STRUCTURAL UNEMPLOYMENT, UNDEREMPLOYMENT, AND SECULAR STAGNATION

Ken-ichi Hashimoto
Yoshiyasu Ono
Matthias Schlegl

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The Institute of Social and Economic Research
Osaka University
6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan
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Ken-ichi Hashimoto†  Yoshiyasu Ono‡  Matthias Schlegl§¶

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Abstract

In this paper, we show that underemployment and not necessarily high unemployment becomes the main measure of economic slack under secular stagnation. Specifically, persistent underemployment occurs in the search and matching model, provided that households derive utility from holding wealth, and quickly dominates the total employment gap under stagnation. Wage and cost shocks can explain movements of unemployment and underemployment in opposite directions, while demand and supply shocks cause co-movements. Our analysis provides new insights into empirical puzzles such as Japan’s seemingly decent employment record and the absence of wage pressures despite low unemployment rates after the Great Recession.

Keywords: Secular stagnation, underemployment, unemployment, labor market frictions, search and matching

JEL Classification: E24, E31, E44, J20, J64

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†Graduate School of Economics, Kobe University; Email: hashimoto@econ.kobe-u.ac.jp
‡Institute of Social and Economic Research, Osaka University; Email: ono@iser.osaka-u.ac.jp
§Department of Economics, Sophia University, Tokyo; Email: m-schlegl-4t5@sophia.ac.jp
¶Corresponding author.
1 Introduction

Macroeconomists conventionally rely on the unemployment rate to measure the slack in the labor market and to analyze its implications for wage growth and inflation. However, such an analysis is incomplete at best and misleading at worst once an economy is constrained by the effective lower bound on the nominal interest rate and suffers from persistent stagnation. Under such circumstances, it is underemployment, a state in which employed workers want to increase working hours at the same wage but are unable to do so, rather than unemployment that provides a more accurate measure of labor market slack.

Japan during its lost decades is the prime example of an economy stuck in a stagnation equilibrium without any natural recovery. Yet, despite persistent deflationary tendencies and a shortfall of production below estimates of potential output due to insufficient aggregate demand, the country has performed surprisingly well in terms of its employment record. Albeit increasing in the immediate aftermath of the asset price crash, the unemployment rate of Japan has remained low by international comparison. As illustrated in panel (a) of Figure 1, it peaked at slightly more than 5% in 2002 and has since declined, the trend being briefly interrupted by the global financial crisis. With 2.4% in 2018, the unemployment rate had reached its lowest level since the early 1990s. Yet, this decline in unemployment has not resulted in significant wage or price pressures.¹

How did the lack of demand manifest itself in the labor market, if not in the unemployment rate? There has been a profound change in the structure of employment in Japan; the emergence of underemployment in the form of an increase in part-time and non-regular employment as well as a secular decline in working hours. Panel (b) of Figure 1 shows the share of part-time employment in contrast to the OECD average.² Having been less than 12% in the 1980s, the share of part-time employees has sharply increased throughout the lost decades. By 2018, it had doubled compared to the 1980s level whereas it only increased by 3 percentage points over all OECD countries. A similar trend can be observed for other forms of non-regular employment (see Japan Institute for Labour Policy and Training, 2015). And while the rise in part-time employment can partly be attributed to an increased desire for flexibility by employees, it primarily reflects the lack of alternative employment opportunities.³ Finally, the decline in average hours worked in the OECD data is also indicative of the rise of underemployment in Japan since 1990. Average annual hours worked have declined steadily, particularly in the 1990s, and have fallen below the OECD average.⁴

¹In fact, wage growth has been dismal with stagnant real and declining nominal wages. Bell and Blanchflower (2021) report that nominal wages have declined by 0.5% on average over the 2000-2016 period, while real wages have been unchanged. In 2017, real wages actually fell by 0.2% despite the historically low unemployment rate.

²OECD definition: “Part-time employment is defined as people in employment (whether employees or self-employed) who usually work less than 30 hours per week in their main job. Employed people are those aged 15 and over who report that they have worked in gainful employment for at least one hour in the previous week or who had a job but were absent from work during the reference week while having a formal job attachment.”

³For illustration, non-regular employees consistently emphasize the “lack of regular employment opportunities” as a major reason for their current work style in surveys conducted by the Japan Institute for Labour Policy and Training (see Japan Institute for Labour Policy and Training, 2015).

⁴The secular decline in working hours is also confirmed by data from the Labour Force Survey of the Statistics Bureau of Japan. Average working hours per week have substantially fallen from a stable level of slightly more than 28 hours in the 1980s to less than 23 hours by the 2010s.
The absence of widespread unemployment has frequently been attributed to the specific features of the Japanese labor market, most notably the traditional practices of long-term employment, which make unemployment in Japan less responsive to fluctuations in spending as opposed to other advanced economies. However, the rise of underemployment and the absence of wage and price pressures despite low unemployment are no longer exclusively observed in Japan. They have become well-documented phenomena in the United States and European countries following the Great Recession as well.

Bell and Blanchflower (2021) develop measures of underemployment for the United States and 26 European countries and find a substantial level of labor market underutilization in the form of underemployment in the years after the financial crisis implying that the unemployment rate does not adequately capture the total slack in the labor market. In fact, unemployment rates have returned to their pre-crisis levels in most countries while underemployment is persistently higher. Surveying the empirical evidence, Blanchflower (2019) argues that “underemployment has replaced unemployment as the main measure of labor market slack” (p.144). In addition, similar to the case of Japan, low unemployment rates have not generated high wage growth in these countries. Using a panel data set for 19 countries from 1998-2016, Bell and Blanchflower (2021) show that their measure of underemployment has a significantly negative effect on wages after 2007, while the unemployment rate is insignificant. The opposite holds before the Great Recession implying a structural break at the time the effective lower bound became a binding constraint in many countries, leading the authors to conclude that “underemployment replaces unemployment as the main influence on wages in the years since the Great Recession”. Other studies, such as Bell and Blanchflower (2018) for the United Kingdom, directly attribute underemployment to a shortage of aggregate demand.

Data sources: (a) Unemployment rate (in percent of the labor force), OECD; (b) Part-time employment rate (in percent of total employment), OECD, Japan (solid) and OECD average (dotted line)

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5We refer to Hashimoto and Raisian (1985) for a description of the Japanese employment system in comparison to the United States before the lost decades. Several studies show that these practices have eroded remarkably little, at least among core workers, during the stagnation decades (see Shimizutani and Yokoyama, 2009; Kambayashi and Kato, 2011; Hamaaki et al., 2012, among others).
These observations pose important questions for researchers and policymakers alike: How does a lack of aggregate demand manifest itself in the labor market? Under what conditions does demand-driven underemployment occur in equilibrium? How do structural unemployment and underemployment interact with each other? How do they respond to macroeconomic shocks and various labor market policies? And how do these variables contribute to the total employment gap in the labor market in the presence of demand shortage?

Existing macroeconomic models are insufficient to address these questions. The canonical model of the labor market does not allow for underemployment as workers are always on their labor supply curves and part-time employment is purely voluntarily, if considered at all. Similarly, models of persistent stagnation do not model the institutional structure of the labor market and unemployment in detail. In this paper, we offer a richer model of stagnation that allows us to explicitly distinguish between structural unemployment and demand-driven underemployment. Using this model, we show that secular stagnation leads to underemployment in the labor market but not necessarily to high unemployment, thereby providing the theoretical counterpart to the empirical analysis described above. Underemployment as a steady state phenomenon can rationalize the lack of inflationary pressures despite low unemployment. We also examine the effects of macroeconomic demand and supply shocks and various microeconomic labor market policies, thereby addressing the aforementioned questions and substantially extending the existing literature.

Specifically, we incorporate a preference for wealth into the standard search and matching model of the labor market. In a setting with infinitely-lived households, such a preference captures the bequest motive within dynasties that is frequently incorporated in overlapping generations models. The preference for wealth creates a strong motive to save in addition to the standard consumption smoothing motive, which can explain why empirically the saving rate of households is increasing in their wealth (see Benhabib and Bisin, 2018; Fagereng et al., 2019). Importantly, the preference for wealth allows for the possibility of a secular stagnation equilibrium as is well-known from contributions such as Ono (1994), Michau (2018) or Schlegl (2018). Households’ desire to save can drive the equilibrium or natural real interest rate into negative territory. Yet, the effective lower bound, which we model as a rate of zero for simplicity, prevents the nominal interest rate from falling sufficiently. This causes excess savings which depress aggregate demand. Combined with downward nominal wage rigidities in the spirit of Schmitt-Grohé and Uribe (2016, 2017), the model economy operates in a secular stagnation equilibrium characterized by deflation and a persistent lack of demand.

The search and matching friction results in structural unemployment in equilibrium as vacancies and job-seekers have to match successfully in a costly process before a position can be filled. In the absence of demand shortage, our model behaves similar to the standard case. This

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6Several recent contributions have analyzed the macroeconomic implications of a preference for wealth in models similar to ours. Kumhof et al. (2015), Michau et al. (2019) and Mian et al. (2020) study the dynamics of wealth and inequality that result from endogenous differences in saving rates based on the wealth preference, while Michau et al. (2018) analyze the dynamics of asset prices and the possibility of rational bubbles in such a framework. Michaillat and Saez (0) incorporate a preference for wealth (relative to the average wealth level) into a New Keynesian model to study business cycle fluctuations and Saez and Stantcheva (2018) employ a similar framework for the analysis of capital taxation. For the interested reader, we refer to Zou (1994) who provides an in-depth discussion of the idea of a preference for wealth in economic thought.
search friction creates an employment gap, but conditional on being employed, households realize their potential working hours and there is no underemployment. Nevertheless, the preference for wealth offers an interesting new transmission channel as various shocks and labor market policies also affect unemployment and other macroeconomic variables by inducing changes in the real interest rate. In contrast, both structural unemployment and underemployment coexist and mutually affect each other in the stagnation equilibrium. Faced with a lack of demand and depressed sales, firms respond by cutting working hours resulting in underemployment, lower output and less consumption. The lack of demand, however, also affects the incentives for job creation as firm ownership becomes less attractive due to lower profits, resulting in an increase in unemployment as well. The total employment gap then exceeds the unemployment rate being quickly dominated by underemployment as stagnation becomes more severe.

We use this model to study the effects of macroeconomic supply and demand shocks as well as microeconomic shocks such as wage and cost shocks under stagnation. The effects of demand and supply shocks are reversed under demand shortage. Higher demand, in the form of government spending, increases working hours and lowers unemployment thereby raising consumption and output. In contrast, a positive productivity shock worsens deflation and leads to both higher unemployment and more underemployment. This is the paradox of toil that is common to models of stagnation. Importantly, these macro shocks result in co-movements of the unemployment rate and the degree of underemployment as have been observed in Japan following the burst of the bubble economy and during the global financial crisis. In contrast, the effects of labor market policies depend on the shape of the wealth preference and the substitution elasticity of the matching function. If the concavity of the preference for wealth is sufficiently low, these policies result in movements of unemployment and underemployment in opposite directions as has been the case in Japan during labor market liberalization policies in the early 2000s and after 2012.

Finally, we use numerical simulations to quantify the relative importance of search-based unemployment and demand-driven underemployment in the stagnation equilibrium. While the standard search and matching mechanism is still at work in the model - in fact, structural unemployment worsens under stagnation - the effects of shocks are typically reflected in stronger variations in underemployment for reasonable parameter calibrations.

Our analysis concludes that secular stagnation causes demand-driven underemployment and not necessarily high unemployment. As long as the economy is stuck in the stagnation equilibrium, wage growth and inflationary pressures do not emerge even when unemployment is low as they are subdued by the prevalence of underemployment. It is then primarily underemployment that responds to economic disturbances and labor market policies under stagnation, which is in line with the developments in Japan since the 1990s. These findings highlight the need for further policy intervention in support of aggregate demand despite a seemingly decent employment situation in terms of the unemployment rate.

Related Literature: Our paper analyzes the interactions of structural unemployment based on search frictions and underemployment under secular stagnation, thereby contributing to and combining two fields of macroeconomics that have so far been treated fairly independently.
There is a rich literature on unemployment as a consequence of frictions in the labor market. Diamond (1982), Mortensen (1982) and Pissarides (1985) developed search and matching models of unemployment, which scholars have since applied in a wide variety of fields.\textsuperscript{7} Merz (1995) and Andolfatto (1996) introduce these labor market frictions into the standard real business cycle model. In this type of model, all goods are produced by successful matches between workers and firms. Equilibrium unemployment then results from frictions when matching vacancies and job-seekers. Variations in working hours are introduced into this matching framework by Fang and Rogerson (2009) as households optimally choose their labor supply based on the consumption versus leisure trade-off. While their study focuses on cross-country differences in working hours resulting from differences in employment to population ratios and hours per worker, our paper contrasts the labor market equilibria in the presence and absence of persistent stagnation, abstracting from voluntary variations in the labor supply.

Another strand of the literature relies on worker and job heterogeneity to explain the increase in non-regular employment in Japan. For instance, in Ariga and Okazawa (2011), non-regular employment and declining labor productivity are the results of permanent productivity shocks to jobs in combination with labor immobility due to differences in training costs among workers. Using the New Keynesian unemployment model with two types of jobs, Kang et al. (2020) relate the rise in part-time employment to an exogenous increase in part-time labor supply, which induces households and firms to optimally re-allocate between full-time and part-time labor. Yet, these voluntary shifts in the composition of the labor force do not constitute underemployment as defined in this paper. In contrast, our model relates underemployment directly to a lack of demand without relying on firm or worker heterogeneity.

Demand-driven variations in working hours are modelled in two related frameworks by Kudoh et al. (2019) and Michaillat and Saez (2015). Kudoh et al. (2019) analyze changes in the composition of firms’ labor demand over the business cycle. While labor market participation, i.e. the extensive margin of the labor supply, is chosen by each individual, working hours conditional on employment are adjusting in response to demand-driven fluctuations in production, which is a features similar to our model. However, these fluctuations occur over the business cycle and are hence purely temporarily in nature, which is in stark contrast to our model which allows for the possibility of goods market disequilibrium and demand-driven underemployment as a steady state phenomenon.

Michaillat and Saez (2015) apply a matching framework to both the labor market and the product market, such that the equilibrium level of sales is determined by a matching process between sellers and buyers. This implies that the price level and labor market tightness, i.e. the ratio of vacancies to job-seekers, have to adjust in order to balance demand and supply in both markets simultaneously. This creates feedback effects from demand fluctuations into the labor market. Under fixed prices, an increase in aggregate demand increases labor demand, which leads to a rise in working hours and lower unemployment.\textsuperscript{8} A similar channel operates in our model as an increase in aggregate demand stimulates both employment and working hours.

\textsuperscript{7}Pissarides (2000) provides the standard textbook for an introduction to models of search and matching frictions in the labor market. Rogerson et al. (2007) survey the related literature.

\textsuperscript{8}Landais et al. (2015b,a) use the same framework to study the effects of various labor market policies.
Yet, while the above-mentioned contributions model structural unemployment based on the search friction and allow for voluntary variations in working hours, they do not consider the possibility of involuntary underemployment in steady state. There are only few contributions that allow for such a phenomenon. Michaillat (2012) introduces rationing into the search and matching model. Rationing unemployment then results from a wage above market clearing level and coexists with structural unemployment. Other contributions, such as Manning (2003) and Ashenfelter et al. (2010), provide a microeconomic view on underemployment based on market concentration and monopsony power of firms. From the firm’s perspective, variations in working hours are a cost-efficient way to avoid hiring and firing costs. In contrast, there is no rationing and firms are perfectly competitive in our model. Underemployment is a macroeconomic phenomenon. Workers want to work additional hours at the same wage, i.e. they are off their labor supply curves, but are constrained by a lack of demand due to the binding effective lower bound on the nominal interest rate in steady state, i.e. secular stagnation.

The secular stagnation hypothesis has been revived by Summers (2013) against the background of the weak recovery following the Great Recession, based on the original idea of Hansen (1939). Proponents of this hypothesis argue that an oversupply of savings at full employment permanently depresses aggregate demand as the zero lower bound on the nominal rate prevents the real interest rate from falling sufficiently to stimulate spending. In the presence of downward nominal wage rigidity, this results in an equilibrium characterized by deflation and a shortfall of output below potential. The oversupply of savings has been modelled among others as a consequence of demographics (Eggertsson et al., 2019), a shortage of safe assets (Caballero and Farhi, 2018) or strong liquidity preferences (Ono, 2001; Illing et al., 2018).9 In this paper, we model the secular stagnation equilibrium based on a preference for wealth in line with the contributions of Ono (1994, Chapter 11; 2015) and Michau (2018).

These models, however, do not consider structural unemployment and abstract from labor market frictions. The labor market gap is either entirely ignored, in case the analysis is based on an endowment economy, or it consists of a demand-driven shortfall of realized working hours only. Hence, these models do not allow for potential interactions of unemployment and underemployment. In contrast, this paper treats both phenomena in an integrated framework, thereby showing that they are inherently interrelated. In our model, unemployment increases under stagnation, which is also consistent with the idea of hysteresis in the labor market.

This paper is organized as follows. Section 2 outlines the features of the model and the different steady state equilibria. Section 3 briefly discusses the implications of the preference of wealth for the standard search and matching model. In section 4, we extensively study the properties of the stagnation steady state and the role of underemployment and structural unemployment for the total employment gap, supported by a numerical simulation of the model. The final section concludes. All proofs are relegated to the mathematical appendix.

9For the open economy versions of these models, we refer to Eggertsson et al. (2016), Caballero et al. (2016) and Ono (2014).

10In addition, Schlegl (2018) provides a treatment of secular stagnation in an economy with land while Michau (2015b) carefully analyzes the effects of helicopter drops of money under stagnation based on a preference for wealth. Michaillat and Saez (2014) also rely on a preference for wealth to obtain an equilibrium with a permanent liquidity trap.
2 The Model

2.1 The representative household

Time is continuous and denoted by \( t \). The infinitely-lived representative household consists of a large number of individuals normalized to unity. Each individual is endowed with one unit of labor and one unit of time, both of which are supplied inelastically as there is no disutility of working. Let \( l_t \in [0,1] \) denote the realized labor supply and \( x_t \in [0,1] \) realized working hours per individual. Slack in the labor market can take two forms: Unemployment occurs when household members are not employed \( (l_t = 0) \). Let \( u_t \) denote the number of unemployed members of the household, and hence the unemployment rate, and \( 1-u_t \) the number of employed ones. Underemployment occurs when realized working hours of the employed \( (l_t = 1) \) fall short of potential working hours, i.e. \( x_t < 1 \). Each household receives wage income of \( w_t(1-u_t)x_t \) and unemployment benefits of \( \ddot{z}_tu_t \), where \( w_t \) is the real wage rate and \( \ddot{z}_t \) benefit payments per unemployed. We follow Merz (1995) and Andolfatto (1996) and assume that the household provides perfect consumption insurance for its members such that consumption is the same for employed and unemployed individuals.

Household assets \( a_t \), in real terms, consist of interest bearing assets (bonds or equities) \( b_t \) and real money holdings \( m_t \) that do not pay interest:

\[
a_t = b_t + m_t. \tag{1}
\]

Let \( r_t \) denote the real interest rate and \( R_t \geq 0 \) the nominal interest rate, which are related via the Fisher Equation by \( R_t = r_t + \pi_t \), where \( \pi_t \) denotes the rate of inflation. Then, real wealth evolves as:\[11\]

\[
\dot{a}_t = r_t a_t + w_t(1-u_t)x_t + \ddot{z}_tu_t - c_t - R_tm_t - \tau_t, \tag{2}
\]

where \( \tau_t \) is a real lump-sum tax, and \( (1-u_t)x_t \in [0,1] \) is effective employment. Note that the household faces opportunity costs \( R_t \) when holding money due to the foregone interest earnings.

As in Michaillat and Saez (2015) and Michau (2018), the lifetime utility of the household is given by

\[
U_0 = \int_0^\infty \left[ \phi(c_t) + \mu(a_t - m_t^S) \right] e^{-\rho t} dt,
\]

where \( \rho > 0 \) is the subjective discount rate. At any point in time, the household derives utility \( \phi(c_t) \) from consuming \( c_t \) with \( \phi'(\cdot) > 0 \) and \( \phi''(\cdot) < 0 \) and utility \( \mu(m_t) \) from holding real money balances \( m_t \), with \( \mu'(\cdot) > 0 \) and \( \mu''(\cdot) < 0 \), and \( \mu'(m) = 0 \) for all \( m \geq \bar{m} \). At \( \bar{m} \), the transaction demand for money is fully satiated. In addition, we introduce a preference for net wealth \( \omega(a_t - m_t^S) \) with \( \omega'(\cdot) > 0 \) and \( \omega''(\cdot) \leq 0 \), where \( m_t^S \) denotes the real money supply. This specification implies that the household considers the money stock a government liability that eventually needs to be redeemed and hence not part of aggregate net wealth, even though each member considers his personal money holdings a part of his wealth. As there is no heterogeneity on the household level, it is only the value of equity holdings (as bonds are in zero net supply)
that affects the wealth preference in steady state.\footnote{Michau (2019b) discusses and justifies the preference for net wealth in contrast to alternative specifications.}

The household maximizes lifetime utility subject to the asset constraint \( (1) \) and the flow budget constraint \( (2) \) for given initial wealth \( a_0 \). Optimal household behavior is described by the Euler Equation, the money demand function and the transversality condition as

\[
\eta c_t = r_t - \rho + \frac{\omega'(b_t + m_t - m^S_t)}{\phi'(c_t)}, \quad (3)
\]

\[
R_t = \frac{\mu'(m_t)}{\phi'(c_t)} \geq 0, \quad (4)
\]

\[
\lim_{t \to \infty} \phi'(c_t)a_t e^{-\rho t} = 0, \quad (5)
\]

where \( \eta \equiv -\phi''(c_t)c_t/\phi'(c_t) \) is the elasticity of the marginal utility with respect to consumption (and the inverse of the intertemporal elasticity of substitution), which we assume constant.\footnote{These equations are obtained from the standard current value Hamiltonian function \( H_t = \phi(c_t) + \mu(m_t) + \omega(a_t - m^S_t) + \lambda_t[r_t a_t + u_t(1 - u_t)x_t + zw_t x_{u_t} - c_t - R_t m_t - \tau_t], \) with control variables \( c_t \) and \( m_t \) and state variable \( a_t \) and where \( \lambda_t \) is the costate variable for \( a_t \).}

The preference for wealth affects the intertemporal allocation of consumption in the Euler Equation \( (3) \). The household has stronger incentives to save since accumulation of wealth becomes an end in itself, representing the bequest motive within dynasties, rather than a mere means to smooth consumption. A lower level of wealth, ceteris paribus, induces the household to choose a steeper consumption path or equivalently higher savings in order to accumulate wealth. In steady state, the preference for wealth creates a wedge between the real interest rate and the time preference rate of the household, which allows for the possibility of a negative natural real interest rate. Optimal money demand in \( (4) \) requires the marginal rate of substitution between money and consumption to equal the opportunity cost of holding money, which are given by the nominal interest rate. The nominal rate equals zero whenever \( m_t \geq \bar{m} \), which allows for the possibility of a liquidity trap in our framework.

### 2.2 Firms

There is a large number of identical firms that produce the consumption good using labor as the only input factor. Each firm requires exactly one worker in the production process and therefore offers one position, which can be filled, in which case the firm is operating, or vacant. When offering this position and searching for workers in the labor market, a search cost \( k \) occurs. The government might however subsidize the searching process at rate \( s \in [0, 1) \) so that the effective search costs for the firm are given by \( (1 - s)k \). Firms that successfully match hire \( l_t = 1 \) workers and produce output \( y_t \) with the linear production function:

\[
y_t = x_t \bar{y} l_t = x_t \bar{y}, \quad (6)
\]

where \( \bar{y} \) denotes labor productivity and \( x_t \) hours per worker. The operating profit of each producing firm is then simply given by \( (\bar{y} - w_t)x_t \). In equilibrium, the number of operating
firms equals $1 - u_t$ and the number of vacant firms is denoted by $v_t$. Total output is therefore determined as
\[ Y_t = (1 - u_t)x_t\bar{y}. \] (7)

Firms face frictions when setting their prices. Specifically, we follow Ono (1994, 2001) and introduce sluggish nominal price adjustment in the goods market via a reduced-form Phillips curve for the inflation rate $\pi_t$. Importantly, the dynamics of the price level in the presence of aggregate demand shortage differ from those with no aggregate demand shortage as follows:
\[ \pi_t = \frac{\dot{P}_t}{P_t} = \begin{cases} g_m & \text{if } x_t = 1, \\ \alpha (x_t - 1) & \text{if } x_t < 1, \end{cases} \] (8)
where $\alpha$ represents the adjustment speed of nominal prices. In the absence of aggregate demand shortage, the dynamics of the price level are as in the standard Money-in-the-Utility framework where the inflation rate is determined by the money growth rate $g_m$ and the price level adjusts to clear the money market. In contrast, we impose a limit on price declines under aggregate demand shortage in order to prevent deflationary wage-price-spirals and to allow for the possibility of a disequilibrium in the goods market in steady state.\(^{14}\)

The asymmetry in the inflation process is a fundamental element of stagnation models including among others the contributions of Schmitt-Grohé and Uribe (2016, 2017), Michau (2018), Illing et al. (2018) and Eggertsson et al. (2019) and typically results from some form of downward nominal wage rigidity that becomes binding in case of demand shortage.\(^{15}\)

### 2.3 Government

The government consists of a fiscal authority and a central bank. The fiscal authority adjusts lump-sum taxes $\tau_t$ to balance the budget while providing benefits $\hat{z}_t u_t$ to the unemployed, subsidizing a fraction $s$ of search costs $k$ of vacant firms and financing government spending $g$ such that
\[ \tau_t + g_m m_t^S = \hat{z}_t u_t + skv_t + g, \] (9)
where $g_m$ denotes the growth rate of the nominal money supply, $g_m m_t^S$ income from seignorage and $v_t$ the number of vacant firms. The government pays unemployment benefits to unemployed workers following the policy rule
\[ \hat{z}_t = zx_t w_t, \] (10)
where $z \in [0, 1)$ denotes the replacement rate that is proportional to labor income. The balanced budget policy implies that government bonds are in zero net supply and interest-bearing assets of the household $b_t$ solely consist of equity holdings in firms. The replacement rate $z$, the firm

\(^{14}\)Note that our model features outright deflation in case of demand shortage. We can generalize the inflation process, for instance by introducing a reference rate of inflation, following Michau (2018), that allows for a positive inflation rate under demand shortage. Here, we implicitly set this reference rate equal to zero to be consistent with the experience of Japan during its lost decades when prices and wages actually declined on average.

\(^{15}\)Otherwise, the possibility of unemployment due to demand shortage is intrinsically avoided. Note that under this assumption, the possibility of no aggregate demand shortage is not eliminated. Ono and Ishida (2014) provide a micro-foundation for such an adjustment process.
subsidy $s$ and public spending $g$ are policy parameters.

The central bank controls the nominal money supply $M^S_t$ by setting the nominal money growth rate $g_m \geq 0$ for a given initial money supply $M^S_0$. The nominal interest rate is endogenously determined in the money market in line with (4). In equilibrium, the household is indifferent between money and other assets as a means of saving. This implies that the real interest rate has to satisfy the Fisher equation

$$r_t = R_t - \pi_t = \begin{cases} \frac{\mu'(m_t)}{\phi'(c_t)} - g_m & \text{if } x_t = 1, \\ \frac{\mu'(m_t)}{\phi'(c_t)} - \alpha (x_t - 1) & \text{if } x_t < 1, \end{cases}$$

where the inflation rate is given in (8). Using the same expression for inflation, the real money supply $m^S_t \equiv M^S_t / P_t$ evolves as

$$\dot{m}_t = m - \pi_t = \begin{cases} 0 & \text{if } x_t = 1, \\ g_m - \alpha (x_t - 1) & \text{if } x_t < 1. \end{cases}$$

Hence, the real money supply is constant in the absence of aggregate demand shortage but expands indefinitely in case of secular stagnation due to the effects of deflation, thereby pushing the economy into a permanent liquidity trap as eventually $m_t > \bar{m}$.

2.4 Labor market

2.4.1 Matching mechanism

Output is produced as a result of successful matches of firms and workers in the labor market. This matching friction causes structural unemployment in equilibrium, even though each household member inelastically supplies one unit of labor. The matching function $F(u_t, v_t)$ determines the number of matches between vacant firms $v_t$ and unemployed $u_t$ with $F(0, v_t) = 0$ and $F(u_t, 0) = 0$. The function $F(u_t, v_t)$ is continuously differentiable, concave, homogeneous of degree one, and increasing with respect to both $u_t$ and $v_t$. Let $\theta_t$ denote the jobs-to-applicants ratio, i.e.,

$$\theta_t \equiv \frac{v_t}{u_t} \in [0, \infty),$$

which measures the tightness of the labor market. A higher value of $\theta_t$ implies more vacancies per job-seeker, which we will refer to as a tighter labor market. Firms are less likely to fill vacancies and unemployed workers are more likely to find employment the tighter the labor market as measured by $\theta_t$.

Formally, the probability that a vacant firm matches with a worker $q(\theta_t)$ is simply given by the relative frequency of matches among all vacancies and satisfies

$$q(\theta_t) \equiv \frac{F(u_t, v_t)}{v_t} = F \left( \frac{1}{\theta_t}, 1 \right), \quad q'(\cdot) < 0, \quad q''(\cdot) > 0, \quad q(\infty) = 0.$$
Similarly, the probability that a worker matches with a firm with a vacancy \( p(\theta_t) \) is given by the relative frequency of matches among all unemployed and satisfies

\[
p(\theta_t) = \frac{F(u_t, v_t)}{u_t} = F(1, \theta_t) = \theta_t q(\theta_t), \quad p'(\cdot) > 0, \quad p''(\cdot) < 0, \quad p(0) = 0. \tag{15}
\]

In each period a fraction \( \delta \) of workers become unemployed, where \( \delta \) constitutes an exogenous separation rate. Hence, the flow into unemployment is given by \( \delta (1 - u_t) \). At the same time, the number of successful matches between vacant firms and unemployed workers is given by \( F(u_t, v_t) = p(\theta_t) u_t \), which follows from (15). Therefore, the dynamics of the unemployment rate are determined by the following law of motion:

\[
\dot{u}_t = \delta (1 - u_t) - p(\theta_t) u_t = \delta - (\delta + p(\theta_t)) u_t. \tag{16}
\]

### 2.4.2 Value functions and Nash bargaining

A set of value functions describes the values of participating in the labor market for firms and workers. The firm value equals the present discounted value of the future profit stream. We denote the value of a vacant firm searching for a worker by \( V_t \), and the value of an operating firm employing a worker by \( J_t \). The Bellman equations for each type are given as follows:

\[
r_t J_t = (\bar{y} - w_t) x_t - \delta [J_t - V_t] + \dot{J}_t, \tag{17}
\]

\[
r_t V_t = -(1 - s) k + q(\theta_t) [J_t - V_t] + \dot{V}_t. \tag{18}
\]

Similarly, we denote the present discounted values of the future income streams associated with employment and unemployment by \( E_t \) and \( U_t \) respectively. The Bellman equations for both types of household members are given by

\[
r_t E_t = w_t x_t - \delta [E_t - U_t] + \dot{E}_t, \tag{19}
\]

\[
r_t U_t = \hat{z}_t + \theta_t q(\theta_t) [E_t - U_t] + \dot{U}_t. \tag{20}
\]

Underemployment (or the working rate \( x_t \)) directly affects \( J_t \) and \( E_t \) through its effects on firm profits and the income of workers, but has no direct effects on \( V_t \) and \( J_t \).

After a vacancy is filled, wages are negotiated between firms and workers through Nash bargaining. Let \( \varepsilon \in (0, 1) \) denote the bargaining power of workers. Wages are set to maximize the joint surplus of workers and firms given by \( (E_t - U_t)^\varepsilon (J_t - V_t)^{1-\varepsilon} \). Using (17) and (19) and taking the derivative with respect to wages, we obtain the following sharing rule of the total matching surplus between the firm and the worker:

\[
(1 - \varepsilon)(E_t - U_t) = \varepsilon (J_t - V_t). \tag{21}
\]

\footnote{This assumption allows us to focus on the effects of underemployment on the job creation channel as it implies that there is no direct effect of underemployment on the job destruction channel of the matching model. For the case of Japan, our focus on job creation is supported by empirical evidence in Ariga and Okazawa (2011) who show that hiring rates have declined substantially during the stagnation period while separation rates appear quite stable.}
Firms can enter the labor market freely. As a consequence, a firm will enter the labor market and post a vacancy as long as its expected value is greater or equal to zero. The free entry condition therefore implies

\[ V_t = 0. \]  
\[ (22) \]

Applying (22) to (18) and using the property \( q'() < 0 \) in (14), the value of an operating firm \( J_t \) satisfies

\[ J(\theta_t) = \frac{(1-s)k}{q(\theta_t)} \quad J(\infty) \to \infty, \quad J'(.) > 0. \]  
\[ (23) \]

The market entry condition requires that the expected value of a filled job equals the search costs associated with a vacancy. The tighter the labor market, the lower the probability of filling a vacancy and the higher the value of an operating firm.

Using (10), (17), (19), (20), (21), (22) and (23), the solution to the Nash bargaining problem is given by (see appendix)

\[ w_t x_t = \psi\left(\bar{y} x_t + (1-s)k\theta_t\right), \]  
\[ (24) \]

where \( \psi \equiv \frac{1}{1-(1-s)z} \) reflects the effective bargaining power of workers, which is increasing in both the bargaining power \( \varepsilon \) and the replacement rate \( z \). All else equal, the negotiated wage \( w_t \) is increasing in effective bargaining power \( \psi \) and effective vacancy cost \( (1-s)k \). In addition, the wage increases in labor market tightness \( \theta_t \) and decreases in effective working hours \( x_t \).

In equilibrium, the household needs to be indifferent between investment in firm equity and other assets. Hence, the return on equity, i.e. the sum of the net dividend yield and valuation changes, has to equal the real interest rate. Formally, applying free entry condition (22) to (17) implies

\[ r_t = \frac{(\bar{y} - w_t)x_t}{J_t} - \delta + \frac{\dot{J}_t}{J_t}. \]  
\[ (25) \]

From (23) and (24), the dividend yield \( (\bar{y} - w_t)x_t/J_t \) represented by the first term of the right-hand side above turns to

\[ \vartheta(\theta_t, x_t) = \frac{(1-s)k\theta_t}{(1-s)k} + \psi p(\theta_t). \]  
\[ (26) \]

Let \( \vartheta(\theta_t, 1) \) denote the dividend yield when there is no underemployment and \( \vartheta(\theta_t, 1) \) its partial derivative with respect to \( \theta_t \). As \( \bar{y} > z \), we have \( \vartheta(0, 1) > 0, \vartheta(\infty, 1) < 0 \) and \( \vartheta(\theta_t, 1) < 0 \). Substituting the time derivative of \( J_t \) derived from (23) and the expression for the dividend yield \( \vartheta(\theta_t, x_t) \) in (26) into (25) gives the law of motion for \( \theta_t \):

\[ \eta_\theta \frac{\dot{\theta}_t}{\theta_t} = \frac{\dot{J}_t}{J_t} = r_t - \vartheta(\theta_t, x_t) + \delta, \]  
\[ (27) \]

where \( \eta_\theta \equiv \frac{\theta_t q'(\theta_t)}{q(\theta_t)} > 0. \) Intuitively, labor market tightness \( \theta_t \) adjusts so that the real interest rate \( r_t \) equals the net return on equity holdings. If \( r_t \) exceeds the net dividend yield \( \vartheta(\theta_t, x_t) - \delta \), labor market tightness \( \theta_t \) has to increase, thereby raising the firm value \( J_t \) to fill the difference and to make it sufficiently attractive to invest in firms. If \( r_t \) is less than \( \vartheta(\theta_t, x_t) - \delta \), on the contrary, \( \theta_t \) must decrease so that \( J_t \) declines, yielding capital losses to firm owners.
2.5 Market clearing conditions

At any point in time, the bond market, money market and equity market are in equilibrium. Bonds are in zero net supply. In the money market, money demand of the household in (4) equals the money supply set by the central bank in (12) so that we have

$$m_t = m_t^S = \frac{M_t^S}{P_t}. \tag{28}$$

In the stock market, the value of equity owned by the household $b_t$ equals the aggregate firm value. Because the number of operating firms is equal to the number of employed workers, $J_t(1 - u_t)$ represents the total supply of equities. Using (23), the value of equity and hence the net wealth of the household $a_t - m_t^S$, with bonds in zero net supply, is given by

$$b_t = (1 - u_t) \frac{(1 - s)k}{q(\theta_t)}. \tag{29}$$

Financial market equilibrium is also represented by the No-Arbitrage conditions (11) and (27) that require the same real return on money, bonds and stocks.

Finally, the goods market equilibrium is derived from the household and government budget equations (see appendix) as

$$(1 - u_t)x_t\bar{y} = c_t + g + k\theta_t u_t, \tag{30}$$

implying that total output is used for private consumption, public spending and total search costs of firms with vacancies $v_t = \theta_t u_t$. Output is reduced below the production capacity $\bar{y}$ due to structural unemployment $u_t$ and demand-driven underemployment $x_t$, when $x_t < 1$. We can interpret $(1 - u_t)\bar{y}$ as a measure of potential output in the presence of structural inefficiencies. By reformulating (30), $x_t$ can be expressed as the ratio between effective demand and potential output. Therefore, $x_t$ constitutes both a measure of underemployment and a measure of effective demand shortage as reflected in the output gap. In the business cycle literature, output fluctuations around the natural level of output are the result of variations in working hours in response to various shocks. In contrast, our framework allows for the possibility of underemployment as a steady state phenomenon, even in the absence of shocks.

2.6 Equilibrium and steady state

The equilibrium of the model is defined as follows:

**Definition 1** An equilibrium is a set of paths for prices $P_0$ and $\{r_t, R_t, w_t, \pi_t\}$ and for quantities $\{c_t, m_t, b_t, a_t, Y_t, y_t, x_t, u_t, v_t, \theta_t\}$ such that

- $\{c_t, m_t, b_t, a_t\}$ solves the consumer’s problem given $\{r_t, R_t, w_t, u_t, \theta_t\}$, $a_0$, $m_0 = M_0/P_0$ and $a_t = b_t + m_t$ in (1);
- firms produce $y_t$ given $\{r_t, w_t, u_t, \theta_t\}$ resulting in aggregate output $Y_t$ given by (7);
- $\{w_t, v_t, u_t, \theta_t\}$ solves the Nash bargaining problem in the labor market where $v_t = \theta_t u_t$ according to (13);
• equilibrium prices $P_0$ and $\{r_t, R_t, \pi_t\}$ are consistent with money market equilibrium, stock market equilibrium as well as the No-Arbitrage relation between money and stocks (11), while in the goods market, inflation and working hours are determined by the non-linear Phillips curve (8) and $x_t \leq 1$, which hold with complementary slackness.

Equations (3), (12), (16) and (27) form an autonomous dynamic system with respect to $c_t, m_t, u_t$ and $\theta_t$, where $r_t, b_t$ and $x_t$ are derived from (11), (29) and (30) as functions of these variables. A steady state of this system is defined as follows:

**Definition 2** In a stationary steady state, consumption, the unemployment rate and labor market tightness are constant, i.e., it has to hold that $\dot{c}_t = 0$, $\dot{u}_t = 0$ and $\dot{\theta}_t = 0$, while the behavior of $\dot{m}_t$ depends on the presence of aggregate demand shortage.

Applying this definition, we derive the general properties of a steady state equilibrium with a particular focus on the effects of underemployment and unemployment as well as the spillovers between the labor market, the goods market and the money market.

**The labor market:** Slack in the labor market manifests itself in two forms, structural unemployment as measured by the unemployment rate $u_t$ and underemployment due to demand shortage when $x_t < 1$. The total employment gap can hence be decomposed as

$$1 - (1 - u_t)x_t = \underbrace{u_t}_{\text{Structural unemployment}} + \underbrace{(1 - u_t)(1 - x_t)}_{\text{Underemployment}}.$$  

In the standard steady state with $x = 1$, the unemployment rate perfectly describes the total employment gap. However, it is an inadequate indicator of the overall slack in the labor market of an economy that suffers from demand shortage.

In steady state, the unemployment rate is determined such that the number of newly unemployed equals the number of newly filled jobs. Substituting $\dot{u}_t = 0$ into equation (16) yields the steady-state unemployment rate, which satisfies

$$u(\theta) = \frac{\delta}{\delta + p(\theta)}, \quad u(0) = 1, \quad u'(.) < 0, \quad u''(.) > 0,$$

where the signs of the derivatives follow from (15). This equation represents the Beveridge curve. A tighter labor market is characterized by a lower structural unemployment rate. In addition, a higher separation rate (ceteris paribus) increases the unemployment rate. There are no direct spillovers from underemployment into unemployment. However, demand shortage indirectly affects the unemployment rate via its effects on labor market tightness.

In contrast, underemployment directly affects the job creation incentives of firms in steady state due to its effect on firm profits. This follows from the No-Arbitrage condition (27) between investment in firms and other assets with $\dot{\theta}_t = 0$ as

$$r = \frac{(1 - \psi)\bar{y}x}{(1 - s)k}q(\theta) - \psi p(\theta) - \delta.$$
A shortfall of working hours under stagnation reduces firm profits and the return on equity. For a given real interest rate \( r \), investment in firms becomes less attractive resulting in market exit. As firms leave the market, few vacancies are posted and labor market tightness declines until the equilibrium in (33) is restored. Unemployment increases as is clear from (32).\(^{17}\) All else equal, the return on equity is increasing in labor productivity \( \bar{y} \) and decreasing in the effective bargaining power of workers \( \psi \), effective search costs \( (1 - s)k \) and the separation rate \( \delta \).

**The goods market:** In goods market equilibrium, consumption demand of the household has to be consistent with the supply of goods available for private consumption. Equivalently, the natural real interest rate, defined by the Euler Equation (3) with \( \dot{c} = 0 \), has to equal the real return on savings, i.e.,

\[
    r = \rho - \frac{\omega'(b)}{\phi'(c)}.
\]

The natural real interest rate is declining in the wealth premium, i.e. the marginal rate of substitution between consumption and wealth. Household net wealth equals the aggregate value of firms’ equity in (29) and is given by

\[
    b(\theta) = (1 - u(\theta)) \frac{(1 - s)k}{q(\theta)}, \quad b(0) = 0, \quad b(\infty) \to \infty, \quad b'(\cdot) > 0.
\]

A tighter labor market implies a higher number of operating firms and a higher valuation of these, both of which raise the total equity value and, hence, household wealth. Ceteris paribus, net wealth also increases in net search costs \( (1 - s)k \) and decreases in the separation rate \( \delta \).

The net income of the household equals total output less entry costs of firms and government spending. Applying (32) to the goods market clearing condition (30), the net supply of goods is a function of labor market tightness and realized working hours given by

\[
    c(\theta, x) = (1 - u(\theta))x\bar{y} - k\theta u(\theta) - g.
\]

Both unemployment and underemployment reduce household income as they lower total production due to their effects on effective employment. In contrast, the net supply of goods shows a hump-shaped pattern with respect to labor market tightness. Initially, total output raises by more than total search costs when the labor market tightens and the net supply of goods increases. At some point, however, the increase in total search costs exceeds the increase in output, thereby lowering the net supply available for household consumption. Following Michaillat and Saez (2015), we refer to the former case as a “slack” steady state and the latter one as a “tight” steady state.\(^{18}\) All else equal, consumption is also decreasing in government spending \( g \), as a larger fraction of output is diverted to public instead of private uses, and declining in \( \delta \) and \( k \) due to higher unemployment and higher search costs respectively.

\(^{17}\)In principal, we could have \( \partial \theta(\theta, x)/\partial \theta > 0 \) for sufficiently small values of \( x \). Yet, such a scenario is not feasible in any steady state of our model as we either have \( x = 1 \) or \( r > 0 \) if \( x < 1 \), effectively putting a lower bound on any feasible \( x \) in steady state.

\(^{18}\)Formally, a steady state with \( dc/d\theta > 0 \) in (36) is referred to as slack and a steady state with \( dc/d\theta < 0 \) as tight. Note, however, that the definition refers to the total (rather than the partial) derivative.
Consider the spillovers from the labor market into the goods market for a given real interest rate. On the one hand, an increase in labor market tightness increases household wealth in (35), which stimulates consumption demand. On the other hand, a tighter labor market affects the net income of the household in (36). If the steady state is “tight”, the supply of goods available to the household falls. Taken together, this results in excess demand in the goods market, which allows firms to increase sales and therefore working hours. The increase in output increases the supply of goods until equilibrium is restored. In contrast, if the steady state is slack, the supply of goods available to the household increases. Taken together, this can result in excess demand or excess supply in the goods market depending on the concavity of the wealth preference. The former gives firms incentives to expand working hours, while firms reduce working hours as they are not able to sell their products due to a lack of demand in the latter case.

The money market: The existence of money imposes a lower bound on the real interest rate. In equilibrium, the return on money in (11) has to be the same as the return on other assets. This implies that

\[
 r = \begin{cases} 
 \mu'(m) - g_m & \text{if } x = 1, \\
 \alpha (1 - x) & \text{if } x < 1.
\end{cases}
\]

Without demand shortage, the real interest rate is determined by the real side of the economy such that the goods and labor markets are simultaneously in equilibrium. The nominal rate then adjusts endogenously in the money market. In contrast, underemployment is associated with persistent deflation and, hence, an expanding real money supply as is clear from (8) and (12). Eventually, when \( m_t > \bar{m} \), the liquidity preference of the household is satiated and the nominal interest rate is at the zero lower bound as \( \mu'(m) = 0 \). The real interest rate is then determined by the rate of deflation and adjustment takes place in the labor and goods markets.

In general equilibrium, unemployment and underemployment are jointly determined such that all markets are in equilibrium. In the steady state without underemployment, the zero lower bound on the nominal interest rate and the downward price rigidity constraint are not binding. In contrast, both constraints are binding in the stagnation steady state and underemployment emerges in addition to structural unemployment.

3 The standard model with a preference for wealth

In this section, we analyze the equilibrium in the absence of demand shortage, i.e. for \( x_t = 1 \). For simplicity, we focus on the case of a constant nominal money supply with \( g_m = 0 \) and hence a steady state with stable prices, but can be generalized as discussed in Illing et al. (2018) or Michau (2018). The model equilibrium without demand shortage can be summarized by a system of three differential equations for \( u_t \), \( \theta_t \) and \( m_t \) as shown in the appendix.\footnote{The transversality condition (5) needs to be satisfied as well, which requires a sufficiently low money growth rate \( g_m \) and a sufficiently low rate of price adjustment \( \alpha \). A sufficient restriction is \( \rho > g_m + \alpha \).}

\footnote{Since \( x_t = 1 \) and \( \dot{x}_t = 0 \), we can derive an expression for the real interest rate \( r_t \) based on the total differential of the goods market clearing condition (36) together with the Euler Equation (3) and the law of motions for \( \theta_t \) and \( u_t \) in (16) and (27).}
The steady state of this equilibrium is characterized by the consistency of (33), (34) and (37) for \( x = 1 \), which requires that the natural real interest rate governing the consumption-saving decision of the household equals the return on equity and the return on money holdings. Formally, this steady state is characterized by

\[
\rho - \frac{\omega'(b(\theta))}{\phi'(c(\theta, 1))} = \frac{(1 - \psi)\bar{\theta}}{(1 - s)k} q(\theta) - \psi p(\theta) - \delta = \frac{\mu'(m)}{\phi'(c(\theta, 1))} \geq 0, \tag{38}
\]

where \( b(\theta) \) is given by (35) and \( c(\theta, 1) \) by (36) with \( x = 1 \).

In this steady state, the real interest rate is jointly determined in the labor and goods markets such that the return on equity is consistent with the natural real interest rate. The nominal rate adjusts to clear the money market so that holding money yields the same real return. However, since \( R \geq 0 \) and \( \pi = 0 \) (due to \( g_m = 0 \)), existence requires that the real interest rate is weakly positive as otherwise an oversupply of savings would occur. In that case, holding real money balances is too attractive to be compatible with asset market equilibrium. As the return on equity is declining in labor market tightness, i.e. \( \vartheta(\theta, 1) < 0 \), there is a unique value \( \tilde{\theta} \) associated with a zero real interest rate. This value \( \tilde{\theta} \) is given by

\[
\tilde{\theta} : \vartheta(\tilde{\theta}, 1) \equiv \frac{(1 - \psi)\bar{\theta}}{(1 - s)k} q(\tilde{\theta}) - \psi p(\tilde{\theta}) = \delta. \tag{39}
\]

Specifically, a steady state with \( x = 1 \) is no longer feasible if \( \theta > \tilde{\theta} \) as this would require a negative real interest rate, which is prevented by the zero lower bound and the restriction on the steady state inflation rate. The following lemma summarizes these existence conditions:

**Lemma 1** Define \( \bar{\theta} \) and \( \tilde{\theta} \) by \( c(\bar{\theta}, 1) = c(\tilde{\theta}, 1) = 0 \) in (36) with \( \bar{\theta} < \tilde{\theta} \) and \( \bar{\theta} \) by \( \vartheta(\bar{\theta}, 1) = \delta \) as in (39). The steady state without demand shortage exists if \( \vartheta(\bar{\theta}, 1) > \rho + \delta > \vartheta(\tilde{\theta}, 1) \) and if

\[
\rho \phi' \left( c(\bar{\theta}, 1) \right) \geq \omega' \left( b(\bar{\theta}) \right), \tag{40}
\]

where \( b(\theta) \) is given by (35).

Lemma 1 requires that the return on equity exceeds the time preference rate of the household for low levels of labor market tightness. Put differently, it has to be sufficiently attractive to create new vacancies when unemployment is high. If there are no incentives to establish firms even in case of high unemployment, existence of a steady state is not guaranteed. This is expressed in the requirement \( \vartheta(\bar{\theta}, 1) - \delta > \rho > \vartheta(\tilde{\theta}, 1) - \delta \). Moreover, the real interest rate can be negative in steady state if the preference for wealth is sufficiently strong. But then, holding money is too attractive due to the zero lower bound on the nominal interest rate in combination with a ceiling on the inflation rate represented by \( g_m = 0 \). This results in an oversupply of savings and asset market disequilibrium at \( x = 1 \), which is not consistent with the existence of a steady state. Hence, Lemma 1 requires a sufficiently weak wealth preference for the steady state without aggregate demand shortage to exist, as expressed in condition (40).

As a corollary, it immediately follows that relative to the standard matching model, the preference for wealth reduces the steady state real interest rate and increases labor market
tightness. To see this, note that without the preference for wealth the real rate is simply given by the time preference rate $\rho$ of the household. Compared to this case, the preference for wealth implies that the household is willing to accept a lower return on his savings which in turn stimulates firm creation resulting in a tighter labor market.

In principle, there could be multiple steady states consistent with (38). Yet, mild technical restrictions on the functional forms of $\phi(\cdot), \omega(\cdot)$ and $q(\cdot)$ are sufficient to establish uniqueness of this steady state. Intuitively, uniqueness requires that an exogenous increase in wealth reduces the household’s propensity to save via a reduction in steady state labor market tightness $\theta^f$. Not only is it the natural economic scenario to expect, it is also technically a very mild condition.\footnote{Technically, we assume that the equation $\omega'(b(\theta) + \xi) = [\rho + \delta - \vartheta(\theta, 1)]\phi'$, with $b(\theta), \vartheta(\theta, 1)$ and $c(\theta, 1)$ given by (26), (29) and (36), defines $\xi$ as a decreasing function of $\theta$ for all admissible values of $\theta$. Totally differentiating this equation with respect to $\theta$ reveals that this assumption must be satisfied provided that the elasticity of the unemployment rate with respect to $\theta$ is sufficiently low.}

We assume that the steady state without underemployment is unique throughout this paper.

Having established existence of the steady state, what about its stability? This is summarized in the following lemma.

**Lemma 2** If the steady state without aggregate demand shortage is unique, then it is also saddle-path stable.

The following proposition summarizes how this steady state is affected by wage and cost shocks (“micro” shocks) as well as demand and supply shocks (“macro” shocks).

**Proposition 1** In the steady state without aggregate demand shortage, variations in the model parameters have the following effects on labor market tightness $\theta^f$, the unemployment rate $u^f$, total output $Y^f$ and consumption $c^f$:

<table>
<thead>
<tr>
<th></th>
<th>$\theta^f$</th>
<th>$u^f$</th>
<th>$Y^f$</th>
<th>$c^f$</th>
<th>$c^f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wage Shock: $\varepsilon, z$</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Cost Shock: $k$</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+/-</td>
</tr>
<tr>
<td>Cost Subsidy: $s$</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Supply Shock: $\bar{y}$</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+/-</td>
</tr>
<tr>
<td>Demand Shock: $g$</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+/-</td>
</tr>
</tbody>
</table>

These effects follow immediately from the previous discussion. An exogenous increase in firm profits, due to lower wages or lower operating costs, increases the dividend yield making investment in firms more attractive than holding other assets. As firms enter the market, labor market tightness increases and unemployment is reduced. This is the job creation channel of the search and matching model. Without a preference for wealth, the real interest rate is pinned
down to a constant subjective discount rate. Then, it is easy to see from the job creation condition in (35) with \( r = \rho \), that wage and cost shocks decrease labor market tightness and increase unemployment, while a cost subsidy or a productivity shock have opposite effects.

Thus, in the absence of demand shortage, our model behaves similar to the standard search frictions model without a preference for wealth (cf. Pissarides, 2000), except in response to demand shocks. The wealth preference creates a new transmission channel as exogenous changes in the firm value or net income feed back into the real interest rate by affecting the wealth premium. In the goods market, the higher real interest rate reduces consumption demand resulting in excess supply. The increase in labor market tightness helps to restore equilibrium as higher wealth and lower unemployment increase consumption demand. Changes in the real interest rate via the wealth premium have a feedback effect into the labor market. The working of this wealth channel can reinforce the job creation channel or work opposite to it depending on the shape of the wealth preference. Yet, it is the job creation channel that dominates the wealth channel in the absence of demand shortage.

Therefore, exogenous increases in the net dividend yield result in higher labor market tightness and lower unemployment. The reduction in unemployment also results in a higher output level as is clear from (7), whereas the overall effect on household consumption is undetermined due to the hump-shaped patterns of the consumption function in (36) allowing for both a “slack” or a “tight” steady state.\(^{22}\)

Finally, demand shocks in the form of higher government expenditures have no effect on the labor market in the standard model without a preference for wealth. In contrast, demand shocks negatively affect labor market tightness and employment in our model. Crowding out of household consumption increases the interest rate, which lowers the firm value, reduces the number of vacancies as firms leave the labor market and increases unemployment.

### 4 Unemployment and underemployment under stagnation

Consider now the case of aggregate demand shortage. Persistent stagnation occurs when saving is too attractive as the nominal interest rate cannot fall below zero. In such a steady state, realized working hours upon employment are below their potential and economic slack manifests itself in both structural unemployment \( u \) and demand-driven underemployment \( x < 1 \).

The steady state of this equilibrium is defined by (33), (34) and (37) for \( x < 1 \). Formally, this steady state is hence characterized by

\[
\rho - \frac{\omega'(b(\theta))}{\phi'(c(\theta, x))} = \frac{(1 - \psi)\bar{y}x}{(1 - s)k} q(\theta) - \psi p(\theta) - \delta = \alpha(1 - x),
\]

where \( b(\theta) \) is given by (35) and \( c(\theta, x) \) by (36).

\(^{22}\)Whether an equilibrium is slack or tight depends on the shape and curvature of the matching function and the steady state level of \( \theta \), which is in turn related to the parameters and the shape of the preference for wealth. Michaillat and Saez (2014) analyze business cycle fluctuations in the slack equilibrium in a model with wealth preferences. This focus seems natural since it implies that an increase in labor market tightness raises total final output \( (\bar{y}(1 - u)) \) by more than total search costs \( (kv = k\theta u) \). Pissarides (2000, Chapter 8) provides a general discussion of the interactions of labor market tightness and consumption in a matching model.
Under stagnation, the real interest rate is determined by the rate of deflation since the nominal interest rate is at the zero lower bound, the transaction demand for money is satiated due to expanding real money balances and the downward price rigidity constraint is binding. Working hours $x^s$ and labor market tightness $\theta^s$ are then jointly determined in the goods and labor markets such that the return on equity and the natural real rate are equal to the return on holding money.

Consider the spillovers between the money and labor markets under stagnation in case of an exogenous increase in underemployment. The shortfall of working hours increases the rate of deflation and reduces firm profits and the return on equity, thereby making money holdings more attractive than investment in firms. Hence, firms leave the labor market and labor market tightness falls until equilibrium is restored. This results in higher unemployment. These spillovers are captured by the second equality in condition (41). It implies the equilibrium relationship

$$x^a(\theta) = \frac{(1-s)k[\psi p(\theta) + \delta + \alpha]}{(1-\psi)\bar{g}q(\theta) + (1-s)k\alpha},$$

which satisfies $\frac{dx^a(\theta)}{d\theta} > 0$ and $x^a(\tilde{\theta}) = 1$, with $\tilde{\theta}$ defined in (39). For a given $\theta$, values of $x > x^a(\theta)$ are associated with a return on equity above the return on money. Hence, there is excessive demand for equity, which makes firms enter the labor market resulting in more vacancies and lower unemployment until equilibrium is restored. Similarly, values of $x < x^a(\theta)$ are associated with a return on equity below the return on money. Hence, there is excessive demand for money, which induces firms to leave the labor market resulting in fewer vacancies and higher unemployment until equilibrium is restored. In addition, this implies that an exogenous reduction in the net dividend yield, for given working hours, leads to firm exit and a reduction in labor market tightness $\theta$, and vice versa.

It also follows from above that the realized return on equity has to be strictly positive in the stagnation steady state. Since the return on equity at $\theta = \tilde{\theta}$ is zero and we have $\theta\phi(\bar{\theta}, x) < 0$, it has to hold that $\theta^s < \tilde{\theta}$ in the stagnation steady state. Consumption $c^s$ is determined by (36) with $x = x^s$ and $\theta = \theta^s$. Non-negativity of $c^s$ and the restriction $x^s \leq 1$ require $\theta \in (\theta^s, \tilde{\theta})$, where $\theta^s$ is defined by $c(\theta^s, x^a(\theta^s)) = 0$ with $x^a(\theta)$ defined in (42). The following lemma summarizes these conditions.

**Lemma 3** Define $\theta^s$ by $c(\theta^s, x^a(\theta^s)) = 0$ in (36) with $x^a(\theta)$ given by (42) and with $\theta^s < \tilde{\theta}$, where $\tilde{\theta}$ is uniquely defined in (39). The secular stagnation steady state with $\theta^s \in (\theta^s, \tilde{\theta})$ and $x^s < 1$ exists if $\rho > \alpha$ and if

$$\rho \phi'(c(\tilde{\theta}, 1)) < \omega'(b(\tilde{\theta})),$$

where $b(\theta)$ is given by (35).

The condition $\rho > \alpha$ establishes a lower bound on the rate of deflation and hence an upper bound on the real interest rate thereby preventing a wage-price spiral that is not consistent with the existence of a well-defined steady state. An excessively high rate of deflation in steady state would imply an expansion of real money balances that violates the transversality condition.

---

23 Note that $\theta$ in Lemma 1 and $\theta^s$ are identical for $g = 0$ and given by $\theta = \theta^s = \theta$. 20
of the household in (5). Existence of the secular stagnation steady state further requires the real interest at $\hat{\theta}$ to be negative. This is exactly opposite to condition (40) in Lemma 1 that establishes the feasibility of the steady state without aggregate demand shortage.\textsuperscript{24} Throughout this paper, we assume that the secular stagnation steady state is unique. This holds under mild restrictions on the functional forms of the utility, production and matching functions.\textsuperscript{25}

Whilst the response of consumption to an increase in labor market tightness in the absence of aggregate demand shortage depends on the functional form of the matching function and the preference for wealth, the secular stagnation steady state necessarily has to be slack if the steady state is unique. This is summarized in the following lemma, which follows immediately from the uniqueness condition of the steady state.

**Lemma 4** If the secular stagnation steady state is unique, then it is slack and an increase in labor market tightness raises steady state consumption $c(\theta^*, x^*(\theta^*))$, i.e. $\frac{dc(\theta, x^*(\theta))}{d\theta} > 0$ in (36) where $x^a(\theta)$ is given by (42).

How about the stability properties of the secular stagnation steady state? We derive the following lemma.

**Lemma 5** If the secular stagnation steady state is unique, then it is also saddle-path stable.

Having analyzed existence and stability of the stagnation steady state, we turn to the relative importance of structural unemployment and demand-driven underemployment and the effects of various shocks. In order to get a better understanding of the model and to quantify these effects, the theoretical analysis is supported by a numerical simulation of the model.

### 4.1 Calibration

We set a time period to be one quarter. We assume a standard CRRA utility function for utility from consumption. A non-homothetic specification for the preference for wealth is empirically most plausible as argued by Mian et al. (2020), implying that the marginal utility of wealth decays more slowly than the marginal utility of consumption. The easiest way to implement this feature is via a constant marginal utility of wealth, i.e. $\omega' = \beta$. This is our benchmark specification. In addition, we consider a constant absolute risk aversion (CARA) utility, where the parameter $\beta$ reflects the weight of the preference in the utility function.\textsuperscript{26} We assume a

\textsuperscript{24} Assuming uniqueness of the steady state with $x = 1$ and the one with $x < 1$, these steady states are mutually exclusive. This no longer holds for $g_m > 0$. Then both steady states are feasible for some parameter calibrations.

\textsuperscript{25} Formally, uniqueness requires that under secular stagnation an exogenous increase in wealth increases the propensity to save of the household and therefore increases the aggregate equity value via an increase in the steady state labor market tightness $\theta^*$. Compared to the previous case, an increase in $\theta$ affects the propensity to save via two additional channels: First, the induced decrease in the real interest rate is dampened due to the weaker response of the return on equity to increases in $\theta$ as a consequence of the prevalence of persistent deflation. Secondly, there is a stronger effect of changes in $\theta$ on the net supply as a consequence of the induced change in realized labor working hours. Both, higher income and a smaller sensitivity of the return on savings relative the case of full employment support the assumption of an increase in the propensity to save in response to an exogenous increase in wealth under stagnation.

\textsuperscript{26} Mian et al. (2020) do not directly specify the preference for wealth but the marginal utility of wealth relative to a logarithmic specification, which reflects the utility from consumption. Their calibration implies that this measure is always strictly positive in $(0, 1)$. Similarly, our benchmark specification implies that this measure is constant at level $\beta$, which we chose in $(0, 1)$.  

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Cobb-Douglas matching function in our baseline specification as is commonly employed in the literature. It implies a constant elasticity $\gamma$ of the job filling rate $q(\theta)$ with respect to labor market tightness. The scale parameter $\kappa$ reflects the matching efficiency. Specifically, we assume the following functions:

$$\phi(c) = \frac{c^{1-\eta_c} - 1}{1 - \eta_c},$$

$$\omega(a - m^s) = \beta(a - m^s) \text{ (benchmark)},$$

$$\omega(a - m^s) = -\frac{\beta}{\eta_a} e^{-\eta_a(a-m^s)} \text{ (alternative)},$$

$$F(u, v) = \kappa u^\gamma v^{1-\gamma}.$$ 

Based on these functional forms, fifteen parameters need to be calibrated. The calibration is summarized in Table 1. We set $\eta_c = 2$ and $\eta_a = 0.5$ to capture the difference in curvature between $\phi(.)$ and $\omega(.)$. The parameter $\beta$ measures the relative importance of the preference for wealth and is chosen to generate a specific level of underemployment in steady state.

In standard models, the time preference rate is calibrated to match the steady state real interest rate. In our set-up, the time preference rate only represents an upper bound on the real rate, the difference reflecting the wealth premium, as can be seen in (34). We assume this upper bound to be equal to 12% on an annual basis, resulting in a quarterly time preference rate of $\rho = 0.03$. Lemma 3 restricts the speed of price adjustment to allow for the possibility of stagnation. This parameter also reflects an upper bound on the rate of deflation as is clear from equation (8). We set $\alpha = 0.005$, which implies a maximum annual deflation rate of 2%.

Without loss of generality, we normalize labor productivity and potential output to $\bar{y} = 1$. Variations in this parameter are used to analyze the effects of supply shocks. Our baseline calibration also assumes $g = 0$ and $s = 0$ and subsequent variations in these parameter are used to illustrate the effects of demand shocks and the introduction of a search cost subsidy.

We set the wage replacement rate for unemployed equal to $z = 0.6$, which follows the calibration in Kitao et al. (2017) for the U.S. and Miyamoto (2011) as well as Kudoh et al. (2019) for Japan. In addition, the OECD reports similar initial net replacement rates for Japan (62%) and the United States (54%) (see Table 3.2 in OECD, 2006). Following Mortensen and Pissarides (1994) and Den Haan et al. (2000), among others, we assume symmetric bargaining power $\epsilon = 0.5$ between workers and firms. This implies an effective bargaining power of $\psi = 0.714$. Contributions such as Shimer (2005), Miyamoto (2011) and Kudoh et al. (2019) have assigned a higher bargaining power to workers of $\varepsilon = 0.72$ and $\varepsilon = 0.6$ respectively. Taken together, our values for $\varepsilon$ and $z$ imply an effective bargaining power of workers of $\psi = 0.714$.

This set of parameters is common to all of our simulation. In contrast, the properties of the search process are chosen to match empirical features of the Japanese labor market.

We calibrate the job separation rate to match the average duration of a job. Miyamoto (2011) reports an average job duration rate of 18 years for Japan during the 1980 to 2010...
Table 1: Parameter calibration

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Calibration</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>CRRA parameter in $\phi(.)$</td>
<td>$\eta_c = 2$</td>
<td>implies $\eta_c &gt; -b^* \omega''(b^<em>) / \omega'(b^</em>)$</td>
</tr>
<tr>
<td>CARA parameter in $\omega(.)$</td>
<td>$\eta_a = 0.5$</td>
<td>(used in the alternative specification of the wealth preference)</td>
</tr>
<tr>
<td>Scale parameter in $\omega(.)$</td>
<td>$\beta$</td>
<td>calibrated to generate a specific $x^*$ in steady state</td>
</tr>
<tr>
<td>Time preference rate $\rho$</td>
<td>$\rho = 0.03$</td>
<td>upper bound on the real interest rate: $r \leq 0.12$</td>
</tr>
<tr>
<td>Price adjustment speed $\alpha$</td>
<td>$\alpha = 0.005$</td>
<td>maximum rate of deflation: $\pi \geq -0.02$</td>
</tr>
<tr>
<td>Potential income</td>
<td>$\bar{y} = 1$</td>
<td>normalization, used to model supply shocks</td>
</tr>
<tr>
<td>Public spending</td>
<td>$g = 0$</td>
<td>used to model demand shocks</td>
</tr>
<tr>
<td>Search cost subsidy $s$</td>
<td>$s = 0$</td>
<td>policy variable</td>
</tr>
<tr>
<td>Replacement rate $z$</td>
<td>$z = 0.6$</td>
<td>cf. Kitao et al. (2017), Kudoh et al. (2019)</td>
</tr>
<tr>
<td>Bargaining power</td>
<td>$\varepsilon = 0.5$</td>
<td>cf. Mortensen and Pissarides (1994), Den Haan et al. (2000)</td>
</tr>
<tr>
<td>Bargaining power (eff.) $\psi$</td>
<td>$\psi = 0.714$</td>
<td>$\psi = \varepsilon (1 - (1 - \varepsilon) z)^{-1}$</td>
</tr>
<tr>
<td>Job separation rate $\delta$</td>
<td>$\delta = 0.015$</td>
<td>average job duration of 16.7 years</td>
</tr>
<tr>
<td>Cost of a vacancy $k$</td>
<td>$k = 0.431$</td>
<td>match $r = 5%$ (yearly) in (33) for $\bar{\theta} = 0.85$ and $x = 1$</td>
</tr>
<tr>
<td>Elasticity of $q(\theta)$</td>
<td>$\gamma = 0.5$</td>
<td>Hosios (1990) efficiency condition, $\gamma = \varepsilon$</td>
</tr>
<tr>
<td>Scale parameter $\kappa$</td>
<td>$\kappa = 0.449$</td>
<td>match $\bar{u} = 0.035$ and $\bar{\theta} = 0.85$</td>
</tr>
</tbody>
</table>

Note: The parameter $\beta$ of the wealth preference is chosen to generate a specific output gap in steady state. Details for the calibration of the other parameters are provided in the text.

period, while more recent data by the Japan Institute for Labour Policy and Training is closer to 12 years. We hence choose $\delta = 0.015$ for Japan, implying an average job duration of 16.7 years.\textsuperscript{28} Japan reports an average unemployment rate of $\bar{u} = 3.5\%$ over the 1980-2018 period (see Figure 1). In addition, data on the active job-openings-to-applicants ratio for the period 1980-2018 is available from the Japan Institute for Labour Policy and Training. We use the average value of $\bar{\theta} = 0.85$ over this period as a measure of steady state labor market tightness. Based on this data, the Beveridge curve in (32) implies an average job-finding rate $p(\bar{\theta}) = 0.41$ on a quarterly basis, which is in line with Miyamoto (2011), who reports a quarterly rate of 0.426 for Japan. We calibrate the cost parameter $k$ to yield an annual return on equity of 5% in the absence of aggregate demand shortage using the steady state expression (33) with $x = 1$. This yields an estimated vacancy cost of $k = 0.431$ based on data for Japan.\textsuperscript{29}

As is standard in the literature (cf. Shimer, 2005; Miyamoto, 2011; Kudoh et al., 2019, among others), we use the Hosios (1990) efficiency criterion and calibrate the parameter $\gamma$ in the matching function equal to the bargaining power parameter, i.e. $\gamma = \varepsilon = 0.5$, so that the only inefficiencies emerge from the zero lower bound and the price rigidity. The scale parameter $\kappa$ in the matching function is then calculated to match the average job finding rate $p(\theta)$. This

\textsuperscript{28}Japan’s job separation rate is substantially lower than the separation rate in the United States, reflecting among others institutional features such as the traditional system of lifetime employment.

\textsuperscript{29}This value may be slightly higher compared to the existing literature. Miyamoto (2011) calibrate a value of $k = 0.31$ for Japan. Yet, this is based on the assumption of a higher bargaining power of workers. We also can attain lower values by assuming a higher bargaining power of workers. For instance, for $\varepsilon = 0.6$, the respective value becomes $k = 0.29$. 

23
implies a value of $\kappa = 0.449$, which is in line with the values reported in Miyamoto (2011) and Kudoh et al. (2019) based on a similar approach.

In addition, we modify this calibration to conduct two robustness checks. In robustness check I, we adjust the calibration to match the corresponding features of the U.S. labor market with a higher separation rate and a higher steady state unemployment rate. In robustness check II, we use a more general matching function with a constant elasticity of substitution between vacancies and unemployed following Den Haan et al. (2000). Our main conclusions are robust to these modifications. All results are reported in detail in the appendix.

4.2 The employment gap under stagnation

How does worsening stagnation affect the labor and goods markets? Consider the effects of an exogenous increase in the desire to save, i.e. the steady state marginal utility of wealth $\omega'(\cdot)$, in condition (41). Higher desired savings result in excess supply in the goods market to which firms respond by reducing working hours and production as they are no longer able to sell all of their products. Hence, realized working hours fall reducing firm profits and increasing deflation. As a consequence, firms leave the market resulting in fewer vacancies and higher unemployment. The higher unemployment and lower working hours reduce household income. In equilibrium, the drop in net income has to be sufficiently strong to balance the higher desired savings. Hence, worsening stagnation leads to both higher unemployment and underemployment. In addition, the real wage, consumption and output fall. This is summarized in the following proposition.

**Proposition 2** As stagnation worsens, following an exogenous increase in the desire to save, both unemployment and underemployment increase while total output, household consumption and the real wage decline in steady state.

As a consequence of rising unemployment and underemployment, the total employment gap in (31) increases as stagnation worsens with the unemployment rate itself being an inadequate indicator of the overall slack in the labor market. Using the steady state expression for the unemployment rate in (32), we rewrite decomposition (31) in shares of the steady state employment gap as

$$1 = \frac{\delta}{\delta + p(\theta)(1 - x)} + \frac{p(\theta)(1 - x)}{\delta + p(\theta)(1 - x)},$$

(44)

The relative importance of underemployment is determined by the behavior of $p(\theta)(1 - x)$ in response to (exogenous) variations in $(1 - x)$. Underemployment becomes relatively more important if this term is increasing the more an economy suffers from stagnation and relatively less important otherwise. The share of underemployment in the total employment gap increases under stagnation if

$$x'(\theta) \frac{p(\theta)}{p'(\theta)} > 1 - x(\theta),$$

(45)

where $x(\theta)$ is defined in (42) with $x'(\theta) > 0$. It is easy to see that this inequality has to hold when $x$ is sufficiently close to unity. As an economy falls into stagnation and the degree of
### Table 2: Steady state with underemployment

<table>
<thead>
<tr>
<th>Shortfall of working hours $1 - x^s$ (targeted)</th>
<th>2 %</th>
<th>5 %</th>
<th>10 %</th>
<th>20 %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unemployment rate under stagnation $u^s$</td>
<td>3.47</td>
<td>3.52</td>
<td>3.62</td>
<td>3.84</td>
</tr>
<tr>
<td>Total employment gap in equation (31)</td>
<td>5.3</td>
<td>8.3</td>
<td>13.2</td>
<td>23.1</td>
</tr>
<tr>
<td>Underemployment share in equation (44)</td>
<td>35.0</td>
<td>57.7</td>
<td>72.6</td>
<td>83.4</td>
</tr>
</tbody>
</table>

| Difference in unemployment rates $u^s - u^f$   | 0.04| 0.11| 0.22 | 0.49 |
| Shortfall of consumption $c^s/c^f - 1$         | -2.0| -5.1| -10.2| -20.5|
| Shortfall of production $Y^s/Y^f - 1$          | -2.0| -5.1| -10.1| -20.4|

Rows: (1) Targeted shortfall of working hours below potential, i.e. $1 - x^s$. Generated by adjusting the scale parameter $\beta$ in the marginal utility of wealth. (2) Unemployment rate (in %). (3) Total employment gap (in %), see equation (31). (4) Share of underemployment in the total employment gap (in %), see equation (44). (5) Difference of unemployment rate relative to standard model without demand shortage (in %). (6) Consumption shortfall relative to the standard model without demand shortage (in %). (7) Consumption shortfall relative to the standard model without demand shortage (in %).

As the degree of demand shortage worsens, the total employment gap widens considerably as both unemployment and underemployment increase and the economy moves along the Beveridge curve in (32). This is in stark contrast to the standard case where a stronger desire for savings reduces the real interest rate, which increases labor market tightness and results in a lower unemployment rate.

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30 These values are in fact barely affected by the specification of the matching function and the wealth preference. This is in stark contrast to the effects of parameter variations discussed later.

31 This is the appropriate benchmark for comparisons since changes in the strength of the wealth preference also affect the standard steady state. A stronger wealth preference, i.e. a stronger desire to save, reduces the real interest rate, which increases labor market tightness and results in a lower unemployment rate.
to save reduces unemployment. However, the spillovers from demand shortage to the unemployment rate are quantitatively weak. Even at a 20% shortfall of working hours, structural unemployment is only about half a percentage point higher than in the alternative steady state. Underemployment quickly dominates in the total employment gap accounting for already one third of the gap at a 2% shortfall of working hours and more than half at a 5% shortfall.

Finally, we document strong spillover effects from underemployment in the labor market into macroeconomic variables in the goods market. It is the shortfall of working hours rather than the increase in the unemployment rate that accounts almost completely for the shortfall in consumption and production under stagnation.

To further illustrate the importance of underemployment, Figure 2 shows the decomposition of the total employment gap (solid line) into structural unemployment (dashed line) and demand-driven underemployment (shaded area), based on equation (31), for the baseline calibration as well as for the two alternative calibrations described in the appendix. Variations in the employment gap are generated by variations in the scale parameter of the wealth preference representing exogenous changes in the desire to save.

In all cases, underemployment rises steeply once stagnation occurs and quickly dominates in the employment gap. In contrast, the response of structural unemployment is weak, albeit spillover effects from demand shortage into unemployment are slightly stronger using the calibration based on the U.S. labor market.\textsuperscript{32} Hence, we conclude that demand-driven underemployment provides a more accurate measure of the slack in the labor and goods markets under stagnation than the unemployment rate.

\textsuperscript{32}In fact, throughout our simulations, it is only when the economy suffers from a substantial degree of demand shortage that the unemployment rate becomes more responsive to further variations in the wealth preference parameter. The reason is that the No-Arbitrage condition (42) establishes a lower bound on realized working hours. Once the economy operates close to this value, the derivative $x'(\theta)$ becomes sufficiently small such that the inequality in (45) no longer holds.
4.3 Effects of micro and macro shocks under stagnation

Before looking at the properties of the stagnation steady state in response to parameter variations, we establish a helpful intermediate result about the effects of an exogenous increase in working hours \( x \) in the goods market for a given level of labor market tightness \( \theta \). On the one hand, an increase in working hours increases output and hence net income of the household \( c(\theta, x) \). The propensity to save increases as the wealth premium rises. On the other hand, higher working hours reduce the rate of deflation making savings less attractive. If the intertemporal elasticity of substitution for consumption \( \eta_c^{-1} \) is sufficiently small, the first effect dominates and the increase in working hours results in excess supply in the goods market. This is summarized in the following lemma.

**Lemma 6** For a sufficiently small intertemporal elasticity of substitution \( \eta_c^{-1} \), an exogenous increase (decrease) in working hours creates excess supply (demand) in the goods market for a given \( \theta \). Formally,

\[
- \frac{\omega'(b)\phi''(c)(1-u(\theta))\bar{y}}{\phi'(c)^2} > \alpha \quad \text{if} \quad \eta_c > \frac{\alpha}{\rho}.
\]

Since \( \rho > \alpha \) under stagnation, this is a weak restriction on the intertemporal substitution elasticity for consumption and we will assume that it holds throughout the following analysis. Note that this condition holds irrespective of the properties of the preference for wealth.

The properties of the stagnation steady state differ substantially from those of the standard model and are summarized in the following proposition.

**Proposition 3** In the secular stagnation steady state, variations of the model parameters have the following effects on labor market tightness \( \theta^s \), the unemployment rate \( u^s \), realized working hours \( x^s \), total output \( Y^s \) and consumption \( c^s \):

<table>
<thead>
<tr>
<th></th>
<th>( \theta^s )</th>
<th>( u^s )</th>
<th>( x^s )</th>
<th>( Y^s )</th>
<th>( c^s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wage Shock: ( \varepsilon, z )</td>
<td>-/+</td>
<td>+/-</td>
<td>+/-</td>
<td>+/-</td>
<td>+/-</td>
</tr>
<tr>
<td>Cost Shock: ( k )</td>
<td>+/-</td>
<td>+/-</td>
<td>+/-</td>
<td>+/-</td>
<td>+/-</td>
</tr>
<tr>
<td>Cost Subsidy: ( s )</td>
<td>+/-</td>
<td>+/-</td>
<td>+/-</td>
<td>+/-</td>
<td>+/-</td>
</tr>
<tr>
<td>Supply Shock: ( \bar{y} )</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Demand Shock: ( g )</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

In addition, we quantify these effects for our baseline calibration. The respective tables for the alternative calibrations can be found in the appendix. All effects are shown for the stagnation steady state with 5% underemployment, i.e. \( x^s = 0.95 \), before the parameter variation.

**Wage and cost shocks:** Wage and cost shocks create rich interactions between the standard job creation channel and the wealth channel. Unlike in the previous case, the job creation channel no longer necessarily dominates the wealth channel, which is why most of the effects are
undetermined. These shocks can qualitatively result in either co-movements of unemployment and underemployment or movements in opposite directions. In fact, the relative strength of the transmission channel depends on the strength and curvature of the wealth preference, the substitution elasticity between vacancies and unemployed in the matching function, the degree of demand shortage and the steady state properties of the net supply function in (36).

For illustration, consider the effects of an increase in the replacement rate $z$, which increases the effective bargaining power of workers $\psi$ in the negotiation process. The higher negotiated real wage reduces firm profits and the return on equity in (41). Firm ownership becomes less attractive relative to money and firms leave the labor market. Firm exit results in a reduction in labor market tightness and higher unemployment for given working hours. This is the standard job creation channel. However, there are spillovers into the goods market. These depend on the response of the wealth premium to changes in labor market tightness. On the one hand, consumption demand decreases as household wealth declines following the reduction in firm value. The associated increase in the wealth premium makes the household more willing to save. The strength of this channel depends on the strength and curvature of the wealth preference and the shape of the matching function, which affects the response of wealth to changes in labor market tightness. On the other hand, higher unemployment and a lower number of firms affect household income in (36), which can increase or decrease in response to changes in labor market tightness for a given $x$. This depends on the degree of demand shortage $x$ and on whether the steady state without demand shortage is “tight” or “slack”.

Consider two extreme cases. First, assume a linear wealth preference, a low degree of demand shortage and a slack steady state with $\partial c(\theta, 1)/\partial \theta > 0$. In this case, the reduction in labor market tightness following the higher replacement rate reduces household income and therefore increases the wealth premium resulting in excess demand in the goods market. Excess demand allows firms to sell more products, which is why working hours increase and underemployment declines. The increase in working hours reduces excess demand, which follows from Lemma 6, restoring goods market equilibrium and mitigating the initial drop in labor market tightness. In addition, the rate of deflation falls and so does the real interest rate, which stimulates consumption. Hence, the higher replacement rate results in higher unemployment but fewer underemployment and higher household consumption.

Assume instead a sufficiently concave wealth preference or a high degree of demand shortage or a tight steady state with $\partial c(\theta, 1)/\partial \theta < 0$. In this case, the reduction in labor market tightness has the opposite effect as it reduces the wealth premium resulting in higher desired savings and excess supply in the goods market. Firms respond to the lower demand by cutting production, which is why working hours decrease and underemployment rises. The reduction in working hours is necessary to restore goods market equilibrium, but now reinforces the initial drop in labor market tightness as the return on holding money rises due to higher deflation. Household consumption drops substantially in response to both a higher real interest rate and lower wealth. Taken together, the higher replacement rate results in higher unemployment, more underemployment and lower household consumption.

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33 This channel does not operate in the extreme case of a linear wealth preference and hence a constant marginal utility of wealth.
Table 3: Steady state effects of wage and cost shocks

<table>
<thead>
<tr>
<th>Change in:</th>
<th>Benefits $z \uparrow$</th>
<th>Subsidy $s$</th>
<th>Cost $k \uparrow$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Linear</td>
<td>CARA</td>
<td>Linear</td>
</tr>
<tr>
<td>Underemployment $\Delta(1-x^*)$</td>
<td>-0.15</td>
<td>1.43</td>
<td>0.11</td>
</tr>
<tr>
<td>Unemployment $\Delta u^*$</td>
<td>0.23</td>
<td>0.27</td>
<td>-0.18</td>
</tr>
<tr>
<td>Employment gap in (31)</td>
<td>0.07</td>
<td>1.63</td>
<td>-0.06</td>
</tr>
<tr>
<td>Consumption $\Delta c^<em>/c^</em>$ (in %)</td>
<td>0.01</td>
<td>-1.70</td>
<td>-0.01</td>
</tr>
<tr>
<td>Production $\Delta Y^<em>/Y^</em>$ (in %)</td>
<td>-0.08</td>
<td>-1.78</td>
<td>0.07</td>
</tr>
<tr>
<td>Unemployment $\Delta u^f$</td>
<td>0.23</td>
<td>0.22</td>
<td>-0.17</td>
</tr>
<tr>
<td>Consumption $\Delta c^f/c^f$ (in %)</td>
<td>-0.15</td>
<td>-0.15</td>
<td>0.10</td>
</tr>
<tr>
<td>Production $\Delta Y^f/Y^f$ (in %)</td>
<td>-0.24</td>
<td>-0.23</td>
<td>0.18</td>
</tr>
</tbody>
</table>

Policies: (i) Increase in the replacement rate from $z = 0.6$ to $z = 0.65$. (ii) Introduction of a search cost subsidy of $s = 10\%$. (iii) 1% increase in vacancy costs $k$.

Rows: (1) Change in underemployment (in pp). (2) Change in unemployment rate (in pp). (3) Change in total employment gap (in pp). (4) Change in private consumption (in %). (5) Change in production (in %). (6) Change in unemployment rate in the absence of demand shortage (in pp). (7) Change in consumption in the absence of demand shortage (in %). (8) Change in production in the absence of demand shortage (in %).

Table 3 summarizes the effects of wage and cost shocks, specifically an increase in the replacement rate $z$ by 5%, the introduction of a $s = 10\%$ search cost subsidy and a 1% increase in the vacancy cost parameter $k$, for two specifications of the wealth preference. The first column for each shock show the results for a linear wealth preference, corresponding to the first scenario above, the second column refer to the CARA specification, reflecting the second scenario.

All parameter variations in Table 3 have the same qualitative effects on labor market tightness and unemployment as in the standard case. Their effects on unemployment under stagnation are also quantitatively similar to the case without demand shortage.

In fact, wage shocks unanimously reduce labor market tightness and increase structural unemployment under stagnation. Cost shocks have additional effects as they directly affect the firm value in (23) and hence the wealth premium and the saving patterns of the household, unless the marginal utility of wealth is constant. Higher entry costs for firms $k$ also affect the net supply in (36). As a consequence, the combined effect of these changes is unclear and labor market tightness could in principle increase, following higher costs. However, this requires a sufficiently strong wealth channel such that the spillover from the goods into the labor market dominates the job creation channel. This does not occur in our simulations, all of which support the conventional search and matching channel in the labor market, but could be generated for a very strong curvature of the utility from wealth in combination with a substantial amount of underemployment in steady state and a low degree of complementarity of vacancies and unemployed in the matching function.\(^{34}\)

\(^{34}\)Note that with a constant marginal utility of wealth, as in our baseline scenario, the effect of introducing a subsidy or changing the cost parameter $k$ on unemployment is unambiguous and follows directly from Lemma 6.
While the effects of wage and cost shocks on the unemployment rate under stagnation are quantitatively and qualitatively similar to those of the standard model, their effects on underemployment depend critically on the strength of the wealth preference as explained above. In our baseline specification with a constant marginal utility of wealth, the response of underemployment is in stark contrast to the unemployment rate with underemployment declining in response to wage and cost shocks and increasing following the introduction of a cost subsidy. Hence, unemployment and underemployment move in opposite directions and the response of the total employment gap is indeterminate. In contrast, our alternative specification of a CARA wealth preference results in co-movements of underemployment and unemployment as wage and cost shocks do reduce working hours whereas a cost subsidy improves underemployment. As explained above, the reason is that a reduction in labor market tightness results in excess demand in the goods market for a given degree of underemployment in our baseline specification, whereas it results in excess supply in the alternative case.

Our simulations show that the response of underemployment, and hence the spillovers into the goods market, depends on the degree of demand shortage, the curvature of the wealth preference and the substitution elasticity of vacancies and unemployed in the matching function. Specifically, we document co-movements of underemployment and unemployment when the degree of stagnation is small, vacancies and unemployed are closer substitutes in the matching process and for a stronger curvature of the utility from wealth. In contrast, underemployment and unemployment move in opposite directions following wage and costs shocks when stagnation is sufficiently severe, for a higher degree of complementarity of vacancies and unemployed and for a lower concavity of the utility from wealth. The effects of wage and cost shocks on consumption and output follow directly from this analysis. In fact, for a constant marginal utility of wealth, consumption and working hours have to change in the same direction.

The spillover effects from the labor market into consumption and output are weak in our baseline specification but can be quite substantial in the alternative case of a CARA specification. To conclude, the response of the economy to variations in wage and cost factors under stagnation is dependent on several factors related to the functional form of the wealth preference and the matching function allowing for both co-movement of unemployment and underemployment or movements in opposite directions. Labor market policies that focus exclusively on reducing the unemployment rate might in fact increase the total employment gap via their effect on underemployment and should only be employed with great caution.

Supply and demand shocks: In contrast to wage and cost shocks, the effects of supply and demand shocks are unambiguous and result in co-movements of structural unemployment and underemployment in steady state. These shocks have opposite effects on both labor market variables and macroeconomic variables under stagnation compared to the standard case. Their qualitative effects are independent of the shape of the wealth preference and the matching function.

35Note that the steady state without aggregate demand shortage is slack in our simulation so that $x$ increases following a wage shock in case of a constant marginal utility of wealth in Table 3.

36Any increase (decrease) in working hours directly lowers (increases) the real interest rate, which requires an increase (a reduction) in the wealth premium. With $\omega'(b) = \beta$, consumption has to increase (decrease).
Table 4: Steady state effects of supply and demand shocks

<table>
<thead>
<tr>
<th>Change in:</th>
<th>Productivity $\bar{y}$ ↑</th>
<th>Spending $g$ ↑</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Linear</td>
<td>CARA</td>
</tr>
<tr>
<td>Underemployment $\Delta(1 - x^s)$</td>
<td>1.02</td>
<td>1.03</td>
</tr>
<tr>
<td>Unemployment $\Delta u^s$</td>
<td>(+)0</td>
<td>(+)0</td>
</tr>
<tr>
<td>Employment gap in (31)</td>
<td>0.99</td>
<td>1.00</td>
</tr>
<tr>
<td>Consumption $\Delta c^s/c^s$ (in %)</td>
<td>-0.09</td>
<td>-0.10</td>
</tr>
<tr>
<td>Production $\Delta Y^s/Y^s$ (in %)</td>
<td>-0.09</td>
<td>-0.10</td>
</tr>
<tr>
<td>Unemployment $\Delta u^f$</td>
<td>-0.02</td>
<td>-0.02</td>
</tr>
<tr>
<td>Consumption $\Delta c^f/c^f$ (in %)</td>
<td>1.03</td>
<td>1.03</td>
</tr>
<tr>
<td>Production $\Delta Y^f/Y^f$ (in %)</td>
<td>1.02</td>
<td>1.02</td>
</tr>
</tbody>
</table>

Shocks: (i) 1% increase in productivity (or potential output) $\bar{y}$. (ii) Increase in public spending from $g = 0$ to $g = 0.01$ (i.e. 1% of potential output).

Rows: (1) Change in underemployment (in pp). (2) Change in unemployment rate (in pp). (3) Change in total employment gap (in pp). (4) Change in private consumption (in %). (5) Change in production (in %). (6) Change in unemployment rate in the absence of demand shortage (in pp). (7) Change in consumption in the absence of demand shortage (in %). (8) Change in production in the absence of demand shortage (in %).

function. The reversal of the effects compared to the standard model is a direct consequence of the shortage of aggregate demand under stagnation.

Table 4 summarizes the effects of supply and demand shocks, specifically a 1% increase in productivity $\bar{y}$ and an increase in government spending to $g = 0.01$ (i.e. 1% of potential output).

Consider the effects of the increase in potential output $\bar{y}$. This increases the net income of the household and hence his desire to save, resulting in excess supply in the goods market. Firms respond to reduced sales by cutting working hours resulting in hour underemployment. In fact, the decline in working hours overcompensates the effect of the increase in productivity so that firm profits drop and firms exit the market resulting in a reduction in labor market tightness. As a consequence, both unemployment and underemployment under stagnation increase, which reduces household consumption and total output, thereby overcompensating the effect of the higher productivity. This “paradox of toil” is a common occurrence in stagnation models and in stark contrast to the responses of consumption and output in standard models.

An increase in aggregate demand via higher government spending results in a tighter labor market with both a lower unemployment rate and higher realized working hours. The reason is as follows. Higher public spending creates additional demand to which firms respond by increasing production. Higher firm profits and less deflation make investment in firms more attractive. As more firms enter the market, labor market tightness increases and unemployment declines. This in turn stimulates consumption. Hence, there is a crowding-in effect of government spending, which is in stark contrast to the crowding-out effect in the standard model.\[注\]

Note that this is not related to the notion of deficit spending. In fact, any increase in government spending is budget neutral as the government runs a balanced budget each period by equation (9).
Quantitatively, these “macro” shocks cause substantial variations in working hours, consumption and output. However, their spillover effects into the unemployment rate are weak as fluctuations in the total employment gap are overwhelmingly driven by changes in underemployment, further highlighting the insufficiency of the unemployment rate to capture the total slack in the goods market.

In light of these finding, the simultaneous increase in unemployment and underemployment in Japan during the early 1990s and after 2007 can be attributed to the occurrence of severe negative demand shocks associated with the burst of the bubble economy and the global financial crisis.

5 Conclusion

We have presented a model of secular stagnation that distinguishes between structural unemployment and demand-driven underemployment as a manifestation of economic slack in the labor market. In a permanent liquidity trap, the unemployment rate becomes an insufficient and potentially misleading indicator of the extent of the aggregate demand problem leading to inadequate policy conclusions. Under stagnation, the lack of demand can express itself in various forms of underemployment, such as part-time or non-regular employment. These provide better indicators of the stance of the macroeconomy and the further need for policies in support of demand.

Our findings also suggest that traditional labor market policies aiming to improve the supply side might still succeed in lowering unemployment, but at the same time could contribute to more widespread underemployment in a stagnating economy. Put differently, albeit these policies can succeed in creating new jobs, these are non-regular ones such as part-time jobs. As a consequence, increases in the employment rate are not reflected in an expansion in total output but might in fact be hurtful for both output and consumption. This insight helps to explain the continued sluggishness in the Japanese economy despite the seemingly decent employment record. Examples of such policies include reductions in unemployment benefits, job-creation subsidies to firms or policies aimed at reducing the bargaining power of workers in wage negotiations. These policies should therefore be used only with great caution. In contrast, our model highlights the need for policies that improve aggregate demand as such measures both create new jobs and decrease the degree of underemployment.

References


Appendix: Robustness checks

We summarize the result for two alternative calibrations in this appendix. In robustness check I, we match the model to the U.S. labor market data, using the same approach as in the main text. For robustness check II, we employ a different specification of the matching function, specifically a general CES function following Den Haan et al. (2000). The associated changes in the calibration for both robustness checks are summarized in Table 5. Otherwise the calibration in Table 1 remains unchanged.

<table>
<thead>
<tr>
<th>Table 5: Changes in parameter calibration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Robustness check I: Calibration based on U.S. data</td>
</tr>
<tr>
<td>Job separation rate $\delta = 0.1$</td>
</tr>
<tr>
<td>Cost of a vacancy $k = 0.361$</td>
</tr>
<tr>
<td>Scale parameter $\kappa = 1.487$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Robustness check II: CES matching function</th>
</tr>
</thead>
<tbody>
<tr>
<td>CES parameter $\gamma = 1.27$</td>
</tr>
<tr>
<td>Scale parameter $\kappa = 0.778$</td>
</tr>
</tbody>
</table>

Robustness check I: U.S. labor market data

First, we calibrate the matching process based on U.S. data. Shimer (2005) estimates an average job duration of 2.5 years in the U.S. for the period 1951 to 2003, which is substantially shorter than in the respective value for Japan. We use this estimate to calibrate a quarterly separation rate of $\delta = 0.1$. Based on the average unemployment rate in the United States over the 1980-2018 period of $\bar{u} = 6.3\%$, the Beveridge curve in (32) then allows us to calculate the average job-finding rate $p(\theta)$ implied by the model. The implied value of 1.49 (on a quarterly basis) is close to the ones in Shimer (2005), who estimates the monthly job-finding rate to be equal to 0.45, which implies an average quarterly value of 1.35.

In contrast to Japan, there is no comparable data on labor market tightness available for the United States over an equally long time period. We therefore follow Shimer (2005) and target a steady state value of $\bar{\theta} = 1$, which we use to calibrate the scale parameter $\kappa = 1.487$ in the matching function to be in line with the average job finding rate $p(\theta)$.

Finally, the cost parameter $k = 0.361$ is chosen to yield an annual return on equity of 5% in the absence of aggregate demand shortage using the steady state expression (33) with $x = 1$ and the average labor market tightness value.

Table 6 summarizes the properties of the stagnation steady state for different targeted values of the working hour shortfall, which are generated by variations in the strength of the wealth preference. Tables 7 and 8 illustrate the effects of parameter variations on underemployment and the unemployment in the stagnation steady state.

This calibration results in a higher steady state unemployment rate and slightly stronger
Table 6: Steady state with underemployment

<table>
<thead>
<tr>
<th>Shortfall of working hours $1 - x^*$ (targeted)</th>
<th>2 %</th>
<th>5 %</th>
<th>10 %</th>
<th>20 %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unemployment rate under stagnation $u^*$</td>
<td>6.33</td>
<td>6.43</td>
<td>6.6</td>
<td>6.99</td>
</tr>
<tr>
<td>Total employment gap in equation (31)</td>
<td>8.2</td>
<td>11.1</td>
<td>16.0</td>
<td>25.6</td>
</tr>
<tr>
<td>Underemployment share in equation (44)</td>
<td>22.8</td>
<td>42.1</td>
<td>58.6</td>
<td>72.7</td>
</tr>
<tr>
<td>Difference in unemployment rates $u^* - u^f$</td>
<td>0.07</td>
<td>0.17</td>
<td>0.35</td>
<td>0.77</td>
</tr>
<tr>
<td>Shortfall of consumption $c^*/c^f - 1$</td>
<td>-2.1</td>
<td>-5.2</td>
<td>-10.5</td>
<td>-20.9</td>
</tr>
<tr>
<td>Shortfall of production $Y^*/Y^f - 1$</td>
<td>-2.1</td>
<td>-5.2</td>
<td>-10.4</td>
<td>-20.7</td>
</tr>
</tbody>
</table>

Rows: (1) Targeted shortfall of working hours below potential, i.e. $1 - x^*$. Generated by adjusting the scale parameter $\beta$ in the marginal utility of wealth. (2) Unemployment rate (in %). (3) Total employment gap (in %), see equation (31). (4) Share of underemployment in the total employment gap (in %), see equation (44). (5) Difference of unemployment rate relative to standard model without demand shortage (in %). (6) Consumption shortfall relative to the standard model without demand shortage (in %). (7) Consumption shortfall relative to the standard model without demand shortage (in %).

Table 7: Steady state effects of wage and cost shocks

<table>
<thead>
<tr>
<th>Change in:</th>
<th>Benefits $z \uparrow$</th>
<th>Subsidy $s$</th>
<th>Cost $k \uparrow$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Linear</td>
<td>CARA</td>
<td>Linear</td>
</tr>
<tr>
<td>Underemployment $\Delta(1 - x^*)$</td>
<td>-0.28</td>
<td>0.11</td>
<td>0.2</td>
</tr>
<tr>
<td>Unemployment $\Delta u^*$</td>
<td>0.41</td>
<td>0.43</td>
<td>-0.32</td>
</tr>
<tr>
<td>Employment gap in (31)</td>
<td>0.13</td>
<td>0.51</td>
<td>-0.12</td>
</tr>
<tr>
<td>Consumption $\Delta c^<em>/c^</em>$ (in %)</td>
<td>0.02</td>
<td>-0.40</td>
<td>-0.02</td>
</tr>
<tr>
<td>Production $\Delta Y^<em>/Y^</em>$ (in %)</td>
<td>-0.08</td>
<td>-1.78</td>
<td>0.07</td>
</tr>
<tr>
<td>Unemployment $\Delta u^f$</td>
<td>0.41</td>
<td>0.41</td>
<td>-0.31</td>
</tr>
<tr>
<td>Consumption $\Delta c^f/c^f$ (in %)</td>
<td>-0.28</td>
<td>-0.27</td>
<td>0.20</td>
</tr>
<tr>
<td>Production $\Delta Y^f/Y^f$ (in %)</td>
<td>-0.44</td>
<td>-0.44</td>
<td>0.34</td>
</tr>
</tbody>
</table>

Policies: (i) Increase in the replacement rate from $z = 0.6$ to $z = 0.65$. (ii) Introduction of a search cost subsidy of $s = 10\%$. (iii) 1% increase in vacancy costs $k$.

Rows: (1) Change in underemployment (in pp). (2) Change in unemployment rate (in pp). (3) Change in total employment gap (in pp). (4) Change in private consumption (in %). (5) Change in production (in %). (6) Change in unemployment rate in the absence of demand shortage (in pp). (7) Change in consumption in the absence of demand shortage (in %). (8) Change in production in the absence of demand shortage (in %).

Spillovers from demand shortage into the labor market as also illustrated in Figure 2. Yet, underemployment continues to be the decisive factor in explaining variations in the total employment gap as well as the shortfall of consumption and production relative to the steady state without demand shortage.

In addition, our conclusions on “micro” (i.e. wage and cost) shocks and “macro” (i.e. supply
Table 8: Steady state effects of supply and demand shocks

<table>
<thead>
<tr>
<th>Change in:</th>
<th>Productivity $\bar{y}$ ↑</th>
<th>Spending $g$ ↑</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Linear CARA</td>
<td>Linear CARA</td>
</tr>
<tr>
<td>Underemployment $\Delta(1 - x^s)$</td>
<td>1.02</td>
<td>-1.13</td>
</tr>
<tr>
<td>Unemployment $\Delta u^s$</td>
<td>(+)0</td>
<td>(-)0</td>
</tr>
<tr>
<td>Employment gap in (31)</td>
<td>0.95</td>
<td>-1.10</td>
</tr>
<tr>
<td>Consumption $\Delta c^s/c^s$ (in %)</td>
<td>-0.09</td>
<td>0.10</td>
</tr>
<tr>
<td>Production $\Delta Y^s/Y^s$ (in %)</td>
<td>-0.09</td>
<td>1.23</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.28</td>
</tr>
<tr>
<td>Unemployment $\Delta u^f$</td>
<td>-0.03</td>
<td>(+)0</td>
</tr>
<tr>
<td>Consumption $\Delta c^f/c^f$ (in %)</td>
<td>1.05</td>
<td>(-)0</td>
</tr>
<tr>
<td>Production $\Delta Y^f/Y^f$ (in %)</td>
<td>1.04</td>
<td>(-)0</td>
</tr>
</tbody>
</table>

Shocks: (i) 1% increase in productivity (or potential output) $\bar{y}$. (ii) Increase in public spending from $g = 0$ to $g = 0.01$ (i.e. 1% of potential output).
Rows: (1) Change in underemployment (in pp). (2) Change in unemployment rate (in pp). (3) Change in total employment gap (in pp). (4) Change in private consumption (in %). (5) Change in production (in %). (6) Change in unemployment rate in the absence of demand shortage (in pp). (7) Change in consumption in the absence of demand shortage (in %). (8) Change in production in the absence of demand shortage (in %).

and demand) shocks continue to hold. The latter result in co-movements of unemployment and underemployment, while the effects of the former depend on the strength and curvature of the wealth preference as well as the shape of the matching function.

Robustness check II: CES matching function

For the second robustness check, we employ the following general CES matching function

$$F(u, v) = \kappa \left( u^{-\gamma} + v^{-\gamma} \right)^{-1/\gamma},$$

where we use the same calibration as in Den Haan et al. (2000) with an elasticity parameter of $\gamma = 1.27$.\(^{38}\) Using this specification, we pursue the same approach as in the main text and calibrate the scale parameter $\kappa$ to match the average job finding rate in Japan.

The steady state with this alternative matching function is almost identical to the one depicted in Table 2 on therefore omitted here. Tables 9 and 10 illustrate the effects of parameter variations on underemployment and the unemployment in the stagnation steady state.

Compared to Tables 3 and 4 in the main text, there are only minor quantitative and no qualitative changes of the responses of the model variables to various parameter variations emphasizing again that it is predominantly the functional form of the wealth preference that determines the response to “micro” shocks. Demand shortage manifests itself predominantly in underemployment with few spillovers to the unemployment rate.

\(^{38}\)Note that the elasticity of substitution between vacancies and unemployed in the matching function is given by $(1 + \gamma)^{-1}$. Higher values of $\gamma$ imply that vacancies and unemployed are complements in producing successful matches, while lower values imply a higher degree of substitutability.
Table 9: Steady state effects of wage and cost shocks

<table>
<thead>
<tr>
<th>Change in:</th>
<th>Benefits $z \uparrow$</th>
<th>Subsidy $s$</th>
<th>Cost $k \uparrow$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Linear CARA</td>
<td>Linear CARA</td>
<td>Linear CARA</td>
</tr>
<tr>
<td>Underemployment $\Delta(1 - x^s)$</td>
<td>-0.21 1.13</td>
<td>0.13 1.38</td>
<td>-0.03 -0.16</td>
</tr>
<tr>
<td>Unemployment $\Delta u^s$</td>
<td>0.27 0.30</td>
<td>-0.19 -0.17</td>
<td>0.02 0.02</td>
</tr>
<tr>
<td>Employment gap in (31)</td>
<td>0.06 1.37</td>
<td>-0.06 1.17</td>
<td>-0.01 -0.14</td>
</tr>
<tr>
<td>Consumption $\Delta c^s/c^s$ (in %)</td>
<td>0.02 -1.43</td>
<td>-0.01 -1.36</td>
<td>(+)0 0.14</td>
</tr>
<tr>
<td>Production $\Delta Y^s/Y^s$ (in %)</td>
<td>-0.06 -1.50</td>
<td>0.06 -1.28</td>
<td>0.01 0.15</td>
</tr>
<tr>
<td>Unemployment $\Delta u^f$</td>
<td>0.26 0.25</td>
<td>-0.18 -0.18</td>
<td>0.02 0.02</td>
</tr>
<tr>
<td>Consumption $\Delta c^f/c^f$ (in %)</td>
<td>-0.19 -0.18</td>
<td>0.11 0.11</td>
<td>-0.03 -0.03</td>
</tr>
<tr>
<td>Production $\Delta Y^f/Y^f$ (in %)</td>
<td>-0.27 -0.26</td>
<td>0.19 0.19</td>
<td>-0.02 -0.02</td>
</tr>
</tbody>
</table>

Policies: (i) Increase in the replacement rate from $z = 0.6$ to $z = 0.65$. (ii) Introduction of a search cost subsidy of $s = 10\%$. (iii) 1% increase in vacancy costs $k$.

Rows: (1) Change in underemployment (in pp). (2) Change in unemployment rate (in pp). (3) Change in total employment gap (in pp). (4) Change in private consumption (in %). (5) Change in production (in %). (6) Change in unemployment rate in the absence of demand shortage (in pp). (7) Change in consumption in the absence of demand shortage (in %). (8) Change in production in the absence of demand shortage (in %).

Table 10: Steady state effects of supply and demand shocks

<table>
<thead>
<tr>
<th>Change in:</th>
<th>Productivity $\bar{y} \uparrow$</th>
<th>Spending $g \uparrow$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Linear CARA</td>
<td>Linear CARA</td>
</tr>
<tr>
<td>Underemployment $\Delta(1 - x^s)$</td>
<td>1.02 1.03</td>
<td>-1.11 -1.24</td>
</tr>
<tr>
<td>Unemployment $\Delta u^s$</td>
<td>(+)0 (+)0</td>
<td>-0.02 -0.03</td>
</tr>
<tr>
<td>Employment gap in (31)</td>
<td>0.99 1.00</td>
<td>-1.09 -1.22</td>
</tr>
<tr>
<td>Consumption $\Delta c^s/c^s$ (in %)</td>
<td>-0.09 -0.10</td>
<td>0.10 0.24</td>
</tr>
<tr>
<td>Production $\Delta Y^s/Y^s$ (in %)</td>
<td>-0.09 -0.10</td>
<td>1.19 1.34</td>
</tr>
<tr>
<td>Unemployment $\Delta u^f$</td>
<td>-0.02 -0.02</td>
<td>(+)0 (+)0</td>
</tr>
<tr>
<td>Consumption $\Delta c^f/c^f$ (in %)</td>
<td>1.03 1.03</td>
<td>-1.05 -1.05</td>
</tr>
<tr>
<td>Production $\Delta Y^f/Y^f$ (in %)</td>
<td>1.02 1.02</td>
<td>(-)0 (-)0</td>
</tr>
</tbody>
</table>

Shocks: (i) 1% increase in productivity (or potential output) $\bar{y}$. (ii) Increase in public spending from $g = 0$ to $g = 0.01$ (i.e. 1% of potential output).

Rows: (1) Change in underemployment (in pp). (2) Change in unemployment rate (in pp). (3) Change in total employment gap (in pp). (4) Change in private consumption (in %). (5) Change in production (in %). (6) Change in unemployment rate in the absence of demand shortage (in pp). (7) Change in consumption in the absence of demand shortage (in %). (8) Change in production in the absence of demand shortage (in %).
For Online Publication: Mathematical appendix

**Derivation: Wage rate (24) under Nash bargaining**

From the Bellman equations (19) and (20), we have
\[(r_t + \delta + \theta_t q(\theta_t)) [E_t - U_t] = (1 - z)w_t x_t + [E_t - U_t]. \tag{A.1}\]

Using the optimal sharing rule (21), the above equation is rewritten as
\[(r_t + \delta + \theta_t q(\theta_t)) \frac{\varepsilon}{1 - \varepsilon} J_t = (1 - z)w_t x_t + \frac{\varepsilon}{1 - \varepsilon} \dot{J}_t. \tag{A.2}\]

Using the Bellman equation (17) and the free entry condition (22), the value of an operating firm is given as
\[\dot{J}_t = (r_t + \delta) J_t - (\bar{y} - w_t) x_t. \tag{A.3}\]

Substituting this equation for \(\dot{J}_t\) into (A.2) gives
\[\theta_t q(\theta_t) \frac{\varepsilon}{1 - \varepsilon} J_t = (1 - z)w_t x_t - \frac{\varepsilon}{1 - \varepsilon}(\bar{y} - w_t) x_t. \tag{A.4}\]

Using expression (23) for the value of an operating firm \(J_t\), the above equation gives the wage rate under Nash bargaining as equation (24).

**Derivation: Goods market clearing condition (30)**

Substituting the dynamics of the money supply (12) and the government budget constraint (9) into the flow of funds constraint (2) gives
\[\dot{b}_t = r_t b_t + w_t x_t(1 - u_t) - c_t - skv_t - g. \tag{B.1}\]

Differentiating \(b_t = J_t(1 - u_t)\) from the asset market clearing condition (29) with respect to time, we also have
\[\dot{b}_t = \dot{J}_t(1 - u_t) - J_t \dot{u}_t. \tag{B.2}\]

Substituting (B.2) and \(b_t = J_t(1 - u_t)\) into equation (B.1) gives
\[\dot{J}_t(1 - u_t) - J_t \dot{u}_t = r_t J_t(1 - u_t) + w_t x_t(1 - u_t) - c_t - g - skv_t. \tag{B.3}\]

Using the dynamic equations (16) and (A.3) to substitute for \(\dot{u}_t\) and \(\dot{J}_t\) in (B.3), we get
\[(1 - u_t)\bar{y} x_t = J_t p(\theta_t) u_t + c_t + g + skv_t. \tag{B.4}\]

Finally, using expression (23) for the value of an operating firm \(J_t\) and the identities \(v_t \equiv \theta_t u_t\) and \(p(\theta_t) \equiv \theta_t q(\theta_t)\), equation (B.4) gives the goods market clearing condition as equation (30).

**Derivation: Dynamic system when \(x_t = 1\)**

Without aggregate demand shortage, we have \(x_t = 1\) and \(\dot{x}_t = 0\). The goods market clearing condition (36) then defines consumption at any time \(t\) as a function of the unemployment rate \(u_t\) and the labor market tightness \(\theta_t\) as
\[c_t(\theta_t, u_t) = (1 - u_t)\bar{y} - k\theta_t u_t - g. \tag{C.1}\]
where the partial derivatives are given by $c_u = -(\bar{y} + k\theta) < 0$ and $c_\theta = -ku < 0$. Differentiating this function with respect to $t$ implies

$$\dot{c}_t = -\dot{u}_t (\bar{y} + k\theta_t) - ku_t \dot{\theta}_t. \quad (C.2)$$

Using (3), (16) and (27) to substitute for $\dot{c}_t$, $\dot{u}_t$ and $\dot{\theta}_t$, we derive the following expression for the real interest rate $r_t$ at any time $t$:

$$r_t = \frac{c_t}{\eta_c} \left[ r_t - \rho + \frac{\omega'(b_t)}{\phi'(c_t)} \right] = -\frac{ku_t \theta_t}{\eta_{\theta_t}} [r_t - \vartheta(\theta_t, 1) + \delta] - (\bar{y} + k\theta_t)[\delta(1 - u_t) - p(\theta_t)u_t],$$

$$r_t(\theta_t, u_t) = \varphi_t \left[ \rho - \frac{\omega'(b_t)}{\phi'(c_t)} \right] + (1 - \varphi_t) [\vartheta(\theta_t, 1) - \delta] - \tilde{\varphi}_t \left[ \delta(1 - u_t) - p(\theta_t)u_t \right], \quad (C.3)$$

where

$$\varphi_t = \frac{c_t/\eta_c}{c_t/\eta_c + ku_t \theta_t/\eta_{\theta_t}}, \quad \tilde{\varphi}_t = \frac{\bar{y} + k\theta_t}{c_t/\eta_c + ku_t \theta_t/\eta_{\theta_t}}.$$

From (C.1), $c_t$ is a function of $\theta_t$ and $u_t$. Hence, the real rate $r_t$ is also a function of $\theta_t$ and $u_t$ as $b_t$ depends on the same variables by (21). Using these expressions, we can rewrite the model as a system of three differential equations in $m_t$, $u_t$ and $\theta_t$:

$$\dot{m}_t = \left[ g_m - \frac{\mu'(m_t)}{\phi'(c_t(\theta_t, u_t))} + r_t(\theta_t, u_t) \right] m_t, \quad (C.4)$$

$$\dot{u}_t = (1 - u_t)\delta - p(\theta_t)u_t, \quad (C.5)$$

$$\dot{\theta}_t = \frac{\theta_t}{\eta_{\theta_t}} [r_t(\theta_t, u_t) - \vartheta(\theta_t, 1) + \delta]. \quad (C.6)$$

**Proof: Lemma 1**

Reformulate the first equation of (38) to define the function $H(\theta)$ as

$$H(\theta) = \rho + \delta + \psi p(\theta) - \frac{(1 - \psi)\bar{y}}{(1 - s)k} q(\theta) - \frac{\omega' \left( \frac{(1 - u(\theta)) (1 - s)k}{q(\theta)} \right)}{\phi' \left( \frac{(1 - u(\theta))\bar{y} - k\theta u(\theta) - g}{q(\theta)} \right)}. \quad (D.1)$$

The function $c(\theta, 1) = (1 - u(\theta))\bar{y} - k\theta u(\theta) - g$ in (36) with $x = 1$ defines consumption as a continuous function of $\theta$ with a hump-shaped pattern that attains a maximum at a strictly positive value.\(^{39}\) Define $\bar{\theta}$ and $\theta$ by $c(\bar{\theta}, 1) = c(\theta, 1) = 0$ with $\bar{\theta} < \theta$. Note that $\bar{\theta} = 0$ for $g = 0$. Then $c(\theta, 1) \geq 0$ and $H(\theta)$ is a continuous function of $\theta$ for $\theta \in (\bar{\theta}, \theta)$. As $\omega'(\cdot) < \infty$, we have

$$H(\bar{\theta}) = \rho + \delta + \psi p(\bar{\theta}) - \frac{(1 - \psi)\bar{y}}{(1 - s)k} q(\bar{\theta}) = \rho + \delta - \vartheta(\bar{\theta}, 1),$$

$$H(\theta) = \rho + \delta + \psi p(\theta) - \frac{(1 - \psi)\bar{y}}{(1 - s)k} q(\theta) = \rho + \delta - \vartheta(\bar{\theta}, 1).$$

Note that $H(\bar{\theta}) < H(\theta)$ since $\vartheta(\bar{\theta}, 1) < 0$. Without any other restrictions, a steady state with $H(\theta^f) = 0$ always exists for $\theta^f \in (\bar{\theta}, \theta)$ if $H(\bar{\theta}) < 0 < H(\theta)$ or equivalently if

$$\vartheta(\bar{\theta}, 1) - \delta > \rho > \vartheta(\bar{\theta}, 1) - \delta. \quad (D.2)$$

Money market equilibrium requires a positive real interest rate in steady state. Define $\bar{\theta}$ by

\(^{39}\)We ensure this by choosing the parameters $g$ and $k$ sufficiently low relative to $\bar{y}$.
\[ \phi(\bar{\theta}, 1) = \delta \text{ as in (39), which implies a zero net return on equity and hence a real interest rate of zero. Since } \phi_\theta(\theta, 1) < 0, \text{ equilibrium requires } \theta^* \leq \bar{\theta}. \text{ The steady state without aggregate demand shortage exists in } (\bar{\theta}, \bar{\theta}) \text{ if the following condition holds:}^{40} \]

\[ H(\bar{\theta}) \geq H(\theta^*) = 0 \iff \rho \phi' \left( c(\bar{\theta}, 1) \right) \geq \omega' \left( b(\bar{\theta}) \right). \quad (D.3) \]

Conditions (D.2) and (D.3) are the necessary existence conditions of Lemma 1. Uniqueness of the steady state requires \( H'(\theta) > 0 \) for all \( \theta \) with \( H(\theta) = 0 \). This derivative is given by:

\[ H'(\theta) = \frac{\omega''(b) \phi'(c)}{\phi''(c)} \frac{\rho \phi'}{\omega''(b)} \phi'(c) - b'(\theta) + \frac{\omega''(b) \phi''(c)}{\phi''(c)} \frac{dc}{d\theta}(\theta, 1). \quad (D.4) \]

We assume that \( H'(\theta^*) > 0 \) so that the steady state with \( x = 1 \) is unique.

**Proof: Lemma 2**

The dynamic system is given by (C.4), (C.5) and (C.6). The unemployment rate \( u_t \) is a predetermined state variables, while labor market tightness \( \theta_t \) (via vacancies \( v_t \)) and the money supply (for \( x = 1 \)) are control variables and can jump. Stability of the dynamic system therefore requires one negative and two positive eigenvalues. The Jacobian of this system evaluated at the steady state with \( x = 1 \) is given by

\[
\begin{bmatrix}
-\frac{\mu''(m)}{\sigma'(c)} m & \left[ \frac{\mu'(m)\phi''(c)c_u + r_u}{m} \right] m & \left[ \frac{\mu'(m)\phi''(c)c_u + r_u}{\sigma'(c)^2} \right] m \\
0 & -[\delta + p(\theta)] & -p'(\theta) u \\
0 & \frac{\partial}{\partial \theta} r_u & \frac{\partial}{\partial \theta} (r_\theta - \theta(\theta, 1))
\end{bmatrix}.
\]

(E.1)

where \( r_\chi \) denotes the partial derivative of the real rate defined in (C.1) with respect to variable \( \chi \). The eigenvalues of this system solve the following characteristic equation:

\[ \Omega(\lambda) = \left[ -\frac{\mu''(m)}{\sigma'(c)} m - \lambda \right] \left[ -[\delta + p(\theta)] - \lambda \right] \frac{\partial}{\partial \theta} r_u \frac{\partial}{\partial \theta} (r_\theta - \theta(\theta, 1)) - \lambda = 0. \quad (E.2) \]

It is clear that \( \lambda_1 = -\mu''(m)m/\phi'(c) > 0 \) is one solution. The other two eigenvalues solve

\[ \lambda^2 - \left( \frac{\partial}{\partial \theta} [r_\theta - \theta(\theta, 1)] - \delta + p(\theta) \right) \lambda + \frac{\partial}{\partial \theta} r_u p'(\theta) u - \frac{\partial}{\partial \theta} [\delta + p(\theta)] (r_\theta - \theta(\theta, 1)) = 0. \quad (E.3) \]

We can recover the sign of the eigenvalues from

\[ \lambda_2 \lambda_3 = \frac{\partial}{\partial \theta} \left[ r_u p'(\theta) u - [\delta + p(\theta)] (r_\theta - \theta(\theta, 1)) \right]. \quad (E.4) \]

From (C.3), we get the following expression for the partial derivative of \( r_t \) with respect to \( u_t \) and \( \theta_t \) in steady state:

\[ r_u = -\varphi \left[ \frac{\omega''(b)\phi'(c)b_u - \omega'(b)\phi''(c)c_u}{\phi'(c)^2} \right] + \tilde{\varphi} [\delta + p(\theta)], \]

\[ r_\theta = -\varphi \left[ \frac{\omega''(b)\phi'(c)b_\theta - \omega'(b)\phi''(c)c_\theta}{\phi'(c)^2} \right] + (1 - \varphi) \theta(\theta, 1) + \tilde{\varphi} p'(\theta) u. \]

\(^{40}\)We assume a constant nominal money supply and hence zero inflation, i.e. \( g_m = 0 \). For \( g_m > 0 \), \( \bar{\theta} \) becomes a function of \( g_m \) as the real interest rate cannot fall below \( -g_m \).
Also note that the unemployment rate in steady state is a function of labor market tightness from equation (32) with \( du/d\theta \equiv u'(\theta) = -p'(\theta)u(\theta)/(\delta + p(\theta)) \). Using this expression, we can relate the partial derivatives \( c_u \) and \( c_p \) to the total derivative \( dc(\theta, 1)/d\theta = c'(\theta, 1) \) in steady state as follows (and equivalently for \( b(\theta) \)):{41}

\[
\frac{dc(\theta, u)}{d\theta} \equiv \frac{dc(\theta, 1)}{d\theta} = c_\theta + c_u u'(\theta) = c_\theta - c_u \frac{p'(\theta)u(\theta)}{\delta + p(\theta)},
\]

\[
\frac{db(\theta, u)}{d\theta} \equiv b'(\theta) = b_\theta + b_u u'(\theta) = b_\theta - b_u \frac{p'(\theta)u(\theta)}{\delta + p(\theta)}.
\]

Using these properties, we rewrite the above equation as

\[
\lambda_2 \lambda_3 = \frac{\theta}{\eta_\theta} \left[ -\varphi \left[ \frac{\omega''(b)\phi'(c)b_u - \omega'(b)\phi''(c)c_u}{\phi'(c)^2} \right] p'(\theta)u + \varphi[\delta + p(\theta)] p'(\theta)u \right] -[\delta + p(\theta)] \left[ -\varphi \left[ \frac{\omega''(b)\phi'(c)b_u - \omega'(b)\phi''(c)c_u}{\phi'(c)^2} \right] + (1 - \varphi)\theta(\theta, 1) + \varphi p'(\theta)u - \theta_\theta \right],
\]

\[
= \frac{\theta}{\eta_\theta} \left[ -\varphi \left[ \frac{\omega''(b)\phi'(c)b_u - \omega'(b)\phi''(c)c_u}{\phi'(c)^2} \right] p'(\theta)u \right] +[\delta + p(\theta)] \left[ \varphi \left[ \frac{\omega''(b)\phi'(c)b_u - \omega'(b)\phi''(c)c_u}{\phi'(c)^2} \right] + \varphi\theta(\theta, 1) \right],
\]

\[
= \frac{\theta}{\eta_\theta} \left[ -\varphi[\delta + p(\theta)] \left[ \frac{\omega''(b)}{\phi'(c)} \theta(\theta, 1) - b'(\theta) - \varphi \left[ \frac{\omega''(b)}{\phi'(c)} \theta(\theta, 1) - b'(\theta) + \frac{c_\theta - c_u p'(\theta)u}{\delta + p(\theta)} + \theta_\theta \right] \right] \right],
\]

The term in the brackets of the last expression above is identical to expression (D.4) above. Hence, we can rewrite this equation as

\[
\lambda_2 \lambda_3 = -\frac{\theta}{\eta_\theta} \varphi[\delta + p(\theta)] H'(\theta) < 0.
\] (E.5)

If the steady state is unique, it holds that \( H'(\theta) > 0 \). It then follows that \( \lambda_2 \lambda_3 < 0 \), which implies one negative and one positive eigenvalue. Together with \( \lambda_1 > 0 \), the dynamic system has one negative and two positive eigenvalues and therefore exhibits saddle-path stability around the steady state without demand surplus.

**Proof: Proposition 1**

Use (38) to define the function \( H(\theta, \chi) \), where \( \chi \) denotes any parameter in the model, as follows:

\[
H(\theta, \chi) = \rho + \delta + \psi p(\theta) - \frac{(1 - \psi)\bar{y}}{(1 - s)\bar{q}} q(\theta) - \frac{\omega'(1 - u(\theta))(1 - s)^k}{\phi'(c(\theta, 1))},
\] (F.1)

with \( H(\theta^f) = 0 \) and \( H'(\theta^f) > 0 \) in a unique steady state where \( H_\theta \) denotes the derivative with respect to \( \theta \) evaluated in steady state. \( c(\theta, 1) \) is defined in (36). The effects of changes in a

---

{41} Note that \( d\theta(\theta, 1)/d\theta \equiv \theta_\theta(\theta, 1) = \vartheta_\theta \) in the absence of aggregate demand shortage.
parameter $\chi$ on the labor market tightness $\theta$ can be recovered from (F.1) as

$$\frac{d\theta}{d\chi} = \frac{H_\chi}{H_\theta}. \quad \text{(F.2)}$$

It holds that

$$H_e = \frac{(\bar{y}q(\theta) + (1 - s)kp(\theta) \partial \phi}{(1 - s)k} > 0,$$

$$H_z = \frac{(\bar{y}q(\theta) + (1 - s)kp(\theta) \partial \phi}{(1 - s)k} > 0,$$

$$H_s = \frac{\omega''(b)(1 - u(\theta))k}{\phi'(c)q(\theta)} - \frac{(1 - \psi)\bar{y}q(\theta)}{(1 - s)^2k} < 0,$$

$$H_k = -\frac{\omega''(b)(1 - u(\theta))(1 - s)}{\phi'(c)q(\theta)} - \frac{\omega'(b)\phi'(c)\theta u(\theta)}{\phi'(c)^2} + \frac{(1 - \psi)\bar{y}q(\theta)}{(1 - s)k^2} > 0,$$

$$H_y = \omega'(b)\phi''(c)(1 - u(\theta)) - \frac{(1 - \psi)\bar{y}q(\theta)}{(1 - s)k} < 0,$$

$$H_g = -\frac{\omega'(b)\phi''(c)}{\phi'(c)^2} > 0.$$

This implies the following relationship between the steady state labor market tightness and the model parameters:

$$\theta^f = \theta(\varepsilon, z, s, k, \bar{y}, g). \quad \text{(F.3)}$$

From the Beveridge curve in (32), it immediately follows that the effects on the unemployment rate are opposite to those on the labor market tightness, i.e.

$$u^f = u(\varepsilon, z, s, k, \bar{y}, g). \quad \text{(F.4)}$$

In addition, total output $Y = (1 - u(\theta))\bar{y}$ is affected by the model parameters in the same way as the labor market tightness, except that changes in productivity are reinforced, i.e.

$$Y^f = Y(\varepsilon, z, s, k, \bar{y}, g). \quad \text{(F.5)}$$

Finally, the effects on household consumption can be derived from the goods market clearing condition (36). Specifically, it holds that

$$\frac{dc(\theta, 1)}{d\chi} = \frac{dc(\theta, 1)}{d\theta} \frac{d\theta}{d\chi} + \frac{dc(\theta, 1)}{d\chi}, \quad \text{(F.6)}$$

where $\frac{dc}{d\chi} = 0$ for $\chi = \varepsilon, z, s$, $\frac{dc}{d\chi} = -\theta u$ for $\chi = k$, $\frac{dc}{d\chi} = 1 - u$ for $\chi = \bar{y}$ and $\frac{dc}{d\chi} = -1$ for $\chi = g$. Moreover, $dc(\theta, 1)/d\theta = -u'(\theta)(\bar{y} + k\theta) - ku(\theta)$ and its sign is not uniquely determined. For $dc(\theta, 1)/d\theta > 0$ (“slack” steady state), the sign of the effects of all parameter changes are identical to those of the labor market tightness. For $dc(\theta, 1)/d\theta < 0$ (“tight” steady state), the sign of the effects of changes in $\varepsilon, z$ and $s$ is opposite to those on $\theta$, while the effects of variations in $k, \bar{y}$ and $g$ are indeterminate.
Proof: Lemma 3 and Lemma 4

Equation (41) and (42) define the function $G(\theta)$ as

$$G(\theta) = \rho - \frac{\omega' \left( (1 - u(\theta)) \frac{(1 - s)k}{q(\theta)} \right)}{\phi' \left( (1 - u(\theta)) \bar{y} - k\bar{u}(\theta) - g \right)} - \alpha (1 - x^a(\theta)), \quad (G.1)$$

where $x^a(\theta)$ is the increasing part of the asset market equilibrium curve given by (42). A well-defined steady state requires non-negative consumption. Define $\tilde{\theta}^s$ as the smallest value of $\theta$ such that $c(\tilde{\theta}^s, x^a(\tilde{\theta}^s)) = 0$, where $c(\theta, x^a(\theta)) = x^a(\theta)(1 - u(\theta))\bar{y} - k\bar{u}(\theta) - g$. Note that $\tilde{\theta}^s = \bar{\theta} = 0$ for $g = 0$, where $\bar{\theta}$ is defined in the proof of Lemma 1. In addition, let $\bar{\theta}$ be defined as before by $\bar{\theta}(1) = \delta$ in (39), which implies $x^a(\bar{\theta}) = 1$ in (42). We then have

$$G(\tilde{\theta}^s) = \rho - \alpha (1 - x^a(\tilde{\theta}^s)) = \rho - \alpha \frac{\bar{\theta}(1) - \delta}{\alpha + \frac{(1 - s)k}{q(\tilde{\theta}^s)} > 0},$$

$$G(\bar{\theta}) = \rho - \frac{\omega' \left( (1 - u(\bar{\theta})) \frac{(1 - s)k}{q(\bar{\theta})} \right)}{\phi' \left( (1 - u(\bar{\theta}))\bar{y} - k\bar{u}(\bar{\theta}) - g \right)} = \rho - \frac{\omega' \left( b(\bar{\theta}) \right)}{\phi' \left( c(\bar{\theta}, 1) \right)}.$$  

Note that $G(\tilde{\theta}^s) > 0$ if $\rho > \alpha$ since the ratio $\frac{\bar{\theta}(1) - \delta}{\alpha + \frac{(1 - s)k}{q(\tilde{\theta}^s)}}$ is always strictly smaller than unity.

Existence of the secular stagnation steady state then requires $G(\bar{\theta}) < 0$ or equivalently

$$\rho \phi' \left( c(\bar{\theta}, 1) \right) < \omega' \left( b(\bar{\theta}) \right). \quad (G.2)$$

Whenever full employment is not feasible, we have $G(\bar{\theta}) = H(\bar{\theta}) < 0$ by Lemma 1. It then follows that there exists $\tilde{\theta}^s$ such that $G(\tilde{\theta}^s) = 0$.

Uniqueness of the steady state requires $G'(\theta) < 0$ for all $\theta$ with $G(\theta) = 0$. This derivative is given by

$$G'(\theta) = \frac{\omega'(b)}{\phi'(c)} \left[ -b'(\theta) + \frac{\omega'(b) \phi''(c)}{\omega'(b) \phi'(c)} \frac{dc(\theta, x^a(\theta))}{d\theta} + \frac{\phi'(c) \omega''(b)}{\omega'(b) \alpha} \frac{dx^a(\theta)}{d\theta} \right]. \quad (G.3)$$

Throughout this paper, we assume that $G'(\tilde{\theta}^s) < 0$ so that the steady state is unique. Uniqueness in turn implies that the secular stagnation steady state is slack. Reformulating $G'(\theta) < 0$ implies

$$\frac{\omega'(b) \phi''(c) \frac{dc(\theta, x^a(\theta))}{d\theta}}{\omega''(b) \phi'(c)} > b'(\theta) - \alpha \frac{\phi'(c) \frac{dx^a(\theta)}{d\theta}}{\omega''(b) \phi'(c)} > 0, \quad (G.4)$$

since $\frac{dx^a(\theta)}{d\theta} > 0$ in steady state, which can be seen in (42). It follows that $\frac{dc(\theta, x^a(\theta))}{d\theta} > 0$.

Proof: Lemma 5

In the stagnation steady state, the liquidity preference of the household is satiated such that $R_t = 0$ and hence $\tau_t = R_t - \pi_t = \alpha (1 - x_t)$. The goods market clearing condition (36) defines realized working hours $x_t$ at any time $t$ as a function of $c_t$, $\theta_t$ and $u_t$ as

$$x_t = c_t + k\theta_t u_t + g \frac{1 - x_t}{1 - u_t} \bar{y}, \quad (H.1)$$

\[42\] We ensure non-negativity of consumption in steady state by choosing the parameters $g$ and $k$ sufficiently low relative to $\bar{y}$.  

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where the partial derivatives satisfy \( x_c > 0, x_\theta > 0 \) and \( x_u > 0 \). Define \( \hat{m}_t \equiv 1/m_t \) with \( \hat{m} = 0 \) in steady state. The dynamic system is given by the following differential equations for \( \hat{m}_t, c_t, \theta_t \) and \( u_t \):

\[
\dot{\hat{m}}_t = \alpha (x_t - 1) \hat{m}_t, \\
\dot{c}_t = \left( \alpha (1 - x_t) - \rho + \frac{\omega'(b(\theta_t, u_t))}{\phi'(c_t)} \right) \frac{c_t}{\eta_c}, \\
\dot{\theta}_t = \left[ \alpha (1 - x_t) - \vartheta(\theta_t, x_t) + \delta \right] \frac{\theta_t}{\eta_\theta}, \\
\dot{u}_t = \delta (1 - u_t) - p(\theta_t) u_t,
\]

where \( b(\theta_t, u_t) \) and \( \vartheta(\theta_t, x_t) \) are given by (29) and (26) respectively. Let \( \theta_\theta \) and \( \vartheta_x \) denote the partial derivatives of \( \vartheta(\theta_t, x_t) \) with respect to \( \theta_t \) and \( x_t \). Since the price level cannot adjust freely, the money supply grows with the rate of deflation and is hence a predetermined variable. In addition, the unemployment rate is a state variable whereas consumption and labor market tightness are jump variables. Stability of the system therefore requires two positive and two negative eigenvalues. The Jacobian of this system evaluated at the secular stagnation steady state is as follows:

\[
\begin{bmatrix}
\alpha (x - 1) & \alpha x_c \hat{m} \\
0 & \frac{c}{\eta_c} \left[ -\alpha x_c - \frac{\omega'(b) \phi''(c)}{\phi'(c)^2} \right] - \lambda \\
0 & \frac{\theta}{\eta_\theta} \left[ -\alpha x_c - \vartheta_x x_c \right] - \vartheta_x \theta \\
0 & 0 - p'(\theta) u \\
\end{bmatrix}
\begin{bmatrix}
\alpha x_\theta \hat{m} \\
\alpha x_u \hat{m} \\
\alpha x_u \hat{m} \\
-\delta - p(\theta) \end{bmatrix} = 0.
\]

The eigenvalues of this system solve the following characteristic equation:

\[
\Omega(\lambda) = [\alpha (x - 1) - \lambda].
\]

\[
\begin{bmatrix}
\frac{c}{\eta_c} \left[ -\alpha x_c + \frac{\omega'(b) \phi''(c)}{\phi'(c)^2} \right] - \lambda \\
\frac{\theta}{\eta_\theta} \left[ -\alpha x_c + \theta \right] - \vartheta_x \theta \\
0 - p'(\theta) u \\
-\delta - p(\theta) + \lambda \end{bmatrix} = 0.
\]

It is clear that \( \lambda_1 = \alpha (x - 1) < 0 \) is one solution. The other eigenvalues solve

\[
\hat{\Omega}(\lambda) = p'(\theta) u \cdot \begin{bmatrix}
-\frac{c}{\eta_c} \left[ \alpha x_c + \frac{\omega'(b) \phi''(c)}{\phi'(c)^2} \right] - \lambda \\
\frac{\theta}{\eta_\theta} \left[ \alpha + \vartheta_x \right] - \vartheta_x \theta \\
- \frac{c}{\eta_c} \left[ \alpha x_\theta - \frac{\omega'(b) \phi''(c)}{\phi'(c)^2} \right] - \lambda \\
- \frac{\theta}{\eta_\theta} [\alpha + \vartheta_x] x_c - \frac{\theta}{\eta_\theta} \left[ \alpha x_\theta + \vartheta_x \theta \right] - \vartheta_x \theta - \lambda \\
\end{bmatrix} = 0.
\]

We rewrite this expression as

\[
\hat{\Omega}(\lambda) = -\lambda^3 + A_2 \lambda^2 - A_1 \lambda + A_0 = 0,
\]

where

\[
A_2 = -\frac{c}{\eta_c} \left[ \alpha x_c + \frac{\omega'(b) \phi''(c)}{\phi'(c)^2} \right] - \frac{\theta}{\eta_\theta} (\alpha x_\theta + \vartheta_x \theta + \vartheta_\theta) - \delta - p(\theta),
\]

\[
A_1 = -p'(\theta) u \frac{\theta}{\eta_\theta} (\alpha + \vartheta_x) x_u + [\delta + p(\theta)] \left[ \frac{c}{\eta_c} \left[ \alpha x_c + \frac{\omega'(b) \phi''(c)}{\phi'(c)^2} \right] + \frac{\theta}{\eta_\theta} (\alpha x_\theta + \vartheta_x \theta + \vartheta_\theta) \right] + \frac{c}{\eta_c} \frac{\theta}{\eta_\theta} (\alpha x_c + \vartheta_x x_c) (\alpha x_\theta - \frac{\omega''(b) \phi'(c)}{\phi'(c)}),
\]

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\[ A_0 = p'(\theta)u \frac{c}{\eta_c \eta_\theta} (\alpha + \vartheta) \left[ \left( \alpha x_c + \frac{\omega'(b)\varphi''(c)}{\varphi'(c)^2} \right) x_u - \left( \alpha x_u - \frac{\omega''(b)\varphi(c)}{\varphi'(c)} \right)x_c \right] \]

\[-[\delta + p(\theta)] \frac{c}{\eta_c \eta_\theta} \left[ \left( \alpha x_c + \frac{\omega'(b)\varphi''(c)}{\varphi'(c)^2} \right) x_u + (\alpha + \vartheta) x_c \right] \left( \alpha x_\theta + \vartheta x_\theta + \vartheta \right) - (\alpha + \vartheta) x_c \left( \alpha x_\theta - \frac{\omega''(b)\varphi(c)}{\varphi'(c)} \right). \]

Noting that \( u'(\theta) = -p'(\theta)u/[\delta + p(\theta)] \) and \( b'(\theta) = b_\theta + b_u u'(\theta) \), we can modify \( A_0 \) as follows:

\[ A_0 = [\delta + p(\theta)] \frac{c}{\eta_c \eta_\theta} \left[ \frac{p'(\theta)u}{\delta + p(\theta)} (\alpha + \vartheta) \left( \alpha x_c x_u + \frac{\omega'(b)\varphi''(c)}{\varphi'(c)^2} x_u - \alpha x_u x_c + \frac{\omega''(b)\varphi(c)}{\varphi'(c)} x_c \right) \right. \]

\[ -\left( \alpha x_c + \frac{\omega'(b)\varphi''(c)}{\varphi'(c)^2} \right) (\alpha x_\theta + \vartheta x_\theta + \vartheta \theta) + (\alpha + \vartheta) x_c \left( \alpha x_\theta - \frac{\omega''(b)\varphi(c)}{\varphi'(c)} \right) \right] \]

\[ = [\delta + p(\theta)] \frac{c}{\eta_c \eta_\theta} \left( \alpha + \vartheta x_c \right) \left[ -\frac{\omega'(b)\varphi''(c)}{\varphi'(c)^2} x_u u'(\theta) - \frac{\omega''(b)\varphi(c)}{\varphi'(c)} x_c u'(\theta) - \frac{\omega''(b)\varphi(c)}{\varphi'(c)} x_c + x_c \alpha x_\theta \right. \]

\[ -\alpha x_c \left( x_\theta + \frac{\vartheta}{\alpha + \vartheta} \right) - \frac{\omega'(b)\varphi''(c)}{\varphi'(c)^2} \left( x_\theta + \frac{\vartheta}{\alpha + \vartheta} \right) \right] \]

\[ = [\delta + p(\theta)] \frac{c}{\eta_c \eta_\theta} \left( \alpha + \vartheta x_c \right) \left[ -\frac{\omega'(b)\varphi''(c)}{\varphi'(c)^2} \left( x_u u'(\theta) + x_\theta + \frac{\vartheta}{\alpha + \vartheta} \right) \right. \]

\[ \left. \frac{\frac{\omega''(b)\varphi(c)}{\varphi'(c)} \alpha x_\theta}{\alpha + \vartheta} \right] \]

Using the partial derivatives of \( x \) defined in (H.1), we can rewrite the term in brackets of the last equation above as follows:

\[ \frac{x_u}{x_c} u'(\theta) + \frac{x_\theta}{x_c} + \frac{\vartheta}{\alpha + \vartheta} = (\tilde{y}x + k\theta)u'(\theta) + ku - (1-u)\tilde{y} \frac{\vartheta}{\alpha + \vartheta} \]

\[ = (\tilde{y}x + k\theta)u'(\theta) + ku - (1-u)\tilde{y} \frac{dx^\alpha(\theta)}{d\theta} = \frac{dc(\theta, x^\alpha(\theta))}{d\theta}, \]

where \( \frac{dx^\alpha(\theta)}{d\theta} = -\frac{\vartheta}{(\alpha + \vartheta)} \) follows from the No-Arbitrage condition (42) and \( c(\theta, x^\alpha(\theta)) \) is defined in (36). It then follows that \( A_0 \) can be rewritten as

\[ A_0 = [\delta + p(\theta)] \frac{c}{\eta_c \eta_\theta} \left( \alpha + \vartheta x_c \right) \omega'(b) \frac{\varphi''(c)}{\varphi'(c)} \left[ -b'(\theta) + \frac{\omega'(b)\varphi''(c)}{\varphi'(c)} \frac{dc(\theta, x^\alpha(\theta))}{d\theta} - \frac{\vartheta}{\omega''(b)\varphi'(c)} \alpha x_\theta \right] \]

Using the expressions for \( \vartheta, \vartheta \) and \( \frac{dx^\alpha(\theta)}{d\theta} \), it is easy to see that the last part of this term is identical to expression (G.3). Hence, we can rewrite this equation as

\[ A_0 = [\delta + p(\theta)] \frac{c}{\eta_c \eta_\theta} \left( \alpha + \vartheta x_c \right) G'(\theta). \] (H.10)

If the secular stagnation steady state is unique, it holds that \( G'(\theta) < 0 \) and hence \( A_0 < 0 \). Since \( A_0 \) can be rewritten as the product of the three remaining eigenvalues, i.e. \( A_0 = \lambda_2 \lambda_3 \lambda_4 < 0 \), we either have one or three additional negative eigenvalues.

Stability requires two positive and two negative eigenvalues. Since \( A_2 = \lambda_2 + \lambda_3 + \lambda_4 \) and \( A_1 = \lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4 \), we require either \( A_2 > 0 \) and/or \( A_1 < 0 \).

Suppose \( A_2 > 0 \), then stability immediately follows. Suppose instead \( A_2 \leq 0 \). This implies

\[ -\frac{c}{\eta_c} \left( \alpha x_c + \frac{\omega'(b)\varphi''(c)}{\varphi'(c)^2} \right) - \frac{\vartheta}{\eta_\theta} (\alpha + \vartheta x_\theta \leq \delta + p(\theta) + \frac{\vartheta}{\eta_\theta} \vartheta, \] (H.11)
which we use to show that $A_1 < 0$ in this case, from which stability follows. Reformulate $A_1$ as

$$A_1 = [\delta + p(\theta)] \frac{\theta}{\eta_\theta} (\alpha + \vartheta_x) \left( \frac{-p'(\theta) u}{\delta + p(\theta)} x_u + x_\theta + \frac{\vartheta_\theta}{\alpha + \vartheta_x} \right) + [\delta + p(\theta)] \frac{c}{\eta_c} \left( \alpha x_c + \frac{\omega'(\theta)}{\phi'(c)} \right)$$

$$+ \frac{c \theta}{\eta_c \eta_\theta} \left( \alpha x_c + \frac{\omega'(\theta)}{\phi'(c)} \right) \vartheta_\theta + (\alpha + \vartheta_x) \left( \frac{\omega'(\theta)}{\phi'(c)} \right) x_\theta - x_c \left( \alpha x_c - \frac{\omega'(\theta) b_\theta}{\phi'(c)} \right).$$

$$= \frac{c \theta}{\eta_c \eta_\theta} \left( \alpha x_c + \frac{\omega'(\theta)}{\phi'(c)} \right) \vartheta_\theta + (\alpha + \vartheta_x) \left( \frac{\omega'(\theta)}{\phi'(c)} \right) x_\theta - x_c \left( \alpha x_c - \frac{\omega'(\theta) b_\theta}{\phi'(c)} \right).$$

For sufficiently high values of $\eta_c$, specifically for all $\eta_c > \frac{a}{p}$, it holds that $\left( \frac{c \alpha x_c - \omega'(\theta)}{\phi'(c)} \right) < 0$. We formally show this in Lemma 6. It then follows that for $A_2 < 0$, we have

$$A_1 = -[\delta + p(\theta)] \frac{\theta}{\eta_\theta} (\alpha + \vartheta_x) x_c \frac{dc(\theta, x^a(\theta))}{d\theta} - \frac{c}{\eta_c} \alpha x_c - \frac{\omega'(\theta)}{\phi'(c)} + \frac{\theta}{\eta_\theta} (\alpha + \vartheta_x) x_\theta \left( \frac{c}{\eta_c} \alpha x_c - \frac{\omega'(\theta)}{\phi'(c)} \right)$$

$$+ \frac{c \theta}{\eta_c \eta_\theta} \left( \alpha + \vartheta_x \right) \left( \frac{\omega'(\theta)}{\phi'(c)} \right) x_\theta + \frac{\theta}{\eta_\theta} (\alpha + \vartheta_x) x_\theta \left( \frac{c}{\eta_c} \alpha x_c - \frac{\omega'(\theta)}{\phi'(c)} \right)$$

$$= -[\delta + p(\theta)] \frac{\theta}{\eta_\theta} (\alpha + \vartheta_x) x_c \frac{dc(\theta, x^a(\theta))}{d\theta} - \frac{c}{\eta_c} \alpha x_c - \frac{\omega'(\theta)}{\phi'(c)} + \frac{\theta}{\eta_\theta} (\alpha + \vartheta_x) x_\theta \left( \frac{c}{\eta_c} \alpha x_c - \frac{\omega'(\theta)}{\phi'(c)} \right)$$

$$+ \frac{c \theta}{\eta_c \eta_\theta} \left( \alpha + \vartheta_x \right) \left( \frac{\omega'(\theta)}{\phi'(c)} \right) x_\theta + \frac{\theta}{\eta_\theta} (\alpha + \vartheta_x) x_\theta \left( \frac{c}{\eta_c} \alpha x_c - \frac{\omega'(\theta)}{\phi'(c)} \right)$$

Since $\frac{dc(\theta, x^a(\theta))}{d\theta} > 0$ by Lemma 4 in the stagnation steady state, all three terms are negative and we have $A_1 < 0$. Therefore, we cannot have three negative eigenvalues, which together with $A_0 < 0$ and $\lambda_1 > 0$ implies that there are exactly two positive and two negative eigenvalues. It follows that the dynamic system is saddle path stable around the stagnation steady state.

**Proof: Proposition 2**

Use the steady state conditions (41) to define $G(\theta)$, where $x^a(\theta)$ is given by (42) as $x < 1$ as follows:

$$G(\theta) = \rho - \frac{\omega'(\theta)}{\phi'(c(\theta, x^a(\theta)))).} - \alpha(1 - x^a(\theta)).$$

(I.1)
We model worsening stagnation by exogenous increases in the steady state marginal utility of wealth $\omega'(\cdot)$, e.g., via modifying the scale parameter of the wealth preference relative to the preference for consumption. This implies a stronger desire to save and a decline in the natural real interest rate. Recall from before that $G'(\theta) < 0$ in steady state.

The effects of worsening stagnation on labor market tightness is given by

$$\frac{d\theta}{d\omega'} = \frac{\omega' G'(\theta)}{\phi'(c(\theta, x^a(\theta)))} < 0.$$  \hspace{1cm} (I.2)

It then follows immediately from (32), (42) and (7) that

$$\frac{du}{d\omega'} > 0, \quad \frac{dx}{d\omega'} < 0, \quad \frac{dY}{d\omega'} < 0.$$  \hspace{1cm} (I.3)

The effect on consumption follows from (I.1). In steady state, $G'(\theta) = 0$. As $\frac{dx}{d\omega'} < 0$, it necessarily follows that $\frac{dc}{d\omega'} < 0$. \hspace{1cm} (I.4)

Finally, from (24) and (42), the real wage can be rewritten as

$$w_t = \psi \left( \bar{y} + \frac{(1 - \psi)(\bar{y} + (1 - s)k\alpha)}{\psi(\theta) + \delta + \alpha} \right),$$  \hspace{1cm} (I.5)

which gives

$$\frac{dw}{d\omega'} = \frac{\psi(\delta + \alpha)[(1 - \psi)(\bar{y} + (1 - s)k\alpha) - (1 - s)k\alpha\psi'\theta q'(\theta)]}{\psi(\theta) + \delta + \alpha^2} \frac{d\theta}{d\omega'} < 0,$$  \hspace{1cm} (I.6)

where the negative sign results from $p'(\theta) > 0$ in (15), $q'(\theta) < 0$ in (16) and $\frac{d\theta}{d\omega'} < 0$ in (I.2). Expressions (I.2), (I.3), (I.4) and (I.6) establish Proposition 2.

**Proof: Lemma 6**

We want to derive the conditions under which, in the stagnation steady state, we have

$$\Upsilon(\theta) \equiv \frac{-\omega'(b)\phi'(c)(1 - u)\bar{y}}{\phi'(c)^2} - \alpha > 0.$$  \hspace{1cm} (J.1)

Consider the following modifications using the definition of $\eta_c$ and $c$

$$\Upsilon(\theta) = \frac{\omega'(b) (1 - u)\bar{y} \eta_c}{c} - \alpha = \frac{(1 - u)\bar{y}}{c} \left[ \rho \eta_c - \alpha (1 - x) \eta_c - \alpha \frac{c}{1 - u} \frac{\bar{y}}{\bar{y}} \right]$$

$$= \frac{(1 - u)\bar{y}}{c} \left[ \rho \eta_c - \alpha (1 - x) \eta + \alpha (1 - u)\bar{y}x - k\theta - g ight]$$

$$= \frac{(1 - u)\bar{y}}{c} \left[ (\rho - \alpha)\eta_c + \alpha x (\eta_c - 1) + \alpha k\theta + g \right].$$  \hspace{1cm} (J.2)

It is easy to see that $\Upsilon(\theta) > 0$ for $\eta_c \geq 1$. Suppose $\eta_c < 1$, then $x = 1$ establishes a lower bound on the right hand side:

$$\Upsilon(\theta) > \frac{(1 - u)\bar{y}}{c} \left[ \rho \eta_c - \alpha + \alpha \frac{k\theta + g}{1 - u} \bar{y} \right] > \frac{(1 - u)\bar{y}}{c} \left[ \rho \eta_c - \alpha \right].$$  \hspace{1cm} (J.3)
It follows directly that \( \eta_c > \frac{a}{\rho} \) is a sufficient condition for \( \Upsilon(\theta) > 0 \) or

\[
-\frac{\omega'(b)\phi''(c)(1-u)\bar{y}}{\phi'(c)^2} > \alpha \quad \text{if} \quad \eta_c > \frac{a}{\rho}.
\]  

(J.4)

**Proof: Proposition 3**

(i) **Effects on labor market tightness and unemployment:** Use the steady state conditions (41) and (42) to define \( G(\theta, \chi) \), where \( x^a(\theta) \) is given by (42) as \( x < 1 \) and \( \chi \) denotes any parameter as follows:

\[
G(\theta, \chi) = \rho - \frac{\omega' \left( (1 - u(\theta))(1-s)k \right)}{\phi'(c(x^a(\theta))))} - \alpha (1 - x^a(\theta)),
\]  

(K.1)

with \( G(\theta^a, \chi) = 0 \) and \( G_{\theta^a} < 0 \) in steady state where \( G_{\theta^a} \) denotes the total derivative with respect to \( \theta \) evaluated in steady state. The negative sign follows from the uniqueness of the steady state. \( c(\theta, x^a(\theta)) \) is given by (36) with \( x = x^a(\theta) \). The effects of changes in a parameter \( \chi \) on the labor market tightness \( \theta \) can be recovered from (K.1) as

\[
\frac{d\theta}{d\chi} = -\frac{G_{\chi}}{G_{\theta^a}}.
\]  

(K.2)

In addition, we derive the following partial derivatives from \( x^a(\theta, \chi) \) in (42):

\[
x_x > 0, \quad x_z > 0, \quad x_s < 0, \quad x_k > 0, \quad x_y < 0, \quad x_g = 0, \quad x + \bar{y}x_{\bar{y}} > 0.
\]  

(K.3)

Using (K.1) to (K.3) and Lemma 6, we derive the following properties:

\[
\frac{d\theta}{dz} = -\frac{\omega'(b)\phi''(c)(1-u)\bar{y}}{\phi'(c)^2} + \alpha \frac{x_z}{G_{\theta^a}}, \quad \frac{d\theta}{d\bar{y}} = -\frac{\omega'(b)\phi''(c)(1-u)\bar{y}}{\phi'(c)^2} + \alpha \frac{x_y}{G_{\theta^a}}.
\]  

(K.4)

\[
\frac{d\theta}{ds} = -\frac{\omega''(b)b(\theta)}{(1-s)\phi'(c)G_{\theta^a}} - \left[ \frac{\omega'(b)\phi''(c)(1-u)\bar{y}}{\phi'(c)^2} + \alpha \right] \frac{x_s}{G_{\theta^a}}.
\]  

(K.5)

\[
\frac{d\theta}{dk} = \frac{\omega''(b)b(\theta)}{k\phi'(c)G_{\theta^a}} + \frac{\omega'(b)\phi''(c)\theta u}{\phi'(c)^2} - \left[ \frac{\omega'(b)\phi''(c)(1-u)\bar{y}}{\phi'(c)^2} + \alpha \right] \frac{x_k}{G_{\theta^a}}.
\]  

(K.6)

\[
\frac{d\theta}{dy} = -\frac{\omega'(b)\phi''(c)(1-u)x}{\phi'(c)^2} + \omega'(b)\phi''(c)\bar{y}G_{\theta^a} - \alpha \frac{x_g}{G_{\theta^a}} < 0,
\]  

(K.7)

\[
\frac{d\theta}{dg} = \frac{\omega'(b)\phi''(c)}{\phi'(c)^2} \frac{1}{G_{\theta^a}} > 0.
\]  

(K.8)

Expressions (K.4) to (K.9) imply the following response of labor market tightness to parameter variations. The response of the unemployment rate is opposite as is clear from the Beveridge curve in (32):

\[
\theta^s = \theta(\varepsilon, z, s, k, \bar{y}, g).
\]  

(K.10)
(ii) Effects on realized working hours: From the asset market equilibrium curve (42), we recover the effects of parameter changes on realized working hours as follows:

\[
\frac{dx^a(\theta, \chi)}{d\chi} = x^a_{\chi} + \frac{dx^a(\theta)}{d\theta} \frac{d\theta}{d\chi} = x^a_{\chi} - \frac{dx^a(\theta)}{d\theta} \frac{G_X}{G_{\theta^*}} = \left[ G_{\theta^*} - \frac{dx^a(\theta)}{d\theta} \frac{G_X}{x_{\chi}} \right] x_{\chi}^{-1},
\]

where \(\frac{dx^a(\theta)}{d\theta} > 0\) and \(G_X\) are the partial derivatives of \(G(\theta, \chi)\) with respect to any parameter \(\chi\). Using the expressions above for \(\frac{d\theta}{d\chi}\) and \(\frac{d\theta x^a(\theta)}{d\theta} = (1 - u)\vec{y} \frac{dx^a(\theta)}{d\theta} - (\vec{y} x^a(\theta) + k\theta)u'(\theta) - ku\) from (36), we get the following results:

\[
\frac{dx}{d\bar{e}} = \left[ -\omega'(b) \frac{b'}{\bar{e}}(c) - \omega'(b) \frac{\phi''(c)}{\phi'(c)} \frac{k u + (\vec{y} x + k\theta)u'(\theta)}{x_{\chi}} \right] x_{\chi}^{-1},
\]

(13)

\[
\frac{dx}{ds} = \left[ -\omega'(b) \frac{b'}{\phi'(c)} - \omega'(b) \frac{\phi''(c)}{\phi'(c)} \frac{k u + (\vec{y} x + k\theta)u'(\theta)}{x_{\chi}} \right] x_{\chi}^{-1},
\]

(14)

\[
\frac{dx}{ds} = \left[ \omega'(b) \frac{dx^a(\theta)}{d\theta} - \omega'(b) \frac{\phi''(c)}{\phi'(c)} \frac{k u + (\vec{y} x + k\theta)u'(\theta)}{x_{\chi}} \right] x_{\chi}^{-1},
\]

(15)

From \(G_{\theta^*} < 0\), we derive the following upper bound for the first term in (K.17):

\[
-\omega'(b) \frac{b'}{\phi'(c)} < \omega'(b) \phi''(c) \frac{dx^a(\theta)}{d\theta} \frac{dx^a(\theta)}{d\theta} - \alpha \frac{dx^a(\theta)}{d\theta}.
\]

(16)

Since \(x_g < 0\) and \(G_{\theta^*} < 0\), from (K.17) it follows that

\[
\frac{dx}{dy} < \omega'(b) \frac{\phi''(c)}{\phi'(c)} \frac{dx^a(\theta)}{d\theta} - \alpha \frac{dx^a(\theta)}{d\theta} - \omega'(b) \frac{\phi''(c)}{\phi'(c)} \frac{[dx^a(\theta) (1 - u)] x_{\chi}}{x_{\gamma}} (x + \vec{y} x_{\gamma}) - \frac{dx^a(\theta)}{d\theta}^{-1} \frac{dx^a(\theta)}{d\theta}^{-1} x_{\gamma}^{-1} G_{\theta^*},
\]

(17)

\[
\frac{dx}{dy} = \frac{dx^a(\theta)}{d\theta} \frac{dy}{dx^a(\theta)} x_{\gamma} > 0.
\]

(18)

(19)

Taken together, we have derived the following effects of parameter changes in (K.13) to (K.19):

\[
x^a = x(\varepsilon, z, s, k, \vec{y}, g).
\]

(20)

(iii) Effects on total output: The effects of parameter changes on total output follow from the total differential of \(Y(\theta) = (1 - u(\theta))x^a(\theta)\vec{y}\) as

\[
\frac{dY}{d\chi} = -x^a(\theta)\vec{y} \frac{dx}{d\chi} + (1 - u)\vec{y} \frac{dx}{d\chi} + Y, \quad (K.21)
\]

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where $Y_\chi = (1 - u)x$ for $\chi = \bar{y}$ and zero otherwise. It then follows that these effects are given by

\[
\frac{dY}{dx} = (1 - u)\bar{y}x + \left[ (1 - u)\bar{y} \frac{dx^a(\theta)}{d\theta} - x\bar{y}u'(\theta) \right] \frac{d\theta}{d\bar{y}}, \tag{K.22}
\]

\[
\frac{dY}{dz} = (1 - u)\bar{y}x + \left[ (1 - u)\bar{y} \frac{dx^a(\theta)}{d\theta} - x\bar{y}u'(\theta) \right] \frac{d\theta}{dz}, \tag{K.23}
\]

\[
\frac{dY}{ds} = (1 - u)\bar{y}x + \left[ (1 - u)\bar{y} \frac{dx^a(\theta)}{d\theta} - x\bar{y}u'(\theta) \right] \frac{d\theta}{ds}, \tag{K.24}
\]

\[
\frac{dY}{dk} = (1 - u)\bar{y}x + \left[ (1 - u)\bar{y} \frac{dx^a(\theta)}{d\theta} - x\bar{y}u'(\theta) \right] \frac{d\theta}{dk}, \tag{K.25}
\]

\[
\frac{dY}{dy} = (1 - u)\bar{y} \frac{dx}{dy} - x\bar{y}u'(\theta) \frac{d\theta}{dy} + (1 - u)x^a(\theta)
\]

As above, we apply an upper bound on $-\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}$ that results from $G_{\theta^*} < 0$. Since $x_\bar{y} < 0$ and $G_{\theta^*} < 0$, it follows that

\[
\frac{dY}{dy} < \left[ -\left(1 - u\right) \left( \bar{y} + \frac{x^a(\theta)}{x_\bar{y}} \left( \frac{\omega'(b)\phi''(c)}{\phi'(c)^2} \frac{dx^a(\theta)}{d\theta} + \frac{dx^a(\theta)}{d\theta} \frac{d\theta}{d\bar{y}} \right) + \alpha \frac{d\chi}{d\bar{y}} \left( \bar{y}u'(\theta) + \frac{dx^a(\theta)}{d\theta} \frac{d\theta}{d\bar{y}} \right) \right)
\]

\[
\frac{dY}{dy} = \left[ -\left(1 - u\right) \left( \bar{y} + \frac{x^a(\theta)}{x_\bar{y}} \right) - \alpha \bar{y} \left( 1 - u \frac{dx^a(\theta)}{d\theta} - u'(\theta)x^a(\theta) \right) \right] \frac{d\chi}{d\bar{y}} + \left(1 - u\right) \frac{dx^a(\theta)}{d\theta} - u'(\theta)x^a(\theta)
\]

\[
\frac{dY}{dy} = \frac{p'(\theta)}{p(\theta)} \left( \frac{dx^a(\theta)}{d\theta} + \frac{dx^a(\theta)}{d\theta} \right) \frac{p(\theta)}{d\bar{y}} + \frac{dx^a(\theta)}{d\theta} \frac{d\theta}{d\bar{y}} > 0. \tag{K.27}
\]

Taken together, we have derived the following effects of parameter changes from (K.22) to (K.27):

\[
Y^s = Y(\bar{y}, \bar{y}, \bar{y}, s, k, \bar{y}, g). \tag{K.28}
\]

(iv) Effects on consumption: Finally, the effects on consumption are given by the differential of (36) with $x = x^a(\theta)$ given by (42) using the expressions above as

\[
\frac{dc}{d\chi} = \left[ -[\bar{y}x + k\theta]u'(\theta) - ku \right] \frac{d\theta}{d\bar{y}} + (1 - u)\bar{y} \frac{dx}{d\chi} + c\chi = \frac{dx^a(\theta)}{d\theta} \frac{d\theta}{d\bar{y}} + (1 - u)\bar{y}x + c\chi. \tag{K.29}
\]
where $\frac{dL}{dt} > 0$, $c_z = c_z = c_z = 0$, $c_k = -\theta u < 0$, $c_g = (1 - u)x > 0$ and $c_g = -1 < 0$.

\[
\frac{dc}{dz} = \frac{d\left(\frac{dc}{dx}\right)}{dz} + (1 - u)\tilde{g}x = \left[(\tilde{g}x + k\theta)u'(\theta) + ku\right]\alpha - (1 - u)\frac{\omega''(b)b'(\theta)}{\phi'(c)} \frac{dx}{G_\theta},
\]

(K.30)

\[
\frac{dc}{dz} = \frac{d\left(\frac{dc}{dx}\right)}{dz} + (1 - u)\tilde{g}x = \left[(\tilde{g}x + k\theta)u'(\theta) + ku\right]\alpha - (1 - u)\frac{\omega''(b)b'(\theta)}{\phi'(c)} \frac{dx}{G_\theta},
\]

(K.31)

\[
\frac{dc}{ds} = \frac{d\left(\frac{dc}{dx}\right)}{ds} + (1 - u)\tilde{g}x,
\]

(K.32)

\[
\frac{dc}{dk} = \frac{d\left(\frac{dc}{dx}\right)}{dk} + (1 - u)\tilde{g}x - u\theta.
\]

(K.33)

\[
\frac{dc}{dy} = (1 - u)x \left[1 - \frac{\omega'(b)\phi''(c)dc}{\phi'(c)^2G_\theta} \frac{dc}{dz}\right] + \left[(\tilde{g}x + k\theta)u'(\theta) + ku\right]\alpha - (1 - u)\frac{\omega''(b)b'(\theta)}{\phi'(c)} \frac{x}{G_\theta},
\]

\[
= \frac{(1 - u)x}{\phi'(c)^2G_\theta} \phi''(c) \frac{dc}{dx}(\tilde{g}x + x) - \frac{dc}{dz}(\tilde{g}x + x) + \left(1 - u\right)\tilde{g}x + x \right] \frac{\omega''(b)b'(\theta)}{\phi'(c)} \frac{1}{G_\theta} < 0,
\]

(K.34)

\[
\frac{dc}{dy} = -\left[\frac{dc}{dz} \frac{dc}{dx}(\tilde{g}x + x) \right] \frac{1}{G_\theta} > 0.
\]

(K.35)

Taken together, (K.30) to (K.35) imply the following effects of parameter changes:

\[
c = c\left(\frac{\varepsilon}{\tilde{g}}, \frac{z}{\tilde{g}}, \frac{s}{\tilde{g}}, \frac{k}{\tilde{g}}, \frac{\tilde{g}}{\tilde{g}}, \frac{g}{\tilde{g}}\right).
\]

(K.36)