ISSN (Print) 0473-453X
Discussion Paper No. 1095
ISSN (Online) 2435-0982

SERIAL VICKREY MECHANISM

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July 2020

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Serial Vickrey mechanism[1]

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July 30, 2020

Abstract

We study an assignment market where multiple heterogenous objects are sold to unit demand agents who have general preferences accommodating imperfect transferability of utility and income effects. In such a model, there is a minimum price equilibrium. We establish the structural characterizations of minimum price equilibria and employ these results to design the " Serial Vickrey mechanism," that finds a minimum price equilibrium in a finite number of steps. The Serial Vickrey mechanism introduces the objects one by one, and requires agents to report finite-dimensional prices in finitely many times. Besides, the Serial Vickrey mechanism also has nice dynamic incentive properties.

Keywords: The assignment market, minimum price equilibrium, general preferences, structural characterizations, Serial Vickrey mechanism, dynamic incentive compatibility

JEL Classification: C63, C70, D44

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[1]The preliminary version of this article is presented at 2017, 2019 Economic Design Conferences, SING 13, 2018 SPMID conference, PET 2018, the 14th Meeting of the Society for Social Choice, the 19th SAET, 2019 APIOC, 2019 Nanjing International Conference on Theory of Games and Economic Behavior, and seminars at City University of Hong Kong, ITAM, and Maastricht. We thank Tommy Andersson, Itai Ashlagi, Lawrence Ausubel, Vincent Crawford, Chuangyin Dang, Jean-Jacques Herings, Fuhito Kojima, Scott Duke Kominers, Andy Mackenzie, Debasis Mishra, James Schummer, Ning Sun, Dolf Talman, Alex Teytelboym, Ryan Tierney, William Thomson, and Rakesh Vohra for their helpful comments. We gratefully acknowledge financial support from the Joint Usage/Research Center at ISER, International Joint Research Promotion Program (Osaka University), and Grant-in-aid for Research Activity, Japan Society for the Promotion of Science (15J01287, 15H03328, 15H05728, 19K13653).

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1 Introduction

In auction and matching theory, quasi-linearity of preferences is commonly assumed.[1] It means that utility can be perfectly transferred among agents or the payment for a good exhibits no income effect for its demand. Quasi-linearity largely simplifies analysis and makes the duality of linear programming techniques applicable to auction and matching problems (Vohra, 2011).[2] Nevertheless, the frictionless world with perfectly transferred utility is rather ideal. Quasi-linearity holds only in restricted environments such as those where payments are negligibly small compared to incomes or there are no budget constraints.

In the spectrum license auctions in OECD countries, regarded as one of the most important applications of the theory, bidders often borrow to pay for the huge bids and face non-linearity of borrowing cost, which makes their preferences non-quasi-linear (Klemperer, 2004). In practical housing markets, income effects and budget constraints also prevail (Zhou and Serizawa, 2018). Distortional taxes in transactions make utility imperfectly transferred (Fleiner et al. 2019).

These practical limitations motivate many researchers to generate new techniques to extend the results and gain new insights for auction and matching theory for *general preferences* that accommodate imperfect transferability of utility and income effects. Such examples include:

• Properties of equilibria or stable outcomes in matching models (Crawford and Knoer, 1981; Kelso and Crawford, 1982; Quinzi, 1984; Demange and Gale, 1985; Caplin and Leady, 2014; Alaei et al. 2016; Fleiner et al. 2019; Schlegel, 2020)

• Efficient, strategy-proof, and fair rules in the assignment market (Sun and Yang, 2003; Andersson and Svensson, 2014, 2018; Morimoto and Serizawa, 2015; Kazumura et al. 2020a)

• Properties of equilibria or stable outcomes in matching models with price controls or hierarchy constraints (Andersson and Svensson, 2014, 2018; Herings, 2018; Kojima et al. 2020)

• Mechanism design or optimal auction design without quasi-linearity (Baisa, 2017; Noldeke and Samuelson, 2018; Kazumura et al. 2020b)

• The theoretical foundation for empirical research (Galichon et al. 2019)

This paper works on one of most prominent models in the matching theory, the assignment market: Multiple heterogenous objects are to be sold to several agents where payments can continuously change, and agents have unit-demand general preferences. We draw on the above first two bullet directions, but go

[1] See Myerson (1980) and Shapley and Shubik (1972) for example.

[2] The recent development of (discrete) convex analysis and tropical geometry techniques relies on quasi-linearity, e.g., Murota (2003) and Baldwin and Klemperer (2019).

beyond them by studying efficient and dynamic incentive-compatible mechanisms with finite-dimensional information revelation of agents.

In our settings, when general preferences satisfies standard assumptions of monotonicity and finiteness, there is a minimum price equilibrium (MPE), whose supporting price (vector) is coordinate-wise minimum among all equilibrium prices (Demange and Gale, 1985). The MPE attracts particular attention since the MPE rule, mapping each general preference profile to an MPE, is the only rule satisfying efficiency and strategy-proofness, and also has desirable revenue-maximazation property.[3] However, the MPE rules unrealistically require agents to report their whole general preferences, and have neither information of how an MPE is implemented, nor any dynamic incentive property.

We design a dynamic mechanism that finds an MPE, by asking each agent to report a finite-dimensional price whose coordinate is an "indifference price" in finitely many times, instead of requiring each agent to report her full preference. An agent's *indifference price* of an object is her willingness to pay for the object, evaluated from her provisionally assigned bundle (object-payment pair). A quasi-linear preference is presented by an agent's valuations of the objects, or simply a finite-dimensional price. Thus, reporting indifference prices naturally generalizes the information elicitation of quasi-linear preferences to general preferences.

We proceed the mechanism design by providing three structural characterizations that show the inner connections between an (arbitrary) equilibrium and an MPE. Proposition 1 establishes the first characterization, the "connectedness" property of the MPE. It says the equivalence of the three conditions: (i) An equilibrium price is an MPE price; (ii) each object is connected via agents' demands; and (iii) each agent gets either a connected object or nothing.

To show the second and third characterizations, we also show as Lemma 1 that (i) to find an MPE from an equilibrium, the object reassignment and price adjustment are within the unconnected objects and agents, i.e., objects and agents without connectedness property, and (ii) the MPE prices of the unconnected objects are bounded below by the indifference prices of the connected agents. Then we build a "I pay others' indifference prices (IPOIP) process," to find an MPE from an equilibrium. Given a candidate of the MPE assignment, the IPOIP process recursively raises the prices of objects in that assignment from their lower bounds.

Theorem 1 establishes the second characterization: (i) the price and assignments of the connected objects in an equilibrium are the same as the MPE, and (ii) the MPE price of the unconnected objects in an equilibrium is the coordinate-wise minimum among the IPOIP prices of all candidate assignments. Theorem 2 establishes the third characterization that if IPOIP process stops raising the prices at some point for a candidate assignment, then the candidate is an MPE

[3] See Morimoto and Serizawa (2015), and Kazumura et al. (2020a) for details.

assignment, and the price at that point is the MPE price.

Then, we propose a "Serial Vickrey (SV) mechanism," by exploring the above structural results. The SV mechanism introduces objects sequentially, and based on an MPE for k objects, it employs the "SV sub-mechanism," to find an MPE for $k+1$ objects in a finite number of steps, established as Theorem 3. When introducing the first object, the SV sub-mechansim coincides with the second-price auction mechanism. In general, given an MPE for k objects, the SV sub-mechanism contains two stages. Stage 1 uses the first structural result to construct an equilibrium for $k+1$ by the "E-generating mechanism," and identify the agents and objects needed to adjust their assignments and prices in Stage 2, by the "connected-agent identifying mechanism." Stage 2 explores the second and third structural results and their by-products to conduct object reassignment and price adjustment by "the MPE-adjustment mechanism."

Finally, we study the incentive compatibility of the SV mechanism. We observe that the rule induced by the SV mechanism coincides with the MPE rule so it is strategy-proof. We show as Proposition 5 that given an MPE for k objects under the true preferences, revealing the true preferences is a dominant-strategy equilibrium in the normal game forms induced by both the E-generating mechanism and MPE-adjustment mechanism. We establish as Theorem 4 that given an MPE for k objects under the true preferences, revealing the true preferences is a dominant-strategy equilibrium in the normal game form induced by the SV sub-mechanism for $k+1$ objects. Theorem 4 is neither implied by the strategy-proofness of the MPE rule since the number of objects changes, nor by the aggregation of two incentive-compatible mechanisms within an SV sub-mechanism. Besides, we remark that in the SV mechanism, agents' welfare is always increasing with the number of introduced objects.

All our results and the corresponding proof techniques are novel to the existing works for general preferences. We leave the details of this point to the next section.

The remainder is organized as follows: Section 2 discusses the related literature. Section 3 defines the model and MPEs. Section 4 graphically illustrates the indifference price and the MPE. Section 5 gives the structural characterizations. Section 6 presents the SV mechanism. Section 7 analyzes the incentives. Section 8 discusses the SV mechanism as concluding remarks.

2 Related literature

The efficient and incentive-compatible mechanism has been studied in the assignment market for quasi-linear preferences. There are three remarkable results: (i) The MPE price coincides with Vickrey-Clarkes-Groves (VCG) payment (Leonard, 1983); (ii) Assuming integer valuations of agents, the auction mechanisms of Demange et al. (1986), Mishra and Parkes (2010), Andersson and Erlanson (2013), and Liu and Bagh (2019), yield MPEs; (iii) The approximate auction mechanisms of Crawford and Knoer (1981) and Demange et al. (1986) find a price that deviates from the MPE price within certain boundary. For auctions in (ii) and (iii), the increment\decrement is pre-fixed and agents reveal information of their demand sets. However, insights in (i), (ii), and (iii) cannot be extended to general preferences: There is no parallel VCG-type payment for the MPE price. In Appendix D, we exemplify that when conducted in the general preference settings, auctions in (ii) and (iii) either largely overshoot or undershoot the MPE price beyond boundary estimated in the quasi-linear environments.[4] These observations justify the use of indifference prices as the information elicitation for general preferences: it is almost impossible to propose a price adjustment process by pre-fixing an increment/decrement and only eliciting the information of agents' demand sets.

For general preferences, the equivalence between equilibria and stable outcomes, and the lattice property of equilibria are widely studied, e.g., Quinzi (1984), Demange and Gale (1985), Fleiner et al. (2019), and Schlegel (2020). These properties are qualitatively different from our structural results. Exceptions are Caplin and Leahy (2014, 2020) and Alaei et al. (2016), who also study the structural properties of the equilibria.

In the same model as ours, Caplin and Leahy (2014) characterize the MPE price by the graph-allocation structure, which initiates Caplin and Leahy (2020) to employ the homotopy method to study the equilibrium changes in response to parameter changes. Alaei et al. (2016) characterize the MPE price by a recursive system that computes the minimum and maximum price equilibria in all smaller markets with different numbers of agents and objects. These papers use their structural results to obtain the MPEs. In contrast, the SV mechanism iteratively finds an MPE, which largely improves computations, and also has appealing dynamic incentive properties. In Section 8, we show the models where the SV mechanism efficiently finds an MPE, but their results do not, and detail the applications where the SV mechanism finds certain type of MPEs, but theirs do not.

The existence of equilibria for general preferences is commonly proved by two methods. The first one is to operate the Keslo-Crawford type adjustment process to get an approximate equilibrium with the discreteized payments and then take the limit argument, the existence is shown, e.g., Kelso and Crawford (1982), Herings (2018), and Kojima et al. (2020). The second one is to establish the existence under the piece-wise linear functions, and show the existence for general utility functions by the point-to-point convergence argument. This method is used by Alkan and Gale (1990) and conveys the proof idea of using the "Scarf Lemma",

[4] In our model, the cumulative offer process of Hatfield and Milgrom (2005) coincides with the approximate auction mechanism, so it fails to approximate an MPE either.

e.g., Quinzi (1984). The SV mechanism is different from these methods.

Finally, we discuss how our results differ from Andersson and Svensson (2018), Noldeke and Samuelson (2018), and Galichon et al. (2019). In the assignment market with price controls, Andersson and Svensson (2018) construct a finite ascending-price sequence that finds an "minimum rationing price equilibrium." This sequence terminates at an MPE price in our model. It is not clear how to identify two adjacent prices in the sequence in finitely many steps so their construction is different from the SV mechanism. Noldeke and Samuelson (2018) study the duality relationship without quasi-linearity in the one-to-one two-sided matching model, and characterize the duality of implementable profiles and assignments via the Galois connection. The Galois connection is a pair of mappings, with no information of how to find a stable outcome so it is different from the SV mechanism. Galichon et al. (2019) provide the theoretical model with potential application to structural estimation of imperfectly transferable utilities in the one-to-one two-sided matching model. They define an aggregate equilibrium (AE) that can be characterized by a system of equations in terms of matching pairs. Their proposed AE is different from the MPE, and the corresponding characterization does not hold for the MPE.

3 The model and minimum price equilibrium

There are a finite set of agents N and a finite set of heterogenous objects M where $|N| = n$ and $|M| = m$.[5] Not receiving an object is called receiving object 0. Let $L \equiv M \cup \{0\}$.

Each agent has **unit demand**: She receives at most one object. The **bundle** for agent i is a pair $z_i \equiv (x_i, t_i) \in L \times \mathbb{R}$, in which agent i receives an object $x_i \in L$ and pays $t_i \in \mathbb{R}$. Each agent i has a complete and transitive preference R_i over $L \times \mathbb{R}$. Let P_i and I_i be the associated strict and indifference relations. Assume the following properties of preferences.

Money monotonicity: For each $x_i \in L$, each pair $t_i, t_j \in \mathbb{R}$, if $t_i < t'_i$, $(x_i, t_i)\, P_i\, (x_i, t'_i)$.

Finiteness: For each $t_i \in \mathbb{R}$, each pair $x_i, x_j \in L$, there is $t_j \in \mathbb{R}$ such that $(x_i, t_i)\, I_i\, (x_j, t_j)$.

Money monotonicity states that for a given object, a lower payment makes the agent better off. Finiteness says that there is no object that is infinitely good or bad. A preference R_i is **general** if it satisfies the above two properties. Let $\mathcal{R}^G$ be the class of general preferences and $R \equiv (R_i)_{i\in N} \in (\mathcal{R}^G)^n$ be a preference profile.

[5]Let $|\cdot|$ denote the cardinality of a set.

An assignment market is summarized by (N, M, R). *We fix N and M throughout the paper, and fix a preference profile below until we analyze the incentive issues in Section 7.*

An **allocation** $z \in [L \times \mathbb{R}]^n$ is a list of agents' bundles such that except for object 0, no two agents receive the same object, i.e., $z \equiv (z_i)_{i \in N} = ((x_i, t_i))_{i \in N}$ such that for each pair $i, j \in N$, if $x_i \neq 0$ and $i \neq j$, then $x_i \neq x_j$. We denote the set of allocations by Z.

Let $p \equiv (p_x)_{x \in M} \in \mathbb{R}^m_+$ be a price (vector). Without loss of generality, assume the price of object 0 and all reserve prices of objects are zero. Agent $i's$ **demand set at price** p is defined as $D_i(p) \equiv \{x \in L : (x, p_x) \, R_i \, (y, p_y), \forall y \in L\}$.

Definition 1: A pair $(z, p) \in Z \times \mathbb{R}^m_+$ is an **equilibrium** if

$$\text{for each } i \in N, \; x_i \in D_i(p) \text{ and } t_i = p_{x_i}, \tag{E-i}$$

$$\text{for each } y \in M, \text{ if for each } i \in N, \; x_i \neq y, \text{ then } p_y = 0. \tag{E-ii}$$

(E-i) says that each agent i receives a bundle z_i consisting of a demanded object x_i and a payment t_i equal to the price of x_i. (E-ii) says that prices of unassigned objects are zero. Let W and $\mathcal{P}$ be the sets of equilibria and equilibrium prices.

Fact 1 (Alkan and Gale, 1990; Demange and Gale, 1985): There is an equilibrium and the set of equilibrium prices is a complete lattice.

Therefore, among all equilibrium prices, there is a unique coordinate-wise minimum price $p^{\min} \in \mathcal{P}$. A **minimum price equilibrium** (**MPE**) is an equilibrium supported by $p^{\min}$. Let $W^{\min} \subseteq W$ be the set of MPEs. Since indifferences in preferences are admitted, MPE allocations may not be unique, but they are welfare-equivalent: for each agent i, each MPE brings her the same welfare, i.e., for each pair $(z, p^{\min}), (z', p^{\min}) \in W^{\min}$, $z_i \, I_i \, z'_i$.

4 Illustration of indifference price and MPE

The following graphic tools illustrate the "indifference price" defined below, and an MPE. These illustrations also help us understand the concepts and results presented in the next sections.

· **The indifference price (IP)**

By money monotonicity and finiteness, for each $z_i \in L \times \mathbb{R}$, each $y \in L$, there is a *unique* payment $V_i(y; z_i) \in \mathbb{R}$ such that two bundles z_i and $(y, V_i(y; z_i))$ are welfare-equivalent, i.e., $(y, V_i(y; z_i)) \, I_i \, z_i$. We interpret $V_i(y; z_i)$ as agent $i's$ **indifference price (IP) of object** y **at bundle** z_i. *The indifference price works as the proxy for agents' preference elicitation.*

Suppose that there are two agents i and j, and two objects, A and B. Figure 1 illustrates the indifference prices for two preferences.

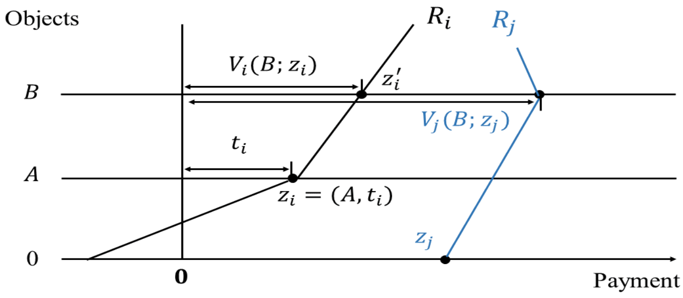

Figure 1: Illustration of IPs $V_i(B; z_i)$, and $V_j(B; z_j)$

Three horizontal lines correspond to objects 0, A, and B. For each point on one of the three lines, the distance between that point and the vertical line denotes the payment, e.g., z_i denotes the bundle consisting of object A and payment t_i. By money monotonicity, moving leftward along the same line makes the agent better off, e.g., $(0, A)\, P_i\, z_i$. The agent is indifferent between bundles connected by an indifference curve, e.g., for agent i, $z_i\, I_i\, z_i'$, and for agent j, $z_j\, I_j\, (B, V_j(B; z_j))$.

The IP of object B at z_i is $V_i(B; z_i)$ since $(B, V_i(B; z_i))\, I_i\, z_i$. Similarly, the IP of object B at z_j is $V_j(B; z_j)$ since $(B, V_j(B; z_j))\, I_j\, z_j$.

· **The minimum price equilibrium (MPE) for general preferences**

We illustrate an MPE for three agents, $1, 2$, and 3, and two objects, A and B, and a preference profile where the (real and dotted) lines in purple describe R_1, lines in red describe R_2, and the line in black describes R_3.

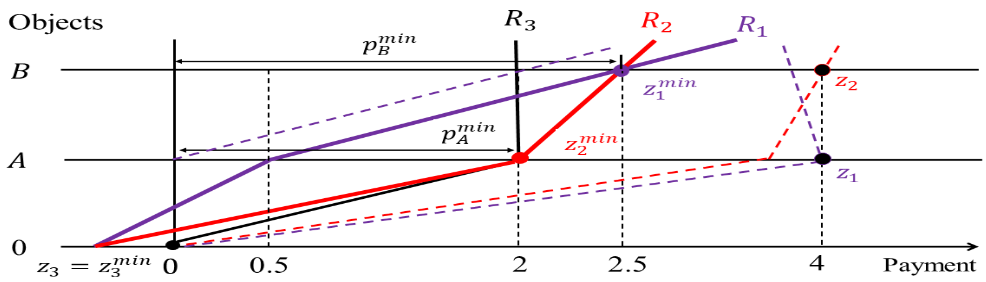

Figure 2: Illustration of an MPE $(z^{\min}, p^{\min})$ for general preferences

We show that $(z^{\min}, p^{\min})$ is an MPE. At $p^{\min} = (p_A^{\min}, p_B^{\min})$, $D_1(p^{\min}) = \{B\}$, $D_2(p^{\min}) = \{A, B\}$, and $D_3(p^{\min}) = \{0, A\}$ so each agent i receives an object from her demand set at $z^{\min}$ and no object is unassigned. Thus (E-i) and (E-ii) holds, and $(z^{\min}, p^{\min})$ is an equilibrium.

To see why $p^{\min}$ is an MPE price, let $p' = (p'_A, p'_B)$ be an equilibrium price. We show $p' \geq p^{\min}$. Since there are three agents and two objects, by (E-i), one agent must demand and be assigned object 0 at p'. Thus, $p'_A \geq 2 = p^{\min}_A$ or $p'_B \geq 2.5 = p^{\min}_B$. In case of $p'_A \geq 2$ and $p'_B < 2.5$, $D_1(p') = D_2(p') = \{B\}$, contradicting (E-i). In case of $p'_A < 2$ and $p'_B \geq 2.5$, $D_2(p') = D_3(p') = \{A\}$, contradicting (E-i). Thus $p'_A \geq p^{\min}_A$ and $p'_B \geq p^{\min}_B$.

For quasi-linear preferences, any equilibrium allocation is an MPE allocation, but this statement is not true for general preferences. Let $p = (4, 4)$. It is easy to see that (z, p) is an equilibrium where agent 1 gets A and 2 gets B. However, at $p^{\min}$, there is no equilibrium allocation such that 1 gets A and 2 gets B.

5 The structural characterizations

This section establishes structural characterizations of MPEs. The first result studies when an equilibrium coincides with an MPE (Proposition 1). The second and third results disclose a dynamic relation between an equilibrium and an MPE (Theorems 1 and 2). These characterizations are fundamental to design the mechanism that finds an MPE in Section 6.

5.1 Characterization by connectedness

First, we introduce two concepts: "connected object" and "connected agent."

Definition 2: An object $x \in M$ is **connected** at $(z, p) \in Z \times \mathbb{R}^m_+$ if (i) x is unassigned or (ii) there is a sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of Λ distinct agents ($\Lambda > 1$) that forms a **demand connectedness path (DCP)** such that
(ii-1) $x_{i_1} = 0$ or $p_{x_{i_1}} = 0$,
(ii-2) $x_{i_\Lambda} = x$,
(ii-3) for each $\lambda \in \{2, \cdots, \Lambda\}$, $x_{i_\lambda} \neq 0$ and $p_{x_{i_\lambda}} > 0$, and
(ii-4) for each $\lambda \in \{1, \cdots, \Lambda - 1\}$, $\{x_{i_\lambda}, x_{i_{\lambda+1}}\} \in D_{i_\lambda}(p)$.

Definition 2 says that an object x is connected in two cases. In Case (i), object x is unassigned. In Case (ii), there is a sequence of distinct agents such that the first agent receives an object with zero price (ii-1); the last agent receives object x (ii-2); each agent who is not the first agent gets an object with a positive price (ii-3); each agent who is not the last agent demands his assigned object and the object assigned to her successive agent (ii-4). These agents form a demand connected path to object x. Example 1 illustrates.

Example 1 (Figure 2): At $(z^{\min}, p^{\min})$, object B is connected by a DCP formed by agents 1, 2, and 3 where $\Lambda = 3$, $i_1 = 3$, $i_2 = 2$, and $i_3 = i_\Lambda = 1$: (i) $x^{\min}_3 = 0$,

(ii) $x_1^{\min} = B$, (iii) $x_2^{\min} = A$, $p_A^{\min} = 2$, and $x_1^{\min} = B$, $p_B^{\min} = 2.5$, and (iv) $\{0, A\} \in D_3(p^{\min})$ and $\{A, B\} \in D_2(p^{\min})$. (i) to (iv) corresponds to (ii-1) to (ii-4) in Definition 2.

Definition 3: An agent $i \in N$ is **connected** at $(z, p) \in Z \times \mathbb{R}_+^m$ if x_i is connected or $x_i = 0$.

Definition 3 says that an agent is connected if she gets either a connected object or object 0. Let N_C and M_C be the sets of connected agents and objects. Let $N_U \equiv N \backslash N_C$ and $M_U \equiv M \backslash M_C$ be the sets of unconnected agents and objects. Example 2 illustrates.

Example 2 (Figure 2): At the MPE $(z^{\min}, p^{\min})$, $N = N_C = \{1, 2, 3\}$ and $M = M_C = \{A, B\}$. Let $p = (4, 4)$. At the equilibrium (z, p), $N_C = \{3\}$, $N_U = \{1, 2\}$, and $M = M_U = \{A, B\}$.

We give a further remark of connected objects and agents.

Remark 1: (i) Since unassigned objects are connected, the existence of connected objects does not imply the existence of connected agents.
(ii) The connectedness of objects is not applicable to object 0. Thus the existence of connected agents does not imply the existence of connected objects.
(iii) By Definition 2(ii-1), an agent who gets an object with zero price is connected.
(iv) A connected agent exists $\iff$ Some agent gets an object with zero price.[6]

Proposition 1 characterizes the MPE by connected agents and objects.

Proposition 1: Let (z, p) be an equilibrium, and N_C and M_C be defined at (z, p). Then, the following are equivalent:
(i) $p = p^{\min}$, (ii) $N = N_C$, and (iii) $M = M_C$.

The proof of Proposition 1 is relegated to Appendix A.1. Proposition 1 gives three equivalent conditions to judge whether an equilibrium is an MPE, i.e., (i) the equilibrium price is an MPE price, or (ii) all the agents are connected, or (iii) all the objects are connected. Example 2 also illustrates Proposition 1.

5.2 Characterizations by I-pay-others'-indifference-prices process

In this section, we relate an MPE to an arbitrary equilibrium via a dynamic process. First, we introduce the notation of "the ranking of indifference prices of some object." Consider an object x, an allocation z and a group N' of agents.

[6] "$\Leftarrow$" comes from Remark 1(iii). For "$\Rightarrow$", by contradiction, suppose that for each $i \in N$, $p_{x_i} > 0$. Since $N_C \neq \emptyset$, for each $i \in N_C$, $p_{x_i} > 0$. Thus for an arbitrary connected agent i, there is a sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of distinct agents satisfying Definition 2, with $x_{i_1} = 0$ or $p_{x_{i_1}} = 0$, contradicting that for each $i \in N$, $p_{x_i} > 0$.

We ask each agent $i \in N'$ to report her IP $V_i(x; z_i)$ of x at z_i. Let $C^h(x; z_{N'})$ be the **h-th highest IP of x from z among N' agents**. Let $C^h_+(x; z_{N'}) \equiv \max\{C^h(x; z_{N'}), 0\}$. If $N' = N$, we write z instead of z_N. In case of $N' = \emptyset$, let $C^h_+(x; z_{N'}) = 0$. Example 3 illustrates.

Example 3 (Figure 2): Let $N' = \{1, 3\}$, $x = A$, and $z = z^{\min}$. Since $V_3(A; z_3^{\min}) = 2 > 0.5 = V_1(A; z_1^{\min}) > 0$, then $C^2_+(A; z^{\min}_{\{1,3\}}) = C^2(A; z^{\min}_{\{1,3\}}) = V_1(A; z_1^{\min}) = 0.5$.

Lemma 1 shows what properties of an equilibrium can preserve at an MPE.

Lemma 1: Let (z, p) be an equilibrium and N_U and M_U be defined at (z, p). Let $(z^{\min}, p^{\min})$ be an MPE. Then,
(i) $|N_U| = |M_U|$,
(ii) for each $x \in M_U$, $C^1_+(x; z_{N_C}) \leq p^{\min}_x < p_x$, and
(iii) for each $i \in N_U$, $x_i^{\min} \in M_U$.

The proof of Lemma 1 is relegated to Appendix A.2. Lemma 1(i) and 1(iii) say that to obtain an MPE from an equilibrium, we only need to reallocate unconnected objects among unconnected agents. Lemma 1(ii) says that for each unconnected object x, its MPE price is upper bounded by its equilibrium price p_x and lower bounded by the maximum value $C^1_+(x; z_{N_C})$ of reported indifference prices of object x by connected agents at their equilibrium allocation z_{N_C}. Lemma 1(i) is confirmed by M_U and N_U at the equilibrium (z, p) in Example 2. Lemma 1(ii) and (iii) are confirmed by comparing $(z^{\min}, p^{\min})$ and (z, p) in Example 2.

Second, we define the "I-pay-others'-indifference-prices process." Let μ be an **assignment** or bijection, from M_U to N_U, and μ_i be the associated object assigned to agent $i \in N_U$. Let Ω be the set of all such assignments.

Definition 4: Let (z, p) be an equilibrium and N_U and M_U be defined at (z, p). Let $\mu \in \Omega$. The **$k-$I pay-others'-indifference-prices (IPOIP) process for μ** is defined as follows: For each $x \in M_U$ and each $i \in N_U$,
(i) $\overline{p}^0_x \equiv C^1_+(x; z_{N_C})$ and $\overline{z}^0_i \equiv (\mu_i, \overline{p}^0_{\mu_i})$, and
(ii) for each $s = 1, \cdots k$,

$$\overline{p}^s_x \equiv C^1(x; \overline{z}^{s-1}_{N_U}) \text{ and } \overline{z}^s_i \equiv (\mu_i, \overline{p}^s_{\mu_i}).$$

The $k-$IPOIP process for a given assignment μ contains k rounds. It updates the prices and agents' bundles recursively: First, set the starting price of each unconnected object x as $\overline{p}^0_x = C^1_+(x; z_{N_C})$. Each unconnected agent i is assigned an unconnected object μ_i and gets the bundle $\overline{z}^0_i = (\mu_i, \overline{p}^0_{\mu_i})$. Each unconnected agent i reports her IP of each unconnected object x at $\overline{z}^0_i$, i.e., $V_i(x; \overline{z}^0_i)$. Then the price of x is updated to the maximum value of all reported IPs, i.e., $\overline{p}^1_x = C^1_+(x; \overline{z}^0_{N_U})$. Each unconnected agent i keeps the same object μ_i but pays an updated price $\overline{p}^1_{\mu_i}$, i.e., $\overline{z}^1_i = (\mu_i, \overline{p}^1_{\mu_i})$. In the same manner, the price of each unconnected object x is updated to $\overline{p}^2_x = C^1_+(x; \overline{z}^1_{N_U})$ and so forth. The formation of agent's tentative payment $\overline{p}_x^s$ is in the spirit of the VCG payment. Example 4 illustrates a one-round IPOIP process.

Example 4 (Figure 2): In Example 2, recall $N_C = \{3\}$, $N_U = \{1,2\}$, and $M_U = \{A,B\}$ at the equilibrium (z,p). Consider an assignment μ such that agent 1 gets B and 2 gets A. In this case $\overline{p}_A^0 = C_+^1(A; z_{N_C}) = C_+^1(A; z_3) = 2$, $\overline{p}_B^0 = C_+^1(B; z_3) = 2$, $\overline{z}_1^0 = (B,2)$ and $\overline{z}_2^0 = (A,2)$. Then $\overline{p}_A^1 = C_+^1(x; \overline{z}_{\{1,2\}}^0) = \max\{V_1(A; \overline{z}_1^0); V_2(A; \overline{z}_2^0)\} = 2$ and $\overline{p}_B^1 = \max\{V_1(B; \overline{z}_1^0); V_2(B; \overline{z}_1^0)\} = 2.5$.

Let $\overline{p}^s(\mu)$ be the price generated in round s in a $k-$IPOIP process for μ. It is easy to see that prices in the process are non-decreasing. Formally, we have:

Fact 2: Let (z,p) be an equilibrium and N_U and M_U be defined at (z,p). Let $\mu \in \Omega$. In the $k-$IPOIP process for μ, for each $s = 1, \cdots, k$, we have $\overline{p}^0(\mu) \leq \overline{p}^1(\mu) \leq \cdots \leq \overline{p}^s(\mu)$.

Now we give the characterizations of MPEs in terms of IPOIP processes.

Theorem 1: Let (z,p) be an equilibrium, and N_C, M_C, N_U, and M_U be defined at (z,p). Let $p^{\min}$ be the MPE price. Then the following holds.
(i) For each $x \in M_C$, $p_x^{\min} = p_x$, and for each $i \in N_C$, z_i is an MPE bundle.
(ii) There is $\mu \in \Omega$ such that
(ii-1) μ is an MPE assignment among unconnected agents, and
(ii-2) for each $x \in M_U$, $p_x^{\min} = \overline{p}_x^{|M_U|-1}(\mu) = \min\limits_{\mu' \in \Omega} \overline{p}_x^{|M_U|-1}(\mu')$.

The proof of Theorem 1 is relegated to Appendix A.3. The proof sketch is as follows. First, we show Theorem 1(i). We argue that if the prices of connected objects decrease, these objects will be "overdemanded" at the MPE price. Thus, their MPE price should be the same as the given equilibrium prices. Then by Lemma 1(ii), connected agents could keep the same equilibrium bundles as their MPE bundles.

Second, we show Theorem 1(ii). For (ii-1), by Fact 1 and Theorem 1(i), there must exist an MPE assignment that assign the unconnected objects to the unconnected agents. For (ii-2), it contains two Steps.

Step 1 shows that if μ is an MPE assignment, then at each round of the IPOIP process, at least one unconnected object's price reaches its MPE price, and it never increases in the later round. Since there are $|M_U|$ unconnected objects, starting from round 0, the IPOIP process for μ finds the MPE price for unconnected objects at most by $|M_U| - 1$ rounds. Thus, for each unconnected object x, $p_x^{\min} = \overline{p}_x^{|M_U|-1}(\mu)$. Step 2 shows that for any non-MPE assignment μ', at each round of the IPOIP process, at least one unconnected object's price exceeds its MPE price and by Fact 2, it never decreases in the later round. Since there are $|M_U|$ unconnected objects, for each unconnected object x, $p_x^{\min} \leq \overline{p}_x^{|M_U|-1}(\mu')$.

If all the objects are connected at the given equilibrium, then Theorem 1 coincides with Proposition 1. The novelty of Theorem 1 deals with the case where there are unconnected agents at the given equilibrium.

Remark 2: Given $\mu' \in \Omega$, $\overline{p}^{|M_U|-1}(\mu')$ is generally not an equilibrium price for unconnected objects. Thus, Theorem 1(ii-2) is different from the meet operation of equilibrium prices that generates a new equilibrium price, by their lattice property.

Intuitively, if μ is an MPE assignment and $\overline{p}^s$ obtained at some round s in the $k-$IPOIP process for μ is an MPE price, then in the later rounds, the price remains unchanged, i.e., $\overline{p}^s = \cdots = \overline{p}^k$. Theorem 2 surprisingly shows that for any assignment μ, if the prices in two adjacent rounds $s-1$ and s of the IPOIP process for μ remain unchanged, i.e., $\overline{p}^{s-1} = \overline{p}^s$, then $\overline{p}^{s-1}$ is an MPE price and μ is an MPE assignment of unconnected objects and agents.

Theorem 2: Let (z, p) be an equilibrium, and N_U and M_U be defined at (z, p). Let $p^{\min}$ be an MPE price. Let $\mu \in \Omega$. In the $|M_U|-$IPOIP process for μ, the following are equivalent:
(i) There is some $s \leq |M_U|$ such that $\overline{p}^{s-1} = \overline{p}^s$;
(ii) μ is an MPE assignment and $\overline{p}^{s-1}$ is the MPE price of unconnected agents and objects.

The proof of Theorem 2 is relegated to Appendix A.4. The proof sketch is as follows. First we show (ii) $\Rightarrow$ (i). As argued in Step 1 of the proof Theorem 1(ii-2), when $\overline{p}^{s-1}$ is the MPE price of unconnected objects and μ is an MPE assignment of unconnected agents, $\overline{p}^{s-1}$ remain unchanged in the later round of the IPOIP process so $\overline{p}^{s-1} = \overline{p}^s$.

Next, we show (i) $\Rightarrow$ (ii). Since the outcome of the IPOIP process may not assign unconnected agents bundles in their demand sets, we define weak connected objects and agents by relaxing (ii-4) in Definition 2: Agents on the path are indifferent between their bundles and the successive agents' bundles. The proof contains five steps. Step 1 shows that all the unconnected objects are weakly connected at $(\mu, \overline{p}^{s-1})$. Steps 2 and 3 show that the price of unconnected objects generated by the IPOIP process is bounded above by their MPE price. Step 4 shows that each unconnected agent i weakly prefer $(\mu_i, \overline{p}^{s-1}_{\mu_i})$ to the bundles consisting of connected objects paired with their MPE price. Step 5 concludes that (ii) holds.

In Example 4, for the assignment μ where agent 1 gets B and 2 gets A, it is easy to see $\overline{p}^1 = \overline{p}^2$. By Theorem 2, $\overline{p}^1 = p^{\min}$ and μ is an MPE assignment.

6 Serial Vickrey mechanisms

6.1 Sketch of Serial Vickrey mechanisms

Based on the obtained structural characterizations in Section 5, we design a "**Serial Vickrey (SV) mechanism**," that finds an MPE in a finite number of steps. The SV mechanism introduces objects one by one and sequentially employs "**SV sub-mechanism**," to find an MPE for $k+1$ objects, based on an MPE for k objects.

Definition 5: The **SV mechanism** is defined as follows. Each agent is initially assigned $(0, 0)$. Introduce object sequentially by its index, $1, 2, \cdots$.

Step $k(\geq 1)$ introduces object k, and run the SV sub-mechanism for k objects, which is described in Definition 6. If $k = m$, stop at the output of the SV sub-mechansim. Otherwise, go to **Step** $k + 1$.

Remark 3: By Fact 1, the MPE price for the assignment market (N, M, R) is unique so how we index the objects in M does not matter in the SV mechanism.

The central part of the SV mechanism is the SV sub-mechanism. When the first object is introduced, it coincides with the second-price auction mechanism. Now we define the SV sub-mechanism for $k + 1$ objects where $0 \leq k < m$.

Definition 6: Let $(z^{\min}, p^{\min})$ be an MPE for k objects. The **SV sub-mechanism for** $k + 1$ **objects** is defined as follows.

Stage 1: Operate the "**E-generating mechanism**," (Definition 8, Section 6.2), to generate an equilibrium (z, p) for $k + 1$ objects from $(z^{\min}, p^{\min})$. Then run the "N_C**-identifying mechanism,**" (Definition 9, Section 6.2), to identify the set of connected agents N_C at (z, p). If all agents are connected, i.e., $N_C = N$, then terminate at (z, p). Otherwise, identify M_U and N_U, and go to Stage 2.

Stage 2: Operate the "**MPE-adjustment mechanism**," (Definition 11, Section 6.3), to identify the MPE allocation of unconnected agents N_U.

In the SV sub-mechanism, agents report finite-dimensional prices composed of indifference prices in finitely many times. We will establish:

Theorem 3: Given an MPE for k objects, the SV sub-mechanism finds an MPE for $k + 1$ objects via agents' reports of finite-dimensional prices in finitely many times.

We prove Theorem 3 via Propositions 2 and 3 in Section 6.2, and Proposition 4 in Section 6.3, which shows the properties of the E-generating mechanism, N_C-identifying mechanism, and the MPE-adjustment mechanism, respectively. As a direct outcome, we have:

Corollary 1: The SV mechanism finds an MPE in a finite number of steps via agents' reports of finite-dimensional prices in finitely many times.

6.2 Stage 1 of SV sub-mechanism

We first introduce a process to identify the demand connected path for a given object, which is key to establish the E-generating mechanism.

Definition 7: Demand-connectedness-path-finding (DCP-finding) process for object x Let $x \in M$ be a connected object that is assigned to agent i, i.e., $x_i = x$, at $(z, p) \in Z \times \mathbb{R}^m_+$.
Phase 1: *Round 1:* Set $N_1 \equiv \{i\}$.
If $p_x = 0$, stop the process.
If $p_x > 0$, set $N_2 \equiv \{j \in N \backslash N_1 : x \in D_j(p)\}$ and go to Round 2.
Round $s(\geq 2)$*:* Set $L_s \equiv \{y \in L : x_j = y \text{ for some } j \in N_s\}$.
If there is $y \in L_s$ s.t. $y = 0$ or $p_y = 0$, stop Phase 1 and go to Phase 2.
Otherwise, set $N_{s+1} \equiv \{j \in N \backslash \cup_{k=1}^s N_k : D_j(p) \cap L_s \neq \emptyset\}$ and go to Round $s+1$.[7]

Phase 2: Let S be the final round of the process. Then, construct a sequence $\{i_s\}_{s=1}^S$ of distinct agents as follows: (i) Choose $i_1 \in N_S$ such that $x_{i_1} = 0$ or $p_{x_{i_1}} = 0$, and (ii) for each $j \in \{2, \cdots, S\}$, choose $i_j \in N_{S+1-j}$ such that $x_{i_j} \in L_{S+1-j}$ and $x_{i_j} \in D_{i_{j-1}}(p)$.

The DCP-finding process works as follows: pick a connected object x that is assigned agent i. In Phase 1, if the price of x is zero, we are done and it contains a trivial DCP. If not, i.e., $p_x > 0$, we collect the demanders N_2 of x by excluding agent i. Since x is connected and $p_x > 0$, $N_2 \neq \emptyset$. If there is some agent in N_2, say, agent j, who obtains an object x_j with zero price, then agent j is connected to x by her demand, i.e., $\{x, x_j\} \in D_j(p)$ and we are done. Otherwise, we collect the set L_2 of objects assigned to agents in N_2, and repeat the process till some agent obtains an object with zero price. In Phase 2, we trace back from the agent who gets an object with zero price to object x via the DCP. Example 5 illustrates.

Example 5 (Figure 2): Consider a DCP-finding process for object B at $(z^{\min}, p^{\min})$. Recall that agent 1 gets B, i.e., $i = 1$ and $x_1 = B$.

In Phase 1, at Round 1, we set $N_1 = \{1\}$. Since $p_B^{\min} > 0$, we need to find all the agents who demand B, except for agent 1. It is just the agent 2 so $N_2 = \{2\}$. Then we come to Round 2 and collect agent $2's$ assigned object, i.e., $L_2 = \{A\}$. Since $p_A^{\min} > 0$, we need to find all the agents who demand A, except for agents 1 and 2. It is just the agent 3 so $N_3 = \{3\}$. Since $x_3^{\min} = 0$. We stop Phase 1 and go to Phase 2.

In this case $S = 3$. Phase 2 constructs a DCP consisting of a sequence of agents $\{3, 2, 1\}$, i.e., $i_1 = 3$, $i_2 = 2$, and $i_3 (= i_S) = 1$.

The following result summarizes the property of DCP-finding process.

Lemma 2: Let $(z, p) \in Z \times \mathbb{R}^m_+$. Let $x \in M$ be an assigned connected object.
(i) Phase 1 of DCP-finding process stops in a finite number of rounds.
(ii) The sequence $\{i_s\}_{s=1}^S$ of distinct agents of Phase 2 is a DCP for object x.

By the finiteness of N, Lemma 2(i) holds. By construction, the sequence $\{i_s\}_{s=1}^S$ in Lemma 2(ii) satisfies (ii-1) to (ii-4) in Definition 2. The demand sets

[7] In such a case, since for each $y \in L_s$, y is connected and $p_y > 0$, $N_{s+1} \neq \emptyset$.

used to identify the DCPs can be derived from the agents' reported indifference prices at the given allocation.

Now we are ready to propose the E-generating mechanism.

Definition 8: **E-generating mechanism** Let $(z^{\min}, p^{\min})$ be an MPE of k objects. Introduce a new object y, i.e., the object indexed by number $k+1$.

Phase 1: Each agent i reports $V_i(y; z_i^{\min})$, and compute $C^1(y; z^{\min})$.

If $C^1(R, y; z^{\min}) \leq 0$, set (z, p) as

(a) $p_y = 0$, and $p = (p^{\min}, p_y)$, and (b) for each $i \in N$, $z_i = z_i^{\min}$.

Otherwise, go to Phase 2.

Phase 2: Arbitrarily select an agent i with the highest IP of y at $z_i^{\min}$, i.e., $V_i(y; z_i^{\min}) = C^1(y; z^{\min})$. Run the DCP-finding process for $x_i^{\min}$ at $(z^{\min}, p^{\min})$, and obtain a DCP $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ for $x_i^{\min}$. Set (z, p) as

(a) $p_y = C_+^2(y; z^{\min})$, and $p = (p^{\min}, p_y)$,

(b) $z_{i_\Lambda} = (y, p_y)$,

(c) for each $i_l \in \{i_\lambda\}_1^{\Lambda-1}$, $z_{i_l} = z_{i_{l+1}}^{\min}$, and

(d) for each $j \in N \backslash \{i_\lambda\}_1^{\Lambda}$, $z_j = z_j^{\min}$.

The essence of E-generating mechanism is Phase 2. It works as follows. When object y is introduced, each agent reports her IP of y at $z_i^{\min}$, i.e., $V_i(y; z_i^{\min})$. We assign the new object to an arbitrary agent i whose has the highest IP, but ask her to pay the second highest IP (in the spirit of the second-price auction). We identify a DCP for agent $i's$ assignment $x_i^{\min}$ at $(z^{\min}, p^{\min})$. After agent i is assigned object y, we alternate the bundles of agents on the identified DCP from the object with zero price. All other agents remain their bundles at $z^{\min}$. Such an construction generates an equilibrium for $k+1$ objects. Example 6 illustrates.

Example 6 (Figures 2 and 3): Given $(z^{\min}, p^{\min})$ in Figure 2**,** we introduce object C and illustrate the E-generating mechanism by Figure 3.

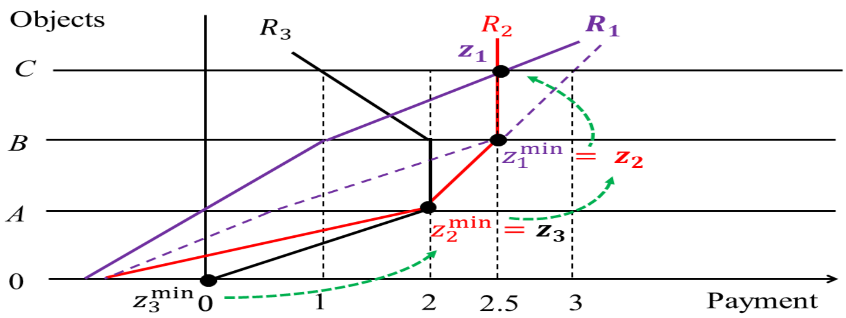

Figure 3: Illustration of an E-generating mechanism

In Phase 1, each agent i reports her IP for C at $z_i^{\min}$: $V_1(C; z_1^{\min}) = 3$, $V_2(C; z_2^{\min}) = 2.5$, and $V_3(C; (0,0)) = 1$. Since $C^1(C; z^{\min}) = V_1(C; z_1^{\min}) > 0$, go to Phase 2.

In Phase 2, we set $p_C = C_+^2(C; z^{\min}) = 2.5$ and $p = (p_A^{\min}, p_B^{\min}, p_C)$. We assign object C to agent 1, but ask her to pay $C_+^2(C; z^{\min})$. Thus, agent 1 is assigned $(C, 2.5)$. Recall Example 2 for the DCP of object B, which consists of agents 3, 2, and 1. We alternate the bundles of agents 3 and 2 along this path: $z_3 = z_2^{\min} = (A, 2)$ and $z_2 = z_1^{\min} = (B, 2.5)$.

In Example 6, the constructed (z, p) is an equilibrium. Formally, we can show:.

Proposition 2 (Property of E-generating mechanism): Let $(z^{\min}, p^{\min})$ be an MPE for k objects. Introduce object y. Then the E-generating mechanism finds an equilibrium (z, p) for $k+1$ objects in a finite number of phases.

The proof of Proposition 2 is relegated to Appendix B.1. The following mechanism identifies the set of connected agents N_C at some given equilibrium.

Definition 9: **N_C-identifying mechanism** Let (z, p) be an equilibrium.

Round 1: Let $N_1 \equiv \{i \in N : p_{x_i} = 0\}$. If $N_1 = \emptyset$, set $N^* = \emptyset$ and stop the process. Otherwise, go to Round 2.

Round $s(\geq 2)$: Let

$$M^{s-1} \equiv \{y \in M \backslash \{x_i : i \in \cup_{k=1}^{s-1} N_k\} : p_y > 0,\ y \in D_i(p) \backslash \{x_i\} \text{ for some } i \in N_{s-1}\}.$$

If $M^{s-1} = \emptyset$, set $N^* = \cup_{k=1}^{s-1} N_k$ and stop the process.

Otherwise, let $N_s \equiv \{i \in N : x_i = y \text{ for some } y \in M^{t-1}\}$, and go to Round $s+1$.

In words, N_C-identifying mechanism works as follows: at the equilibirum (z, p), we collect a set N_1 of the agents who get objects with zero prices. Then we collect a set M^1 of objects in the demand sets of agents in N_1, except for their assigned objects. Since objects in M^1 have positive prices, they must be assigned to some agents. Then we identify the set N_2 of agents who are assigned objects from M^1, and repeat the process. The collection of the identified agents in N_1, N_2,..., are connected agents. Formally, we get the result below.

Proposition 3 (Property of N_C-identifying mechanism): Let (z, p) be an equilibrium. Then the N_C-identifying mechanism stops in a finite number of rounds and N^* is the set of connected agents at (z, p).

The proof of Proposition 3 is relegated to Appendix B.2. To confirm Proposition 3, in Figure 2, at (z, p), agent 3 is the only connected agent so $N_1 = N^* = \{3\}$. Once N_C is identified, M_C, N_U, and M_U can be immediately obtained.

6.3 Stage 2 of SV sub-mechanism

Theorems 1 and 2 imply that by conducting a IPOIP process for each assignment in Ω, we can find an MPE from an equilibrium. However, it is possible to find an MPE (i) without conducting IPOIP processes for all assignments in Ω, and (ii) while reducing the number of rounds during a IPOIP process. Such ideas are embodied into Stage 2 of SV sub-mechanism.

Theorem 1 and Lemma 1(ii) imply that some assignments can be disqualified as the MPE assignments before and after conducting the IPOIP process. They reduce the number of assignments needed to be operated via the IPOIP processes. As direct outcomes of Theorem 1 and Lemma 1(ii), we have:

Fact 3 (Disqualification before and after IPOIP process): Let (z, p) be an equilibrium. Let agent i and object x be unconnected at (z, p).
(i) If $V_i(x; z_i) \leq C^1_+(x; z_{N_C}) = \overline{p}^0_x$, then no MPE assignment gives x to i.
(ii) Let $\mu \in \Omega$ and $\overline{z}^{|M_U|-1}(\mu)$ be the outcome of the $(|M_U| - 1)-$IPOIP process for μ. If $V_i(x; \overline{z}^{|M_U|-1}_i(\mu)) < C^1_+(x; z_{N_C}) = \overline{p}^0_x$, no MPE assignment gives x to i.

The implication of Fact 3 is as follows. Given an equilibrium generated by Stage 1 of SV sub-mechanism, if it is not an MPE, we use Fact 3(i) to examine assignments in Ω. Let DQ_0 be the **set of initial disqualified assignments**.[8] We remove DQ_0 from Ω and $\Omega^{*0} = \Omega \backslash DQ_0$ be the initial candidate set.

After operating the IPOIP process for some μ, we obtain its outcome $\overline{z}^{|M_U|-1}(\mu)$. Using Fact 3(ii), we examine assignments in Ω again. Let $DQ(\mu)$ be the **set of disqualified assignments** after the IPOIP process for μ.[9] We remove $DQ(\mu)$ from Ω. Whenever we conduct IPOIP process for some μ, we remove $DQ(\mu)$ from the current candidate set of MPE assignments.

Example 7 illustrates Fact 3 by introducing agent 4 to Figure 3 in Example 6.

Example 7 (Figure 3): Suppose that $V_4(A; (0, 0)) = V_4(B; (0, 0)) = 1$, and $V_4(C; (0, 0)) = 2$. Let $z'_1 = z_1 = (C, 2.5)$, $z'_2 = z_2 = (B, 2.5)$, $z'_3 = z_3 = (A, 2)$, $z'_4 = (0, 0)$, and $p' = p$ where z and p are depicted in Example 6. It is easy to see that (z', p') is an equilibrium for the assignment market with agents $\{1, 2, 3, 4\}$ and objects $\{A, B, C\}$. At (z', p'), $N'_U = \{1, 2, 3\}$, $N'_C = \{4\}$, $M'_U = M$.

In this case, $\Omega = \{(A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B), (C, B, A)\}$. For each $x \in M'_U$, $C^1_+(x; z'_{N'_C}) = V_4(x; z'_4)$.

By $V_1(A; z'_1) < C^1_+(A; z'_{N_C})$, $V_1(B; z'_1) = C^1_+(B; z'_{N_C})$, and Fact 3(i), agent 1 never gets A or B at any MPE so $DQ_0 = \{\mu \in \Omega : \mu_1 = A, B\}$. We remove DQ_0 from Ω.

Let $\mu = (C, A, B)$ and assume $\overline{p}^{|M_U|-1}(\mu) = (2, 2, 2)$. Then, $\overline{z}^{|M_U|-1}_3(\mu) = (B, 2)$. By $V_3(C; \overline{z}^{|M_U|-1}_3(\mu)) = 1 < 2 = C^1_+(A; z'_{N_C})$ and Fact 3(ii), agent 3 never gets C at any MPE. Since $(B, A, C) \in DQ_0$ is removed, $DQ(\mu) = \{(A, B, C)\}$ is further removed from a candidate set after the IPOIP process for μ.

[8] $DQ_0 \equiv \{\mu \in \Omega : \exists i \in N_U, V_i(\mu_i; z) < \overline{p}^0_{\mu_i}\}$ where $\overline{p}^0_{\mu_i} = C^1_+(x; z_{N_C})$.

[9] $DQ(\mu) \equiv \{\mu' \in \Omega : \exists i \in N_U, \exists x \in M_U$ s.t. $\mu'_i = x$ and $V_i(x; \overline{z}^{|M_U|-1}_i(\mu)) < \overline{p}^0_x\}$ where $\overline{p}^0_x = C^1_+(x; z_{N_C})$

Before showing how to reduce the number of rounds during a IPOIP process, we introduce the following concept.

Definition 10: Let $\mu \in \Omega$ and $p' \in \mathbb{R}^{|M_U|}$. Assignment μ **survives** in the $k-$IPOIP process against p' if $\overline{p}^k(\mu) \leq p'$ and for some $x \in M_U$, $\overline{p}^k_x(\mu) = \overline{p}^0_x$.

In other words, an assignment μ does not survive in the $k-$IPOIP process against p' if there is some $x \in M_U$ whose price $\overline{p}^k(\mu)$ exceeds p'_x or the price $\overline{p}^k(\mu)$ is detached from the initial price $\overline{p}^0$ coordinate-wisely. By Lemma 1(ii) and Theorems 1 and 2, we have:

Fact 4 (Disqualification during IPOIP process): If an assignment $\mu \in \Omega$ does not survive in the $|M_U|-$IPOIP process against any equilibrium price p, then μ is not an MPE assignment.

Putting Facts 3 and 4 together, we present the following mechanism.

Definition 11: MPE-adjustment mechanism Let (z, p) be an equilibrium, and N_U and M_U be defined at (z, p).

Session 0*:* Identify Ω. Each unconnected agent i reports $V_i(x; z_i)$ of each unconnected object x. Set $\Omega^{*0} \equiv \Omega \backslash DQ_0$, $\mu^{*0} \equiv (x_i)_{i \in N_U}$, and $p^{*0} \equiv (p_x)_{x \in M_U}$. Then, go to Session 1.

Session $s(\geq 1)$: Choose $\mu_s \in \Omega^{*s}$ and conduct the $|M_U|-$IPOIP process for μ_s. Set $\Omega^{*1} \equiv \Omega^{*0}$, $p^{*1} = p^{*0}$, $\mu^{*1} = \mu^{*0}$, and choose $\mu_1 = \mu^{*0}$. Then, one of following three cases occurs:

Case 1 : If $\overline{p}^r(\mu_s) = \overline{p}^{r-1}(\mu_s)$ at round $r \leq |M_U|$, then stop the process at (z^*, p^*) such that $\mu^* \equiv \mu_s$ and $p^* \equiv \overline{p}^r(\mu_s)$.

Case 2: If Case 1 fails to hold, but μ_s survives the $|M_U|-$IPOIP process against p^{*s}, then set $\mu^{*s+1} \equiv \mu_s$ and $p^{*s+1} \equiv \overline{p}^{|M_U|-1}(\mu_s)$. Collect the disqualified set $DQ(\mu_s)$, and set $\Omega^{*s+1} \equiv \Omega^{*s} \backslash (DQ(\mu_s) \cup \{\mu_s\})$. Then, go to Session $s + 1$.

Case 3: If both Case 1 and Case 2 fail to hold, set $\mu^{*s+1} \equiv \mu^{*s}$, $p^{*s+1} \equiv p^{*s}$, and $\Omega^{*s+1} \equiv \Omega^{*s} \backslash \{\mu_s\}$. Then, go to Session $s + 1$.

The MPE-adjustment mechanism works as follows. Given an equilibrium (z, p), we identify the unconnected agents N_U and objects M_U. In Session 0, each unconnected agent i reports her IP $V_i(x; z_i)$ at z_i to each unconnected object x. Using Fact 3(i), we obtain the initially candidate set of assignments Ω^{*0}. Then we set the equilibrium assignment and price at (z, p) as the initial reference assignment and price, i.e., $\mu^{*0} \equiv (x_i)_{i \in N_U}$ and $p^{*0} \equiv (p_x)_{x \in M_U}$.

In Session 1, set $\Omega^{*1} \equiv \Omega^{*0}$, $\mu^{*1} = \mu^{*0}$, and $p^{*1} = p^{*0}$. We choose $\mu_1 = \mu^{*0}$ and run the $|M_U|-$IPOIP process for μ_1. We meet one of three cases. In Case 1, if at round $r \leq |M_U|$, the price remain the same as round $r - 1$, i.e., $\overline{p}^r(\mu_1) = \overline{p}^{r-1}(\mu_1)$, by Theorem 2, μ_1 and $\overline{p}^r(\mu_1)$ are the MPE for unconnected agents and objects. Case 2 deals with the situation where Case 1 fails to hold, but μ_1 survives against p^{*1}. Then we update the reference assignment and price by $\mu^{*2} \equiv \mu_1$ and $p^{*2} \equiv$

$\overline{p}^{|M_U|-1}(\mu_1)$. Using Fact 3(ii), we collect the disqualified assignments $DQ(\mu_1)$ for μ_1 and remove $DQ(\mu_1)$ and μ_1 from Ω^{*1}. Set $\Omega^{*2} \equiv \Omega^{*1} \backslash (DQ(\mu_1) \cup \{\mu_1\})$ and proceed to Session 2. If neither Case 1 nor Case 2 holds, using Fact 4, we keep the reference assignment and price unchanged. We just remove μ_1 from Ω^{*1} and set $\Omega^{*2} \equiv \Omega^{*1} \backslash \{\mu_1\}$. Then we proceed Session 2, and repeat above process. Formally, we have the following result.

Proposition 4 (**Property of MPE-adjustment mechanism**): Let (z, p) be an equilibrium. The MPE-adjustment mechanism generates a sequence $\{(\mu^{*t}, p^{*t}\}_{t=0}^{T}$ such that
(i) $T < +\infty$, and for each $t = 1, \cdots, T$, $p^{*t} \leq p^{*t-1}$.
(ii) μ^{*T} and p^{*T} are MPE assignment and price of unconnected agents and objects at (z, p).

The proof of Proposition 4 is relegated to Appendix B.3. Propositions 2, 3, and 4 together establish Theorem 3. Notably, if the assignment of a given equilibrium (z, p) is just an MPE assignment, then the MPE-adjustment mechanism stops at Session 1. This happens for certain classes of preferences detailed in Section 8.

7 Incentive compatibilities

In this section, we investigate the incentive properties of the SV mechanism. A *rule* f is a mapping from the set of general preference profiles $(\mathcal{R}^G)^n$ to the set of allocations Z. It assigns each agent i with a bundle $f_i(R)$ at each preference profile R. A rule f is *strategy-proof* if no agent can gain from misreporting her preference, i.e., for each $R \in (\mathcal{R}^G)^n$, each $i \in N$, and each $R'_i \in \mathcal{R}^G$, $f_i(R_i, R_{-i}) \, R_i \, f_i(R'_i, R_{-i})$.

In our model, the *MPE rule* that assigns each general preference profile an MPE is strategy-proof (Demange and Gale, 1985; Morimoto and Serizawa, 2015). Let f^{SV} be the rule that selects the outcome of the SV mechanism. By Theorem 3, f^{SV} coincides with the MPE rule.

Fact 5: The rule f^{SV} is strategy-proof on the set of general preference profiles.

Fact 5 says that in the normal game form induced by f^{SV} where agents' actions are revealing their preferences, truthfully revealing is a dominant-strategy equilibrium.

The SV mechanism is decomposed into m steps, i.e., m SV sub-mechanisms. Each SV sub-mechanism is decomposed into two stages, i.e., the E-generating and MPE-adjustment mechanisms. We show that these sub-mechanisms are also incentive compatible. In other words, even if agents are not fully rational and their perspectives are limited to the step or the stage where they are interacting, agents have incentives to reveal true preferences. Stage-wise and step-wise incentive compatibilities are remarkable properties of the SV mechanism.

We first study the cases in which agents' perspectives are limited to stages, and analyze the incentive properties of the E-generating and MPE-adjustment mechanisms. Note that the outcomes of these mechanisms depend both on the revealed preferences and allocations obtained from previous steps or stages. Thus, we need to introduce some notations.

For each $k \in \mathbb{N}_+$, let $Z^k \equiv \{z \in Z : \forall i \in N, x_i \in \{0, 1, \ldots, k\}\}$ be the set of allocations for k objects (together with object 0). If $k = m$, then $Z^k = Z$.

Given $k, k' \in \mathbb{N}_+$, an *augmented rule* from k to k', $g^{k \to k'} : (\mathcal{R}^G)^n \times Z^k \to Z^{k'}$, associates each preference profile R and an allocation z with k objects, to an allocation $g^{k \to k'}(R, z)$ with k' objects. Let $g_i^{k \to k'}(\cdot; \cdot)$ be the bundle assigned to agent i. Given an allocation $z \in Z^k$, $g^{k \to k'}(\cdot; z)$ induces a normal game form. If all agents' actions are revealing their preferences, $(g^{k \to k'}, R)$ forms a revelation game.

Definition 12: Let $k, k' \in \mathbb{N}_+$, $R \in (\mathcal{R}^G)^n$, and $z \in Z^k$. Revealing R is a (weakly) **dominant-strategy from** z in $g^{k \to k'}(\cdot; z)$ if for each $i \in N$, each $R'_i \in \mathcal{R}^G$, and each $R_{-i} \in (\mathcal{R}^G)^{n-1}$, $g_i^{k \to k'}(R_i, R_{-i}; z) \, R_i \, g_i^{k \to k'}(R'_i, R_{-i}; z)$.

Consider the case of $k = 0$ and $k' = |M|$. Then $Z^0 = \{\mathbf{0}\}$ where $\mathbf{0} \equiv (0, 0)_{i \in N}$. Let $g^{0 \to |M|}(\cdot; \mathbf{0}) = f^{SV}(\cdot)$. Then revealing R is a dominant-strategy equilibrium from $\mathbf{0}$ in $g^{0 \to |M|}(\cdot; \mathbf{0})$ if and only if $f^{SV}(\cdot)$ is strategy-proof.

Let $g^{sub1}(\cdot; \cdot)$ be the augmented rule from k to $k + 1$ such that if $z \in Z^k$ is an MPE allocation for R, $g^{sub1}(R; z)$ selects the outcome of the E-generating mechanism in the SV sub-mechanism for $k + 1$ objects. Notice that $g^{sub1}(\cdot; z)$ depends on both the preferences revealed in the E-generating mechanism and recorded DCPs at $z \in Z^k$, which depends on the preferences revealed in Step k of the SV mechanism.

Let $g^{sub2}(\cdot; \cdot)$ be the augmented rule from $k + 1$ to $k + 1$ such that if $z \in Z^{k+1}$ is an equilibrium allocation for R, $g^{sub2}(R; z)$ selects the outcome of the MPE-adjustment mechanism in the SV sub-mechanism for $k + 1$ objects. Notice that $g^{sub2}(\cdot; z)$ depends both on the preferences revealed in the MPE-adjustment mechanism and the recorded N_C, N_U, M_C and M_U at $z \in Z^{k+1}$, which depends on the preferences revealed in Stage 1 in the SV sub-mechanism for $k + 1$ objects

Proposition 5: Let $0 \leq k < |M|$ and $R \in (\mathcal{R}^G)^n$.
(i) For each $z \in Z^k$, if z is an MPE allocation for R, then revealing R is a dominant-strategy equilibrium from z in $g^{sub1}(\cdot; z)$.
(ii) For each $z \in Z^{k+1}$, if z is an equilibrium allocation for R, then revealing R is a dominant-strategy equilibrium from z in $g^{sub2}(\cdot; z)$.

The proof of Proposition 5 is relegated to Appendix C.1. Proposition 5 states "stage-wise strategy-proofness" of the SV sub-mechanism. In Stage 1, when a new object is introduced, no agent can gain by misreporting. In Stage 2, at the equilibrium obtained in stage 1, no agent can gain by misreporting. Thus, even if agents' perspectives are myopic and limited to the stages where they are currently interacting, agents have incentives to reveal their true preferences.

Finally, we investigate the cases in which agents' perspectives are beyond stages but limited to steps, and show that the SV sub-mechanism, the composition of above two mechanisms, is also incentive compatible. Let $g^{sub}(\cdot;\cdot)$ be the augmented rule from k to $k+1$ such that if $z \in Z^k$ is an MPE allocation for R, $g^{sub}(R;z)$ coincides with the outcome of the SV sub-mechanism for $k+1$ objects. Notice that $g^{sub}(\cdot;z)$ depends on both the preferences revealed in the SV sub-mechanism for $k+1$ objects, and the recorded information on DCPs at $z \in Z^k$ which depends on the preferences revealed in Step k in the SV mechanism.

Theorem 4: Let $0 \leq k < |M|$ and $R \in (\mathcal{R}^G)^n$. For each $z \in Z^k$, if z is an MPE allocation for R, then revealing R is a dominant-strategy equilibrium from z in $g^{sub}(\cdot;z)$.

The proof of Theorem 4 is relegated to Appendix C.2. Theorem 4 states "step-wise strategy-proofness" of the SV sub-mechanism. Theorem 4 is not implied by Fact 5, since at each step, the SV sub-mechanism involves a different number of objects. As discussed below, it is not implied Proposition 5 either.

To see how Theorem 4 works, we discuss its sketch proof idea. We fix an agent i and show that she cannot benefit from misreporting. We first give some facts and then construct three lemmas, as the preliminary results. Fact C.1 extends Fact 5 by introducing reserve price. Fact C.5 shows that given an MPE, the MPE allocation of any subset of agents that includes all unconnected agents, is an MPE with reserve price of the subeconomy that excludes the remaining agents and sets the reserve price of each object as the maximum value of all reported IPs by those excluded agents at the given MPE. Facts C.2 to C.4, Fact C.6, and three lemmas show how prices are adjusted and objects are reallocated, and identify the set of agents and objects that preserve the connectedness or unconnectedness properties at the outcome of Stage 1, when agent i misreports.

We complete the proof by considering two cases. In Case I, agent i is connected (at the outcome of Stage 1) under truthful reporting. We show that agent i is always connected even if she misreports. By Proposition 5(i), this implies that agent i never gains by misreporting in Stage 1. Thus, by Theorem 1(i), she cannot gain from misreporting.

In Case II, agent i is unconnected under truthful reporting. By misreporting, agent i can indirectly influence the outcome of Stage 2 by making the set of unconnected agents unchanged, expanded or shrunk at the outcomes of Stage 1. Such impacts invalidate Proposition 5(ii). We develop new techniques that employ Fact C.5 so that Fact C.1 implies that agent i cannot gain from misreporting.

We end this section by discussing the welfare property of the SV mechanism. First, agents' welfare is non-decreasing with the number of introduced objects in the SV mechanism. This result follows Proposition 4 and the fact that agents are better off at the MPEs with a larger number of objects (Demange and Gale, 1985). Second, agents' welfare is non-decreasing stage-wise within the SV sub-mechanism. By construction, agents' welfare at the outcome of Stage 1 is bounded below by that at the given MPE. By Lemma 1, agents' welfare at the outcome of Stage 2 is bounded below by that at the equilibrium obtained in Stage 1.

8 Concluding remarks

We conclude by providing some further discussions related to the applications, generality, and dynamic incentive property of the SV mechanism.

• **Application of the SV mechanism to restricted preference settings**

Quasi-linearity is widely assumed in the multi-object auction models such as Demange et al. (1986), Mishra and Parkes (2010), and Andersson and Erlanson (2013). Assuming quasi-linearity simplifies Stage 2 of the SV sub-mechanism. Any equilibrium assignment is an MPE assignment of unconnected objects, i.e., μ^{*0} is just an MPE assignment so the MPE-adjustment mechanism stops only after Session 1. Different from the above-mentioned three papers, the SV mechanism does not require that agents have integer valuations and the increment/decrement is unitary in the auctions. In contrast, agents are allowed to have arbitrary real numbers as their valuations of objects.

The Alonso-type (discrete) housing market is well-studied in the urban economics. In such a market, houses are identical but different only in locations. Agents have the same utility functions and at each payment, agents commonly prefer houses with shorter distance to the city center than those with longer distance. Agents are distinguished only by their incomes and consuming houses exhibits positive income effects. In this model, all the equilibrium assignment is positively assortative, i.e., agents with higher incomes obtain houses nearer to the city center (Zhou and Serizawa, 2018). Such an equilibrium property simplifies the SV sub-mechanism in the same way as the quasi-linearity.

However, there is no such simplification when we apply the structural results in Caplin and Leahy (2014) and Alaei et al. (2016) to find an MPE in above restricted settings.

• **The assignment market without outside option**

In the assignment market without outside option, there is no object 0 and each agent is exactly assigned one object ($|N| \leq |M|$). This model is also called the task-assignment model analyzed by, e.g., Sun and Yang (2003), Andersson (2007), and Noldeke and Samuelson (2018).

In such settings, equilibria and MPEs are defined as "envy-free allocations," and "fair and optimal allocations" (Sun and Yang, 2003). Theorems 1 and 2, together with MPE-adjustment mechanism, identify a fair and optimal allocation.

Andersson (2007) provides a continuous-time auction for the fair and optimal allocation for quasi-linear preferences. However, when preferences are general, the continuous-time auction may not be well defined (Crawford and Knoer, 1981). We remark that the structural result in Alaei et al. (2016) to find an MPE depends on the existence of object 0, so it fails to identify the fair and optimal allocation.

• **Alternative incentive compatibility concepts**

Li (2017) introduces a notion of "obvious strategy-proofness," which requires that in the game form induced by the mechanism, along the equilibrium path, by comparing the maximum payoff among all deviations with the minimum payoff by following the truth-telling strategy, no agent have incentive to deviate. Obvious strategy-proofness implies that even if agents' cognitive powers or perspectives are limited, they have incentives to reveal their true preferences. Li (2017) demonstrates in an auction model with one object that the ascending auction is obvious strategy-proof. However, nontrivial obvious strategy-proof mechanisms exist only in limited environments. For example, Ashlagi and Gonczarowski (2018) show that in matching models, no stable matching mechanisms is obviously strategy-proof for any side of the market. The SV mechanism is not obvious strategy-proof, but still incentivizes agents with limited perspectives to reveal their true preferences (step-wise and stage-wise incentive strategy-proofness).

Ausubel (2006) and Sun and Yang (2014) study the auction mechanisms for heterogenous objects where agents have multi-unit demand quasi-linear preferences. They show that sincere bidding forms an ex-post perfect equilibria (EXPE) in the game form induced by their mechanisms. EXPE is stronger than step-wise and stage-wise strategy-proofness in that it is off-path incentive compatible. On the other hand, step-wise and stage-wise strategy-proofness is stronger than EXPE in that it gives on-path dominant strategy of agents. Thus, our incentive notions are independent of EXPE.

There are two promising venues for future research. The first is to study the empirical implications of SV mechanism. The techniques developed by Galichon et al. (2019) may help. The second is to extend the insights of the current structural characterizations and SV mechanism to more general models such as the exchange economy or the trading networks.

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Appendix

Part A gives the proofs of results in Section 5. Part B gives the proofs of results in Section 6. Part C gives the proofs of results in Section 7.

Appendix A: Proofs of Proposition 1, Lemma 1, Theorems 1 and 2

Definition A.1: (i) A non-empty set $M' \subseteq M$ of objects is **overdemanded at** p if $|\{i \in N : D_i(p) \subseteq M'\}| > |M'|$.
(ii) A non-empty set $M' \subseteq M$ of objects is (**weakly**) **underdemanded at** p if

$$[\forall x \in M', \, p_x > 0] \Rightarrow |\{i \in N : D_i(p) \cap M' \neq \emptyset\}| \, (\leq) < |M'| \, .$$

Fact A.1 (Mishra and Talman, 2010; Morimoto and Serizawa, 2015). p is an equilibrium price vector $\Longleftrightarrow$ no set is overdemanded and no set is underdemanded at p.

Fact A.2 (Alkan and Gale, 1990; Morimoto and Serizawa, 2015). p is an MPE price $\Longleftrightarrow$ no set is overdemanded and no set is weakly underdemanded at p.

Fact A.3: Let $(z, p) \in W$ and M_C be defined at (z, p). Let $M' \subseteq M_C$ be such that $M' \neq \emptyset$ and for each $x \in M'$, $p_x > 0$. Then, $|\{i \in N : D_i(p) \cap M' \neq \emptyset\}| > |M'|$.

Proof: Since $(z, p) \in W$, and for each $x \in M'$, $p_x > 0$, then by Fact A.1, $|\{i \in N : D_i(p) \cap M' \neq \emptyset\}| \geq |M'|$. To show ">", we proceed by contradiction. Suppose that $|\{i \in N : D_i(p) \cap M' \neq \emptyset\}| = |M'|$. Then, by $M' \subseteq M_C$,

$$\text{for each } i \in N \text{ such that } D_i(p) \cap M' \neq \emptyset,\ x_i \in M', \text{ and } i \in N_C. \qquad (*)$$

Let $i \in N$ such that $x_i \in M'$. Then by $(*)$, $i \in N_C$. By $x_i \in M'$, $p_{x_i} > 0$. By Definition 2, there is a sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of Λ distinct agents such that
(a) $x_{i_1} = 0$ or $p_{x_{i_1}} = 0$,
(b) for each $\lambda \in \{2, \cdots, \Lambda\}$, $x_{i_\lambda} \neq 0$ and $p_{x_{i_\lambda}} > 0$,
(c) $x_{i_\Lambda} = x_i$, and
(d) for each $\lambda \in \{1, \cdots, \Lambda - 1\}$, $\{x_{i_\lambda}, x_{i_{\lambda+1}}\} \in D_{i_\lambda}(p)$.

Claim: Let $l = 1, \cdots, \Lambda - 1$ and $N(l) \equiv \{i_{\Lambda-1}, \cdots, i_{\Lambda-l}\}$. Then, for each $j \in N(l)$, $x_j \in M'$.

Step 1: The Claim holds for $l = 1$.

By (c), $x_{i_\Lambda} = x_i \in M'$. By (d), $D_{i_{\Lambda-1}}(p) \cap M' \neq \emptyset$. Thus by $(*)$, $x_{i_{\Lambda-1}} \in M'$.

Induction hypothesis: The Claim holds for s such that $1 \leq s < \Lambda - 1$.

Step 2: The Claim holds for $l = s + 1$.

By induction hypothesis, $x_{i_{\Lambda-s}} \in M'$. By (d), $x_{i_{\Lambda-s}} \in D_{i_{\Lambda-(s+1)}}(p)$. Thus $D_{i_{\Lambda-(s+1)}}(p) \cap M' \neq \emptyset$. Thus by $(*)$, $x_{i_{\Lambda-(s+1)}} \in M'$.

Let $l = \Lambda - 1$. The above Claim implies that for each $j \in \{i_1, \cdots, i_{\Lambda-1}\}$, $x_j \in M'$. If $|\Lambda| > |M'|$, then the feasibility condition is violated. If $|\Lambda| \leq |M'|$, then by $x_{i_1} \in M'$, $p_{x_{i_1}} > 0$ and (a) is violated. Thus $|\{i \in N : D_i(p) \cap M' \neq \emptyset\}| = |M'|$ does not hold. **Q.E.D.**

Given $p \in \mathbb{R}^m_+$ and $M' \subseteq M$, let $p_{M'} \equiv (p_x)_{x \in M'}$ and $p_{M \setminus M'} \equiv (p_x)_{x \in M \setminus M'}$.

A.1 Proof of Proposition 1

Proof: We prove Proposition 1 by showing (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(i).

Step 1: (i)$\Longrightarrow$(ii), i.e., $p = p^{\min} \Longrightarrow N = N_C$

Obviously $N_C \subseteq N$. For each $i \in N$, if $p_{x_i} = 0$, by Definition 3 and Remark 1(iii), x_i is connected so $i \in N_C$. If $p_{x_i} > 0$, by Corollary 2 in Morimoto and Serizawa (2015) and Definition 3, $i \in N_C$. Thus $N \subseteq N_C$. Thus $N = N_C$.

Step 2: (ii)$\Longrightarrow$(iii), i.e., $N = N_C \Longrightarrow M = M_C$.

Obviously $M_C \subseteq M$. For each $x \in M$, if x is assigned, by $N = N_C$ and Definition 3, $x \in M_C$. If x is unassigned, by Definition 2, $x \in M_C$. Thus $M \subseteq M_C$. Thus $M = M_C$.

Step 3: (iii)$\Longrightarrow$(i), i.e., $M = M_C \Longrightarrow p = p^{\min}$.

Since $(z,p) \in W$, then $p \geq p^{\min}$. To prove $p = p^{\min}$, by contradiction, suppose that there is a non-empty set $M' \subseteq M$ such that for each $x \in M'$, $p_x > p_x^{\min} \geq 0$. Since $M = M_C$, then $M' \subseteq M^C$. Since $(z,p) \in W$, by Fact A.3, $|\{i \in N : D_i(p) \cap M' \neq \emptyset\}| > |M'|$. Thus, for each $i \in N$ such that $D_i(p) \cap M' \neq \emptyset$, by $p_{M'}^{\min} < p_{M'}$, $D_i(p^{\min}) \subseteq M'$. Thus

$$\left|\{i \in N : D_i(p^{\min}) \subseteq M'\}\right| > |M'|.$$

Thus M' is overdemanded at $p^{\min}$, violating Fact A.2. Thus $p = p^{\min}$. **Q.E.D.**

A.2 Proof of Lemma 1

Proof: **Part (i):** By Definition 2, for each $x \in M_U$, $p_x > 0$. By $(z,p) \in W$, for each $x \in M_U$, there is $i \in N$ such that $x_i = x$. By Definition 3, $i \in N \backslash N_C = N_U$. Thus $|M_U| \leq |N_U|$.

If there is $i \in N_U$ such that $x_i = 0$, then by Definition 3 and Remark 1(iii), $i \in N_C$, a contradiction. Thus, for each $i \in N_U$, $x_i \in M$, and by Definition 2, $x_i \notin M_C$. Thus $x_i \in M \backslash M_C = M_U$. Thus $|N_U| \leq |M_U|$. Thus $|M_U| = |N_U|$.

Part (ii): Step 1: For each $x \in M_U$, $p_x^{\min} < p_x$.

We proceed by contradiction. Suppose that there is a non-empty set $M' \subseteq M_U$ such that for each $x \in M'$, $p_x^{\min} = p_x$. By Definition 2, for each $x \in M'$, $p_x^{\min} = p_x > 0$.

If there is $i \in N_C$ such that $D_i(p) \cap M' \neq \emptyset$, then by Definition 3, for $j \in N$ such that $x_j \in D_i(p) \cap M'$, $j \in N_C$. Thus, for each $i \in N_C$, $D_i(p) \cap M' = \emptyset$. Thus by Definition 2, $x_j \in M_C$, contradicting $x_j \in M' \subseteq M_U$. Thus, by $p \geq p^{\min}$ and $p_{M'}^{\min} = p_{M'}$,

$$\text{for each } i \in N_C,\ D_i(p^{\min}) \cap M' = \emptyset. \qquad (*)$$

Since $p_{M_U \backslash M'}^{\min} < p_{M_U \backslash M'}$ and $p_{M'}^{\min} = p_{M'}$, then

$$\text{for each } i \in N_U \text{ such that } x_i \in M_U \backslash M',\ D_i(p^{\min}) \cap M' = \emptyset. \qquad (**)$$

Thus,

$$\begin{aligned} & \left|\{i \in N : D_i(p^{\min}) \cap M' \neq \emptyset\}\right| \\ = & \left|\{i \in N \backslash N_C : D_i(p^{\min}) \cap M' \neq \emptyset\}\right| && \text{by } (*) \\ = & \left|\{i \in N_U : D_i(p^{\min}) \cap M' \neq \emptyset\}\right| \\ \leq & |N_U| - |\{i \in N_U : x_i \in M_U \backslash M'\}| && \text{by } (**) \\ = & |\{i \in N_U : x_i \in M'\}| = |M'|. \end{aligned}$$

Thus M' is weakly underdemanded, violating Fact A.2.

Step 2: For each $x \in M_U$, $p_x^{\min} \geq C_+^1(x; z_{N_C})$.

We proceed by contradiction. Suppose that there is a non-empty set $M' \subseteq M_U$ such that for each $x \in M'$, $0 \leq p_x^{\min} < C_+^1(x; z_{N_C})$.

Case 1: For each $x \in M_C$, $p_x^{\min} = p_x$.

For each $i \in N_U$ and each $x \in M_C \cup \{0\}$,

$$z_i^{\min} \underset{\text{Def of Equilibrium}}{R_i} (x_i, p_{x_i}^{\min}) \underset{\text{Step 1}}{P_i} z_i \underset{\text{Def of Equilibrium}}{R_i} (x, p_x) = (x, p_x^{\min}).$$

Thus, for each $i \in N_U$, $D_i(p^{\min}) \cap (M_C \cup \{0\}) = \emptyset$ and thus $D_i(p^{\min}) \subseteq M_U$.

Since for each $x \in M'$, $0 \leq p_x^{\min} < C_+^1(x; z_{N_C})$, then there is $i \in N_C$ such that $V_i(x; z_i) = C_+^1(x; z_{N_C}) > 0$, and so by $p_{M_C} = p_{M_C}^{\min}$, $D_i(p^{\min}) \subseteq M_U$. Thus,

$$\left|\{i \in N : D_i(p^{\min}) \subseteq M_U\}\right| \geq 1 + |N_U| \underset{\text{(i)}}{>} |M_U|\,.$$

Thus M' is overdemanded, violating Fact A.2.

Case 2: There is a non-empty set $M'' \subseteq M_C$ such that $0 \leq p_x^{\min} < p_x$.

For each $i \in N_U$, since $p_{M_C \setminus M''}^{\min} = p_{M_C \setminus M''}$, by the same reasoning as in Case 1, $D_i(p^{\min}) \cap (M_C \cup \{0\} \setminus M'') = \emptyset$ and thus $D_i(p^{\min}) \subseteq M_U \cup M''$.

By the construction of $M'' \subseteq M_C$ and Fact A.3, $|\{i \in N_C : D_i(p) \cap M'' \neq \emptyset\}| > |M''|$. For each $i \in N_C$ with $D_i(p) \cap M'' \neq \emptyset$, by $p_{M_C \setminus M''}^{\min} = p_{M_C \setminus M''}$ and for each $x \in M''$, $p_x > p_x^{\min}$, $D_i(p^{\min}) \subseteq M'' \cup M_U$. Thus,

$$\begin{aligned}\left|\{i \in N : D_i(p^{\min}) \subseteq M'' \cup M_U\}\right| &\geq |N_U| + |\{i \in N_C : D_i(p) \cap M'' \neq \emptyset\}| \\ &\underset{\text{Lemma 1(i)}}{>} |M_U| + |M''| = |\{M'' \cup M_U\}|\end{aligned}$$

Thus $M'' \cup M_U$ is overdemanded, violating Fact A.2.

Part (iii) First, we show the following claim:

Claim A.1: For each $x \in M_C$, $p_x = p_x^{\min}$.

By contradiction, suppose that there is a non-empty set $M' \subseteq M_C$ such that for each $x \in M'$, $p_x > p_x^{\min} \geq 0$.

For each $i \in N_C$ and each $x \in M_U$, $z_i \, R_i \, (x, C_+^1(x; z_{N_C})) \underset{\text{Lemma 1(ii)}}{R_i} (x, p_x^{\min})$.

Thus, for each $i \in N_C$ with $D_i(p) \cap M' \neq \emptyset$, by $p_{M'} > p_{M'}^{\min}$, $D_i(p^{\min}) \cap M_U = \emptyset$.

By Fact A.3, $|\{i \in N_C : D_i(p) \cap M' \neq \emptyset\}| > |M'|$. Thus, for each $i \in N_C$ with $D_i(p) \cap M' \neq \emptyset$, by $D_i(p^{\min}) \cap M_U = \emptyset$, $p_{M'} < p_{M'}^{\min}$, and $p_{M_C \setminus M'} = p_{M_C \setminus M'}^{\min}$, $D_i(p^{\min}) \subseteq M'$. Thus $\left|\{i \in N_C : D_i(p^{\min}) \subseteq M'\}\right| > |M'|$, violating Fact A.2.

Thus, for each $i \in N_U$ and each $x \in M_C \cup \{0\}$,

$$z_i^{\min} \underset{\text{Def of Equilibrium}}{R_i} (x_i, p_{x_i}^{\min}) \underset{\text{Step 1 in (ii)}}{P_i} z_i \underset{\text{Def of Equilibrium}}{R_i} (x, p_x) \underset{\text{Claim A.1}}{=} (x, p_x^{\min})\,.$$

Thus, for each $i \in N_U$, $x_i^{\min} \in M_U$. **Q.E.D.**

A.3 Proof of Theorem 1

First, we propose two lemmas.

Lemma A.1: Let $(z, p) \in W$, and $(z^{\min}, p^{\min}) \in W^{\min}$. Let N_C and M_U be defned at (z, p).Then there is $x \in M_U$ such that $p_x^{\min} = C_+^1(x; z_{N_C})$.

Proof: Let M_C and N_U be the sets of connected objects and unconnected agents at (z, p), respectively. We proceed by contradiction. Suppose that for each $x \in M_U$, $p_x^{\min} > C_+^1(R_{N_C}, x; z)$. By Lemma 1(ii), for each $i \in N_C$, $D_i(p^{\min}) \cap M_U = \emptyset$. Thus,

$$\begin{aligned} & \left|\{i \in N : D_i(p^{\min}) \cap M_U \neq \emptyset\}\right| \\ = & \left|\{i \in N_U : D_i(p^{\min}) \cap M_U \neq \emptyset\}\right| \\ \leq & \left|N_U\right| \underset{\text{Lemma 1(i)}}{=} \left|M_U\right|. \end{aligned}$$

Thus, M_U is weakly underdemanded, contradicting Fact A.2. Thus there is $x \in M_U$ such that $p_x^{\min} = C_+^1(R_{N_C}, x; z)$. **Q.E.D.**

Lemma A.2: Let μ be an MPE assignment. In the IPOIP process for μ,
(i) for each $x \in M_U$ and each $s = 1, \cdots$, $\overline{p}_x^s(\mu) \leq p_x^{\min}$, and
(ii) **f**or each $x \in M_U$ and each $s = 1, \cdots$, if $\overline{p}_x^{s-1}(\mu) = p_x^{\min}$, then $\overline{p}_x^s(\mu) = \overline{p}_x^{s-1}(\mu)$.

Proof: Part (i) We prove by induction.
Step 1: For each $x \in M_U$, $\overline{p}_x^1(\mu) \leq p_x^{\min}$.

For each $x \in M_U$, by Lemma 1(ii), $\overline{p}_x^0 \equiv C_+^1(R_{N_C}, x; z^0) \leq p_x^{\min}$. Thus, for each $x \in M_U$ and each $i \in N_C$, $V_i(y; z_i) \leq \overline{p}_x^0 \leq p_x^{\min}$. For each $x \in M_U$ and each $j \in N_U$,

$$V_j(x; \overline{z}_j^0(\mu)) \underset{\overline{p}^0_{\mu(j)} \leq p^{\min}_{\mu(j)}}{\leq} V_j(x; (\mu(j), p^{\min}_{\mu(j)})) \underset{\text{Def of EquilibRIum}}{\leq} p_x^{\min}.$$

Thus, for each $x \in M_U$, $\overline{p}_x^1(\mu) = C_+^1(x; \overline{z}_{N_U}^0(\mu)) \leq p_x^{\min}$.
Induction hypothesis: For some $s \geq 1$, and for each $x \in M_U$, $\overline{p}_x^s(\mu) \leq p_x^{\min}$.
Step 2: For each $x \in M_U$, $\overline{p}_x^{s+1}(\mu) \leq p_x^{\min}$.

For each $x \in M_U$, by Lemma 1(ii), $\overline{p}_x^0 = C_+^1(R_{N_C}, x; z^0) \leq p_x^{\min}$. Thus, for each $x \in M_U$ and each $i \in N_C$, $V_i(y; z_i) \leq \overline{p}_x^0 \leq p_x^{\min}$.

For each $x \in M_U$ and each $j \in N_U$,

$$V_j(x; \overline{z}_j^s(\mu)) \underset{\overline{p}^s_{\mu(j)} \leq p^{\min}_{\mu(j)}}{\leq} V_j(x; (\mu(j), p^{\min}_{\mu(j)})) \underset{\text{Def of Equilibrium}}{\leq} p_x^{\min}.$$

Thus, for $x \in M_U$, $\overline{p}_x^{s+1}(\mu) \equiv C_+^1(x; \overline{z}_{N_U}^s(\mu)) \leq p_x^{\min}$.
(ii) Let $x \in M_U$ and $s \in \mathbb{N}^+$ be such that $\overline{p}_x^{s-1}(\mu) = p_x^{\min}$. By Lemma A.1 and Fact 2, $\overline{p}_x^{s-1}(\mu) \leq \overline{p}_x^s(\mu) = p_x^{\min}$. Thus, $\overline{p}_x^s(\mu) = \overline{p}_x^{s-1}(\mu)$. **Q.E.D.**

Part (i) of Theorem 1: We only show that for each $i \in N_C$, $z_i = z_i^{\min}$. With Claim A.1 in the proof of Lemma 1(iii), we establish Theorem 1(i).

For each $i \in N_C$ and each $x \in M_U$,

$$z_i \, R_i \, (x, C_+^1(x; z_{N_C})) \underset{\text{Lemma 1(ii)}}{R_i} (x, p_x^{\min}).$$

and for each $y \in M_C \cup \{0\}$,

$$z_i \underset{\text{Def of Equilibrium}}{R_i} (y, p_y) = (y, p_y^{\min}).$$

Thus for each $i \in N_C$ and each $x \in L$, $z_i \, R_i \, (x, p_x^{\min})$ and thus $z_i \, R_i \, z_i^{\min}$. Note that

$$z_i^{\min} \, R_i \, (x_i, p_{x_i}^{\min}) \, I_i \, (x_i, p_{x_i}) = z_i.$$

Thus $z_i^{\min} \, I_i \, z_i$. By Lemma 1(iii), for each $i \in N_C$, $x_i^{\min} \in M_C \cup \{0\}$. Thus, for each $i \in N_C$, we can set $z_i = z_i^{\min}$ while let unassigned objects at (z, p) remain unassigned at $(z^{\min}, p^{\min})$.

Part (ii) of Theorem 1: Step 1: Let μ be an MPE assignment. Then $\overline{p}^{|M_U|-1}(\mu) = p_{M_U}^{\min}$.

First, we show Claim A.2.

Claim A.2: For each $s = 0, 1, \cdots$, let $M_s \equiv \{x \in M_U : \overline{p}_x^s(\mu) = p_x^{\min}\}$ and $N_s \equiv \{i \in N_U : \mu_i \in M_s\}$. Then, for each $s = 0, 1, \cdots$,
(a) $|M_s| = |N_s|$, and (b) if $M_U \backslash M_s \neq \emptyset$, then $M_{s+1} \supsetneq M_s$.

By Definition, for each $s = 0, 1, \cdots$, (a) holds. Thus, we show only (b).

Let $M_U \backslash M_s \neq \emptyset$. By Step 1-2, $M_{s+1} \supseteq M_s$. Suppose that $M_{s+1} = M_s$. By Lemma A.2(i), for each $x \in M_U \backslash M_s$, $p_x^{\min} > \overline{p}_x^s(\mu) \geq \overline{p}_x^0$. Thus, by $M_{s+1} = M_s$, for each $x \in M_U \backslash M_s$, $\overline{p}_x^0 \leq \overline{p}_x^{s+1}(\mu) < p_x^{\min}$. By Lemma 1(ii), for each $i \in N_C$, $D_i(p^{\min}) \cap (M_U \backslash M_s) = \emptyset$.

If $i \in N_s$, then for each $x \in M_U \backslash M_s$,

$$V_i(x, z_i^{\min}) \underset{i \in N_s}{=} V_i(x, \overline{z}_i^s(\mu)) \leq C_+^1(x; \overline{z}_{N_U}^s(\mu)) = \overline{p}_x^{s+1}(\mu) < p_x^{\min}.$$

Thus, for each $i \in N_s$, $D_i(p^{\min}) \cap (M_U \backslash M_s) = \emptyset$.

Since for each $i \in N_C \cup N_s$, $D_i(p^{\min}) \cap (M_U \backslash M_s) = \emptyset$, then

$$\{i \in N : D_i(p^{\min}) \cap (M_U \backslash M_s) \neq \emptyset\} = \{i \in N_U \backslash N_s : D_i(p^{\min}) \cap (M_U \backslash M_s) \neq \emptyset\}.$$

and so

$$\left|\{i \in N_U \backslash N_s : D_i(p^{\min}) \cap (M_U \backslash M_s) \neq \emptyset\}\right| \leq |N_U \backslash N_s| \underset{\text{Lemma 1(i) and (a)}}{=} |M_U \backslash M_s| \, .$$

Thus $M_U \backslash M_s$ is weakly underdemanded, contradicting Fact A.2.

Now we complete the proof of Step 1. If $M_U \backslash M_0 = \emptyset$, then for each $x \in M_U$, $\overline{p}_x^0(\mu) = p_x^{\min}$, and so Lemma A.2(ii) implies that for each $x \in M_U$, $\overline{p}_x^{|M_U|-1}(\mu) = p_x^{\min}$. Thus, assume $M_U \backslash M_0 \neq \emptyset$. Then, Claim A.2 says that as s increases, M_s expands strictly until $\{x \in M_U : \overline{p}_x^s(\mu) = p_x^{\min}\} = M_U$. Since Lemma A.1 implies $M_0 \neq \emptyset$, M_s expands strictly at most $|M_U| - 1$ times. Thus, $\{x \in M_U : p_x^{\min} = \overline{p}_x^{|M_U|-1}(\mu)\} = M_U$, i.e., for $x \in M_U$, $\overline{p}_x^{|M_U|-1}(\mu) = p_x^{\min}$.

Step 2: Let $\mu' \in \Omega$ be a non-MPE assignment. Then, $p_{M_U}^{\min} \leq p^{|M_U|-1}(\mu')$.

Definition A.2: Let $x_{N_U}^{\min}$ be an MPE assignment, $\mu' \in \Omega$, and $i \in N_U$. A sequence $\{\sigma^l(i)\}_{l=1}^d$ of distinct agents ($1 \leq d \leq n$) is called a **trading cycle from** i **in** μ' if (i) $\sigma^1(i) = i$, and (ii) for each $l \in \{1, \cdots, d-1\}$, ${\mu'}_{\sigma^{l+1}(i)} = x_{\sigma^l(i)}^{\min}$ and ${\mu'}_{\sigma^1(i)} = x_{\sigma^d(i)}^{\min}$.

Step 2-1: Let $i \in N_U$, and $\{\sigma^l(i)\}_{l=1}^d$ be a trading cycle from i in μ' and $s \geq 0$. If $\overline{p}_{x_{\mu'(i)}^{\min}}^s(\mu') \geq p_{x_{\mu'(i)}^{\min}}^{\min}$, then for each $j \in \{\sigma^1(i), \cdots, \sigma^d(i)\}$, $\overline{p}_{x_j^{\min}}^{s+d}(\mu') \geq p_{x_j^{\min}}^{\min}$.

If $d = 1$, then Step 2-1 trivially holds. In the following, let $d \geq 2$. In the proof, without loss of generality, we assume that $i = \sigma^1(i) = 1$, $\sigma^2(i) = 2, \cdots, \sigma^d(i) = d$. Then, $\mu'_2 = x_1^{\min}$, $\mu'_3 = x_2^{\min}, \cdots, \mu'_d = x_{d-1}^{\min}$, $\mu'_1 = x_d^{\min}$ and $\overline{p}_{x_d^{\min}}^s(\mu') \geq p_{x_d^{\min}}^{\min}$. We inductively show that for each $j \in \{1, \cdots, d\}$, $\overline{p}_j^{s+d}(\mu') \geq p_j^{\min}$. Note that

$$
\begin{aligned}
\overline{p}_{x_1^{\min}}^{s+1}(\mu') &= C_+^1(x_1^{\min}; \overline{z}_{N_U}^s(\mu')) \\
&\geq V_1(x_1^{\min}, \overline{z}_1^s(\mu')) && (1) \\
&= V_1(x_1^{\min}, (x_d^{\min}, \overline{p}_{x_d^{\min}}^s(\mu'))) && \text{by } \mu'(1) = x_d^{\min} \\
&\geq V_1(x_1^{\min}, (x_d^{\min}, p_{x_d^{\min}}^{\min}(\mu'))) && \text{by } \overline{p}_{x_d^{\min}}^s(\mu') \geq \overline{p}_{x_d^{\min}}^{\min}(\mu') \\
&\geq p_{x_1^{\min}}^{\min}. && \text{Def of Equilibrium}
\end{aligned}
$$

Thus, $\overline{p}_{x_1^{\min}}^{s+1}(\mu') \geq p_{x_1^{\min}}^{\min}$.

Let $j \in \{1, \cdots, d\}$, and assume that $\overline{p}_j^{s+j}(\mu') \geq p_j^{\min}$. Then, by similar reasoning as above but replacing $\mu'_1 = x_d^{\min}$ and $\overline{p}_{x_d^{\min}}^s(\mu') \geq p_{x_d^{\min}}^{\min}$ by $\mu'_{j+1} = x_j^{\min}$ and $\overline{p}_j^{s+j}(\mu') \geq p_j^{\min}$, respectively, $\overline{p}_{j+1}^{s+j+1}(\mu') \geq p_{j+1}^{\min}$ holds. Thus, for each $k \in \{1, \cdots, d\}$, $\overline{p}_{x_k^{\min}}^{s+k}(\mu') \geq p_{x_k^{\min}}^{\min}$. Thus, by Fact 2, for each $j \in \{1, \cdots, d\}$, $\overline{p}_j^{s+d}(\mu') \geq p_j^{\min}$.

Step 2-2: Let $x_{N_U}^{\min}$ be an MPE assignment and $\mu' \in \Omega$. Let $\{N_l(\mu')\}_{l \in K}$ be a partition of N_U such that $K \equiv \{1, \cdots, k\}$, and for each $l \in K$, agents in $N_l(\mu')$ form a trading cycle.[10] Let $L_0 \equiv \{l \in K : \text{there is } i \in N_l(\mu') \text{ s.t. } p_{x_i^{\min}}^{\min} = \overline{p}_{x_i^{\min}}^0\}$ and $M_0 \equiv \{x \in M_U : \text{there is } i \in \underset{r \in L_0}{\cup} N_r(\mu') \text{ s.t. } \mu'_i = x\}$. For each $s = 1, 2, \cdots$,

[10]Precisely, there is $i \in N_l(\mu')$ such that there is a sequence $\{\sigma^l(i)\}_{l=1}^{|N_l(\mu')|}$ of distinct agents forming a trading cycle from i in μ'

let $L_s \equiv \{l \in K$:there are $i \in N_l(\mu')$ and $j \in \underset{r\in L_{s-1}}{\cup} N_r(\mu')$ s.t. $z_j^{\min} I_j z_i^{\min}\}$ and $M_s \equiv \{x \in M_U$:there is $i \in \underset{r\in L_s}{\cup} N_r(\mu')$ s.t. $\mu'_i = x\}$. Then, for each $s = 0, 1, \cdots$,

$$\text{(a)} \left|\underset{r\in L_s}{\cup} N_r(\mu')\right| = |M_s|\text{, and (b) if } K\backslash L_s \neq \emptyset\text{, then } L_{s+1} \supsetneq L_s.$$

By definition, for each $s = 0, 1, \cdots$, (a) holds. Thus, we show only (b).

Let $K\backslash L_s \neq \emptyset$. By Definition, $L_{s+1} \supseteq L_s$. Suppose that $L_{s+1} = L_s$. By $K\backslash L_s \neq \emptyset$, and Lemma 1(i), $M_U\backslash M_s \neq \emptyset$. For each $x \in M_U\backslash M_s$, by $L_{s+1} = L_s$, $x \notin M_0$ and so by Lemma 1(ii), $p_x^{\min} > \overline{p}_x^0$. Thus, by Lemma 1(ii), for each $i \in N_C$, $D_i(p^{\min}) \cap (M_U\backslash M_s) = \emptyset$.

Let $i \in \underset{r\in L_s}{\cup} N_r(\mu')$ and $x \in M_U\backslash M_s$. Let $j \in N_U$ be such that $\mu'(j) = x$, i.e., $z_j^{\min} = (x, p_x^{\min})$. By $x \in M_U\backslash M_s$ and $L_{s+1} = L_s$, $j \notin \underset{r\in L_{s+1}}{\cup} N_r(\mu')$. Thus, by $i \in \underset{r\in L_s}{\cup} N_r(\mu')$, $z_i^{\min} I_i z_j^{\min}$ does not hold. By the definition of equilibrium, $z_i^{\min} R_i z_j^{\min}$ and so $z_i^{\min} P_i z_j^{\min}$, i.e., $x \notin D_i(p^{\min})$. Thus for each $i \in \underset{r\in L_s}{\cup} N_r(\mu')$, $D_i(p^{\min}) \cap (M_U\backslash M_s) = \emptyset$.

Since for each $i \in \underset{r\in L_s}{\cup} N_r(\mu') \cup N_C$, $D_i(p^{\min}) \cap (M_U\backslash M_s) = \emptyset$, then

$$\{i \in N : D_i(p^{\min}) \cap (M_U\backslash M_s) \neq \emptyset\} = \{i \in N_U\backslash \underset{r\in L_s}{\cup} N_r(\mu') : D_i(p^{\min}) \cap (M_U\backslash M_s) \neq \emptyset\},$$

and so

$$\begin{aligned} &\left|\{i \in N_U\backslash \underset{r\in L_s}{\cup} N_r(\mu') : D_i(p^{\min}) \cap (M_U\backslash M_s) \neq \emptyset\}\right| \\ \leq\ &\left|N_U\backslash \underset{r\in L_s}{\cup} N_r(\mu')\right| \underset{\text{Lemma 1(i) and (a)}}{=} |M_U\backslash M_s|. \end{aligned}$$

Thus, $M_U\backslash M_s$ is weakly underdemanded, contradicting Fact A.2. Thus, (b) $L_{s+1} \supsetneq L_s$ holds.

Now we complete the proof of Step 2. By the finiteness of N_U, Step 2-2 implies that there is $q \in \{0, \cdots, k\}$ such that $L_q = K$. Let $d_0 \equiv \max_{l\in L_0} |N_l(\mu')|$ and for each $r = 1, \cdots, q$, $d_r \equiv \max_{l\in L_r\backslash L_{r-1}} |N_l(\mu')|$. By Fact B.4, $L_0 \neq \emptyset$.

If $q = 0$, then $d_0 \leq |N_U| = |M_U|$. Thus, by Step 2-1 and Fact 2, at round $d_0 - 1$, for each $x \in M_U$, $p_x^{\min} \leq \overline{p}_x^{d_0-1}(\mu') \leq \overline{p}_x^{|M_U|-1}(\mu')$. If $q > 0$, by Step 2-1, at round $d_0 - 1$, for each $x \in M_0$, $p_x^{\min} \leq \overline{p}_x^{d_0-1}(\mu')$ and there is $y \in M_1\backslash M_0$ such that $p_y^{\min} \leq \overline{p}_y^{d_0-1}(\mu')$. By Step 2-1, at round $d_0 + d_1 - 1$, for each $x \in M_1\backslash M_0$, $p_x^{\min} \leq \overline{p}_x^{d_0+d_1-1}(\mu')$. By Fact 2, for each $x \in M_0$, $p_x^{\min} \leq \overline{p}_x^{d_0+d_1-1}(\mu')$. Thus for each $x \in M_1$, $p_x^{\min} \leq \overline{p}_x^{d_0+d_1-1}(\mu')$. By induction argument, at round $D \equiv \sum_{i=0}^{q} d_i - 1$, for each $x \in M_U$, $p_x^{\min} \leq \overline{p}_x^D(\mu')$. Since $D \equiv \sum_{i=0}^{q} d_i \leq |N_U| = |M_U|$, then for each $x \in M_U$, $p_x^{\min} \leq \overline{p}_x^{d_0-1}(\mu') \leq \overline{p}_x^{|M_U|-1}(\mu')$. Thus Step 2 holds.

Step 3: Completion of the proof

By Fact 1, there is $\mu \in \Omega$ such that μ is an MPE assignments. By Step 1, for each $x \in M_U$, $p_x^{\min} = \overline{p}_x^{|M_U|-1}(\mu)$. By Step 2, for each $\mu' \in \Omega\backslash\{\mu\}$ and each $x \in M_U$, $p_x^{\min} \leq p_x^{|M_U|-1}(\mu')$. Thus Theorem 1(ii-2) holds. **Q.E.D.**

A.4 Proof of Theorem 2

By Lemma A.2(ii), (ii) implies (i). Thus, we only show that (i) implies (ii). Let $\mu \in \Omega$ and $s \leq |M_U|$ be such that $\overline{p}_{M_U}^{s-1}(\mu) = \overline{p}_{M_U}^{s}(\mu)$.

First, we introduce a weak variant of connectedness.

Definition A.3: Let $(z,p) \in Z \times \mathbb{R}^m$. An agent $i \in N$ is **weakly connected** at p if there is a sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of Λ distinct agents such that
(i) $x_{i_1} = 0$ or $p_{x_{i_1}} = 0$,
(ii) for each $\lambda \in \{2, \cdots, \Lambda\}$, $x_{i_\lambda} \neq 0$ and $p_{x_{i_\lambda}} > 0$,
(iii) $x_{i_\Lambda} = x_i$, and
(iv) for each $\lambda \in \{1, \cdots, \Lambda-1\}$, $z_{i_\lambda} \, I_{i_\lambda} \, z_{i_{\lambda+1}}$.

Definition A.3 is weaker than Definition 3 since the weak connectedness does not require that for each $\lambda \in \{1, \cdots, \Lambda-1\}$, $\{x_{i_\lambda}, x_{i_{\lambda+1}}\} \subseteq D_{i_\lambda}(p)$, but instead only $z_{i_\lambda} \, I_{i_\lambda} \, z_{i_{\lambda+1}}$.

Definition A.4: Let $(z,p) \in Z \times \mathbb{R}^m$. An object $x \in M$ is **weakly connected** at p if (i) x is assigned to a weakly connected agent or (ii) x is unassigned.

Let $(z,p) \in W$, and N_C and M_C be defined at (z,p). Then agents in N_C and objects in M_C are all weakly connected.

Step 1: For each $x \in M_U$, x is a weakly connected object at $(\overline{p}^{s-1}(\mu), p_{M_C})$.

Let M' be the set of weakly connected objects in M_U at $(\overline{p}^{s-1}(\mu), p_{M_C})$. To prove $M' = M_U$, we proceed by contradiction. Suppose that $M_U\backslash M' \neq \emptyset$. Let $N' \equiv \{i \in N_U : \mu_i \in M'\}$. Then, N' is the set of weakly connected agents in N_U at $(\overline{p}^{s-1}(\mu), p_{M_C})$, and $|M'| = |N'|$. Then by Lemma 1(i) and $|M'| = |N'|$, $N_U\backslash N' \neq \emptyset$.

If there is $x \in M_U\backslash M'$ such that $\overline{p}_x^{s-1}(\mu) = C_+^1(x; z_{N_C})$, then either $\overline{p}_x^{s-1}(\mu) = 0$ or there is some $j \in N_C$ such that $(x, \overline{p}_x^{s-1}(\mu)) \, I_j \, z_j$, contradicting $x \in M_U\backslash M'$. Thus, for each $x \in M_U\backslash M'$, $\overline{p}_x^{s-1}(\mu) \neq C_+^1(x; z_{N_C})$. Thus, by Fact 2, for each $x \in M_U\backslash M'$, $\overline{p}_x^{s-1}(\mu) \geq C_+^1(x; z_{N_C})$ and so $\overline{p}_x^{s-1}(\mu) > C_+^1(x; z_{N_C}) \geq 0$.

Let $x \in M_U\backslash M'$. Note that $\overline{p}_x^{s-1}(\mu) \equiv C_+^1(x; \overline{z}_{N_U}^{s-1}(\mu)) \geq C^1(x; \overline{z}_{N'}^{s-1}(\mu))$. Suppose $\overline{p}_x^{s-1}(\mu) = C^1(x; \overline{z}_{N'}^{s-1}(\mu))$. Then, there is $i \in N'$ such that $\overline{p}_x^{s-1}(\mu) = V_i(x; \overline{z}_i^{s-1}(\mu))$. By $i \in N'$ and $\overline{p}_x^{s-1}(\mu) = V_i(x; \overline{z}_i^{s-1}(\mu))$, x is a weakly connected object at $(\overline{p}^{s-1}(\mu), p_{M_C})$, contradicting $x \in M_U\backslash M'$. Thus, for each $x \in M_U\backslash M'$, $\overline{p}_x^{s-1}(\mu) > C^1(x; \overline{z}_{N'}^{s-1}(\mu))$.

Let s' be the earliest round in the IPOIP process such that there is $x \in M_U \backslash M'$ such that $\overline{p}_x^{s'}(\mu) = \overline{p}_x^{s-1}(\mu)$. Then, by Fact 2 and $s' \leq s-1$,

$$\text{for each } s'' < s' \text{ and each } y \in M_U \backslash M',\ \overline{p}_y^{s''}(\mu) < \overline{p}_y^{s'}(\mu) \leq \overline{p}_y^{s-1}(\mu). \qquad (*)$$

Since for each $y \in M_U \backslash M'$, $\overline{p}_y^{s-1}(\mu) > C_+^1(x; z_{N_C})$, then $s' \geq 1$.

To derive a contradiction to $(*)$, we first show the following claim.

Claim A.3: Let $i \in N_U$, $x \in M_U$, $s' \leq s-1$, and $V_i(x, \overline{z}_i^{s'-1}(\mu)) = \overline{p}_x^{s-1}(\mu)$. Then $\overline{p}_{\mu_i}^{s'-1}(\mu) = \overline{p}_{\mu_i}^{s-1}(\mu)$.

Note that

$$\overline{p}_x^{s-1}(\mu) = V_i(x, \overline{z}_i^{s'-1}(\mu)) \underset{\text{Fact2 \& } s' \leq s-1}{\leq} V_i(x, \overline{z}_i^{s-1}(\mu)) \underset{i \in N_U}{\leq} C_+^1(x; \overline{z}_{N_U}^{s-1}(\mu)) = \overline{p}_x^s(\mu).$$

Thus, by $\overline{p}^{s-1}(\mu) = \overline{p}^s(\mu)$, $V_i(x, \overline{z}_i^{s'-1}(\mu)) = V_i(x, \overline{z}_i^{s-1}(\mu))$. Since $\overline{z}_i^{s'-1}(\mu) = (\mu_i, \overline{p}_{\mu_i}^{s'-1}(\mu))$ and $\overline{z}_i^{s-1}(\mu) = (\mu_i, \overline{p}_{\mu_i}^{s-1}(\mu))$, then $\overline{p}_{\mu_i}^{s'-1}(\mu) = \overline{p}_{\mu_i}^{s-1}(\mu)$.

By the definition of IPOIP process and $s' \geq 1$, there is $i \in N_U$ such that $V_i(x, \overline{z}_i^{s'-1}(\mu)) = \overline{p}_x^{s'}(\mu) = \overline{p}_x^{s-1}(\mu)$. Note that for each $x \in M_U \backslash M'$, $\overline{p}_x^{s-1}(\mu) > C^1(x; \overline{z}_{N'}^{s''}(\mu))$. By Fact 2, for each $s'' \leq s-1$, $\overline{p}_x^{s-1}(\mu) > C^1(x; \overline{z}_{N'}^{s''}(\mu))$. Thus, $i \notin N'$ and so $i \in N_U \backslash N'$, and $\mu_i \in M_U \backslash M'$. By Claim A.3, $\overline{p}_{\mu_i}^{s'-1}(\mu) = \overline{p}_{\mu_i}^{s-1}(\mu)$, contradicting $(*)$.

Thus $M_U \backslash M' \neq \emptyset$ fails to hold, i.e., $M_U = M'$.

Step 2: Let $M_0 \equiv \{x \in M_U : \overline{p}_x^{s-1}(\mu) = C_+^1(x; z_{N_C})\}$. Then $M_0 \neq \emptyset$.

By Definitions A.3 and A.4 and Step 1, there is no $x \in M_U \backslash M_0$ that is weakly connected to some $y \in M_C$ at $(\overline{p}^{s-1}(\mu), p_{M_C})$. Thus, $M_0 \neq \emptyset$ just follows Step 1.

Step 3: For each $x \in M_U$, $\overline{p}_x^{s-1}(\mu) \leq p_x^{\min}$.

If $M_U = M_0$, by Lemma 1(ii), Step 3 trivially holds. Thus, let $M_U \backslash M_0 \neq \emptyset$.

Let $M' \equiv \{x \in M_U : \forall\, x \in M', \overline{p}_x^{s-1}(\mu) > p_x^{\min}\}$. To show $M' = \emptyset$, we proceed by contradiction. Suppose that $M' \neq \emptyset$.

Let $N' \equiv \{i \in N_U : \mu_i \in M'\}$. By Definition, $|N'| = |M'|$. By Lemma 1(i) and $|M'| = |N'|$, $|N_U \backslash N'| = |M_U \backslash M'| \neq \emptyset$. By Step 2, $M_U \backslash M' \supseteq M_0 \neq \emptyset$ and so $N_U \backslash N' \neq \emptyset$.

For each $i \in N_U$ and each $x \in L \backslash M_U$,

$$z_i^{\min}\, R_i\, (x_i, p_{x_i}^{\min}) \underset{\text{Lemma 1(ii)\&(iii)}}{P_i} (x_i, p_{x_i}) = z_i \underset{\text{Def of Equilibrium}}{R_i} (x, p_x) \underset{\text{Theorem 1(i)}}{=} (x, p_x^{\min}).$$

and so $D_i(p^{\min}) \subseteq M_U$.

For each $i \in N_U$ and each $y \in M_U \backslash M'$,

$$\begin{aligned} \overline{z}_i^{s-1}(\mu) \quad & R_i\, (y, \overline{p}_y^{s-1}(\mu)) && \text{by } \overline{p}_y^{s-1}(\mu) = \overline{p}_y^s(\mu) \geq V_i(y, \overline{z}_i^{s-1}(\mu)) \\ & R_i\, (y, p_y^{\min}). && \text{by } \overline{p}_y^{s-1}(\mu) \leq p_y^{\min} \end{aligned}$$

For each $i \in N'$ and each $y \in M_U \backslash M'$,

$$z_i^{\min} \underset{\text{Def of Equilibrium}}{R_i} (\mu_i, p_{\mu_i}^{\min}) \underset{p_{\mu_i}^{\min} < \overline{p}_{\mu_i}^{s-1}(\mu)}{P_i} \overline{z}_i^{s-1}(\mu) \underset{N' \subseteq N_U}{R_i} (y, p_y^{\min}).$$

Thus, for each $i \in N'$, by $D_i(p^{\min}) \subseteq M_U$, $D_i(p^{\min}) \subseteq M'$. Thus,

$$\left|\{i \in N_U : D_i(p^{\min}) \subseteq M'\}\right| \geq |N'| = |M'|.$$

By Step 1, for each $x \in M'$, x is weakly connected at $(\overline{p}^{s-1}(\mu), p_{M_C})$ and $\overline{p}_x^{s-1}(\mu) > p_x^{\min} \geq 0$. Then by $N_U \backslash N' \neq \emptyset$, there is $i \in N_U \backslash N'$ and $x' \in M'$ such that $\overline{z}_i^{s-1}(\mu)\, I_i\, (x', \overline{p}_{x'}^{s-1}(\mu))$. Thus for each $y \in M_U \backslash M'$,

$$z_i^{\min} \underset{\text{Def of Equilibrium}}{R_i} (x', p_{x'}^{\min}) \underset{p_{x'}^{\min} < \overline{p}_{x'}^{s-1}(\mu)}{P_i} (x', \overline{p}_{x'}^{s-1}(\mu))\, I_i\, \overline{z}_i^{s-1}(\mu) \underset{i \in N_U \backslash N'}{R_i} (y, p_y^{\min}).$$

Thus $y \notin D_i(p^{\min})$. By $D_i(p^{\min}) \subseteq M_U$, $D_i(p^{\min}) \subseteq M'$ so

$$|M'| < |N'| + 1 \leq \left|\{i \in N_U : D_i(p^{\min}) \subseteq M'\}\right|,$$

contradicting Fact A.2.

Step 4: For each $i \in N_U$ and each $x \in L \backslash M_U$, $V_i(x; \overline{z}_i^{s-1}(\mu)) \leq p_x^{\min}$.

Let $i \in N_U$ and $x \in L \backslash M_U$. By Lemma 1(iii), $x_i^{\min} \in M_U$. Thus,

$$V_i(x_i^{\min}; \overline{z}_i^{s-1}(\mu)) \leq C_+^1(x_i^{\min}; \overline{z}^{s-1}(\mu)) = \overline{p}_{x_i^{\min}}^s(\mu) \underset{\overline{p}^{s-1}(\mu) = \overline{p}^s(\mu)}{=} \overline{p}_{x_i^{\min}}^{s-1}(\mu).$$

Thus, $\overline{z}_i^{s-1}(\mu)\, R_i\, (x_i^{\min}, \overline{p}_{x_i^{\min}}^{s-1}(\mu))$. Note

$$(x_i^{\min}, \overline{p}_{x_i^{\min}}^{s-1}(\mu)) \underset{\text{Step 3}}{R_i} (x_i^{\min}, p_{x_i^{\min}}^{\min}) = z_i^{\min} \underset{\text{Def. of Equilibrium}}{R_i} (x, p_x^{\min}).$$

Thus, by $\overline{z}_i^{s-1}(\mu)\, R_i\, (x_i^{\min}, \overline{p}_{x_i^{\min}}^{s-1}(\mu))$, $\overline{z}_i^{s-1}(\mu)\, R_i\, (x, p_x^{\min})$, i.e., $V_i(x; \overline{z}_i^{s-1}(\mu)) \leq p_x^{\min}$.

Step 5: $((\overline{z}^{s-1}(\mu), z_{N_C}), (\overline{p}^{s-1}(\mu), p_{M_C})) \in W^{\min}$

By Lemma 1(ii) and Theorem 1(i), for each $i \in N_C$, (E-i) holds. For each $i \in N_U$ and each $x \in M_U$, $V_i(x; \overline{z}_i^{s-1}(\mu)) \leq C_+^1(x; \overline{z}^{s-1}(\mu)) = \overline{p}_x^s(\mu) = \overline{p}_x^{s-1}(\mu)$, and for each $x \in L \backslash M_U$,

$$V_i(x; \overline{z}_i^{s-1}(\mu)) \underset{\text{Step 5}}{\leq} p_x^{\min} \underset{\text{Theorem 1(i)}}{=} p_x.$$

Thus (E-i) holds. (E-ii) holds obviously. Thus $((\overline{z}_{N_U}^{s-1}(\mu), z_{N_C}), (\overline{p}_{M_U}^{s-1}(\mu), p_{M_C})) \in W$. By Theorem 1(i), Step 3, and Fact 1, $p^{\min} = (\overline{p}_{M_U}^{s-1}(\mu), p_{M_C})$. Thus Step 5 holds. **Q.E.D.**

Appendix B: Proofs of Propositions 2, 3, and 4

B.1 Proof of Proposition 2

Proof: Let (z, p) be the output of the E-generating mechanism. We prove that (z, p) is an equilibrium. Let $M(k)$ be the set of k objects and let $L(k) = M(k) \cup \{0\}$.
Mechanism stops at Phase 1: In this case, $C^1(y; z^{\min}) \leq 0$ and $z = z^{\min}$. Since $z = z^*$, then for each $i \in N$ and each $x \in L(k)$,

$$z_i = z_i^{\min} \underset{\text{Def of Equilibrium}}{R_i} (x, p_x^{\min}) = (x, p_x)$$

and for y, by $C^1(y; z^{\min}) \leq 0$ and $p_y = 0$, $z_i = z_i^{\min} R_i (y, C^1(y; z^{\min})) R_i (y, p_y)$. Thus, (z, p) satisfies (E-i). It is straightforward that (z, p) satisfies (E-ii).
Mechanism stops at Phase 2: In this case, $C^1(y; z^{\min}) > 0$, and there is $i \in N'$ such that $z_i^* = (y, C_+^2(y; z^{\min}))$. For each $x \in L(k) \cup \{y\}$,

$$z_i \underset{C_+^2(y;z^{\min}) \leq C_+^1(y;z^{\min})}{R_i} z_i^{\min} \underset{\text{Def of Equilibrium}}{R_i} (x, p_x^{\min}) = (x, p_x).$$

For each $j \in N \backslash \{i\}$ and each $x \in L(k)$, by Definition 8

$$z_j \, R_j \, z_j^{\min} \underset{\text{Def of Equilibrium}}{R_j} (x, p_x^{\min}) = (x, p_x),$$

and for y, by $V_j(y; z_j^{\min}) \leq C_+^2(y; z^{\min}) = p_y$, $z_j \, R_j \, z_j^{\min} \, R_j \, (y, p_y)$.

Thus, (z, p) satisfies (E-i). Unassigned objects at $M(k)$ remain unassigned with zero prices, and $p_{x_{i_1}} = p_{x_{i_1}}^{\min} = 0$. Thus (z, p) satisfies (E-ii). **Q.E.D.**

B.2 Proof of Proposition 3

Let T be the final round of the process. By the finiteness of agents and objects, $T < +\infty$.

In the following, we show $N^* = N_C$. If $N_C = \emptyset$, by Remark 1(iii), there is no agent $i \in N$ such that $p_{x_i} = 0$. Thus, the mechanism stops at $N_1' = \emptyset$, i.e., $N_C = \emptyset$. Let $N_C \neq \emptyset$.

First, we show that $\overset{T}{\underset{k=1}{\cup}} N_k' \subseteq N_C$. By Remark 1(iii), there is some $i \in N$ such that $p_{x_i} = 0$. Thus, $N_1' \neq \emptyset$ and $N_1' \subseteq N_C$. If $T = 2$, i.e., $N_T' = \emptyset$, then $\overset{2}{\underset{k=1}{\cup}} N_k' \subseteq N_C$. Let $T > 2$. Thus $N_2' \neq \emptyset$. By the definition of N_2', for each $i \in N_2'$, $p_{x_i} > 0$ and there is $j \in N_1$ such that $x_i \in D_j(p)$. By Definition 3, $N_2' \subseteq N_C$. By induction argument, for each $t = 1, \cdots, T-1$, $N_t' \neq \emptyset$ and $N_t' \subseteq N_C$. Recall that $N_T' = \emptyset$. Thus $\overset{T}{\underset{k=1}{\cup}} N_k' \subseteq N_C$.

Then, we show that $\overset{T}{\underset{k=1}{\cup}} N_k' = N_C$. We proceed by contradiction. Suppose that there is $i \in N_C \backslash \overset{T}{\underset{k=1}{\cup}} N_k'$. Then $i \notin N_1'$ and $p_{x_i} > 0$. By Definition 2, there is a sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of $\Lambda(\Lambda \geq 2)$ distinct agents such that (a) $x_{i_1} = 0$ or $p_{x_{i_1}} = 0$, (b) for each $\lambda \in \{2, \cdots, \Lambda\}$, $x_{i_\lambda} \neq 0$ and $p_{x_{i_\lambda}} > 0$, (c) $x_{i_\Lambda} = x_i$, and (d) for each $\lambda \in \{1, \cdots, \Lambda - 1\}$, $\{x_{i_\lambda}, x_{i_{\lambda+1}}\} \in D_{i_\lambda}(p)$.

By Definition 6, (a) implies $i_1 \in N'_1$. By (d), there is $i \in \{i_2, \cdots, i_\Lambda\}$ such that $x_i \in D_j(p)$ for some $j \in N'_1$, e.g., $i = i_2$. Thus, $i \in N'_2$ and $N'_2 \neq \emptyset$. Let $i_{l_1} \in \{i_2, \cdots, i_\Lambda\}$ be such that there is no $l' > l_1$ such that $i_{l'} \in N'_2$, i.e., agent i_{l_1} is the agent who belongs to N'_2 with the largerst index in $\{i_\lambda\}_{\lambda=2}^{\Lambda}$. By (d), there $i \in \{i_{l_1+1}, \cdots, i_\Lambda\}$ such that $x_i \in D_j(p)$ for some $j \in N'_2$, e.g., $i = i_{l_1+1}$. Thus $i \in N'_3$ and $N'_3 \neq \emptyset$. By same reasoning, we can select $i_{l_2} \in \{i_{l_1+1}, \cdots, i_\Lambda\}$ such that $i_{l_2} \in N'_3$ with the largest index in $\{i_\lambda\}_{\lambda=l_1+1}^{\Lambda}$. Repeating such argument, we can show $i = i_\Lambda \in \bigcup_{k=1}^{T} N'_k$, contradicting $i \in N_C \setminus \bigcup_{k=1}^{T} N'_k$. **Q.E.D.**

B.3 Proof of Proposition 4

Proposition 4 trivially holds for $M_U = \emptyset$. In the following, let $M_U \neq \emptyset$.

Part (i): $T < +\infty$ comes from the finiteness of M_U and Ω. By the construction of the MPE-adjustment mechanism, for each $t = 1, \cdots, T$, $p^{*t} \leq p^{*t-1}$.

Part (ii): **Claim B.1**: (a) for each $t < T$, μ^{*t} is not an MPE assignment and (b) μ^{*T} is an MPE assignment.

(a): By contradiction, suppose there is $t < T$ such that μ^{*t} is an MPE assignment. W.o.l.g. assume that there is no $t' < t$ such that $\mu^{*t'}$ is an MPE assignment. By Fact 4, for each $t' < t$, μ^{*t} can succeed in the $|M_U|$ −IPOIP process for μ^{*t} against $p^{*t'}$. Thus the MPE-adjustment mechanism terminates at $t < T$, a contradiction. (b): A direct outcome of Theorem 2.

By Claim B.1, and Theorems 1 and 2, (ii) holds. **Q.E.D.**

Appendix C: Proofs of Proposition 5 and Theorem 4

Let $r \in \mathbb{R}_+^m$ be the reserve price vectors. A pair $(z, p) \in Z \times \mathbb{R}_+^m$ is an **equilibrium with reserve price** r if (i) (E-i) holds and (ii) for each $y \in M$, $p_y \geq r_y$, and if $p_y > r_y$ then there is some agent $i \in N$ such that $y = x_i$.

By Demange and Gale (1985), there is an equilibrium with reserve price r and the set of equilibrium prices with reserve price r is a complete lattice. Thus, there is an MPE with reserve price r. Let the MPE rule with reserve price r be a mapping from each preference profile to an MPE with reserve price r.

Fact C.1 (Demange and Gale, 1985): The MPE rule with reserve price r is strategy-proof on the set of general preference profiles.

For each $R \in (\mathcal{R}^G)^n$, let $W(R)$ and $W^{\min}(R)$ be the set of equilibria and that of MPEs for R. For each $0 \leq k \leq m$, let $W(k, R)$ and $W^{\min}(k, R)$ be the corresponding notions and $M(k)$ be the set of objects in the assignment market with k objects. Fact C.2 follows Definition 6 of E-generating mechanism.

Fact C.2: Let $R \in (\mathcal{R}^G)^n$ and $(z^{\min}, p^{\min}) \in W^{\min}(k, R)$. Let $i \in N$, $R'_i \in \mathcal{R}^G$, and $R' = (R'_i, R_{-i}) \in (\mathcal{R}^G)^n$. Let (z', p') be the outcome of E-generating mechanism for R'. Let $i^* \in N \backslash \{i\}$ be such that $x'_{i^*} = y$. Then
(i) $z'_{i^*} \, R_{i^*} \, z^{\min}_{i^*}$ and for each $j \in N \backslash \{i^*\}$, $z'_j \, I_j \, z^{\min}_j$,
(ii) $p'_{M(k)} = p^{\min}$, and
(iii) for each $x \in M(k)$ such that $p^{\min}_x > 0$, there is $j \in N \backslash \{i^*\}$ such that $x'_j = x$.

C.1 Proof of Proposition 5

Part (i) Let $R \in (\mathcal{R}^G)^n$, $(z^{\min}, p^{\min}) \in W^{\min}(k, R)$, and $i \in N$. Let (z, p) and (z', p') be the outcomes of E-generating mechanism for R and $R' = (R'_i, R_{-i})$, i.e, $z = g^{sub1}(R; z^{\min})$ and $z' = g^{sub1}(R'; z^{\min})$. We show $z_i \, R_i \, z'_i$.

By contradiction, suppose that $z'_i \, P_i \, z_i$. By Fact C.2(i), $x'_i = y$ and so $p'_{k+1} = C^1_+(y; z^{\min}_{N \backslash \{i\}})$. In case of $x_i = y$, by $V_i(y; z^{\min}_i) = C^1_+(y; z^{\min})$, $p_{k+1} = C^2_+(y; z^{\min}) = C^1_+(; z^{\min}_{N \backslash \{i\}})$. Thus $z'_i = z_i$, contradicting $z'_i \, P_i \, z_i$. In case of $x_i \in M(k)$, by $V_i(y; z^{\min}_i) \leq C^2_+(y; z^{\min}) \leq C^1_+(; z^{\min}_{N \backslash \{i\}}) = p'_{k+1}$, $z_i \, I_i \, z^{\min}_i \, R_i \, (y, p'_y) = z'_i$, contradicting $z'_i \, P_i \, z_i$. Thus, $z_i \, R_i \, z'_i$.

Part (ii) Let $R \in (\mathcal{R}^G)^n$ and $z \in Z^{k+1}$ be an equilibirum allocation. Let N_C and N_U be defined at z for R. If $i \in N_C$, then agent i does not participate in the MPE-adjustment mechansim and keeps the same allocation as z_i, and so incentive property holds trivially. Thus, let $i \in N_U$.

Let $z_{N_U} = g^{sub2}(R; z)$ and $z'_{N_U} = g^{sub2}(R'; z)$ be outcomes of MPE-adjustment mechanism for R and $R' = (R'_i, R_{-i})$. Since z_{N_U} is an MPE for (N_U, M_U, R_{N_U}) with $r' = (C^1_+(x; z^{\min}_{N_C}))_{x \in M_U}$, and that z'_{N_U} is an MPE for (N_U, M_U, R'_{N_U}) with r', by Fact C.1, $z_i \, R_i \, z'_i$. **Q.E.D.**

C.2 Proof of Theorem 4

The proof contains three steps. Step 1 gives additional four facts. Step 2 establishes three lemmas based on the above facts, together with Facts C.1 and C.2. Step 3 completes the proof.

Step 1: Construction of Facts C.3 to C.6

Fact C.3: Let $R \in (\mathcal{R}^G)^n$, $(z, p) \in W(R)$, $N' \subseteq N$, and $M' \subseteq M$. Let N_C be defined at (z, p).
(i) Let $(z_{N'}, p_{M'})$ be an MPE in $(N', M', R_{N'})$. Then $N' \subseteq N_C$.
(ii) Let $i \in N$, $R'_i \in \mathcal{R}^G$, $R' = (R'_i, R_{-i}) \in (\mathcal{R}^G)^n$ and $(z, p) \in W^{\min}(R')$. Then $i \in N_C$.

Proof: (i) Let $x \in M'$ be such that $p_x > 0$ for $(N', M', R_{N'})$. Since $(z_{N'}, p_{M'})$ is an MPE in $(N', M', R_{N'})$, there is a DCP $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of agents to x in $(N', M', R_{N'})$. Note that for each $\lambda = 1, \ldots, \Lambda$,

$$\{y \in M' : \forall y' \in M', (y, p_y) \, R_\lambda \, (y', p_{y'})\} \subseteq \{y \in M : \forall y' \in M, (y, p_y) \, R_\lambda \, (y', p_{y'})\}.$$

Thus, the DCP $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of agents to x is also a DCP in (N, M, R). Thus $x \in N_C$. Thus, $M' \subseteq M_C$ and so $N' \subseteq N_C$.
(ii) If $x_i = 0$ or $p_{x_i} = 0$, by definition, $i \in N_C$. Thus, let $p_{x_i} > 0$. By $(z, p) \in W^{\min}(R')$, there is a DCP $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of agents to x_i at (z, p) in (N, M, R'). By $R_{-i} = R'_{-i}$ and $i \notin \{i_\lambda\}_{\lambda=1}^{\Lambda-1}$, the agents in $\{i_\lambda\}_{\lambda=1}^{\Lambda-1}$ have the same demands at p in (N, M, R) as in (N, M, R'). Thus, $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ is also a DCP to x_i at the same pair (z, p) in (N, M, R). Thus $i \in N_C$. **Q.E.D.**

Fact C.4 follows the definition of DCP (Definition 2).

Fact C.4: For $x \in M_C$ such that $p_x > 0$ and each DCP $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of agents to x, $\{i_\lambda\}_{\lambda=1}^{\Lambda} \subseteq N_C$.

Given $N' \subseteq N$, $M' \subseteq M$, $R'_{N'} \in (\mathcal{R}^G)^{N'}$ and $r \in \mathbb{R}_+^{|M'|}$, let $Z(N', M', R'_{N'}, r)$ and $Z^{\min}(N', M', R'_{N'}, r)$ denote the sets of equilibrium and MPE allocations for $(N', M', R'_{N'})$ with reserve price r, respectively. Let $W(\cdot,\cdot,\cdot,\cdot)$ and $W^{\min}(\cdot,\cdot,\cdot,\cdot)$ be the sets of equilibria and MPEs similarly defined for $(N', M', R'_{N'})$ with reserve price r. When $r = \mathbf{0}$ or $N' = N$ or $M' = M$, we just omit writing r or N' or M'. Recall that $Z(R)$ is the set of equilibrium allocations for (N, M, R) with $r = \mathbf{0}$. Given $N' \subseteq N$, denote

$$Z_{N'}^{\min}(R) \equiv \{z_{N'} : \exists z_{N\setminus N'} \text{ such that } (z_{N'}, z_{N\setminus N'}) \in Z^{\min}(R)\}.$$

The following fact is easy to see.

Fact C.5: Let $R \in (\mathcal{R}^G)^n$ and $(z, p) \in W(R)$. Let N_C be defined at (z, p) for R. Let $N' \subseteq N_C$, $N'' = N\setminus N'$ and $M'' = \{x_i : i \in N''\}$. Then

$$Z_{N''}^{\min}(R) = Z^{\min}(N'', M'', R_{N''}, r) \text{ where } r_x = C_+^1(R, x; z_{N'}) \text{ for each } x \in M''.$$

Fact C.6: Let $R \in (\mathcal{R}^G)^n$ and $(z, p) \in W(R)$. Let N_U be defined at (z, p). Let $N' \subseteq \{i \in N : x_i \in M \text{ and } p_{x_i} > 0\}$. If for each $j \in N\setminus N'$, $D_j(p) \cap \{x_i : i \in N'\} = \emptyset$, then $N' \subseteq N_U$.

Proof: By contradiction, suppose that there is $k \in N' \cap N_C$. Then, by $x_k \in M_C$, $p_{x_k} > 0$ and Definition 2, there is a DCP of agents $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ to x_k satisfying (a) $p_{x_{i_1}} = 0$, $i_\Lambda = k$, and $x_{i_\Lambda} = x_k$, and (b) for each $\lambda \in \{1, \cdots, \Lambda - 1\}$, $\{x_{i_\lambda}, x_{i_{\lambda+1}}\} \subseteq D_{i_\lambda}(p)$. By (b), for agent $i_{\Lambda-1}$, $\{x_k, x_{i_{\Lambda-1}}\} \subseteq D_{i_{\Lambda-1}}(p)$. Since $k \in N'$, $x_k \in D_{i_{\Lambda-1}}(p)$ and $D_j(p) \cap \{x_i : i \in N'\} = \emptyset$ for each $j \in N\setminus N'$, we have $i_{\Lambda-1} \notin N\setminus N'$, ie., $i_{\Lambda-1} \in N'$. Repeating the same argument, we can show $\{i_\lambda : \lambda = 1, ..., \Lambda\} \subseteq N'$. By (a), we have $p_{x_{i_1}} = 0$, contradicting that for each $i \in N'$, $x_i \in M$ and $p_{x_i} > 0$. **Q.E.D.**

Step 2: **Construction of Lemmas C.1 to C.3**

Lemma C.1: Let $R \in (\mathcal{R}^G)^n$ and $(z^*, p^{\min}) \in W^{\min}(k, R)$. Let (z, p) be the outcome of E-generating mechanism for R. Let N_C, N_U, M_C, and M_U be defined at (z, p) for R. Let $i^* \in N$ be such that $x_{i^*} = k + 1$.

(i) $(z_{N_C\backslash\{i^*\}}, p_{M_C\backslash\{k+1\}}) \in W(N_C\backslash\{i^*\}, M_C\backslash\{k+1\}, R_{N_C\backslash\{i^*\}})$ and $(z_{N_C}, p_{M_C}) \in W^{\min}(N_C, M_C, R_{N_C})$.
(ii) If (a) $C^1_+(R, k+1; z^*) > C^2_+(R, k+1; z^*)$, or
(b) $\left|\{j \in N_C\backslash\{i^*\} : V_j(k+1; z^*_j) = C^2_+(R, k+1; z^*)\}\right| \leq 1$,
then $(z_{N_C\backslash\{i^*\}}, p_{M_C\backslash\{k+1\}}) \in W^{\min}(N_C\backslash\{i^*\}, M_C\backslash\{k+1\}, R_{N_C\backslash\{i^*\}})$.

Proof: **Part (i)**: By Facts C.2(i) and C.2(ii), (E-i) holds for $(N_C\backslash\{i^*\}, M_C\backslash\{k+1\}, R_{N_C\backslash\{i^*\}})$. By Fact C.2(iii), (E-ii) holds for $(N_C\backslash\{i^*\}, M_C\backslash\{k+1\}, R_{N_C\backslash\{i^*\}})$. Thus $(z_{N_C\backslash\{i^*\}}, p_{M_C\backslash\{k+1\}}) \in W(N_C\backslash\{i^*\}, M_C\backslash\{k+1\}, R_{N_C\backslash\{i^*\}})$. By the same reasoning, $(z_{N_C}, p_{M_C}) \in W(N_C, M_C, R_{N_C})$.

Let $x \in M_C$ be such that $p_x > 0$ and $\{i_\lambda\}^\Lambda_{\lambda=1}$ be a DCP $\{i_\lambda\}^\Lambda_{\lambda=1}$ of agents to x. Then by Fact C.4, $\{i_\lambda\}^\Lambda_{\lambda=1} \subseteq N_C$. Thus, each $x \in M_C$ such that $p_x > 0$ is also connected in (N_C, M_C, R_{N_C}). Thus by Proposition 1, $(z_{N_C}, p_{M_C}) \in W^{\min}(N_C, M_C, R_{N_C})$.

Part (ii) Let $x \in M_C\backslash\{k+1\}$ be such that $p_x > 0$ and $\{i_\lambda\}^\Lambda_{\lambda=1}$ be a DCP $\{i_\lambda\}^\Lambda_{\lambda=1}$ of agents to x. Then by Fact C.4, $\{i_\lambda\}^\Lambda_{\lambda=1} \subseteq N_C$. Note that to establish that $(z_{N_C\backslash\{i^*\}}, p_{M_C\backslash\{k+1\}}) \in W^{\min}(N_C\backslash\{i^*\}, M_C\backslash\{k+1\}, R_{N_C\backslash\{i^*\}})$, by Proposition 1 and Part (i), we only need to show $i^* \notin \{i_\lambda\}^\Lambda_{\lambda=1}$, which implies $\{i_\lambda\}^\Lambda_{\lambda=1} \subseteq N_C\backslash\{i^*\}$, in each of Case (a) and Case (b). In the following, we show $\{i_\lambda\}^\Lambda_{\lambda=1} \subseteq N_C\backslash\{i^*\}$ in Case (a).

Assume $C^1_+(R, k+1; z^*) > C^2_+(R, k+1; z^*)$. Then by $V_{i^*}(k+1; z^*_{i^*}) = C^1_+(R, k+1; z^*)$ and $x_{i^*} = k+1$, we have: $V_{i^*}(k+1; z^*_{i^*}) > C^2_+(R, k+1; z^*) = t_{i^*}$ and $z_{i^*} = (k+1, t_{i^*})\, P_{i^*}\, z^*_{i^*}$. Thus, $D_{i^*}(p) = \{k+1\}$. By Definition 2(ii-4), $i^* \notin \{i_\lambda\}^\Lambda_{\lambda=1}$.

For Case (b), if $\left|\{j \in N_C\backslash\{i^*\} : V_j(k+1; z^*_j) = C^2_+(R, k+1; z^*)\}\right| = 0$. it is straightforward to see $i^* \notin \{i_\lambda\}^\Lambda_{\lambda=1}$. For the other case, the same reasoning of Case (a) works. **Q.E.D.**

Lemma C.2: Let $R \in (\mathcal{R}^G)^n$ and $(z^*, p^{\min}) \in W^{\min}(k, R)$. Let $i \in N$, $R'_i \in \mathcal{R}^G$ and $R' = (R'_i, R_{-i}) \in (R^G)^n$. Let (z, p) and (z', p') be the outcomes of E-generating mechanism for R and R'. Let N_C, N_U, M_C, and M_U be defined at (z, p) for R. Let $i^* \in N$ be such that $x_{i^*} = k+1$.
(i) For each $j \in N_C\backslash\{i^*\}$, $x'_j \in M_C \cup \{0, k+1\}$.
(ii) For each $x \in M_U\backslash\{k+1\}$, there is $j \in N_U \cup \{i^*\}$ such that $x'_j = x$.

Proof: **Part (i)**: Let $j \in N_C\backslash\{i^*\}$. By contradition, suppose that $x'_j \notin M_C \cup \{0, k+1\}$. By $x'_j \neq k+1$, we have $x'_j \in M_U\backslash\{k+1\}$. By $x'_j \neq k+1$ and Fact C.2(ii), $p_{x'_j} = p^{\min}_{x'_j} = p'_{x'_j}$. Thus, by Lemma 1(ii), and $x'_j \in M_U$, we have $V_j(x'_j; z^*_j) < p_{x'_j} = p'_{x'_j}$. On the other hand, by Fact C.2(i), $(x'_j, p'_{x'_j}) = z'_j\, R_j\, z^*_j$. Thus, $V_j(x'_j; z^*_j) \geq p'_x$. This is a contradiction.

Part (ii): Let $x \in M_U\backslash\{k+1\}$. By Lemma 1(ii) and $x \in M_U$, $p_x > 0$. By Fact C.2(ii) and $x \neq k+1$, $p_x = p^{\min}_x = p'_x$. Thus, $p'_x > 0$. By Fact C.2(iii) and Part (i), there is $j \in N_U \cup \{i^*\}$ such that $x'_j = x$. **Q.E.D.**

Lemma C.3: Let $R \in (\mathcal{R}^G)^n$ and $(z^*, p^{\min}) \in W^{\min}(k, R)$. Let $i \in N$, $R'_i \in R^G$ and $R' = (R'_i, R_{-i}) \in (\mathcal{R}^G)^n$. Let (z, p) and (z', p') be respectively the outcomes of some E-generating mechanisms for R and R'. Let N_C, N_U, M_C, and M_U be defined at (z, p) for R. Let N'_C, N'_U, M'_C, and M'_U be defined at (z', p') for R'. Let $i^* \in N$ be such that $x_{i^*} = k + 1$.

(i) Let $i^* \neq i$, $k + 1 \in M_C$ and $p_{k+1} = C^1_+(R, k + 1; z^*) > 0$. Then $(z, p) \in W^{\min}(M(k + 1), R)$.

(ii) Let $i \in N_C$ and $i^* \neq i$.

(ii-1) Let $V'_i(k + 1; z^*_i) = C^1_+(R, k + 1; z^*) > 0$. Then

$$V'_i(k + 1; z^*_i) = p'_{k+1} = C^1_+(R, k + 1; z^*) = C^1_+(R', k + 1; z^*) = V_{i^*}(k + 1; z^*_{i^*}) > 0.$$

(ii-2) Let $V'_i(k + 1; z^*_i) = C^1_+(R, k + 1; z^*) > 0$. Then $(z', p') \in W^{\min}(N, M(k + 1), R')$.

(ii-3) Let $V'_i(k + 1; z^*_i) > C^1_+(R, k + 1; z^*) > 0$. Let $R''_i \in R^G$ be such that for each $x \in M(k)$, $V''_i(x; \cdot) = V'_i(x; \cdot)$ and $V''_i(k + 1; z^*_i) = C^1_+(R, k + 1; z^*)$. Let $R'' \equiv (R''_i, R_{-i})$. Then $(z', p') \in W^{\min}(N, M(k + 1), R'')$.

(iii) Let $i^* \in N_U$ and $p'_{k+1} > C^1_+(R_{N_C}, k + 1; z^*)$. Then $N_C = N'_C$, $M_C = M'_C$, $M_U = M'_U$, and $N_U = N'_U$.

Proof: **Part (i-1)** By Proposition 2, $(z, p) \in W(M(k + 1), R)$. To establish $(z, p) \in W^{\min}(M(k + 1), R)$, by Proposition 1, we only need to show $N_C = N$.

By Fact C.2(ii), we have: (1) $p_{M(k)} = p^{\min}$. Since z is generated by E-generating mechanism for R and $x_{i^*} = k + 1$, there is a sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ to $x^*_{i^*}$ such that (2) $i_\Lambda = i^*$, (3) $x^*_{i_1} = 0$ or $p^{\min}_{x^*_{i_1}} = 0$, and (4) for each $\lambda \in \{1, ..., \Lambda - 1\}$, $\{x^*_{i_\lambda}, x^*_{i_{\lambda+1}}\} \in D_{i_\lambda}(p^{\min})$ and $x_{i_\lambda} = x^*_{i_{\lambda+1}}$. By $x_{i^*} = k+1$, $V_{i^*}(k+1; z^*_{i^*}) = C^1_+(R', k+1; z^*) = p_{k+1}$. Thus by (2), we have: (5) $x_{i^*} = x_{i_\Lambda} = k + 1$ and (6) $V_{i_\Lambda}(k + 1; z^*_{i_\Lambda}) = p_{k+1}$. In the next two paragraphs, we show that $\{i_\lambda\}_{\lambda=1}^{\Lambda} \subseteq N_C$.

By $k + 1 \in M_C$, $i_\Lambda \in N_C$, and there is a DCP $\{i'_\lambda\}_{\lambda=1}^{\Lambda'}$ to $k + 1$ in (z, p). Note

$$(k + 1, p_{k+1}) \underset{(6)}{I_{i_\Lambda}} z^*_{i_\Lambda} = (x^*_{i_\Lambda}, p^{\min}_{x^*_{i_\Lambda}}) \underset{(1)\ \&\ (4)}{=} (x_{i_{\Lambda-1}}, p_{x_{i_{\Lambda-1}}}).$$

Thus, by (1), (4) and (5), we have $\{x_{i_{\Lambda-1}}, x_{i_\Lambda}\} \subseteq D_{i_\Lambda}(p)$. Thus, $\{i'_\lambda\}_{\lambda=1}^{\Lambda'} \cup \{i_{\Lambda-1}\}$ is a DCP to $x_{i_{\Lambda-1}} = x^*_{i^*}$ in (z, p). Thus, $i_{\Lambda-1} \in N_C$.

By (4), we have: $\{x^*_{i_{\Lambda-1}}, x^*_{i_\Lambda}\} \subseteq D_{i_{\Lambda-1}}(p^{\min})$, $x^*_{i_{\Lambda-1}} = x_{i_{\Lambda-2}}$ and $x^*_{i_\Lambda} = x_{i_{\Lambda-1}}$. Thus, by (1), $\{x_{i_{\Lambda-2}}, x_{i_{\Lambda-1}}\} \in D_{i_{\Lambda-1}}(p)$. Thus, $\{i'_\lambda\}_{\lambda=1}^{\Lambda'} \cup \{i_{\Lambda-1}, i_{\Lambda-2}\}$ is a DCP to $x_{i_{\Lambda-2}} = x^*_{i_{\Lambda-1}}$ in (z, p). Thus, $i_{\Lambda-2} \in N_C$. Similarly, we have that for each $\lambda = \Lambda - 3, \ldots, 1$, $i_\lambda \in N_C$. Thus, $\{i_\lambda\}_{\lambda=1}^{\Lambda} \subseteq N_C$.

Finally, we show that for each $j \notin \{i_\lambda\}_{\lambda=1}^{\Lambda}$, $j \in N_C$. Let $j \notin \{i_\lambda\}_{\lambda=1}^{\Lambda}$. Note that $x_j = x^*_j$, and so by $(z^*, p^{\min}) \in W^{\min}(k, R)$ and Proposition 1, there is a DCP $\{i''_\lambda\}_{\lambda=1}^{\Lambda''}$ to $x_j = x^*_j$ in $(z^*, p^{\min})$. If $\{i''_\lambda\}_{\lambda=1}^{\Lambda''} \cap \{i_\lambda\}_{\lambda=1}^{\Lambda} = \emptyset$, since for each λ,

$x_{i''_\lambda} = x^*_{i''_\lambda} \neq k+1$, by (1), $\{i''_\lambda\}_{\lambda=1}^{\Lambda''}$ is also a DCP to $x_j = x^*_j$ in (z,p) and so $j \in N_C$. Thus, assume $\{i''_\lambda\}_{\lambda=1}^{\Lambda''} \cap \{i_\lambda\}_{\lambda=1}^{\Lambda} \neq \emptyset$. Then, there is $\lambda' \in \{1,\ldots,\Lambda''\}$ such that $i''_{\lambda'} \in \{i_\lambda\}_{\lambda=1}^{\Lambda}$ and for any $\lambda'' > \lambda'$, $i''_{\lambda''} \notin \{i_\lambda\}_{\lambda=1}^{\Lambda}$. Let λ'' be such that $i_{\lambda''} = i''_{\lambda'}$. Note that for any $\lambda''' > \lambda'$, $x_{i''_{\lambda'''}} = x^*_{i''_{\lambda'''}}$, and that $x_{i_{\lambda''}} = x_{i''_{\lambda'}} \, I_{i''_{\lambda'}} \, x^*_{i''_{\lambda'}} \, I_{i''_{\lambda'}} \, x^*_{i''_{\lambda'+1}} = x_{i''_{\lambda'+1}}$, and $\{x_{i''_{\lambda'+1}}, x_{i_{\lambda''}}\} \subseteq D_{i'_{\lambda'}}(p)$. Thus, the sequence $\{i'_\lambda\}_{\lambda=1}^{\Lambda'} \cup \{i_\lambda\}_{\lambda=\lambda''}^{\Lambda} \cup \{i''_\lambda\}_{\lambda=\lambda'+1}^{\Lambda'}$ is a DCP to x_j in (z,p) and so $j \in N_C$.

Part (ii-1): By $x_{i^*} = k+1$, $V_{i^*}(k+1; z^*_{i^*}) = C^1_+(R, k+1; z^*)$. By $V'_i(k+1; z^*_i) = C^1_+(R, k+1; z^*) > 0$,

$$V'_i(k+1; z^*_i) = C^1_+(R', k+1; z^*) = C^1_+(R, k+1; z^*) = V_{i^*}(k+1; z^*_{i^*}) > 0,$$

and so by $i \neq i^*$, $p'_{k+1} = C^2_+(R', k+1; z^*) = C^1_+(R, k+1; z^*)$. Thus, we have:

$$V_i^{'}(k+1; z^*_i) = p'_{k+1} = C^1_+(R, k+1; z^*) = C^1_+(R', k+1; z^*) = V_{i^*}(k+1; z^*_{i^*}) > 0.$$

Part (ii-2): By Proposition 2, $(z', p') \in Z(N, M(k+1), R')$. Thus, if p' is an MPE price for $(N, M(k+1), R')$, $(z', p') \in W^{\min}(N, M(k+1), R')$. Let z'' be such that $z''_{i^*} = (k+1, p'_{k+1})$ and for each $j \in N\backslash\{i^*\}$, $z''_j = z_j$. We show $(z'', p') \in W^{\min}(N, M(k+1), R')$.

First, we show $(z'', p') \in W(N, M(k+1), R')$. By Part (ii-1) and Fact C.2(ii), we have: (1) $p'_{M(k)} = p^{\min} = p_{M(k)}$ and $p'_{k+1} \geq p_{k+1}$. By construction, unassigned objects at (z,p) remain unassigned at (z'', p'). Thus (E-ii) holds. By (1), for each $j \in N\backslash\{i^*\}$, $x''_j = x_j \in D_j(p')$. By Part (ii-1) and $z''_{i^*} = (k+1, p'_{k+1})$, $z''_{i^*} \, I_{i^*} \, z^*_{i^*}$. Thus $x''_{i^*} = k+1 \in D_{i^*}(p')$. Thus (E-i) holds. Thus, $(z'', p') \in W(N, M(k+1), R')$.

Next, we show that $(z'', p') \in W^{\min}(N, M(k+1), R')$. Let N''_C, N''_U, M''_C, and M''_U be defined at (z'', p') for R'. Note that we only need to show $x''_{i^*} = k+1 \in M''_C$, which, by Part (i) and Part (ii-1), implies $(z'', p') \in W^{\min}(N, M(k+1), R')$.

By $i \in N_C$, there is a DCP $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of agents to x_i at (z,p) for R. By $i^* \in N\backslash\{i\}$ and $i_\Lambda = i$, $i^* \neq i_\Lambda$. Thus, there are the two cases below.

Case 1: $i^* \notin \{i_\lambda\}_{\lambda=1}^{\Lambda-1}$ Since $R'_{-i} = R_{-i}$, $i_\Lambda = i \neq i^*$, $i^* \notin \{i_\lambda\}_{\lambda=1}^{\Lambda-1}$, and $z''_j = z_j$ for each $j \in N\backslash\{i^*\}$, by (1), $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ is also a DCP to x_i at (z'', p') for R'. Thus $i \in M'_C$. By $i \neq i^*$, $z_i \, I'_i \, z^*_i$. Thus, by Part (ii-1), $z''_i = z_i \, I'_i \, z^*_i \, I'_i \, (k+1, p'_{k+1}) = z''_{i^*}$. Thus, $\{x''_i, x''_{i^*}\} \in D_{i_\Lambda}(p')$, and $\{i_\lambda\}_{\lambda=1}^{\Lambda} \cup \{i^*\}$ is a DCP to $k+1$ at (z'', p') for R'. Thus $k+1 \in M''_C$.

Case 2: $i^* \in \{i_\lambda\}_{\lambda=1}^{\Lambda-1}$ Let $\Lambda' \leq \Lambda - 1$ be such that $i^* = i_{\Lambda'}$. Then, the subsequence $\{i_\lambda\}_{\lambda=1}^{\Lambda'}$ of $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ is a DCP to $x_{i^*} = k+1$ at (z,p) for R. For each $\lambda \in \{1,\ldots,\Lambda'-1\}$, by $i_\lambda \neq i^*$, $z''_{i_\lambda} = z_{i_\lambda}$. Thus by (1), $\{i_\lambda\}_{\lambda=1}^{\Lambda'}$ is also a DCP to $x''_{i^*} = x_{i^*} = k+1$ at (z'', p') for R'. Thus $k+1 \in M''_C$.

Part (ii-3): By $V''_i(k+1; z^*_i) = C^1_+(R, k+1; z^*)$, there is an outcome (z'', p'') of E-generating mechanism for R'' such that $x''_i = k+1$. By $V''_i(k+1; z^*_i) = C^1_+(R, k+1; z^*) > 0$ and Part (ii-2), $(z'', p'') \in W^{\min}(N, M(k+1), R'')$. To show

$(z', p') \in W^{\min}(N, M(k+1), R'')$, we need to show that $p' = p''$, $z''_i = z'_i$, and for each $j \in N\backslash\{i\}$, $z'_j \, I_j \, z''_j$.

By $i \neq i^*$ and $V'_i(k+1; z^*_i) > C^1_+(R, k+1; z^*) > 0$,

$$p'_{k+1} = C^2_+(R', k+1; z^*) = C^1_+(R, k+1; z^*).$$

By Part (ii-1) and $V''_i(k+1; z^*_i) = C^1_+(R, k+1; z^*)$, $p''_{k+1} = C^1_+(R, k+1; z^*)$. Thus, $p'_{k+1} = p''_{k+1}$. By Fact C.2(ii), $p'_{M(k)} = p^{\min} = p''_{M(k)}$. Thus, $p' = p''$.

By $V'_i(k+1; z^*_i) > C^1_+(R, k+1; z^*) = C^2_+(R', k+1; z^*)$, $x'_i = k+1 = x''_i$. Thus, $z'_i = z''_i$. By Fact C.2(i), for each $j \in N\backslash\{i\}$, $z'_j \, I_j \, z^*_j \, I_j \, z''_j$.

Part (iii): By Fact C.2(ii), $p_{M(k)} = p^{\min} = p'_{M(k)}$. Since $p'_{k+1} > C^1_+(R_{N_C}, k+1; z^*)$, we have: (1) there is some $i' \in N_U$ such that $x'_{i'} = k+1$.

By Lemma C.2(i), $i^* \in N_U$, and (1), we have: (2) for each $j \in N_C$, $x'_j \in M_C \cup \{0\}$.

By Lemma C.1(i), $i^* \in N_U$, and $x_{i^*} = k+1$, $(z_{N_C}, p_{M_C}) \in W^{\min}(N_C, M_C, R_{N_C})$. By $p_{M_C} = p'_{M_C}$, (2), Fact C.2(i) and C.2 (iii), $(z'_{N_C}, p'_{M_C}) \in W^{\min}(N_C, M_C, R'_{N_C})$. Thus, by Proposition 1 and Fact C.3, we have: (3) $N_C \subseteq N'_C$ and $M_C \subseteq M'_C$.

By Fact C.2(ii), Lemma 1(ii), and Lemma C.2(ii), for each $x \in M_U$, $p'_x > 0$ and x is assigned to some $j \in N_U$ at (z', p') for R'. Since $p'_{k+1} > C^1_+(R_{N_C}, k+1; z^*)$ and $p'_{M(k)} = p_{M(k)}$, $\{i \in N : D_i(p') \cap M_U \neq \emptyset)\} = N_U$. Thus, $N_U \subseteq N'_U$ and $M_U \subseteq M'_U$. By (3), $N_C = N'_C$, $M_C = M'_C$, $M_U = M'_U$, and $N_U = N'_U$. **Q.E.D.**

Step 3: Completion of the proof

Let $R \in (\mathcal{R}^G)^n$, $(z^*, p^{\min}) \in W^{\min}(k, R)$, and $i \in N$. Let (z, p) and (z', p') be the outcomes of E-generating mechanism for R and $R' \equiv (R'_i, R_{-i})$. Let N_C, N_U, M_C, and M_U be defined at (z, p) for R, and N'_C, N'_U, M'_C, and M'_U be defined at (z', p') for R'. Let $(\widehat{z}, \widehat{p})$ and $(\widetilde{z}, \widetilde{p})$ be the outcomes of MPE-adjustment mechanism for R and R' from (z, p) and (z', p'), respectively. Thus $\widehat{z} = f_{SV}(R; z; k+1)$ and $\widetilde{z} = f_{SV}(R'; z; k+1)$.

Assume for each $j \in N$, $V_j(k+1; z^*_j) \leq 0$. Then $p_{k+1} = 0$ and $(z^*, p) \in W^{\min}(M(k+1), R)$ where $p = (p^{\min}, 0)$. In case $V'_i(k+1; z^*_i) \leq 0$, $z'_i = z^*_i = z_i$ holds. By Theorem 1(i) and Definition 11, $\widetilde{z}_i = z'_i = z_i = \widehat{z}_i$. In case $V'_i(k+1; z^*_i) > 0$, $z'_i = (k+1, 0)$ and $i \in N'_C$ hold. Thus, by $V_i(k+1; z^*_i) \leq 0$, $\widehat{z}_i = z_i \, R_i \, z'_i = \widetilde{z}_i$. In the following, assume $C^1_+(R, k+1; z^*) > 0$. Then there is $i^* \in N$ such that $x_{i^*} = k+1$. We show $\widehat{z}_i \, R_i \, \widetilde{z}_i$ by considering two cases where $i \in N_C$ in Case I and $i \in N_U$ in Case II.

Case I: $i \in N_C$

By Theorem 1(i) and Definition 11, for agent i, $\widehat{z}_i = z_i$. We conclude that $i \in N'_C$ at (z', p') for R' for each subcase. This implies that i does not participate the MPE-adjustment mechanism from (z', p') so that by $\widehat{z}_i = z_i$ and Fact C.2(i), $\widehat{z}_i = z_i \, R_i \, z^*_i \, I_i \, z'_i = \widetilde{z}_i$.

Case I-1: $i \neq i^*$.

Case I-1-1: $V_i'(k+1; z_i^*) < C_+^1(R, k+1; z^*)$

Case I-1-1-1-1: $C_+^1(R, k+1; z^*) > C_+^2(R, k+1; z^*)$ or $\left|\{j \in N_C \backslash \{i^*, i\} : V_j(k+1; z_j^*) = C_+^2(R, k+1; z^*)\}\right| = 0$.

Since $V_i'(k+1; z_i^*) < C_+^1(R, k+1; z^*)$ and since $C_+^1(R, k+1; z^*) > C_+^2(R, k+1; z^*)$ or $\left|\{j \in N \backslash \{i, i^*\} : V_j(k+1; z_j^*) = C_+^2(R, k+1; z^*)\}\right| = 0$, we have: (1) for each $j \in N_C \backslash \{i^*\}$, $x_j' \neq k+1$. By Lemma C.1(ii), we have: (2) $(z_{N_C \backslash \{i^*\}}, p_{M_C \backslash \{k+1\}}) \in W^{\min}(N_C \backslash \{i^*\}, M_C \backslash \{k+1\}, R_{N_C \backslash \{i^*\}})$. By Lemma C.2(ii) and (1), we have: (3) for each $j \in N_C \backslash \{i^*\}$, $x_j' \in (M_C \backslash \{k+1\}) \cup \{0\}$.

By Fact C.2(i) and (ii), $p_{M_C \backslash \{k+1\}} = p'_{M_C \backslash \{k+1\}}$ and for each $j \in N_C \backslash \{i^*\}$, $z_j' \, I_j \, z_j$. By $R_{N_C \backslash \{i^*\}} = R'_{N_C \backslash \{i^*\}}$, (2), (3), and Fact C.2(iii), $(z'_{N_C \backslash \{i^*\}}, p'_{M_C \backslash \{k+1\}}) \in W^{\min}(N_C \backslash \{i^*\}, M_C \backslash \{k+1\}, R'_{N_C \backslash \{i^*\}})$. Thus, by $i \neq i^*$, Proposition 1 and Fact C.3, $i \in N_C'$ at (z', p') for R'.

Case I-1-1-2: $C_+^1(R, k+1; z^*) = C_+^2(R, k+1; z^*)$ and $\left|\{j \in N_C \backslash \{i^*, i\} : V_j(k+1; z_j^*) = C_+^2(R, k+1; z^*)\}\right| \geq 1$.

Since $p_{k+1} = C_+^2(R, k+1; z^*) = C_+^1(R, k+1; z^*) > 0$, $i^* \in N_C$, and $x_{i^*} = k+1$, by Lemma C.3(i), $(z, p) \in W^{\min}(M(k+1), R)$. Thus, by Proposition 1, $N = N_C$.

By $\left|\{j \in N \backslash \{i, i^*\} : V_j(k+1; z_j^*) = C_+^2(R, k+1; z^*)\}\right| \geq 1$, there is $j \in N_C = N \backslash \{i, i^*\}$ such that $V_j(k+1; z_j^*) = C_+^2(R, k+1; z^*)$. Thus, $p'_{k+1} = C_+^2(R', k+1; z^*) = C_+^2(R, k+1; z^*)$. By Fact C.2(i) and (ii), $p'_{M(k)} = p_{M(k)}$. Thus $p = p'$. By $V_i'(k+1; z_i^*) < C_+^1(R, k+1; z^*)$, $x_i' \neq k+1$. Since for each $j \in N \backslash \{i\}$, $z_j' \, I_j \, z_j$, by $p' = p$, $(z', p') \in W^{\min}(M(k+1), R)$. Since $(z', p') \in W(M(k+1), R)$, by Fact C.3, $i \in N_C'$ at (z', p') for R'.

Case I-1-2: $V_i'(k+1; z_i^*) = C_+^1(R, k+1; z^*)$

Since $i \neq i^*$, $i \in N_C$, and $V_i'(k+1; z_i^*) = C_+^1(R, k+1; z^*) > 0$, by Lemma C.3(ii-2), $(z', p') \in W^{\min}(N, M(k+1), R')$. Thus, $i \in N_C'$ at (z', p') for R'.

Case I-1-3: $V_i'(k+1; z_i^*) > C_+^1(R, k+1; z^*)$

Let R_i'' be such that for each $x \in M(k)$, $V_i''(x; \cdot) = V_i'(x; \cdot)$ and $V_i''(k+1; z_i^*) = C_+^1(R, k+1; z^*)$. Let $R'' \equiv (R_i'', R_{-i})$. Since $i \neq i^*$, $i \in N_C$, and $V_i'(k+1; z_i^*) > C_+^1(R, k+1; z^*) > 0$, by Lemma C.3(ii-3), $(z', p') \in W^{\min}(N, M(k+1), R'')$. Thus, by $(z', p') \in Z(R')$, and Fact C.3(ii), $i \in N_C'$ at (z', p') for R'.

Case I-2: $i = i^* \in N_C$.

Assume $C_+^2(R, k+1; z^*) = 0$. If $V_i'(k+1; z_i^*) \leq 0$, by Definition 8, $z' = z^* = \widetilde{z}$. Since $V_i(k+1; z_i^*) > 0$, then $\widehat{z}_i = z_i \, P_i \, z_i' = \widetilde{z}_i$. If $V_i'(k+1; z_i^*) > 0$, then $z_i' = z_i = (k+1, 0)$. Thus $i \in N_C'$ at (z', p') for R'.

In the following, assume $C_+^2(R, k+1; z^*) > 0$. Then there is $j \in N_C \backslash \{i\}$ such that $V_j(k+1; z_j^*) = C_+^2(R, k+1; z^*)$.

Case I-2-1: $V_i'(k+1; z_i^*) > C_+^2(R, k+1; z^*)$

Case I-2-1-1: $V_i(k+1; z_i^*) = C_+^2(R, k+1; z^*)$

Since $x_i = k+1$, $V_i(k+1; z_i^*) = C_+^1(R, k+1; z^*) = C_+^2(R, k+1; z^*) > 0$. Since $j \in N_C$, by Lemma 3(ii-2), $(z, p) \in W^{\min}(R)$.

Next we show $(z', p') \in W^{\min}(R)$. By Proposition 2, $(z', p') \in W(R')$. Since $V_i'(k+1; z_i^*) > C_+^2(R, k+1; z^*)$, $p'_{k+1} = C_+^2(R, k+1; z^*) = p_{k+1}$. By Fact C.2(ii), $p'_{M(k)} = p^{\min} = p_{M(k)}$. Thus $p' = p$. By Fact C.2(i), for each $j \in N \backslash \{i\}$, $z_j \, I_j \, z_j'$. Together with $z_i' = z_i$, we have $(z', p') \in W(R)$. Since $p = p'$ is an MPE price for R, $(z', p') \in W^{\min}(R)$.

By $(z', p') \in W^{\min}(R)$, $(z', p') \in W(R)$, and Fact C.3(ii), $i \in N_C'$ at (z', p') for R'.

Case I-2-1-2: $V_i(k+1; z_i^*) > C_+^2(R, k+1; z^*)$

Since $V_i'(k+1; z_i^*) > C_+^2(R, k+1; z^*)$, we have: (1) $x_i' = k+1$. By $x_i = k+1$, $j \in N_C \backslash \{i\}$, and Lemma C.1(i), we have: (2) $(z_{N_C \backslash \{i^*\}}, p_{M_C \backslash \{k+1\}}) \in W^{\min}(N_C \backslash \{i^*\}, M_C \backslash \{k+1\}, R_{N_C \backslash \{i^*\}})$. By Lemma C.2(i) and (1), we have: (3) for each $j \in N_C \backslash \{i\}$, $x_j' \in (M_C \backslash \{k+1\}) \cup \{0\}$.

By Fact C.2(i) and (ii), $p_{M_C \backslash \{k+1\}} = p'_{M_C \backslash \{k+1\}}$ and for each $j \in N_C \backslash \{i\}$, $z_j' \, I_j \, z_j^* \, I_j \, z_j$. By $R_{N_C \backslash \{i\}} = R'_{N_C \backslash \{i\}}$, (2), (3), Fact C.2(iii), $(z'_{N_C \backslash \{i\}}, p'_{M_C \backslash \{k+1\}}) \in W^{\min}(N_C \backslash \{i\}, M_C \backslash \{k+1\}, R'_{N_C \backslash \{i\}})$. By Fact C.3(i), $N_C \backslash \{i\} \subseteq N_C'$. Thus $j \in N_C'$. By $j \in N_C'$, $V_j(k+1; z_j^*) = C_+^2(R, k+1; z^*)$, and $z_j' \, I_j \, z_j^*$, $i \in N_C'$ at (z', p') for R'.

Case I-2-2: $V_i'(k+1; z_i^*) = C_+^2(R, k+1; z^*)$

In this case, $V_j(k+1; z_j^*) = C_+^2(R, k+1; z^*) = C_+^1(R', k+1; z^*)$. Since $j \neq i$, $j \in N_C$, and $x_i = k+1$ at (z, p) for R, by Lemma C.3(ii-2), $(z', p') \in W^{\min}(N, M(k+1), R')$. Thus $i \in N_C'$ at (z', p') for R'.

Case I-2-3: $V_i'(k+1; z_i^*) < C_+^2(R, k+1; z^*)$

Let R_i'' be such that for each $x \in M(k)$, $V_i''(x; \cdot) = V_i(x; \cdot)$ and $V_i''(k+1; z_i^*) = C_+^2(R, k+1; z^*)$. Let $R'' \equiv (R_i'', R_{-i})$.

First we show that $(z, p'') \in Z(R'')$ where $p''_{k+1} = C_+^2(R'', k+1; z^*)$ and $p''_{M(k)} = p_{M(k)}$. Since $V_i''(k+1; z_i^*) = V_j(k+1; z_j^*) = C_+^2(R, k+1; z^*)$, we have $p''_{k+1} = C_+^2(R'', k+1; z^*) = C_+^2(R, k+1; z^*) = p_{k+1}$. Thus $p'' = p$. Since $R_{-i} = R''_{-i}$ and $j \in N_C$, then j is connected at (z, p'') for R''. By $V_j(k+1; z_j^*) = C_+^2(R, k+1; z^*)$ and $x_i = k+1$, i and $k+1$ are also connected at (z, p'') for R''. Since $V_i''(k+1; z_i^*) = C_+^1(R'', k+1; z^*)$, by Lemma C.3(ii-3), $(z, p'') \in W^{\min}(R'')$.

Then we consider the following two scenarios.

Case I-2-3-1: $\left|\{k \in N \backslash \{i, j\} : V_k(k+1; z_k^*) \geq C_+^2(R, k+1; z^*)\}\right| \geq 1$

We show $(z', p') \in W^{\min}(R'')$. By $V_j(k+1; z_j^*) = C_+^2(R, k+1; z^*)$, $V_i'(k+1; z_i^*) < C_+^2(R, k+1; z^*)$, and $\left|\{k \in N \backslash \{i, j\} : V_k(k+1; z_k^*) \geq C_+^2(R, k+1; z^*)\}\right| \geq 1$, $p'_{k+1} = C_+^2(R', k+1; z^*) = C_+^2(R, k+1; z^*)$. Thus $p'_{k+1} = p_{k+1}$. By Fact C.2(ii) and $p = p''$, $p'_{M(k)} = p_{M(k)} = p''_{M(k)}$. Thus, $p' = p''$. By Proposition 2,

$(z', p') \in W(R')$. Thus for each $k \in N\backslash\{i\}$, $x'_k \in D_k(p')$. Since $V'_i(k+1; z^*_i) < C^2_+(R, k+1; z^*)$, $x'_i \neq k+1$ and by Fact C.2(i), $z'_i \, I_i \, z^*_i$. By the construction of R''_i, $x'_i \in D''_i(p')$. Thus, $(z', p') \in W(R'')$. Since p'' is an MPE price for R'' and $p'' = p'$, $(z', p') \in W^{\min}(R'')$. Since $(z', p') \in W(R')$, by Fact C.3(ii), $i \in N'_C$ at (z', p') for R'.

Case I-2-3-2: $\left|\{k \in N\backslash\{i,j\} : V_k(k+1; z^*_k) \geq C^2_+(R, k+1; z^*)\}\right| = 0$

Let $\widehat{z}''$ such that $\widehat{z}''_j = (k+1, p''_{k+1})$ and for each $i' \in N\backslash\{j\}$, $\widehat{z}''_{i'} = z'_{i'}$. By Fact C.2(i), for each $i' \in N\backslash\{j\}$, $z^*_i \, I_i \, z'_{i'} = \widehat{z}''_{i'}$. Since $V_j(k+1; z^*_j) = C^2_+(R, k+1; z^*) = p''_{k+1}$, $(\widehat{z}'', p'') \in W(R'')$. Thus, $(\widehat{z}'', p'') \in W^{\min}(R'')$. Note that for each $i' \in N\backslash\{i,j\}$, $V_{i'}(k+1; z^*_{i'}) < C^2_+(R, k+1; z^*) = p''_{k+1}$. By Lemma C.1(ii) and (1), $(\widehat{z}''_{N\backslash\{j\}}, p''_{M(k)}) = (z'_{N\backslash\{j\}}, p'_{M(k)}) \in W^{\min}(N\backslash\{j\}, M(k), R''_{N\backslash\{j\}})$. By the construction of R''_i, $(z'_{N\backslash\{j\}}, p'_{M(k)}) \in W^{\min}(N\backslash\{j\}, M(k), R'_{N\backslash\{j\}})$. By Fact C.3(ii), $i \in N'_C$ at (z', p') for R'.

Case II: $i \in N_U$

Case II-1: $i^* \in N_C$

By $i \in N_U$ and $i^* \in N_C$, $i \neq i^*$. Thus, by $V_{i^*}(k+1; z^*_i) = C^1_+(R, k+1; z^*)$, we have: $(*1)$ $V_i(k+1; z^*_i) \leq C^2_+(R, k+1; z^*)$. By $i^* \in N_C$ and $z^*_{i^*} = (k+1, C^2_+(R, k+1; z^*))$, we have: $(*2)$ there is $j \in N_C$ such that $V_j(k+1; z^*_j) = C^2_+(R, k+1; z^*)$. By $i \in N_U$ and $R_{-i} = R'_{-i}$, we have: $R_{N_C} = R'_{N_C}$ and $R'_{N_U} = (R'_i, R_{N_U\backslash\{i\}})$.

By contradiction, suppose $C^2_+(R, k+1; z^*) = C^1_+(R, k+1; z^*)$. Then $p_{k+1} = C^1_+(R, k+1; z^*) > 0$. By $i^* \in N_C$, $k+1 \in M_C$. Thus, by $i \neq i^*$ and Lemma C.3(i), $(z, p) \in W^{\min}(M(k+1), R)$ and so by Proposition 1, $N = N_C$, contradicting $i \in N_U$. Thus we have: $(*3)$ $C^2_+(R, k+1; z^*) < C^1_+(R, k+1; z^*) = V_{i^*}(k+1; z^*_{i^*})$.

Case II-1-1: $V'_i(k+1; z^*_i) \leq C^2_+(R, k+1; z^*)$

By Fact C.5 and Theorem 1, we have $\widehat{z}_{N_U} \in Z^{\min}(N_U, M_U, R_{N_U}, \widehat{r})$ where for each $x \in M_U$, $\widehat{r}_x = C^1_+(R_{N_C}, x; z^*)$, and $\widetilde{z}_{N'_U} \in Z^{\min}(N'_U, M'_U, R'_{N_U}, \widetilde{r})$ where for each $x \in M'_U$, $\widetilde{r}_x = C^1_+(R_{N'_C}, x; z^*)$.

Thus if $N_U = N'_U$ and $M_U = M'_U$, then $N_C = N'_C$ and $\widehat{r} = \widetilde{r}$, and so by $i \in N_U = N'_U$, $R'_{N_U} = (R'_i, R_{N_U\backslash\{i\}})$, and Fact C.1, $\widehat{z}_i \, R_i \, \widetilde{z}_i$. Thus we need to show $N_U = N'_U$ and $M_U = M'_U$.

By $i \in N_U$, $(*1)$, $(*2)$ and $V'_i(k+1; z^*_i) \leq C^2_+(R, k+1; z^*)$,

$$p'_{k+1} = C^2_+(R', k+1; z^*) = C^2_+(R, k+1; z^*) = p_{k+1}.$$

Thus, by $(*3)$, $x'_{i^*} = x_{i^*} = k+1 \in M_C$ and $z'_{i^*} = z_{i^*}$. By Fact C.2(ii), $p'_{M(k)} = p^{\min}_{M(k)} = p_{M(k)}$. Thus, by $p'_{k+1} = p_{k+1}$, we have: (1) $p = p'$.

By Lemma C.2(i) and $x'_{i^*} = k+1 \in M_C$, we have: (2) $\{x'_j\}_{j \in N_C} \subseteq M_C \cup \{0\}$. By $z_{i^*} = z'_{i^*}$ and Fact C.2(i), we have: (3) for each $j \in N_C$, $z'_j \, I_j \, z^*_j \, I_j \, z_j$. Thus, by

(1), we have: (4) for each $j \in N_C$, $\{x_j, x'_j\} \subseteq D_j(p) = D_j(p')$. Thus,

$$\begin{aligned}\text{Lemma C.1(i)} \;&\Rightarrow\; (z_{N_C}, p_{M_C}) \in W^{\min}(N_C, M_C, R_{N_C}) \\ &\Rightarrow\; (z_{N_C}, p'_{M_C}) \in W^{\min}(N_C, M_C, R'_{N_C}) \text{ by (1), } R_{N_C} = R'_{N_C} \\ &\Rightarrow\; (z'_{N_C}, p'_{M_C}) \in W^{\min}(N_C, M_C, R'_{N_C}) \text{ by (2), (4)}\end{aligned}$$

Thus, by $N_C \subseteq N$, $M_C \subseteq M$ and Fact C.3(i), we have: $N_C \subseteq N'_C$, which implies (5) $N_U \supseteq N'_U$.

In the following, we show $N_U \subseteq N'_U$.

By Lemma C.2(ii) and $x_{i^*} = x'_{i^*} = k+1 \in M_C$, we have $M_U \subseteq \{x'_j\}_{j \in N_U}$. Thus, by $|M_U| = |N_U|$ (Lemma 1(i)), we have: (6) $M_U = \{x'_j\}_{j \in N_U}$.

Let $j \in N_C$. By Lemma 1(ii), for each $x \in M_U$, $z_j \, P_j \, (x, p_x)$. By (1) and (3), for each $x \in M_U$, $z'_j \, I_j \, z_j \, P_j \, (x, p_x) = (x, p'_x)$. Thus, we have: (7) for each $j \in N_C$, $D_j(p') \cap M_U = \emptyset$.

For each $j \in N_U$, by (1) and (6), $p'_{x'_j} = p_{x'_j} > 0$. Thus, by (6), (7) and Fact C.6, we have $N_U \subseteq N'_U$.

Thus, by (5), $N_U = N'_U$, and so by (6), $M_U = M'_U$.

Case II-1-2: $V'_i(k+1; z^*_i) > C^2_+(R, k+1; z^*)$

By $i^* \in N_C$, $k+1 \in M_C$, Fact C.5, and Theorem 1, $\widehat{z}_{N_U \cup \{i^*\}} \in Z^{\min}(N_U \cup \{i^*\}, M_U \cup \{k+1\}, R_{N_U \cup \{i^*\}}, \widehat{r})$ where $\widehat{r}_x = C^1_+(R_{N_C \backslash \{i^*\}}, x; z^*)$ for each $x \in M_U \cup \{k+1\}$, and $\widetilde{z}_{N'_U} \in Z^{\min}(N'_U, M'_U, R'_{N_U}, \widetilde{r})$ where $\widetilde{r}_x = C^1_+(R_{N'_C}, x; z^*)$ for each $x \in M'_U$.

Thus if $N_U \cup \{i^*\} = N'_U$ and $M_U \cup \{k+1\} = M'_U$, then $N_C \backslash \{i^*\} = N'_C$ and $\widehat{r} = \widetilde{r}$, and so by $i \in N_U \cup \{i^*\} = N'_U$, $R'_{N_U \cup \{i^*\}} = (R'_i, R_{N_U})$, and Fact C.1, $\widehat{z}_i \, R_i \, \widetilde{z}_i$. Thus we need to show $N_U \cup \{i^*\} = N'_U$ and $M_U \cup \{k+1\} = M'_U$.

By (∗3), and $V'_i(k+1; z^*_i) > C^2_+(R, k+1; z^*)$,

$$p'_{k+1} = C^2_+(R', k+1; z^*) \geq \min\{V'_i(k+1; z^*_i), V_{i^*}(k+1; z^*_{i^*})\} > C^2_+(R, k+1; z^*) = p_{k+1}.$$

Thus we have: (1) for each $j \in N_C \backslash \{i^*\}$, $V_j(k+1; z^*_j) \leq p_{k+1} < p'_{k+1}$. Thus, by Fact C.2(ii), we have: (2) $p'_{k+1} > p_{k+1}$ and $p'_{M(k)} = p^{\min} = p_{M(k)}$.

By Lemma C.2(i), for each $j \in N_C \backslash \{i^*\}$, $x'_j \in M_C \cup \{0, k+1\}$. By (1) and Definition 8, for each $j \in N_C \backslash \{i^*\}$, $x'_j \neq k+1$. Thus we have: (3) $\{x'_j\}_{j \in N_C \backslash \{i^*\}} \subseteq (M_C \backslash \{k+1\}) \cup \{0\}$. By Fact C.2(i), we have: (4) for each $j \in N_C \backslash \{i^*\}$, $z'_j \, I_j \, z^*_j \, I_j \, z_j$. Thus, by (2) and (4), we have: (5) for each $j \in N_C \backslash \{i^*\}$, $\{x_j, x'_j\} \subseteq D_j(p') \subseteq D_j(p)$. Recall $R_{N_C \backslash \{i^*\}} = R'_{N_C \backslash \{i^*\}}$. Thus,

$$\begin{aligned}&\text{Lemma C.1(ii) (by (∗3), condition (a) holds)} \\ \Rightarrow\;& (z_{N_C \backslash \{i^*\}}, p_{M_C \backslash \{k+1\}}) \in W^{\min}(N_C \backslash \{i^*\}, M_C \backslash \{k+1\}, R_{N_C \backslash \{i^*\}}) \\ \Rightarrow\;& (z_{N_C \backslash \{i^*\}}, p'_{M_C \backslash \{k+1\}}) \in W^{\min}(N_C \backslash \{i^*\}, M_C \backslash \{k+1\}, R'_{N_C \backslash \{i^*\}}) \text{ by (2)} \\ \Rightarrow\;& (z'_{N_C \backslash \{i^*\}}, p'_{M_C \backslash \{k+1\}}) \in W^{\min}(N_C \backslash \{i^*\}, M_C \backslash \{k+1\}, R'_{N_C \backslash \{i^*\}}) \text{ by (3), (5)}\end{aligned}$$

Thus, by $N_C\backslash\{i^*\} \subseteq N$, $M_C\backslash\{k+1\} \subseteq M$ and Fact C.3(i), we have: $N_C\backslash\{i^*\} \subseteq N'_C$, which implies (6) $N_U \cup \{i^*\} \supseteq N'_U$.

In the following, we show $N_U \cup \{i^*\} \subseteq N'_U$.

By $p'_{k+1} > p_{k+1}$, $k+1$ must be assigned. Thus by (3) and Lemma C.2(ii), for each $x \in M_U \cup \{k+1\}$, there is $j \in N_U \cup \{i^*\}$ such that $x'_j = x$, which implies $M_U \cup \{k+1\} \subseteq \{x'_j\}_{j \in N_U}$. Thus, by $|M_U| = |N_U|$ (Lemma 1(i)), $i^* \in N_C$, and $k+1 \in M_C$, $|M_U \cup \{k+1\}| = |N_U \cup \{i^*\}|$. Thus we have: (7) $M_U \cup \{k+1\} = \{x'_j\}_{j \in N_U \cup \{i^*\}}$.

Let $j \in N_C\backslash\{i^*\}$. Then by Lemma 1(ii), for each $x \in M_U$, $z_j \, P_j \, (x, p_x)$. Thus, by (2) and (3), for each $x \in M_U \subseteq M(k)$, $z'_j \, I_j \, z_j \, P_j \, (x, p_x) = (x, p'_x)$ and by (1), $z'_j \, P_j \, (k+1, p'_{k+1})$. Thus, we have: (8) for each $j \in N_C\backslash\{i^*\}$, $D_j(p') \cap (M_U \cup \{k+1\}) = \emptyset$.

For each $j \in N_U \cup \{i^*\}$, by (2) and (7), $p'_{x'_j} = p_{x'_j} > 0$. By (7), (8) and Fact C.6, we have $N_U \cup \{i^*\} \subseteq N'_U$.

Thus, by (6), $N_U \cup \{i^*\} = N'_U$, and so by (7), $M_U \cup \{k+1\} = M'_U$.

Case II-2: $i^* \in N_U$.

By $i^* \in N_U$ and $x_{i^*} = k+1$, there is $i' \in N_U\backslash\{i^*\}$ such that

$$V_{i'}(k+1; z^*_{i'}) = C^2_+(R, k+1; z^*) \leq V_{i^*}(k+1; z^*_{i^*}) \qquad (*4)$$

By $i^* \in N_U$ and $x_{i^*} = k+1$, we have: $(*5)$ $C^2_+(R, k+1; z^*) > C^1_+(R_{N_C}, k+1; z^*)$. To see $(*5)$, by contradiction, suppose not, i.e., $C^2_+(R, k+1; z^*) = C^1_+(R_{N_C}, k+1; z^*)$. In case of $C^2_+(R, k+1; z^*) = 0$, by Definition 3, $i^* \in N_C$, contradicting $i^* \in N_U$. In case of $C^2_+(R, k+1; z^*) > 0$, there is $j \in N_C$ such that $V_j(k+1; z^*_j) = C^1_+(R_{N_C}, k+1; z^*)$. Thus, $k+1$ is connected by agent j's demand and so by $j \in N_C$, $i^* \in N_C$, contradicting $i^* \in N_U$. By $(*4)$ and $(*5)$, we have: $(*6)$ $V_{i^*}(k+1; z^*_{i^*}) \geq V_{i'}(k+1; z^*_{i'}) > C^1_+(R_{N_C}, k+1; z^*)$.

The proof are divided into four cases, Cases II-2-1, II-2-2, II-2-3, and II-2-4. We group them in two parts. Part A treats Cases II-2-1, II-2-2, and II-2-3. Part B treats Case II-2-4.

Part A: By Fact C.5 and Theorem 1, $\widehat{z}_{N_U} \in Z^{\min}(N_U, M_U, R_{N_U}, \widehat{r})$ where $\widehat{r}_x = C^1_+(R_{N_C}, x; z^*)$ for each $x \in M_U$, and $\widetilde{z}_{N'_U} \in Z^{\min}(N'_U, M'_U, R'_{N_U}, \widetilde{r})$ where $\widetilde{r}_x = C^1_+(R_{N'_C}, x; z^*)$ for each $x \in M'_U$.

Thus if $N_U = N'_U$ and $M_U = M'_U$, then $N_C = N'_C$ and $\widehat{r} = \widetilde{r}$, and so by $i \in N_U = N'_U$, $R'_{N_U} = (R'_i, R_{N_U\backslash\{i\}})$, and Fact C.1, $\widehat{z}_i \, R_i \, \widetilde{z}_i$. Thus we need to show $N_U = N'_U$ and $M_U = M'_U$.

By $i^* \in N_U$ and $x_{i^*} = k+1$, if $p'_{k+1} > C^1_+(R_{N_C}, k+1; z^*)$, then by Lemma C.3(iii), $N_U = N'_U$ and $M_U = M'_U$. Thus, in Cases II-2-1, II-2-2, and II-2-3, we show that $p'_{k+1} > C^1_+(R_{N_C}, k+1; z^*)$, respectively.

Case II-2-1: $i \neq i^*$ and $i \neq i'$

By $i' \in N_U \backslash \{i^*\}$, $i' \neq i^*$. Thus,

$$\begin{aligned} p'_{k+1} &= C^2_+(R', k+1; z^*) \underset{i'\neq i^*, i\neq i^*,\ i\neq i'}{\geq} \min\{V_{i'}(k+1; z^*_{i'}), V_{i^*}(k+1; z^*_{i^*})\} \\ &\underset{(*6)}{>} C^1_+(R_{N_C}, k+1; z^*). \end{aligned}$$

Case II-2-2: (a) $i = i^*$ or $i = i'$, and

(b) $\left|\{j \in N_U \backslash \{i', i^*\} : V_j(k+1; z^*_j) > C^1_+(R_{N_C}, k+1; z^*)\}\right| \geq 1$

By (b), there is $j \in N_U \backslash \{i', i^*\}$ such that (1) $V_j(k+1; z^*_{i^*}) > C^1_+(R_{N_C}, k+1; z^*)$. In case of $i = i^*$, by $i' \in N_U \backslash \{i^*\}$, we have $i \neq i'$, and

$$\begin{aligned} p'_{k+1} &= C^2_+(R', k+1; z^*) \underset{j\neq i', i'\neq i,\ j\neq i}{\geq} \min\{V_{i'}(k+1; z^*_{i'}), V_j(k+1; z^*_{i^*})\} \\ &\underset{(1),(*6)}{>} C^1_+(R_{N_C}, k+1; z^*). \end{aligned}$$

We can treat the case of $i = i'$ by the same way, and so we omit it.

Case II-2-3: (a) and (c) $V'_i(k+1; z^*_i) > C^1_+(R_{N_C}, k+1; z^*)$.

In case of $i = i^*$, by $i' \in N_U \backslash \{i^*\}$, we have $i \neq i'$ and $i' \neq i^*$, and

$$\begin{aligned} p'_{k+1} &= C^2_+(R', k+1; z^*) \underset{i'\neq i^*, i\neq i^*, i\neq i'}{\geq} \min\{V_{i'}(k+1; z^*_{i'}), V_{i^*}(k+1; z^*_i)\} \\ &\underset{(c),(*6)}{>} C^1_+(R_{N_C}, k+1; z^*). \end{aligned}$$

We can treat the case of $i = i'$ by the same way, and so we omit it.

Part B: This part treats Case II-2-4.

Case II-2-4: (a), (d) $\left|\{j \in N_U \backslash \{i', i^*\} : V_j(k+1; z^*_j) > C^1_+(R_{N_C}, k+1; z^*)\}\right| = 0$,

and (e) $V'_i(k+1; z^*_i) \leq C^1_+(R_{N_C}, k+1; z^*)$.

By (a), $i = i'$ or $i = i^*$. We consider only the case of $i = i'$ here. We can treat the case of $i = i^*$ by the same way, and so we omit it.

By Fact C.5 and Theorem 1, $\widehat{z}_{N_U} \in Z^{\min}(N_U, M_U, R_{N_U}, \widehat{r})$ where for each $x \in M_U$, $\widehat{r}_x = C^1_+(R_{N_C}, x; z^*)$, and if $i^* \in N'_C$ and $x'_{i^*} = k+1$, then $\widetilde{z}_{N'_U \cup \{i^*\}} \in Z^{\min}(N'_U \cup \{i^*\}, M'_U \cup \{k+1\}, R'_{N'_U \cup \{i^*\}}, \widetilde{r})$ where for each $x \in M'_U \cup \{k+1\}$, $\widetilde{r}_x = C^1_+(R_{N \backslash [N'_U \cup \{i^*\}]}, x; z^*)$.

Thus if $i^* \in N'_C$, $x'_{i^*} = k+1$, $N_U = N'_U \cup \{i^*\}$ and $M_U = M'_U \cup \{k+1\}$, then $N_C = N \backslash [N'_U \cup \{i^*\}]$ and $\widehat{r} = \widetilde{r}$, and so by $i \in N_U = N'_U \cup \{i^*\}$, $R'_{N'_U \cup \{i^*\}} = (R'_i, R_{N' \backslash \{i\}})$, and Fact C.1, $\widehat{z}_i \, R_i \, \widetilde{z}_i$. Thus, we show that $i^* \in N'_C$, $x'_{i^*} = k+1$, $N_U = N'_U \cup \{i^*\}$ and $M_U = M'_U \cup \{k+1\}$.

By (d), we have: (1) $V_j(k+1; z^*_j) \leq C^1_+(R_{N_C}, k+1; z^*)$ for each $j \in N_U \backslash \{i', i^*\}$. By $i' \in N \backslash \{i^*\}$, $i = i' \neq i^*$. Thus,

$$V'_i(k+1; z^*_i) \underset{\text{(e)}}{\leq} C^1_+(R_{N_C}, k+1; z^*) \underset{(*6), i\neq i^*}{<} V_{i^*}(k+1; z^*_{i^*}),$$

and so by (1),

$$V_{i^*}(k+1; z^*_{i^*}) = C^1_+(R', k+1; z^*) > C^1_+(R_{N_C}, k+1; z^*) = C^2_+(R', k+1; z^*).$$

By $V_{i^*}(k+1; z^*_{i^*}) = C^1_+(R', k+1; z^*) > C^2_+(R', k+1; z^*)$, we have: (2) $x_{i^*} = x'_{i^*} = k+1$. By $C^2_+(R', k+1; z^*) = C^1_+(R_{N_C}, k+1; z^*)$, $p'_{k+1} = C^1_+(R_{N_C}, k+1; z^*)$. Thus by (2), we have: (3) $i^* \in N'_C$ and $k+1 \in M'_C$. By $p'_{k+1} = C^1_+(R_{N_C}, k+1; z^*)$,

$$p'_{k+1} = C^1_+(R_{N_C}, k+1; z^*) \underset{(*5)}{<} C^2_+(R, k+1; z^*) = p_{k+1}.$$

Thus, by Fact C.2(ii), we have: (4) $p'_{M(k)} = p^{\min} = p_{M(k)}$ and $p'_{k+1} < p_{k+1}$.

By (2), (3), and Lemma C.2(i), we have: (5) $\{x'_j\}_{j \in N_C} \subseteq M_C \cup \{0\}$. By $i^* \in N_U$ and Fact C.2(i), we have: (6) for each $j \in N_C$, $z'_j \, I_j \, z^*_j \, I_j \, z_j$. Thus, by (4), we have: (7) for each $j \in N_C$, $\{x_j, x'_j\} \subseteq D_j(p) \subseteq D_j(p')$. By $i \in N_U$, $R_{N_C} = R'_{N_C}$. By $i^* \in N_U$, $x_{i^*} = k+1 \in M_U$ and (4), $p_{M_C} = p'_{M_C}$. Thus,

$$\begin{aligned}
& \text{Lemma C.1(i)} \\
\Rightarrow\ & (z_{N_C}, p_{M_C}) \in W^{\min}(N_C, M_C, R_{N_C}) \\
\Rightarrow\ & (z_{N_C}, p'_{M_C}) \in W^{\min}(N_C, M_C, R'_{N_C}) \quad p_{M_C} = p'_{M_C},\ R_{N_C} = R'_{N_C} \\
\Rightarrow\ & (z'_{N_C}, p'_{M_C}) \in W^{\min}(N_C, M_C, R'_{N_C}) \quad (5),\ (7)
\end{aligned}$$

Thus, by $N_C \subseteq N$, $M_C \subseteq M(k)$ and Fact C.3(i), we have: $N_C \subseteq N'_C$. Thus by (3), $N_C \cup \{i^*\} \subseteq N'_C$, which implies (8) $N'_U \cup \{i^*\} \subseteq N_U$.

In the following we show $N_U \backslash \{i^*\} \subseteq N'_U$, which implies $N'_U \cup \{i^*\} \supseteq N_U$.

By Lemma C.2(ii) and (2), we have $M_U \backslash \{k+1\} \subseteq \{x'_j\}_{j \in N_U \backslash \{i^*\}}$. Thus, by $|M_U| = |N_U|$ (Lemma 1(i)), $i^* \in N_U$, and $x_{i^*} = k+1 \in M_U$, we have: (9) $M_U \backslash \{k+1\} = \{x'_j\}_{j \in N_U \backslash \{i^*\}}$.

Note that for each $j \in N_C$ and each $x \in M_U \backslash \{k+1\}$,

$$z'_j \underset{(6)}{I_j} z_j \underset{\text{Lemma 1(ii)}}{P_j} (x, p_x) \underset{(4)}{=} (x, p'_x).$$

Also note that for each $x \in M_U \backslash \{k+1\}$,

$$z'_{i^*} \underset{(2)}{=} (k+1, p'_{k+1}) \underset{(4)}{P_{i^*}} (k+1, p_{k+1}) \underset{x_{i^*}=k+1}{=} z_{i^*} \, R_{i^*} \, (x, p_x) \underset{(4)}{=} (x, p'_x).$$

Thus we have: (10) for each $j \in N_C \cup \{i^*\}$, $D_j(p') \cap (M_U \backslash \{k+1\}) = \emptyset$.

By (2), (4) and (9), for each $j \in N_U \backslash \{i^*\}$, $p'_{x'_j} = p_{x'_j} > 0$. Thus, by (9), (10) and Fact C.6, $N_U \backslash \{i^*\} \subseteq N'_U$. Thus by (8), $N_U = N'_U \cup \{i^*\}$, and so by (9), $M_U = M'_U \cup \{k+1\}$. **Q.E.D.**

Appendix D: Difficulties with DGS auctions under general preferences

We construct two examples below to show that the "exact DGS auction" and "approximate DGS auction" substantially overshoot the MPE prices when general preferences are considered. Examples can be similarly constructed to show that when the price increment is larger than the measurement of agents' valuation, the auction in Mishra and Parkes (2009) substantially undershoots the MPE price, and the auctions in Andersson and Erlanson (2013) and Liu and Bagh (2019) either substantially overshoots or undershoots the MPE price.

The exact DGS auction:

The exact DGS auction finds the MPE price in a finite number of steps if (i) agents have quasi-linear preferences and (ii) the price increment is equal to the measurement unit of agents' valuations, e.g., both are integers (Demange et al. 1986). The following example shows that even if only (ii) fails, i.e., the price increment is larger than the measurement unit of agents' valuations, the exact DGS auction generates an outcome whose prices are higher than the MPE price and fail to approximate it. Note that (ii) often fails to hold for general preferences.

Consider the case of two agents, 1 and 2, and two objects, A and B. Receiving object 0 means receiving nothing. Agents have quasi-linear preferences. Let $V^i(x)$ denote agent i's valuation over $x = 0, A, B$. Let

$$\begin{aligned} V_1(0) &= 0, \quad V_1(A) = 9.2, \quad V_1(B) = 9.8, \\ V_2(0) &= 0, \quad V_2(A) = 9.1, \quad V_2(B) = 9.6. \end{aligned}$$

Let p_A and p_B be the prices of A and B, and $p \equiv (p_A, p_B)$. The price of object 0 is zero. Agent i's demand set at p is: $D_i(p) \equiv \{x \in \{0, A, B\} : V_i(x) - p_x \geq V_i(y) - p_y, y \in \{0, A, B\}\}$. Since the MPE price for this value profile coincides with the Vickrey payment, the MPE price is $p^{\min} = (0, 0.5)$.

The DGS auction starts from $p = (0, 0)$, the reserve prices, with an integer increment. At $p = (0, 0)$, both agents demand only object B. Since only object B is overdemanded (also MOD), then increase only p_B by one unit. At $p = (0, 1)$, both agents demand only object A (A is overdemanded, also MOD), and so increase only p_A by one unit. Again, at $p = (1, 1)$, both agents demand only object B. Similarly, the price of each object alternatively increases at least to $(9, 9)$ and the auction terminates at $(10, 10)$. The price $(9, 9)$ could be possibly treated as the outcome of the auction since $(9, 9)$ is an approximate equilibrium price[11] that is coordinate-wisely closest to $(10, 10)$, given the unit increment. Still $(9, 9)$ substantially overshoots $p^{\min} = (0, 0.5)$.

[11] Let ε be the increment in the auction. Agent i's ε−demand set at p is given by $D_i^{\varepsilon}(p) \equiv \{x \in L : (x, p_x)\, R_i\, (0, 0) \text{ and } (x, p_x)\, R_i\, (y, p_y + \varepsilon),\ l' \in M\}$. When $\varepsilon = 0$, $D_i(p) = D_i(p')$. A pair $(z, p) \in Z \times \mathbb{R}_+^m$ is *an approximate equilibrium (for* ε*)* if (i) for each $i \in N$, $x_i \in D_i^{\varepsilon}(p)$ and $p_{x_i} = t_i$, and (ii) for each $y \in M$, if for each $i \in N$, $x_i \neq y$, then $p_y = 0$.

The approximate DGS auction:

The approximate DGS auction works as follows. Agents are called bid on objects one by one, according to some exogenously given queue. If an agent bids on an unassigned object, she becomes committed to that object at the reserve price. If an agent bids on an assigned object at some price, the price of that object is increased by a price increment, and the agent becomes committed to that object at the increased price. Simultaneously, the agent to whom the object had been assigned becomes uncommitted and occupies the first position among the remaining uncommitted agents. If an agent bids on no object, she drops out of the auction. The auction terminates when all uncommitted agents drop out.

The approximate DGS auction obtains an outcome where the prices derive from the (exact) MPE price, coordinate-wise, by at most $k \cdot \delta$ (δ: the price increment; k: the minimum of the numbers of agents and objects), if agents have quasi-linear preferences (Demange et al. 1986). The following example shows that if agents have general preferences, the outcome prices of an approximate DGS auction lie outside the estimation in the quasi-linear setting.

Consider the case of three agents, 1, 2, and 3, and two objects, A and B. Agents are called in the order 1, 2, and 3. Let $\delta \equiv 1$ and agents' preferences satisfy the standard assumptions (See Section 3), and in addition:
For agent 1, $(0, 0)\, I_1\, (A, 0.3)\, I_1\, (B, 20.4)$;
For agent 2, $(A, 0)\, I_2\, (B, -1)$, $(0, -20)\, I_2\, (A, 5)\, I_2\, (B, 20.4)$ and
$(0, 0)\, I_2\, (A, 20.2)\, I_2\, (B, 20.6)$;
For agent 3, $(0, -21)\, I_3\, (A, 0.5)\, I_3\, (B, 20.4)$ and $(0, 0)\, I_3\, (A, 20.6)\, I_3\, (B, 20.8)$,

For the above preference profile, the MPE price is $p^{\min} = (0.5, 20.4)$.

The approximate DGS auction starts from $p = (0, 0)$, the reserve prices. First, agent 1 is called on and demands object B, and committed to B at price 0. Second, agent 2 is called, and since agent 2 demands object A at $p = (0, 1)$, she then bids A and is committed to A at price 0. Third, agent 3 is called on, and since agent 3 demands object B at $p = (1, 1)$, she then bids B and is committed to B at price 1. Then, agent 1 becomes uncommitted. Since she is the only uncommitted bidder, agent 1 is called on, and since she demands object B at $p = (1, 2)$, agent 1 then bids B and is committed to B at price 2. Agent 3 thus becomes uncommitted. Since she is the only uncommitted bidder, agent 3 is called on to bid. Note that agents 1 and 3 alternatively bid on object B until its price reaches 20. Since agent 1 is committed to object B at price 20, agent 3 is called on; because agent 3 demands object A at $p = (1, 21)$, he then bids A and is committed to it at price 1. By similar reasoning, agents 2 and 3 alternatively bid on object A until its price reaches 20 but stop bidding at 21. The outcome price of object A, i.e., 20, overshoots its $p_A^{\min} = 0.5$, much more than $k \cdot \delta = 2$. Given k and δ, the set of preference profiles for such undesirable deviations is non-negligible.