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STRUCTURAL UNEMPLOYMENT, UNDEREMPLOYMENT, AND SECULAR STAGNATION

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Structural unemployment, underemployment, and secular stagnation*

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Abstract

In this paper, we show that underemployment and not necessarily high unemployment becomes the main measure of economic slack in the labor market under secular stagnation. Specifically, involuntary underemployment in the form of a persistent shortfall of working hours occurs in the search and matching model, provided that households derive utility from holding wealth, and quickly dominates the total employment gap under stagnation. Conventional policy measures aimed at reducing unemployment may increase the labor market gap through their effects on underemployment and should be used with caution. In contrast, positive demand shocks improve unemployment and working hours, while increases in potential output worsen both ("paradox of toil"). Our analysis provides new insights into empirical puzzles such as Japan's seemingly decent employment record during its lost decades.

Keywords: Secular stagnation, underemployment, unemployment, labor market frictions, search and matching

***JEL* Classification**: E24, E31, E44, J20, J64

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1 Introduction

Macroeconomists conventionally rely on the unemployment rate as the main measure of slack in the labor market and its implications for wage and price dynamics. However, such an analysis is incomplete at best and potentially misleading once an economy is constrained by the effective lower bound on the nominal interest rate and suffers from persistent stagnation. Under such circumstances, it can be underemployment, a state in which employed workers want to increase working hours at the same wage but are unable to do so, rather than unemployment that provides a more accurate measure of labor market slack.

Japan is the prime example of an economy stuck in a stagnation equilibrium without any natural recovery. However, Japan's economy has performed surprisingly well in terms of its employment record during the lost decades as the unemployment rate has remained low by international comparison peaking at slightly more than 5% in 2002 as illustrated in panel (a) of Figure 1. Unemployment in 2018 had reached its lowest level since the early 1990s. Yet, this decline has not resulted in significant wage or price pressures.[1] The absence of widespread unemployment is often attributed to the specific features of the Japanese labor market, most notably the traditional practices of long-term employment, which make unemployment in Japan less responsive to fluctuations in spending.[2] Then how, if not in unemployment, did the lack of demand manifest itself in the labor market?

During the stagnation period the structure of Japan's labor market has changed profoundly with a substantial rise in underemployment in the form of part-time and non-regular employment and declining working hours. Panel (b) of Figure 1 shows the share of part-time employment in total employment in contrast to the OECD average. Having been less than 12% in the 1980s, the share of part-time employees has sharply increased throughout the lost decades, primarily reflecting the lack of alternative employment opportunities.[3] By 2018, it had doubled compared to the 1980s level whereas it only increased by 3 percentage points over all OECD countries. A similar trend can be observed for other forms of non-regular employment (see Japan Institute for Labour Policy and Training, 2015). Finally, the steady decline in average hours worked in the OECD data is also indicative of the rise of underemployment in Japan since 1990.[4]

Rising underemployment and the absence of wage and price pressures despite low unemployment are no longer exclusively observed in Japan, but have become well-documented phenomena in several European countries after the Great Recession as well.[5] Surveying the empirical evi-

[1] In fact, wage growth has been dismal with stagnant real and declining nominal wages. Bell and Blanchflower (2021) report that nominal wages have declined by 0.5% on average over the 2000-2016 period, while real wages have been unchanged. In 2017, real wages actually fell by 0.2% despite the historically low unemployment rate.

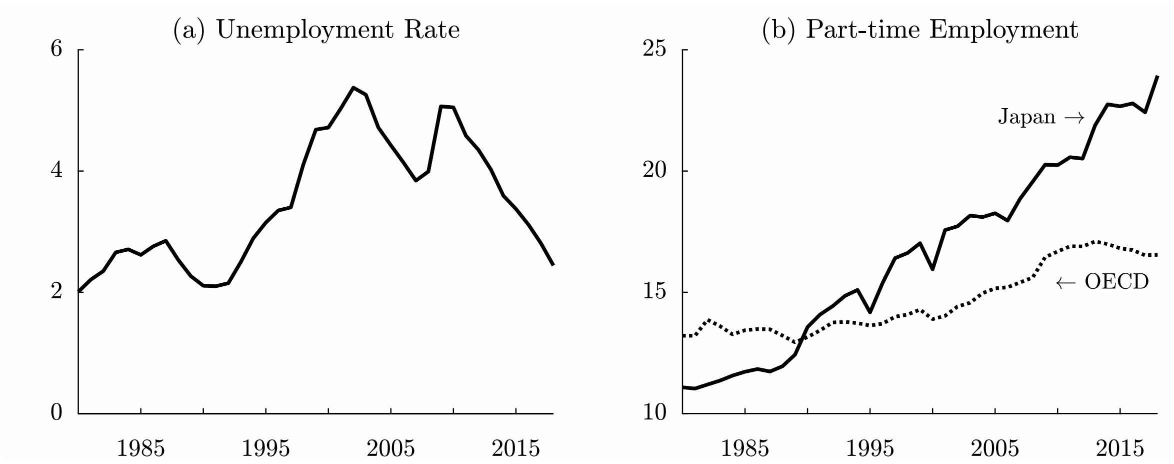
[2] We refer to Hashimoto and Raisian (1985) for a description of the Japanese employment system in comparison to the United States *before* the lost decades. Several studies show that these practices have eroded remarkably little, at least among core workers, during the stagnation decades (see Shimizutani and Yokoyama, 2009; Kambayashi and Kato, 2011; Hamaaki et al., 2012, among others).

[3] For illustration, non-regular employees consistently emphasize the "lack of regular employment opportunities" as a major reason for their current work style in surveys conducted by the Japan Institute for Labour Policy and Training (see Japan Institute for Labour Policy and Training, 2015).

[4] The secular decline in working hours is confirmed by data from the Labour Force Survey of the Statistics Bureau of Japan. Average working hours per week have substantially fallen from a stable level of slightly more than 28 hours in the 1980s to less than 23 hours by the 2010s.

[5] Bell and Blanchflower (2021) document a substantial level of labor market underutilization in the form of

Figure 1: Labor market developments in Japan, 1980-2018



Data sources: (a) Unemployment rate (in percent of the labor force), OECD; (b) Part-time employment rate (in percent of total employment), OECD, Japan (solid) and OECD average (dotted line); OECD definiton: "Part-time employment is defined as people in employment (whether employees or self-employed) who usually work less than 30 hours per week in their main job."

dence, Blanchflower (2019, p.144) argues that "underemployment has replaced unemployment as the main measure of labor market slack".

In this paper, we propose a model of secular stagnation that distinguishes between structural unemployment and demand-driven underemployment. Using this model, we show that secular stagnation leads to underemployment in the labor market but not necessarily to high unemployment, thereby providing a rationalization for the puzzling case of Japan and the theoretical counterpart to the empirical analysis described above. Specifically, we incorporate a preference for wealth into the standard search and matching model of the labor market. In a setting with infinitely-lived households, such a preference captures the bequest motive within dynasties that is frequently incorporated in overlapping generations models. The preference for wealth creates a strong motive to save in addition to the standard consumption smoothing motive, which can explain why empirically the saving rate of households is increasing in their wealth (see Benhabib and Bisin, 2018; Fagereng et al., 2019).[6] Importantly, the preference for wealth allows for the possibility of a secular stagnation equilibrium as is well-know from contributions such as Ono

underemployment after the financial crisis for the United States and 26 European countries. In fact, unemployment rates have returned to their pre-crisis levels in most countries while underemployment is persistently higher. In addition, using panel data for 19 countries from 1998-2016, the authors show that underemployment has a significantly negative effect on wages after 2007, while the unemployment rate is insignificant. The opposite holds before the Great Recession implying a structural break at the time the effective lower bound became binding. The authors hence conclude that "underemployment replaces unemployment as the main influence on wages in the years since the Great Recession". Other studies, such as Bell and Blanchflower (2018) for the United Kingdom, directly attribute underemployment to a shortage of aggregate demand.

[6]Several recent contributions have analyzed the macroeconomic implications of a preference for wealth in models similar to ours. Kumhof et al. (2015), Michau et al. (2020) and Mian et al. (2021) study the dynamics of wealth and inequality that result from endogenous differences in saving rates based on the wealth preference, while Michau et al. (2018) analyze the dynamics of asset prices and the possibility of rational bubbles in such a framework. Michaillat and Saez (2021) incorporate a preference for wealth (relative to the average wealth level) into a New Keynesian model to study business cycle fluctuations and Saez and Stantcheva (2018) employ a similar framework for the analysis of capital taxation. For the interested reader, we refer to Zou (1994) who provides an in-depth discussion of the idea of a preference for wealth in economic thought.

(1994), Michau (2018) or Schlegl (2018). High desired savings can drive the equilibrium real interest rate into negative territory. Yet, the effective lower bound, which we model as a rate of zero for simplicity, prevents the nominal interest rate from falling sufficiently. This causes excess savings which depress aggregate demand. Combined with downward nominal wage rigidities in the spirit of Schmitt-Grohé and Uribe (2016, 2017), the model economy operates in a secular stagnation equilibrium characterized by deflation and a persistent lack of demand.

In the absence of demand shortage, our model behaves similar to the standard search and matching model except that the preference for wealth creates a new transmission channel for shocks via induced changes in the real interest rate. Structural unemployment occurs in equilibrium as vacancies and job-seekers have to match in a costly process before a position can be filled, but conditional on being employed, households realize their potential working hours and there is no underemployment. The tightness of the labor market affects both the intertemporal consumption decision of the household via changes in the wealth premium and market entry and exit of firms via changes in returns on firm ownership. In equilibrium, labor market tightness is determined such that optimal household and firm behavior are compatible with each other. The interest rate then adjusts endogenously in the money market.

In the stagnation equilibrium, in contrast, both structural unemployment and underemployment coexist and mutually affect each other. When desired savings are sufficiently strong, the zero lower bound on the nominal interest rate becomes a binding constraint and the real interest rate is determined by the rate of deflation. Faced with a lack of demand and depressed sales, firms respond by cutting working hours resulting in underemployment, lower output and less consumption. The lack of demand, however, also affects the incentives for job creation. Firm ownership becomes less attractive due to lower profits ("job creation channel"), while at the same time holding money becomes more attractive due to increased deflation ("deflation channel"). Both channels reduce job creation incentives for firms and result in an increase in unemployment. The total employment gap then exceeds the unemployment rate.

We calibrate this model to match characteristics of Japan's labor market and quantify the relative importance of unemployment and underemployment under stagnation. For all of our specifications, underemployment quickly dominates the total employment gap as stagnation becomes more severe. This finding is confirmed in several robustness, which alter the workings of the job creation and deflation channels. It is only in the case of an increasing job destruction rate under stagnation that unemployment rises substantially. While empirical evidence points towards a relatively stable job destruction rate in Japan, this job destruction channel is likely of relevance in explaining labor market patterns in other economies. In addition, we examine the effects of labor market policies and demand and supply shocks using numerical simulations. Conventional policies aimed at reducing unemployment, specifically a reduction in unemployment benefits and the introduction of a search cost subsidy, might increase the total employment gap under stagnation via their effects on underemployment depending on the shape of the wealth preference. If the concavity of the preference for wealth is sufficiently low, these policies result in movements of unemployment and underemployment in opposite directions as has been the case in Japan during labor market liberalization policies in the early 2000s and after 2012. The effects of demand and supply shocks are reversed under demand shortage. Higher demand, in the form of government spending, increases working hours and lowers unemployment thereby raising consumption and output. In contrast, a positive productivity shock worsens both. This is equivalent to the paradox of toil that is common to models of stagnation. These shocks result in co-movements of unemployment and underemployment as have been observed in Japan during the 1990s and the financial crisis.

Our analysis concludes that secular stagnation causes demand-driven underemployment and not necessarily high unemployment. As long as the economy is stuck in the stagnation equilibrium, wage growth and inflationary pressures do not emerge even when unemployment is low as they are subdued by the prevalence of underemployment. It is then primarily underemployment that responds to economic disturbances and labor market policies under stagnation, which is in line with the developments in Japan since the 1990s. These findings highlight the need for further policy intervention in support of aggregate demand despite a seemingly decent employment situation in terms of the unemployment rate.

Related Literature: Our paper analyzes the interactions of structural unemployment based on search frictions and demand-driven underemployment under secular stagnation, thereby contributing to two fields of macroeconomics that have so far been treated fairly independently.

The canonical search and matching model of the labor market (see Diamond, 1982; Mortensen, 1982; Pissarides, 1985, 2000, among others) has initiated a rich literature on structural unemployment as a consequence of frictions in the labor market.[7] Variations in working hours are introduced into this framework by Fang and Rogerson (2009) as households optimally choose their labor supply based on the consumption versus leisure trade-off. Workers are always on their labor supply curves and part-time and non-regular employment is purely voluntarily. Their rise in Japan is, for instance, explained by productivity shocks in combination with labor immobility due to differences in training costs among workers (see Ariga and Okazawa, 2011) or by an exogenous increase in part-time labor supply inducing households and firms to optimally reallocate between full-time and part-time jobs (see Kang et al., 2020). These voluntary shifts in the composition of the labor force do not constitute underemployment as defined in this paper.[8] In addition, underemployment in our model does not rely on firm or worker heterogeneity.

Other contributions, such as Manning (2003) and Ashenfelter et al. (2010), provide a microeconomic view on underemployment based on market concentration and monopsony power of firms. From the firm's perspective, variations in working hours are a cost-efficient way to avoid hiring and firing costs. In contrast, firms are perfectly competitive with free market entry and exit in our model. Underemployment is a macroeconomic phenomenon. Workers want to work additional hours at the same wage, but are constrained by a lack of demand.

Demand-driven variations in working hours are modelled in a related framework by Kudoh et al. (2019) who analyze changes in the composition of firms' labor demand over the business

[7] Rogerson et al. (2007) survey the related literature.

[8] In fact, we abstract from voluntary variations in working hours and assume an inelastic labor supply such that all variations in working hours are involuntarily. Our model can be extended to allow for fluctuations in working hours based on the consumption versus leisure choice as well. Yet, this makes the model more complex without substantially altering the results.

cycle. While labor market participation is chosen by each individual, working hours conditional on employment are adjusting in response to demand-driven fluctuations in production, which is a features similar to our model. However, these fluctuations occur over the business cycle and are hence purely temporarily in nature, which is in stark contrast to our model which allows for the possibility of demand-driven underemployment as a steady state phenomenon due to the binding effective lower bound on the nominal interest rate, i.e. secular stagnation.

The secular stagnation hypothesis, first proposed by Hansen (1939), argues that an oversupply of savings at full employment permanently depresses aggregate demand as the zero lower bound on the nominal rate prevents the real interest rate from falling sufficiently to stimulate spending. In the presence of downward nominal wage rigidity, this results in an equilibrium with a persistent output gap. The oversupply of savings has been modelled among others as a consequence of demographics (Eggertsson et al., 2019), a shortage of safe assets (Caballero and Farhi, 2018) or strong liquidity preferences (Ono, 2001; Illing et al., 2018).

In this paper, we rely on a preference for wealth in line with the contributions of Ono (1994, Chapter 11) and Michau (2018), which, however, do not consider structural unemployment and abstract from labor market frictions.[9] The labor market gap is either abstracted from or it consists of a demand-driven shortfall of working hours only. We extend these models by allowing for potential interactions of unemployment and underemployment in an integrated framework.

Several approaches have been proposed in the literature to model involuntary unemployment in the search and matching model, which are related to our concept of underemployment. Michaillat (2012) shows how wage rigidities and decreasing marginal returns to labor can result in rationing unemployment in addition to frictional unemployment. The former occurs only during recessions following negative technology shocks which reduce the marginal product of labor below the wage to which firms optimally respond by reducing employment. While Michaillat (2012) focuses only on the supply side and business cycle variations, we complement this analysis by modelling the demand side allowing for the possibility of persistent demand shortage. Interestingly, a similar mechanism creates spillovers from underemployment to unemployment in our model. Underemployment occurs when the downward price rigidity becomes binding. Then, firm profits and the marginal product of labor fall due to lower working hours resulting in higher structural unemployment.

Similarly to our model, Michaillat and Saez (2022) also rely on a preference for wealth allowing for an equilibrium with a permanent liquidity trap. Keynesian unemployment occurs in addition to frictional unemployment when the real interest rate, which is implicitly set by the central bank, is excessively high and depresses demand. Higher unemployment reduces desired savings of households thereby restoring equilibrium. The same channel is at work in our model as the real interest rate is excessively high under stagnation.[10] However, our model also explicitly considers firm behavior under free market entry. Then working hours and unemployment are jointly determined to reconcile optimal household and optimal firm behavior for a given real

[9] In addition, Schlegl (2018) discusses secular stagnation in an economy with land while Michau (2022) carefully analyzes the effects of helicopter drops of money under stagnation based on a preference for wealth.

[10] We assume money supply targeting instead of interest rate targeting, which is why underemployment only occurs in the stagnation equilibrium when the zero lower bound is binding in the money market. Implicitly, we hence assume that the central bank always sets the optimal interest rate, unless it is constrained from doing so.

interest rate. Adjustment then occurs via a combination of fewer working hours and higher unemployment. Moreover, we show that the first effect is quantitatively more important using extensive numerical simulations.[11]

This paper is organized as follows. Section 2 presents a general equilibrium model of underemployment. Section 3 analyzes the steady state equilibria. In section 4, we discuss the calibration of the model to Japanese labor market data and then extensively study the properties of underemployment under stagnation in section 5, supported by a numerical simulations of the model. The final section concludes. All proofs are in the mathematical appendix.

2 A general equilibrium model of underemployment

2.1 The representative household

Time is continuous and denoted by t. The infinitely-lived representative household consists of a large number of individuals normalized to unity. Each individual is endowed with one unit of labor and one unit of time, both of which are supplied inelastically as there is no disutility of working. Let $l_t \in [0,1]$ denote the realized labor supply and $x_t \in [0,1]$ realized working hours per individual. Slack in the labor market can take two forms: *Unemployment* occurs when household members are not employed ($l_t = 0$). Let u_t denote the number of unemployed members of the household, and hence the unemployment rate, and $1-u_t$ the number of employed ones. *Underemployment* occurs when realized working hours of the employed ($l_t = 1$) fall short of potential working hours, i.e. $x_t < 1$. Each household receives wage income of $w_t(1-u_t)x_t$ and unemployment benefits of $\hat{z}_t u_t$, where w_t is the real wage rate and $\hat{z}_t$ benefit payments per unemployed. We follow Merz (1995) and Andolfatto (1996) and assume that the household provides perfect consumption insurance for its members so that consumption is the same for employed and unemployed individuals.

Household assets a_t, in real terms, consist of interest bearing assets (bonds or equities) b_t and real money holdings m_t that do not pay interest:

$$a_t = b_t + m_t. \tag{1}$$

Let r_t denote the real interest rate and $R_t \geq 0$ the nominal interest rate, which are related via the Fisher Equation by $R_t = r_t + \pi_t$, where π_t denotes the rate of inflation. Then, real wealth evolves as:[12]

$$\dot{a}_t = r_t a_t + w_t(1-u_t)x_t + \hat{z}_t u_t - c_t - R_t m_t - \tau_t, \tag{2}$$

where τ_t is a real lump-sum tax, and $(1-u_t)x_t \in [0,1]$ is effective employment. There are

[11] Michaillat and Saez (2015) apply a matching framework to both the labor market and the product market so that the price level and labor market tightness are jointly determined in both markets. This creates feedback effects from demand fluctuations into the labor market. Under fixed prices, an increase in demand leads to a rise in working hours and therefore lower unemployment. Yet, they also do not consider the optimal market entry and exit decision of firms thereby not allowing for a distinction between unemployment and idle time. Landais et al. (2015b,a) use the same framework to study the effects of labor market policies.

[12] The nominal flow budget equation is $\dot{A}_t = R_t P_t b_t + W_t(1-u_t)x_t + \hat{Z}_t u_t - P_t c_t - P_t \tau_t$, where $A_t(\equiv P_t a_t)$ denotes total asset holdings. Using $a_t = m_t + b_t$ and $r_t \equiv R_t - \pi_t$, where $\pi_t \equiv \dot{P}_t / P_t$, we obtain (2).

opportunity costs R_t when holding money due to the foregone interest earnings.

The household derives utility from consumption c_t, from its holdings of liquid real money balances m_t and from its net wealth holdings. Its lifetime utility is given by

$$\mathcal{U}_0 = \int_0^\infty [\phi(c_t) + \mu(m_t) + \omega(a_t - m_t^S)]e^{-\rho t}dt,$$

where $\rho > 0$ is the discount rate and the flow utility functions satisfy $\phi' > 0$, $\phi'' < 0$, $\mu' > 0$, $\mu'' < 0$ and $\omega' > 0$ and $\omega'' \leq 0$. We assume that there exists $\bar{m}$ for which the transaction demand for money becomes fully satiated, i.e. $\mu'(m_t) = 0$ for all $m_t \geq \bar{m}$, to allow for a liquidity trap. Our specification of a preference for net wealth $\omega(a_t - m_t^S)$, where m_t^S denotes the real money supply, implies that the household considers the money stock a government liability that eventually needs to be redeemed and hence not part of aggregate net wealth, even though each member considers his personal money holdings a part of his wealth. As there is no heterogeneity on the household level, it is only the value of equity holdings (as bonds are in zero net supply) that affects the wealth preference in steady state.[13]

The household maximizes lifetime utility subject to the asset constraint (1) and the flow budget constraint (2) for given initial wealth a_0. Optimal household behavior is described by the Euler Equation, the money demand function and the transversality condition as

$$\eta_c \frac{\dot{c}_t}{c_t} = r_t - \rho + \frac{\omega'(b_t + m_t - m_t^S)}{\phi'(c_t)}, \tag{3}$$

$$R_t = \frac{\mu'(m_t)}{\phi'(c_t)} \geq 0, \tag{4}$$

$$\lim_{t\to\infty} \phi'(c_t)a_t e^{-\rho t} = 0, \tag{5}$$

where $\eta_c \equiv -\phi''(c_t)c_t/\phi'(c_t)$ is the elasticity of the marginal utility with respect to consumption (and the inverse of the intertemporal elasticity of substitution), which we assume constant.[14]

The preference for wealth affects the intertemporal allocation of consumption (3), providing additional incentives for saving since accumulation of wealth becomes an end in itself. A lower level of wealth, ceteris paribus, induces the household to choose a steeper consumption path or equivalently higher savings in order to accumulate wealth. In steady state, the preference for wealth creates a wedge between the real interest rate and the time preference rate of the household, which allows for the possibility of a negative natural real interest rate. Optimal money demand in (4) requires the marginal rate of substitution between money and consumption to equal the opportunity cost of holding money. The nominal rate is at the zero lower bound whenever $m_t \geq \bar{m}$, which allows for the possibility of a liquidity trap in our framework.

[13] Michau (2019) discusses and justifies the preference for *net* wealth in contrast to alternative specifications.

[14] These equations are obtained from the standard current value Hamiltonian function $\mathcal{H}_t = \phi(c_t) + \mu(m_t) + \omega(a_t - m_t^S) + \lambda_t[r_t a_t + w_t(1 - u_t)x_t + z w_t x_t u_t - c_t - R_t m_t - \tau_t]$, with control variables c_t and m_t and state variable a_t and where λ_t is the costate variable for a_t.

2.2 The labor market

We assume the standard search and matching model of the labor market with free market entry and exit and Nash bargaining between workers and firms as a mechanism for wage determination. The government conducts two types of labor market policies. It pays benefits $\hat{z}_t$ to unemployed workers following the policy rule

$$\hat{z}_t = z w_t x_t, \tag{6}$$

where $z \in [0, 1)$ denotes the replacement rate that is proportional to labor income. In addition, it pays a search cost subsidy $s \in (0, 1)$ to support firms when posting vacancies.

2.2.1 Matching mechanism

Output is produced when firms and workers successfully match in the labor market. The matching function $F(u_t, v_t)$ describes the number of matches between vacant firms v_t and unemployed u_t. $F(u_t, v_t)$ is continuously differentiable, concave, homogeneous of degree one, increasing with respect to both u_t and v_t, and satisfies $F(0, v_t) = 0$ and $F(u_t, 0) = 0$. Let θ_t denote the jobs-to-applicants ratio, i.e.,

$$\theta_t \equiv \frac{v_t}{u_t} \in [0, \infty), \tag{7}$$

which measures the tightness of the labor market. A higher value of θ_t implies more vacancies per job-seeker, which we will refer to as a tighter labor market. Firms are less likely to fill vacancies and unemployed workers are more likely to find employment the tighter the labor market as measured by θ_t. Formally, the probability that a vacant firm matches with a worker $q(\theta_t)$ is simply given by the relative frequency of matches among all vacancies and satisfies

$$q(\theta_t) \equiv \frac{F(u_t, v_t)}{v_t} = F\left(\frac{1}{\theta_t}, 1\right), \quad q'(\cdot) < 0, \quad q''(\cdot) > 0, \quad q(\infty) = 0. \tag{8}$$

Similarly, the probability that a worker matches with a firm with a vacancy $p(\theta_t)$ is given by the relative frequency of matches among all unemployed and satisfies

$$p(\theta_t) \equiv \frac{F(u_t, v_t)}{u_t} = F(1, \theta_t) = \theta_t q(\theta_t), \quad p'(\cdot) > 0, \quad p''(\cdot) < 0, \quad p(0) = 0. \tag{9}$$

In each period a fraction δ of workers become unemployed, where δ constitutes an exogenous separation rate.[15] Hence, the flow into unemployment is given by $\delta(1 - u_t)$. At the same time, the number of successful matches between vacant firms and unemployed workers is given by $F(u_t, v_t) = p(\theta_t)u_t$, which follows from (9). Therefore, the dynamics of the unemployment rate are given by

$$\dot{u}_t = \delta(1 - u_t) - p(\theta_t)u_t. \tag{10}$$

[15] This assumption allows us to focus on the effects of underemployment on the job creation channel as it implies that there is no direct effect of underemployment on the job destruction rate. For the case of Japan, our focus on job creation is supported by empirical evidence in Ariga and Okazawa (2011) who show that hiring rates have declined substantially during the stagnation period while separation rates appear quite stable. We discuss the importance of this assumption for our results in detail in section 5.1.

2.2.2 Firm behavior under free market entry and exit

There is a large number of identical firms that produce the consumption good using labor as the only input factor. Each firm requires exactly one worker in the production process and therefore offers one position, which can be filled, in which case the firm is operating, or vacant. When offering this position and searching for workers in the labor market, a search cost k occurs. The government might provide a search cost subsidy at rate $s \in [0, 1)$ reducing effective search costs for the firm to $(1 - s)k$. Firms that successfully match hire $l_t = 1$ workers and produce output y_t with the linear production function

$$y_t = \bar{y} x_t l_t = \bar{y} x_t, \tag{11}$$

where $\bar{y}$ denotes labor productivity and x_t hours per worker. The operating profit of each producing firm is given by $(\bar{y} - w_t)x_t$, which is increasing in x_t.

The value functions of an operating firm J_t and a vacant firm posting a vacancy V_t, which describe the values of participating in the labor market for firms, are hence given by the Bellman equations

$$r_t J_t = (\bar{y} - w_t)x_t - \delta[J_t - V_t] + \dot{J}_t, \tag{12}$$

$$r_t V_t = -(1 - s)k + q(\theta_t)[J_t - V_t] + \dot{V}_t. \tag{13}$$

Firms can freely enter and exit the labor market without costs. As a consequence, a firm will enter the labor market and post a vacancy as long as its expected value is greater or equal to zero. This free entry condition therefore implies in equilibrium that

$$V_t = 0. \tag{14}$$

From (13), it follows immediately that the value of an operating firm J_t has to satisfy

$$J(\theta_t) = \frac{(1 - s)k}{q(\theta_t)}, \quad J(\infty) \to \infty, \quad J'(\cdot) > 0. \tag{15}$$

The free entry condition requires that the expected value of a filled job equals the search costs associated with a vacancy. The tighter the labor market, the lower the probability of filling a vacancy and the higher the value of an operating firm. Applying the free entry condition (14) and the equilibrium firm value (15) to (12) implies the following differential equation for labor market tightness:

$$\eta_{\theta_t} \frac{\dot{\theta}_t}{\theta_t} \equiv \frac{\dot{J}_t}{J_t} = r_t - \frac{(\bar{y} - w_t)x_t}{(1 - s)k} q(\theta_t) + \delta, \tag{16}$$

where $\eta_{\theta_t} \equiv -\frac{\theta_t q'(\theta_t)}{q(\theta_t)} > 0$ is the elasticity of the matching probability of a vacant firm with respect to θ_t. Note that η_{θ_t} is constant for a Cobb-Douglas matching function. Equation (16) represents the job creation condition of the matching model describing optimal firm behavior for a given real wage. Intuitively, labor market tightness θ_t adjusts so that the real interest rate r_t equals the return on firm ownership. If the real rate exceeds the net profit yield, firms leave the labor market leading to a drop in labor market tightness which in turn allows for capital gains on firm ownership over time. If the real rate is below the net profit yield, jobs are created as firms enter the labor market leading to a rise in tightness and capital losses over time.

2.2.3 Nash bargaining

The real wage is set once workers and firms have successfully matched. While several wage setting mechanisms are consistent with wage determination in this environment, we follow the standard Pissarides (2000) assumption of Nash bargaining between firms and workers.

Wages are set to maximize the joint surplus of workers and firms. We derive the solution to this problem in Appendix A. The equilibrium real wage is determined as:

$$w_t x_t = \psi \left(\bar{y} x_t + (1-s) k \theta_t \right), \tag{17}$$

where $\psi \in (0,1)$ reflects the effective bargaining power of workers.[16] All else equal, the negotiated wage w_t increases in workers' bargaining power ψ, vacancy cost $(1-s)k$ and labor market tightness θ_t, but decreases in working hours x_t. Using (17), the job creation condition (16) can be rewritten as

$$\eta_{\theta_t} \frac{\dot{\theta}_t}{\theta_t} = r_t - \frac{(1-\psi)\bar{y} x_t}{(1-s)k} q(\theta_t) + \psi p(\theta_t) + \delta. \tag{18}$$

2.3 Government

The government consists of a central bank and a fiscal authority. The central bank controls the nominal money supply M_t^S by setting the nominal money growth rate $g_m \geq 0$ for a given initial money supply M_0^S. The real money supply m_t^s evolve as

$$\frac{\dot{m}_t^s}{m_t^s} = g_m - \pi_t, \tag{19}$$

where $\pi_t \equiv \frac{\dot{P}_t}{P_t}$ denotes the rate of inflation. The nominal interest rate is endogenously determined in the money market in line with (4).

The fiscal authority adjusts lump-sum taxes τ_t to balance the budget while providing benefits $\hat{z}_t u_t$ to the unemployed according to the policy rule in (6), subsidizing a fraction s of search costs k of vacant firms and financing government spending g, which is a policy parameter to model exogenous changes in demand, such that

$$\tau_t + g_m m_t^S = \hat{z}_t u_t + s k v_t + g, \tag{20}$$

where g_m denotes the growth rate of the nominal money supply, $g_m m_t^S$ income from seignorage and v_t the number of vacant firms. Government bonds are in zero net supply so that interest-bearing assets of the household b_t solely consist of firm equity.

[16]Specifically, ψ is defined in Appendix A as $\psi \equiv \frac{\varepsilon}{1-(1-\varepsilon)z}$, where ε denotes the coefficient on the surplus of workers in the joint surplus function and z the replacement rate in payments to unemployed workers in (6).

2.4 Aggregation and market clearing

Without demand shortage ($x_t = 1$), the dynamics of the price level P_t are as in the standard Money-in-the-Utility framework where the steady state inflation rate is determined by the money growth rate g_m and the price level adjusts to clear the money market. As a result of the search matching mechnism real wage $w_t(= W_t/P_t)$ is detemined, and therefore, nominal wage W_t follows commodity price P_t.

Turning to the case with demand shortage, because the number of operating firms equals $1 - u_t$ as each operating firm hires exactly one worker and working hours are $x_t(< 1)$, total output is

$$Y_t = (1 - u_t)x_t\bar{y}, \tag{21}$$

based on the linear production function in (11). We can interpret $(1 - u_t)\bar{y}$ as a measure of potential output in the presence of structural unemployment. Total output is reduced below the production capacity $\bar{y}$ due to structural unemployment u_t and idle working hours $(1 - x_t)$ if $x_t < 1$, and then underemployment is $(1 - u_t)(1 - x_t)$. Total output is used for private consumption, public spending and total search costs of firms with vacancies. In the goods market, we hence have

$$(1 - u_t)x_t\bar{y} = c_t + g + k\theta_t u_t, \tag{22}$$

which we derive formally in Appendix B by combining the budget equations of the household and the government. By reformulating (22), we can express x_t as the ratio between effective demand and potential output. Therefore, x_t constitutes both a measure of underemployment and a measure of effective demand shortage as reflected in the output gap.[17]

We assume downward price rigidity when underemployment occurs ($x < 1$) –i.e., there is a lower bound of the time change rate of prices P_t, which is $\alpha(x_t - 1)$, in order to prevent deflationary wage-price-spirals and to allow for the possibility of a stagnation steady state.[18] P_t declines but its speed cannot be lower than $\alpha(x_t - 1)$. These properties are summarized as follows:

$$\pi_t \left(= \frac{\dot{P}_t}{P_t} \right) = \alpha(x_t - 1) \quad \text{for } x_t < 1. \tag{23}$$

In order for the steady state to be supported, g_m must be higher than $\alpha(x_t - 1)$:

$$g_m > \alpha(x_t - 1). \tag{24}$$

The asymmetry in the inflation process is a fundamental element of stagnation models including among others the contributions of Schmitt-Grohé and Uribe (2016, 2017), Michau (2018), Illing et al. (2018) and Eggertsson et al. (2019) and typically results from some form of downward nominal wage rigidity that becomes binding in case of demand shortage.

At any point in time, financial markets are in equilibrium. Bonds are in zero net supply. In

[17] In the business cycle literature, output fluctuations around the natural level of output are the result of variations in working hours in response to various shocks. In contrast, our framework allows for the possibility of underemployment as a steady state phenomenon, even in the absence of shocks.

[18] Our model features outright deflation under stagnation to be consistent with the experience of Japan during its lost decades when prices and wages declined on average.

the money market, money demand of the household in (4) equals the money supply set by the central bank, i.e. $m_t = m_t^S \equiv \frac{M_t^S}{P_t}$. The aggregate firm value is given by $(1-u_t)J_t$, which in equilibrium equals the value of equity holdings of the household b_t. Using (15), the net wealth of the household $b_t = a_t - m_t^S$ is therefore given by

$$b_t = (1-u_t)\frac{(1-s)k}{q(\theta_t)}. \tag{25}$$

Financial market equilibrium also satisfies the job creation condition (18) and the Fisher equation

$$r_t = R_t - \pi_t = \frac{\mu'(m_t)}{\phi'(c_t)} - \alpha(x_t - 1) \quad \text{for } x_t < 1, \tag{26}$$

where the inflation rate is given in (23) and the nominal interest rate in (4), so that households are indifferent between equity, bonds and money as a means of saving. In equilibrium, the return on money has to be the same as the return on other assets and real money holdings evolve as

$$\frac{\dot{m}_t}{m_t} = g_m - \alpha(x_t - 1) \;\; > 0 \text{ for } x_t < 1. \tag{27}$$

Hence, real money holdings are constant in the absence of aggregate demand shortage but expand indefinitely in case of secular stagnation due to the effects of deflation, thereby pushing the economy into a permanent liquidity trap as eventually $m_t > \bar{m}$ and the household's transaction demand becomes satiated.

This concludes the exposition of our model. In the next section, we analyze and contrast the different possibilities for steady state equilibria. For simplicity, we will assume a constant nominal money supply

$$g_m = 0,$$

but our analysis can easily be generalized as discussed in Illing et al. (2018) or Michau (2018).

3 Occurrence of underemployment in steady state

Equations (3), (10), (18) and (27) form an autonomous dynamic system with respect to c_t, u_t, θ_t and m_t, where x_t, b_t and r_t are derived from (22), (25) and (26) as functions of these variables. The equilibrium of the model is defined as follows:

An ***equilibrium*** *is a set of paths for quantities* $\{c_t, m_t, b_t, a_t, Y_t, y_t, x_t, u_t, v_t, \theta_t\}$ *and for prices* P_0 *and* $\{r_t, R_t, w_t, \pi_t\}$ *such that*

- $\{c_t, m_t, b_t, a_t\}$ *solves the consumer's problem given* $\{r_t, R_t, w_t, u_t, \theta_t\}$, a_0, $m_0 = M_0/P_0$ *and* $a_t = b_t + m_t$ *in* (1);
- *firms produce* y_t *given* $\{r_t, w_t, u_t, \theta_t\}$ *resulting in aggregate output* Y_t *given by* (21);
- $\{w_t, v_t, u_t, \theta_t\}$ *solves the Nash bargaining problem in the labor market under free market entry and exit of firms where* $v_t = \theta_t u_t$ *according to* (7);

- *equilibrium prices P_0 and $\{r_t, R_t, \pi_t\}$ are consistent with money market equilibrium, equity market equilibrium as well as the No-Arbitrage relation between money and equities* (26), *while in the goods market, inflation and working hours are determined by the non-linear Phillips curve* (23) *and $x_t \leq 1$, which hold with complementary slackness.*

In a **stationary steady state**, consumption, the unemployment rate and labor market tightness are constant, i.e. $\dot{c}_t = 0$, $\dot{u}_t = 0$ and $\dot{\theta}_t = 0$, while the behavior of $\dot{m}_t$ depends on the presence of aggregate demand shortage.

Consider first the labor market. In steady state, the number of newly unemployed equals the number of newly filled jobs. From (10) with $\dot{u}_t = 0$, the unemployment rate satisfies

$$u(\theta) = \frac{\delta}{\delta + p(\theta)}, \quad u(0) = 1, \quad u'(.) < 0, \quad u''(.) > 0. \tag{28}$$

This is the standard Beveridge curve. A tighter labor market is characterized by lower structural unemployment due to a higher probability of finding a job, i.e. $p'(\theta) > 0$ in (9). There are no direct spillovers from underemployment into unemployment or job destruction.

In contrast, underemployment directly affects the job creation incentives of firms. Under free market entry and exit, the real interest rate equals the return on equity, which in steady state consists of the profit yield net of the separation rate. From (18) with $\dot{\theta}_t = 0$, it has to hold that

$$r = \frac{(1-\psi)\bar{y}x}{(1-s)k} q(\theta) - \psi p(\theta) - \delta. \tag{29}$$

This relationship is illustrated by the blue curve in Figure 2 for $x = 1$ as a decreasing function of θ. A tighter labor market, ceteris paribus, reduces the return on equity as both the firm value in (15) and the negotiated real wage in (17) increase. If the real rate is above the return on equity, firm ownership is less attractive than other assets and firms leave the labor market resulting in a drop in labor market tightness and higher unemployment until equilibrium is restored (and vice versa). There is a unique value $\tilde{\theta}$ for which the return on equity is zero given $x = 1$, defined by

$$\tilde{\theta}: \quad \frac{(1-\psi)\bar{y}}{(1-s)k} q(\tilde{\theta}) - \psi p(\tilde{\theta}) = \delta. \tag{30}$$

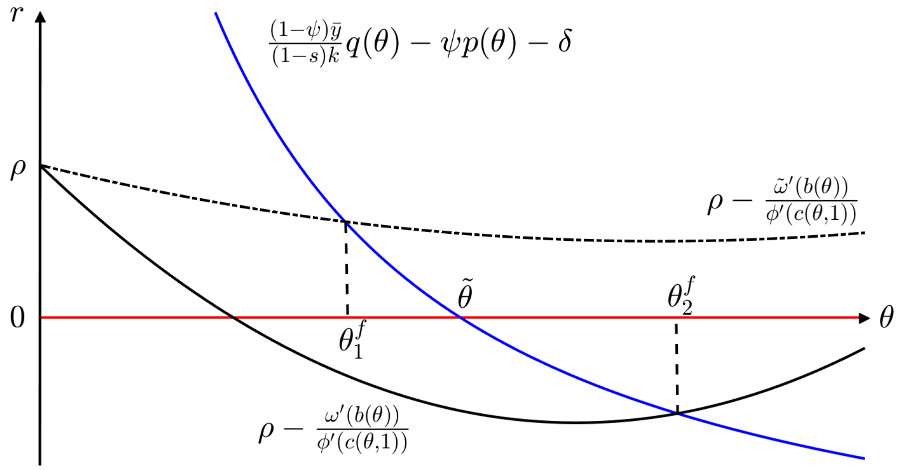
In the goods market, the supply of goods available for household consumption, i.e. total output less entry costs of firms and government spending in (22), is given by

$$c(\theta, x) = (1 - u(\theta))x\bar{y} - k\theta u(\theta) - g. \tag{31}$$

It shows a hump-shaped pattern with respect to labor market tightness for $x = 1$. Initially, total output rises by more than search costs as unemployment decreases. At some point, however, the increase in total search costs exceeds the increase in output, thereby lowering household income. Following Michaillat and Saez (2015), we refer to the former case as a "slack" steady state and the latter one as a "tight" steady state.[19]

[19] Formally, a steady state with $\frac{dc}{d\theta} > 0$ in (31) is referred to as slack and a steady state with $\frac{dc}{d\theta} < 0$ as tight. Note, however, that the definition refers to the total (rather than the partial) derivative.

Figure 2: Steady state without underemployment



Notes: The blue curve illustrates the return on firm equity in (29) as a function of labor market tightness θ, the black curves the natural real rate consistent with optimal household behavior in (32) for two specifications of the wealth preference. The red line represents the zero lower bound on the nominal and real (as $g_m = 0$) interest rates. Labor market tightness is determined by the intersection of the two curves for $x = 1$. Such a steady state is only feasible if $\theta^f < \tilde{\theta}$.

Consumption demand of the household has to be consistent with the supply of goods available for private consumption. For a given real interest rate, it follows from the Euler Equation (3) with $\dot{c}_t = 0$ that

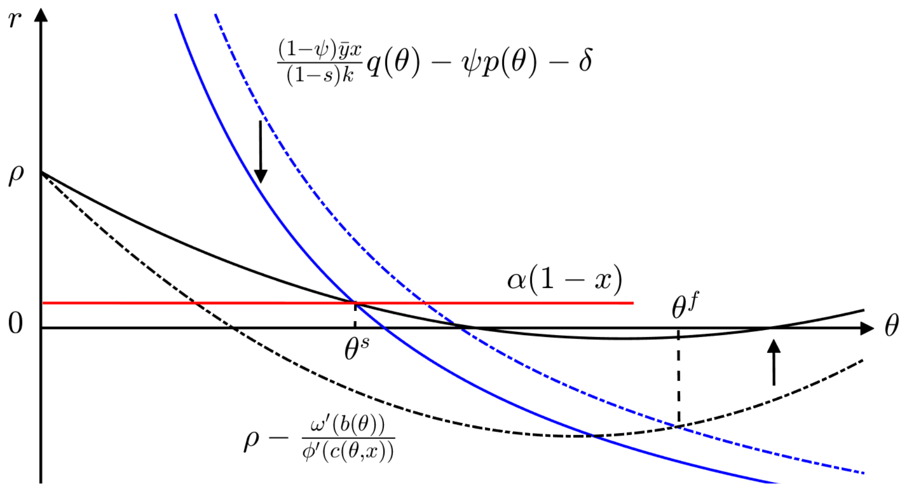
$$r = \rho - \frac{\omega'(b(\theta))}{\phi'(c(\theta, x))}, \tag{32}$$

where $c(\theta, x)$ is given by (31) and $b(\theta)$ by (25) and (28) with $b'(\theta) > 0$ as a tighter labor market implies both a larger number of operating firms and a higher valuation of them. In the standard model, the steady state interest rate is simply determined by the time preference rate ρ. In contrast, the wealth premium, i.e. the marginal rate of substitution between consumption and wealth, creates a wedge between the time preference rate and the real interest rate as the household saves for the purpose of accumulating wealth. We refer to the interest rate for which consumption demand equals the supply of goods as the "natural" real interest rate. It is represented by the black curves in Figure 2 for two specifications of the preference for wealth, ω' and $\tilde{\omega}'$, representing a strong and weak desire to save respectively.

If the real rate is above the natural rate, demand is depressed and there is excess supply in the goods market. For $x = 1$, adjustment occurs via changes in θ. A drop in labor market tightness affects the goods market via two channels. On the one hand, household wealth decreases, which reduces consumption demand. On the other hand, the income of the household in (31) increases in a tight steady state and falls in a slack steady state. In the former case, the wealth premium increases which reduces demand and increases desired savings thereby worsening the excess supply. In the latter case, the combined effect is unclear and depends on the concavity of the wealth preference.[20]

[20] In Michaillat and Saez (2022) a fall in labor market tightness and higher unemployment unambiguously reduce desired savings of the household. This is because of their assumption of a preference for relative wealth, i.e. wealth relative to the average wealth across households, implying a constant marginal utility of wealth in steady state, and a slack steady state such that $c'(\theta) > 0$ in (31) for $x = 1$. As a consequence, an excessively

Figure 3: Steady state with underemployment



Notes: The shift in of the blue curve illustrates the effects of underemployment on the return on firm equity in (29), the shift of the black curve the effects on the natural real rate in (32). The red line represents the rate of deflation for a given x. Underemployment and labor market tightness are jointly determined such that (33) with $x < 1$ holds.

In general equilibrium, unemployment and underemployment are jointly determined such that the optimal firm behavior in (29) is consistent with the optimal household behavior in (32). As illustrated in Figure 2, the real interest rate is determined by the simultaneous equilibrium in goods and labor markets. The nominal rate then adjusts endogenously in the money market. However, if the wealth preference is sufficiently strong, the steady state requires a negative real interest rate, which is not feasible due to the zero lower bound resulting in an oversupply of savings and asset market disequilibrium, as holding money becomes too attractive. This occurs when the steady state with $x = 1$ requires labor market tightness to be above $\tilde{\theta}$ as defined in (30). In Figure 2, the steady state with θ_2^f is therefore not feasible.

Then, an involuntary drop in working hours is necessary to reconcile optimal household and firm behavior and underemployment occurs. This stagnation steady state is characterized by persistent deflation and expanding real money balances as is clear from (27), such that eventually, when $m_t > \bar{m}$, the liquidity preference of the household is satiated and the nominal interest rate is at the zero lower bound as $\mu'(m) = 0$.[21] The real interest rate is then determined by the rate of deflation, represented by the red line in Figure 3, and adjustment takes place in the labor and goods markets via both unemployment and underemployment.

In the labor market, there are fewer vacancies as firms leave the market since holding money is more attractive than equity. This results in a drop in labor market tightness and higher unemployment. However, this adjustment is not sufficient to establish goods market equilibrium as well, which can easily be seen in Figure 3. There is insufficient demand in the goods market, which depresses firm sales and profits. Firms respond by reducing working hours. Underemployment occurs which reduces household income as employed members work fewer hours. This

high real interest rate results in demand-driven "Keynesian" unemployment, which eliminates the excess supply.

[21]The transversality condition (5) needs to be satisfied as well, which requires a sufficiently low money growth rate g_m and a sufficiently low rate of price adjustment α. A sufficient restriction is $\rho > g_m + \alpha$. Recall that we focus on the case of $g_m = 0$ and hence assume $\rho > \alpha$.

in turn reduces desired savings as the wealth premium falls, which is represented by the upward shift of the black curve in panel (b) of Figure 3 for given θ. The drop in working hours feeds back into the labor market. As the marginal product of labor falls and firm profits drop, investment in firms becomes less attractive resulting in further market exit. As firms leave the market, fewer vacancies are posted and labor market tightness declines. Unemployment increases as is clear from (28).[22] This is represented by the shift of the blue curve to the left in Figure 3.

In the stagnation steady state, unemployment and underemployment are determined such that both optimal household and firm behavior are consistent with the excessively high real interest rate caused by deflation. This is a substantial extension of the existing literature on involuntary unemployment where adjustment occurs through involuntary variations in unemployment only. Our analysis shows that such an adjustment is not sufficient in an integrated framework that takes both the supply and the demand side into account.

Formally, the two possible steady states of our model are expressed in the following general equilibrium condition combining (29), (32) and the steady state expression of (26):

$$\rho - \frac{\omega'(b(\theta))}{\phi'(c(\theta, x))} = \frac{(1-\psi)\bar{y}x}{(1-s)k} q(\theta) - \psi p(\theta) - \delta = \begin{cases} \dfrac{\mu'(m)}{\phi'(c, x)} & \text{if } x = 1, \\ \alpha(1-x) & \text{if } x < 1. \end{cases} \tag{33}$$

The following lemmata summarize their existence and stability conditions:

Lemma 1 *The steady state without demand shortage exists if*

$$\rho\phi'\left(c(\tilde{\theta}, 1)\right) \geq \omega'\left(b(\tilde{\theta})\right), \tag{34}$$

where $b(\theta)$ is given by (25) *in combination with* (28) *and $\tilde{\theta}$ is uniquely defined by* (30). *If the steady state without demand shortage is unique, then it is also saddle-path stable.*

At $\tilde{\theta}$ the return on equity is zero for $x = 1$ and hence $\theta > \tilde{\theta}$ in steady state is not feasible as this would require a negative real interest rate. Hence, Lemma 1 requires a sufficiently weak wealth preference for the steady state without aggregate demand shortage to exist, which can easily be seen in Figure 2.[23] In principle, there could be multiple steady states consistent with (33). Yet, mild technical restrictions on the functional forms of $\phi(.)$, $\omega(.)$ and $q(.)$ are sufficient to establish uniqueness of this steady state.[24] We assume that this steady state with $x = 1$ is unique throughout this paper.

Consider now the stagnation steady state, which is illustrated in Figure 3.

[22]This spillover channel from underemployment to unemployment is similar to the occurrence of rationing unemployment in Michaillat (2012) due to the effects of negative technology shocks on the marginal product of workers. Yet, we do not rely on shocks as underemployment occurs endogenously in our model.

[23]In addition, it needs to hold that $\frac{(1-\psi)\bar{y}}{(1-s)k}q(\underline{\theta}) - \psi p(\underline{\theta}) > \rho + \delta > \frac{(1-\psi)\bar{y}}{(1-s)k}q(\bar{\theta}) - \psi p(\bar{\theta})$ where $\underline{\theta}$ and $\bar{\theta}$ are defined by $c(\underline{\theta}, 1) = c(\bar{\theta}, 1) = 0$ in (31) with $\underline{\theta} < \bar{\theta}$. This is a purely technical restriction that ensures that creating new vacancies is sufficiently attractive when unemployment is high. If there are no incentives to establish firms even in case of high unemployment, existence of a steady state is not guaranteed.

[24]Intuitively, uniqueness requires that an exogenous increase in wealth reduces the household's propensity to save via a reduction in labor market tightness θ^f. Not only is it the natural economic scenario to expect, it is also technically a very mild condition. Specifically, this steady state is always unique provided that the elasticity of the unemployment rate with respect to θ is sufficiently low.

Lemma 2 *The secular stagnation steady state with $\theta^s < \tilde{\theta}$ and $x^s < 1$ exists if $\rho > \alpha$ and if*

$$\rho\phi'\left(c(\tilde{\theta}, 1)\right) < \omega'\left(b(\tilde{\theta})\right), \tag{35}$$

where $b(\theta)$ is given by (25) *in combination with* (28) *and $\tilde{\theta}$ is uniquely defined by* (30)*. If the secular stagnation steady state is unique, then it is also saddle-path stable.*

Under stagnation, working hours x^s and labor market tightness θ^s are jointly determined in the goods and labor markets such that optimal household and firm behavior are consistent with the real interest rate given by the rate of deflation. Since the return on equity at $\theta = \tilde{\theta}$ is zero and declining in θ, it has to hold that $\theta^s < \tilde{\theta}$ in the stagnation steady state. Consumption is then determined by (31) with $x = x^s$ and $\theta = \theta^s$.[25] The condition $\rho > \alpha$ establishes a lower bound on the rate of deflation thereby preventing a wage-price spiral that is not consistent with the existence of a well-defined steady state. An excessively high rate of deflation would imply an expansion of real money balances that violates the transversality condition (5). Lemma 2 further requires the real rate at $\tilde{\theta}$ to be negative. This is exactly opposite to condition (34) in Lemma 1.[26] Again, we assume that the stagnation steady state is unique. This holds under mild restrictions on the functional forms of the utility, production and matching functions.[27]

The second equality in condition (33) for $x < 1$ implies that firms optimally adjust working hours and employment such that the return on equity is equalized to the rate of deflation. This is the modified job creation condition under stagnation and implies the equilibrium relationship

$$x(\theta) = \frac{(1-s)k\left[\psi p(\theta) + \delta + \alpha\right]}{(1-\psi)\bar{y}q(\theta) + (1-s)k\alpha}, \tag{36}$$

which satisfies $\frac{dx(\theta)}{d\theta} > 0$ and $x(\tilde{\theta}) = 1$, with $\tilde{\theta}$ defined in (30). For $x > x(\theta)$, the return on equity exceeds the return on money, which makes firms enter the labor market resulting in more vacancies and lower unemployment until equilibrium is restored. Similarly, for $x < x(\theta)$, the return on equity is below the return on money and firms leave the labor market resulting in fewer vacancies and higher unemployment until equilibrium is restored.

Two corollaries follow immediately from the analysis above. First, the preference for wealth reduces the steady state real interest rate relative to the standard matching model, in which the real rate is simply given by the time preference rate ρ. In the absence of demand shortage, the lower real rate stimulates firm creation resulting in a tighter labor market and a lower

[25] To ensure non-negativity of consumption under stagnation, we impose the technical restriction on spending g and the matching function such that $c(\theta^s, x^s) \geq 0$ exists.

[26] Assuming uniqueness of the steady state with $x = 1$ and the one with $x < 1$, these steady states are mutually exclusive. This no longer holds for $g_m > 0$. Then both steady states are feasible for some parameter calibrations.

[27] Formally, uniqueness requires that under secular stagnation an exogenous increase in wealth increases the propensity to save of the household and therefore increases the aggregate equity value via an increase in the steady state labor market tightness θ^s. Compared to the previous case, an increase in θ affects the propensity to save via two additional channels: First, the induced decrease in the real interest rate is dampened due to the weaker response of the return on equity to increases in θ as a consequence of the prevalence of persistent deflation. Secondly, there is a stronger effect of changes in θ on the net supply as a consequence of the induced change in realized labor working hours. Both, higher income and a smaller sensitivity of the return on savings relative the to case of full employment support the assumption of an increase in the propensity to save in response to an exogenous increase in wealth under stagnation.

unemployment rate. Second, the secular stagnation steady state necessarily has to be slack if it is unique, i.e. an increase in labor market tightness raises steady state consumption $c(\theta^s, x^s)$. This is in contrast to the steady state without demand shortage which can be slack or tight depending on the functional form of the matching function and the preference for wealth.

Having analyzed existence and stability of the stagnation steady state, we turn to the relative importance of structural unemployment and demand-driven underemployment and their sensitivity to labor market policies as well as demand and supply shocks. In order to get a better understanding of the model and to quantify these effects, the theoretical analysis is supported by a calibration and numerical simulation of the model.

4 Calibration

We set a time period to be one quarter. We assume CRRA utility for consumption and a non-homothetic specification for the preference for wealth, which is empirically most plausible as argued by Mian et al. (2021), implying that the marginal utility of wealth decays more slowly than the marginal utility of consumption. In our benchmark specification we implement this feature via a constant marginal utility, i.e. $\omega' = \beta$. Alternatively, we consider a CARA utility function. The parameter β reflects the weight of the wealth preference.[28] We rely on a Cobb-Douglas matching function in our baseline specification, which implies a constant elasticity $\eta_\theta = \gamma$ of the job filling rate $q(\theta)$. The scale parameter κ reflects the matching efficiency. Specifically, we assume the following functions:

$$\phi(c) = \frac{c^{1-\eta_c} - 1}{1 - \eta_c},$$

$$\omega(a - m^s) = \beta(a - m^s) \quad \text{(benchmark)},$$

$$\omega(a - m^s) = -\frac{\beta}{\eta_a} e^{-\eta_a(a-m^s)} \quad \text{(alternative)},$$

$$F(u, v) = \kappa u^\gamma v^{1-\gamma}.$$

Fifteen parameters need to be calibrated. The calibration is summarized in Table 1. We set $\eta_c = 2$ and $\eta_a = 0.5$ to capture the difference in curvature between $\phi(.)$ and $\omega(.)$ and choose β to generate a specific level of underemployment in steady state. The time preference rate ρ represents an upper bound on the real rate as can be seen in (32). We assume $\rho = 12\%$ on an annual basis, resulting in a quarterly time preference rate of $\rho = 0.03$. Lemma 2 restricts the speed of price adjustment to allow for the possibility of stagnation. We set $\alpha = 0.005$, which implies a maximum annual deflation rate of 2%. Without loss of generality, labor productivity is normalized to $\bar{y} = 1$. Variations in this parameter are used to analyze the effects of supply shocks. Our baseline calibration also assumes $g = 0$ and $s = 0$ and subsequent variations in these parameter are used to illustrate the effects of demand shocks and the introduction of a

[28] Mian et al. (2021) do not directly specify the preference for wealth but the marginal utility of wealth relative to a logarithmic specification, which reflects the utility from consumption. Their calibration implies that this measure is always strictly positive in $(0, 1)$. Similarly, our benchmark specification implies that this measure is constant at level β, which we chose in $(0, 1)$.

Table 1: Parameter calibration

Parameter	Calibration	Comment
CRRA parameter in $\phi(.)$	$\eta_c = 2$	implies $\eta_c > -b^s\omega''(b^s)/\omega'(b^s)$
CARA parameter in $\omega(.)$	$\eta_a = 0.5$	(used in the alternative specification of the wealth preference)
Scale parameter in $\omega(.)$	β	calibrated to generate a specific x^s in steady state
Time preference rate	$\rho = 0.03$	upper bound on the real interest rate: $r \leq 0.12$
Price adjustment speed	$\alpha = 0.005$	maximum rate of deflation: $\pi \geq -0.02$
Potential income	$\bar{y} = 1$	normalization, used to model supply shocks
Public spending	$g = 0$	used to model demand shocks
Search cost subsidy	$s = 0$	policy variable
Replacement rate	$z = 0.6$	cf. Kitao et al. (2017), Kudoh et al. (2019)
Bargaining power	$\varepsilon = 0.5$	cf. Mortensen and Pissarides (1994), Den Haan et al. (2000)
Bargaining power (eff.)	$\psi = 0.714$	$\psi = \varepsilon(1-(1-\varepsilon)z)^{-1}$
Job separation rate	$\delta = 0.015$	average job duration of 16.7 years
Cost of a vacancy	$k = 0.431$	match $r = 5\%$ (yearly) in (29) for $\bar{\theta} = 0.85$ and $x = 1$
Elasticity of $q(\theta)$	$\gamma = 0.5$	Hosios (1990) efficiency condition, $\gamma = \varepsilon$
Scale parameter	$\kappa = 0.449$	match $\bar{u} = 0.035$ and $\bar{\theta} = 0.85$

Note: The parameter β of the wealth preference is chosen to generate a specific output gap in steady state. Details for the calibration of the other parameters are provided in the text.

search cost subsidy. We set the wage replacement rate for unemployed equal to $z = 0.6$ in (6), which follows the calibration in Kitao et al. (2017) for the U.S. and Miyamoto (2011) as well as Kudoh et al. (2019) for Japan. In addition, the OECD reports an initial net replacement rates of 62% for Japan (see Table 3.2 in OECD, 2006).[29] Following Mortensen and Pissarides (1994) and Den Haan et al. (2000), among others, we assume symmetric bargaining power $\varepsilon = 0.5$ between workers and firms. This implies an effective bargaining power of $\psi = 0.714$. Contributions such as Shimer (2005), Miyamoto (2011) and Kudoh et al. (2019) have assigned a higher bargaining power to workers of $\varepsilon = 0.72$ and $\varepsilon = 0.6$ respectively.

The matching process is calibrated to replicate empirical features of the Japanese labor market. We set the job separation rate δ to match the average duration of a job. Miyamoto (2011) reports an average job duration rate of 18 years for Japan during the 1980 to 2010 period, while more recent data by the Japan Institute for Labour Policy and Training is closer to 12 years. We hence choose $\delta = 0.015$ for Japan, implying an average job duration of 16.7 years.[30] Japan reports an average unemployment rate of $\bar{u} = 3.5\%$ over the 1980-2018 period (Figure 1). In addition, data on the active job-openings-to-applicants ratio for the period 1980-2018 is available from the Japan Institute for Labour Policy and Training. We use the average value

[29] It should be noted that there is substantial variation in the literature when it comes to calibrating this parameter with values ranging between 0.2 and 0.7. Studies such as Shimer (2005) assign a slightly smaller value ($z = 0.4$) to the United States. Following our methodology, such a smaller value would result in a higher cost parameter k. The results are, however, robust to this change in the calibration.

[30] Japan's job separation rate is substantially lower than the separation rate in the United States, reflecting among others institutional features such as the traditional system of lifetime employment.

of $\bar{\theta} = 0.85$ over this period as a measure of steady state labor market tightness. Based on this data, the Beveridge curve in (28) implies an average job-finding rate $p(\bar{\theta}) = 0.41$ on a quarterly basis, which is in line with Miyamoto (2011), who reports a quarterly rate of 0.426 for Japan. We calibrate the cost parameter k to yield an annual return on equity of 5% in the absence of demand shortage based on (29) with $x = 1$. This yields a vacancy cost of $k = 0.431$.[31]

As is standard in the literature (cf. Shimer, 2005; Miyamoto, 2011; Kudoh et al., 2019, among others), we use the Hosios (1990) efficiency criterion and calibrate the parameter γ in the matching function equal to the bargaining power parameter, i.e. $\gamma = \varepsilon = 0.5$, so that the only inefficiencies emerge from the zero lower bound and the price rigidity. The scale parameter κ in the matching function is then calculated to match the average job finding rate $p(\theta)$. This implies a value of $\kappa = 0.449$, which is in line with the values reported in Miyamoto (2011) and Kudoh et al. (2019) based on a similar approach.

We conduct two robustness checks in the appendix. In robustness check I, we adjust the calibration to match the corresponding features of the U.S. labor market with a higher separation rate and a higher steady state unemployment rate. In robustness check II, we use a more general matching function with a constant elasticity of substitution between vacancies and unemployed following Den Haan et al. (2000). Our main conclusions are robust to these modifications.

5 Unemployment and underemployment under stagnation

5.1 The employment gap under stagnation

In the stagnation steady state underemployment emerges in addition to structural unemployment. The total employment gap in the labor market can be decomposed as

$$\underbrace{1-(1-u_t)x_t}_{\text{Employment gap}} = \underbrace{u_t}_{\substack{\text{Structural}\\ \text{unemployment}}} + \underbrace{(1-u_t)(1-x_t)}_{\text{Underemployment}}. \tag{37}$$

While the unemployment rate captures the total employment gap in the standard model when $x = 1$, it is an inadequate indicator of the slack in the labor market under stagnation.

How do firms optimally respond to the emergence of demand shortage? Consider the effects of an exogenous increase in the desire to save, i.e. the marginal utility of wealth $\omega'(.)$, under stagnation in (33). Higher desired savings result in excess supply in the goods market to which firms respond by reducing working hours and production as they are no longer able to sell all of their products. This affects unemployment via two channels. First, the reduction in firm profits induced by lower sales makes it less attractive for firms to enter the recruitment market (job creation channel). Second, deflation increases, which makes money holdings more attractive (deflation channel). Both effects imply that firms post fewer job openings, which results in fewer vacancies and higher unemployment. Higher unemployment and lower working

[31] This value may be slightly higher compared to the existing literature. Miyamoto (2011) calibrate a value of $k = 0.31$ for Japan. Yet, this is based on the assumption of a higher bargaining power of workers. We also can attain lower values by assuming a higher bargaining power of workers. For instance, for $\varepsilon = 0.6$, the respective value becomes $k = 0.29$.

Table 2: Steady state with underemployment

Shortfall of working hours $1 - x^s$ (targeted)	2 %	5 %	10 %	20 %
Unemployment rate under stagnation u^s	3.47	3.52	3.62	3.84
Total employment gap in equation (37)	5.40	8.34	13.26	23.07
Underemployment share in employment gap	35.7	57.8	72.7	83.4
Difference in unemployment rates $u^s - u^f$	0.04	0.11	0.22	0.49
Shortfall of consumption $c^s/c^f - 1$	-2.0	-5.1	-10.2	-20.5
Shortfall of production $Y^s/Y^f - 1$	-2.0	-5.1	-10.1	-20.4

Rows: (1) Targeted shortfall of working hours below potential, i.e. $1 - x^s$. Generated by adjusting the scale parameter β in the marginal utility of wealth. (2) Unemployment rate (in %). (3) Total employment gap (in %), see equation (37). (4) Share of underemployment in the total employment gap (in %). (5) Difference of unemployment rate relative to standard model without demand shortage (in %). (6) Consumption shortfall relative to the standard model without demand shortage (in %). (7) Consumption shortfall relative to the standard model without demand shortage (in %).

hours reduce household income, the fall of which has to be sufficiently strong to reduce the higher desired savings. In addition, the real wage, consumption and output fall. The following proposition summarizes these effects.

Proposition 1 *As stagnation worsens, due to an exogenous increase in the marginal utility of wealth, both unemployment and underemployment increase while total output, household consumption and the real wage decline in steady state.*

The employment gap widens as stagnation worsens due to rising unemployment and underemployment. The relative response of working hours and unemployment to worsening stagnation depends on the slope of the Beveridge curve in (28) relative to the slope of the job creation condition (36) as neither is directly affected by variations in the wealth premium. The steeper the Beveridge curve and the flatter the job creation curve under stagnation, the stronger the response of unemployment to demand shortage:

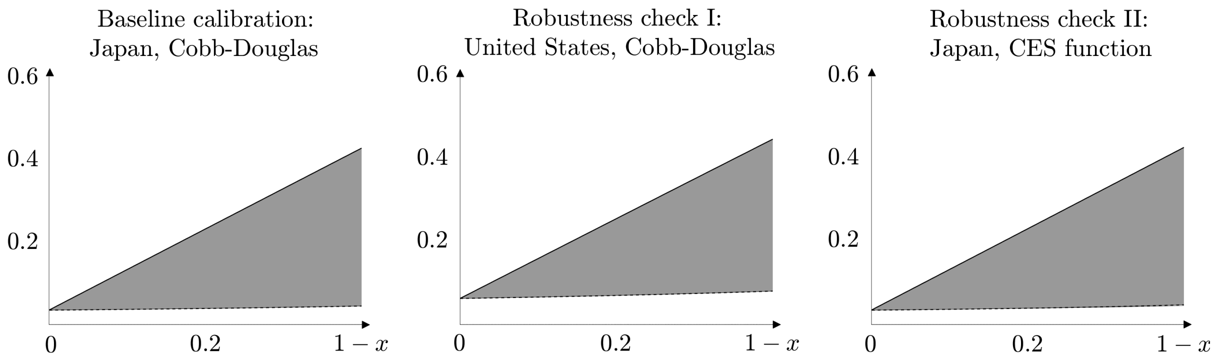
$$\frac{du^s}{dx^s} = \frac{u'(\theta)}{x'(\theta)} < 0 \tag{38}$$

Table 2 summarizes the properties of the stagnation steady state for the baseline calibration.[32] The parameter β is calibrated to target a specific shortfall of working hours, which is illustrated in the first row.[33] All other properties follow from this target. Rows (2) to (4) show the unemployment rate, the total employment gap and the share of underemployment. Rows (5) to (7) contrast the stagnation case to the standard case with $x = 1$ assuming that the

[32] These values are in fact barely affected by the specification of the matching function and the wealth preference. This is in stark contrast to the effects of parameter variations discussed later.

[33] Note that this is equivalently to specifying a negative value for the natural real interest rate for $x = 1$ (which cannot be obtained due to the binding lower bound on the nominal interest rate).

Figure 4: Employment gap under stagnation



Note: This figure shows the decomposition of the employment gap $1-(1-u)x$ among the unemployment rate u (dashed line) and underemployment $(1-u)(1-x)$ (shaded area) as a function of the degree of demand shortage $(1-x)$ (generated by variations in the scale parameter β of the wealth preference). The baseline calibration is described in the main text, robustness checks I (based on U.S. data) and II (based on a CES matching function) are discussed in the appendix.

negative real interest rate could be attained.[34] Row (5) shows the difference in unemployment rates, while rows (6) and (7) contain the shortfall of consumption and production.

As demand shortage worsens, the total employment gap widens considerably with both unemployment and underemployment increasing. This is in stark contrast to the standard case where a stronger desire to save reduces unemployment. However, the spillovers from demand shortage to the unemployment rate are quantitatively weak. Even at a 20% shortfall of working hours, structural unemployment is only about half a percentage point higher than in the alternative steady state. Underemployment quickly dominates in the total employment gap accounting for already one third of the gap at a 2 % shortfall of working hours and more than half at a 5 % shortfall. In addition, we document strong spillover effects from underemployment in the labor market into macroeconomic variables in the goods market. It is the shortfall of working hours rather than the increase in the unemployment rate that accounts almost completely for the shortfall in consumption and production under stagnation.

For further illustration, Figure 4 shows the decomposition of the steady state employment gap into unemployment and underemployment (shaded area) for the baseline calibration and two alternative calibrations. These are based on U.S. labor market data (I), which most notably features a higher job separation rate δ, and a different specification of the matching function (II), described in detail in the appendix. In all cases, underemployment rises steeply once stagnation occurs and quickly dominates in the employment gap. In contrast, the response of structural unemployment is weak, albeit spillover effects from demand shortage into unemployment are slightly stronger using the calibration based on the U.S. labor market. It is demand-driven underemployment that provides a more accurate measure of the slack in the labor and goods markets under stagnation.

We next analyze the robustness of these findings. For a given degree of demand shortage

[34] This is the appropriate benchmark for comparisons since changes in the strength of the wealth preference also affect the standard steady state. A stronger wealth preference, i.e. a stronger desire to save, reduces the real interest rate, which increases labor market tightness and results in a lower unemployment rate.

Table 3: Comparative statics for the steady state unemployment rate

Calibration:	Baseline	$k = 0.2$	$\psi = 0.5$	$\gamma = 0.2$	$\gamma = 0.8$	$\delta = 0.03$	$\alpha = 0.025$
Unemployment rate u^s	3.52	2.41	2.25	3.71	3.35	6.96	3.52
Total employment gap	8.3	7.3	7.1	8.5	8.2	11.6	8.3
Underemployment share	57.7	67.0	68.5	56.5	59.1	40.1	57.7
Increase in unemployment	0.10	0.07	0.06	0.17	0.04	0.19	0.10

Parameter variations are in comparison to the baseline calibration in Table 1 for a steady state with a targeted 5% shortfall of working hours below potential. Rows: (1) Unemployment rate (in %). (2) Total employment gap (in %), see equation (37). (3) Share of underemployment in the total employment gap (in %). (4) Increase in unemployment rate (in %) when shortfall of working hours increases to 10%.

$1 - x$, equilibrium labor market tightness and unemployment are affected by the parameters in job creation condition (36) and the Beveridge curve (28) as well as the shape of the matching function, but independent of the wealth preference. Table 3 summarizes the effects of some parameter changes in comparison to our baseline calibration for a given 5% shortfall of working hours (corresponding to the second column in Table 2). For instance, higher firm productivity $\bar{y}$ or lower vacancy cost k are associated with a tighter labor market and hence a lower unemployment rate as, all else equal, they imply a higher return on firm ownership and hence more incentives for posting job openings.[35] Similarly, a weaker bargaining power of workers ψ results in a lower unemployment rate via the same channel. The curvature parameter γ in the matching function affects the unemployment rate for given working hours via both the job creation condition and the Beveridge curve, with lower values of γ resulting in a higher unemployment rate compared to our baseline calibration (and vice versa).[36] A higher job separation rate δ, while reducing the return on equity, has the additional effect of shifting the Beveridge curve, thereby directly increasing the unemployment rate. This is why variations in δ lead to substantial changes in the level of unemployment. Finally, more rapid price adjustment implies a higher rate of deflation, which makes holding money more attractive relative to firm ownership. Hence, unemployment increases though the magnitude is quantitatively discernible.

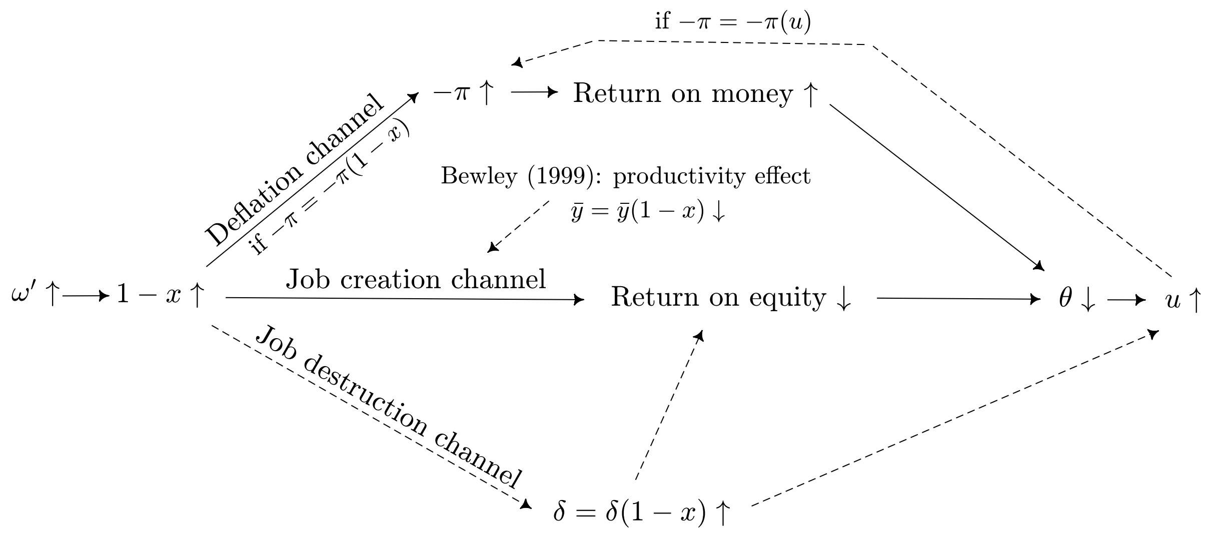
While variations in these parameters can substantially alter the steady state level of unemployment, they barely affect the magnitude of the spillover from demand-driven underemployment into the unemployment rate. This is illustrated in the last row of Table 3, which shows the increase in the unemployment rate under worsening stagnation, specifically under the assumption that the shortfall of working hours increases from 5% to 10%. None of the parameter variations generates a substantial response of unemployment, implying that the lack of demand is primarily reflected in variations in underemployment.

Next, we modify the workings of the transmission channels that determine the spillovers

[35] In fact, only the ratio of these parameters, i.e. effective vacancy cost relative to labor productivity, affects the composition of the steady state employment gap of our model, which is why only one parameter variation is shown in Table 2.

[36] In addition, the matching efficiency κ affects the equilibrium level of unemployment. However, by rewriting the job creation condition and the Beveridge curve in terms of matching efficiency units, it is easy to show that an increase in κ is equivalent to a proportional reduction in δ and α.

Figure 5: Transmission channels



from underemployment into unemployment. These are illustrated by the solid arrows in Figure 5. On the one hand, an increase in underemployment following a stronger desire to save (via the wealth preference) reduces the return on equity as firm profits decline, which reduces job openings resulting in higher unemployment. This is the job creation channel. On the other hand, underemployment worsens deflation which makes holding money more attractive further reducing the incentives for job creation. This is the deflation channel. While we provide a microfoundation for the job creation channel, the deflation channel is based on the reduced-form specification of the Phillips curve in (23).

We conduct four robustness checks to analyse the sensitivity of our results to the specification of the Phillips curve. These are summarized in part A of Table 4, which reports the unemployment rate for a given degree of underemployment under stagnation in comparison to our baseline specification. Our first modification simply assumes a constant rate of deflation under stagnation, effectively shutting down the deflation channel so that unemployment is only affected via the job creation channel.[37] This results in a slightly weaker response of the unemployment rate to worsening stagnation, even though the effect is barely noticeable. In turn, this implies that our baseline results are not driven by the deflation but rather by the job creation channel. We then strengthen the deflation channel by assuming a stronger response of deflation to underemployment with a Phillips curve coefficient of $\alpha = 0.025$, which is a fivefold increase relative to the baseline calibration. While the stronger response of deflation to underemployment results in a stronger spillover into unemployment, the effect is quantitatively small yielding only an additional 0.03 percentage points increase in unemployment despite a 20% shortfall of working hours.[38] Next, we drop our assumption that deflation is driven by underemployment

[37]In Table 4, we assume price stability under stagnation to avoid a discontinuity at $x = 1$. Assuming a positive, but constant rate of deflation results in a higher steady state level of unemployment, but does not yield stronger spillover effects since the deflation channel still remains shut down.

[38]We could consider even larger values of α although this requires some adjustments of other parameters to ensure that the existence conditions in Lemma 2 are met. Our simulations show that even unrealistically large

Table 4: Transmission channels

Shortfall of working hours $1 - x^s$ (targeted)	2%	5%	10%	20%
Unemployment rate with baseline calibration	3.47	3.52	3.62	3.84
A. Deflation channel: $-\pi_t = \alpha_1(1 - x_t) + \alpha_2(u_t - \tilde{u})$				
Constant deflation rate: $\alpha_1 = 0$, $\alpha_2 = 0$	3.47 (-0)	3.52 (-0)	3.62 (-0)	3.83 (-0.01)
Steeper Phillips curve: $\alpha_1 = 0.025$, $\alpha_2 = 0$	3.47 (+0)	3.53 (+0.01)	3.63 (+0.01)	3.87 (+0.03)
Unemployment-based: $\alpha_1 = 0$, $\alpha_2 = 1$	3.47 (+0)	3.53 (+0)	3.63 (+0.01)	3.86 (+0.02)
Based on employment gap: $\alpha_1 = 0.025$, $\alpha_2 = 1$	3.47 (+0)	3.53 (+0.01)	3.64 (+0.02)	3.90 (+0.06)
B. Job creation channel: $\bar{y} = y_0 - y_1(1 - x_t)$				
Weak spillover effects: $y_0 = 1$, $y_1 = 0.1$	3.47 (+0)	3.53 (+0.01)	3.64 (+0.02)	3.88 (+0.04)
Strong spillover effects: $y_0 = 1$, $y_1 = 1$	3.50 (+0.03)	3.61 (+0.09)	3.81 (+0.19)	4.29 (+0.45)
C. Job destruction channel: $\delta = \delta_0 + \delta_1(1 - x_t)$				
Weak spillover effects: $\delta_0 = 0.015$, $\delta_1 = 0.1$	3.92 (+0.46)	4.68 (+1.16)	5.99 (+2.37)	8.81 (+4.97)
Strong spillover effects: $\delta_0 = 0.015$, $\delta_1 = 1$	7.97 (+4.50)	14.66 (+11.14)	25.43 (+21.81)	44.88 (+41.04)

Note: Numbers in brackets show the change in the unemployment rate relative to the baseline case. $\tilde{u}$ is the unemployment rate when $\theta = \tilde{\theta}$ and $x = 1$.

and instead assume a traditional Phillips curve with a (strong) response of deflation to unemployment. This specification creates a feedback loop from unemployment into the attractiveness of holding money and thereby into the job creation incentives of firms resulting in further unemployment as represented by the upper dotted arrow in Figure 5. Despite its appeal, the quantitative contribution of this channel is limited even under the extreme assumption of a one-on-one response of deflation to the unemployment rate. Our final specification combines the two previous ones allowing for a strong response of deflation to both underemployment and unemployment. Naturally, spillovers are strongest under this specification. However, their quantitative importance is still limited, further highlighting the robustness of our results to the specification of the Phillips curve.

When it comes to the job creation channel, our baseline specification assumes a linear production function with constant labor productivity $\bar{y}$ in (11). Yet, the literature suggests

values of α add quantitatively little to the response of unemployment. For instance, a value of $\alpha = 0.1$ only adds another 0.1 percentage points of unemployment for a 20% shortfall of working hours.

that there might be increasing returns to working hours. One factor, put forward among others by Bewley (1999), is the negative effect of shorter hours on workers' morale. If involuntary underemployment reduces workers' motivation and lowers their productivity, firms will be more reluctant to cut working hours resulting in a potentially stronger response of unemployment instead of underemployment by reinforcing the job creation channel. This is illustrated by the dotted arrow in the center of Figure 5. While we do not provide a full microfoundation for this effect, a reduced-form analysis shows that the additional effect of such a mechanism on unemployment under stagnation is relatively weak. Specifically, part B of Table 4 reports the unemployment rate for a given degree of underemployment under the assumption that labor productivity is linearly decreasing with a shortfall of working hours. We consider the cases of a weak response of productivity (0.1 percentage point reduction for each percentage point shortfall of working hours) and a strong response (1 percentage point drop). Yet, even in the latter case, the additional response of unemployment is substantially smaller compared to the increase in underemployment, even for a substantial degree of demand shortage.

The main reason why these modifications of the deflation and job creation channels have not resulted in a significantly stronger response of the unemployment rate is that their effects on unemployment work only indirectly through changes in labor market tightness. Put differently, the degree of stagnation does not affect the steady state Beveridge curve. Instead, the economy moves to a new steady state on the same Beveridge curve as stagnation worsens.

In our framework, the stability of the Beveridge curve is a consequence of the assumption of a constant job separation rate. As our primary motivation is to explain the absence of unemployment during Japan's stagnation period, this assumption is intended to reflect the institutional structure of the Japanese labor market, in particular the system of lifetime employment for core workers and the seniority-based system of remuneration and promotions. Houseman and Osawa (1998) and Ariga and Okazawa (2011) provide empirical support for the relative stability of Japan's job destruction rate over time, particularly in comparison to the U.S. economy.

In economies with more flexible firing practices, job separations are likely to increase under stagnation. This affects the unemployment rate in two ways as illustrated by the dashed arrows in the lower part of Figure 5. A higher job destruction rate δ reduces firm profits, thereby indirectly raising the unemployment rate via fewer job openings. In addition, the higher inflow into unemployment shifts the Beveridge curve directly contributing to higher unemployment. Relying again on a reduced-form specification, part C of Table 4 shows the spillovers effects associated with such a systematic variation in the job separation rate. Starting from the baseline calibration of $\delta = 0.015$ in the absence of stagnation, the two scenarios assume a sensitivity of the separation rate of 0.1 (weak scenario) and 1 (strong scenario) with respect to underemployment implying job separation rates of 0.025 and 0.115 for a 10% shortfall of work hours respectively. Even in the weak scenario, the additional effect on the unemployment rate is substantial, adding 2.4 percentage point additional unemployment at the 10% shortfall level compared to the baseline model.

Hence, while this model offers an explanation of the seemingly paradox case of Japan's labor market during its lost decades, its application to other advanced economies requires more caution. The assumption of an exogenous separation rate seems to be well-justified considering the institutional structure of the Japanese labor market and allows for the isolation of the effects of persistent stagnation on the job creation incentives of firms. Yet, its potential interactions with job destruction through shifts of the Beveridge curve, whose microfoundation is well beyond the scope of the analysis in this paper, are likely to be of importance when applying this framework to economies with a different institutional structure of the labor market.

5.2 Comparative statics: Labor market policies

The following proposition summarizes the steady state effects of two traditional labor market policies, variations in the replacement rate z and the introduction of a search cost subsidy s.

Proposition 2 *Labor market policies have the following effects on labor market tightness θ, the unemployment rate u, working hours x, output Y and consumption c in steady state:*

Stagnation steady state:	θ	u	x	Y	c
Replacement rate z	-	+	+/-	+/-	+/-
Search cost subsidy: s	+/-	+/-	+/-	+/-	+/-
Standard steady state:	θ	u	x	Y	c
Replacement rate z	-	+	0	-	+/-
Search cost subsidy s	+	-	0	+	+/-

Consider first the standard case. A lower replacement rate z increases firm profits as the effective bargaining power of workers ψ falls. Firms enter the labor market and unemployment is reduced. This is the job creation channel. The tighter labor market affects both household wealth due to a higher firm valuation in (15) and net income in (31) creating a novel transmission channel through changes in the wealth premium. The working of the wealth channel can reinforce the job creation channel or work opposite to it depending on the shape of the wealth premium. Yet, it is the job creation channel that dominates in the absence of demand shortage. This can easily be seen graphically in Figure 2 as the lower replacement rate shifts the return on equity upwards without affecting optimal household behavior directly. A search cost subsidy has the additional effect of directly increasing the firm value, thereby stimulating consumption demand.[39] Both policies result in a tighter labor market with a lower unemployment rate. Lower unemployment in turn increases output as is clear from (21), whereas the overall effect on consumption is undetermined due to the hump-shaped patterns of (31).[40]

[39]Graphically, introducing a cost subsidy leads to an upwards shift in both the return on equity and the natural rate in Figure 2, but the shift in the former always dominates the shift in the latter.

[40]Michaillat and Saez (2022) analyze business cycle fluctuations in the slack equilibrium in a model with a wealth preference. This focus seems natural since it implies that an increase in labor market tightness raises total final output ($\bar{y}(1-u)$) by more than total search costs ($kv = k\theta u$). Pissarides (2000, Chapter 8) provides a general discussion of the interactions of labor market tightness and consumption in a matching model.

Under stagnation, the job creation channel no longer necessarily dominates the wealth channel, which is why the effects of labor market policies are ambiguous. In fact, the relative strength of the transmission channels depends on the behavior of the wealth premium, which is related to the curvature of the wealth preference and the shape of the consumption function in (31).

As before, a lower replacement rate z increases firm profits leading to market entry, a tighter labor market and less unemployment for given working hours. These affect the goods market via changes in the wealth premium. On the one hand, consumption demand is stimulated by the higher firm valuation.[41] On the other hand, lower unemployment and a larger number of firms affect the amount of production allocated to consumption, which can increase or decrease depending on the shape of (31) for a given x.

Assume the extreme case of a linear wealth preference, a low degree of demand shortage and a slack steady state. In this case, the increase in labor market tightness results in a higher wealth premium, which increases desired savings and reduces consumption demand. Firms respond to the lower demand by cutting production, which is why working hours drop and underemployment rises. The reduction in working hours reduces desired savings and thereby eliminates the excess supply.[42] It also mitigates the initial rise in labor market tightness. Deflation also increases and so does the real interest rate, which reduces consumption.

Assume instead a concave wealth preference or a high degree of demand shortage or a tight steady state. In this case, the increase in labor market tightness has the opposite effect as it reduces the wealth premium. This stimulates consumption demand and allows firms to increase production and sales, which is why working hours increase and underemployment declines. The increase in working hours reduces excess demand and reinforces the initial rise in labor market tightness as holding money becomes less attractive due to lower deflation. Household consumption rises substantially in response to both a lower real interest rate and higher wealth.

Table 5 shows the effects of an increase in the replacement rate z by 5% as well as the effects of introducing a $s = 10\%$ search cost subsidy for two specifications of the wealth preference. The first column for each policy shows the results for a linear wealth preference, corresponding to the first scenario above, the second column refers to the CARA specification, reflecting the second scenario. We assume 5% underemployment in the steady state before the policy.[43]

All parameter variations in Table 5 have the same qualitative effects on labor market tightness and unemployment as in the standard case. Their effects on unemployment under stagnation are also quantitatively similar to the case without demand shortage. A higher replacement rate reduces labor market tightness and increases structural unemployment under stagnation. As the cost subsidy also affects the firm value in (15) and hence the wealth premium and saving

[41] The strength of this channel depends on the strength and curvature of the wealth preference and the shape of the matching function, which affects the response of wealth to changes in labor market tightness. This channel does not operate in the extreme case of a linear wealth preference and hence a constant marginal utility of wealth.

[42] Strictly speaking, this only holds for a sufficiently small intertemporal elasticity of substitution for consumption η_c^{-1}. Specifically, $\eta_c > \frac{\alpha}{\rho}$ is a sufficient (but not necessary) condition for an increase (decrease) in working hours to result in excess supply (demand) in the goods market for a given θ as shown in Appendix F. This is a weak restriction and we will assume that it holds throughout our analysis.

[43] Tables 5 and 6 quantify the effects of parameter variations for our baseline calibration. The respective tables for the alternative calibrations can be found in the appendix. All effects are shown for the stagnation steady state with $x^s = 0.95$ before the parameter variation.

Table 5: Steady state effects of labor market policies

Change in:	Benefits $z \uparrow$ Linear	Benefits $z \uparrow$ CARA	Subsidy s Linear	Subsidy s CARA
Underemployment $\Delta(1-x^s)$	-0.15	1.43	0.11	1.27
Unemployment Δu^s	0.23	0.27	-0.18	-0.16
Employment gap in (37)	0.07	1.63	-0.06	1.08
Consumption $\Delta c^s/c^s$ (in %)	0.01	-1.70	-0.01	-1.27
Production $\Delta Y^s/Y^s$ (in %)	-0.08	-1.78	0.07	-1.18
Unemployment Δu^f	0.23	0.22	-0.17	-0.18
Consumption $\Delta c^f/c^f$ (in %)	-0.15	-0.15	0.10	0.12
Production $\Delta Y^f/Y^f$ (in %)	-0.24	-0.23	0.18	0.18

Policies: (i) Increase in the replacement rate from $z = 0.6$ to $z = 0.65$. (ii) Introduction of a search cost subsidy of $s = 10\%$.
Rows: (1) Change in underemployment (in pp). (2) Change in unemployment rate (in pp). (3) Change in total employment gap (in pp). (4) Change in private consumption (in %). (5) Change in production (in %). (6) Change in unemployment rate in the absence of demand shortage (in pp). (7) Change in consumption in the absence of demand shortage (in %). (8) Change in production in the absence of demand shortage (in %).

patterns of the household, the combined effect of this policy is unclear and labor market tightness could in principle drop, following lower costs. However, this requires a sufficiently strong wealth channel and does not occur in our simulations. It could, however, result from a sufficiently strong curvature of the wealth utility in combination with substantial underemployment and a low degree of complementarity of vacancies and unemployed in the matching function.

While the effects of these policies on the unemployment rate under stagnation are quantitatively and qualitatively similar to those of the standard model, their implications for underemployment depend critically on the shape of the wealth premium as explained above. In our baseline specification with a constant marginal utility of wealth, the response of underemployment is in stark contrast to the unemployment rate with underemployment declining in response to higher unemployment benefits and increasing following the introduction of a cost subsidy. Hence, unemployment and underemployment move in opposite directions and the response of the total employment gap is indeterminate. In contrast, our alternative specification of a CARA wealth preference results in co-movements of underemployment and unemployment.

In general, we document co-movements of underemployment and unemployment when the degree of stagnation is small, vacancies and unemployed are closer substitutes in the matching process and for a stronger curvature of the utility from wealth. In contrast, underemployment and unemployment move in opposite directions following wage and costs shocks when stagnation is sufficiently severe, for a higher degree of complementarity of vacancies and unemployed and for a lower concavity of the wealth preference.[44] The effects of wage and cost shocks on consumption

[44] Note that the steady state without aggregate demand shortage is slack in our simulation so that x increases following higher wages in case of a constant marginal utility of wealth in Table 5.

and output follow directly from this analysis. In fact, for a constant marginal utility of wealth, consumption and working hours have to change in the same direction.[45] The spillover effects from the labor market into consumption and output are weak in our baseline specification but can be quite substantial in the alternative case of a CARA specification.

To conclude, the effects of traditional labor market policies under stagnation are ambiguous and depend on several factors related to the functional form of the wealth preference and the matching function. Policies that focus exclusively on reducing the unemployment rate might in fact increase the total employment gap via their effect on underemployment and should only be employed with great caution.

5.3 Comparative statics: Demand and supply shocks

The following proposition summarizes the steady state effects of demand and supply shocks in the form of variations in government spending g and labor productivity $\bar{y}$.

Proposition 3 *Exogenous variations in demand and supply have the following effects on labor market tightness θ, the unemployment rate u, working hours x, output Y and consumption c in steady state:*

Stagnation steady state:	θ	u	x	Y	c
Supply shock $\bar{y}$	-	+	-	-	-
Demand shock g	+	-	+	+	+
Standard steady state:	θ	u	x	Y	c
Supply shock $\bar{y}$	+	-	0	+	+/-
Demand shock g	-	+	0	-	+/-

Supply and demand shocks unambiguously result in co-movements of structural unemployment and underemployment under stagnation. These shocks have opposite effects on both labor market variables and macroeconomic variables under stagnation compared to the standard case. Their qualitative effects are independent of the shape of the wealth preference and the matching function. The reversal of the effects compared to the standard model is a direct consequence of the shortage of aggregate demand under stagnation.

Higher labor productivity $\bar{y}$ raises household income and the desire to save in the goods market resulting in lower demand. Firms respond to reduced sales by cutting working hours. In fact, the decline in working hours overcompensates the effect of the increase in productivity so that firm profits drop and firms exit the market. Labor market tightness falls and both unemployment and underemployment increase. This reduces household consumption and total output, thereby overcompensating the effect of the higher productivity. This "paradox of toil" is a common occurrence in stagnation models and in stark contrast to the responses of consumption

[45] Any increase (decrease) in working hours directly lowers (increases) the real interest rate, which requires an increase (a reduction) in the wealth premium. With $\omega'(b) = \beta$, consumption has to increase (decrease).

Table 6: Steady state effects of supply and demand shocks

Change in:	Productivity $\bar{y}$ ↑ Linear	Productivity $\bar{y}$ ↑ CARA	Spending g ↑ Linear	Spending g ↑ CARA
Underemployment $\Delta(1-x^s)$	1.02	1.03	-1.11	-1.26
Unemployment Δu^s	(+)0	(+)0	-0.02	-0.02
Employment gap in (37)	0.99	1.00	-1.09	-1.24
Consumption $\Delta c^s/c^s$ (in %)	-0.09	-0.10	0.09	0.26
Production $\Delta Y^s/Y^s$ (in %)	-0.09	-0.10	1.19	1.36
Unemployment Δu^f	-0.02	-0.02	(+)0	(+)0
Consumption $\Delta c^f/c^f$ (in %)	1.03	1.03	-1.05	-1.05
Production $\Delta Y^f/Y^f$ (in %)	1.02	1.02	(-)0	(-)0

Shocks: (i) 1% increase in productivity (or potential output) $\bar{y}$. (ii) Increase in public spending from $g = 0$ to $g = 0.01$ (i.e. 1% of potential output).
Rows: (1) Change in underemployment (in pp). (2) Change in unemployment rate (in pp). (3) Change in total employment gap (in pp). (4) Change in private consumption (in %). (5) Change in production (in %). (6) Change in unemployment rate in the absence of demand shortage (in pp). (7) Change in consumption in the absence of demand shortage (in %). (8) Change in production in the absence of demand shortage (in %).

and output in the standard model where higher productivity increases firm profits making market entry more attractive to firms.

Higher demand via higher government spending g results in a tighter labor market with both a lower unemployment rate and higher realized working hours. The reason is as follows. Higher public spending creates additional demand to which firms respond by increasing production. Higher firm profits and less deflation make investment in firms more attractive. As more firms enter the market, labor market tightness increases and unemployment declines. This in turn stimulates consumption. Hence, there is a crowding-in effect of government spending.[46] In contrast, the preference for wealth creates a crowding out effect of government spending in the standard model as higher government spending increases the real interest rate, which reduces the number of vacancies as firms leave the labor market and increases unemployment.[47]

Table 6 summarizes the effects of supply and demand shocks, specifically a 1% increase in productivity $\bar{y}$ and an increase in government spending to $g = 0.01$ (i.e. 1% of potential output).

Quantitatively, these macro shocks cause substantial variations in working hours, consumption and output. However, their spillover effects into the unemployment rate are weak as fluctuations in the total employment gap are overwhelmingly driven by changes in underemployment, further highlighting the insufficiency of the unemployment rate to capture the total slack in the labor and goods market of a stagnating economy.

[46] Note that this is not related to the notion of deficit spending. In fact, any increase in government spending is budget neutral as the government runs a balanced budget each period by equation (20).

[47] Note that this is a novel transmission channel for shocks which extends the standard search and matching model, where demand variations have no effect on firm creation in steady state as the real interest rate is determined by the time preference rate ρ.

6 Conclusion

We have presented a model of secular stagnation that distinguishes between structural unemployment and demand-driven underemployment as a manifestation of economic slack in the labor market. In a permanent liquidity trap, the unemployment rate becomes an insufficient and potentially misleading indicator of the extent of the aggregate demand problem leading to inadequate policy conclusions. Under stagnation, the lack of demand can express itself in various forms of underemployment, such as part-time or non-regular employment. These provide better indicators of the stance of the economy and the need for policies in support of demand.

Our findings also suggest that traditional labor market policies aiming to improve the supply side might still succeed in lowering unemployment, but at the same time could contribute to more widespread underemployment in a stagnating economy. Put differently, albeit these policies can succeed in creating new jobs, these are non-regular ones such as part-time jobs. As a consequence, increases in the employment rate are not reflected in an expansion in total output but might in fact be hurtful for both output and consumption. This insight helps to explain the continued sluggishness in the Japanese economy despite the seemingly decent employment record. Examples of such policies include reductions in unemployment benefits, job-creation subsidies to firms or policies aimed at reducing the bargaining power of workers in wage negotiations. These policies should therefore be used only with great caution. In contrast, our model highlights the need for policies that improve aggregate demand as such measures both create new jobs and decrease the degree of underemployment.

References

Andolfatto, David, "Business Cycles and Labor-market Search," *American Economic Review*, 1996, *86* (1), 112–132.

Ariga, Kenn and Ryosuke Okazawa, "Labor Immobility in Japan: Its Causes and Consequences," in K. Hamada, A. Kashyap, and D. E. Weinstein, eds., *Japan's Bubble, Deflation, and Long-term Stagnation*, MIT Press, Cambridge, 2011, chapter 8, pp. 263–308.

Ashenfelter, Orley C., Henry Farber, and Michael R. Ransom, "Labor Market Monopsony," *Journal of Labor Economics*, 2010, *28* (2), 203–210.

Bell, David N. F. and David G. Blanchflower, "Underemployment and the Lack of Wage Pressure in the UK," *National Institute Economic Review*, 2018, *243* (1), R53–R61.

_ **and** _ , "Underemployment in the United States and Europe," *ILR Review*, 2021, *74* (1), 56–94.

Benhabib, Jess and Alberto Bisin, "Skewed Wealth Distributions: Theory and Empirics," *Journal of Economic Literature*, 2018, *56* (4), 1261–1291.

Bewley, Truman F., *Why Wages Don't Fall During a Recession*, Harvard University Press, 1999.

Blanchflower, David G., *Not Working: Where Have All the Good Jobs Gone?*, Princeton University Press, 2019.

Caballero, Ricardo J. and Emmanuel Farhi, "The Safety Trap," *Review of Economic Studies*, 2018, *85* (1), 223–274.

Diamond, Peter A., "Wage Determination and Efficiency in Search Equilibrium," *Review of Economic Studies*, 1982, *49* (2), 217–227.

Eggertsson, Gauti B., Neil R. Mehrotra, and Jacob Robbins, "A Model of Secular Stagnation: Theory and Quantitative Evaluation," *American Economic Journal: Macroeconomics*, 2019, *11* (1), 1–48.

Fagereng, Andreas, Martin Blomhoff Holm, Benjamin Moll, and Gisle Natvik, "Saving Behavior Across the Wealth Distribution: The Importance of Capital Gains," Working Paper, Oslo University and Princeton University 2019.

Fang, Lei and Richard Rogerson, "Policy Analysis in a Matching Model with Intensive and Extensive Margins," *International Economic Review*, 2009, *50* (4), 1153–1168.

Haan, Wouter J. Den, Garey Ramey, and Joel Watson, "Job Destruction and Propagation of Shocks," *American Economic Review*, 2000, *90* (3), 482–498.

Hamaaki, Junya, Masahiro Hori, Saeko Maeda, and Keiko Murata, "Changes in the Japanese Employment System in the Two Lost Decades," *Industrial and Labor Relations Review*, 2012, *65* (4), 810–846.

Hansen, Alvin, "Economic Progress and Declining Population Growth," *American Economic Review*, 1939, *29* (1), 1–15.

Hashimoto, Masanori and John Raisian, "Employment Tenure and Earnings Profiles in Japan and the United States," *American Economic Review*, 1985, *75* (4), 721–735.

Hosios, Arthur J., "On the Efficiency of Matching and Related Models of Search and Unemployment," *Review of Economic Studies*, 1990, *57* (2), 279–298.

Houseman, Susan and Machiko Osawa, "What is the Nature of Part-Time Work in the United States and Japan?," in Colette Fagan and Jacqueline O'Reilly, eds., *Part-Time Prospects: An International Comparison of Part-time Work in Europe, North America and the Pacific Rim*, Routledge, 1998, chapter 12, pp. 232–251.

Illing, Gerhard, Yoshiyasu Ono, and Matthias Schlegl, "Credit Booms, Debt Overhang and Secular Stagnation," *European Economic Review*, 2018, *108*, 78–104.

Japan Institute for Labour Policy and Training, "Trends in Non-regular Employment in Japan and Analysis of Several Related Themes," in "Labor Situation in Japan and Its Analysis: Detailed Exposition 2014/2015," Japan Institute for Labour Policy and Training, 2015, chapter 1, pp. 1–21.

Kambayashi, Ryo and Takao Kato, "The Japanese Employment System after the Bubble Burst: New Evidence," in K. Hamada, A. Kashyap, and D. E. Weinstein, eds., *Japan's Bubble, Deflation and Long-term Stagnation*, MIT Press, Cambridge, 2011, chapter 7, pp. 217–261.

Kang, Hyunju, Jaevin Park, and Hyunduk Suh, "The Rise of Part-time Employment in the Great Recession: Its Causes adn Macroeconomic Effects," *Journal of Macroeconomics*, 2020, *66.*

Kitao, Sagiri, Lars Ljungqvist, and Thomas J. Sargent, "A Life-cycle Model of Trans-Atlantic Employment Experiences," *Review of Economic Dynamics*, 2017, *25*, 320–349.

Kudoh, Noritaka, Hiroaki Miyamoto, and Masaru Sasaki, "Employment and Hours over the Business Cycle in a Model with Search Frictions," *Review of Economic Dynamics*, 2019, *31*, 436–461.

Kumhof, Michael, Romain Rancière, and Pablo Winant, "Inequality, Leverage, and Crises," *American Economic Review*, 2015, *105* (3), 1217–1245.

Landais, Camille, Pascal Michaillat, and Emmanuel Saez, "A Macroeconomic Approach to Optimal Unemployment Insurance: Application," *American Economic Journal: Economic Policy*, 2015, *10* (2), 182–216.

_ , _ , and _ , "A Macroeconomic Approach to Optimal Unemployment Insurance: Theory," *American Economic Journal: Economic Policy*, 2015, *10* (2), 152–181.

Manning, Alan, *Monopsony in Motion*, Princeton University Press, 2003.

Merz, Monika, "Search in the Labor Market and the Real Business Cycle," *Journal of Monetary Economics*, 1995, *36* (2), 269–300.

Mian, Atif, Ludwig Straub, and Amir Sufi, "Indebted Demand," *Quarterly Journal of Economics*, 2021, *136* (4), 2243–2307.

Michaillat, Pascal, "Do Matching Frictions Explain Unemployment? Not in Bad Times," *American Economic Review*, 2012, *102* (4), 1721–1750.

_ and Emmanuel Saez, "Aggregate Demand, Idle Time, and Unemployment," *The Quarterly Journal of Economics*, 2015, *130* (2), 507–569.

_ and _ , "Resolving New Keynesian Anomalies with Wealth in the Utility Function," *The Review of Economics and Statistics*, 2021, *103* (2), 197–215.

_ and _ , "An Economical Business-Cycle Model," *Oxford Economic Papers*, 2022, *74* (2), 382–411.

Michau, Jean-Baptiste, "Secular Stagnation: Theory and Remedies," *Journal of Economic Theory*, 2018, *176*, 552–618.

_ , "The Preference for Net Wealth," 2019. Working Paper, Ecole Polytechnique.

_ , "Helicopter Drops of Money under Secular Stagnation," 2022. Working Paper, Ecole Polytechnique.

_ , Yoshiyasu Ono, and Matthias Schlegl, "Wealth Preference and Rational Bubbles," CESifo Working Paper No. 7148 2018.

_ , _ , and _ , "Wealth Preference and Inequality," 2020. unpublished manuscript.

Miyamoto, Hiroaki, "Cyclical Behavior of Unemployment and Vacancies in Japan," *Japan and the World Economy*, 2011, *23* (3), 214–225.

Mortensen, Dale T., "The Matching Process as a Noncooperative Bargaining Game," in "The Economics of Information and Uncertainty" NBER Chapters, National Bureau of Economic Research, Inc, May 1982, pp. 233–258.

_ and Christopher A. Pissarides, "Job Creation and Job Destruction in the Theory of Unemployment," *Review of Economic Studies*, 1994, *61* (3), 397–415.

OECD, "OECD Employment Outlook," Technical Report, Organization for Economic Cooperation and Development 2006.

Ono, Yoshiyasu, *Money, Interest, and Stagnation - Dynamic Theory and Keynes's Economics*, Oxford: Clarendon Press, 1994.

_ , "A Reinterpretation of Chapter 17 of Keynes's General Theory: Effective Demand Shortage under Dynamic Optimization," *International Economic Review*, 2001, *42* (1), 207–236.

Pissarides, Christopher A., "Short-run Equilibrium Dynamics of Unemployment Vacancies, and Real Wages," *American Economic Review*, 1985, *75* (4), 676–690.

_ , *Equilibrium Unemployment Theory*, 2nd ed., MIT Press, Cambridge, MA, 2000.

Rogerson, Richard, Robert Shimer, and Randall Wright, "Search-theoretic Models of the Labor Market: A Survey," *Journal of Economic Literature*, 2007, *43*, 959–988.

Saez, Emmanuel and Stefanie Stantcheva, "A Simpler Theory of Optimal Capital Taxation," *Journal of Public Economics*, 2018, *162*, 120–142.

Schlegl, Matthias, "Secular Stagnation in an Economy with Land," ISER Discussion Paper No. 1032, Institute of Social and Economic Research, Osaka University 2018.

Schmitt-Grohé, Stephanie and Martín Uribe, "Downward Nominal Wage Rigidity, Currency Pegs, and Involuntary Unemployment," *Journal of Political Economy*, 2016, *124* (5), 1466–1514.

_ **and** _ , "Liquidity Traps and Jobless Recoveries," *American Economic Journal: Macroeconomics*, 2017, *9* (1), 165–204.

Shimer, Robert, "The Cyclical Behavior of Equilibrium Unemployment and Vacancies," *American Economic Review*, 2005, *95* (1), 25–49.

Shimizutani, Satoshi and Izumi Yokoyama, "Japan's Long-term Employment Practice Survived? Developments since the 1990s," *Industrial and Labor Relations Review*, 2009, *62* (3), 313–326.

Zou, Heng-fu, "'The Spirit of Capitalism' and Long-Run Growth," *European Journal of Political Economy*, 1994, *10* (2), 279–293.

Appendix: Robustness checks

We summarize the result for two alternative calibrations in this appendix. In robustness check I, we match the model to the U.S. labor market data, using the same approach as in the main text. For robustness check II, we employ a different specification of the matching function, specifically a general CES function following Den Haan et al. (2000). The associated changes in the calibration for both robustness checks are summarized in Table 7. Otherwise the calibration in Table 1 remains unchanged.

Table 7: Changes in parameter calibration

Robustness check I: Calibration based on U.S. data		
Job separation rate	$\delta = 0.1$	average job duration of 2.5 years
Cost of a vacancy	$k = 0.361$	match $r = 5\%$ (yearly) in (29) for $\bar{\theta} = 1$ and $x = 1$
Scale parameter	$\kappa = 1.487$	match $\bar{u} = 0.063$ and $\bar{\theta} = 1$
Robustness check II: CES matching function		
CES parameter	$\gamma = 1.27$	following Den Haan et al. (2000)
Scale parameter	$\kappa = 0.778$	match $\bar{u} = 0.035$ and $\bar{\theta} = 0.85$

Robustness check I: U.S. labor market data

First, we calibrate the matching process based on U.S. data. Shimer (2005) estimates an average job duration of 2.5 years in the U.S. for the period 1951 to 2003, which is substantially shorter than in the respective value for Japan. We use this estimate to calibrate a quarterly separation rate of $\delta = 0.1$. Based on the average unemployment rate in the United States over the 1980-2018 period of $\bar{u} = 6.3\%$, the Beveridge curve in (28) then allows us to calculate the average job-finding rate $p(\theta)$ implied by the model. The implied value of 1.49 (on a quarterly basis) is close to the ones in Shimer (2005), who estimates the monthly job-finding rate to be equal to 0.45, which implies an average quarterly value of 1.35.

In contrast to Japan, there is no comparable data on labor market tightness available for the United States over an equally long time period. We therefore follow Shimer (2005) and target a steady state value of $\bar{\theta} = 1$, which we use to calibrate the scale parameter $\kappa = 1.487$ in the matching function to be in line with the average job finding rate $p(\theta)$.

Finally, the cost parameter $k = 0.361$ is chosen to yield an annual return on equity of 5% in the absence of aggregate demand shortage using the steady state expression (29) with $x = 1$ and the average labor market tightness value.

Table 8 summarizes the properties of the stagnation steady state for different targeted values of the working hour shortfall, which are generated by variations in the strength of the wealth preference. Tables 9 and 10 illustrate the effects of parameter variations on underemployment and the unemployment in the stagnation steady state.

This calibration results in a higher steady state unemployment rate and slightly stronger

Table 8: Steady state with underemployment

Shortfall of working hours $1 - x^s$ (targeted)	2 %	5 %	10 %	20 %
Unemployment rate under stagnation u^s	6.33	6.43	6.6	6.99
Total employment gap in equation (37)	8.2	11.1	16.0	25.6
Underemployment share	22.8	42.1	58.6	72.7
Difference in unemployment rates $u^s - u^f$	0.07	0.17	0.35	0.77
Shortfall of consumption $c^s/c^f - 1$	-2.1	-5.2	-10.5	-20.9
Shortfall of production $Y^s/Y^f - 1$	-2.1	-5.2	-10.4	-20.7

Rows: (1) Targeted shortfall of working hours below potential, i.e. $1 - x^s$. Generated by adjusting the scale parameter β in the marginal utility of wealth. (2) Unemployment rate (in %). (3) Total employment gap (in %), see equation (37). (4) Share of underemployment in the total employment gap (in %). (5) Difference of unemployment rate relative to standard model without demand shortage (in %). (6) Consumption shortfall relative to the standard model without demand shortage (in %). (7) Consumption shortfall relative to the standard model without demand shortage (in %).

Table 9: Steady state effects of labor market policies

	Benefits $z \uparrow$		Subsidy s	
Change in:	Linear	CARA	Linear	CARA
Underemployment $\Delta(1 - x^s)$	-0.28	0.11	0.2	0.46
Unemployment Δu^s	0.41	0.43	-0.32	-0.31
Employment gap in (37)	0.13	0.51	-0.12	0.14
Consumption $\Delta c^s/c^s$ (in %)	0.02	-0.40	-0.02	-0.31
Production $\Delta Y^s/Y^s$ (in %)	-0.15	-0.57	0.13	-0.16
Unemployment Δu^f	0.41	0.41	-0.31	-0.32
Consumption $\Delta c^f/c^f$ (in %)	-0.28	-0.27	0.20	0.20
Production $\Delta Y^f/Y^f$ (in %)	-0.44	-0.44	0.34	0.34

Policies: (i) Increase in the replacement rate from $z = 0.6$ to $z = 0.65$. (ii) Introduction of a search cost subsidy of $s = 10\%$.
Rows: (1) Change in underemployment (in pp). (2) Change in unemployment rate (in pp). (3) Change in total employment gap (in pp). (4) Change in private consumption (in %). (5) Change in production (in %). (6) Change in unemployment rate in the absence of demand shortage (in pp). (7) Change in consumption in the absence of demand shortage (in %). (8) Change in production in the absence of demand shortage (in %).

spillovers from demand shortage into the labor market as also illustrated in Figure 4. Yet, underemployment continues to be the decisive factor in explaining variations in the total employment gap as well as the shortfall of consumption and production relative to the steady state without demand shortage.

Table 10: Steady state effects of supply and demand shocks

Change in:	Productivity $\bar{y}\uparrow$ Linear	Productivity $\bar{y}\uparrow$ CARA	Spending $g\uparrow$ Linear	Spending $g\uparrow$ CARA
Underemployment $\Delta(1-x^s)$	1.02	1.02	-1.13	-1.17
Unemployment Δu^s	(+)0	(+)0	-0.04	-0.04
Employment gap in (37)	0.95	0.96	-1.10	-1.13
Consumption $\Delta c^s/c^s$ (in %)	-0.09	-0.09	0.10	0.14
Production $\Delta Y^s/Y^s$ (in %)	-0.09	-0.09	1.23	1.28
Unemployment Δu^f	-0.03	-0.03	(+)0	(+)0
Consumption $\Delta c^f/c^f$ (in %)	1.05	1.05	-1.10	-1.10
Production $\Delta Y^f/Y^f$ (in %)	1.04	1.04	(-)0	(-)0

Shocks: (i) 1% increase in productivity (or potential output) $\bar{y}$. (ii) Increase in public spending from $g = 0$ to $g = 0.01$ (i.e. 1% of potential output).
Rows: (1) Change in underemployment (in pp). (2) Change in unemployment rate (in pp). (3) Change in total employment gap (in pp). (4) Change in private consumption (in %). (5) Change in production (in %). (6) Change in unemployment rate in the absence of demand shortage (in pp). (7) Change in consumption in the absence of demand shortage (in %). (8) Change in production in the absence of demand shortage (in %).

In addition, our conclusions on the effects of labor market policies and supply and demand shocks continue to hold. The latter result in co-movements of unemployment and underemployment, while the effects of the former depend on the strength and curvature of the wealth preference as well as the shape of the matching function.

Robustness check II: CES matching function

For the second robustness check, we employ the following general CES matching function

$$F(u,v) = \kappa\left(u^{-\gamma} + v^{-\gamma}\right)^{-1/\gamma},$$

where we use the same calibration as in Den Haan et al. (2000) with an elasticity parameter of $\gamma = 1.27$.[48] Using this specification, we pursue the same approach as in the main text.

The steady state with this alternative matching function is almost identical to the one depicted in Table 2 on therefore omitted here. Tables 11 and 12 illustrate the effects of parameter variations on underemployment and the unemployment in the stagnation steady state.

Compared to Tables 5 and 6 in the main text, there are only minor quantitative and no qualitative changes of the responses of the model variables to various parameter variations emphasizing again that it is predominantly the functional form of the wealth preference that determines the response to "micro" shocks. Demand shortage manifests itself predominantly in underemployment with few spillovers to the unemployment rate.

[48] Note that the elasticity of substitution between vacancies and unemployed in the matching function is given by $(1+\gamma)^{-1}$. Higher values of γ imply that vacancies and unemployed are complements in producing successful matches, while lower values imply a higher degree of substitutability.

Table 11: Steady state effects of labor market policies

	Benefits $z \uparrow$		Subsidy s	
Change in:	Linear	CARA	Linear	CARA
Underemployment $\Delta(1 - x^s)$	-0.21	1.13	0.13	1.38
Unemployment Δu^s	0.27	0.30	-0.19	-0.17
Employment gap in (37)	0.06	1.37	-0.06	1.17
Consumption $\Delta c^s/c^s$ (in %)	0.02	-1.43	-0.01	-1.36
Production $\Delta Y^s/Y^s$ (in %)	-0.06	-1.50	0.06	-1.28
Unemployment Δu^f	0.26	0.25	-0.18	-0.18
Consumption $\Delta c^f/c^f$ (in %)	-0.19	-0.18	0.11	0.11
Production $\Delta Y^f/Y^f$ (in %)	-0.27	-0.26	0.19	0.19

Policies: (i) Increase in the replacement rate from $z = 0.6$ to $z = 0.65$. (ii) Introduction of a search cost subsidy of $s = 10\%$.
Rows: (1) Change in underemployment (in pp). (2) Change in unemployment rate (in pp). (3) Change in total employment gap (in pp). (4) Change in private consumption (in %). (5) Change in production (in %). (6) Change in unemployment rate in the absence of demand shortage (in pp). (7) Change in consumption in the absence of demand shortage (in %). (8) Change in production in the absence of demand shortage (in %).

Table 12: Steady state effects of supply and demand shocks

	Productivity $\bar{y} \uparrow$		Spending $g \uparrow$	
Change in:	Linear	CARA	Linear	CARA
Underemployment $\Delta(1 - x^s)$	1.02	1.03	-1.11	-1.24
Unemployment Δu^s	(+)0	(+)0	-0.02	-0.03
Employment gap in (37)	0.99	1.00	-1.09	-1.22
Consumption $\Delta c^s/c^s$ (in %)	-0.09	-0.10	0.10	0.24
Production $\Delta Y^s/Y^s$ (in %)	-0.09	-0.10	1.19	1.34
Unemployment Δu^f	-0.02	-0.02	(+)0	(+)0
Consumption $\Delta c^f/c^f$ (in %)	1.03	1.03	-1.05	-1.05
Production $\Delta Y^f/Y^f$ (in %)	1.02	1.02	(-)0	(-)0

Shocks: (i) 1% increase in productivity (or potential output) $\bar{y}$. (ii) Increase in public spending from $g = 0$ to $g = 0.01$ (i.e. 1% of potential output).
Rows: (1) Change in underemployment (in pp). (2) Change in unemployment rate (in pp). (3) Change in total employment gap (in pp). (4) Change in private consumption (in %). (5) Change in production (in %). (6) Change in unemployment rate in the absence of demand shortage (in pp). (7) Change in consumption in the absence of demand shortage (in %). (8) Change in production in the absence of demand shortage (in %).

For Online Publication: Mathematical appendix

Appendix A: Derivation of the wage rate (17) under Nash bargaining

Let E_t and U_t denote the value functions of employed and unemployed household members. The respective Bellman equations are given by

$$r_t E_t = w_t x_t - \delta[E_t - U_t] + \dot{E}_t, \tag{A.1}$$

$$r_t U_t = \hat{z}_t + \theta_t q(\theta_t)[E_t - U_t] + \dot{U}_t. \tag{A.2}$$

We assume that the household takes unemployment payments $\hat{z}_t$ as given and does not internalize the policy rule (6) of the government. Firms and workers set wages to maximize their joint surplus given by

$$(E_t - U_t)^{\varepsilon}(J_t - V_t)^{1-\varepsilon}, \tag{A.3}$$

where $\varepsilon \in (0,1)$ denotes the bargaining power of workers. Using the Bellman equations for operating firms (12) and employed workers (A.1) in (A.2) and taking the derivative with respect to the real wage, the optimal sharing rule of the total matching surplus is given by

$$(1-\varepsilon)(E_t - U_t) = \varepsilon(J_t - V_t). \tag{A.4}$$

From the Bellman equations (A.1) and (A.2), we have

$$(r_t + \delta + \theta_t q(\theta_t))\,[E_t - U_t] = w_t x_t - \hat{z}_t + [E_t \dot{-} U_t]. \tag{A.5}$$

Using the optimal sharing rule (A.4), the above equation is rewritten as

$$(r_t + \delta + \theta_t q(\theta_t))\,\frac{\varepsilon}{1-\varepsilon} J_t = w_t x_t - \hat{z}_t + \frac{\varepsilon}{1-\varepsilon}\dot{J}_t. \tag{A.6}$$

Using the Bellman equation (12) and the free entry condition (14), the value of an operating firm is given as

$$\dot{J}_t = (r_t + \delta)J_t - (\bar{y} - w_t)x_t. \tag{A.7}$$

Substituting this equation for $\dot{J}_t$ into (A.6) gives

$$\theta_t q(\theta_t)\frac{\varepsilon}{1-\varepsilon} J_t = w_t x_t - \hat{z}_t - \frac{\varepsilon}{1-\varepsilon}(\bar{y} - w_t)x_t. \tag{A.8}$$

Using expression (15) for the value of an operating firm J_t and the policy rule (6) for $\hat{z}_t$, this equation gives the wage rate under Nash bargaining as

$$w_t x_t = \frac{\varepsilon}{1-(1-\varepsilon)z}\left[\bar{y}x_t + (1-s)k\theta_t\right]. \tag{A.9}$$

Define $\psi \equiv \frac{\varepsilon}{1-(1-\varepsilon)z}$ as the effective bargaining power of workers and (A.9) becomes equation (17) for the negotiated wage in the main text.

Appendix B: Derivation of the goods market clearing condition (22)

Substituting the dynamics of the money supply (27) and the government budget constraint (20) into the flow of funds constraint of the household (2) gives

$$\dot{b}_t = r_t b_t + w_t x_t(1-u_t) - c_t - skv_t - g. \tag{B.1}$$

Differentiating $b_t = J_t(1-u_t)$ from the asset market clearing condition (25) with respect to time, we also have

$$\dot{b}_t = \dot{J}_t(1-u_t) - J_t\dot{u}_t. \tag{B.2}$$

Substituting (B.2) and $b_t = J_t(1-u_t)$ into equation (B.1) gives

$$\dot{J}_t(1-u_t) - J_t\dot{u}_t = r_tJ_t(1-u_t) + w_tx_t(1-u_t) - c_t - g - sk_tv_t. \tag{B.3}$$

Using the dynamic equations (10) and (A.7) to substitute for $\dot{u}_t$ and $\dot{J}_t$ in (B.3), we get

$$(1-u_t)\bar{y}x_t = J_tp(\theta_t)u_t + c_t + g + sk_tv_t. \tag{B.4}$$

Finally, using expression (15) for the value of an operating firm J_t and the identities $v_t \equiv \theta_tu_t$ and $p(\theta_t) \equiv \theta_tq(\theta_t)$, equation (B.4) gives the goods market clearing condition as equation (22) in the main text.

Appendix C: Proof of Lemma 1

For notational brevity, from (15) and (17), define the dividend yield $(\bar{y}-w_t)x_t/J_t$ as

$$\vartheta(\theta_t, x_t) \equiv \frac{(\bar{y}-w_t)x_t}{J_t} = \frac{(1-\psi)\bar{y}x_t}{(1-s)k}q(\theta_t) - \psi p(\theta_t). \tag{C.1}$$

Let $\vartheta(\theta_t, 1)$ denote the dividend yield when there is no underemployment and $\vartheta_\theta(\theta_t, 1)$ its partial derivative with respect to θ_t. As $\bar{y} > z$, we have $\vartheta(0,1) > 0$, $\vartheta(\infty, 1) < 0$ and $\vartheta_\theta(\theta_t, 1) < 0$.

Derivation of the dynamic system when $x_t = 1$:

Without aggregate demand shortage, we have $x_t = 1$ and $\dot{x}_t = 0$. The goods market clearing condition (31) then defines consumption at any time t as a function of the unemployment rate u_t and the labor market tightness θ_t as

$$c_t(\theta_t, u_t) = (1-u_t)\bar{y} - k\theta_tu_t - g, \tag{C.2}$$

where the partial derivatives are given by $c_u = -(\bar{y}+k\theta) < 0$ and $c_\theta = -ku_t < 0$. Differentiating this function with respect to t implies

$$\dot{c}_t = -\dot{u}_t(\bar{y}+k\theta_t) - ku_t\dot{\theta}_t. \tag{C.3}$$

Using (3), (10) and (18) to substitute for $\dot{c}_t$, $\dot{u}_t$ and $\dot{\theta}_t$, we derive the following expression for the real interest rate r_t at any time t:

$$\frac{c_t}{\eta_c}\left[r_t - \rho + \frac{\omega'(b_t)}{\phi'(c_t)}\right] = -\frac{ku_t\theta_t}{\eta_{\theta_t}}\left[r_t - \vartheta(\theta_t, 1) + \delta\right] - (\bar{y}+k\theta_t)[\delta(1-u_t) - p(\theta_t)u_t],$$

$$r_t(\theta_t, u_t) = \varphi_t\left[\rho - \frac{\omega'(b_t)}{\phi'(c_t)}\right] + (1-\varphi_t)\left[\vartheta(\theta_t, 1) - \delta\right] - \tilde{\varphi}_t\left[\delta(1-u_t) - p(\theta_t)u_t\right], \tag{C.4}$$

where

$$\varphi_t = \frac{c_t/\eta_c}{c_t/\eta_c + ku_t\theta_t/\eta_{\theta_t}} \quad , \quad \tilde{\varphi}_t = \frac{\bar{y}+k\theta_t}{c_t/\eta_c + ku_t\theta_t/\eta_{\theta_t}}.$$

From (C.2), c_t is a function of θ_t and u_t. Hence, the real rate r_t is also a function of θ_t and u_t as b_t depends on the same variables by (A.4). Using these expressions, we can rewrite the model as a system of three differential equations in m_t, u_t and θ_t:

$$\dot{m}_t = \left[g_m - \frac{\mu'(m_t)}{\phi'(c_t(\theta_t, u_t))} + r_t(\theta_t, u_t) \right] m_t, \tag{C.5}$$

$$\dot{u}_t = (1 - u_t)\delta - p(\theta_t)u_t, \tag{C.6}$$

$$\dot{\theta}_t = \frac{\theta_t}{\eta_{\theta_t}} [r_t(\theta_t, u_t) - \vartheta(\theta_t, 1) + \delta]. \tag{C.7}$$

Existence:

Reformulate the first equation of (33) to define the function $H(\theta)$ as

$$H(\theta) = \rho + \delta + \psi p(\theta) - \frac{(1-\psi)\bar{y}}{(1-s)k} q(\theta) - \frac{\omega'\left((1-u(\theta))\frac{(1-s)k}{q(\theta)}\right)}{\phi'\left((1-u(\theta))\bar{y} - k\theta u(\theta) - g\right)}. \tag{C.8}$$

The function $c(\theta, 1) = (1 - u(\theta))\bar{y} - k\theta u(\theta) - g$ in (31) with $x = 1$ defines consumption as a continuous function of θ with a hump-shaped pattern that attains a maximum at a strictly positive value.[49] Define $\underline{\theta}$ and $\bar{\theta}$ by $c(\underline{\theta}, 1) = c(\bar{\theta}, 1) = 0$ with $\underline{\theta} < \bar{\theta}$. Note that $\underline{\theta} = 0$ for $g = 0$. Then $c(\theta, 1) \geq 0$ and $H(\theta)$ is a continuous function of θ for $\theta \in (\underline{\theta}, \bar{\theta})$. As $\omega'(.) < \infty$, we have

$$H(\underline{\theta}) = \rho + \delta + \psi p(\underline{\theta}) - \frac{(1-\psi)\bar{y}}{(1-s)k} q(\underline{\theta}) = \rho + \delta - \vartheta(\underline{\theta}, 1),$$

$$H(\bar{\theta}) = \rho + \delta + \psi p(\bar{\theta}) - \frac{(1-\psi)\bar{y}}{(1-s)k} q(\bar{\theta}) = \rho + \delta - \vartheta(\bar{\theta}, 1).$$

Note that $H(\underline{\theta}) < H(\bar{\theta})$ since $\vartheta_\theta(\theta, 1) < 0$. Without any other restrictions, a steady state with $H(\theta^f) = 0$ always exists for $\theta^f \in (\underline{\theta}, \bar{\theta})$ if $H(\underline{\theta}) < 0 < H(\bar{\theta})$ or equivalently if

$$\vartheta(\underline{\theta}, 1) - \delta > \rho > \vartheta(\bar{\theta}, 1) - \delta. \tag{C.9}$$

Money market equilibrium requires a positive real interest rate in steady state. Define $\tilde{\theta}$ by $\vartheta(\tilde{\theta}, 1) = \delta$ as in (30), which implies a zero net return on equity and hence a real interest rate of zero. Since $\vartheta_\theta(\theta, 1) < 0$, equilibrium requires $\theta^f \leq \tilde{\theta}$. The steady state without aggregate demand shortage exists in $(\underline{\theta}, \tilde{\theta})$ if the following condition holds:[50]

$$H(\tilde{\theta}) \geq H(\theta^f) = 0 \quad \leftrightarrow \quad \rho\phi'\left(c(\tilde{\theta}, 1)\right) \geq \omega'\left(b(\tilde{\theta})\right). \tag{C.10}$$

Conditions (C.9) and (C.10) are the necessary existence conditions of Lemma 1. Uniqueness of the steady state requires $H'(\theta) > 0$ for all θ with $H(\theta) = 0$. This derivative is given by:

$$H'(\theta) = \frac{\omega''(b)}{\phi'(c)} \left[-\frac{\phi'(c)}{\omega''(b)} \vartheta_\theta(\theta, 1) - b'(\theta) + \frac{\omega'(b)}{\omega''(b)} \frac{\phi''(c)}{\phi'(c)} \frac{dc(\theta, 1)}{d\theta} \right]. \tag{C.11}$$

We assume that $H'(\theta^f) > 0$ so that the steady state with $x = 1$ is unique.

[49] We ensure this by choosing the parameters g and k sufficiently low relative to $\bar{y}$.

[50] We assume a constant nominal money supply and hence zero inflation, i.e. $g_m = 0$. For $g_m > 0$, $\tilde{\theta}$ becomes a function of g_m as the real interest rate cannot fall below $-g_m$.

Stability:

The dynamic system is given by (C.5), (C.6) and (C.7). The unemployment rate u_t is a predetermined state variables, while labor market tightness θ_t (via vacancies v_t) and the money supply (for $x = 1$) are control variables and can jump. Stability of the dynamic system therefore requires one negative and two positive eigenvalues. The Jacobian of this system evaluated at the steady state with $x = 1$ is given by

$$\begin{bmatrix} -\frac{\mu''(m)}{\phi'(c)}m & \left[\frac{\mu'(m)\phi''(c)c_u}{\phi'(c)^2} + r_u\right]m & \left[\frac{\mu'(m)\phi''(c)c_\theta}{\phi'(c)^2} + r_\theta\right]m \\ 0 & -[\delta + p(\theta)] & -p'(\theta)u \\ 0 & \frac{\theta}{\eta_\theta}r_u & \frac{\theta}{\eta_\theta}[r_\theta - \vartheta_\theta(\theta,1)] \end{bmatrix}, \tag{C.12}$$

where r_χ denotes the partial derivative of the real rate defined in (C.2) with respect to variable χ. The eigenvalues of this system solve the following characteristic equation:

$$\Omega(\lambda) = \left[-\frac{\mu''(m)}{\phi'(c)}m - \lambda\right] \cdot \begin{vmatrix} -[\delta + p(\theta)] - \lambda & -p'(\theta)u \\ \frac{\theta}{\eta_\theta}r_u & \frac{\theta}{\eta_\theta}[r_\theta - \vartheta_\theta(\theta,1)] - \lambda \end{vmatrix} = 0. \tag{C.13}$$

It is clear that $\lambda_1 = -\mu''(m)m/\phi'(c) > 0$ is one solution. The other two eigenvalues solve

$$\lambda^2 - \left(\frac{\theta}{\eta_\theta}[r_\theta - \vartheta_\theta(\theta,1)] - \delta - p(\theta)\right)\lambda + \frac{\theta}{\eta_\theta}r_u p'(\theta)u - \frac{\theta}{\eta_\theta}[\delta + p(\theta)](r_\theta - \vartheta_\theta(\theta,1)) = 0. \tag{C.14}$$

We can recover the sign of the eigenvalues from

$$\lambda_2\lambda_3 = \frac{\theta}{\eta_\theta}\left[r_u p'(\theta)u - [\delta + p(\theta)](r_\theta - \vartheta_\theta)(\theta,1)\right]. \tag{C.15}$$

From (C.4), we get the following expression for the partial derivative of r_t with respect to u_t and θ_t in steady state:

$$r_u = -\varphi\left[\frac{\omega''(b)\phi'(c)b_u - \omega'(b)\phi''(c)c_u}{\phi'(c)^2}\right] + \tilde{\varphi}[\delta + p(\theta)],$$

$$r_\theta = -\varphi\left[\frac{\omega''(b)\phi'(c)b_\theta - \omega'(b)\phi''(c)c_\theta}{\phi'(c)^2}\right] + (1-\varphi)\vartheta_\theta(\theta,1) + \tilde{\varphi}p'(\theta)u.$$

Also note that the unemployment rate in steady state is a function of labor market tightness from equation (28) with $du/d\theta \equiv u'(\theta) = -p'(\theta)u(\theta)/(\delta + p(\theta))$. Using this expression, we can relate the partial derivatives c_u and c_θ to the total derivative $dc(\theta,1)/d\theta = c'(\theta,1)$ in steady state as follows (and equivalently for $b(\theta)$):[51]

$$\frac{dc(\theta,u)}{d\theta} \equiv \frac{dc(\theta,1)}{d\theta} = c_\theta + c_u u'(\theta) = c_\theta - c_u\frac{p'(\theta)u(\theta)}{\delta + p(\theta)},$$

$$\frac{db(\theta,u)}{d\theta} \equiv b'(\theta) = b_\theta + b_u u'(\theta) = b_\theta - b_u\frac{p'(\theta)u(\theta)}{\delta + p(\theta)}.$$

Using these properties, we rewrite the above equation as

$$\lambda_2\lambda_3 = \frac{\theta}{\eta_\theta}\left[-\varphi\left[\frac{\omega''(b)\phi'(c)b_u - \omega'(b)\phi''(c)c_u}{\phi'(c)^2}\right]p'(\theta)u + \tilde{\varphi}[\delta + p(\theta)]p'(\theta)u\right.$$

[51]Note that $d\vartheta(\theta,1)/d\theta \equiv \vartheta_\theta(\theta,1) = \vartheta_\theta$ in the absence of aggregate demand shortage.

$$-[\delta+p(\theta)]\left(-\varphi\left[\frac{\omega''(b)\phi'(c)b_\theta-\omega'(b)\phi''(c)c_\theta}{\phi'(c)^2}\right]+(1-\varphi)\vartheta_\theta(\theta,1)+\tilde{\varphi}p'(\theta)u-\vartheta_\theta\right)\bigg],$$

$$=\frac{\theta}{\eta_\theta}\left[-\varphi\left[\frac{\omega''(b)\phi'(c)b_u-\omega'(b)\phi''(c)c_u}{\phi'(c)^2}\right]p'(\theta)u\right.$$

$$\left.+[\delta+p(\theta)]\left(\varphi\left[\frac{\omega''(b)\phi'(c)b_\theta-\omega'(b)\phi''(c)c_\theta}{\phi'(c)^2}\right]+\varphi\vartheta_\theta(\theta,1)\right)\right],$$

$$=\frac{\theta}{\eta_\theta}\varphi[\delta+p(\theta)]\left[\frac{\omega''(b)}{\phi'(c)}\left(b_\theta-b_u\frac{p'(\theta)u}{\delta+p(\theta)}\right)-\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\left(c_\theta-c_u\frac{p'(\theta)u}{\delta+p(\theta)}\right)+\vartheta_\theta(\theta,1)\right],$$

$$=\frac{\theta}{\eta_\theta}\varphi[\delta+p(\theta)]\left[\frac{\omega''(b)}{\phi'(c)}b'(\theta)-\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\frac{dc(\theta,1)}{d\theta}+\vartheta_\theta(\theta,1)\right],$$

$$=-\frac{\theta}{\eta_\theta}\varphi[\delta+p(\theta)]\frac{\omega''(b)}{\phi'(c)}\left[-\frac{\phi'(c)}{\omega''(b)}\vartheta_\theta(\theta,1)-b'(\theta)+\frac{\omega'(b)}{\phi'(c)}\frac{\phi''(c)}{\omega''(b)}\frac{dc(\theta,1)}{d\theta}\right].$$

The term in the brackets of the last expression above is identical to expression (C.11) above. Hence, we can rewrite this equation as

$$\lambda_2\lambda_3=-\frac{\theta}{\eta_\theta}\varphi[\delta+p(\theta)]H'(\theta)<0. \tag{C.16}$$

If the steady state is unique, it holds that $H'(\theta)>0$. It then follows that $\lambda_2\lambda_3<0$, which implies one negative and one positive eigenvalue. Together with $\lambda_1>0$, the dynamic system has one negative and two positive eigenvalues and therefore exhibits saddle-path stability around the steady state without demand shortage.

Appendix D: Proof of Lemma 2

Existence:

Reformulate the second equation of (33) to define the function $G(\theta)$ as

$$G(\theta)=\rho-\frac{\omega'\left((1-u(\theta))\frac{(1-s)k}{q(\theta)}\right)}{\phi'\left(x(\theta)(1-u(\theta))\bar{y}-k\theta u(\theta)-g\right)}-\alpha(1-x(\theta)), \tag{D.1}$$

where $x(\theta)$ is the increasing part of the asset market equilibrium curve given by (36). A well-defined steady state requires non-negative consumption. Define $\underline{\theta}^s$ as the smallest value of θ such that $c(\underline{\theta}^s,x(\underline{\theta}^s))=0$, where $c(\theta,x(\theta))\equiv x(\theta)(1-u(\theta))\bar{y}-k\theta u(\theta)-g$. Note that $\underline{\theta}^s=\underline{\theta}=0$ for $g=0$, where $\underline{\theta}$ is defined in the proof of Lemma 1.[52] In addition, let $\tilde{\theta}$ by defined as before by $\vartheta(\tilde{\theta},1)=\delta$ in (30), which implies $x(\tilde{\theta})=1$ in (36). We then have

$$G(\underline{\theta}^s)=\rho-\alpha(1-x(\underline{\theta}^s))=\rho-\alpha\frac{\vartheta(\underline{\theta}^s,1)-\delta}{\alpha+\frac{(1-\psi)\bar{y}}{(1-s)k}q(\underline{\theta}^s)}>0,$$

$$G(\tilde{\theta})=\rho-\frac{\omega'\left((1-u(\tilde{\theta}))\frac{(1-s)k}{q(\tilde{\theta})}\right)}{\phi'\left((1-u(\tilde{\theta}))\bar{y}-k\tilde{\theta}u(\tilde{\theta})-g\right)}=\rho-\frac{\omega'\left(b(\tilde{\theta})\right)}{\phi'\left(c(\tilde{\theta},1)\right)}.$$

[52] We ensure non-negativity of consumption in steady state by choosing the parameters g and k sufficiently low relative to $\bar{y}$.

Note that $G(\underline{\theta}^s) > 0$ if $\rho > \alpha$ since the ratio $\frac{\vartheta(\underline{\theta}^s,1)-\delta}{\alpha+\frac{(1-\psi)\bar{y}}{(1-s)k}q(\underline{\theta}^s)}$ is always strictly smaller than unity. Existence of the secular stagnation steady state then requires $G(\tilde{\theta}) < 0$ or equivalently

$$\rho\phi'\left(c(\tilde{\theta},1)\right) < \omega'\left(b(\tilde{\theta})\right). \tag{D.2}$$

Whenever full employment is not feasible, we have $G(\tilde{\theta}) = H(\tilde{\theta}) < 0$ by Lemma 1. It then follows that there exists θ^s such that $G(\theta^s) = 0$.

Uniqueness of the steady state requires $G'(\theta) < 0$ for all θ with $G(\theta) = 0$. This derivative is given by

$$G'(\theta) = \frac{\omega''(b)}{\phi'(c)}\left[-b'(\theta) + \frac{\omega'(b)}{\omega''(b)}\frac{\phi''(c)}{\phi'(c)}\frac{dc(\theta,x(\theta))}{d\theta} + \frac{\phi'(c)}{\omega''(b)}\alpha\frac{dx(\theta)}{d\theta}\right]. \tag{D.3}$$

Throughout this paper, we assume that $G'(\theta^s) < 0$ so that the steady state is unique. Uniqueness in turn implies that the secular stagnation steady state is slack. Reformulating $G'(\theta) < 0$ implies

$$\frac{\omega'(b)}{\omega''(b)}\frac{\phi''(c)}{\phi'(c)}\frac{dc(\theta,x(\theta))}{d\theta} > b'(\theta) - \alpha\frac{\phi'(c)}{\omega''(b)}\frac{dx(\theta)}{d\theta} > 0, \tag{D.4}$$

since $\frac{dx(\theta)}{d\theta} > 0$ in steady state, which can be seen in (36). It follows that $\frac{dc(\theta,x(\theta))}{d\theta} > 0$.

Stability:

In the stagnation steady state, the liquidity preference of the household is satiated such that $R_t = 0$ and hence $r_t = R_t - \pi_t = \alpha(1-x_t)$. The goods market clearing condition (31) defines realized working hours x_t at any time t as a function of c_t, θ_t and u_t as

$$x_t = \frac{c_t + k\theta_t u_t + g}{(1-u_t)\bar{y}}, \tag{D.5}$$

where the partial derivatives satisfy $x_c > 0$, $x_\theta > 0$ and $x_u > 0$. Define $\tilde{m}_t \equiv 1/m_t$ with $\tilde{m} = 0$ in steady state. The dynamic system is given by the following differential equations for $\tilde{m}_t$, c_t, θ_t and u_t:

$$\dot{\tilde{m}}_t = \alpha\left(x_t - 1\right)\tilde{m}_t, \tag{D.6}$$

$$\dot{c}_t = \left[\alpha(1-x_t) - \rho + \frac{\omega'(b(\theta_t,u_t))}{\phi'(c_t)}\right]\frac{c_t}{\eta_c}, \tag{D.7}$$

$$\dot{\theta}_t = \left[\alpha(1-x_t) - \vartheta(\theta_t,x_t) + \delta\right]\frac{\theta_t}{\eta_{\theta_t}}, \tag{D.8}$$

$$\dot{u}_t = \delta(1-u_t) - p(\theta_t)u_t, \tag{D.9}$$

where $b(\theta_t,u_t)$ and $\vartheta(\theta_t,x_t)$ are given by (25) and (C.1) respectively. Let ϑ_θ and ϑ_x denote the partial derivatives of $\vartheta(\theta_t,x_t)$ with respect to θ_t and x_t. Since the price level cannot adjust freely, the money supply grows with the rate of deflation and is hence a predetermined variable. In addition, the unemployment rate is a state variable whereas consumption and labor market tightness are jump variables. Stability of the system therefore requires two positive and two negative eigenvalues. The Jacobian of this system evaluated at the secular stagnation steady state is as follows:

$$\begin{bmatrix} \alpha(x-1) & \alpha x_c\tilde{m} & \alpha x_\theta\tilde{m} & \alpha x_u\tilde{m} \\ 0 & \frac{c}{\eta_c}\left[-\alpha x_c - \frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\right] & \frac{c}{\eta_c}\left[-\alpha x_\theta + \frac{\omega''(b)b_\theta}{\phi'(c)}\right] & \frac{c}{\eta_c}\left[-\alpha x_u + \frac{\omega''(b)b_u}{\phi'(c)}\right] \\ 0 & \frac{\theta}{\eta_\theta}\left[-\alpha x_c - \vartheta_x x_c\right] & \frac{\theta}{\eta_\theta}\left[-\alpha x_\theta - \vartheta_x x_\theta - \vartheta_\theta\right] & \frac{\theta}{\eta_\theta}\left[-\alpha x_u - \vartheta_x x_u\right] \\ 0 & 0 & -p'(\theta)u & -[\delta + p(\theta)] \end{bmatrix}. \tag{D.10}$$

The eigenvalues of this system solve the following characteristic equation:

$$\Omega(\lambda) = [\alpha(x-1) - \lambda] \cdot$$

$$\begin{vmatrix} \frac{c}{\eta_c}\left[-\alpha x_c - \frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\right] - \lambda & \frac{c}{\eta_c}\left[-\alpha x_\theta + \frac{\omega''(b)b_\theta}{\phi'(c)}\right] & \frac{c}{\eta_c}\left[-\alpha x_u + \frac{\omega''(b)b_u}{\phi'(c)}\right] \\ \frac{\theta}{\eta_\theta}\left[-\alpha x_c - \vartheta_x x_c\right] & \frac{\theta}{\eta_\theta}\left[-\alpha x_\theta - \vartheta_x x_\theta - \vartheta_\theta\right] - \lambda & \frac{\theta}{\eta_\theta}\left[-\alpha x_u - \vartheta_x x_u\right] \\ 0 & -p'(\theta)u & -[\delta + p(\theta)] - \lambda \end{vmatrix} = 0. \quad \text{(D.11)}$$

It is clear that $\lambda_1 = \alpha(x-1) < 0$ is one solution. The other eigenvalues solve

$$\tilde{\Omega}(\lambda) = p'(\theta)u \cdot \begin{vmatrix} -\frac{c}{\eta_c}\left[\alpha x_c + \frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\right] - \lambda & -\frac{c}{\eta_c}\left[\alpha x_u - \frac{\omega''(b)b_u}{\phi'(c)}\right] \\ -\frac{\theta}{\eta_\theta}\left[\alpha + \vartheta_x\right] x_c & -\frac{\theta}{\eta_\theta}\left[\alpha + \vartheta_x\right] x_u \end{vmatrix}$$

$$-[\delta + p(\theta) + \lambda] \cdot \begin{vmatrix} -\frac{c}{\eta_c}\left[\alpha x_c + \frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\right] - \lambda & -\frac{c}{\eta_c}\left[\alpha x_\theta - \frac{\omega''(b)b_\theta}{\phi'(c)}\right] \\ -\frac{\theta}{\eta_\theta}\left[\alpha + \vartheta_x\right] x_c & -\frac{\theta}{\eta_\theta}\left[\alpha x_\theta + \vartheta_x x_\theta + \vartheta_\theta\right] - \lambda \end{vmatrix} = 0. \quad \text{(D.12)}$$

We rewrite this expression as

$$\tilde{\Omega}(\lambda) = -\lambda^3 + A_2\lambda^2 - A_1\lambda + A_0 = 0, \quad \text{(D.13)}$$

where

$$A_2 = -\frac{c}{\eta_c}\left(\alpha x_c + \frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\right) - \frac{\theta}{\eta_\theta}\left(\alpha x_\theta + \vartheta_x x_\theta + \vartheta_\theta\right) - \delta - p(\theta),$$

$$A_1 = -p'(\theta)u\frac{\theta}{\eta_\theta}\left(\alpha + \vartheta_x\right)x_u + [\delta + p(\theta)]\left[\frac{c}{\eta_c}\left(\alpha x_c + \frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\right) + \frac{\theta}{\eta_\theta}\left(\alpha x_\theta + \vartheta_x x_\theta + \vartheta_\theta\right)\right]$$

$$+\frac{c}{\eta_c}\frac{\theta}{\eta_\theta}\left(\alpha x_c + \frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\right)\left(\alpha x_\theta + \vartheta_x x_\theta + \vartheta_\theta\right) - \frac{c}{\eta_c}\frac{\theta}{\eta_\theta}\left(\alpha + \vartheta_x\right)x_c\left(\alpha x_\theta - \frac{\omega''(b)b_\theta}{\phi'(c)}\right),$$

$$A_0 = p'(\theta)u\frac{c}{\eta_c}\frac{\theta}{\eta_\theta}\left(\alpha + \vartheta_x\right)\left[\left(\alpha x_c + \frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\right)x_u - \left(\alpha x_u - \frac{\omega''(b)b_u}{\phi'(c)}\right)x_c\right]$$

$$-[\delta + p(\theta)]\frac{c}{\eta_c}\frac{\theta}{\eta_\theta}\left[\left(\alpha x_c + \frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\right)\left(\alpha x_\theta + \vartheta_x x_\theta + \vartheta_\theta\right) - \left(\alpha + \vartheta_x\right)x_c\left(\alpha x_\theta - \frac{\omega''(b)b_\theta}{\phi'(c)}\right)\right].$$

Noting that $u'(\theta) = -p'(\theta)u/[\delta + p(\theta)]$ and $b'(\theta) = b_\theta + b_u u'(\theta)$, we can modify A_0 as follows:

$$\begin{aligned} A_0 &= [\delta + p(\theta)]\frac{c}{\eta_c}\frac{\theta}{\eta_\theta}\left[\frac{p'(\theta)u}{\delta + p(\theta)}\left(\alpha + \vartheta_x\right)\left(\alpha x_c x_u + \frac{\omega'(b)\phi''(c)}{\phi'(c)^2}x_u - \alpha x_u x_c + \frac{\omega''(b)b_u}{\phi'(c)}x_c\right)\right. \\ &\quad \left. - \left(\alpha x_c + \frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\right)\left(\alpha x_\theta + \vartheta_x x_\theta + \vartheta_\theta\right) + \left(\alpha + \vartheta_x\right)x_c\left(\alpha x_\theta - \frac{\omega''(b)b_\theta}{\phi'(c)}\right)\right] \\ &= [\delta + p(\theta)]\frac{c}{\eta_c}\frac{\theta}{\eta_\theta}\left(\alpha + \vartheta_x\right)\left[-\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}x_u u'(\theta) - \frac{\omega''(b)b_u}{\phi'(c)}x_c u'(\theta) - \frac{\omega''(b)b_\theta}{\phi'(c)}x_c + x_c\alpha x_\theta\right. \\ &\quad \left. -\alpha x_c\left(x_\theta + \frac{\vartheta_\theta}{\alpha + \vartheta_x}\right) - \frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\left(x_\theta + \frac{\vartheta_\theta}{\alpha + \vartheta_x}\right)\right] \\ &= [\delta + p(\theta)]\frac{c}{\eta_c}\frac{\theta}{\eta_\theta}\left(\alpha + \vartheta_x\right)x_c\left[-\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\left(\frac{x_u}{x_c}u'(\theta) + \frac{x_\theta}{x_c} + \frac{\vartheta_\theta}{\left(\alpha + \vartheta_x\right)x_c}\right)\right. \\ &\quad \left. -\frac{\omega''(b)}{\phi'(c)}b'(\theta) - \frac{\alpha\vartheta_\theta}{\alpha + \vartheta_x}\right]. \end{aligned}$$

Using the partial derivatives of x defined in (D.5), we can rewrite the term in brackets of the last equation above as follows:

$$\frac{x_u}{x_c}u'(\theta)+\frac{x_\theta}{x_c}+\frac{\vartheta_\theta}{(\alpha+\vartheta_x)\,x_c}=(\bar{y}x+k\theta)u'(\theta)+ku-(1-u)\bar{y}\frac{-\vartheta_\theta}{(\alpha+\vartheta_x)}$$

$$=(\bar{y}x+k\theta)u'(\theta)+ku-(1-u)\bar{y}\frac{dx(\theta)}{d\theta}\equiv\frac{dc(\theta,x(\theta))}{d\theta},$$

where $\frac{dx(\theta)}{d\theta}=-\frac{\vartheta_\theta}{(\alpha+\vartheta_x)}$ follows from the No-Arbitrage condition (36) and $c(\theta,x(\theta))$ is defined in (31). It then follows that A_0 can be rewritten as

$$A_0=[\delta+p(\theta)]\frac{c}{\eta_c}\frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)x_c\frac{\omega''(b)}{\phi'(c)}\left[-b'(\theta)+\frac{\omega'(b)}{\omega''(b)}\frac{\phi''(c)}{\phi'(c)}\frac{dc(\theta,x(\theta))}{d\theta}-\frac{\phi'(c)}{\omega''(b)}\frac{\alpha\vartheta_\theta}{\alpha+\vartheta_x}\right].$$

Using the expressions for ϑ_x, ϑ_θ and $\frac{dx(\theta)}{d\theta}$, it is easy to see that the last part of this term is identical to expression (D.3). Hence, we can rewrite this equation as

$$A_0=[\delta+p(\theta)]\frac{c}{\eta_c}\frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)x_cG'(\theta). \tag{D.14}$$

If the secular stagnation steady state is unique, it holds that $G'(\theta)<0$ and hence $A_0<0$. Since A_0 can be rewritten as the product of the three remaining eigenvalues, i.e. $A_0=\lambda_2\lambda_3\lambda_4<0$, we either have one or three additional negative eigenvalues.

Stability requires two positive and two negative eigenvalues. Since $A_2=\lambda_2+\lambda_3+\lambda_4$ and $A_1=\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4$, we require either $A_2>0$ and/or $A_1<0$.

Suppose $A_2>0$, then stability immediately follows. Suppose instead $A_2\le 0$. This implies

$$-\frac{c}{\eta_c}\left(\alpha x_c+\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\right)-\frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)x_\theta\le\delta+p(\theta)+\frac{\theta}{\eta_\theta}\vartheta_\theta, \tag{D.15}$$

which we use to show that $A_1<0$ in this case, from which stability follows. Reformulate A_1 as

$$\begin{aligned}
A_1=&[\delta+p(\theta)]\frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)\left(\frac{-p'(\theta)u}{\delta+p(\theta)}x_u+x_\theta+\frac{\vartheta_\theta}{\alpha+\vartheta_x}\right)+[\delta+p(\theta)]\frac{c}{\eta_c}\left(\alpha x_c+\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\right)\\
&+\frac{c}{\eta_c}\frac{\theta}{\eta_\theta}\left[\left(\alpha x_c+\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\right)\vartheta_\theta+\left(\alpha+\vartheta_x\right)\left[\left(\alpha x_c+\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\right)x_\theta-x_c\left(\alpha x_\theta-\frac{\omega''(b)b_\theta}{\phi'(c)}\right)\right]\right]\\
=&[\delta+p(\theta)]\frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)x_c\left(\frac{x_u}{x_c}u'(\theta)+\frac{x_\theta}{x_c}+\frac{\vartheta_\theta}{x_c\left(\alpha+\vartheta_x\right)}\right)+[\delta+p(\theta)]\frac{c}{\eta_c}\left(\alpha x_c+\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\right)\\
&+\frac{c}{\eta_c}\frac{\theta}{\eta_\theta}\left(\alpha x_c+\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\right)\vartheta_\theta+\frac{c}{\eta_c}\frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)\left[\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}x_\theta+\frac{\omega''(b)b_\theta}{\phi'(c)}x_c\right]\\
=&-[\delta+p(\theta)]\frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)x_c\frac{dc(\theta,x(\theta))}{d\theta}+\frac{c}{\eta_c}\alpha x_c\left(\delta+p(\theta)+\frac{\theta}{\eta_\theta}\vartheta_\theta\right)\\
&+\frac{c}{\eta_c}\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\left(\delta+p(\theta)+\frac{\theta}{\eta_\theta}\vartheta_\theta\right)+\frac{c}{\eta_c}\frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)\left(\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}x_\theta+\frac{\omega''(b)b_\theta}{\phi'(c)}x_c\right)\\
=&-[\delta+p(\theta)]\frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)x_c\frac{dc(\theta,x(\theta))}{d\theta}+\left(\delta+p(\theta)+\frac{\theta}{\eta_\theta}\vartheta_\theta\right)\left(\frac{c}{\eta_c}\alpha x_c-\frac{\omega'(b)}{\phi'(c)}\right)\\
&+\frac{c}{\eta_c}\frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)\left(\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}x_\theta+\frac{\omega''(b)b_\theta}{\phi'(c)}x_c\right).
\end{aligned}$$

For sufficiently high values of η_c, specifically for all $\eta_c > \frac{\alpha}{\rho}$, it holds that $\left(\frac{c}{\eta_c}\alpha x_c - \frac{\omega'(b)}{\phi'(c)}\right) < 0$. It then follows that for $A_2 \leq 0$, we have

$$
\begin{aligned}
A_1 \leq & -[\delta+p(\theta)]\frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)x_c\frac{dc(\theta,x(\theta))}{d\theta} - \left(\frac{c}{\eta_c}\alpha x_c - \frac{\omega'(b)}{\phi'(c)} + \frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)x_\theta\right)\left(\frac{c}{\eta_c}\alpha x_c - \frac{\omega'(b)}{\phi'(c)}\right) \\
& + \frac{c}{\eta_c}\frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)\left(\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}x_\theta + \frac{\omega''(b)b_\theta}{\phi'(c)}x_c\right) \\
= & -[\delta+p(\theta)]\frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)x_c\frac{dc(\theta,x(\theta))}{d\theta} - \left(\frac{c}{\eta_c}\alpha x_c - \frac{\omega'(b)}{\phi'(c)}\right)^2 - \frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)x_\theta\frac{c}{\eta_c}\alpha x_c \\
& + \frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)x_\theta\frac{\omega'(b)}{\phi'(c)} - \frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)\frac{\omega'(b)}{\phi'(c)}x_\theta + \frac{c}{\eta_c}\frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)\frac{\omega''(b)b_\theta}{\phi'(c)}x_c \\
= & -[\delta+p(\theta)]\frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)x_c\frac{dc(\theta,x(\theta))}{d\theta} - \left(\frac{c}{\eta_c}\alpha x_c - \frac{\omega'(b)}{\phi'(c)}\right)^2 - \frac{c}{\eta_c}\frac{\theta}{\eta_\theta}\left(\alpha+\vartheta_x\right)x_c\left(x_\theta - \frac{\omega''(b)b_\theta}{\phi'(c)}\right).
\end{aligned}
$$

Since $\frac{dc(\theta,x(\theta))}{d\theta} > 0$ in the stagnation steady state, which follows from the uniqueness condition, all three terms are negative and we have $A_1 < 0$. Therefore, we cannot have three negative eigenvalues, which together with $A_0 < 0$ and $\lambda_1 > 0$ implies that there are exactly two positive and two negative eigenvalues. It follows that the dynamic system is saddle path stable around the stagnation steady state.

Appendix E: Proof of Proposition 1

Use the steady state conditions (33) with $x < 1$ to define $G(\theta)$, where $x(\theta)$ is given by (36), as follows:

$$G(\theta) = \rho - \frac{\omega'(.)}{\phi'(c(\theta, x(\theta)))} - \alpha(1 - x(\theta)). \tag{E.1}$$

We model worsening stagnation by exogenous increases in the steady state marginal utility of wealth $\omega'(.)$, e.g. via modifying the scale parameter of the wealth preference relative to the preference for consumption. This implies a stronger desire to save and a decline in the natural real interest rate. Recall from before that $G'(\theta) < 0$ in steady state.

The effects of worsening stagnation on labor market tightness is given by

$$\frac{d\theta}{d\omega'} = \frac{\omega' G'(\theta)}{\phi'(c(\theta, x(\theta)))} < 0 \ . \tag{E.2}$$

It then follows immediately from (28), (36) and (21) that

$$\frac{du}{d\omega'} > 0 \ , \quad \frac{dx}{d\omega'} < 0 \ , \quad \frac{dY}{d\omega'} < 0 \ . \tag{E.3}$$

The effect on consumption follows from (E.1). In steady state, $G(\theta) = 0$. As $\frac{dx}{d\omega'} < 0$, it necessarily follows that

$$\frac{dc}{d\omega'} < 0 \ . \tag{E.4}$$

Finally, from (17) and (36), the real wage can be rewritten as

$$w_t = \psi\left(\bar{y} + \frac{(1-\psi)\bar{y}p(\theta) + (1-s)k\alpha\theta}{\psi p(\theta) + \delta + \alpha}\right) \ , \tag{E.5}$$

which gives

$$\frac{dw}{d\omega'} = \frac{\psi(\delta+\alpha)[(1-\psi)\bar{y}p'(\theta)+(1-s)k\alpha] - (1-s)k\alpha\psi^2\theta^2 q'(\theta)}{[\psi p(\theta)+\delta+\alpha]^2} \frac{d\theta}{d\omega'} < 0 \ , \tag{E.6}$$

where the negative sign results from $p'(\theta) > 0$ in (9), $q'(\theta) < 0$ in (10) and $\frac{d\theta}{d\omega'} < 0$ in (E.2). Expressions (E.2), (E.3), (E.4) and (E.6) establish Proposition 1.

Appendix F: Effects of exogenous variations in x

We want to derive the conditions under which an exogenous increase (decrease) in x creates excess supply (demand) in the goods market for a given θ, i.e.

$$\Upsilon(\theta) \equiv -\frac{\omega'(b)\phi''(c)(1-u)\bar{y}}{\phi'(c)^2} - \alpha > 0. \tag{F.1}$$

Consider the following modifications using the definition of η_c and c

$$\begin{aligned}\Upsilon(\theta) = \frac{\omega'(b)}{\phi'(c)}\frac{(1-u)\bar{y}\eta_c}{c} - \alpha &= \frac{(1-u)\bar{y}}{c}\left[\rho\eta_c - \alpha(1-x)\eta_c - \alpha\frac{c}{(1-u)\bar{y}}\right] \\ &= \frac{(1-u)\bar{y}}{c}\left[\rho\eta_c - \alpha(1-x)\eta_c - \alpha\frac{(1-u)\bar{y}x - ku\theta - g}{(1-u)\bar{y}}\right] \\ &= \frac{(1-u)\bar{y}}{c}\left[(\rho-\alpha)\eta_c + \alpha x(\eta_c - 1) + \alpha\frac{ku\theta+g}{(1-u)\bar{y}}\right]. \end{aligned} \tag{F.2}$$

It is easy to see that $\Upsilon(\theta) > 0$ for $\eta_c \geq 1$. Suppose $\eta_c < 1$, then $x = 1$ establishes a lower bound on the right hand side:

$$\Upsilon(\theta) > \frac{(1-u)\bar{y}}{c}\left[\rho\eta_c - \alpha + \alpha\frac{ku\theta+g}{(1-u)\bar{y}}\right] > \frac{(1-u)\bar{y}}{c}\left[\rho\eta_c - \alpha\right]. \tag{F.3}$$

It follows directly that $\eta_c > \frac{\alpha}{\rho}$ is a sufficient condition for $\Upsilon(\theta) > 0$ or

$$-\frac{\omega'(b)\phi''(c)(1-u)\bar{y}}{\phi'(c)^2} > \alpha \ \text{ if } \ \eta_c > \frac{\alpha}{\rho}. \tag{F.4}$$

Appendix G: Proof of Proposition 2 and Proposition 3

Standard steady state:

Use (33) with $x = 1$ to define the function $H(\theta, \chi)$, where χ denotes any parameter in the model, as follows:

$$H(\theta,\chi) = \rho + \delta + \psi p(\theta) - \frac{(1-\psi)\bar{y}}{(1-s)k}q(\theta) - \frac{\omega'\left((1-u(\theta))\frac{(1-s)k}{q(\theta)}\right)}{\phi'(c(\theta,1))}, \tag{G.1}$$

with $H(\theta^f) = 0$ and $H'(\theta^f) > 0$ in a unique steady state where H_θ denotes the derivative with respect to θ evaluated in steady state. $c(\theta, 1)$ is defined in (31). The effects of changes in a parameter χ on the labor market tightness θ can be recovered from (G.1) as

$$\frac{d\theta}{d\chi} = -\frac{H_\chi}{H_\theta}. \tag{G.2}$$

It holds that

$$H_z = \frac{(\bar{y}q(\theta) + (1-s)kp(\theta)}{(1-s)k}\frac{\partial\psi}{\partial z} > 0,$$

$$H_s = \frac{\omega''(b)(1-u(\theta))k}{\phi'(c)q(\theta)} - \frac{(1-\psi)\bar{y}q(\theta)}{(1-s)^2 k} < 0,$$

$$H_{\bar{y}} = \frac{\omega'(b)\phi''(c)(1-u(\theta))}{\phi'(c)^2} - \frac{(1-\psi)q(\theta)}{(1-s)k} < 0,$$

$$H_g = -\frac{\omega'(b)\phi''(c)}{\phi'(c)^2} > 0.$$

This implies the following relationship between the steady state labor market tightness and the model parameters:

$$\theta^f = \theta(\underbrace{z}_{-}, \underbrace{s}_{+}, \underbrace{\bar{y}}_{+}, \underbrace{g}_{-}). \tag{G.3}$$

From the Beveridge curve in (28), it immediately follows that the effects on the unemployment rate are opposite to those on the labor market tightness, i.e.

$$u^f = u(\underbrace{z}_{+}, \underbrace{s}_{-}, \underbrace{\bar{y}}_{-}, \underbrace{g}_{+}). \tag{G.4}$$

In addition, total output $Y = (1-u(\theta))\bar{y}$ is affected by the model parameters in the same way as the labor market tightness, except that changes in productivity are reinforced, i.e.

$$Y^f = Y(\underbrace{z}_{-}, \underbrace{s}_{+}, \underbrace{\bar{y}}_{+}, \underbrace{g}_{-}). \tag{G.5}$$

Finally, the effects on household consumption can be derived from the goods market clearing condition (31). Specifically, it holds that

$$\frac{dc(\theta,1)}{d\chi} = \frac{dc(\theta,1)}{d\theta}\frac{d\theta}{d\chi} + \frac{\partial c(\theta,1)}{\partial\chi}, \tag{G.6}$$

where $\frac{\partial c}{\partial\chi} = 0$ for $\chi = z, s$, $\frac{\partial c}{\partial\chi} = 1-u$ for $\chi = \bar{y}$ and $\frac{\partial c}{\partial\chi} = -1$ for $\chi = g$. Moreover, $dc(\theta,1)/d\theta = -u'(\theta)(\bar{y}+k\theta) - ku(\theta)$ and its sign is not uniquely determined. For $dc(\theta,1)/d\theta > 0$ ("slack" steady state), the sign of the effects of all parameter changes are identical to those of the labor market tightness. For $dc(\theta,1)/d\theta < 0$ ("tight" steady state), the sign of the effects of changes in ε, z and s is opposite to those on θ, while the effects of variations in $\bar{y}$ and g are indeterminate.

Stagnation steady state:

(i) Effects on labor market tightness and unemployment: Use the steady state conditions (33) with $x < 1$ and (36) to define $G(\theta,\chi)$, where $x(\theta)$ is given by (36) as $x < 1$ and χ denotes any parameter as follows:

$$G(\theta,\chi) = \rho - \frac{\omega'\left((1-u(\theta))\frac{(1-s)k}{q(\theta)}\right)}{\phi'(c(\theta,x(\theta)))} - \alpha(1-x(\theta)), \tag{G.7}$$

with $G(\theta^s,\chi) = 0$ and $G_{\theta^s} < 0$ in steady state where G_{θ^s} denotes the total derivative with respect to θ evaluated in steady state. The negative sign follows from the uniqueness of the steady state. $c(\theta, x(\theta))$ is given by (31) with $x = x(\theta)$. The effects of changes in a parameter χ

on the labor market tightness θ can be recovered from (G.7) as

$$\frac{d\theta}{d\chi} = -\frac{G_\chi}{G_{\theta^s}}. \tag{G.8}$$

In addition, we derive the following partial derivatives from $x(\theta, \chi)$ in (36):

$$x_z > 0\ ,\ x_s < 0\ ,\ x_{\bar{y}} < 0\ ,\ x_g = 0\ ,\ x + \bar{y}x_{\bar{y}} > 0. \tag{G.9}$$

Using (G.7) to (G.9) and assuming that $\eta_c > \frac{\alpha}{\rho}$ (see Appendix F), we derive the following properties:

$$\frac{d\theta}{dz} = -\left[\frac{\omega'(b)\phi''(c)(1-u)\bar{y}}{\phi'(c)^2} + \alpha\right]\frac{x_z}{G_{\theta^s}} < 0, \tag{G.10}$$

$$\frac{d\theta}{ds} = -\frac{\omega''(b)b(\theta)}{(1-s)\phi'(c)}\frac{1}{G_{\theta^s}} - \left[\frac{\omega'(b)\phi''(c)(1-u)\bar{y}}{\phi'(c)^2} + \alpha\right]\frac{x_s}{G_{\theta^s}}, \tag{G.11}$$

$$\frac{d\theta}{d\bar{y}} = -\frac{\omega'(b)\phi''(c)(1-u)}{\phi'(c)^2}\frac{x + \bar{y}x_{\bar{y}}}{G_{\theta^s}} - \alpha\frac{x_{\bar{y}}}{G_{\theta^s}} < 0, \tag{G.12}$$

$$\frac{d\theta}{dg} = \frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\frac{1}{G_{\theta^s}} > 0. \tag{G.13}$$

Expressions (G.10) to (G.13) imply the following response of labor market tightness to parameter variations. The response of the unemployment rate is opposite as is clear from the Beveridge curve in (28):

$$\theta^s = \theta(\underbrace{z}_{-}, \underbrace{s}_{?}, \underbrace{\bar{y}}_{-}, \underbrace{g}_{+}), \tag{G.14}$$

$$u^s = u(\underbrace{z}_{+}, \underbrace{s}_{?}, \underbrace{\bar{y}}_{+}, \underbrace{g}_{-}). \tag{G.15}$$

(ii) Effects on realized working hours: From the asset market equilibrium curve (36), we recover the effects of parameter changes on realized working hours as follows:

$$\frac{dx(\theta,\chi)}{d\chi} = x_\chi + \frac{dx(\theta)}{d\theta}\frac{d\theta}{d\chi} = x_\chi - \frac{dx(\theta)}{d\theta}\frac{G_\chi}{G_{\theta^s}} = \left[G_{\theta^s} - \frac{dx(\theta)}{d\theta}\frac{G_\chi}{x_\chi}\right]\frac{x_\chi}{G_{\theta^s}}, \tag{G.16}$$

where $\frac{dx(\theta)}{d\theta} > 0$ and G_χ are the partial derivatives of $G(\theta, \chi)$ with respect to any parameter χ. Using the expressions above for $\frac{d\theta}{d\chi}$ and $\frac{dc(\theta,x(\theta))}{d\theta} = (1-u)\bar{y}\frac{dx(\theta)}{d\theta} - (\bar{y}x(\theta) + k\theta)u'(\theta) - ku$ from (31), we get the following results:

$$\frac{dx}{dz} = \left[\frac{-\omega''(b)b'(\theta)}{\phi'(c)} - \frac{\omega'(b)}{\phi'(c)}\frac{\phi''(c)}{\phi'(c)}\left[ku + (\bar{y}x + k\theta)u'(\theta)\right]\right]\frac{x_z}{G_{\theta^s}}, \tag{G.17}$$

$$\frac{dx}{ds} = \left[\frac{\omega''(b)}{\phi'(c)}\left[-\frac{dx(\theta)}{d\theta}\frac{b(\theta)}{x_s(1-s)} - b'(\theta)\right] - \frac{\omega'(b)}{\phi'(c)}\frac{\phi''(c)}{\phi'(c)}\left[ku + (\bar{y}x + k\theta)u'(\theta)\right]\right]\frac{x_s}{G_{\theta^s}}, \tag{G.18}$$

$$\begin{aligned}\frac{dx}{d\bar{y}} &= \left[-\frac{\omega''(b)b'(\theta)}{\phi'(c)} - \frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\left[ku + (\bar{y}x + k\theta)u'(\theta) + \frac{dx(\theta)}{d\theta}\frac{(1-u)x(\theta)}{x_{\bar{y}}}\right]\right]\frac{x_{\bar{y}}}{G_{\theta^s}} \\ &= \left[-\frac{\omega''(b)b'(\theta)}{\phi'(c)} - \frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\left[\frac{dx(\theta)}{d\theta}\frac{(1-u)}{x_{\bar{y}}}(x + \bar{y}x_{\bar{y}}) - \frac{dc(\theta, x(\theta))}{d\theta}\right]\right]\frac{x_{\bar{y}}}{G_{\theta^s}}.\end{aligned} \tag{G.19}$$

From $G_{\theta^s} < 0$, we derive the following upper bound for the first term in (G.19):

$$-\frac{\omega''(b)b'(\theta)}{\phi'(c)} < -\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\frac{dc(\theta, x(\theta))}{d\theta} - \alpha\frac{dx(\theta)}{d\theta}.$$

Since $x_{\bar{y}} < 0$ and $G_{\theta^s} < 0$, from (G.19) it follows that

$$\frac{dx}{d\bar{y}} < \left[-\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\frac{dc(\theta,x(\theta))}{d\theta} - \alpha\frac{dx(\theta)}{d\theta} - \frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\left[\frac{dx(\theta)}{d\theta}\frac{(1-u)}{x_{\bar{y}}}(x+\bar{y}x_{\bar{y}}) - \frac{dc(\theta,x(\theta))}{d\theta}\right]\right]\frac{x_{\bar{y}}}{G_{\theta^s}}$$

$$= -\frac{dx(\theta)}{d\theta}\left[\alpha + \frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\frac{(1-u)}{x_{\bar{y}}}(x+\bar{y}x_{\bar{y}})\right]\frac{x_{\bar{y}}}{G_{\theta^s}} < 0, \tag{G.20}$$

$$\frac{dx}{dg} = \frac{dx(\theta)}{d\theta}\frac{d\theta}{dg} > 0. \tag{G.21}$$

Taken together, we have derived the following effects of parameter changes in (G.17) to (G.21):

$$x^s = x(\underbrace{z}_{?}, \underbrace{s}_{?}, \underbrace{\bar{y}}_{-}, \underbrace{g}_{+}). \tag{G.22}$$

(iii) Effects on total output: The effects of parameter changes on total output follow from the total differential of $Y(\theta) = (1-u(\theta))x(\theta)\bar{y}$ as

$$\frac{dY}{d\chi} = -x(\theta)\bar{y}\frac{du}{d\chi} + (1-u)\bar{y}\frac{dx}{d\chi} + Y_\chi, \tag{G.23}$$

where $Y_\chi = (1-u)x$ for $\chi = \bar{y}$ and zero otherwise. It then follows that these effects are given by

$$\frac{dY}{dz} = \underbrace{(1-u)\bar{y}x_z}_{+} + \underbrace{\left[(1-u)\bar{y}\frac{dx(\theta)}{d\theta} - x\bar{y}u'(\theta)\right]\frac{d\theta}{dz}}_{-}, \tag{G.24}$$

$$\frac{dY}{ds} = \underbrace{(1-u)\bar{y}x_s}_{-} + \underbrace{\left[(1-u)\bar{y}\frac{dx(\theta)}{d\theta} - x\bar{y}u'(\theta)\right]\frac{d\theta}{ds}}_{+/-}, \tag{G.25}$$

$$\frac{dY}{d\bar{y}} = (1-u)\bar{y}\frac{dx}{d\bar{y}} - x\bar{y}u'(\theta)\frac{d\theta}{d\bar{y}} + (1-u)x(\theta)$$
$$= \left[-(1-u)\left(\bar{y}+\frac{x}{x_{\bar{y}}}\right)\frac{\omega''(b)b'(\theta)}{\phi'(c)} + \alpha x(\theta)\left(\bar{y}u'(\theta) + \frac{dx(\theta)}{d\theta}\frac{1-u}{x_{\bar{y}}}\right)\right.$$
$$\left. -(1-u)\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\left(\bar{y}+\frac{x}{x_{\bar{y}}}\right)(ku + k\theta u'(\theta))\right]\frac{x_{\bar{y}}}{G_{\theta^s}}.$$

As above, we apply an upper bound on $-\frac{\omega''(b)b'(\theta)}{\phi'(c)}$ that results from $G_{\theta^s} < 0$. Since $x_{\bar{y}} < 0$ and $G_{\theta^s} < 0$, it follows that

$$\frac{dY}{d\bar{y}} < \left[-(1-u)\left(\bar{y}+\frac{x(\theta)}{x_{\bar{y}}}\right)\left[\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\frac{dc(\theta,x(\theta))}{d\theta} + \alpha\frac{dx(\theta)}{d\theta}\right] + \alpha x\left(\bar{y}u'(\theta) + \frac{dx(\theta)}{d\theta}\frac{1-u}{x_{\bar{y}}}\right)\right.$$
$$\left. -(1-u)\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\left(\bar{y}+\frac{x(\theta)}{x_{\bar{y}}}\right)(ku + k\theta u'(\theta))\right]\frac{x_{\bar{y}}}{G_{\theta^s}}$$
$$= \left[-(1-u)\left(\bar{y}+\frac{x(\theta)}{x_{\bar{y}}}\right)\frac{\omega'(b)\phi''(c)}{\phi'(c)^2}\bar{y}\left((1-u)\frac{dx(\theta)}{d\theta} - u'(\theta)x(\theta)\right)\right.$$
$$\left. -\alpha\bar{y}\left((1-u)\frac{dx(\theta)}{d\theta} - x(\theta)u'(\theta)\right)\right]\frac{x_{\bar{y}}}{G_{\theta^s}}$$

$$= -\underbrace{\left[(1-u)\frac{dx(\theta)}{d\theta} - u'(\theta)x(\theta)\right]}_{+}\bar{y}\underbrace{\left[(1-u)\left(\bar{y}+\frac{x(\theta)}{x_{\bar{y}}}\right)\frac{\omega'(b)\phi''(c)}{\phi'(c)^2} + \alpha\right]}_{+}\underbrace{\frac{x_{\bar{y}}}{G_{\theta^s}}}_{+} < 0, \tag{G.26}$$

$$\frac{dY}{dg} = \left[\frac{p'(\theta)}{p(\theta)}ux(\theta) + \frac{dx(\theta)}{d\theta}\right]\frac{p(\theta)\bar{y}}{\delta + p(\theta)}\frac{d\theta}{dg} > 0. \tag{G.27}$$

Taken together, we have derived the following effects of parameter changes from (G.24) to (G.27):

$$Y^s = Y(\underbrace{z}_{?}, \underbrace{s}_{?}, \underbrace{\bar{y}}_{-}, \underbrace{g}_{+}). \tag{G.28}$$

(iv) Effects on consumption: Finally, the effects on consumption are given by the differential of (31) with $x = x(\theta)$ given by (36) using the expressions above as

$$\frac{dc}{d\chi} = \left[-[\bar{y}x + k\theta]u'(\theta) - ku\right]\frac{d\theta}{d\chi} + (1-u)\bar{y}\frac{dx}{d\chi} + c_\chi = \frac{dc(\theta, x(\theta))}{d\theta}\frac{d\theta}{d\chi} + (1-u)\bar{y}x_\chi + c_\chi, \tag{G.29}$$

where $\frac{dc(\theta,x(\theta))}{d\theta} > 0$, $c_z = c_s = 0$, $c_{\bar{y}} = (1-u)x > 0$ and $c_g = -1 < 0$.

$$\frac{dc}{dz} = \underbrace{\frac{dc(\theta, x(\theta))}{d\theta}\frac{d\theta}{dz}}_{-} + \underbrace{(1-u)\bar{y}x_z}_{+} = \left[[(\bar{y}x + k\theta)u'(\theta) + ku]\,\alpha - (1-u)\bar{y}\frac{\omega''(b)b'(\theta)}{\phi'(c)}\right]\frac{x_z}{G_{\theta^s}}, \tag{G.30}$$

$$\frac{dc}{ds} = \underbrace{\frac{dc(\theta, x(\theta))}{d\theta}\frac{d\theta}{ds}}_{+/-} + \underbrace{(1-u)\bar{y}x_s}_{-}, \tag{G.31}$$

$$\begin{aligned}
\frac{dc}{d\bar{y}} &= (1-u)x\left[1 - \frac{\omega'(b)\phi''(c)}{\phi'(c)^2 G_{\theta^s}}\frac{dc(\theta, x(\theta))}{d\theta}\right] + \left[[(\bar{y}x + k\theta)u'(\theta) + ku(\theta)]\,\alpha - (1-u)\bar{y}\frac{\omega''(b)b'(\theta)}{\phi'(c)}\right]\frac{x_{\bar{y}}}{G_{\theta^s}} \\
&= (1-u)x\frac{\phi'^2\alpha\frac{dx^a(\theta)}{d\theta} - \omega''(b)b'(\theta)\phi'(c)}{\phi'(c)^2 G_{\theta^s}} + \left[\left[(1-u)\frac{dx(\theta)}{d\theta}\bar{y} - \frac{dc(\theta, x(\theta))}{d\theta}\right]\alpha - (1-u)\bar{y}\frac{\omega''(b)b'(\theta)}{\phi'(c)}\right]\frac{x_{\bar{y}}}{G_{\theta^s}} \\
&= \left[\alpha\left((1-u)\frac{dx^a(\theta)}{d\theta}(\bar{y}x_{\bar{y}} + x) - \frac{dc(\theta, x(\theta))}{d\theta}x_{\bar{y}}\right) - (1-u)(\bar{y}x_{\bar{y}} + x)\frac{\omega''(b)b'(\theta)}{\phi'(c)}\right]\frac{1}{G_{\theta^s}} \\
&= \left[\underbrace{-\alpha\frac{dc(\theta, x(\theta))}{d\theta}x_{\bar{y}}}_{+\text{ as } x_{\bar{y}}<0} + \underbrace{(1-u)(\bar{y}x_{\bar{y}} + x)}_{+\text{ as } \bar{y}x_{\bar{y}}+x>0}\left(\alpha\frac{dx^a(\theta)}{d\theta} - \frac{\omega''(b)b'(\theta)}{\phi'(c)}\right)\right]\frac{1}{G_{\theta^s}} < 0,
\end{aligned} \tag{G.32}$$

$$\frac{dc}{dg} = -\left[\alpha\frac{dx^a(\theta)}{d\theta} - \frac{\omega''(b)b'(\theta)}{\phi'(c)}\right]\frac{1}{G_{\theta^s}} > 0. \tag{G.33}$$

Taken together, (G.30) to (G.33) imply the following effects of parameter changes:

$$c = c(\underbrace{z}_{?}, \underbrace{s}_{?}, \underbrace{\bar{y}}_{-}, \underbrace{g}_{+}). \tag{G.34}$$