],
   Text["(iii)'", {1.1, 2.2}], Text["(iii)", {0.6, 1.5}],
   Text["(ii)", {1.28, 1.4}], Text["(i)", {0.88, 0.18}]},
 PlotRange → {0, 3}, LabelStyle → (FontSize → 14), AxesLabel → {"p_A", "p_B"},
 AspectRatio → 1, PlotStyle → DotDashed]
```




```
Clear[a, b, δ, t, k]
```

We now turn to Firm B's reaction function in the three cases.
As before, we start with an example by setting a = 0, b = 1, t = 1, and $\delta = 1/2$.

```
a = 0
b = 1
δ = 1 / 2
t = 1
k = 1
```

```
0
```

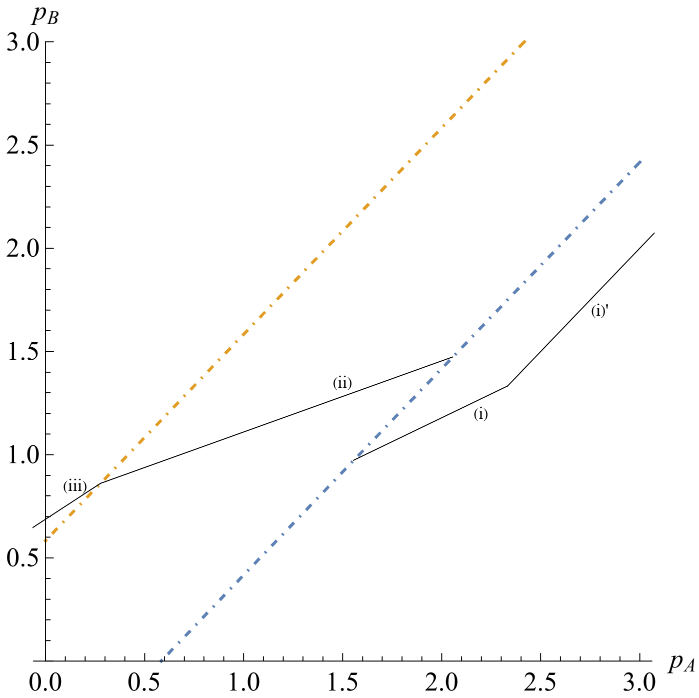
```
1
```

$\frac{1}{2}$

```
1
```

```
1
```

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
   x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3},
  Epilog → {Line[{c1dbL, c1dbR}], Line[{c1bL, c1bR}], Line[{c2bL, c2bR}],
    Line[{c3bL, c3bR}], Text["(iii)", {0.15, 0.85}], Text["(ii)", {1.5, 1.35}],
    Text["(i)", {2.2, 1.2}], Text["(i)'", {2.8, 1.7}]}, PlotRange → {0, 3},
  LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"}, AspectRatio → 1, PlotStyle → DotDashed]
```



We find that for some p_A, there are two local optimal prices for Firm B. We can show that the multiplicity of local optimal prices always appears.

```
Clear[a, b, δ, t, k]
```

To show the multiplicity, we check the locations of the three endpoints: The right-hand endpoint in (ii) (c2bR) and the left-hand and right-hand endpoints in (i) (c1bL and c1bR) (see below)

c2bR
c1bL
c1bR

$$\left\{\frac{1}{9}(a-b)\,t\,(-18+(-4+3a+3b)\,\delta),\ -\frac{1}{18}(a-b)\,t\,(-4\,(-9+\delta)+3a\,(-3+\delta)+3b\,(-3+\delta))\right\}$$

$$\left\{\frac{2}{9}(a-b)\,t\,(-9+4\,\delta),\ -\frac{1}{18}(a-b)\,t\,(36+9a\,(-1+\delta)+9b\,(-1+\delta)-28\,\delta)\right\}$$

$$\left\{\frac{1}{9}(a-b)\,(2+a+b)\,t\,(-9+4\,\delta),\ -\frac{2}{9}(a-b)\,t\,(9+(-7+a+b)\,\delta)\right\}$$

First, we compare the elements of the right-hand endpoint in (ii) and the left-hand endpoint in (i):

$$\text{Factor}\left[\frac{1}{9}(a-b)\,t\,(-18+(-4+3a+3b)\,\delta)-\frac{2}{9}(a-b)\,t\,(-9+4\,\delta)\right]$$

$$\text{Factor}\left[-\frac{1}{18}(a-b)\,t\,(-4\,(-9+\delta)+3a\,(-3+\delta)+3b\,(-3+\delta))-\left(-\frac{1}{18}(a-b)\,t\,(36+9a\,(-1+\delta)+9b\,(-1+\delta)-28\,\delta)\right)\right]$$

$$\frac{1}{3}(a-b)\,(-4+a+b)\,t\,\delta$$

$$\frac{1}{3}(a-b)\,(-4+a+b)\,t\,\delta$$

The outcome means that for any a∈[0,1], b∈[0,1] (a≥b), t, and δ, the right-hand endpoint in (ii) is located above the left-hand endpoint in (i) as in the above Figure.

Second, we compare the p_A-elements of the right-hand endpoint in (ii) and the right-hand end-point in (i):

$$\text{Factor}\left[\frac{1}{9}(a-b)\,t\,(-18+(-4+3a+3b)\,\delta)-\left(\frac{1}{9}(a-b)\,(2+a+b)\,t\,(-9+4\,\delta)\right)\right]$$

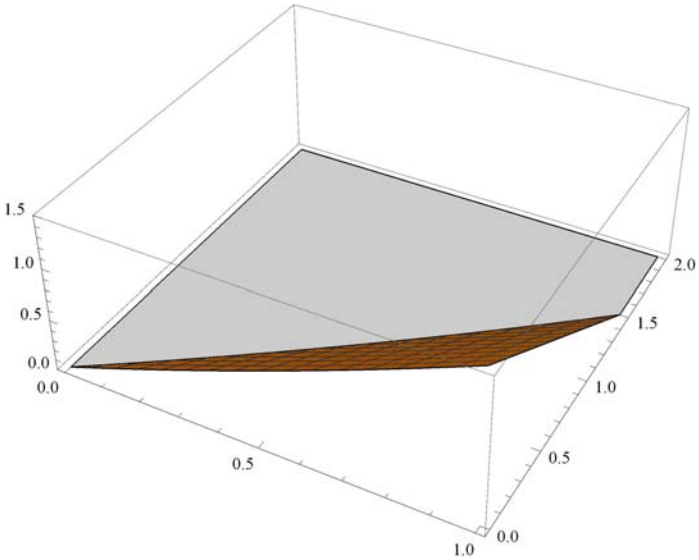
$$-\frac{1}{9}(a-b)\,t\,(-9a-9b+12\,\delta+a\,\delta+b\,\delta)$$

The p_A-element of the right-hand endpoint in (ii) larger than that of the right-hand endpoint in (i) if and only if

$$\frac{12\,\delta}{9-\delta}>a+b$$

If $\frac{12\,\delta}{9-\delta}\le a+b$, we simply compare the reaction function in (ii) and the reaction function in (i); If $\frac{12\,\delta}{9-\delta}>a+b$, in addition to the previous comparison, we also compare the reaction function in (ii) and the reaction function in (i)’.

```
Plot3D[12 δ/(9 - δ) - g, {δ, 0, 1}, {g, 0, 2}, PlotRange → {0, 1.5}]
```



We need to find the global optimal price of Firm B, p_B, when there are two local optima for a given p_A. There is a price p_A such that choosing the reaction function in (i) and choosing the reaction function in (ii) are indifferent for Firm B. This p_A is the threshold in which choosing the reaction function in (i) is preferred by Firm B if p_A is larger than the threshold p_A, otherwise, choosing the reaction function in (ii) is preferred by Firm B. We need to find the threshold value of p_A.

To check the threshold value of p_A for Firm B's reaction function, we derive the profits under cases (i) and (ii).
The interior profit of firm B under case (ii) for p_A is

```
Factor[pB (1 - z) + δf 1/9 (b - a) t (8 - 4 a + a^2 - 4 b + 2 a b + b^2 - 8 z - 2 a z - 2 b z + 10 z^2) /.
   z → 3 (pB - pA)/(2 (b - a) t (3 + δc)) + ((a + b) (3 - δc) + 2 δc)/(2 (3 + δc)) /.
   {pB → 1/(2 (9 + 3 δc - 5 δf)) (-(a - b) t (18 + 6 δc - 8 δf + 4 δc δf +
         a (-9 + δc^2 + 8 δf - 4 δc δf) + b (-9 + δc^2 + 8 δf - 4 δc δf)) + (9 + 3 δc - 10 δf) pA)}]
```

$$\frac{1}{72\,(a-b)\,t\,(-9+2\,\delta)}\Big(324\,a^2 t^2 - 324\,a^3 t^2 + 81\,a^4 t^2 - 648\,a\,b\,t^2 + 324\,a^2 b\,t^2 + 324\,b^2 t^2 + 324\,a\,b^2 t^2 - 162\,a^2 b^2 t^2 - 324\,b^3 t^2 + 81\,b^4 t^2 + 288\,a^2 t^2 \delta - 36\,a^3 t^2 \delta - 18\,a^4 t^2 \delta - 576\,a\,b\,t^2 \delta + 36\,a^2 b\,t^2 \delta + 288\,b^2 t^2 \delta + 36\,a\,b^2 t^2 \delta + 36\,a^2 b^2 t^2 \delta - 36\,b^3 t^2 \delta - 18\,b^4 t^2 \delta - 16\,a^2 t^2 \delta^2 - 24\,a^3 t^2 \delta^2 + 9\,a^4 t^2 \delta^2 + 32\,a\,b\,t^2 \delta^2 + 24\,a^2 b\,t^2 \delta^2 - 16\,b^2 t^2 \delta^2 + 24\,a\,b^2 t^2 \delta^2 - 18\,a^2 b^2 t^2 \delta^2 - 24\,b^3 t^2 \delta^2 + 9\,b^4 t^2 \delta^2 - 324\,a\,t\,p_A + 162\,a^2 t\,p_A + 324\,b\,t\,p_A - 162\,b^2 t\,p_A + 216\,a\,t\,\delta\,p_A - 90\,a^2 t\,\delta\,p_A - 216\,b\,t\,\delta\,p_A + 90\,b^2 t\,\delta\,p_A + 81\,p_A^2\Big)$$

The interior profit of firm B under case (i) for p_B

$$\text{Factor}\left[p_B\,(1-z)+\delta f\left(\frac{1}{18}\,(b-a)\,t\,(-4+a+b+2z)^2\right)\,/.\right.$$
$$z\to\left(-3a^2t+3b^2t-2at\delta c+2a^2t\delta c+2bt\delta c-2b^2t\delta c-3p_A+3p_B\right)/(2\,(a-b)\,t\,(-3+\delta c))\,/.$$
$$\left.p_B\to\left(-\left(-2a+a^2-(-2+b)\,b\right)t\left(9+2\delta c^2-4\delta f+\delta c\,(-9+2\delta f)\right)+(-9+3\delta c+2\delta f)\,p_A\right)/\left(2\,(-9+3\delta c+\delta f)\right)\right]$$

$$-\frac{1}{8\,(a-b)\,t\,(-9+4\delta)}\left(-36a^2t^2+36a^3t^2-9a^4t^2+72abt^2-36a^2bt^2-36b^2t^2-36ab^2t^2+18a^2b^2t^2+36b^3t^2-9b^4t^2+16a^2t^2\delta-16a^3t^2\delta+4a^4t^2\delta-32abt^2\delta+16a^2bt^2\delta+16b^2t^2\delta+16ab^2t^2\delta-8a^2b^2t^2\delta-16b^3t^2\delta+4b^4t^2\delta+36atp_A-18a^2tp_A-36btp_A+18b^2tp_A-16at\delta p_A+8a^2t\delta p_A+16bt\delta p_A-8b^2t\delta p_A-9p_A^2\right)$$

We derive the threshold value of p_B by finding p_B that equalizes the above two profits:

$$\text{FullSimplify}\Big[\text{Solve}\Big[\Big\{\frac{1}{72\,(a-b)\,t\,(-9+2\delta)}\left(324a^2t^2-324a^3t^2+81a^4t^2-648abt^2+324a^2bt^2+324b^2t^2+324ab^2t^2-162a^2b^2t^2-324b^3t^2+81b^4t^2+288a^2t^2\delta-36a^3t^2\delta-18a^4t^2\delta-576abt^2\delta+36a^2bt^2\delta+288b^2t^2\delta+36ab^2t^2\delta+36a^2b^2t^2\delta-36b^3t^2\delta-18b^4t^2\delta-16a^2t^2\delta^2-24a^3t^2\delta^2+9a^4t^2\delta^2+32abt^2\delta^2+24a^2bt^2\delta^2-16b^2t^2\delta^2+24ab^2t^2\delta^2-18a^2b^2t^2\delta^2-24b^3t^2\delta^2+9b^4t^2\delta^2-324atp_A+162a^2tp_A+324btp_A-162b^2tp_A+216at\delta p_A-90a^2t\delta p_A-216bt\delta p_A+90b^2t\delta p_A+81p_A^2\right)==$$
$$-\frac{1}{8\,(a-b)\,t\,(-9+4\delta)}\left(-36a^2t^2+36a^3t^2-9a^4t^2+72abt^2-36a^2bt^2-36b^2t^2-36ab^2t^2+18a^2b^2t^2+36b^3t^2-9b^4t^2+16a^2t^2\delta-16a^3t^2\delta+4a^4t^2\delta-32abt^2\delta+16a^2bt^2\delta+16b^2t^2\delta+16ab^2t^2\delta-8a^2b^2t^2\delta-16b^3t^2\delta+4b^4t^2\delta+36atp_A-18a^2tp_A-36btp_A+18b^2tp_A-16at\delta p_A+8a^2t\delta p_A+16bt\delta p_A-8b^2t\delta p_A-9p_A^2\right)\Big\},\,p_A\Big]\Big]$$

$$\left\{\left\{p_A\to-\frac{1}{18\delta}t\,(-9+4\delta)\left(8a\delta-3a^2\delta-8b\delta+3b^2\delta+3\,(-9+2\delta)\sqrt{\frac{(a-b)^2\,(-4+a+b)^2\,\delta^2}{(-9+2\delta)\,(-9+4\delta)}}\right)\right\},\right.$$
$$\left.\left\{p_A\to\frac{1}{18\delta}t\,(-9+4\delta)\left(-8a\delta+3a^2\delta+8b\delta-3b^2\delta+3\,(-9+2\delta)\sqrt{\frac{(a-b)^2\,(-4+a+b)^2\,\delta^2}{(-9+2\delta)\,(-9+4\delta)}}\right)\right\}\right\}$$

We can easily show that the former outcome is negative. So, we use the latter one.
We simplify the expression of the latter outcome, and obtain the following p_B:

$$p_A\to\frac{t\,(9-4\delta)\,(b-a)\left(3\,(4-a-b)\sqrt{\frac{9-2\delta}{9-4\delta}}-(8-3a-3b)\right)}{18}\quad\text{(pa1)}$$

We rewrite the locations of the two endpoints: The right-hand endpoint in (ii) and the right-hand endpoint in (i) (see below)

```
c2bR
c1bR
```

$$\left\{\frac{1}{9}\,(a-b)\,t\,(-18+(-4+3a+3b)\,\delta),\,-\frac{1}{18}\,(a-b)\,t\,(-4\,(-9+\delta)+3a\,(-3+\delta)+3b\,(-3+\delta))\right\}$$

$$\left\{\frac{1}{9}\,(a-b)\,(2+a+b)\,t\,(-9+4\delta),\,-\frac{2}{9}\,(a-b)\,t\,(9+(-7+a+b)\,\delta)\right\}$$

We check the condition that the derived p_A (pa1) is smaller than the p_A-element of the right-hand endpoints in (i).

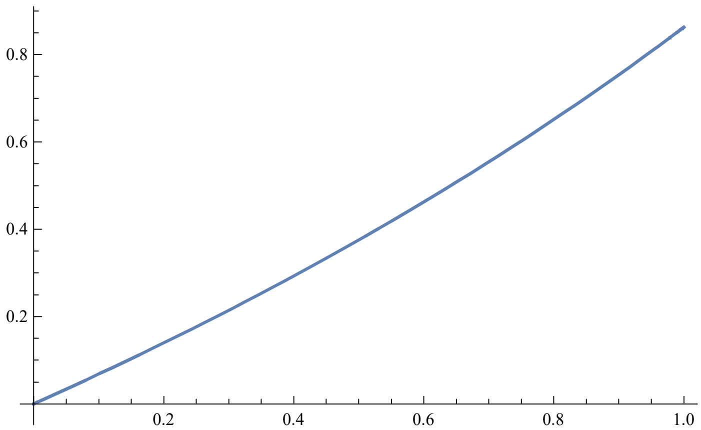
```
Simplify[
 Factor[1/9 (a - b) (2 + a + b) t (-9 + 4 δ) - (t (9 - 4 δ) (b - a) (3 (4 - a - b) Sqrt[(9-2 δ)/(9-4 δ)] - (8 - 3 a - 3 b)))/18]]
```

$$\frac{1}{18}(a-b)\,t\left(12-b+a\left(-1+3\sqrt{\frac{9-2\delta}{9-4\delta}}\right)-12\sqrt{\frac{9-2\delta}{9-4\delta}}+3\,b\sqrt{\frac{9-2\delta}{9-4\delta}}\right)(-9+4\delta)$$

This is positive if the following inequality holds

$$(a+b)\left(-1+3\sqrt{\frac{9-2\delta}{9-4\delta}}\right)>12\left(\sqrt{\frac{9-2\delta}{9-4\delta}}-1\right)$$

```
Plot[12 (Sqrt[(9 - 2 δ)/(9 - 4 δ)] - 1)/(-1 + 3 Sqrt[(9 - 2 δ)/(9 - 4 δ)]), {δ, 0, 1}]
```



If $(a+b)\left(-1+3\sqrt{\frac{9-2\delta}{9-4\delta}}\right)>12\left(\sqrt{\frac{9-2\delta}{9-4\delta}}-1\right)$ holds, the derived p_A (pa1) is on the interval between the left-hand and right-hand endpoints in (i). Otherwise, the reaction function in (ii) is always better than the interval between the left-hand and right-hand endpoints in (i).

Now we derive the profit of firm B under case (i)’ for p_A

```
Factor[pB (1 - z) + δf (1/18 (b - a) t (-4 + a + b + 2 z)^2) /.
  z → (-3 a^2 t + 3 b^2 t - 2 a t δc + 2 a^2 t δc + 2 b t δc - 2 b^2 t δc - 3 pA + 3 pB) / (2 (a - b) t (-3 + δc)) /.
 pB → pA - ((b - a) t ((a + b) (3 - 2 δc) + 2 δc))/3]
```

$$\frac{1}{18}\left(18\,a^2\,t-18\,b^2\,t-4\,a\,t\,\delta-4\,a^2\,t\,\delta-a^3\,t\,\delta+4\,b\,t\,\delta-a^2\,b\,t\,\delta+4\,b^2\,t\,\delta+a\,b^2\,t\,\delta+b^3\,t\,\delta+18\,p_A\right)$$

We derive the threshold value of p_A by finding p_A that equalizes the two profits in cases (ii) and (i)’:

$$\text{FullSimplify}\Big[\text{Solve}\Big[\Big\{\frac{1}{72\,(a-b)\,t\,(-9+2\,\delta)}\,\big(324\,a^2\,t^2 - 324\,a^3\,t^2 + 81\,a^4\,t^2 - 648\,a\,b\,t^2 + 324\,a^2\,b\,t^2 + 324\,b^2\,t^2 + 324\,a\,b^2\,t^2 - 162\,a^2\,b^2\,t^2 - 324\,b^3\,t^2 + 81\,b^4\,t^2 + 288\,a^2\,t^2\,\delta - 36\,a^3\,t^2\,\delta - 18\,a^4\,t^2\,\delta - 576\,a\,b\,t^2\,\delta + 36\,a^2\,b\,t^2\,\delta + 288\,b^2\,t^2\,\delta + 36\,a\,b^2\,t^2\,\delta + 36\,a^2\,b^2\,t^2\,\delta - 36\,b^3\,t^2\,\delta - 18\,b^4\,t^2\,\delta - 16\,a^2\,t^2\,\delta^2 - 24\,a^3\,t^2\,\delta^2 + 9\,a^4\,t^2\,\delta^2 + 32\,a\,b\,t^2\,\delta^2 + 24\,a^2\,b\,t^2\,\delta^2 - 16\,b^2\,t^2\,\delta^2 + 24\,a\,b^2\,t^2\,\delta^2 - 18\,a^2\,b^2\,t^2\,\delta^2 - 24\,b^3\,t^2\,\delta^2 + 9\,b^4\,t^2\,\delta^2 - 324\,a\,t\,p_A + 162\,a^2\,t\,p_A + 324\,b\,t\,p_A - 162\,b^2\,t\,p_A + 216\,a\,t\,\delta\,p_A - 90\,a^2\,t\,\delta\,p_A - 216\,b\,t\,\delta\,p_A + 90\,b^2\,t\,\delta\,p_A + 81\,p_A^2\big) == \frac{1}{18}\,\big(18\,a^2\,t - 18\,b^2\,t - 4\,a\,t\,\delta - 4\,a^2\,t\,\delta - a^3\,t\,\delta + 4\,b\,t\,\delta - a^2\,b\,t\,\delta + 4\,b^2\,t\,\delta + a\,b^2\,t\,\delta + b^3\,t\,\delta + 18\,p_A\big)\Big\},\ p_A\Big]\Big]$$

$$\Big\{\Big\{p_A \to \frac{1}{9}\,(a-b)\,t\left(-9\,a - 9\,(2+b) - 4\,\delta + 5\,(a+b)\,\delta + 18\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\,\delta}} - 4\,\delta\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\,\delta}}\right)\Big\},$$
$$\Big\{p_A \to \frac{1}{9}\,(a-b)\,t\left(-9\,(2+a+b) + (-4+5\,a+5\,b)\,\delta - 18\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\,\delta}} + 4\,\delta\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\,\delta}}\right)\Big\}\Big\}$$

We pick up the first outcome as the threshold p_A.

$$p_A \to \frac{1}{9}\,(a-b)\,t\left(-9\,a - 9\,(2+b) - 4\,\delta + 5\,(a+b)\,\delta + 18\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\,\delta}} - 4\,\delta\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\,\delta}}\right) \quad \text{(pa2)}$$

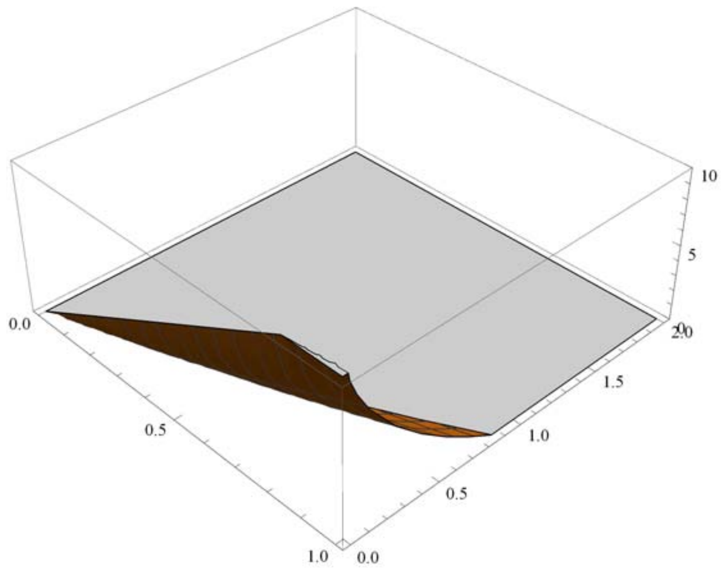
We check the condition that the derived p_A (pa2) is above the p_A-element of the right-hand endpoints in (i).

$$\text{Simplify}\Big[\text{Factor}\Big[\frac{1}{9}\,(a-b)\,t\left(-9\,a - 9\,(2+b) - 4\,\delta + 5\,(a+b)\,\delta + 18\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\,\delta}} - 4\,\delta\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\,\delta}}\right) - \frac{1}{9}\,(a-b)\,(2+a+b)\,t\,(-9+4\,\delta)\Big]\Big]$$

$$\frac{1}{9}\,(a-b)\,t\left(-12\,\delta + a\,\delta + b\,\delta + 18\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\,\delta}} - 4\,\delta\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\,\delta}}\right)$$

By setting a+b by g, we check the value between the large parentheses:

$$\texttt{Plot3D}\left[-\left(-12\,\delta+g\,\delta+18\sqrt{\frac{(-6+g)\,(g)\,\delta}{-9+2\,\delta}}-4\,\delta\sqrt{\frac{(-6+g)\,(g)\,\delta}{-9+2\,\delta}}\right),\ \{\delta,0,1\},\{g,0,2\},\ \texttt{PlotRange}\to\{0,10\}\right]$$



We check the condition that the value between the large parentheses is negative:

$$\texttt{Solve}\left[\left(-12\,\delta+g\,\delta+18\sqrt{\frac{(-6+g)\,(g)\,\delta}{-9+2\,\delta}}-4\,\delta\sqrt{\frac{(-6+g)\,(g)\,\delta}{-9+2\,\delta}}\right)==0,\ g\right]$$

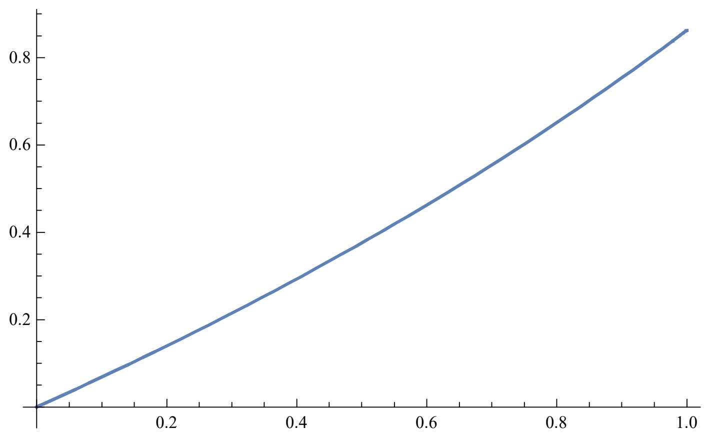
$$\left\{\left\{g\to\frac{12\left(-9+\delta-\sqrt{81-54\,\delta+8\,\delta^2}\right)}{-36+7\,\delta}\right\},\ \left\{g\to\frac{12\left(-9+\delta+\sqrt{81-54\,\delta+8\,\delta^2}\right)}{-36+7\,\delta}\right\}\right\}$$

If $g=a+b<\dfrac{12\left(-9+\delta+\sqrt{81-54\,\delta+8\,\delta^2}\right)}{-36+7\,\delta}$,

the derived p_A (pa2) is above the p_A-element of the right-hand endpoints in (i).

This coincides with the following condition that the derived p_A (pa1) is above the p_A-element of the right-hand endpoints in (i).

$$\texttt{Plot}\left[\frac{12\left(-9+\delta+\sqrt{81-54\,\delta+8\,\delta^2}\right)}{-36+7\,\delta},\ \{\delta,0,1\}\right]$$



If $(a+b)>\dfrac{12\left(-9+\delta+\sqrt{81-54\,\delta+8\,\delta^2}\right)}{-36+7\,\delta}$ holds, the threshold p_B is on the line segment between the left-hand and right-hand endpoints in (i). The threshold is given as

$$p_A \to \frac{t\,(9-4\,\delta)\,(b-a)\left(3\,(4-a-b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}-(8-3\,a-3\,b)\right)}{18}$$

The reaction function of Firm B in (i) is

$$p_B \to \frac{1}{2\,(9-3\,\delta c-\delta f)}\left(((b-a)\,(2-a-b))\,t\left(9+2\,\delta c^2-4\,\delta f+\delta c\,(-9+2\,\delta f)\right)+(9-3\,\delta c-2\,\delta f)\,p_A\right)$$

We substitute p_A into p_B:

```
Simplify[pB →
  (((b - a) (2 - a - b)) t (9 + 2 δc^2 - 4 δf + δc (-9 + 2 δf)) + (9 - 3 δc - 2 δf) pA) / (2 (9 - 3 δc - δf)) /.
   pA → t (9 - 4 δ) (b - a) (3 (4 - a - b) Sqrt[(9-2 δ)/(9-4 δ)] - (8 - 3 a - 3 b)) / 18]
```

$$p_B \to \frac{1}{36\,(9-4\,\delta)}(-a+b)\,t\left(\left(-8+3\,a+3\,b-3\,(-4+a+b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)(9-5\,\delta)\,(9-4\,\delta)-18\,(-2+a+b)\left(9-13\,\delta+4\,\delta^2\right)\right)$$

We can define the jumping point of Firm B's reaction function in (i) as c1jb1

```
c1jb1 = {t (9 - 4 δ) (b - a) (3 (4 - a - b) Sqrt[(9-2 δ)/(9-4 δ)] - (8 - 3 a - 3 b)) / 18,
  1/(36 (9 - 4 δ)) (-a + b) t
   ((-8 + 3 a + 3 b - 3 (-4 + a + b) Sqrt[(9 - 2 δ)/(9 - 4 δ)]) (9 - 5 δ) (9 - 4 δ) - 18 (-2 + a + b) (9 - 13 δ + 4 δ^2))}
```

$$\left\{\frac{1}{18}(-a+b)\,t\left(-8+3\,a+3\,b+3\,(4-a-b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)(9-4\,\delta),\ \frac{1}{36\,(9-4\,\delta)}(-a+b)\,t\left(\left(-8+3\,a+3\,b-3\,(-4+a+b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)(9-5\,\delta)\,(9-4\,\delta)-18\,(-2+a+b)\left(9-13\,\delta+4\,\delta^2\right)\right)\right\}$$

The reaction function of Firm B in (ii) is

$$p_B \to \frac{1}{2\,(9+3\,\delta c-5\,\delta f)}\left(-(a-b)\,t\left(18+6\,\delta c-8\,\delta f+4\,\delta c\,\delta f+a\left(-9+\delta c^2+8\,\delta f-4\,\delta c\,\delta f\right)+b\left(-9+\delta c^2+8\,\delta f-4\,\delta c\,\delta f\right)\right)+(9+3\,\delta c-10\,\delta f)\,p_A\right)$$

We substitute p_A into p_B:

```
Simplify[pB → 1/(2 (9 + 3 δc - 5 δf))
   (-(a - b) t (18 + 6 δc - 8 δf + 4 δc δf + a (-9 + δc^2 + 8 δf - 4 δc δf) + b (-9 + δc^2 + 8 δf - 4 δc δf)) +
    (9 + 3 δc - 10 δf) pA) /. pA → t (9 - 4 δ) (b - a) (3 (4 - a - b) Sqrt[(9-2 δ)/(9-4 δ)] - (8 - 3 a - 3 b)) / 18]
```

$$p_B \to \frac{1}{126}(-a+b)\,t\left(-36\,(-5+a+b)+\left(-8+3\,a+3\,b-3\,(-4+a+b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)(9-4\,\delta)\right)$$

We can define the jumping point of Firm B's reaction function in (ii) as c2jb1

$$c2jb1 = \left\{\frac{t\,(9-4\delta)\,(b-a)\left(3\,(4-a-b)\sqrt{\frac{9-2\delta}{9-4\delta}}-(8-3a-3b)\right)}{18},\right.$$
$$\left.\frac{1}{126}(-a+b)\,t\left(-36\,(-5+a+b)+\left(-8+3a+3b-3\,(-4+a+b)\sqrt{\frac{9-2\delta}{9-4\delta}}\right)(9-4\delta)\right)\right\}$$

$$\left\{\frac{1}{18}(-a+b)\,t\left(-8+3a+3b+3\,(4-a-b)\sqrt{\frac{9-2\delta}{9-4\delta}}\right)(9-4\delta),\right.$$
$$\left.\frac{1}{126}(-a+b)\,t\left(-36\,(-5+a+b)+\left(-8+3a+3b-3\,(-4+a+b)\sqrt{\frac{9-2\delta}{9-4\delta}}\right)(9-4\delta)\right)\right\}$$

If $(a+b) \le \frac{12\left(-9+\delta+\sqrt{81-54\delta+8\delta^2}\right)}{-36+7\delta}$ holds, the threshold p_A is on the half-line starting from the right-hand endpoints in (i), that is, case (i)'. The threshold is given as

$$p_A \to \frac{1}{9}(a-b)\,t\left(-9a-9\,(2+b)-4\delta+5\,(a+b)\,\delta+18\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\delta}}-4\delta\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\delta}}\right)$$

The reaction functions of Firm B in (i)' and (ii) are

$$p_B \to p_A - \frac{(b-a)\,t\,((a+b)\,(3-2\delta c)+2\delta c)}{3}$$

$$p_B \to \left((b-a)\,t\left(2\,(9+3\delta c-4\delta f+2\delta c\,\delta f)+(a+b)\left(-9+\delta c^2+8\delta f-4\delta c\,\delta f\right)\right)+(9+3\delta c-10\delta f)\,p_A\right)\Big/\left(2\,(9+3\delta c-5\delta f)\right)$$

We substitute p_A into p_B:

```
Simplify[pB → pA - ((b - a) t ((a + b) (3 - 2 δc) + 2 δc))/3 /. pA → 1/9 (a - b) t
   (-9 a - 9 (2 + b) - 4 δ + 5 (a + b) δ + 18 Sqrt[((-6 + a + b) (a + b) δ)/(-9 + 2 δ)] - 4 δ Sqrt[((-6 + a + b) (a + b) δ)/(-9 + 2 δ)])]
Simplify[pB → ((b - a) t (2 (9 + 3 δc - 4 δf + 2 δc δf) + (a + b) (-9 + δc^2 + 8 δf - 4 δc δf)) +
      (9 + 3 δc - 10 δf) pA) / (2 (9 + 3 δc - 5 δf)) /. pA → 1/9 (a - b) t
   (-9 a - 9 (2 + b) - 4 δ + 5 (a + b) δ + 18 Sqrt[((-6 + a + b) (a + b) δ)/(-9 + 2 δ)] - 4 δ Sqrt[((-6 + a + b) (a + b) δ)/(-9 + 2 δ)])]
```

$$p_B \to -\frac{1}{9}(a-b)\,t\left(18-2\delta+a\delta+b\delta-18\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\delta}}+4\delta\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\delta}}\right)$$

$$p_B \to \frac{1}{9}(a-b)\,t\left(-18+2\delta+2a\delta+2b\delta+9\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\delta}}-7\delta\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\delta}}\right)$$

We can define the jumping point of Firm B's reaction function in (i) as c1jb2

$$\mathbf{c1jb2} = \left\{\frac{1}{9}(a-b)\,t\left(-9a-9(2+b)-4\delta+5(a+b)\delta+18\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}-4\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right),\ -\frac{1}{9}(a-b)\,t\left(18-2\delta+a\delta+b\delta-18\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}+4\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right)\right\}$$

$$\left\{\frac{1}{9}(a-b)\,t\left(-9a-9(2+b)-4\delta+5(a+b)\delta+18\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}-4\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right),\ -\frac{1}{9}(a-b)\,t\left(18-2\delta+a\delta+b\delta-18\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}+4\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right)\right\}$$

Also, we can define the jumping point of Firm B's reaction function in (ii) as c2jb2

$$\mathbf{c2jb2} = \left\{\frac{1}{9}(a-b)\,t\left(-9a-9(2+b)-4\delta+5(a+b)\delta+18\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}-4\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right),\ \frac{1}{9}(a-b)\,t\left(-18+2\delta+2a\delta+2b\delta+9\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}-7\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right)\right\}$$

$$\left\{\frac{1}{9}(a-b)\,t\left(-9a-9(2+b)-4\delta+5(a+b)\delta+18\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}-4\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right),\ \frac{1}{9}(a-b)\,t\left(-18+2\delta+2a\delta+2b\delta+9\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}-7\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right)\right\}$$

As before, we now show various examples of Firm B's true reaction function for different values of a, b, δ, t, and k.

```
a = 0
b = 1
δ = 1 / 2
t = 1
k = 2
```

0

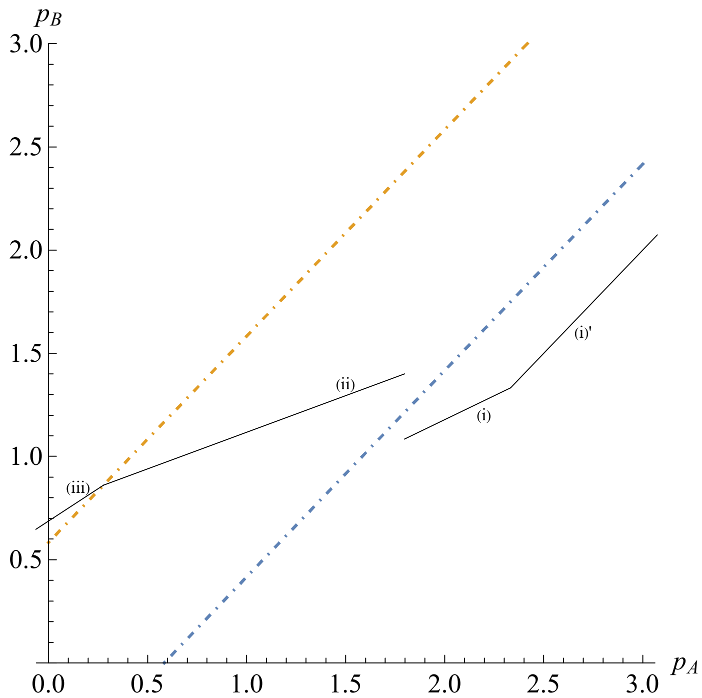
1

$\frac{1}{2}$

1

2

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
   x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3},
  Epilog → {If[a + b > (12 (-9 + δ + Sqrt[81 - 54 δ + 8 δ^2]))/(-36 + 7 δ), Line[{c1dbL, c1dbR}], Line[{c1jb2, c1dbR}]],
    If[a + b > (12 (-9 + δ + Sqrt[81 - 54 δ + 8 δ^2]))/(-36 + 7 δ), Line[{c1jb1, c1bR}], Line[{{0, 0}, {0, 0}}] ],
    If[a + b > (12 (-9 + δ + Sqrt[81 - 54 δ + 8 δ^2]))/(-36 + 7 δ), Line[{c2bL, c2jb1}], Line[{c2bL, c2jb2}]],
    Line[{c3bL, c3bR}], Text["(iii)", {0.15, 0.85}], Text["(ii)", {1.5, 1.35}],
    Text["(i)", {2.2, 1.2}], Text["(i)'", {2.7, 1.6}]},
  PlotRange → {0, 3}, LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"},
  AspectRatio → 1, PlotStyle → DotDashed]
```




```
a = 0
b = 1 - 0.1376881861101862`
δ = 1
t = 1
k = 2
```

0

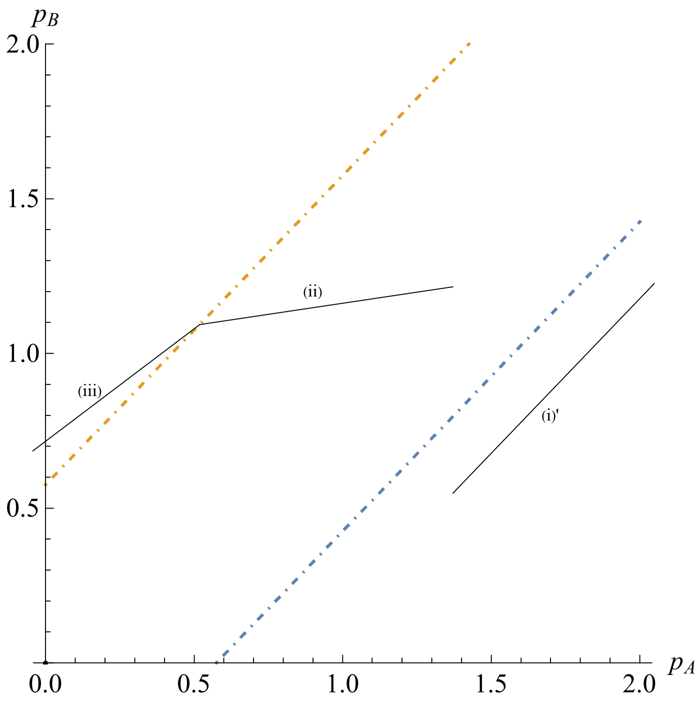
0.862312

1

1

2

```
Plot[{x - (-3 a² t + 3 b² t - 4 a t δc + 3 a² t δc + 4 b t δc - 3 b² t δc) / 6,
  x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 2},
 Epilog → {If[a + b > 12 (-9 + δ + Sqrt[81 - 54 δ + 8 δ²]) / (-36 + 7 δ), Line[{c1dbL, c1dbR}], Line[{c1jb2, c1dbR}]],
   If[a + b > 12 (-9 + δ + Sqrt[81 - 54 δ + 8 δ²]) / (-36 + 7 δ), Line[{c1jb1, c1bR}], Line[{{0, 0}, {0, 0}}] ],
   If[a + b > 12 (-9 + δ + Sqrt[81 - 54 δ + 8 δ²]) / (-36 + 7 δ), Line[{c2bL, c2jb1}], Line[{c2bL, c2jb2}]],
   Line[{c3bL, c3bR}], Text["(iii)", {0.15, 0.88}], Text["(ii)", {0.9, 1.2}],
   Text["(i)'", {1.7, 0.8}]}, PlotRange → {0, 2}, LabelStyle → (FontSize → 14),
 AxesLabel → {"pA", "pB"}, AspectRatio → 1, PlotStyle → DotDashed]
```

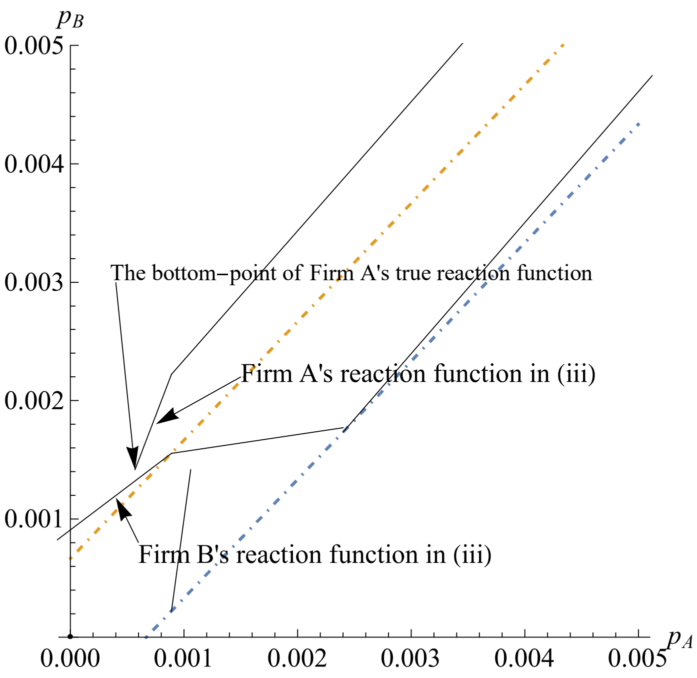


We now show an example in which there is no intersection between the two firms' reaction functions.

```
a = 0
b = 0.001
δ = 1
t = 1
k = 2
```

```
0

0.001

1

1

2
```

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
  x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 0.005},
 Epilog → {If[a + b > 12 (-9 + δ + √(81 - 54 δ + 8 δ^2)) / (-36 + 7 δ), Line[{c1dbL, c1dbR}], Line[{c1jb2, c1dbR}]],
   If[a + b > 12 (-9 + δ + √(81 - 54 δ + 8 δ^2)) / (-36 + 7 δ), Line[{c1jb1, c1bR}], Line[{{0, 0}, {0, 0}}] ],
   If[a + b > 12 (-9 + δ + √(81 - 54 δ + 8 δ^2)) / (-36 + 7 δ), Line[{c2bL, c2jb1}], Line[{c2bL, c2jb2}]],
   Line[{c3bL, c3bR}], Text["(iii)", {0.15, 0.85}], Text["(ii)", {1.5, 1.35}],
   Text["(i)", {2.2, 1.2}], Text["(i)'", {2.7, 1.6}], , , , Line[{c1aL, c1aR}],
   If[ a + b < 2 (-18 - δ + 6 √(81 - 54 δ + 8 δ^2)) / (36 - 7 δ), Line[{c2aL, c2ja1}], Line[{c2aL, c2ja2}]],
   If[a + b < 2 (-18 - δ + 6 √(81 - 54 δ + 8 δ^2)) / (36 - 7 δ), Line[{c3ja1, c3aR}], Line[{{0, 0}, {0, 0}}]],
   If[a + b < 2 (-18 - δ + 6 √(81 - 54 δ + 8 δ^2)) / (36 - 7 δ), Line[{c3daL, c3daR}], Line[{c3ja2, c3daR}]],
   Text[Style["Firm A's reaction function in (iii)", FontSize → 14],
    {0.0015, 0.0021}, {-1, -1}], Arrow[{{0.0004, 0.003}, c3ja1}],
   Arrow[{{0.0015, 0.0022}, {0.00075, 0.0018}}], Arrow[{{0.0006, 0.0008}, {0.0004, 0.00118}}],
   Text[Style["The bottom-point of Firm A's true reaction function", FontSize → 12],
    {0.00035, 0.00308}, {-1, 0}],
   Text[Style["Firm B's reaction function in (iii)", FontSize → 14], {0.0006, 0.00083},
    {-1, 1}], Text["(ii)", {1.2, 1.4}], Text["(i)", {0.75, 0.18}]},
 PlotRange → {0, 0.005}, LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"},
 AspectRatio → 1, PlotStyle → DotDashed]
```



We now turn to the pricing equilibrium in the first period. First, we show that no equilibrium exists in (Case iii).
To this end, we show that the lowest point of Firm A's reaction function in (iii) is above Firm B's reaction function in (iii), as shown in the previous figure. If this property holds, Firm A's reaction function in (iii) never passes through the reaction function of Firm B in (iii) because the slope of Firm A's reaction function in (iii) is steeper than that of Firm B in (iii).

```
Clear[a, b, δ, t, k]
δf = δ
δc = δ
```

δ

δ

The lowest point of Firm A's "true" reaction function in (iii) is point "c3ja1" if $\text{if } a + b < \frac{2\left(-18-\delta+6\sqrt{81-54\,\delta+8\,\delta^2}\right)}{36-7\,\delta}$, otherwise, point "c3ja2". We show the two points in the following:

```
c3ja1
c3ja2
```

$$\left\{\frac{1}{36}\,(-a+b)\,t\left(-2\,(9-5\,\delta)+3\,(2+a+b)\,(9-5\,\delta)\,\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}-3\,(a+b)\,(3+\delta)\right),\right.$$
$$\left.\frac{1}{18}\,(-a+b)\,t\left(-2-3\,a-3\,b+3\,(2+a+b)\,\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)(9-4\,\delta)\right\}$$

$$\left\{\frac{1}{9}\,(a-b)\,t\left(-18+a\,\delta+b\,\delta+2\,\sqrt{\left(-8+a^2+2\,b+b^2+2\,a\,(1+b)\right)\delta\,(-9+2\,\delta)}\right),\right.$$
$$\left.\frac{1}{9}\,(-a+b)\,t\left(9\,(4-a-b)-(6-5\,a-5\,b)\,\delta-2\,\sqrt{(2-a-b)\,(4+a+b)\,(9-2\,\delta)\,\delta}\right)\right\}$$

The following is the reaction function of Firm B in (iii).

$$pb\to\frac{(9-3\,\delta c+10\,\delta f)\,pa}{2\,(9-3\,\delta c+5\,\delta f)}+$$
$$((b-a)\,t\,(2\,(3-\delta c)\,(3-\delta c+3\,\delta f)-(a+b)\,((3-\delta c)\,(3-2\,\delta c)+5\,(1-\delta c)\,\delta f)))\,/$$
$$(2\,(9-3\,\delta c+5\,\delta f))$$

We substitute p_A-element of point "c3ja1" into the reaction function of Firm B in (iii).

$$\text{Simplify}\left[pb\to\frac{(9-3\,\delta c+10\,\delta f)\,pa}{2\,(9-3\,\delta c+5\,\delta f)}+\right.$$
$$((b-a)\,t\,(2\,(3-\delta c)\,(3-\delta c+3\,\delta f)-(a+b)\,((3-\delta c)\,(3-2\,\delta c)+5\,(1-\delta c)\,\delta f)))\,/$$
$$(2\,(9-3\,\delta c+5\,\delta f))\;/.\;pa\to\frac{1}{36}\,(-a+b)\,t$$
$$\left.\left(-2\,(9-5\,\delta)+3\,(2+a+b)\,(9-5\,\delta)\,\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}-3\,(a+b)\,(3+\delta)\right)\;/.\;\{\delta f\to\delta,\ \delta c\to\delta\}\right]$$

$$pb\to\frac{1}{72\,(9+2\,\delta)}\,(-a+b)\,t\left((9+7\,\delta)\left(3\,(2+a+b)\,(9-5\,\delta)\,\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}-3\,(a+b)\,(3+\delta)+2\,(-9+5\,\delta)\right)+\right.$$
$$\left.36\left(18+6\,\delta-4\,\delta^2+a\left(-9+4\,\delta+3\,\delta^2\right)+b\left(-9+4\,\delta+3\,\delta^2\right)\right)\right)$$

We check if this derived value of p_B is smaller than p_B-element of point "c3ja1". If the derived value of p_B is actually smaller than p_B-element of point "c3ja1", Firm A's "true" reaction function in (iii) does not pass through the reaction function of Firm B in (iii).
The difference between the derived value of p_B and the p_B-element of c3ja1:

```
Simplify[Factor[
  1/(72 (9 + 2 δ)) (-a + b) t ((9 + 7 δ) (3 (2 + a + b) (9 - 5 δ) Sqrt[(9 - 2 δ)/(9 - 4 δ)] - 3 (a + b) (3 + δ) + 2 (-9 + 5 δ)) +
      36 (18 + 6 δ - 4 δ^2 + a (-9 + 4 δ + 3 δ^2) + b (-9 + 4 δ + 3 δ^2))) -
   1/18 (-a + b) t (-2 - 3 a - 3 b + 3 (2 + a + b) Sqrt[(9 - 2 δ)/(9 - 4 δ)]) (9 - 4 δ)]]
```

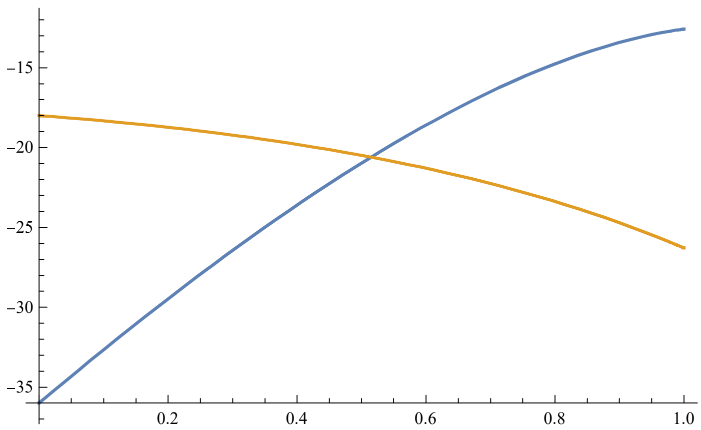
$$\frac{1}{24\,(9+2\delta)}(a-b)\,t\,(-3+\delta)\left(3a\left(21-27\sqrt{\frac{9-2\delta}{9-4\delta}}+\delta+\sqrt{\frac{9-2\delta}{9-4\delta}}\,\delta\right)+3b\left(21+\delta-27\sqrt{\frac{-9+2\delta}{-9+4\delta}}+\delta\sqrt{\frac{-9+2\delta}{-9+4\delta}}\right)+2\left(63+23\delta-81\sqrt{\frac{-9+2\delta}{-9+4\delta}}+3\delta\sqrt{\frac{-9+2\delta}{-9+4\delta}}\right)\right)$$

We rearrange this value, and obtain the following:

$$\frac{1}{24\,(9+2\delta)}(b-a)\,t\,(3-\delta)\left(\left((2\,(63+23\delta))-6\,(27-\delta)\sqrt{\frac{9-2\delta}{9-4\delta}}\right)-3\,(a+b)\left((27-\delta)\sqrt{\frac{9-2\delta}{9-4\delta}}-(21+\delta)\right)\right)$$

We check the values of the first and the second terms in the largest parentheses

```
Plot[{(2 (63 + 23 δ)) - 6 (27 - δ) Sqrt[(9 - 2 δ)/(9 - 4 δ)], -3 ((27 - δ) Sqrt[(9 - 2 δ)/(9 - 4 δ)] - (21 + δ))}, {δ, 0, 1}]
```



Both values are negative. Therefore, the difference between the derived value of p_B and the p_B-element of c3ja1 is negative. That is, Firm A's "true" reaction function in (iii) does not pass through the reaction function of Firm B in (iii) if $a + b < \frac{2\left(-18-\delta+6\sqrt{81-54\delta+8\delta^2}\right)}{36-7\delta}$.

We substitute p_A-element of c3ja2 into the reaction function of Firm B in (iii).

```
Simplify[
 pb → 1/9 (a - b) t (-18 + a δ + b δ + 2 Sqrt[(-8 + a^2 + 2 b + b^2 + 2 a (1 + b)) δ (-9 + 2 δ)]) /. pa → 1/36 (-a + b)
     t (-2 (9 - 5 δ) + 3 (2 + a + b) (9 - 5 δ) Sqrt[(9 - 2 δ)/(9 - 4 δ)] - 3 (a + b) (3 + δ)) /. {δf → δ, δc → δ}]
```

$$pb \to \frac{1}{9}(a-b)\,t\left(-18+a\,\delta+b\,\delta+2\sqrt{\left(-8+a^2+2\,b+b^2+2\,a\,(1+b)\right)\delta\,(-9+2\,\delta)}\right)$$

We check if this derived value of p_B is smaller than p_B-element of point "c3ja2". If the derived value of p_B is actually smaller than p_B-element of point "c3ja2", Firm A's "true" reaction function in (iii) does not pass through the reaction function of Firm B in (iii).
The difference between the derived value of p_B and the p_B-element of c3ja2:

```
Simplify[Factor[1/9 (a - b) t (-18 + a δ + b δ + 2 Sqrt[(-8 + a^2 + 2 b + b^2 + 2 a (1 + b)) δ (-9 + 2 δ)]) -
   1/9 (-a + b) t (9 (4 - a - b) - (6 - 5 a - 5 b) δ - 2 Sqrt[(2 - a - b) (4 + a + b) (9 - 2 δ) δ])]]
```

$$\frac{1}{3}(a-b)\,t\,(6-2\,\delta+a\,(-3+2\,\delta)+b\,(-3+2\,\delta))$$

We rearrange this value, and obtain the following:

$$-\frac{1}{3}(b-a)\,t\,(6-2\,\delta+(a+b)\,(-3+2\,\delta))$$

The above is negative. The reason is as follows. 6-2 δ+(a+b) (-3+2 δ) is decreasing in (a+b), hence it is minimized when a+b=2, and then 6-2 δ+2(-3+2 δ)=2δ>0.

Therefore, the difference between the derived value of p_B and the p_B-element of c3ja2 is negative. That is, Firm A's "true" reaction function in (iii) does not pass through the reaction function of Firm B in (iii) if $a+b \geq \frac{2\left(-18-\delta+6\sqrt{81-54\,\delta+8\,\delta^2}\right)}{36-7\,\delta}$.

Thus, for any a, b, and δ, Firm A's "true" reaction function in (iii) does not pass through the reaction function of Firm B in (iii). This shows that the (pure-strategy) pricing equilibrium in period 1 does not exist in (Case iii).

Since (Case i) is symmetric to (Case iii), we can apply the same logic to conclude that the pricing equilibrium in the first period does not exist in (Case i).

We now turn to (Case ii), and identify conditions under which the pricing equilibrium in the first period exists in this case.
We pick up the reaction functions of the firms in (Case ii), without considering if they are in (Case ii). We derive the intersection between the reaction functions:

```
FullSimplify[Solve[{pa ==
    ((9 + 3 δc - 10 δf) pb + (b - a) t ((a + b) (9 - δc^2 - 8 δf + 4 δc δf) + 2 (3 δc + δc^2 + 4 δf - 2 δc δf))) /
     (2 (9 + 3 δc - 5 δf)),
   pb == ((9 + 3 δc - 10 δf) pa + (b - a) t (18 + 6 δc - 8 δf + 4 δc δf + (a + b) (-9 + δc^2 + 8 δf - 4 δc δf))) /
     (2 (9 + 3 δc - 5 δf))}, {pa, pb}]]
```

$$\left\{\left\{pa \to \frac{1}{-81+33\delta}(a-b)\,t\left(27\,(2+a+b)-6\,(-3+4a+4b)\,\delta+(-20+9a+9b)\,\delta^2\right),\; pb \to -\frac{1}{-81+33\delta}(a-b)\,t\left(27\,(-4+a+b)-6\,(-5+4a+4b)\,\delta+(2+9a+9b)\,\delta^2\right)\right\}\right\}$$

From the result, if there is an intersection between the reaction functions of Firms A and B in (Case ii), the equilibrium p_B is the following:

$$\text{Eq pb}: -\frac{1}{-81+33\delta}(a-b)\,t\left(27\,(-4+a+b)-6\,(-5+4a+4b)\,\delta+(2+9a+9b)\,\delta^2\right)$$

We can easily show that under the prices the realized z satisfies (a+b)/4<z<(2+a+b)/4 (Case ii).

We rewrite the jump point of Firm A's reaction function (c3ja1 and c3ja2):

```
c3ja1
c3ja2
```

$$\left\{\frac{1}{36}(-a+b)\,t\left(-2\,(9-5\delta)+3\,(2+a+b)\,(9-5\delta)\sqrt{\frac{9-2\delta}{9-4\delta}}-3\,(a+b)\,(3+\delta)\right),\; \frac{1}{18}(-a+b)\,t\left(-2-3a-3b+3\,(2+a+b)\sqrt{\frac{9-2\delta}{9-4\delta}}\right)(9-4\delta)\right\}$$

$$\left\{\frac{1}{9}(a-b)\,t\left(-18+a\delta+b\delta+2\sqrt{\left(-8+a^2+2b+b^2+2a\,(1+b)\right)\delta\,(-9+2\delta)}\right),\; \frac{1}{9}(-a+b)\,t\left(9\,(4-a-b)-(6-5a-5b)\,\delta-2\sqrt{(2-a-b)\,(4+a+b)\,(9-2\delta)\,\delta}\right)\right\}$$

If those values of p_B in the two jump points are larger than the equilibrium p_B in (Case ii), the equilibrium point is stable.

First, we compare p_B at the intersection and at c3ja1:

```
Simplify[Factor[1/18 (-a + b) t (-2 - 3 a - 3 b + 3 (2 + a + b) Sqrt[(9 - 2 δ)/(9 - 4 δ)]) (9 - 4 δ) -
   (-1/(-81 + 33 δ) (a - b) t (27 (-4 + a + b) - 6 (-5 + 4 a + 4 b) δ + (2 + 9 a + 9 b) δ^2))]]
```

$$\frac{1}{-486+198\delta}(a-b)\,t\left(3a\left(27\left(-7+9\sqrt{\frac{9-2\delta}{9-4\delta}}\right)-3\left(-53+69\sqrt{\frac{9-2\delta}{9-4\delta}}\right)\delta+\left(-26+44\sqrt{\frac{9-2\delta}{9-4\delta}}\right)\delta^2\right)+3b\left(27\left(-7+9\sqrt{\frac{9-2\delta}{9-4\delta}}\right)-3\left(-53+69\sqrt{\frac{9-2\delta}{9-4\delta}}\right)\delta+\left(-26+44\sqrt{\frac{9-2\delta}{9-4\delta}}\right)\delta^2\right)+2\left(81\left(-7+9\sqrt{\frac{9-2\delta}{9-4\delta}}\right)+\left(297-621\sqrt{\frac{9-2\delta}{9-4\delta}}\right)\delta+2\left(-19+66\sqrt{\frac{9-2\delta}{9-4\delta}}\right)\delta^2\right)\right)$$

Rearranging the above, we obtain

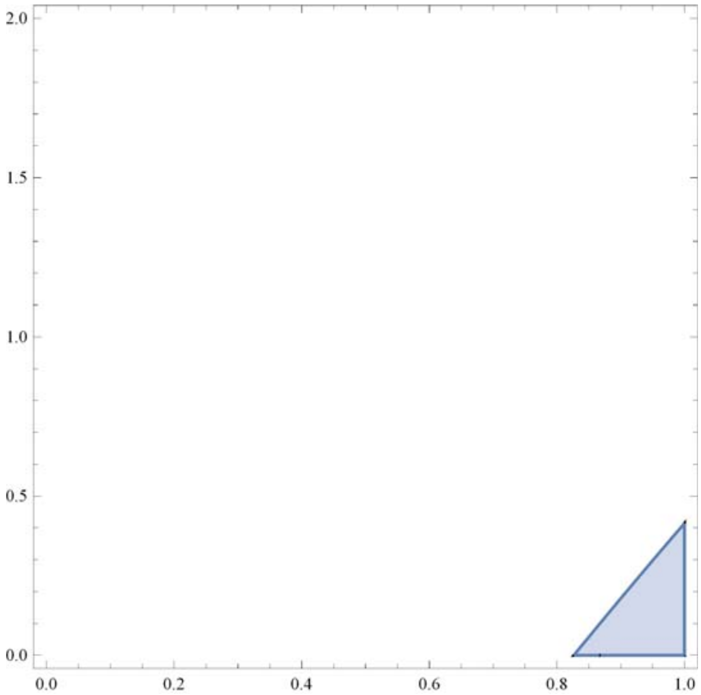
$$\frac{1}{486-198\delta}(b-a)\,t\left(\left(3\left(243-207\delta+44\delta^2\right)\sqrt{\frac{9-2\delta}{9-4\delta}}-(9-2\delta)\,((63-39\delta))\right)(a+b)-(9-2\delta)\,(126-38\delta)+6\left(243-207\delta+44\delta^2\right)\sqrt{\frac{9-2\delta}{9-4\delta}}\right)$$

The sign of this value depends on δ and g≡a+b.

We draw the area in which the above value is negative, that is, the equilibrium point in (Case ii) is not stable.

(Horizontal axis is δ, Vertical axis is g=a+b):

```
RegionPlot[(3 (243 - 207 δ + 44 δ^2) Sqrt[(9 - 2 δ)/(9 - 4 δ)] - (9 - 2 δ) ((63 - 39 δ))) g - (9 - 2 δ) (126 - 38 δ) +
    6 (243 - 207 δ + 44 δ^2) Sqrt[(9 - 2 δ)/(9 - 4 δ)] < 0 ∧ g < (2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]))/(36 - 7 δ), {δ, 0, 1}, {g, 0, 2}]
```



On the blue area,the equilibrium point in (Case ii) is **not stable**.

From the figure, we can find the highest value of δ in which the equilibrium point in (Case ii) is stable for any g=a+b, by solving the following equation with respect to δ.

```
NSolve[-(9 - 2 δ) (126 - 38 δ) + 6 (243 - 207 δ + 44 δ^2) Sqrt[(9 - 2 δ)/(9 - 4 δ)] == 0, δ]
```

{{δ → 4.5}, {δ → 0.826528}}

We find that if δ<0.826, the equilibrium point in area (ii) is stable for any g=a+b.

From the figure, we can find the lowest value of g in which the equilibrium point in area (ii) is stable for any δ, by solving the following equation with respect to g.

```
Solve[(3 (243 - 207 δ + 44 δ^2) Sqrt[(9 - 2 δ)/(9 - 4 δ)] - (9 - 2 δ) ((63 - 39 δ))) g -
    (9 - 2 δ) (126 - 38 δ) + 6 (243 - 207 δ + 44 δ^2) Sqrt[(9 - 2 δ)/(9 - 4 δ)] == 0 /. δ → 1, g]
```

$$\left\{\left\{g \to \frac{77 - 12\sqrt{35}}{3\left(-7 + 2\sqrt{35}\right)}\right\}\right\}$$

```
N[%]
```

$$\{\{g \to 0.414379\}\}$$

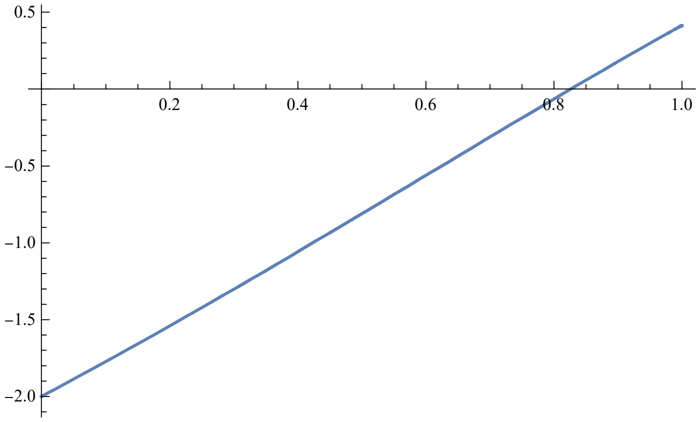
Because of symmetry, the result can be applied to the highest value of g in which the equilibrium point in area (ii) is stable for any δ.

We derive the threshold g as a function of δ.

$$\text{Solve}\left[\left(3\left(243-207\,\delta+44\,\delta^2\right)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}-(9-2\,\delta)\,((63-39\,\delta))\right)g-(9-2\,\delta)\,(126-38\,\delta)+6\left(243-207\,\delta+44\,\delta^2\right)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}==0,\ g\right]$$

$$\left\{\left\{g\to\frac{(126-38\,\delta)\,(9-2\,\delta)-6\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\left(243-207\,\delta+44\,\delta^2\right)}{-(63-39\,\delta)\,(9-2\,\delta)+3\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\left(243-207\,\delta+44\,\delta^2\right)}\right\}\right\}$$

$$\text{Plot}\left[\left(-12\,(3-\delta)\left(1296-1431\,\delta+479\,\delta^2\right)+120\,\delta\,(27-11\,\delta)\sqrt{(9-4\,\delta)\,(9-2\,\delta)}\right)\Big/\left(18\left(1296-1233\,\delta+426\,\delta^2-73\,\delta^3\right)\right),\ \{\delta,0,1\}\right]$$



$$N\left[2-\frac{77-12\sqrt{35}}{3\left(-7+2\sqrt{35}\right)}\right]$$

$$1.58562$$

Thus, we find that if 0.415<g<1.585, the equilibrium point in area (ii) is stable for any δ.

$$\left\{\frac{1}{9}\,(a-b)\,t\left(-18+a\,\delta+b\,\delta+2\sqrt{\left(-8+a^2+2\,b+b^2+2\,a\,(1+b)\right)\delta\,(-9+2\,\delta)}\right),\right.$$
$$\left.\frac{1}{9}\,(-a+b)\,t\left(9\,(4-a-b)-(6-5\,a-5\,b)\,\delta-2\sqrt{(2-a-b)\,(4+a+b)\,(9-2\,\delta)\,\delta}\right)\right\}$$

Second, we compare p_B at the intersection and at c3ja2:

$$\text{Simplify}\left[\text{Factor}\left[\frac{1}{9}(-a+b)\,t\left(9(4-a-b)-(6-5a-5b)\delta-2\sqrt{(2-a-b)(4+a+b)(9-2\delta)\delta}\right)-\left(-\frac{1}{-81+33\delta}(a-b)\,t\left(27(-4+a+b)-6(-5+4a+4b)\delta+(2+9a+9b)\delta^2\right)\right)\right]\right]$$

$$-\frac{1}{-243+99\delta}2(a-b)\,t\left(-324+81b+234\delta-81b\delta-36\delta^2+14b\delta^2+27\sqrt{\left(-8+a^2+2b+b^2+2a(1+b)\right)\delta(-9+2\delta)}-11\delta\sqrt{\left(-8+a^2+2b+b^2+2a(1+b)\right)\delta(-9+2\delta)}+a\left(81-81\delta+14\delta^2\right)\right)$$

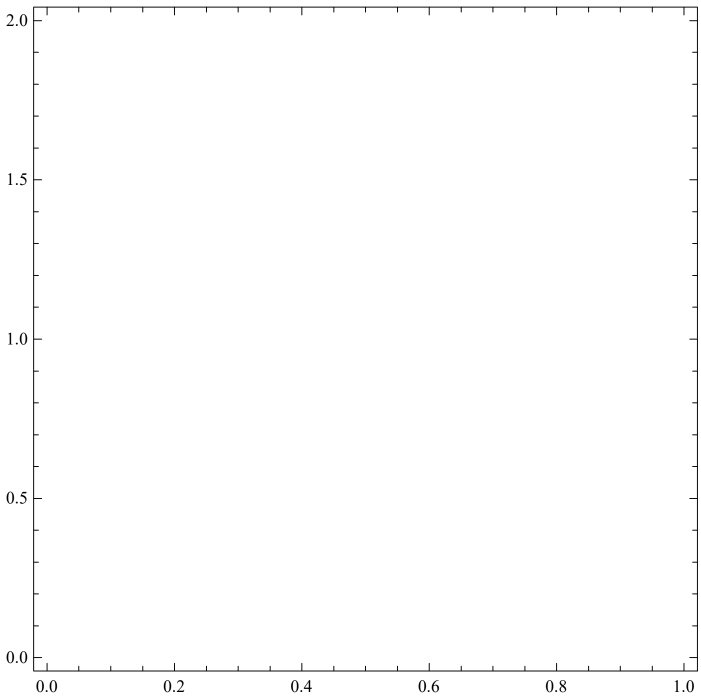
Rearranging the above, we obtain

$$-\frac{1}{243-99\delta}2(b-a)\,t\left(-18(2-\delta)(9-2\delta)+(27-11\delta)\sqrt{(2-(a+b))(4+a+b)\delta(9-2\delta)}+(a+b)(9-2\delta)(9-7\delta)\right)$$

The sign of this value depends on δ and g=a+b.
We draw the area in which the above value is negative, that is, the equilibrium point in (Case ii) is not stable.
(Horizontal axis is δ, Vertical axis is g=a+b):

$$g>\frac{2\left(-18-\delta+6\sqrt{81-54\delta+8\delta^2}\right)}{36-7\delta}$$

$$\text{RegionPlot}\left[\left(-18(2-\delta)(9-2\delta)+(27-11\delta)\sqrt{(2-g)(4+g)\delta(9-2\delta)}+g(9-2\delta)(9-7\delta)\right)>0,\ \{\delta,0,1\},\ \{g,0,2\}\right]$$



This means that Firm B's deviation incentive does not matter.

From the two arguments, we find that if 0.415<g<1.585, the equilibrium point in area (ii) is stable for any δ.

Thus we conclude that (i) if $\delta < 0.826$, the pricing equilibrium exists in (Case ii) for any a + b, and (ii) if $0.415 < a+b < 1.585$, then the pricing equilibrium exists in (Case ii) for any δ.

3. First Period - Locations

The profits of Firms A and B are respectively

```
Simplify[Factor[pa z + δf 1/9 (b - a) t (2 + 2 a + a^2 + 2 b + 2 a b + b^2 - 8 z - 2 a z - 2 b z + 10 z^2) /.
    z → 3 (pb - pa)/(2 (b - a) t (3 + δc)) + ((a + b) (3 - δc) + 2 δc)/(2 (3 + δc)) /. {pa → ((a - b) t
        (-27 (2 + a + b) - 54 δc + 3 (-4 + a + b) δc^2 - 4 (-9 + 3 a (-2 + δc) + 3 b (-2 + δc) - 8 δc) δf)) /
      (81 + 27 δc - 60 δf), pb → -(((a - b) t (-27 (-4 + a + b) + 54 δc + 3 (2 + a + b) δc^2 -
            4 (21 + 3 a (-2 + δc) + 3 b (-2 + δc) + 2 δc) δf)) / (81 + 27 δc - 60 δf))}]]
Simplify[Factor[pb (1 - z) + δf 1/9 (b - a) t (8 - 4 a + a^2 - 4 b + 2 a b + b^2 - 8 z - 2 a z - 2 b z + 10 z^2) /.
    z → 3 (pb - pa)/(2 (b - a) t (3 + δc)) + ((a + b) (3 - δc) + 2 δc)/(2 (3 + δc)) /. {pa → ((a - b) t
        (-27 (2 + a + b) - 54 δc + 3 (-4 + a + b) δc^2 - 4 (-9 + 3 a (-2 + δc) + 3 b (-2 + δc) - 8 δc) δf)) /
      (81 + 27 δc - 60 δf), pb → -(((a - b) t (-27 (-4 + a + b) + 54 δc + 3 (2 + a + b) δc^2 -
            4 (21 + 3 a (-2 + δc) + 3 b (-2 + δc) + 2 δc) δf)) / (81 + 27 δc - 60 δf))}]]
```

$$
-\frac{1}{18\,(27+9\,\delta c-20\,\delta f)^2}(a-b)\,t\,\Big(9\,a^2\left(81+3\,\delta c^3+45\,\delta f-184\,\delta f^2+80\,\delta f^3+3\,\delta c^2\,(-3+7\,\delta f)+\delta c\left(-27+138\,\delta f-88\,\delta f^2\right)\right)+
9\,b^2\left(81+3\,\delta c^3+45\,\delta f-184\,\delta f^2+80\,\delta f^3+3\,\delta c^2\,(-3+7\,\delta f)+\delta c\left(-27+138\,\delta f-88\,\delta f^2\right)\right)-
12\,b\left(-243+18\,\delta c^3+\delta c^2\,(27-57\,\delta f)+216\,\delta f+84\,\delta f^2-80\,\delta f^3+\delta c\left(-162-9\,\delta f+88\,\delta f^2\right)\right)+
4\left(729+108\,\delta c^3-972\,\delta f+216\,\delta f^2+80\,\delta f^3-9\,\delta c^2\,(-72+43\,\delta f)+3\,\delta c\left(405-450\,\delta f+104\,\delta f^2\right)\right)+
6\,a\Big(3\,b\left(81+3\,\delta c^3+45\,\delta f-184\,\delta f^2+80\,\delta f^3+3\,\delta c^2\,(-3+7\,\delta f)+\delta c\left(-27+138\,\delta f-88\,\delta f^2\right)\right)-
2\left(-243+18\,\delta c^3+\delta c^2\,(27-57\,\delta f)+216\,\delta f+84\,\delta f^2-80\,\delta f^3+\delta c\left(-162-9\,\delta f+88\,\delta f^2\right)\right)\Big)\Big)
$$

$$
-\frac{1}{18\,(27+9\,\delta c-20\,\delta f)^2}(a-b)\,t\,\Big(9\,a^2\left(81+3\,\delta c^3+45\,\delta f-184\,\delta f^2+80\,\delta f^3+3\,\delta c^2\,(-3+7\,\delta f)+\delta c\left(-27+138\,\delta f-88\,\delta f^2\right)\right)+
9\,b^2\left(81+3\,\delta c^3+45\,\delta f-184\,\delta f^2+80\,\delta f^3+3\,\delta c^2\,(-3+7\,\delta f)+\delta c\left(-27+138\,\delta f-88\,\delta f^2\right)\right)+
4\left(2916+27\,\delta c^3-1863\,\delta f-1944\,\delta f^2+1280\,\delta f^3+9\,\delta c^2\,(45+16\,\delta f)-18\,\delta c\left(-108+3\,\delta f+56\,\delta f^2\right)\right)+
12\,b\left(-486+9\,\delta c^3+81\,\delta f+636\,\delta f^2-320\,\delta f^3-6\,\delta c^2\,(-9+20\,\delta f)+\delta c\left(-81-423\,\delta f+352\,\delta f^2\right)\right)+
6\,a\Big(3\,b\left(81+3\,\delta c^3+45\,\delta f-184\,\delta f^2+80\,\delta f^3+3\,\delta c^2\,(-3+7\,\delta f)+\delta c\left(-27+138\,\delta f-88\,\delta f^2\right)\right)+
2\left(-486+9\,\delta c^3+81\,\delta f+636\,\delta f^2-320\,\delta f^3-6\,\delta c^2\,(-9+20\,\delta f)+\delta c\left(-81-423\,\delta f+352\,\delta f^2\right)\right)\Big)\Big)
$$

Rearranging them, we obtain the profits of Firms A and B respectively

$$
\frac{(b-a)\,t}{18\,(27+9\,\delta c-20\,\delta f)^2}\Big(9\,(a+b)^2\left(3\,\delta c^3-3\,\delta c^2\,(3-7\,\delta f)-\delta c\left(27-138\,\delta f+88\,\delta f^2\right)+81+45\,\delta f-184\,\delta f^2+80\,\delta f^3\right)+
12\,(a+b)\left(-18\,\delta c^3-3\,\delta c^2\,(9-19\,\delta f)+\delta c\left(162+9\,\delta f-88\,\delta f^2\right)+243-216\,\delta f-84\,\delta f^2+80\,\delta f^3\right)+
4\left(108\,\delta c^3+9\,\delta c^2\,(72-43\,\delta f)+3\,\delta c\left(405-450\,\delta f+104\,\delta f^2\right)+729-972\,\delta f+216\,\delta f^2+80\,\delta f^3\right)\Big)
$$

$$
\frac{1}{18\,(27+9\,\delta c-20\,\delta f)^2}(b-a)\,t\,\Big(9\,(a+b)^2\left(3\,\delta c^3-3\,\delta c^2\,(3-7\,\delta f)-\delta c\left(27-138\,\delta f+88\,\delta f^2\right)+81+45\,\delta f-184\,\delta f^2+80\,\delta f^3\right)+
12\,(a+b)\left(9\,\delta c^3+6\,\delta c^2\,(9-20\,\delta f)-\delta c\left(81+423\,\delta f-352\,\delta f^2\right)-486+81\,\delta f+636\,\delta f^2-320\,\delta f^3\right)+
4\left(27\,\delta c^3+9\,\delta c^2\,(45+16\,\delta f)+18\,\delta c\left(108-3\,\delta f-56\,\delta f^2\right)+2916-1863\,\delta f-1944\,\delta f^2+1280\,\delta f^3\right)\Big)
$$

We define J1, the coefficient of $(a+b)^2$, as follows:

$$\mathrm{J1} = \left(3\,\delta c^3 - 3\,\delta c^2\,(3 - 7\,\delta f) - \delta c\,\left(27 - 138\,\delta f + 88\,\delta f^2\right) + 81 + 45\,\delta f - 184\,\delta f^2 + 80\,\delta f^3\right)$$

$$81 + 3\,\delta c^3 - 3\,\delta c^2\,(3 - 7\,\delta f) + 45\,\delta f - 184\,\delta f^2 + 80\,\delta f^3 - \delta c\,\left(27 - 138\,\delta f + 88\,\delta f^2\right)$$

Differentiating Firm A's profit with respect to a and Firm B's profit with respect to b, we obtain

$$\mathrm{Factor}\Big[D\Big[pa\,z + \delta f\,\frac{1}{9}\,(b - a)\,t\,\left(2 + 2\,a + a^2 + 2\,b + 2\,a\,b + b^2 - 8\,z - 2\,a\,z - 2\,b\,z + 10\,z^2\right)\;/.\; z \to \frac{3\,(pb - pa)}{2\,(b - a)\,t\,(3 + \delta c)} + \frac{((a + b)\,(3 - \delta c) + 2\,\delta c)}{2\,(3 + \delta c)}\;/.\; \{pa \to \left((a - b)\,t\,\left(-27\,(2 + a + b) - 54\,\delta c + 3\,(-4 + a + b)\,\delta c^2 - 4\,(-9 + 3\,a\,(-2 + \delta c) + 3\,b\,(-2 + \delta c) - 8\,\delta c)\,\delta f\right)\right) / (81 + 27\,\delta c - 60\,\delta f),\; pb \to -\left(\left((a - b)\,t\,\left(-27\,(-4 + a + b) + 54\,\delta c + 3\,(2 + a + b)\,\delta c^2 - 4\,(21 + 3\,a\,(-2 + \delta c) + 3\,b\,(-2 + \delta c) + 2\,\delta c)\,\delta f\right)\right) / (81 + 27\,\delta c - 60\,\delta f)\right)\},\; a\Big]\Big]$$

$$\mathrm{Factor}\Big[D\Big[pb\,(1 - z) + \delta f\,\frac{1}{9}\,(b - a)\,t\,\left(8 - 4\,a + a^2 - 4\,b + 2\,a\,b + b^2 - 8\,z - 2\,a\,z - 2\,b\,z + 10\,z^2\right)\;/.\; z \to \frac{3\,(pb - pa)}{2\,(b - a)\,t\,(3 + \delta c)} + \frac{((a + b)\,(3 - \delta c) + 2\,\delta c)}{2\,(3 + \delta c)}\;/.\; \{pa \to \left((a - b)\,t\,\left(-27\,(2 + a + b) - 54\,\delta c + 3\,(-4 + a + b)\,\delta c^2 - 4\,(-9 + 3\,a\,(-2 + \delta c) + 3\,b\,(-2 + \delta c) - 8\,\delta c)\,\delta f\right)\right) / (81 + 27\,\delta c - 60\,\delta f),\; pb \to -\left(\left((a - b)\,t\,\left(-27\,(-4 + a + b) + 54\,\delta c + 3\,(2 + a + b)\,\delta c^2 - 4\,(21 + 3\,a\,(-2 + \delta c) + 3\,b\,(-2 + \delta c) + 2\,\delta c)\,\delta f\right)\right) / (81 + 27\,\delta c - 60\,\delta f)\right)\},\; b\Big]\Big]$$

$$-\frac{1}{18\,(27 + 9\,\delta c - 20\,\delta f)^2}\,t\,\big(2916 + 5832\,a + 2187\,a^2 + 1458\,a\,b - 729\,b^2 + 4860\,\delta c + 3888\,a\,\delta c - 729\,a^2\,\delta c - 486\,a\,b\,\delta c + 243\,b^2\,\delta c + 2592\,\delta c^2 - 648\,a\,\delta c^2 - 243\,a^2\,\delta c^2 - 162\,a\,b\,\delta c^2 + 81\,b^2\,\delta c^2 + 432\,\delta c^3 - 432\,a\,\delta c^3 + 81\,a^2\,\delta c^3 + 54\,a\,b\,\delta c^3 - 27\,b^2\,\delta c^3 - 3888\,\delta f - 5184\,a\,\delta f + 1215\,a^2\,\delta f + 810\,a\,b\,\delta f - 405\,b^2\,\delta f - 5400\,\delta c\,\delta f + 216\,a\,\delta c\,\delta f + 3726\,a^2\,\delta c\,\delta f + 2484\,a\,b\,\delta c\,\delta f - 1242\,b^2\,\delta c\,\delta f - 1548\,\delta c^2\,\delta f + 1368\,a\,\delta c^2\,\delta f + 567\,a^2\,\delta c^2\,\delta f + 378\,a\,b\,\delta c^2\,\delta f - 189\,b^2\,\delta c^2\,\delta f + 864\,\delta f^2 - 2016\,a\,\delta f^2 - 4968\,a^2\,\delta f^2 - 3312\,a\,b\,\delta f^2 + 1656\,b^2\,\delta f^2 + 1248\,\delta c\,\delta f^2 - 2112\,a\,\delta c\,\delta f^2 - 2376\,a^2\,\delta c\,\delta f^2 - 1584\,a\,b\,\delta c\,\delta f^2 + 792\,b^2\,\delta c\,\delta f^2 + 320\,\delta f^3 + 1920\,a\,\delta f^3 + 2160\,a^2\,\delta f^3 + 1440\,a\,b\,\delta f^3 - 720\,b^2\,\delta f^3\big)$$

$$-\frac{1}{18\,(27 + 9\,\delta c - 20\,\delta f)^2}\,t\,\big(-11\,664 + 729\,a^2 + 11\,664\,b - 1458\,a\,b - 2187\,b^2 - 7776\,\delta c - 243\,a^2\,\delta c + 1944\,b\,\delta c + 486\,a\,b\,\delta c + 729\,b^2\,\delta c - 1620\,\delta c^2 - 81\,a^2\,\delta c^2 - 1296\,b\,\delta c^2 + 162\,a\,b\,\delta c^2 + 243\,b^2\,\delta c^2 - 108\,\delta c^3 + 27\,a^2\,\delta c^3 - 216\,b\,\delta c^3 - 54\,a\,b\,\delta c^3 - 81\,b^2\,\delta c^3 + 7452\,\delta f + 405\,a^2\,\delta f - 1944\,b\,\delta f - 810\,a\,b\,\delta f - 1215\,b^2\,\delta f + 216\,\delta c\,\delta f + 1242\,a^2\,\delta c\,\delta f + 10\,152\,b\,\delta c\,\delta f - 2484\,a\,b\,\delta c\,\delta f - 3726\,b^2\,\delta c\,\delta f - 576\,\delta c^2\,\delta f + 189\,a^2\,\delta c^2\,\delta f + 2880\,b\,\delta c^2\,\delta f - 378\,a\,b\,\delta c^2\,\delta f - 567\,b^2\,\delta c^2\,\delta f + 7776\,\delta f^2 - 1656\,a^2\,\delta f^2 - 15\,264\,b\,\delta f^2 + 3312\,a\,b\,\delta f^2 + 4968\,b^2\,\delta f^2 + 4032\,\delta c\,\delta f^2 - 792\,a^2\,\delta c\,\delta f^2 - 8448\,b\,\delta c\,\delta f^2 + 1584\,a\,b\,\delta c\,\delta f^2 + 2376\,b^2\,\delta c\,\delta f^2 - 5120\,\delta f^3 + 720\,a^2\,\delta f^3 + 7680\,b\,\delta f^3 - 1440\,a\,b\,\delta f^3 - 2160\,b^2\,\delta f^3\big)$$

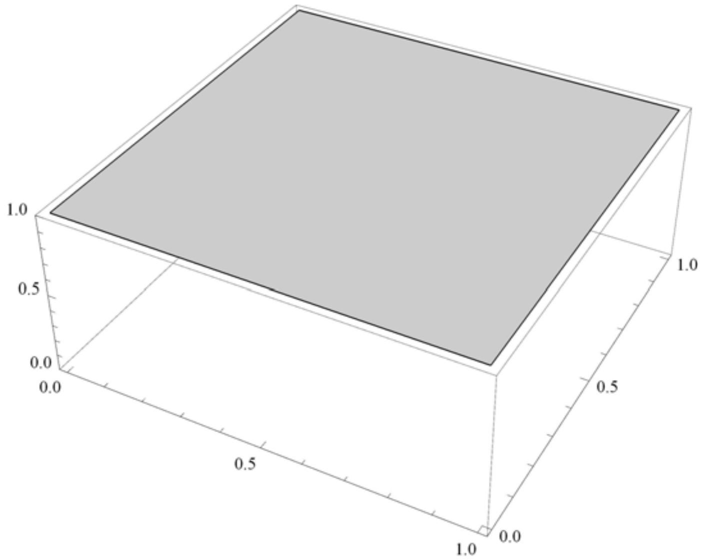
The above first-order derivatives can be rearranged as follows.

$$-\frac{t}{18\,(27 + 9\,\delta c - 20\,\delta f)^2}\,\Big(27\,\mathrm{J1}\,a^2 + \left(24\,\left(9\,\left(9 - \delta c^2\right)\,(3 + 2\,\delta c) - 3\,\left(72 - 3\,\delta c - 19\,\delta c^2\right)\,\delta f - 4\,(21 + 22\,\delta c)\,\delta f^2 + 80\,\delta f^3\right) + 18\,\mathrm{J1}\,b\right)\,a + 4\,\left(27\,(3 + \delta c)\,(3 + 2\,\delta c)^2 - 9\,\left(108 + 150\,\delta c + 43\,\delta c^2\right)\,\delta f + 24\,(9 + 13\,\delta c)\,\delta f^2 + 80\,\delta f^3\right) - 9\,\mathrm{J1}\,b^2\Big)$$

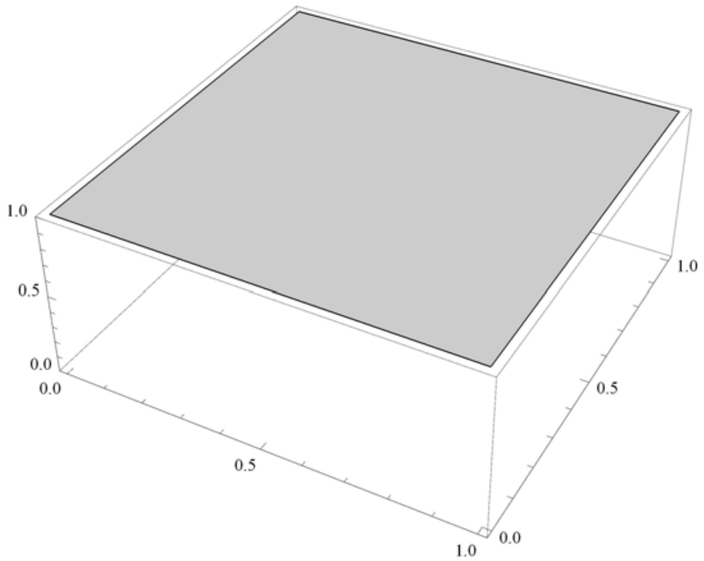
$$\frac{t}{18\,(27 + 9\,\delta c - 20\,\delta f)^2}\,\Big(27\,\mathrm{J1}\,b^2 + \left(24\,\left(-9\,\left(9 - \delta c^2\right)\,(6 + \delta c) + 3\,(27 - \delta c\,(141 + 40\,\delta c))\,\delta f + 4\,(159 + 88\,\delta c)\,\delta f^2 - 320\,\delta f^3\right) + 18\,\mathrm{J1}\,a\right)\,b + 4\,\left(27\,(3 + \delta c)\,(6 + \delta c)^2 - 9\,(207 + 2\,\delta c\,(3 - 8\,\delta c))\,\delta f - 72\,(27 + 14\,\delta c)\,\delta f^2 + 1280\,\delta f^3\right) - 9\,\mathrm{J1}\,a^2\Big)$$

First, we show that the derivative of Firm A's profit monotonically decreases in a. This implies that Firm A's optimal location choice is $a = 0$. To show this, we show that the coefficients to a^2, *a* and the constant term in Firm A's first-order derivative are all positive. For the coefficients to a^2 and a, see the following figures, which show they are both positive.

```
Plot3D[J1, {δc, 0, 1}, {δf, 0, 1}, PlotRange → {0, 1}]
```



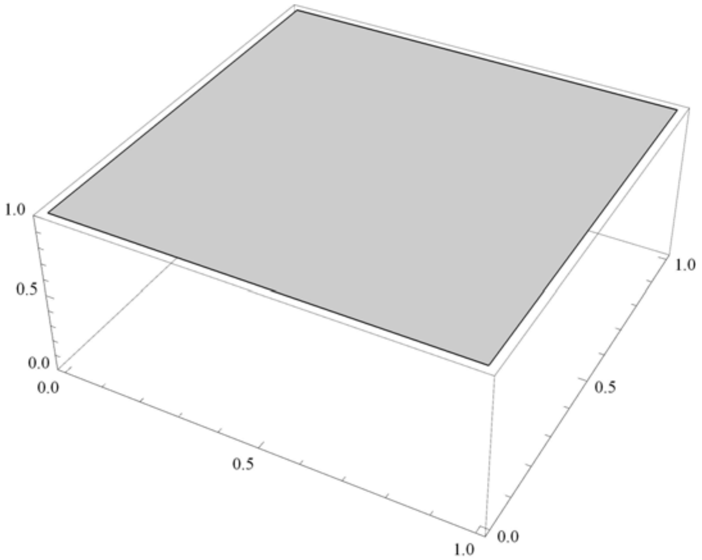

```
Plot3D[24 (9 (9 - δc^2) (3 + 2 δc) - 3 (72 - 3 δc - 19 δc^2) δf - 4 (21 + 22 δc) δf^2 + 80 δf^3),
 {δc, 0, 1}, {δf, 0, 1}, PlotRange → {0, 1}]
```



Those are positive.

For the constant term in Firm A's first-order derivative, notice that it is minimized when $b = 1$. As the next figure shows, the constant term is positive when $b = 1$.

```
Plot3D[4 (27 (3 + δc) (3 + 2 δc)^2 - 9 (108 + 150 δc + 43 δc^2) δf + 24 (9 + 13 δc) δf^2 + 80 δf^3) - 9 J1,
 {δc, 0, 1}, {δf, 0, 1}, PlotRange → {0, 1}]
```



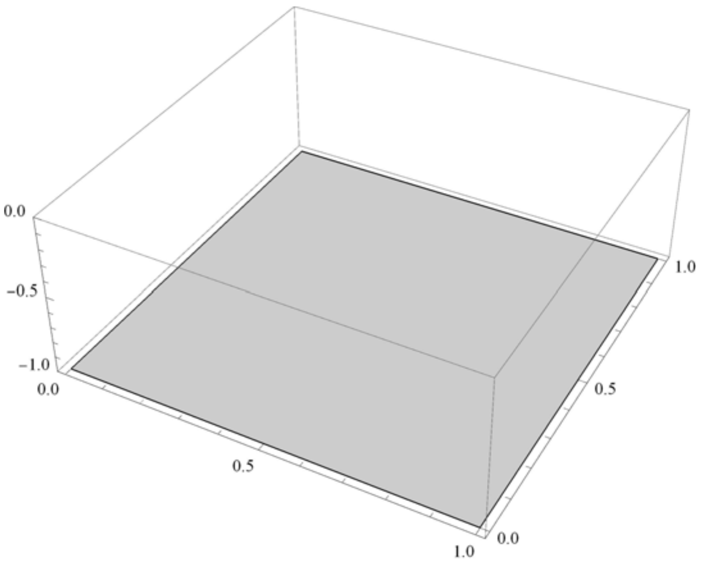
Put together, we have shown that Firm A's profit decreases monotonically in *a*. Thus *a* = 0 is Firm A's optimal location choice.

Next, we turn to Firm B's problem. Differentiating the numerator of the first-order derivative of Firm B's profit with respect to *b*, we obtain

```
54 J1 b +
 (24 (-9 (9 - δc^2) (6 + δc) + 3 (27 - δc (141 + 40 δc)) δf + 4 (159 + 88 δc) δf^2 - 320 δf^3) + 18 J1 a )
```

This is maximized when *a*=*b*=1. At *a*=*b*=1, we can show that this is negative (see the following figure).

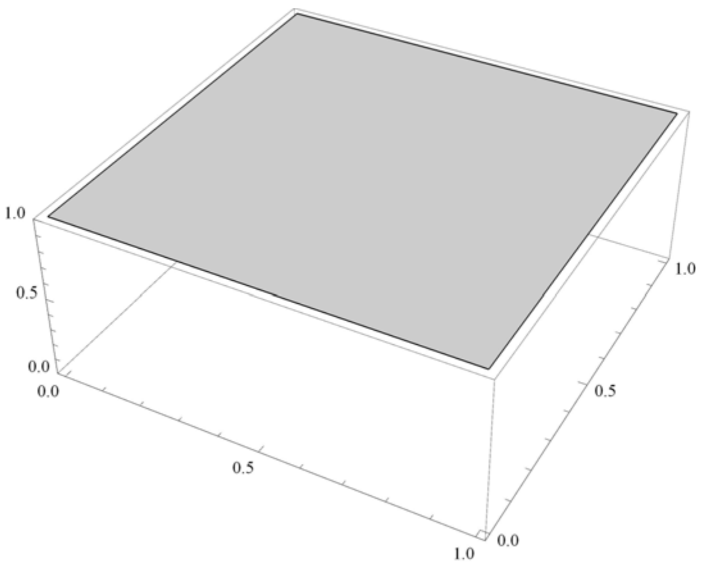
```
Plot3D[
 54 J1 + (24 (-9 (9 - δc^2) (6 + δc) + 3 (27 - δc (141 + 40 δc)) δf + 4 (159 + 88 δc) δf^2 - 320 δf^3) + 18 J1 ),
 {δc, 0, 1}, {δf, 0, 1}, PlotRange → {-1, 0}]
```



Thus the first-order derivative of Firm B's profit decreases monotonically in *b*.

We now show that at *a*=0 and *b*=1, the numerator of the first-order derivative of Firm B's profit is positive (see the following figure). This implies that Firm B's optimal location choice is *b* = 1.

```
Plot3D[
 (27 J1 + (24 (-9 (9 - δc^2) (6 + δc) + 3 (27 - δc (141 + 40 δc)) δf + 4 (159 + 88 δc) δf^2 - 320 δf^3) ) +
   4 (27 (3 + δc) (6 + δc)^2 - 9 (207 + 2 δc (3 - 8 δc)) δf - 72 (27 + 14 δc) δf^2 + 1280 δf^3)),
 {δc, 0, 1}, {δf, 0, 1}, PlotRange → {0, 1}]
```



Therefore, the equilibrium locations are *a*=0, *b*=1.
The rest of Proposition 1 follows by substituting *a* = 0, *b* = 1 into relevant prices and the locations of marginal consumers in the two periods.

Proofs of Propositions 2 and 3.

Because each firm's reaction function consists of three different pieces, we need to derive the 'true' reaction function by checking when the firm's profit obtains a global, rather than, local maximum.

From here on, we assume that firms' discount factor is δ and consumers' discount factor is 0, as stated in Propositions 2 and 3.

```
δf = δ
```

δ

```
δc = 0
```

0

From Firm A's reaction function derived in the proof of Proposition 1, we check the endpoints of each of the three line segments corresponding to the three cases: (i)$0 \le z \le (a+b)/4$, (ii) $(a+b)/4 < z < (2+a+b)/4$, (iii) $(2+a+b)/4 \le z \le 1$.

(Case i): The first endpoint below locates the left-hand side of Firm A's reaction function, and the second endpoint below locates the right-hand side of Firm A's reaction function in the first period.

```
c1aL = Simplify[{1/9 (a - b) (-8 + 5 a + 5 b) t δf, -1/9 (b - a) t ((a + b) (9 - 6 δc + 5 δf) + 6 δc - 8 δf)}]
c1aR = FullSimplify[
  Factor[{1/18 (a - b) t (3 (a + b) (-3 + δc) - 16 δf), ((b - a) t (3 (-2 + a + b) δc + 8 δf))/9}]]
```

$\left\{\frac{1}{9}(a-b)(-8+5a+5b)t\delta,\ -\frac{1}{9}(-a+b)t(-8\delta+(a+b)(9+5\delta))\right\}$

$\left\{-\frac{1}{18}(a-b)t(9(a+b)+16\delta),\ \frac{8}{9}(-a+b)t\delta\right\}$

(Case ii): The first endpoint below locates the left-hand side of Firm A’s reaction function, and the second endpoint below locates the right-hand side of Firm A’s reaction function in the first period.

```
c2aL = FullSimplify[Factor[{-1/18 (a - b) t (3 (a + b) (3 + δc) - 2 (-8 + 3 a + 3 b) δf),
     ((b - a) t (3 (a + b) (2 δc - δf) - 2 (3 δc - 4 δf)))/9}]]
c2aR = Simplify[{-1/18 (a - b) t (3 (2 + a + b) (3 + δc) - 2 (2 + 3 a + 3 b) δf),
    ((b - a) t (3 (a + b) (2 δc - δf) + 2 (9 - δf)))/9}]
```

$\left\{\frac{1}{18}(a-b)t(-9(a+b)+2(-8+3a+3b)\delta),\ \frac{1}{9}(a-b)(-8+3a+3b)t\delta\right\}$

$\left\{-\frac{1}{18}(a-b)t(9(2+a+b)-2(2+3a+3b)\delta),\ \frac{1}{9}(a-b)t(-18+(2+3a+3b)\delta)\right\}$

At this stage, we cannot say if Firm A’s reaction function is also continuous in (Case i) and (Case ii). We will come back to this shortly. Note that the left-hand endpoint in (Case i) corresponds to (a+b)/4=z.

(Case iii): The first endpoint below locates the left-hand side of Firm A’s reaction function, and the second endpoint below locates the right-hand side of Firm A’s reaction function in in the first period.

```
c3aL = Simplify[{1/18 (a - b) t (3 (2 + a + b) (-3 + δc) + 2 (2 + 3 a + 3 b) δf),
    ((b - a) t (2 (9 - 3 δc - δf) + 3 (a + b) (δc - δf)))/9}]
c3aR = Simplify[{2/9 (a - b) t (-9 + 3 δc + (2 + a + b) δf),
    ((b - a) t (4 (9 - 3 δc - δf) - (a + b) (9 - 6 δc + 2 δf)))/9}]
```

$\left\{\frac{1}{18}(a-b)t(-9(2+a+b)+2(2+3a+3b)\delta),\ \frac{1}{9}(a-b)t(-18+(2+3a+3b)\delta)\right\}$

$\left\{\frac{2}{9}(a-b)t(-9+(2+a+b)\delta),\ \frac{1}{9}(-a+b)t(-4(-9+\delta)-(a+b)(9+2\delta))\right\}$

As shown above, the left endpoint of Firm A’s reaction function in (Case iii) coincides with the right endpoint of Firm A’s reaction function in (Case ii). That is, Firm A’s reaction function is continuous in (Case ii) and (Case iii).

(Case iii)’: When z=1, the reaction function of Firm A consists of the segment connecting the following two points.

```
c3daL = Simplify[
  {2/9 (a - b) t (-9 + 3 δc + (2 + a + b) δf), ((b - a) t (4 (9 - 3 δc - δf) - (a + b) (9 - 6 δc + 2 δf)))/9}]
c3daR = Simplify[{2/9 (a - b) t (-9 + 3 δc + (2 + a + b) δf) + k,
   ((b - a) t (4 (9 - 3 δc - δf) - (a + b) (9 - 6 δc + 2 δf)))/9 + k}]
```

$$\left\{\frac{2}{9}(a-b)\,t\,(-9+(2+a+b)\,\delta),\ \frac{1}{9}(-a+b)\,t\,(-4\,(-9+\delta)-(a+b)\,(9+2\,\delta))\right\}$$

$$\left\{k+\frac{2}{9}(a-b)\,t\,(-9+(2+a+b)\,\delta),\ k+\frac{1}{9}(-a+b)\,t\,(-4\,(-9+\delta)-(a+b)\,(9+2\,\delta))\right\}$$

where k is a sufficient large positive number (to keep p_A at the monopoly price leading to $z=1$). Note that the left endpoint of Firm A's reaction function in (Case iii)' coincides with the right endpoint of Firm A's reaction function in (Case iii).

Next, from firm B's reaction function derived the proof of Proposition 1, we check the endpoints of each of the three line segments corresponding to the three cases: (i)0≤z≤(a+b)/4, (ii) (a+b)/4<z<(2+a+b)/4, (iii) (2+a+b)/4≤z≤1.

(Case i): The first endpoint below locates the left-hand side of Firm B's reaction function, and the second endpoint below locates the right-hand side of Firm B's reaction function in the first period.

```
c1bL = FullSimplify[{((b - a) t (18 - 3 (a + b) (δc - δf) - 8 δf))/9,
   -1/18 (a - b) t (3 (-4 + a + b) (-3 + δc) + 2 (-8 + 3 a + 3 b) δf)}]
c1bR = Simplify[{((b - a) t (18 + (a + b) (9 - 6 δc + 2 δf) - 8 δf))/9,
   -2/9 (a - b) t (9 - 3 δc + (-4 + a + b) δf)}]
```

$$\left\{\frac{1}{9}(-a+b)\,t\,(18-8\,\delta+3\,(a+b)\,\delta),\ -\frac{1}{18}(a-b)\,t\,(-9\,(-4+a+b)+2\,(-8+3\,a+3\,b)\,\delta)\right\}$$

$$\left\{\frac{1}{9}(-a+b)\,t\,(18-8\,\delta+(a+b)\,(9+2\,\delta)),\ -\frac{2}{9}(a-b)\,t\,(9+(-4+a+b)\,\delta)\right\}$$

(Case i)': When z=0, The reaction function of Firm B consists of the segment connecting the following two points.

```
c1dbL =
 Simplify[{((b - a) t (18 + (a + b) (9 - 6 δc + 2 δf) - 8 δf))/9, -2/9 (a - b) t (9 - 3 δc + (-4 + a + b) δf)}]
c1dbR = Simplify[{((b - a) t (18 + (a + b) (9 - 6 δc + 2 δf) - 8 δf))/9 + k,
   -2/9 (a - b) t (9 - 3 δc + (-4 + a + b) δf) + k}]
```

$$\left\{\frac{1}{9}(-a+b)\,t\,(18-8\,\delta+(a+b)\,(9+2\,\delta)),\ -\frac{2}{9}(a-b)\,t\,(9+(-4+a+b)\,\delta)\right\}$$

$$\left\{k+\frac{1}{9}(-a+b)\,t\,(18-8\,\delta+(a+b)\,(9+2\,\delta)),\ k-\frac{2}{9}(a-b)\,t\,(9+(-4+a+b)\,\delta)\right\}$$

where k is a sufficient large positive number (to keep p_B at the monopoly price leading to $z=0$). Note that the left-hand endpoint of Firm A's reaction function in case (i)' coincides with the right-hand endpoint of Firm A's reaction function in case (i).

Case (ii): The first endpoint below locates the left-hand side of Firm B's reaction function, and the second endpoint below locates the right-hand side of Firm B's reaction function in the first period.

```
c2bL = FullSimplify[{((b - a) t (6 (1 - a - b) δc + (2 + 3 a + 3 b) δf))/9,
  1/18 (a - b) t (3 (-2 + a + b) (3 + δc) - 2 (2 + 3 a + 3 b) δf)}]
c2bR = Simplify[{((b - a) t (18 + 6 (2 - a - b) δc - (8 - 3 a - 3 b) δf))/9,
  1/18 (a - b) t (3 (-4 + a + b) (3 + δc) - 2 (-8 + 3 a + 3 b) δf)}]
```

$\left\{\frac{1}{9}(-a+b)(2+3a+3b)t\delta, \frac{1}{18}(a-b)t(9(-2+a+b)-2(2+3a+3b)\delta)\right\}$

$\left\{-\frac{1}{9}(a-b)t(18+(-8+3a+3b)\delta), \frac{1}{18}(a-b)t(9(-4+a+b)-2(-8+3a+3b)\delta)\right\}$

We find that the left endpoint of Firm B's reaction function in (Case i) coincides with the right endpoint of Firm B's reaction function in (Case ii). That is, Firm B's reaction function is continuous in (Case i) and (Case ii).

Case (iii): The first endpoint below locates the left-hand side of Firm B's reaction function, and the second endpoint below locates the right-hand side of Firm B's reaction function in the first period.

```
c3bL =
 Simplify[{((b - a) t ((a + b) (9 - 6 δc + 5 δf) - 2 (9 - 3 δc + δf)))/9, -1/9 (a - b) (-2 + 5 a + 5 b) t δf}]
c3bR = FullSimplify[
  Factor[{((b - a) t (8 δf - 3 (a + b) δc))/9, -1/18 (a - b) t (3 (-2 + a + b) (-3 + δc) + 16 δf)}]]
```

$\left\{\frac{1}{9}(-a+b)t(-2(9+\delta)+(a+b)(9+5\delta)), -\frac{1}{9}(a-b)(-2+5a+5b)t\delta\right\}$

$\left\{\frac{8}{9}(-a+b)t\delta, \frac{1}{18}(a-b)t(9(-2+a+b)-16\delta)\right\}$

At this stage, we cannot say if Firm B's reaction function is continuous in (Case iii) and (Case ii). We will come back to this shortly. Note that the right-hand endpoint in (Case iii) corresponds to (2+a+b)/4=z.

We now turn to the pricing equilibrium in the first period. To this end, we need to find the 'true' reaction function for each firm, by checking when each firm's reaction leads to a global optimum. Next we check when the two 'true' reaction functions intersect. As before, calculations are quite messy although the logical steps are identical. If necessary, readers can jump directly to the stage where we show that the pricing equilibrium exists only in (Case ii) and for all values of δ.

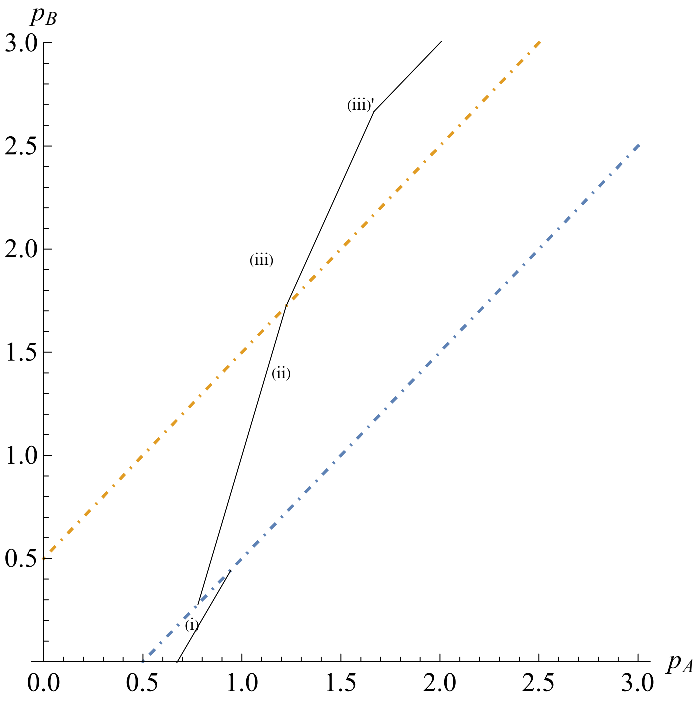
For illustrative purposes, we start with an example where we set a = 0, b = 1, t = 1, and δ = 1/2.

```
a = 0
b = 1
δ = 1 / 2
t = 1
k = 1
```

```
0

1

1
—
2

1

1
```

First, we plot Firm A's reaction function corresponding to the three cases.

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
   x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3},
  Epilog → {Line[{c1aL, c1aR}], Line[{c2aL, c2aR}], , Line[{c3aL, c3aR}],
    Line[{c3daL, c3daR}], Text["(iii)'", {1.6, 2.7}], Text["(iii)", {1.1, 1.95}],
    Text["(ii)", {1.2, 1.4}], Text["(i)", {0.75, 0.18}]}, PlotRange → {0, 3},
  LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"}, AspectRatio → 1, PlotStyle → DotDashed]
```



As shown above, for some p_B, there are two local optimal prices for Firm A.
We can show that the multiplicity of local optimal prices always appears.

```
Clear[a, b, δ, t, k]
```

To show the multiplicity, we check the locations of the three endpoints: The left-hand endpoint in (ii) (c2aL), the left-hand and right-hand endpoints (c1aL and c1aR) in (i) (see below)

c2aL
c1aL
c1aR

$$\left\{\frac{1}{18}(a-b)\,t\,(-9(a+b)+2(-8+3a+3b)\,\delta),\ \frac{1}{9}(a-b)(-8+3a+3b)\,t\,\delta\right\}$$

$$\left\{\frac{1}{9}(a-b)(-8+5a+5b)\,t\,\delta,\ -\frac{1}{9}(-a+b)\,t\,(-8\delta+(a+b)(9+5\delta))\right\}$$

$$\left\{-\frac{1}{18}(a-b)\,t\,(9(a+b)+16\delta),\ \frac{8}{9}(-a+b)\,t\,\delta\right\}$$

First, we compare the elements of the left-hand endpoint in (ii) and the right-hand endpoint in (i):

Factor$\left[\frac{1}{18}(a-b)\,t\,(-9(a+b)+2(-8+3a+3b)\,\delta)-\left(-\frac{1}{18}(a-b)\,t\,(9(a+b)+16\delta)\right)\right]$

Factor$\left[\frac{1}{9}(a-b)(-8+3a+3b)\,t\,\delta-\frac{8}{9}(-a+b)\,t\,\delta\right]$

$$\frac{1}{3}(a-b)(a+b)\,t\,\delta$$

$$\frac{1}{3}(a-b)(a+b)\,t\,\delta$$

The outcome means that for any a∈[0,1], b∈[0,1] (a≥b), t, and δ, the left-hand endpoint in (ii) is located below the right-hand endpoint in (i) as in the above Figure.

Second, we compare the elements of the left-hand endpoint in (ii) and the left-hand endpoint in (i):

Factor$\left[\frac{1}{18}(a-b)\,t\,(-9(a+b)+2(-8+3a+3b)\,\delta)-\frac{1}{9}(a-b)(-8+5a+5b)\,t\,\delta\right]$

Factor$\left[\frac{1}{9}(a-b)(-8+3a+3b)\,t\,\delta-\left(-\frac{1}{9}(-a+b)\,t\,(-8\delta+(a+b)(9+5\delta))\right)\right]$

$$-\frac{1}{18}(a-b)(a+b)\,t\,(9+4\delta)$$

$$-\frac{1}{9}(a-b)(a+b)\,t\,(9+2\delta)$$

The outcome means that for any a∈[0,1], b∈[0,1] (a≥b), t, and δ, the left-hand endpoint in (ii) is located above the left-hand endpoint in (i) as in the above Figure.

We need to find the global optimal price of Firm A, p_A, when there are two local optima for a given p_B. There is a price p_B such that choosing the reaction function in (ii) and choosing the reaction function in (i) are indifferent for Firm A. This p_B is the threshold for which choosing the reaction function in (ii) is preferred by Firm A if p_B is larger than this threshold p_B, otherwise, choosing the reaction function in (i) is preferred by Firm A. We need to find the threshold value of p_B.

To check the threshold value of p_B for Firm A's reaction function, we derive the profits under cases (ii) and (i).

The interior profit of firm A under case (ii) for p_B is

```
Factor[pA z + δf 1/9 (b - a) t (2 + 2 a + a^2 + 2 b + 2 a b + b^2 - 8 z - 2 a z - 2 b z + 10 z^2) /.
  z → (3 (pB - pA))/(2 (b - a) t (3 + δc)) + ((a + b) (3 - δc) + 2 δc)/(2 (3 + δc)) /. {pA →
   ((9 + 3 δc - 10 δf) pB + (b - a) t ((a + b) (9 - δc^2 - 8 δf + 4 δc δf) + 2 (3 δc + δc^2 + 4 δf - 2 δc δf))) /
    (2 (9 + 3 δc - 5 δf))}]
```

$$-\frac{1}{72\,(a-b)\,t\,(-9+5\,\delta)}\big(-81\,a^4\,t^2+162\,a^2\,b^2\,t^2-81\,b^4\,t^2-144\,a^2\,t^2\,\delta-36\,a^4\,t^2\,\delta+288\,a\,b\,t^2\,\delta-144\,b^2\,t^2\,\delta+72\,a^2\,b^2\,t^2\,\delta-36\,b^4\,t^2\,\delta+16\,a^2\,t^2\,\delta^2+48\,a^3\,t^2\,\delta^2+36\,a^4\,t^2\,\delta^2-32\,a\,b\,t^2\,\delta^2-48\,a^2\,b\,t^2\,\delta^2+16\,b^2\,t^2\,\delta^2-48\,a\,b^2\,t^2\,\delta^2-72\,a^2\,b^2\,t^2\,\delta^2+48\,b^3\,t^2\,\delta^2+36\,b^4\,t^2\,\delta^2+162\,a^2\,t\,p_B-162\,b^2\,t\,p_B-144\,a\,t\,\delta\,p_B-36\,a^2\,t\,\delta\,p_B+144\,b\,t\,\delta\,p_B+36\,b^2\,t\,\delta\,p_B-81\,p_B^2\big)$$

The interior profit of firm A under case (i) for p_B is

```
Factor[pA z + δf (1/18 (b - a) t (4 + 4 a + a^2 + 4 b + 2 a b + b^2 - 16 z + 10 a z + 10 b z - 20 z^2)) /.
  z → (-3 a^2 t + 3 b^2 t - 2 a t δc + 2 a^2 t δc + 2 b t δc - 2 b^2 t δc - 3 pA + 3 pB) / (2 (a - b) t (-3 + δc)) /.
  pA → 1/(2 (-9 + 3 δc - 5 δf)) ((a - b) t (6 δc - 2 δc^2 + 8 δf + 4 δc δf + b (9 - 9 δc + 2 δc^2 + 5 δf - 5 δc δf) +
       a (9 + 2 δc^2 + 5 δf - δc (9 + 5 δf))) + (-9 + 3 δc - 10 δf) pB)]
```

$$-\frac{1}{8\,(a-b)\,t\,(9+5\,\delta)}\big(9\,a^4\,t^2-18\,a^2\,b^2\,t^2+9\,b^4\,t^2+16\,a^2\,t^2\,\delta+14\,a^4\,t^2\,\delta-32\,a\,b\,t^2\,\delta+16\,b^2\,t^2\,\delta-28\,a^2\,b^2\,t^2\,\delta+14\,b^4\,t^2\,\delta+16\,a^2\,t^2\,\delta^2+5\,a^4\,t^2\,\delta^2-32\,a\,b\,t^2\,\delta^2+16\,b^2\,t^2\,\delta^2-10\,a^2\,b^2\,t^2\,\delta^2+5\,b^4\,t^2\,\delta^2-18\,a^2\,t\,p_B+18\,b^2\,t\,p_B+16\,a\,t\,\delta\,p_B-10\,a^2\,t\,\delta\,p_B-16\,b\,t\,\delta\,p_B+10\,b^2\,t\,\delta\,p_B+9\,p_B^2\big)$$

We derive the threshold value of p_B by finding p_B that equalizes the above two profits:

```
FullSimplify[
 Solve[{-1/(72 (a - b) t (-9 + 5 δ)) (-81 a^4 t^2 + 162 a^2 b^2 t^2 - 81 b^4 t^2 - 144 a^2 t^2 δ - 36 a^4 t^2 δ + 288 a b t^2 δ -
       144 b^2 t^2 δ + 72 a^2 b^2 t^2 δ - 36 b^4 t^2 δ + 16 a^2 t^2 δ^2 + 48 a^3 t^2 δ^2 + 36 a^4 t^2 δ^2 - 32 a b t^2 δ^2 -
       48 a^2 b t^2 δ^2 + 16 b^2 t^2 δ^2 - 48 a b^2 t^2 δ^2 - 72 a^2 b^2 t^2 δ^2 + 48 b^3 t^2 δ^2 + 36 b^4 t^2 δ^2 +
       162 a^2 t pB - 162 b^2 t pB - 144 a t δ pB - 36 a^2 t δ pB + 144 b t δ pB + 36 b^2 t δ pB - 81 pB^2) ==
    -1/(8 (a - b) t (9 + 5 δ)) (9 a^4 t^2 - 18 a^2 b^2 t^2 + 9 b^4 t^2 + 16 a^2 t^2 δ + 14 a^4 t^2 δ - 32 a b t^2 δ + 16 b^2 t^2 δ -
       28 a^2 b^2 t^2 δ + 14 b^4 t^2 δ + 16 a^2 t^2 δ^2 + 5 a^4 t^2 δ^2 - 32 a b t^2 δ^2 + 16 b^2 t^2 δ^2 - 10 a^2 b^2 t^2 δ^2 +
       5 b^4 t^2 δ^2 - 18 a^2 t pB + 18 b^2 t pB + 16 a t δ pB - 10 a^2 t δ pB - 16 b t δ pB + 10 b^2 t δ pB + 9 pB^2)}, pB]]
```

$$\left\{\left\{p_B\to\frac{1}{90\,\delta}\,t\left(-80\,a\,\delta^2+80\,b\,\delta^2+3\,a^2\,\delta\,(9+5\,\delta)-3\,b^2\,\delta\,(9+5\,\delta)+3\left(81-25\,\delta^2\right)\sqrt{-\frac{\left(a^2-b^2\right)^2\delta^2}{-81+25\,\delta^2}}\right)\right\},\right.$$

$$\left.\left\{p_B\to\frac{1}{90\,\delta}\,t\left(-80\,a\,\delta^2+80\,b\,\delta^2+3\,a^2\,\delta\,(9+5\,\delta)-3\,b^2\,\delta\,(9+5\,\delta)+3\sqrt{-\frac{\left(a^2-b^2\right)^2\delta^2}{-81+25\,\delta^2}}\left(-81+25\,\delta^2\right)\right)\right\}\right\}$$

We can easily show that the latter outcome is negative. So, we use the former one.
We simplify the expression of the latter outcome, and obtain the following p_B:

$$p_B\to\frac{t\,(b-a)\,\delta\left((80\,\delta-3\,(a+b)\,(9+5\,\delta))+3\,(a+b)\,\sqrt{81-25\,\delta^2}\right)}{90\,\delta}\quad\text{(pb1)}$$

We rewrite the locations of the two endpoints: The right-hand endpoint in (i), the left-hand endpoints in (ii) (see below)

```
c1aR
c2aL
```

$$\left\{-\frac{1}{18}(a-b)\,t\,(9(a+b)+16\,\delta),\ \frac{8}{9}(-a+b)\,t\,\delta\right\}$$

$$\left\{\frac{1}{18}(a-b)\,t\,(-9(a+b)+2(-8+3a+3b)\,\delta),\ \frac{1}{9}(a-b)(-8+3a+3b)\,t\,\delta\right\}$$

We check the condition that the derived p_B (pb1) is below the p_B-element of the right-hand endpoints in (i).

$$\text{Simplify}\left[\text{Factor}\left[\frac{8}{9}(-a+b)\,t\,\delta-\frac{t\,(b-a)\,\delta\left((80\,\delta-3(a+b)(9+5\,\delta))+3(a+b)\sqrt{81-25\,\delta^2}\right)}{90\,\delta}\right]\right]$$

$$\frac{1}{30}(a-b)(a+b)\,t\left(-9-5\,\delta+\sqrt{81-25\,\delta^2}\right)$$

This is positive.

We also rewrite the location of the endpoint: The left endpoints in (i) (see below)

```
c1aL
```

$$\left\{\frac{1}{9}(a-b)(-8+5a+5b)\,t\,\delta,\ -\frac{1}{9}(-a+b)\,t\,(-8\,\delta+(a+b)(9+5\,\delta))\right\}$$

We check the condition that the derived p_B (pb1) is above the p_B-element of the left endpoints in (i).

$$\text{Simplify}\left[\text{Factor}\left[\frac{t\,(b-a)\,\delta\left((80\,\delta-3(a+b)(9+5\,\delta))+3(a+b)\sqrt{81-25\,\delta^2}\right)}{90\,\delta}-\left(-\frac{1}{9}(-a+b)\,t\,(-8\,\delta+(a+b)(9+5\,\delta))\right)\right]\right]$$

$$-\frac{1}{90}(a-b)(a+b)\,t\left(63+35\,\delta+3\sqrt{81-25\,\delta^2}\right)$$

This is positive.

For the threshold value of p_B (pb1), the point of p_A-element in Firm A's reaction function in (i) is

$$\text{Simplify}\Big[\text{Expand}\Big[p_A\to\frac{1}{2(-9+3\,\delta c-5\,\delta f)}\Big((a-b)\,t\left(6\,\delta c-2\,\delta c^2+8\,\delta f+4\,\delta c\,\delta f+b\left(9-9\,\delta c+2\,\delta c^2+5\,\delta f-5\,\delta c\,\delta f\right)+a\left(9+2\,\delta c^2+5\,\delta f-\delta c\,(9+5\,\delta f)\right)\right)+(-9+3\,\delta c-10\,\delta f)\,p_B\Big)\ /.\ p_B\to\frac{1}{90\,\delta}t\,(b-a)\,\delta\left((80\,\delta-3(a+b)(9+5\,\delta))+3(a+b)\sqrt{81-25\,\delta^2}\right)\Big]\Big]$$

$$p_A\to-\frac{1}{180(9+5\,\delta)}(a-b)\,t\left(160\,\delta\,(9+5\,\delta)+3a\left(-50\,\delta^2+9\left(21+\sqrt{81-25\,\delta^2}\right)+5\,\delta\left(3+2\sqrt{81-25\,\delta^2}\right)\right)+3b\left(-50\,\delta^2+9\left(21+\sqrt{81-25\,\delta^2}\right)+5\,\delta\left(3+2\sqrt{81-25\,\delta^2}\right)\right)\right)$$

$$p_A\to\frac{(b-a)\,t}{180(9+5\,\delta)}\left(160\,\delta\,(9+5\,\delta)+3(a+b)\left((9+5\,\delta)(21-10\,\delta)+(9+10\,\delta)\sqrt{81-25\,\delta^2}\right)\right)$$

The jumping point of Firm A's reaction function in (i) is defined as c1ja

```
c1ja = { (b - a) t / (180 (9 + 5 δ)) (160 δ (9 + 5 δ) + 3 (a + b) ((9 + 5 δ) (21 - 10 δ) + (9 + 10 δ) Sqrt[81 - 25 δ^2])),
  t (b - a) δ ((80 δ - 3 (a + b) (9 + 5 δ)) + 3 (a + b) Sqrt[81 - 25 δ^2]) / (90 δ)}
```

$$\left\{\frac{1}{180\,(9+5\,\delta)}\,(-a+b)\,t\left(160\,\delta\,(9+5\,\delta)+3\,(a+b)\left((21-10\,\delta)\,(9+5\,\delta)+(9+10\,\delta)\sqrt{81-25\,\delta^2}\right)\right),\ \frac{1}{90}\,(-a+b)\,t\left(80\,\delta-3\,(a+b)\,(9+5\,\delta)+3\,(a+b)\sqrt{81-25\,\delta^2}\right)\right\}$$

Also, for the threshold value of p_B (pb1), the point of p_A-element in Firm A's reaction function in (ii) is

```
Simplify[Expand[pA →
    ((9 + 3 δc - 10 δf) pB + (b - a) t ((a + b) (9 - δc^2 - 8 δf + 4 δc δf) + 2 (3 δc + δc^2 + 4 δf - 2 δc δf))) /
    (2 (9 + 3 δc - 5 δf)) /. pB → 1/(90 δ) t (b - a) δ ((80 δ - 3 (a + b) (9 + 5 δ)) + 3 (a + b) Sqrt[81 - 25 δ^2])]]
```

$$p_A \to \frac{1}{180\,(-9+5\,\delta)}\,(a-b)\,t\left(160\,(9-5\,\delta)\,\delta+3\,a\left(50\,\delta^2+9\left(21+\sqrt{81-25\,\delta^2}\right)-5\,\delta\left(39+2\sqrt{81-25\,\delta^2}\right)\right)+3\,b\left(50\,\delta^2+9\left(21+\sqrt{81-25\,\delta^2}\right)-5\,\delta\left(39+2\sqrt{81-25\,\delta^2}\right)\right)\right)$$

```
pA → (b - a) t / (180 (9 - 5 δ)) (160 (9 - 5 δ) δ + 3 (a + b) ((9 - 5 δ) (21 - 10 δ) + (9 - 10 δ) Sqrt[81 - 25 δ^2]))
```

The jumping point of Firm A's reaction function in (ii) is defined as c2ja

```
c2ja = { (b - a) t / (180 (9 - 5 δ)) (160 (9 - 5 δ) δ + 3 (a + b) ((9 - 5 δ) (21 - 10 δ) + (9 - 10 δ) Sqrt[81 - 25 δ^2])),
  t (b - a) δ ((80 δ - 3 (a + b) (9 + 5 δ)) + 3 (a + b) Sqrt[81 - 25 δ^2]) / (90 δ)}
```

$$\left\{\frac{1}{180\,(9-5\,\delta)}\,(-a+b)\,t\left(160\,(9-5\,\delta)\,\delta+3\,(a+b)\left((21-10\,\delta)\,(9-5\,\delta)+(9-10\,\delta)\sqrt{81-25\,\delta^2}\right)\right),\ \frac{1}{90}\,(-a+b)\,t\left(80\,\delta-3\,(a+b)\,(9+5\,\delta)+3\,(a+b)\sqrt{81-25\,\delta^2}\right)\right\}$$

Next we show various examples of Firm A's 'true' reaction function for different values of a, b, t, δ, and k.

```
a = 0
b = 1
δ = 1 / 2
t = 1
k = 2
```

0

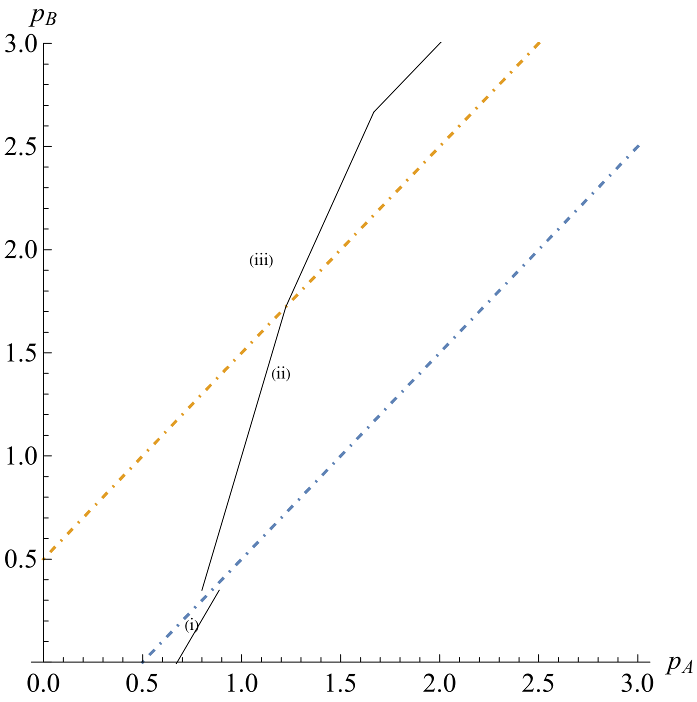
1

$\frac{1}{2}$

1

2

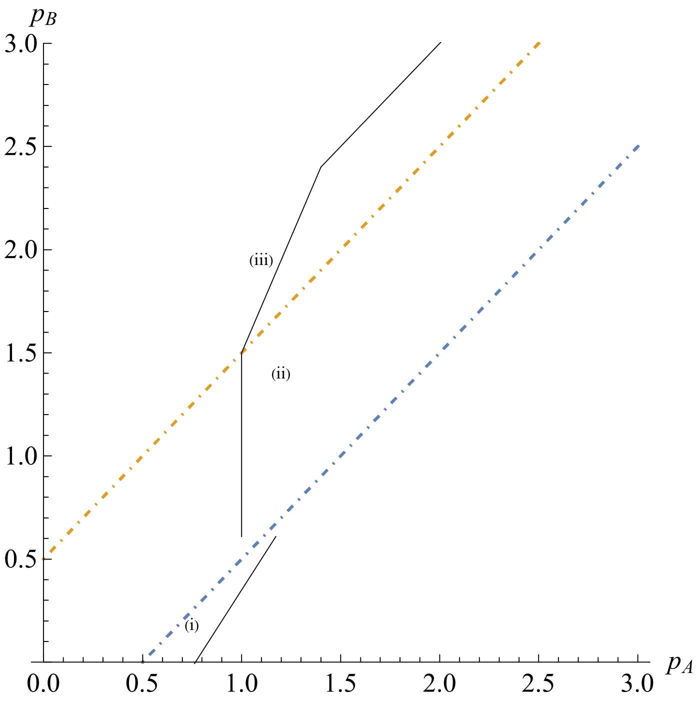
```
Plot[{x - (-3 a² t + 3 b² t - 4 a t δc + 3 a² t δc + 4 b t δc - 3 b² t δc) / 6,
  x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3},
 Epilog → {Line[{c1aL, c1ja}], Line[{c2ja, c2aR}], Line[{c3aL, c3aR}], Line[{c3daL, c3daR}],
   Text["(iii)", {1.1, 1.95}], Text["(ii)", {1.2, 1.4}], Text["(i)", {0.75, 0.18}]},
 PlotRange → {0, 3}, LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"},
 AspectRatio → 1, PlotStyle → DotDashed]
```




```
a = 0
b = 1
δ = 0.9
t = 1
k = 2
```

```
0

1

0.9

1

2
```

```
Plot[{x - (-3 a² t + 3 b² t - 4 a t δc + 3 a² t δc + 4 b t δc - 3 b² t δc) / 6,
  x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3},
 Epilog → {Line[{c1aL, c1ja}], Line[{c2ja, c2aR}], Line[{c3aL, c3aR}], Line[{c3daL, c3daR}],
   Text["(iii)", {1.1, 1.95}], Text["(ii)", {1.2, 1.4}], Text["(i)", {0.75, 0.18}]},
 PlotRange → {0, 3}, LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"},
 AspectRatio → 1, PlotStyle → DotDashed]
```




```
Clear[a, b, δ, t, k]
```

We now turn to Firm B's reaction function in the three cases. As before, we start with an example by setting *a=0*, *b=1*, *t=1*, and *δ=1/2*:

```
a = 0
b = 1
δ = 4 / 5
t = 1
k = 1
```

0

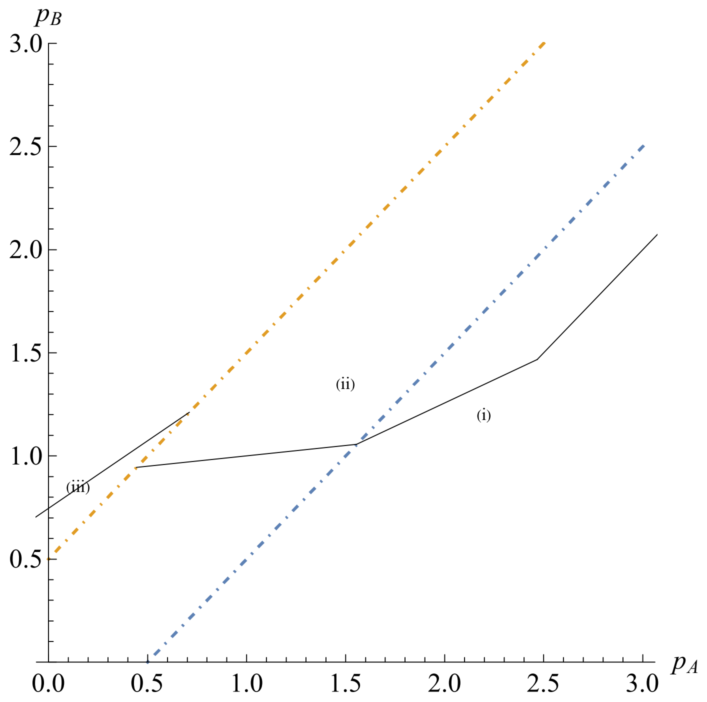
1

$\frac{4}{5}$

1

1

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
  x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3},
 Epilog → {Line[{c1dbL, c1dbR}], Line[{c1bL, c1bR}], Line[{c2bL, c2bR}], Line[{c3bL, c3bR}],
   Text["(iii)", {0.15, 0.85}], Text["(ii)", {1.5, 1.35}], Text["(i)", {2.2, 1.2}]},
 PlotRange → {0, 3}, LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"},
 AspectRatio → 1, PlotStyle → DotDashed]
```



We find that for some p_A, there are two local optimal prices for Firm B. We can show that the multiplicity of local optimal prices always appears.

```
Clear[a, b, δ, t, k]
```

To show the multiplicity, we check the locations of the two endpoints: The left-hand endpoint in (ii) (c2bL) and the left-hand and right-hand endpoints (c3bL and c3bR) in (iii) (see below)

```
c2bL
c3bL
c3bR
```

$$\left\{\frac{1}{9}(-a+b)(2+3a+3b)\,t\,\delta,\ \frac{1}{18}(a-b)\,t\,(9(-2+a+b)-2(2+3a+3b)\,\delta)\right\}$$

$$\left\{\frac{1}{9}(-a+b)\,t\,(-2(9+\delta)+(a+b)(9+5\delta)),\ -\frac{1}{9}(a-b)(-2+5a+5b)\,t\,\delta\right\}$$

$$\left\{\frac{8}{9}(-a+b)\,t\,\delta,\ \frac{1}{18}(a-b)\,t\,(9(-2+a+b)-16\delta)\right\}$$

First, we compare the elements of the left-hand endpoint in (ii) and the right-hand endpoint in (iii):

Factor$\left[\frac{1}{9}(-a+b)(2+3a+3b)\,t\,\delta-\frac{8}{9}(-a+b)\,t\,\delta\right]$

Factor$\left[\frac{1}{18}(a-b)\,t\,(9(-2+a+b)-2(2+3a+3b)\delta)-\left(\frac{1}{18}(a-b)\,t\,(9(-2+a+b)-16\delta)\right)\right]$

$-\frac{1}{3}(a-b)(-2+a+b)\,t\,\delta$

$-\frac{1}{3}(a-b)(-2+a+b)\,t\,\delta$

The outcome means that for any a∈[0,1], b∈[0,1] (a≥b), t, and δ, the right-hand endpoint in (iii) is located above the left-hand endpoint in (ii) as in the above Figure.

Second, we compare the elements of the left-hand endpoint in (ii) and the left-hand endpoint in (iii):

Factor$\left[\frac{1}{9}(-a+b)(2+3a+3b)\,t\,\delta-\frac{1}{9}(-a+b)\,t\,(-2(9+\delta)+(a+b)(9+5\delta))\right]$

Factor$\left[\frac{1}{18}(a-b)\,t\,(9(-2+a+b)-2(2+3a+3b)\delta)-\left(-\frac{1}{9}(a-b)(-2+5a+5b)\,t\,\delta\right)\right]$

$\frac{1}{9}(a-b)(-2+a+b)\,t\,(9+2\delta)$

$\frac{1}{18}(a-b)(-2+a+b)\,t\,(9+4\delta)$

The outcome means that for any a∈[0,1], b∈[0,1] (a≥b), t, and δ, the left-hand endpoint in (ii) is located above the left-hand endpoint in (iii) as in the above Figure.

We need to find the global optimal price of Firm B, p_B, when there are two local optima for a given p_A. There is a price p_A such that choosing the reaction function in (ii) and choosing the reaction function in (iii) are indifferent for Firm B. This p_A is the threshold in which choosing the reaction function in (ii) is preferred by Firm B if p_A is larger than the threshold p_A, otherwise, choosing the reaction function in (iii) is preferred by Firm B. We need to find the threshold value of p_A.

To check the threshold value of p_A for Firm B's reaction function, we derive the profits under cases (ii) and (iii).
The interior profit of firm B under case (ii) for p_A is

Factor$\left[p_B(1-z)+\delta f\,\frac{1}{9}(b-a)\,t\,\left(8-4a+a^2-4b+2ab+b^2-8z-2az-2bz+10z^2\right)\;/.\right.$

$z\to\frac{3(p_B-p_A)}{2(b-a)\,t\,(3+\delta c)}+\frac{((a+b)(3-\delta c)+2\delta c)}{2(3+\delta c)}\;/.$

$\left\{p_B\to\frac{1}{2(9+3\delta c-5\delta f)}\left(-(a-b)\,t\,\left(18+6\delta c-8\delta f+4\delta c\,\delta f+a\left(-9+\delta c^2+8\delta f-4\delta c\,\delta f\right)+b\left(-9+\delta c^2+8\delta f-4\delta c\,\delta f\right)\right)+(9+3\delta c-10\delta f)\,p_A\right)\right\}\Big]$

$$-\frac{1}{72(a-b)\,t\,(-9+5\delta)}\big(-324a^2t^2+324a^3t^2-81a^4t^2+648abt^2-324a^2bt^2-324b^2t^2-324ab^2t^2+162a^2b^2t^2+324b^3t^2-81b^4t^2-288a^2t^2\delta+144a^3t^2\delta-36a^4t^2\delta+576abt^2\delta-144a^2bt^2\delta-288b^2t^2\delta-144ab^2t^2\delta+72a^2b^2t^2\delta+144b^3t^2\delta-36b^4t^2\delta+256a^2t^2\delta^2-192a^3t^2\delta^2+36a^4t^2\delta^2-512abt^2\delta^2+192a^2bt^2\delta^2+256b^2t^2\delta^2+192ab^2t^2\delta^2-72a^2b^2t^2\delta^2-192b^3t^2\delta^2+36b^4t^2\delta^2+324atp_A-162a^2tp_A-324btp_A+162b^2tp_A-216at\delta p_A+36a^2t\delta p_A+216bt\delta p_A-36b^2t\delta p_A-81p_A^2\big)$$

The interior profit of firm B under case (iii) for p_B

```
Factor[pB (1 - z) + δf (1/18 (b - a) t (-18 a + a^2 - 18 b + 2 a b + b^2 + 36 z + 10 a z + 10 b z - 20 z^2)) /.
   z → (-3 a^2 t + 3 b^2 t + 2 a^2 t δc - 2 b^2 t δc - 3 pA + 3 pB)/(2 (a - b) t (-3 + δc)) /. pB → (9 - 3 δc + 10 δf) pA/(2 (9 - 3 δc + 5 δf)) +
    ((b - a) t (2 (3 - δc) (3 - δc + 3 δf) - (a + b) ((3 - δc) (3 - 2 δc) + 5 (1 - δc) δf))) /
     (2 (9 - 3 δc + 5 δf))]
```

$$-\frac{1}{8\,(a-b)\,t\,(9+5\,\delta)}\left(36\,a^2\,t^2-36\,a^3\,t^2+9\,a^4\,t^2-72\,a\,b\,t^2+36\,a^2\,b\,t^2+36\,b^2\,t^2+36\,a\,b^2\,t^2-18\,a^2\,b^2\,t^2-36\,b^3\,t^2+9\,b^4\,t^2+72\,a^2\,t^2\,\delta-56\,a^3\,t^2\,\delta+14\,a^4\,t^2\,\delta-144\,a\,b\,t^2\,\delta+56\,a^2\,b\,t^2\,\delta+72\,b^2\,t^2\,\delta+56\,a\,b^2\,t^2\,\delta-28\,a^2\,b^2\,t^2\,\delta-56\,b^3\,t^2\,\delta+14\,b^4\,t^2\,\delta+36\,a^2\,t^2\,\delta^2-20\,a^3\,t^2\,\delta^2+5\,a^4\,t^2\,\delta^2-72\,a\,b\,t^2\,\delta^2+20\,a^2\,b\,t^2\,\delta^2+36\,b^2\,t^2\,\delta^2+20\,a\,b^2\,t^2\,\delta^2-10\,a^2\,b^2\,t^2\,\delta^2-20\,b^3\,t^2\,\delta^2+5\,b^4\,t^2\,\delta^2-36\,a\,t\,p_A+18\,a^2\,t\,p_A+36\,b\,t\,p_A-18\,b^2\,t\,p_A-4\,a\,t\,\delta\,p_A+10\,a^2\,t\,\delta\,p_A+4\,b\,t\,\delta\,p_A-10\,b^2\,t\,\delta\,p_A+9\,p_A^2\right)$$

We derive the threshold value of p_B by finding p_B that equalizes the above two profits:

```
FullSimplify[
 Solve[{-1/(72 (a - b) t (-9 + 5 δ)) (-324 a^2 t^2 + 324 a^3 t^2 - 81 a^4 t^2 + 648 a b t^2 - 324 a^2 b t^2 - 324 b^2 t^2 -
      324 a b^2 t^2 + 162 a^2 b^2 t^2 + 324 b^3 t^2 - 81 b^4 t^2 - 288 a^2 t^2 δ + 144 a^3 t^2 δ - 36 a^4 t^2 δ + 576 a b t^2 δ -
      144 a^2 b t^2 δ - 288 b^2 t^2 δ - 144 a b^2 t^2 δ + 72 a^2 b^2 t^2 δ + 144 b^3 t^2 δ - 36 b^4 t^2 δ + 256 a^2 t^2 δ^2 -
      192 a^3 t^2 δ^2 + 36 a^4 t^2 δ^2 - 512 a b t^2 δ^2 + 192 a^2 b t^2 δ^2 + 256 b^2 t^2 δ^2 + 192 a b^2 t^2 δ^2 -
      72 a^2 b^2 t^2 δ^2 - 192 b^3 t^2 δ^2 + 36 b^4 t^2 δ^2 + 324 a t pA - 162 a^2 t pA - 324 b t pA + 162 b^2 t pA -
      216 a t δ pA + 36 a^2 t δ pA + 216 b t δ pA - 36 b^2 t δ pA - 81 pA^2) == -1/(8 (a - b) t (9 + 5 δ))
    (36 a^2 t^2 - 36 a^3 t^2 + 9 a^4 t^2 - 72 a b t^2 + 36 a^2 b t^2 + 36 b^2 t^2 + 36 a b^2 t^2 - 18 a^2 b^2 t^2 - 36 b^3 t^2 +
      9 b^4 t^2 + 72 a^2 t^2 δ - 56 a^3 t^2 δ + 14 a^4 t^2 δ - 144 a b t^2 δ + 56 a^2 b t^2 δ + 72 b^2 t^2 δ + 56 a b^2 t^2 δ -
      28 a^2 b^2 t^2 δ - 56 b^3 t^2 δ + 14 b^4 t^2 δ + 36 a^2 t^2 δ^2 - 20 a^3 t^2 δ^2 + 5 a^4 t^2 δ^2 - 72 a b t^2 δ^2 +
      20 a^2 b t^2 δ^2 + 36 b^2 t^2 δ^2 + 20 a b^2 t^2 δ^2 - 10 a^2 b^2 t^2 δ^2 - 20 b^3 t^2 δ^2 + 5 b^4 t^2 δ^2 - 36 a t pA +
      18 a^2 t pA + 36 b t pA - 18 b^2 t pA - 4 a t δ pA + 10 a^2 t δ pA + 4 b t δ pA - 10 b^2 t δ pA + 9 pA^2)}, pA]]
```

$$\left\{\left\{p_A\to\frac{1}{90\,\delta}\,t\left(2\,a\,(27-25\,\delta)\,\delta-3\,a^2\,\delta\,(9+5\,\delta)+3\,b^2\,\delta\,(9+5\,\delta)+2\,b\,\delta\,(-27+25\,\delta)+3\left(81-25\,\delta^2\right)\sqrt{-\frac{(a-b)^2\,(-2+a+b)^2\,\delta^2}{-81+25\,\delta^2}}\right)\right\},\right.$$
$$\left.\left\{p_A\to\frac{1}{90\,\delta}\,t\left(2\,a\,(27-25\,\delta)\,\delta-3\,a^2\,\delta\,(9+5\,\delta)+3\,b^2\,\delta\,(9+5\,\delta)+2\,b\,\delta\,(-27+25\,\delta)+3\sqrt{-\frac{(a-b)^2\,(-2+a+b)^2\,\delta^2}{-81+25\,\delta^2}}\left(-81+25\,\delta^2\right)\right)\right\}\right\}$$

```
pA → 1/(90 δ) t (b - a) δ ( (-54 + 50 δ + 3 (a + b) (9 + 5 δ)) + 3 (2 - (a + b)) √(81 - 25 δ^2) )
```

We can easily show that the latter outcome is negative. So, we use the former one.
We simplify the expression of the latter outcome, and obtain the following p_B:

```
pA → 1/(90 δ) t (b - a) δ ( (-54 + 50 δ + 3 (a + b) (9 + 5 δ)) + 3 (2 - (a + b)) √(81 - 25 δ^2) ) (pa1)
```

We rewrite the locations of the two endpoints: The right endpoint in (iii) and the left endpoint in (ii) (see below)

```
c3bR
c2bL
```

$$\left\{\frac{8}{9}(-a+b)\,t\,\delta,\ \frac{1}{18}(a-b)\,t\,(9\,(-2+a+b)-16\,\delta)\right\}$$

$$\left\{\frac{1}{9}(-a+b)\,(2+3a+3b)\,t\,\delta,\ \frac{1}{18}(a-b)\,t\,(9\,(-2+a+b)-2\,(2+3a+3b)\,\delta)\right\}$$

We show that the derived p_A (pa1) is smaller than the p_A-element of the right-hand endpoints in (iii).

$$\text{Simplify}\left[\text{Factor}\left[\frac{8}{9}(-a+b)\,t\,\delta-\frac{1}{90\,\delta}\,t\,(b-a)\,\delta\left((-54+50\,\delta+3\,(a+b)\,(9+5\,\delta))+3\,(2-(a+b))\sqrt{(81-25\,\delta^2)}\right)\right]\right]$$

$$-\frac{1}{30}(a-b)\,(-2+a+b)\,t\left(-9-5\,\delta+\sqrt{81-25\,\delta^2}\right)$$

This is positive (note that $-9-5\,\delta+\sqrt{81-25\,\delta^2}$ is negative).

We show that the derived p_A (pa1) is larger than the p_A-element of the left-hand endpoints in (ii).

$$\text{Simplify}\left[\text{Factor}\left[\frac{1}{90\,\delta}\,t\,(b-a)\,\delta\left((-54+50\,\delta+3\,(a+b)\,(9+5\,\delta))+3\,(2-(a+b))\sqrt{(81-25\,\delta^2)}\right)-\frac{1}{9}(-a+b)\,(2+3a+3b)\,t\,\delta\right]\right]$$

$$\frac{1}{30}(a-b)\,(-2+a+b)\,t\left(-9+5\,\delta+\sqrt{81-25\,\delta^2}\right)$$

This is positive (note that $-9+5\,\delta+\sqrt{81-25\,\delta^2}$ is positive).

The reaction function of Firm B in (ii) is

$$p_B\to\frac{1}{2\,(9+3\,\delta c-5\,\delta f)}\left(-(a-b)\,t\left(18+6\,\delta c-8\,\delta f+4\,\delta c\,\delta f+a\left(-9+\delta c^2+8\,\delta f-4\,\delta c\,\delta f\right)+b\left(-9+\delta c^2+8\,\delta f-4\,\delta c\,\delta f\right)\right)+(9+3\,\delta c-10\,\delta f)\,p_A\right)$$

We substitute p_A into p_B:

$$\text{Simplify}\left[p_B\to\frac{1}{2\,(9+3\,\delta c-5\,\delta f)}\left(-(a-b)\,t\left(18+6\,\delta c-8\,\delta f+4\,\delta c\,\delta f+a\left(-9+\delta c^2+8\,\delta f-4\,\delta c\,\delta f\right)+b\left(-9+\delta c^2+8\,\delta f-4\,\delta c\,\delta f\right)\right)+(9+3\,\delta c-10\,\delta f)\,p_A\right)\ /.\ p_A\to\frac{1}{90\,\delta}\,t\,(b-a)\,\delta\left((-54+50\,\delta+3\,(a+b)\,(9+5\,\delta))+3\,(2-(a+b))\sqrt{(81-25\,\delta^2)}\right)\right]$$

$$p_B\to\frac{1}{180\,(9-5\,\delta)}(-a+b)\,t\left(90\,(18-8\,\delta+a\,(-9+8\,\delta)+b\,(-9+8\,\delta))+(9-10\,\delta)\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)-3\,(-2+a+b)\sqrt{81-25\,\delta^2}\right)\right)$$

$$\text{c2jb} = \left\{\frac{1}{90\,\delta}\,t\,(b-a)\,\delta\left((-54+50\,\delta+3\,(a+b)\,(9+5\,\delta))+3\,(2-(a+b))\,\sqrt{(81-25\,\delta^2)}\right),\right.$$
$$\frac{1}{180\,(9-5\,\delta)}\,(-a+b)\,t\left(90\,(18-8\,\delta+a\,(-9+8\,\delta)+b\,(-9+8\,\delta))+\right.$$
$$\left.\left.(9-10\,\delta)\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)-3\,(-2+a+b)\,\sqrt{81-25\,\delta^2}\right)\right)\right\}$$

$$\left\{\frac{1}{90}\,(-a+b)\,t\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)+3\,(2-a-b)\,\sqrt{81-25\,\delta^2}\right),\right.$$
$$\frac{1}{180\,(9-5\,\delta)}\,(-a+b)\,t\left(90\,(18-8\,\delta+a\,(-9+8\,\delta)+b\,(-9+8\,\delta))+\right.$$
$$\left.\left.(9-10\,\delta)\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)-3\,(-2+a+b)\,\sqrt{81-25\,\delta^2}\right)\right)\right\}$$

The reaction function of Firm B in (iii) is

$$p_B \to \frac{(9-3\,\delta c+10\,\delta f)\,p_A}{2\,(9-3\,\delta c+5\,\delta f)}+$$
$$((b-a)\,t\,(2\,(3-\delta c)\,(3-\delta c+3\,\delta f)-(a+b)\,((3-\delta c)\,(3-2\,\delta c)+5\,(1-\delta c)\,\delta f)))\,/$$
$$(2\,(9-3\,\delta c+5\,\delta f))$$

We substitute p_A into p_B:

$$\text{Simplify}\left[p_B \to \frac{(9-3\,\delta c+10\,\delta f)\,p_A}{2\,(9-3\,\delta c+5\,\delta f)}+\right.$$
$$((b-a)\,t\,(2\,(3-\delta c)\,(3-\delta c+3\,\delta f)-(a+b)\,((3-\delta c)\,(3-2\,\delta c)+5\,(1-\delta c)\,\delta f)))\,/$$
$$(2\,(9-3\,\delta c+5\,\delta f))\,/.$$
$$\left.p_A \to \frac{1}{90\,\delta}\,t\,(b-a)\,\delta\left((-54+50\,\delta+3\,(a+b)\,(9+5\,\delta))+3\,(2-(a+b))\,\sqrt{(81-25\,\delta^2)}\right)\right]$$

$$p_B \to \frac{1}{180\,(9+5\,\delta)}\,(-a+b)\,t\left(90\,(18\,(1+\delta)-(a+b)\,(9+5\,\delta))+\right.$$
$$\left.(9+10\,\delta)\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)-3\,(-2+a+b)\,\sqrt{81-25\,\delta^2}\right)\right)$$

$$\text{c3jb} = \left\{\frac{1}{90\,\delta}\,t\,(b-a)\,\delta\left((-54+50\,\delta+3\,(a+b)\,(9+5\,\delta))+3\,(2-(a+b))\,\sqrt{(81-25\,\delta^2)}\right),\right.$$
$$\frac{1}{180\,(9+5\,\delta)}\,(-a+b)\,t\left(90\,(18\,(1+\delta)-(a+b)\,(9+5\,\delta))+\right.$$
$$\left.\left.(9+10\,\delta)\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)-3\,(-2+a+b)\,\sqrt{81-25\,\delta^2}\right)\right)\right\}$$

$$\left\{\frac{1}{90}\,(-a+b)\,t\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)+3\,(2-a-b)\,\sqrt{81-25\,\delta^2}\right),\right.$$
$$\frac{1}{180\,(9+5\,\delta)}\,(-a+b)\,t\left(90\,(18\,(1+\delta)-(a+b)\,(9+5\,\delta))+\right.$$
$$\left.\left.(9+10\,\delta)\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)-3\,(-2+a+b)\,\sqrt{81-25\,\delta^2}\right)\right)\right\}$$

```
a = 0
b = 1
δ = 1 / 2
t = 1
k = 2
```

0

1

$\frac{1}{2}$

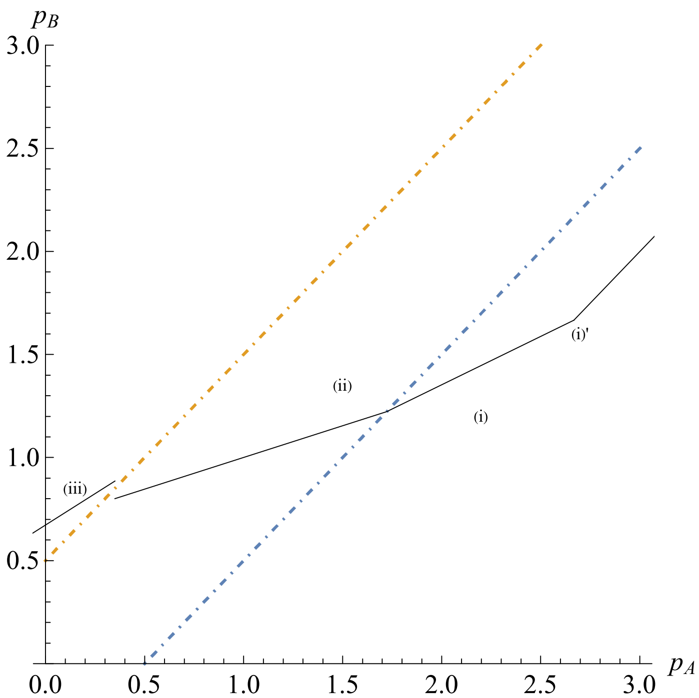
1

2

```
c1dbL
```

$\left\{\frac{1}{9}\ (a-b)\ (2+a+b)\ t\ (-9+4\ \delta),\ -\frac{2}{9}\ (a-b)\ t\ (9+(-7+a+b)\ \delta)\right\}$

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
   x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3},
  Epilog → {Line[{c1dbL, c1dbR}], Line[{c1bR, c1bL}], Line[{c2bR, c2jb}],
    Line[{c3jb, c3bL}], Text["(iii)", {0.15, 0.85}], Text["(ii)", {1.5, 1.35}],
    Text["(i)", {2.2, 1.2}], Text["(i)'", {2.7, 1.6}]}, PlotRange → {0, 3},
  LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"}, AspectRatio → 1, PlotStyle → DotDashed]
```



```
a = 0
b = 1
δ = 1 / 2
t = 1
k = 2
```

0

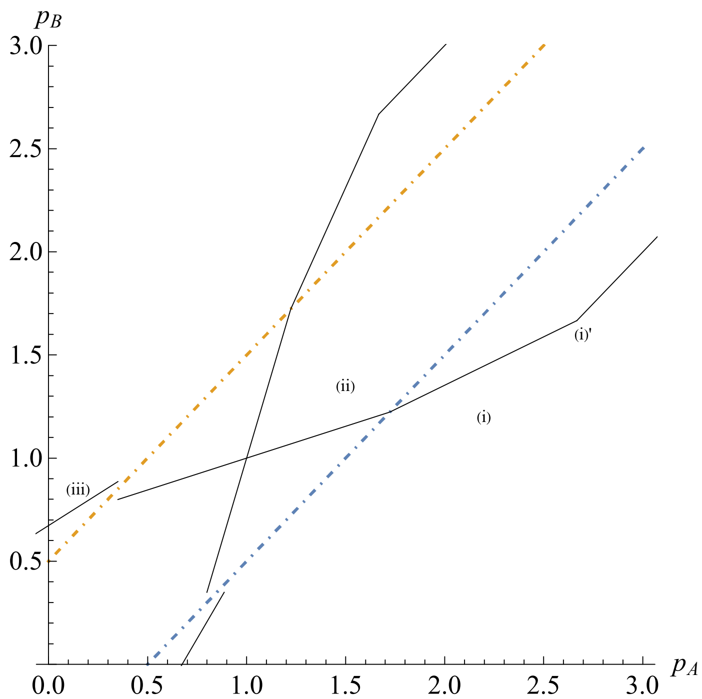
1

$\frac{1}{2}$

1

2

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
   x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3},
  Epilog → {Line[{c1aL, c1ja}], Line[{c2ja, c2aR}], Line[{c3aL, c3aR}],
    Line[{c3daL, c3daR}], , , Line[{c1dbL, c1dbR}], Line[{c1bR, c1bL}],
    Line[{c2bR, c2jb}], Line[{c3jb, c3bL}], , , Text["(iii)", {0.15, 0.85}],
    Text["(ii)", {1.5, 1.35}], Text["(i)", {2.2, 1.2}], Text["(i)'", {2.7, 1.6}]},
  PlotRange → {0, 3}, LabelStyle → (FontSize → 14), AxesLabel → {"p_A", "p_B"},
  AspectRatio → 1, PlotStyle → DotDashed]
```



```
a = 0.95
b = 1
δ = 1 / 2
t = 1
k = 2
```

0.95

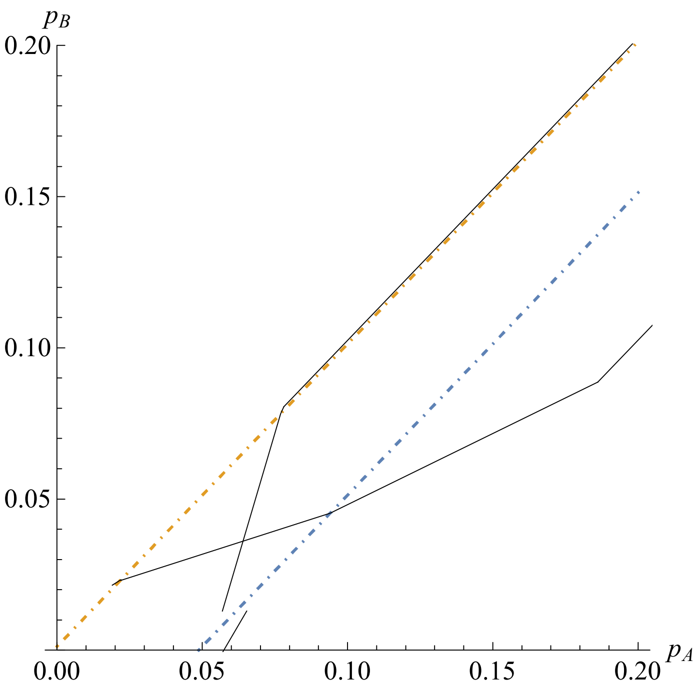
1

$\frac{1}{2}$

1

2

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
   x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 0.2},
 Epilog → {Line[{c1aL, c1ja}], Line[{c2ja, c2aR}], Line[{c3aL, c3aR}],
   Line[{c3daL, c3daR}], , , Line[{c1dbL, c1dbR}], Line[{c1bR, c1bL}],
   Line[{c2bR, c2jb}], Line[{c3jb, c3bL}], , , Text["(iii)", {0.15, 0.85}],
   Text["(ii)", {1.5, 1.35}], Text["(i)", {2.2, 1.2}], Text["(i)'", {2.7, 1.6}]},
 PlotRange → {0, 0.2}, LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"},
 AspectRatio → 1, PlotStyle → DotDashed]
```




```
a = 0.95
b = 1
δ = 1
t = 1
k = 2
```

0.95

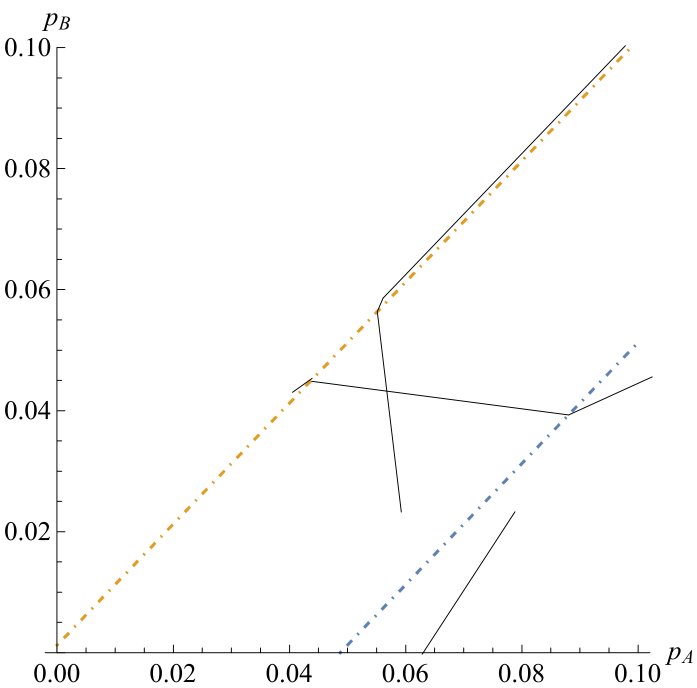
1

1

1

2

```
Plot[{x - (-3 a² t + 3 b² t - 4 a t δc + 3 a² t δc + 4 b t δc - 3 b² t δc) / 6,
  x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 0.1},
 Epilog → {Line[{c1aL, c1ja}], Line[{c2ja, c2aR}], Line[{c3aL, c3aR}],
   Line[{c3daL, c3daR}], , , Line[{c1dbL, c1dbR}], Line[{c1bR, c1bL}],
   Line[{c2bR, c2jb}], Line[{c3jb, c3bL}], , , Text["(iii)", {0.15, 0.85}],
   Text["(ii)", {1.5, 1.35}], Text["(i)", {2.2, 1.2}], Text["(i)'", {2.7, 1.6}]},
 PlotRange → {0, 0.1}, LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"},
 AspectRatio → 1, PlotStyle → DotDashed]
```




```
Clear[a, b, δ, t, k]
```

The above examples suggest that the pricing equilibrium is possible only in (Case ii). We now show that it is indeed the case for general values of (a, b, δ, t, k). To this end, we use the reaction functions corresponding to (Case ii), find the intersection of the two reactions functions, and show that the intersection point is always in (Case ii) for all values of δ.

```
FullSimplify[Solve[{pa ==
    ((9 + 3 δc - 10 δf) pb + (b - a) t ((a + b) (9 - δc² - 8 δf + 4 δc δf) + 2 (3 δc + δc² + 4 δf - 2 δc δf))) /
     (2 (9 + 3 δc - 5 δf)),
   pb == ((9 + 3 δc - 10 δf) pa + (b - a) t (18 + 6 δc - 8 δf + 4 δc δf + (a + b) (-9 + δc² + 8 δf - 4 δc δf))) /
     (2 (9 + 3 δc - 5 δf))}, {pa, pb}]]
```

$$\left\{\left\{pa \to -\frac{(a-b)\ t\ (-9\ (2+a+b)+4\ (3+2\ a+2\ b)\ \delta)}{-27+20\ \delta},\right.\right.$$
$$\left.\left. pb \to \frac{(a-b)\ t\ (-9\ (-4+a+b)+4\ (-7+2\ a+2\ b)\ \delta)}{-27+20\ \delta}\right\}\right\}$$

We can easily show that the derived intersection is always in (Case ii).

We need to show that the intersection is stable even when we consider the reaction functions outside (Case ii).

From the result, if there is an intersection between the reaction functions of Firms A and B in (Case ii), the equilibrium p_B is the following:

$$\text{Eq } p_B : \frac{(a-b)\, t\, (-9\,(-4+a+b)+4\,(-7+2a+2b)\,\delta)}{-27+20\,\delta}$$

We rewrite the jump point of Firm A's reaction function (c1ja):

```
c1ja
```

$$\left\{\frac{1}{180\,(9+5\,\delta)}\,(-a+b)\,t\left(160\,\delta\,(9+5\,\delta)+3\,(a+b)\left((21-10\,\delta)\,(9+5\,\delta)+(9+10\,\delta)\sqrt{81-25\,\delta^2}\right)\right),\ \frac{1}{90}\,(-a+b)\,t\left(80\,\delta-3\,(a+b)\,(9+5\,\delta)+3\,(a+b)\sqrt{81-25\,\delta^2}\right)\right\}$$

If the value of p_B in the jump point is smaller than the equilibrium p_B in (Case ii), the equilibrium point is stable. The difference between them is

$$\text{Simplify}\left[\text{Factor}\left[\frac{(a-b)\,t\,(-9\,(-4+a+b)+4\,(-7+2a+2b)\,\delta)}{-27+20\,\delta}-\frac{1}{90}\,(-a+b)\,t\left(80\,\delta-3\,(a+b)\,(9+5\,\delta)+3\,(a+b)\sqrt{81-25\,\delta^2}\right)\right]\right]$$

$$\frac{1}{90\,(-27+20\,\delta)}(a-b)\,t\left(40\left(81-117\,\delta+40\,\delta^2\right)+a\left(-300\,\delta^2-81\left(1+\sqrt{81-25\,\delta^2}\right)+15\,\delta\left(39+4\sqrt{81-25\,\delta^2}\right)\right)+b\left(-300\,\delta^2-81\left(1+\sqrt{81-25\,\delta^2}\right)+15\,\delta\left(39+4\sqrt{81-25\,\delta^2}\right)\right)\right)$$

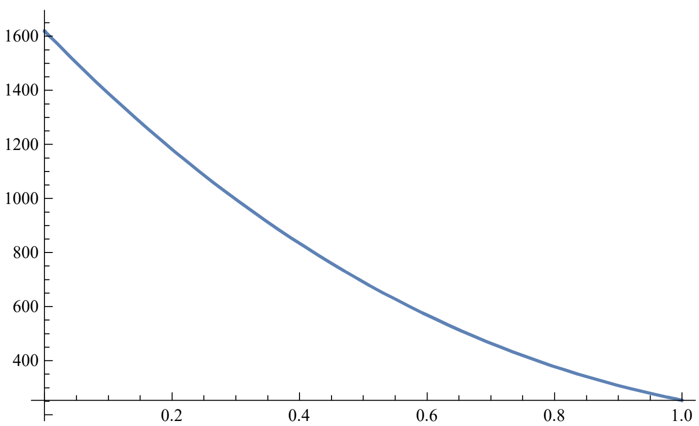
We arrange it, and obtain

$$\frac{1}{90\,(27-20\,\delta)}\,(b-a)\,t\left(40\,(9-5\,\delta)\,(9-8\,\delta)-3\,(a+b)\left((9-5\,\delta)\,(3-20\,\delta)+(27-20\,\delta)\sqrt{81-25\,\delta^2}\right)\right)$$

The sign of this value just depends on δ and g≡a+b.
The value within the largest parentheses is linear in g(=a+b).
This value when g=0 is 40(9-5 δ)(9-8 δ), which positive.
We calculate this value when g=2, and obtain

$$2\,(9-5\,\delta)\,(171-100\,\delta)-6\,(27-20\,\delta)\sqrt{81-25\,\delta^2}$$

$$\text{Plot}\left[2\,(9-5\,\delta)\,(171-100\,\delta)-6\,(27-20\,\delta)\sqrt{81-25\,\delta^2},\ \{\delta, 0, 1\}\right]$$



From the intersection of the reaction functions in (Case ii), the equilibrium p_A is the following:

$$\text{Eq pa} : -\frac{(a-b)\,t\,(-9\,(2+a+b)+4\,(3+2a+2b)\,\delta)}{-27+20\,\delta}$$

We rewrite the jump point of Firm B's reaction function (c3jb):

c3jb

$$\left\{\frac{1}{90}(-a+b)\,t\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)+3\,(2-a-b)\sqrt{81-25\,\delta^2}\right),\right.$$
$$\left.\frac{1}{180\,(9+5\,\delta)}(-a+b)\,t\left(90\,(18\,(1+\delta)-(a+b)\,(9+5\,\delta))+(9+10\,\delta)\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)-3\,(-2+a+b)\sqrt{81-25\,\delta^2}\right)\right)\right\}$$

If the value of p_A in the jump point is smaller than the equilibrium p_A in (Case ii), the equilibrium point is stable. The difference between them is

$$\textbf{Simplify}\left[\textbf{Factor}\left[-\frac{(a-b)\,t\,(-9\,(2+a+b)+4\,(3+2\,a+2\,b)\,\delta)}{-27+20\,\delta}-\frac{1}{90}(-a+b)\,t\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)+3\,(2-a-b)\sqrt{81-25\,\delta^2}\right)\right]\right]$$

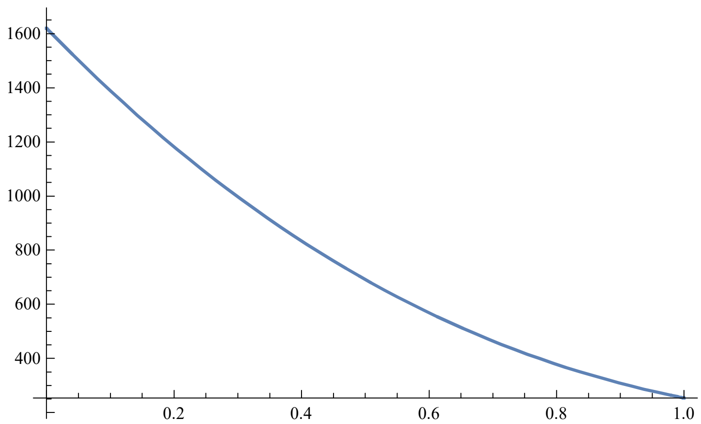
$$\frac{1}{90\,(-27+20\,\delta)}(a-b)\,t\left(2\left(500\,\delta^2-81\left(-19+\sqrt{81-25\,\delta^2}\right)+15\,\delta\left(-117+4\sqrt{81-25\,\delta^2}\right)\right)+a\left(300\,\delta^2+81\left(1+\sqrt{81-25\,\delta^2}\right)-15\,\delta\left(39+4\sqrt{81-25\,\delta^2}\right)\right)+b\left(300\,\delta^2+81\left(1+\sqrt{81-25\,\delta^2}\right)-15\,\delta\left(39+4\sqrt{81-25\,\delta^2}\right)\right)\right)$$

We arrange it, and obtain

$$\frac{(b-a)\,t}{90\,(27-20\,\delta)}\left(2\left((9-5\,\delta)\,(171-100\,\delta)-(81-60\,\delta)\sqrt{81-25\,\delta^2}\right)+3\,(a+b)\left((9-5\,\delta)\,(3-20\,\delta)+(27-20\,\delta)\sqrt{81-25\,\delta^2}\right)\right)$$

The sign of this value just depends on δ and g≡a+b.
The value within the largest parentheses is linear in g(=a+b).
This value when g=0 is $2\left((9-5\,\delta)\,(171-100\,\delta)-(81-60\,\delta)\sqrt{81-25\,\delta^2}\right)$, which positive.
We calculate this value when g=2, and obtain 40 (9-5 δ) (9-8 δ), which is positive.

$$\textbf{Plot}\left[\left\{2\left((9-5\,\delta)\,(171-100\,\delta)-(81-60\,\delta)\sqrt{81-25\,\delta^2}\right)\right\},\ \{\delta,0,1\}\right]$$



We have shown that the equilibrium is given by $a = 0$, $b = 1$ for all values of δ. The rest of Proposition 2 follows by substituting a = 0, b = 1, $\delta_f = \delta$ and $\delta_c = 0$ into relevant prices and the locations of marginal consumers in the two periods.

The following discussion is related to Proposition 3.

Here, we expand the range of g(=a+b) from -1 to 3.

Check the equilibrium locations in which the locations of the firms are restricted within the range [-1/2,3/2].

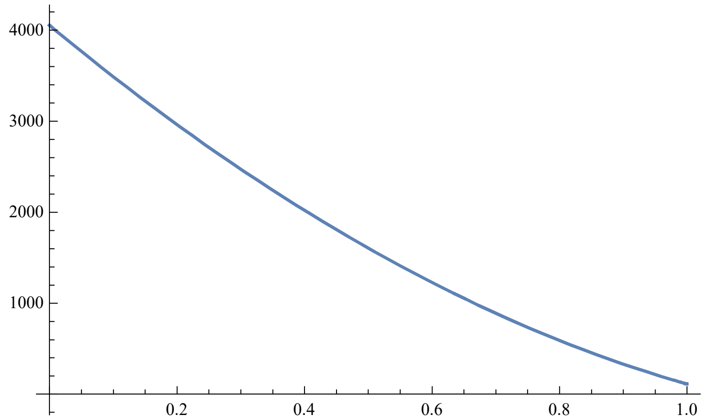
$$\frac{1}{90\,(27-20\,\delta)}\,(b-a)\,t\left(40\,(9-5\,\delta)\,(9-8\,\delta)-3\,(a+b)\left((9-5\,\delta)\,(3-20\,\delta)+(27-20\,\delta)\,\sqrt{81-25\,\delta^2}\right)\right)$$

The sign of this value just depends on δ and “a+b”.

Here we define g≡a+b. The value within the largest parentheses is linear in g(=a+b).

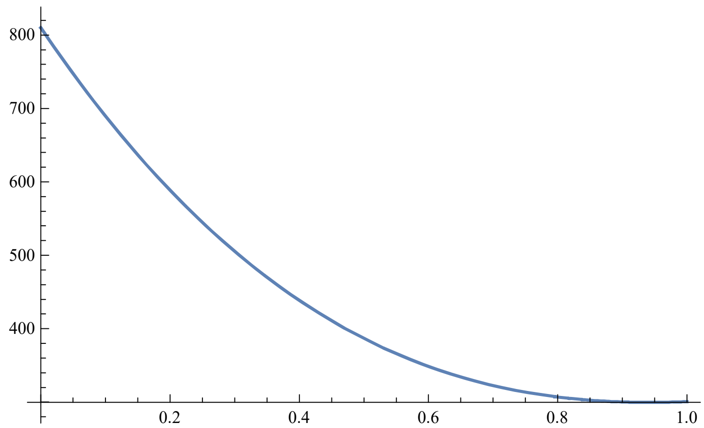
This value when g=-1is

```
Plot[40 (9 - 5 δ) (9 - 8 δ) - 3 * (-1) * ((9 - 5 δ) (3 - 20 δ) + (27 - 20 δ) √(81 - 25 δ²)), {δ, 0, 1}]
```



We calculate this value when g=3, and obtain

```
Plot[40 (9 - 5 δ) (9 - 8 δ) - 3 *3* ((9 - 5 δ) (3 - 20 δ) + (27 - 20 δ) √(81 - 25 δ²)), {δ, 0, 1}]
```



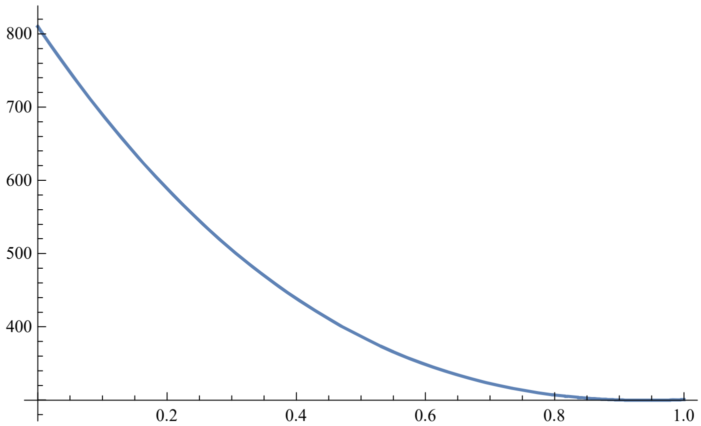
$$\frac{(b-a)\,t}{90\,(27-20\,\delta)}\left(2\left((9-5\,\delta)\,(171-100\,\delta)-(81-60\,\delta)\,\sqrt{81-25\,\delta^2}\right)+3\,(a+b)\left((9-5\,\delta)\,(3-20\,\delta)+(27-20\,\delta)\,\sqrt{81-25\,\delta^2}\right)\right)$$

The sign of this value just depends on δ and “a+b”.

Here we define g≡a+b. The value within the largest parentheses is linear in g(=a+b).

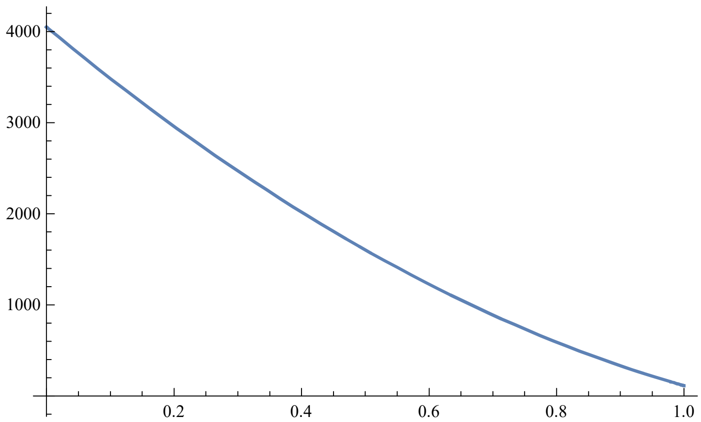
This value when g=-1 is

```
Plot[{2 ((9 - 5 δ) (171 - 100 δ) - (81 - 60 δ) √(81 - 25 δ^2)) +
   3 * (-1) ((9 - 5 δ) (3 - 20 δ) + (27 - 20 δ) √(81 - 25 δ^2))}, {δ, 0, 1}]
```



We calculate this value when g=3 is

```
Plot[{2 ((9 - 5 δ) (171 - 100 δ) - (81 - 60 δ) √(81 - 25 δ^2)) +
   3 * 3 ((9 - 5 δ) (3 - 20 δ) + (27 - 20 δ) √(81 - 25 δ^2))}, {δ, 0, 1}]
```



Check the equilibrium locations in which the locations of the firms are restricted within the range [-1/2,3/2].

The following is the candidate locations of Firms A and B

$$a : -\frac{81 - 99\,\delta f + 20\,\delta f^2}{12\,(27 + 9\,\delta f - 20\,\delta f^2)}$$

$$b : 1 + \frac{81 - 99\,\delta f + 20\,\delta f^2}{12\,(27 + 9\,\delta f - 20\,\delta f^2)}$$

The profit of Firm A in case (ii)

```
Factor[pa z + δf 1/9 (b - a) t (2 + 2 a + a^2 + 2 b + 2 a b + b^2 - 8 z - 2 a z - 2 b z + 10 z^2) /.
   z → (3 (pb - pa))/(2 (b - a) t (3 + δc)) + ((a + b) (3 - δc) + 2 δc)/(2 (3 + δc)) /. {pa → 1/(81 + 27 δc - 60 δf) (a - b) t
      (-27 (2 + a + b) - 54 δc + 3 (-4 + a + b) δc^2 - 4 (-9 + 3 a (-2 + δc) + 3 b (-2 + δc) - 8 δc) δf),
   pb → -1/(81 + 27 δc - 60 δf) (a - b) t (-27 (-4 + a + b) + 54 δc + 3 (2 + a + b) δc^2 -
      4 (21 + 3 a (-2 + δc) + 3 b (-2 + δc) + 2 δc) δf)} /. {δc → 0}]
```

$$-\frac{1}{18\,(-27+20\,\delta f)^2}(a-b)\,t\,\big(2916+2916\,a+729\,a^2+2916\,b+1458\,a\,b+729\,b^2-3888\,\delta f-2592\,a\,\delta f+405\,a^2\,\delta f-2592\,b\,\delta f+810\,a\,b\,\delta f+405\,b^2\,\delta f+864\,\delta f^2-1008\,a\,\delta f^2-1656\,a^2\,\delta f^2-1008\,b\,\delta f^2-3312\,a\,b\,\delta f^2-1656\,b^2\,\delta f^2+320\,\delta f^3+960\,a\,\delta f^3+720\,a^2\,\delta f^3+960\,b\,\delta f^3+1440\,a\,b\,\delta f^3+720\,b^2\,\delta f^3\big)$$

The first-order derivative of Firm A’s profit with respect to *a* is

```
Factor[D[%, a]]
```

$$-\frac{1}{18\,(-27+20\,\delta f)^2}\,t\,\big(2916+5832\,a+2187\,a^2+1458\,a\,b-729\,b^2-3888\,\delta f-5184\,a\,\delta f+1215\,a^2\,\delta f+810\,a\,b\,\delta f-405\,b^2\,\delta f+864\,\delta f^2-2016\,a\,\delta f^2-4968\,a^2\,\delta f^2-3312\,a\,b\,\delta f^2+1656\,b^2\,\delta f^2+320\,\delta f^3+1920\,a\,\delta f^3+2160\,a^2\,\delta f^3+1440\,a\,b\,\delta f^3-720\,b^2\,\delta f^3\big)$$

We substitute the candidate location of Firm B into the first-order derivative:

```
Factor[% /. b → 1 + (81 - 99 δf + 20 δf^2)/(12 (27 + 9 δf - 20 δf^2))]
```

$$-\Big(\big(t\,(-81-324\,a+99\,\delta f-108\,a\,\delta f-20\,\delta f^2+240\,a\,\delta f^2)\,(-255879-78732\,a+65610\,\delta f-69984\,a\,\delta f+420957\,\delta f^2+222588\,a\,\delta f^2-148068\,\delta f^3+14256\,a\,\delta f^3-194000\,\delta f^4-158400\,a\,\delta f^4+91200\,\delta f^5+57600\,a\,\delta f^5)\big)\Big/\big(288\,(-27+20\,\delta f)^2\,(-27-9\,\delta f+20\,\delta f^2)^2\big)\Big)$$

We derive *a* which makes the above first-order derivative equal to zero.

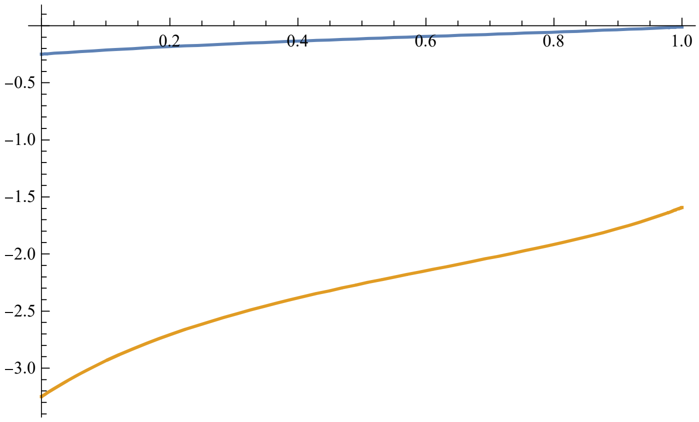
```
Simplify[Solve[% == 0, a]]
```

$$\Big\{\Big\{a\to\frac{81-99\,\delta f+20\,\delta f^2}{12\,(-27-9\,\delta f+20\,\delta f^2)}\Big\},\ \Big\{a\to(255879-65610\,\delta f-420957\,\delta f^2+148068\,\delta f^3+194000\,\delta f^4-91200\,\delta f^5)\Big/\big(36\,(-2187-1944\,\delta f+6183\,\delta f^2+396\,\delta f^3-4400\,\delta f^4+1600\,\delta f^5)\big)\Big\}\Big\}$$

The former *a* coincides with the candidate location of Firm A.

To check the sign of the first-order condition, we draw the values of *a* just derived above.

$$\texttt{Plot}\left[\left\{\frac{81-99\,\delta f+20\,\delta f^2}{12\,\left(-27-9\,\delta f+20\,\delta f^2\right)},\right.\right.$$
$$\left(255\,879-65\,610\,\delta f-420\,957\,\delta f^2+148\,068\,\delta f^3+194\,000\,\delta f^4-91\,200\,\delta f^5\right)/$$
$$\left.\left.\left(36\,\left(-2187-1944\,\delta f+6183\,\delta f^2+396\,\delta f^3-4400\,\delta f^4+1600\,\delta f^5\right)\right)\right\},\ \{\delta f,\ 0,\ 1\}\right]$$



The first-order derivative with $b=1+\frac{81-99\,\delta f+20\,\delta f^2}{12\,(27+9\,\delta f-20\,\delta f^2)}$ is positive if $a<\frac{81-99\,\delta f+20\,\delta f^2}{12\,(-27-9\,\delta f+20\,\delta f^2)}$, otherwise it is negative. Therefore, $a=-\frac{81-99\,\delta f+20\,\delta f^2}{12\,(27+9\,\delta f-20\,\delta f^2)}$ is the best location for Firm A given that Firm B chooses $b=1+\frac{81-99\,\delta f+20\,\delta f^2}{12\,(27+9\,\delta f-20\,\delta f^2)}$.

The profit of Firm B in (Case ii)

$$\texttt{Factor}\left[\text{pb}\,(1-z)+\delta f\,\frac{1}{9}\,(b-a)\,t\,\left(8-4\,a+a^2-4\,b+2\,a\,b+b^2-8\,z-2\,a\,z-2\,b\,z+10\,z^2\right)\ /.\right.$$
$$z\to\frac{3\,(\text{pb}-\text{pa})}{2\,(b-a)\,t\,(3+\delta c)}+\frac{((a+b)\,(3-\delta c)+2\,\delta c)}{2\,(3+\delta c)}\ /.\ \left\{\text{pa}\to\frac{1}{81+27\,\delta c-60\,\delta f}\,(a-b)\,t\right.$$
$$\left(-27\,(2+a+b)-54\,\delta c+3\,(-4+a+b)\,\delta c^2-4\,(-9+3\,a\,(-2+\delta c)+3\,b\,(-2+\delta c)-8\,\delta c)\,\delta f\right),$$
$$\text{pb}\to-\frac{1}{81+27\,\delta c-60\,\delta f}\,(a-b)\,t\,\left(-27\,(-4+a+b)+54\,\delta c+3\,(2+a+b)\,\delta c^2-\right.$$
$$\left.\left.\left.4\,(21+3\,a\,(-2+\delta c)+3\,b\,(-2+\delta c)+2\,\delta c)\,\delta f\right)\right\}\ /.\ \{\delta c\to 0\}\right]$$

$$-\frac{1}{18\,(-27+20\,\delta f)^2}$$
$$(a-b)\,t\,\left(11\,664-5832\,a+729\,a^2-5832\,b+1458\,a\,b+729\,b^2-7452\,\delta f+972\,a\,\delta f+405\,a^2\,\delta f+\right.$$
$$972\,b\,\delta f+810\,a\,b\,\delta f+405\,b^2\,\delta f-7776\,\delta f^2+7632\,a\,\delta f^2-1656\,a^2\,\delta f^2+7632\,b\,\delta f^2-3312\,a\,b\,\delta f^2-$$
$$\left.1656\,b^2\,\delta f^2+5120\,\delta f^3-3840\,a\,\delta f^3+720\,a^2\,\delta f^3-3840\,b\,\delta f^3+1440\,a\,b\,\delta f^3+720\,b^2\,\delta f^3\right)$$

The first-order derivative of Firm B's profit with respect to b is

```
Factor[D[%, b]]
```

$$-\frac{1}{18\,(-27+20\,\delta f)^2}$$
$$t\,\left(-11\,664+729\,a^2+11\,664\,b-1458\,a\,b-2187\,b^2+7452\,\delta f+405\,a^2\,\delta f-1944\,b\,\delta f-810\,a\,b\,\delta f-\right.$$
$$1215\,b^2\,\delta f+7776\,\delta f^2-1656\,a^2\,\delta f^2-15\,264\,b\,\delta f^2+3312\,a\,b\,\delta f^2+$$
$$\left.4968\,b^2\,\delta f^2-5120\,\delta f^3+720\,a^2\,\delta f^3+7680\,b\,\delta f^3-1440\,a\,b\,\delta f^3-2160\,b^2\,\delta f^3\right)$$

We substitute the candidate location of Firm A into the first-order derivative:

$$\texttt{Factor}\left[\% \,/.\, a \to -\frac{81 - 99\,\delta f + 20\,\delta f^2}{12\,(27 + 9\,\delta f - 20\,\delta f^2)}\right]$$

$$\big(t\,(405 - 324\,b + 9\,\delta f - 108\,b\,\delta f - 220\,\delta f^2 + 240\,b\,\delta f^2)\,(334\,611 - 78\,732\,b + 4374\,\delta f - 69\,984\,b\,\delta f - 643\,545\,\delta f^2 + 222\,588\,b\,\delta f^2 + 133\,812\,\delta f^3 + 14\,256\,b\,\delta f^3 + 352\,400\,\delta f^4 - 158\,400\,b\,\delta f^4 - 148\,800\,\delta f^5 + 57\,600\,b\,\delta f^5)\big) \big/ \big(288\,(-27 + 20\,\delta f)^2\,(-27 - 9\,\delta f + 20\,\delta f^2)^2\big)$$

We derive a which makes the above first-order derivative equal to zero:

$$\texttt{Simplify}[\texttt{Solve}[\% == 0, b]]$$

$$\left\{\left\{b \to \frac{405 + 9\,\delta f - 220\,\delta f^2}{324 + 108\,\delta f - 240\,\delta f^2}\right\},\ \left\{b \to -\big((334\,611 + 4374\,\delta f - 643\,545\,\delta f^2 + 133\,812\,\delta f^3 + 352\,400\,\delta f^4 - 148\,800\,\delta f^5) \big/ (36\,(-2187 - 1944\,\delta f + 6183\,\delta f^2 + 396\,\delta f^3 - 4400\,\delta f^4 + 1600\,\delta f^5))\big)\right\}\right\}$$

We rewrite the former *b*:

$$\texttt{Factor}\left[\frac{405 + 9\,\delta f - 220\,\delta f^2}{324 + 108\,\delta f - 240\,\delta f^2} - 1\right]$$

$$-\frac{81 - 99\,\delta f + 20\,\delta f^2}{12\,(-27 - 9\,\delta f + 20\,\delta f^2)}$$

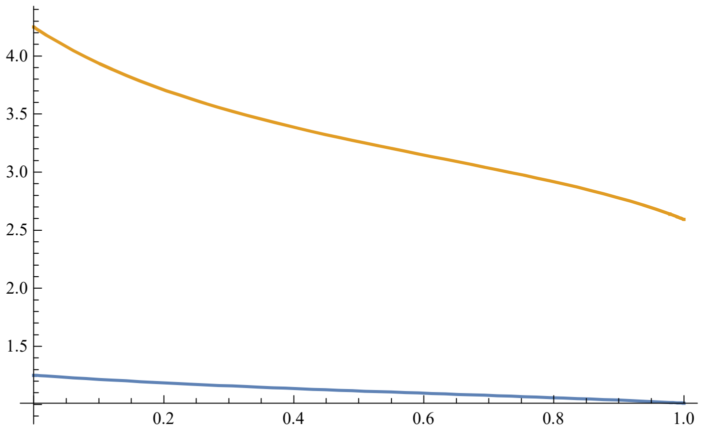
The former *b* is

$$b : 1 + \frac{81 - 99\,\delta f + 20\,\delta f^2}{12\,(27 + 9\,\delta f - 20\,\delta f^2)}$$

The former *b* coincides with the candidate location of Firm B.

To check the sign of the first-order derivative, we draw the values of *b* just derived above.

$$\texttt{Plot}\left[\left\{1 + \frac{81 - 99\,\delta f + 20\,\delta f^2}{12\,(27 + 9\,\delta f - 20\,\delta f^2)},\ -\big((334\,611 + 4374\,\delta f - 643\,545\,\delta f^2 + 133\,812\,\delta f^3 + 352\,400\,\delta f^4 - 148\,800\,\delta f^5) \big/ (36\,(-2187 - 1944\,\delta f + 6183\,\delta f^2 + 396\,\delta f^3 - 4400\,\delta f^4 + 1600\,\delta f^5))\big)\right\}, \{\delta f, 0, 1\}\right]$$



The first-order derivative with a= $-\frac{81-99\,\delta f+20\,\delta f^2}{12\,(27+9\,\delta f-20\,\delta f^2)}$ is positive if b <1+$\frac{81-99\,\delta f+20\,\delta f^2}{12\,(27+9\,\delta f-20\,\delta f^2)}$, otherwise it is negative. Therefore, b= $1 + \frac{81-99\,\delta f+20\,\delta f^2}{12\,(27+9\,\delta f-20\,\delta f^2)}$ is the optimal location choice for Firm B given that Firm A chooses $a = -\frac{81-99\,\delta f+20\,\delta f^2}{12\,(27+9\,\delta f-20\,\delta f^2)}$.