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EFFICIENT AND STRATEGY-PROOF MULTI-UNIT OBJECT ALLOCATION WITH MONEY: (NON)DECREASING MARGINAL VALUATIONS WITHOUT QUASI-LINEARITY

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Efficient and strategy-proof multi-unit object allocation with money: (Non)decreasing marginal valuations without quasi-linearity*

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Abstract

We consider the problem of allocating multiple units of an indivisible object among agents and collecting payments. Each agent can receive multiple units of the object, and his (consumption) bundle is a pair of the units he receives and his payment. An agent's preference over bundles may be non-quasi-linear, which accommodates income effects or soft budget constraints. We show that the generalized Vickrey rule is the only rule satisfying *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* on rich domains with nondecreasing marginal valuations. We further show that if a domain is minimally rich and includes an arbitrary preference exhibiting both decreasing marginal valuations and a positive income effect, then no rule satisfies the same four properties. Our results suggest that in non-quasi-linear environments, the design of an efficient multi-unit auction mechanism is possible only when agents have nondecreasing marginal valuations.

JEL Classification Numbers. D44, D47, D71, D82

Keywords. Efficiency, Strategy-proofness, Non-quasi-linear preferences, Nondecreasing marginal valuations, Decreasing marginal valuations, Constant marginal valuations, Multi-unit auctions

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1 Introduction

1.1 Purpose

The most important goal of many government auctions is to allocate public resources efficiently. The examples of government auctions include spectrum licenses,[1] the rights to vehicle ownership,[2] etc. The winning bids of spectrum license auctions are often larger than annual profits of mega firms such as major mobile operators.[3] The winning bids of vehicle ownership auctions are typically as much as citizens' annual income. An important feature of such large-scale auctions which has been ignored in the literature is the non-quasi-linearity of preferences. Most existing studies on auctions assume quasi-linear preferences, which implies that bidders' payments are less than their cash balances and are so small that income effects are negligible. However, if a payment exceeds a bidder's cash balance, then he will need to borrow cash. The cost of borrowing from financial markets is typically non-linear to the borrowings. This factor makes bidders' preferences non-quasi-linear. Another factor of non-quasi-linearity is complementary goods to utilize the auctioned objects effectively. If a bidder pays a large amount in an auction, he cannot afford to purchase sufficient complementary goods. In such situations, bidders' valuations on the auctioned objects change as their payments change. Thus, the existence of complementary goods causes non-quasi-linearity. In this paper, we take account of non-quasi-linear preferences but focus on multi-unit auctions where objects are identical. Then, we investigate the possibility of designing auction mechanisms that allocate objects efficiently.

1.2 Main results

We consider the problem of allocating multiple units of an indivisible object among agents and collecting payments from them. Each agent can receive multiple units of the object, and his (*consumption*) *bundle* is a pair specifying the units he receives and his payment. Each agent has a (possibly) non-quasi-linear preference over bundles. An *allocation* specifies each agent's bundle. An *allocation rule*, or a *rule* for short, is a function from the set of preference profiles (the *domain*) to the set of allocations. An allocation is *efficient* for a given preference profile if no other allocation makes some agent better off without making any agent worse off or decreasing the auctioneer's revenue. A rule satisfies *efficiency* if it chooses an efficient allocation for each preference profile. A rule satisfies

[1] As of 2020, all OECD countries except for Japan conduct auctions to allocate spectrum licenses.

[2] Singapore Transportation Authority conducts auction to allocate COEs (Certificate of Entitlement), which are necessary to own and use a vehicle in Singapore. For the details, see the following: https://www.onemotoring.com.sg/content/onemotoring/home/buying/upfront-vehicle-costs/certificate-of-entitlement--coe-.html.

[3] For the example of UK spectrum auction, see Ofcom's website: https://www.ofcom.org.uk/.

strategy-proofness if no agent has an incentive to misreport his preference regardless of other agents' reports. A rule satisfies *individual rationality* if no agent is worse off than receiving nothing and making no monetary transfer. A rule satisfies *no subsidy for losers* if an agent who receives nothing cannot receive money. This condition eliminates "fake" agents whose only interest is the participation subsidy. We regard these four properties as desiderata.

A characteristic of a preference is captured by marginal valuations. A preference exhibits *nondecreasing* (resp. *nonincreasing*) *marginal valuations* if a marginal willingness to pay at each bundle is weakly greater than (resp. less than) the marginal willingness to sell at the bundle.[4] A preference exhibits *constant marginal valuations* if a marginal willingness to pay at each bundle coincides with the marginal willingness to sell at the bundle. Because of non-quasi-linearity, both marginal willingness to pay and to sell may vary depending on the payments. The *domain with* nondecreasing (resp. *nonincreasing*) *marginal valuations* is the class of preference profiles which exhibit nondecreasing (resp. nonincreasing) marginal valuations. The *domain with constant marginal valuations* is the class of preference profiles which exhibit constant marginal valuations.

First, we consider the situation where agents' preferences exhibit nondecreasing marginal valuations. Such a situation is typical in some spectrum auctions. An example is the Germany 3G auction held in 2000. As Ausubel (2004) writes, "ten licenses for virtually homogeneous spectrum were offered to the four German mobile phone incumbents." Another example is the Spanish 5G auction held in July 2018. The Spanish government divided the 3.6–3.8 GHz band into 40 small blocks of 5MHz.[5] In those auctions, because of economies of scale, as firms obtain more blocks, they can utilize them better. Thus, it is natural that firms have nondecreasing marginal valuations.

A domain is *rich* if it includes all quasi-linear preferences with constant marginal valuations. We require a domain to be rich as a minimal condition. The *generalized Vickrey rule* is a natural extension of the Vickrey rule for quasi-linear preferences to non-quasi-linear preferences. Our first main result (Theorem 1) is: *on any rich subdomain of the domain with nondecreasing marginal valuations, the generalized Vickrey rule is the only rule satisfying efficiency, strategy-proofness, individual rationality, and no subsidy for losers*.

Most of the literature on multi-unit auctions rather assume preferences exhibiting decreasing (or weaker nonincreasing) marginal valuations.[6] Thus, we next investigate the possibilities of desirable rules on the domain with decreasing marginal valuations. A preference exhibits the *positive income effects* if the demand for the object increases

[4]A preference exhibiting decreasing or increasing marginal valuations can be defined analogously.

[5]Since all the blocks are equally sized and have similar frequencies, as in German 3G auction, the blocks can be regarded as virtually homogeneous.

[6]See Subsection 1.3.1.

as the payment decreases. Our second main result (Theorem 2) is: *if a domain is rich and includes a preference exhibiting both decreasing marginal valuations and a positive income effect, then no rule satisfies efficiency, strategy-proofness, individual rationality, and no subsidy for losers*.[7] Note that the smaller a domain is, the weaker the four desirable properties are, and the higher the possibilities of finding desirable rules are. A rich domain can be "minimal" in that it includes only quasi-linear preferences with constant marginal valuations. Also note that the factor of complementary goods makes the demand for the auctioned object increase as the payment decreases. Thus, preferences exhibiting both decreasing marginal valuations and positive income effects are plausible. Thus, the second result says that if one arbitrary plausible preference is added to a minimally rich domain, it is impossible to design a rule satisfying the desirable properties.

1.3 Related literature

1.3.1 Multi-unit auctions

Most of the vast literature on multi-unit auctions assume quasi-linear preferences with decreasing marginal valuations (Perry and Reny, 2002, 2005; Ausubel, 2004; Ausubel et al., 2014, etc.).[8] Such literature includes two standard texts on auction theory (Milgrom, 2004; Krishna, 2009). Our results are different from this strand of research in covering the case of nondecreasing or nonincreasing marginal valuations in environments of non-quasi-linear preferences.

In the literature on multi-object auctions (Kelso and Crawford, 1982; Gul and Stacchetti, 1999, 2000; Milgrom, 2000; Ausubel, 2004, 2006, etc.), substitutability among objects is the key to the possibility of designing efficient auction mechanisms. Note that substitutability corresponds to nonincreasing marginal valuations in multi-unit auction models. Thus, our results (Theorems 1 and 2) make a striking contrast to the above literature in suggesting that in non-quasi-linear environments, it is possible to design efficient auction mechanisms only when preferences exhibit nondecreasing marginal valuations.

1.3.2 Efficient object allocation

The efficient object allocation problem with monetary transfer has been studied extensively. For quasi-linear preferences, Holmström (1979) and Chew and Serizawa (2007) characterize the Vickrey rule by *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* on rich quasi-linear domains. In particular, the result by Holmström (1979) implies that the Vickrey rule is the only rule satisfying the four desirable properties on the quasi-linear domain with nondecreasing marginal valuations. Thus, our

[7] Precisely, our negative result depends on the number of units. See Section 5.

[8] We also refer to Baranov et al. (2017), who remove the assumption of decreasing marginal valuations while maintaining quasi-linearity in the procurement auction model.

characterization theorem on rich subdomains of the domain with nondecreasing marginal valuations can be seen as an extension of his result to non-quasi-linear domains.

There is a small but expanding literature on the object allocation problem with non-quasi-linear preferences. They are roughly classified into two categories: papers on agents with unit-demand and those on agents with multi-demand, as in ours.

The minimum price Warlasian rule (Demange and Gale, 1985) plays a central role in the papers in the first category. Morimoto and Serizawa (2015) show that it is the only rule on the unrestricted classical domain satisfying *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers*. Zhou and Serizawa (2018) extend the characterization of Morimoto and Serizawa (2015) to the restricted domain where objects are ranked according to the common tiers.

In the single-object environment, the generalized Vickrey rule coincides with the minimum price Walrasian rule, and is characterized by the same four properties (Saitoh and Serizawa, 2008; Sakai, 2008). When preferences exhibit nondecreasing marginal valuations, it is efficient to bundle all the units and allocate them to a single agent. This bundling makes efficient allocations for nondecreasing marginal valuations similar to those of the single-object environment. Thus, our first result seems to be obtained as an application of the characterization for the single-object case. However, we emphasize that our result is not a trivial extension of theirs since several agents may receive the object in our environment.[9]

In contrast, the papers in the latter category typically obtain impossibility theorems. Kazumura and Serizawa (2016) show that in the heterogeneous objects model, if a domain contains a rich unit-demand domain and includes one arbitrary multi-demand preference, then no rule satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers*.

Malik and Mishra (2019) consider dichotomous preferences in the heterogeneous objects environment. They show that on the dichotomous domain, no rule satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy*. They further show that the generalized Vickrey rule is the only rule satisfying the four properties on the dichotomous domain with nonnegative income effects. Since they treat heterogeneous objects, their results are independent of ours.

Baisa (2020) is one of the most closely related papers to ours. He considers the same model as ours, and shows that on the domain with decreasing marginal valuations, positive income effects, and the single-crossing property, there is a rule satisfying *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* if preferences are of one-dimensional types. He also shows that on the same domain no rule satisfies the four properties if preferences are of multi-dimensional types. Both of his results are

[9]Subsection 6.2.1 discusses this point in detail.

independent of ours. We give a detailed discussion on the relationship between his results and ours in Subsection 6.1.

1.4 Organization

The remainder of the paper is organized as follows. Section 2 sets up the model. Section 3 introduces the generalized Vickrey rule. Section 4 provides the characterization theorem. Section 5 provides the impossibility theorem. Section 6 discusses the relationship between our results and the related results obtained by other authors, and explains the difficulty of our proofs. Section 7 concludes. Most proofs are in Appendix, while missing ones can be found in the supplementary material.

2 The model and definitions

There are n agents and m units of an identical object, where $2 \leq n < \infty$ and $2 \leq m < \infty$. We denote the set of agents by $N \equiv \{1, \ldots, n\}$. Let $M \equiv \{0, \ldots, m\}$. Further, given $m' \in M$, let $M(m') \equiv \{0, \ldots, m'\}$.

Each agent $i \in N$ receives $x_i \in M$ units of the object. The amount of money paid by agent i is denoted by $t_i \in \mathbb{R}$. For each agent $i \in N$, his **consumption set** is $M \times \mathbb{R}$, and his (**consumption**) **bundle** is a pair $z_i \equiv (x_i, t_i) \in M \times \mathbb{R}$. Let $\mathbf{0} \equiv (0, 0)$.

2.1 Preferences

Each agent $i \in N$ has a complete and transitive preference relation R_i over $M \times \mathbb{R}$. Let P_i and I_i be the strict and indifference relations associated with R_i, respectively. Throughout this paper, we assume that a preference R_i satisfies the following four properties.

Money monotonicity. For each $x_i \in M$ and each pair t_i,$t'_i \in \mathbb{R}$ with $t_i < t'_i$, we have $(x_i, t_i)\ P_i\ (x_i, t'_i)$.

Object monotonicity. For each pair $x_i, x'_i \in M$ with $x_i > x'_i$ and each $t_i \in \mathbb{R}$, we have $(x_i, t_i)\ P_i\ (x'_i, t_i)$.

Possibility of compensation. For each $z_i \in M \times \mathbb{R}$ and each $x_i \in M$, there is a pair $t_i, t'_i \in \mathbb{R}$ such that $(x_i, t_i)\ R_i\ z_i$ and $z_i\ R_i\ (x_i, t'_i)$.

Continuity. For each $z_i \in M \times \mathbb{R}$, the upper contour set at z_i, $\{z'_i \in M \times \mathbb{R} : z'_i\ R_i\ z_i\}$, and the lower contour set at z_i, $\{z'_i \in M \times \mathbb{R} : z_i\ R_i\ z'_i\}$, are both closed.

A typical class of preferences satisfying the above four properties is denoted by $\mathcal{R}$. For each $i \in N$, each $R_i \in \mathcal{R}$, each $z_i \in M \times \mathbb{R}$, and each $x_i \in M$, possibility of compensation and continuity together imply that there is a payment $V_i(x_i, z_i) \in \mathbb{R}$ such that $(x_i, V_i(x_i, z_i))\ I_i\ z_i$.[10] By money monotonicity, such a payment is unique. We call the payment $V_i(x_i, z_i)$ the **valuation of** x_i **at** z_i **for** R_i, where z_i specifies the bundle at which x_i is evaluated. We define the **net valuation of** x_i **at** z_i **for** R_i as $v_i(x_i, z_i) \equiv V_i(x_i, z_i) - V_i(0, z_i)$. Note that for each $z_i \in M \times \mathbb{R}$, $v_i(0, z_i) = 0$. Moreover, by $V_i(0, \mathbf{0}) = 0$, $v_i(x_i, \mathbf{0}) = V_i(x_i, \mathbf{0})$ for each $x_i \in M$.

Remark 1. Let $i \in N$ and $R_i \in \mathcal{R}$. (i) Let $z_i, z_i' \in M \times \mathbb{R}$ be such that $z_i\ I_i\ z_i'$. Then for each $x_i \in M$, $V_i(x_i, z_i) = V_i(x_i, z_i')$. (ii) Let $z_i \in M \times \mathbb{R}$ be such that $z_i\ I_i\ \mathbf{0}$. Then for each $x_i \in M$, $v_i(x_i, z_i) = v_i(x_i, \mathbf{0})$. (iii) For each $z_i \equiv (x_i, t_i) \in M \times \mathbb{R}$, $t_i = V_i(x_i, z_i)$.

Definition 1. A preference R_i is **quasi-linear** if for each pair $(x_i, t_i), (x_i', t_i') \in M \times \mathbb{R}$ and each $\delta \in \mathbb{R}$, $(x_i, t_i)\ I_i\ (x_i', t_i')$ implies $(x_i, t_i + \delta)\ I_i\ (x_i', t_i' + \delta)$.

Let $\mathcal{R}^Q$ denote the class of quasi-linear preferences.

Remark 2. Let $R_i \in \mathcal{R}^Q$. (i) For each $x_i \in M$, $v_i(x_i, \cdot)$ is independent of a bundle z_i, and we simply write $v_i(x_i)$ instead of $v_i(x_i, z_i)$. (ii) For each pair $(x_i, t_i), (x_i', t_i') \in M \times \mathbb{R}$, $(x_i, t_i)\ R_i\ (x_i', t_i')$ if and only if $v_i(x_i) - t_i \geq v_i(x_i') - t_i'$.

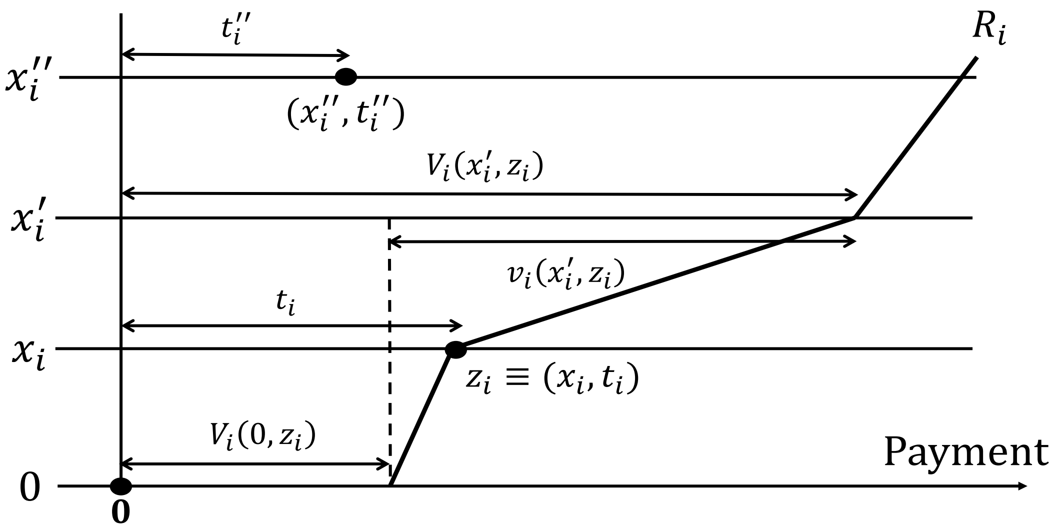

Figure 1: An illustration of the consumption set and indifference curves.

In our paper, the graphical illustrations of the consumption set and indifference curves might help the readers better understand the concepts and discussions. Figure 1 illustrates the consumption set of agent $i \in N$ and an indifference curve of his preference

[10]For the formal proof of the existence of such a payment, see Lemma 1 of Kazumura and Serizawa (2016).

$R_i \in \mathcal{R}$. Each horizontal line corresponds to some consumption level of the object. The intersections of the horizontal lines and the vertical line are the points at which payments are zero. Each point on a horizontal line indicates the amount of money that he pays (if it is on the right side of the vertical line) or receives (if it is on the left side of the vertical line). A solid line is an indifference curve of his preference R_i. By money monotonicity, a bundle is more preferable as it goes to the left on a horizontal line. Thus, $(x_i'', t_i'')\ P_i\ z_i = (x_i, t_i)$.

2.2 Marginal valuations

In the multi-unit object setting with money, the property of marginal valuations determines the characteristic of a preference. Given $R_i \in \mathcal{R}$, $z_i \in M \times \mathbb{R}$, and $x_i \in M \backslash \{m\}$, the **marginal valuation of x_i at z_i for R_i** is $v_i(x_i + 1, z_i) - v_i(x_i, z_i)$.

Definition 2. (i) A preference R_i exhibits **nondecreasing** (resp. **increasing**) **marginal valuations** if for each $z_i \in M \times \mathbb{R}$ and each $x_i \in M \backslash \{0, m\}$,

$$v_i(x_i + 1, z_i) - v_i(x_i, z_i) \geq \text{(resp. } >\text{)}\ v_i(x_i, z_i) - v_i(x_i - 1, z_i).$$

(ii) A preference R_i exhibits **nonincreasing** (resp. **decreasing**) **marginal valuations** if for each $z_i \in M \times \mathbb{R}$ and each $x_i \in M \backslash \{0, m\}$,

$$v_i(x_i + 1, z_i) - v_i(x_i, z_i) \leq \text{(resp. } <\text{)}\ v_i(x_i, z_i) - v_i(x_i - 1, z_i).$$

(iii) A preference R_i exhibits **constant marginal valuations** if for each $z_i \in M \times \mathbb{R}$ and each $x_i \in M \backslash \{0, m\}$,

$$v_i(x_i + 1, z_i) - v_i(x_i, z_i) = v_i(x_i, z_i) - v_i(x_i - 1, z_i).$$

In words, the definition of nondecreasing (resp. nonincreasing) marginal valuations is that for each bundle z_i, the marginal valuation at z_i for R_i is nondecreasing (resp. nonincreasing) in the number of units of the object, and that of constant marginal valuations says that for each bundle z_i, the marginal valuation at z_i for R_i is constant in the number of units. Our definitions of properties of marginal valuations are natural generalizations of the corresponding definitions for quasi-linear preferences.

Let $\mathcal{R}^{ND}$, $\mathcal{R}^{NI}$, and $\mathcal{R}^{C}$ denote the classes of preferences that exhibit nondecreasing, nonincreasing, and constant marginal valuations, respectively. Clearly, we have $\mathcal{R}^{ND} \cap \mathcal{R}^{NI} = \mathcal{R}^{C}$. Further, let $\mathcal{R}^{D}$ and $\mathcal{R}^{I}$ denote the classes of preferences which exhibit decreasing and increasing marginal valuations, respectively. Note that $\mathcal{R}^{D} \subsetneq \mathcal{R}^{NI}$, $\mathcal{R}^{D} \cap \mathcal{R}^{C} = \varnothing$, $\mathcal{R}^{I} \subsetneq \mathcal{R}^{ND}$, $\mathcal{R}^{I} \cap \mathcal{R}^{C} = \varnothing$, and $\mathcal{R}^{D} \cap \mathcal{R}^{I} = \varnothing$.

Graphically, a preference exhibits nondecreasing (resp. nonincreasing) marginal valuations if all indifference curves are convex upward (resp. downward), and a preference exhibits constant marginal valuations if all indifference curves are flat.

Remark 3 (Upward convexity). Let $m' \in M$ with $m' > 0$. Let $i \in N$, $R_i \in \mathcal{R}^{ND}$, and $z_i \in M \times \mathbb{R}$. (i) For each $x_i \in M(m')$, $\frac{x_i}{m'} v_i(m', z_i) \geq v_i(x_i, z_i)$. (ii) If there is $x_i \in M(m') \backslash \{0, m'\}$ such that $\frac{x_i}{m'} v_i(m', z_i) > v_i(x_i, z_i)$, then for each $x'_i \in M(m') \backslash \{0, m'\}$, $\frac{x'_i}{m'} v_i(m', z_i) > v_i(x'_i, z_i)$.

Figure 2 below illustrates Remark 3 in the case where $m' = 3$ and $x_i = 2$.

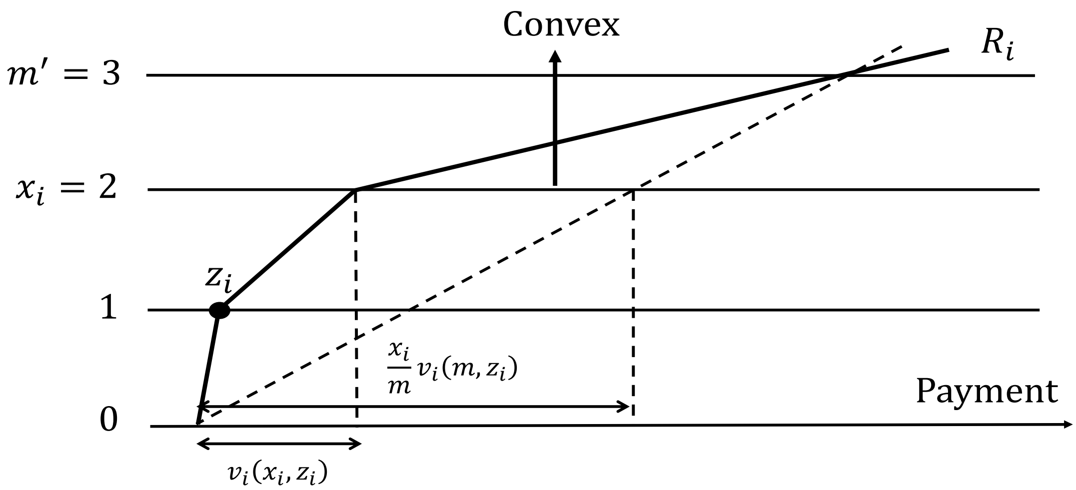

Figure 2: An illustration of Remark 3.

Although Remark 3 is graphically apparent from Figure 2, its formal proof can be found in the supplementary material.

2.3 Allocations and rules

Let $X \equiv \{(x_1, \ldots, x_n) \in M^n : 0 \leq \sum_{i \in N} x_i \leq m\}$. A (**feasible**) **allocation** is an n-tuple $z \equiv (z_1, \ldots, z_n) \equiv ((x_1, t_1), \ldots, (x_n, t_n)) \in (M \times \mathbb{R})^n$ such that $(x_1, \ldots, x_n) \in X$. Let Z denote the set of feasible allocations. We denote the object allocation and the payments at $z \in Z$ by $x \equiv (x_1, \ldots, x_n)$ and $t \equiv (t_1, \ldots, t_n)$, respectively. We write $z \equiv (x, t) \in Z$.

Given $N' \subseteq N$ and $m' \in M$, let

$$X(N', m') \equiv \{x \in X : 0 \leq \sum_{i \in N'} x_i \leq m' \text{ and } x_i = 0 \text{ for each } i \in N \backslash N'\}$$

and $Z(N', m') \equiv \{z \equiv (x, t) \in Z : x \in X(N', m')\}$. These sets correspond to the sets of feasible object allocations and feasible allocations in the reduced economy, where the set of agents is N' and there are m' units of the object, respectively.

We call $\mathcal{R}^n$ a **domain**. The partial list of domains to appear in this paper is as follows:

- The **quasi-linear domain**: $(\mathcal{R}^Q)^n$.
- The **domain with nondecreasing marginal valuations**: $(\mathcal{R}^{ND})^n$.
- The **domain with nonincreasing marginal valuations**: $(\mathcal{R}^{NI})^n$.
- The **domain with constant marginal valuations**: $(\mathcal{R}^C)^n$.

A **preference profile** is an n-tuple $R \equiv (R_1, \ldots, R_n) \in \mathcal{R}^n$. Given $R \in \mathcal{R}^n$ and $N' \subseteq N$, let $R_{N'} \equiv (R_i)_{i \in N'}$ and $R_{-N'} \equiv (R_i)_{i \in N \setminus N'}$. Specifically, for each distinct pair $i, j \in N$, we may write $R_{-i} \equiv (R_k)_{k \in N \setminus \{i\}}$ and $R_{-i,j} \equiv (R_k)_{k \in N \setminus \{i,j\}}$.

An **allocation rule**, or simply a **rule**, on $\mathcal{R}^n$ is a function $f : \mathcal{R}^n \to Z$. With a slight abuse of notation, we may write $f \equiv (x, t)$, where $x : \mathcal{R}^n \to X$ and $t : \mathcal{R}^n \to \mathbb{R}^n$ are the object allocation and the payment rules associated with f, respectively. We denote agent i's outcome bundle under a rule f at a preference profile R by $f_i(R) = (x_i(R), t_i(R))$, where $x_i(R)$ and $t_i(R)$ are the consumption level of the object and the payment made by agent i, respectively.

We now introduce the properties of a rule. The efficiency condition takes the auctioneer's preference into account, and we assume that he is only interested in his revenue. An allocation $z \equiv (x, t) \in Z$ is (**Pareto-**)**efficient** for a given preference profile $R \in \mathcal{R}^n$ if there is no other allocation $z' \equiv (x', t') \in Z$ such that (i) $z'_i \; R_i \; z_i$ for each $i \in N$, (ii) $\sum_{i \in N} t'_i \geq \sum_{i \in N} t_i$, and (iii) some agent has the strict relation in (i) or the inequality in (ii) is strict.

Note that if $R \in (\mathcal{R}^Q)^n$, then an allocation $z \equiv (x, t) \in Z$ is efficient for R if and only if $\sum_{i \in N} v_i(x_i) = \max_{x' \in X} \sum_{i \in N} v_i(x'_i)$. Remark 4 below generalizes this property to non-quasi-linear preferences. Since the (net) valuations depend on the bundles, an efficient allocation under non-quasi-linear preferences should depend on the payments, unlike an efficient allocation under quasi-linear preferences.

Remark 4. Let $R \in \mathcal{R}^n$ and $z \equiv (x, t) \in Z$. Then z is efficient for R if and only if $\sum_{i \in N} v_i(x_i, z_i) = \max_{x' \in X} \sum_{i \in N} v_i(x'_i, z_i)$.

By Remark 4, we obtain another characterization of an efficient allocation under preferences with nonincreasing marginal valuations.

Remark 5. Let $N' \subseteq N$ and $m' \in M$. Let $R_{N'} \in (\mathcal{R}^{NI})^{|N'|}$ and $z \equiv (x, t) \in Z(N', m')$. Then $\sum_{i \in N'} v_i(x_i, z_i) = \max_{x' \in X(N', m')} \sum_{i \in N'} v_i(x'_i, z_i)$ if and only if for each pair $i, j \in N'$ with $x_i < m'$ and $x_j > 0$, we have $V_i(x_i + 1, z_i) - t_i \leq t_j - V_j(x_j - 1, z_j)$. In particular, by Remark 4, z is efficient for R if and only if for each pair $i, j \in N$ with $x_i < m$ and $x_j > 0$, we have $V_i(x_i + 1, z_i) - t_i \leq t_j - V_j(x_j - 1, z_j)$.

The first property states that a rule should select an efficient allocation.

Efficiency. For each $R \in \mathcal{R}^n$, $f(R)$ is efficient for R.

The second property is a dominant strategy incentive compatibility. It states that no agent would be better off by misrepresenting his preference.

Strategy-proofness. For each $R \in \mathcal{R}^n$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $f_i(R)\ R_i\ f_i(R'_i, R_{-i})$.

The third property is a participation constraint. It states that a rule never selects an allocation at which some agent is worse off than if he had received no object and paid nothing.

Individual rationality. For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R)\ R_i\ \mathbf{0}$.

The fourth and fifth properties are both concerned with nonnegativity of payments.

No subsidy. For each $R \in \mathcal{R}^n$ and each $i \in N$, $t_i(R) \geq 0$.

No subsidy for losers. For each $R \in \mathcal{R}^n$ and each $i \in N$, if $x_i(R) = 0$, then $t_i(R) \geq 0$.

Clearly, *no subsidy* implies *no subsidy for losers*.

3 The generalized Vickrey rule

In this section, we introduce the Vickrey rule for quasi-linear preferences (Vickrey, 1961) and its generalization for non-quasi-linear preferences (Saitoh and Serizawa, 2008; Sakai, 2008).

Given $i \in N$, $R_{-i} \in \mathcal{R}^{n-1}$, and $x_i \in M$, define the maximum sum of net valuations at $\mathbf{0}$ for agents other than agent i, given that agent i has already obtained x_i units, as

$$\sigma_i(R_{-i}; x_i) \equiv \max_{x \in X(N\setminus\{i\}, m - x_i)} \sum_{j \in N\setminus\{i\}} v_j(x_j, \mathbf{0}).$$

Note that if $R_{-i} \in (\mathcal{R}^Q)^{n-1}$, then $\sigma_i(R_{-i}; x_i) = \max_{x \in X(N\setminus\{i\}, m-x_i)} \sum_{j \in N\setminus\{i\}} v_j(x_j)$.

We first introduce the Vickrey rule for quasi-linear preferences.

Definition 3. Let $\mathcal{R} \subseteq \mathcal{R}^Q$. A rule $f \equiv (x, t)$ on $\mathcal{R}^n$ is a **Vickrey rule** if the following two conditions hold:

(i) for each $R \in \mathcal{R}^n$,

$$x(R) \in \arg\max_{x \in X} \sum_{i \in N} v_i(x_i),$$

(ii) for each $R \in \mathcal{R}^n$ and $i \in N$,

$$t_i(R) = \sigma_i(R_{-i}; 0) - \sigma_i(R_{-i}; x_i(R)).$$

Condition (i) says that the object is allocated so as to maximize the sum of net valuations, and condition (ii) says that each agent must pay the externality that he imposes on other agents.

Remark 2 (i) states that the net valuations are independent of bundles under quasi-linear preferences, and so we do not have to care about the address of the net valuations in the definition of the Vickrey rule. However, the net valuations may vary depending on payments under non-quasi-linear preferences, and we must pin down net valuations before defining a Vickrey rule on a non-quasi-linear domain. The generalized Vickrey rule picks the net valuations at $\mathbf{0}$, and apply the Vickrey rule to these valuations.

Definition 4. A rule $f \equiv (x, t)$ on $\mathcal{R}^n$ is a **generalized Vickrey rule** if the following two conditions hold:
(i) for each $R \in \mathcal{R}^n$,

$$x(R) \in \arg\max_{x \in X} \sum_{i \in N} v_i(x_i, \mathbf{0}),$$

(ii) for each $R \in \mathcal{R}^n$ and each $i \in N$,

$$t_i(R) = \sigma_i(R_{-i}; 0) - \sigma_i(R_{-i}; x_i(R)).$$

Remark 6. Let $f \equiv (x, t)$ be a generalized Vickrey rule on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$ and $z \equiv (x_i(R), v_i(x_i(R), \mathbf{0}))_{i \in N}$. Note that $v_i(x_i(R), \mathbf{0}) = V_i(x_i(R), \mathbf{0})$ for each $i \in N$ as $V_i(0, \mathbf{0}) = 0$. Thus, for each $i \in N$, $z_i \; I_i \; \mathbf{0}$, and by Remark 1 (ii), $v_i(x_i(R), z_i) = v_i(x_i(R), \mathbf{0})$. Then by Remark 4, the first condition (i) of the generalized Vickrey rule implies that z is efficient for R.

Note that Remark 6 simply states that the allocation $(x_i(R), v_i(x_i(R), \mathbf{0}))_{i \in N}$ is efficient for each preference profile $R \in \mathcal{R}^n$, and it does not imply that the generalized Vickrey rule satisfies *efficiency*.

4 Characterization theorem

In this section, we provide a characterization theorem on domains contained by the domain with nondecreasing marginal valuations.

Our result requires a domain to be rich in the sense defined below.

Definition 5. A domain $\mathcal{R}^n$ is **rich** if $\mathcal{R} \supseteq \mathcal{R}^C \cap \mathcal{R}^Q$. We call the domain $(\mathcal{R}^C \cap \mathcal{R}^Q)^n$ the **minimally rich domain**.

The following theorem states that on any rich subdomain of the domain with nondecreasing marginal valuations, the generalized Vickrey rule is the only rule satisfying *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers*.

Theorem 1. *Let $\mathcal{R}$ be a class of preferences such that $\mathcal{R}^C \cap \mathcal{R}^Q \subseteq \mathcal{R} \subseteq \mathcal{R}^{ND}$. A rule on $\mathcal{R}^n$ satisfies efficiency, strategy-proofness, individual rationality, and no subsidy for losers if and only if it is a generalized Vickrey rule on $\mathcal{R}^n$.*

Note that Theorem 1 immediately implies that the generalized Vickrey rule satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* on any subdomain of the domain with nondecreasing marginal valuations.

Since a domain in Theorem 1 might include non-quasi-linear preferences, the existence of a rule satisfying desirable properties such as *efficiency* and *strategy-proofness* is not trivial. Indeed, Theorem 1 is not an immediate consequence of the existing results on the quasi-linear domain (Holmström, 1979; Chew and Serizawa, 2007). We further emphasize that our result is not a trivial extension of the characterization of the generalized Vickrey rule for the single-object environment (Saitoh and Serizawa, 2008; Sakai, 2008) since we allow preferences to have constant marginal valuations. A detailed discussion can be found in Subsection 6.2.1.

The independence of properties in Theorem 1 is demonstrated in the examples below. In the examples, we fix a rich class of preferences at $\mathcal{R}$ satisfying $\mathcal{R} \subseteq \mathcal{R}^{ND}$.

Example 1 (Dropping efficiency). Let f be the *no-trade rule* on $\mathcal{R}^n$ such that each agent receives $\mathbf{0}$ for each preference profile. Then f satisfies all the properties in Theorem 1 other than *efficiency*.

Example 2 (Dropping strategy-proofness). Let $f \equiv (x, t)$ be the *generalized pay-as-bid rule* on $\mathcal{R}^n$ such that for each preference profile $R \in \mathcal{R}^n$, (i) the object is allocated so as to maximize the sum of net valuations at $\mathbf{0}$, and (ii) each agent has to pay his net valuation of $x_i(R)$ at $\mathbf{0}$. By Remark 6, f satisfies *efficiency*. Further, it satisfies *individual rationality* and *no subsidy for losers*, but violates *strategy-proofness*.

Example 3 (Dropping individual rationality). Let f be the generalized Vickrey rule with fixed and common entry fee $e > 0$ on $\mathcal{R}^n$. Then f satisfies all the properties in Theorem 1 other than *individual rationality*.

Example 4 (Dropping no subsidy for losers). Let f be the generalized Vickrey rule with fixed and common participation subsidy $s < 0$ on $\mathcal{R}^n$. Then f satisfies all the properties in Theorem 1 other than *no subsidy for losers*.

5 Impossibility theorem

In this section, we address the following question: if a rich domain includes some non-quasi-linear preferences with decreasing marginal valuations, then is there a rule satisfying *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* on the domain?

To answer this question formally, we have to introduce a class of preferences which describe a reasonable pattern of income effects. Although our model does not take into account income explicitly, the zero payment can be regarded as initial income. Then the increase of income by $\delta > 0$ induces the shift of the origin of the consumption space to the right by δ. If we fix the origin of the original consumption space, then this shift corresponds to the decrease of payments of all the bundles by δ. Then positive (resp. non-negative) income effect requires that the increase of income (or equivalently, the decrease of payments) by δ increase (resp. do not decrease) the demand of the object.

Definition 6. A preference R_i exhibits the **positive** (resp. **nonnegative**) **income effect** if for each pair $(x_i, t_i), (x'_i, t'_i) \in M \times \mathbb{R}$ with $x_i > x'_i$ and $t_i > t'_i$, and each $\delta \in \mathbb{R}_{++}$, $(x_i, t_i)\ I_i\ (x'_i, t'_i)$ implies $(x_i, t_i - \delta)\ P_i\ (x'_i, t'_i - \delta)$ (resp. $(x_i, t_i - \delta)\ R_i\ (x'_i, t'_i - \delta)$).

Let $\mathcal{R}^{++}$ and $\mathcal{R}^{+}$ denote the classes of preferences that exhibit positive and nonnegative income effects, respectively. Note that $\mathcal{R}^{++} \subsetneq \mathcal{R}^{+}$, $\mathcal{R}^{++} \cap \mathcal{R}^{Q} = \varnothing$, and $\mathcal{R}^{Q} \subsetneq \mathcal{R}^{+}$.

Remark 7. Let $R_i \in \mathcal{R}^{++}$. (i) Let $x_i \in M \backslash \{m\}$ and $h^{+}(\cdot; x_i) : \mathbb{R} \to \mathbb{R}_{++}$ be such that $h^{+}(t_i; x_i) = V_i(x_i + 1, (x_i, t_i)) - t_i$ for each $t_i \in \mathbb{R}$. Then $h^{+}(\cdot; x_i)$ is strictly decreasing in t_i. (ii) Let $x_i \in M \backslash \{0\}$ and $h^{-}(\cdot; x_i) : \mathbb{R} \to \mathbb{R}_{++}$ be such that $h^{-}(t_i; x_i) = t_i - V_i(x_i - 1, (x_i, t_i))$ for each $t_i \in \mathbb{R}$. Then $h^{-}(\cdot; x_i)$ is strictly decreasing in t_i as well.

The proof of Remark 7 can be found in the supplementary material.

We are now ready to state our main result in this section. The following theorem gives a markedly negative answer to the above question: in the case of an odd number of units, if a rich domain includes one arbitrary preference with decreasing marginal valuations and a positive income effect, then no rule satisfies the four desirable properties on the domain.

Theorem 2. *Assume m is odd. Let $R_0 \in \mathcal{R}^{D} \cap \mathcal{R}^{++}$ and $\mathcal{R}$ be a class of preferences satisfying $\mathcal{R} \supseteq (\mathcal{R}^{C} \cap \mathcal{R}^{Q}) \cup \{R_0\}$. No rule on $\mathcal{R}^n$ satisfies efficiency, strategy-proofness, individual rationality, and no subsidy for losers.*

Note that Theorem 2 does not cover the case where m is even. Indeed, if m is even, then for some $R_0 \in \mathcal{R}^{D} \cap \mathcal{R}^{++}$, we find that a rule which we call *inverse-demand-based generalized Vickrey rule* on the domain $((\mathcal{R}^{C} \cap \mathcal{R}^{Q}) \cup \{R_0\})^n$ satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* (Proposition 4 in Appendix D).

Although we relegate the formal definition of the inverse-demand-based generalized Vickrey rule to Appendix D, let us sketch it informally here. Note that when quasi-linear preferences exhibit nonincreasing marginal valuations, valuation functions are equivalent to inverse-demand functions.[11] We adopt the parallel identification to non-quasi-linear preferences with nonincreasing marginal valuations as well, and the inverse-demand-based generalized Vickrey rule applies the Vickrey rule to inverse-demand functions of such preferences. Note that when preferences may not be quasi-linear, the rule does not coincide with the generalized Vickrey rule.

For other $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, no rule on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^n$ satisfies the four desirable properties even if m is an even number (Proposition 5 in Appendix D). In Appendix D, we will identify a necessary and sufficient condition of $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ for the existence of a rule satisfying the desirable properties when m is even and $n = 2$ (Corollary 2 in Appendix D).

An interesting question arising naturally is whether the positive result in the case of an even number of units is robust or not. Although the answer depends on the precise meaning of "robustness", one reasonable way of investigation is to expand the minimally rich domain. To slightly expand the minimally rich domain, we introduce the perturbation of quasi-linear preferences with constant marginal valuations.[12]

Definition 7. Given a quasi-linear preference $R_0 \in \mathcal{R}^C \cap \mathcal{R}^Q$ with constant marginal valuation $v_0 > 0$ and a positive number $\varepsilon > 0$, a preference R_i is the **ε-perturbation of** R_0 if for each $x_i \in M \backslash \{m\}$ and each $t_i \in \mathbb{R}$,

$$|v_0 - (V_i(x_i + 1, (x_i, t_i)) - t_i)| < \varepsilon.$$

Given $R_0 \in \mathcal{R}^C \cap \mathcal{R}^Q$ and $\varepsilon > 0$, let $\mathcal{R}^C(R_0, \varepsilon)$ denote the class of preferences that are ε-perturbation of R_0. Let $\mathcal{R}^C(\varepsilon) \equiv \bigcup_{R_0 \in \mathcal{R}^C \cap \mathcal{R}^Q} \mathcal{R}^C(R_0, \varepsilon)$. Clearly, for each pair $\varepsilon, \varepsilon' > 0$ with $\varepsilon < \varepsilon'$, we have $\mathcal{R}^C(\varepsilon) \subsetneq \mathcal{R}^C(\varepsilon')$, and $\mathcal{R}^C \cap \mathcal{R}^Q \subsetneq \mathcal{R}^C(\varepsilon) \cap \mathcal{R}^Q$. Note that as $\varepsilon \to 0$, $\mathcal{R}^C(\varepsilon) \cap \mathcal{R}^Q$ converges to $\mathcal{R}^C \cap \mathcal{R}^Q$, that is, $\bigcap_{\varepsilon \in \mathbb{R}_{++}} (\mathcal{R}^C(\varepsilon) \cap \mathcal{R}^Q) = \mathcal{R}^C \cap \mathcal{R}^Q$.

The next proposition states that if a domain contains the quasi-linear domain with slightly perturbed constant marginal valuations, and includes one arbitrary preference with decreasing marginal valuations and a positive income effect, then no rule satisfies the desirable properties on the domain *regardless of the number of units*. This means that the positive result in the case of an even number of units is vulnerable.

Proposition 1. *Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ and $\varepsilon \in \mathbb{R}_{++}$. Let $\mathcal{R}$ be a class of preferences satisfying $\mathcal{R} \supseteq (\mathcal{R}^C(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\}$. No rule on $\mathcal{R}^n$ satisfies efficiency, strategy-proofness, individual rationality, and no subsidy for losers.*

[11] For a detailed discussion, see, for example, Chapter 12 of Krishna (2009).

[12] Kazumura et al. (2020a) also introduce perturbation of quasi-linear preferences, although their notion is different from ours.

6 Discussion

6.1 Comparison to Baisa (2020)

We compare our results to the related results obtained by Baisa (2020). He considers the same setting as ours, and obtains the results for preferences exhibiting decreasing marginal valuations. He parametarizes preferences by type. His first two results (Theorems 1 and 2 of Baisa (2020)) state that in two-agent or two-unit case, if preferences are of single-dimensional types, then there is a rule on the domain with decreasing marginal valuations, positive income effects, and the single-crossing property satisfying *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers*.[13]

His results are different from ours (Theorem 1) in many points. Firstly, his results are for the case of two-agent or two-unit. Secondly, they are for the case of preferences of single-dimensional types. This means not only that the planner needs to know that preferences are of single-dimensional types, but also what preference each type has. Our result is free from those assumptions. However, his results involve technical discussions, such as a fixed-point argument, to show that the Vickrey rule can be generalized in a quite novel way so that it satisfies the desirable properties. In contrast, we characterize a natural generalization of the Vickrey rule by only elementary argument. Thus, the two results are mutually and completely independent.

Baisa (2020) also shows that if a class of preferences admits multi-dimensional types, then no rule on the domain satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* (Theorem 3 of Baisa (2020)).[14] This result is different from ours (Theorem 2 and Proposition 1) in terms of domains. The domain of his result includes only non-quasi-linear preferences with decreasing marginal valuations. In contrast, the domain of our result includes the class of quasi-linear preferences with constant marginal valuations, and one preference exhibiting decreasing marginal valuations and a positive income effect. Thus, these two results can be applied to different environments.

6.2 Difficulty of the proofs

We explain the difficulty of the proofs and discuss the factors that make our proofs challenging.

[13] A class of preferences exhibits *single-crossing property* if the marginal valuation at each bundle is increasing in a type.

[14] To be precise, he shows that no rule satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no deficit*. A rule f on $\mathcal{R}^n$ satisfies *no deficit* if for each $R \in \mathcal{R}^n$, we have $\sum_{i \in N} t_i(R) \geq 0$. However, his proof can be directly applied to the proof of the impossibility theorem stated in the body.

6.2.1 Characterization theorem

As we explained in Subsection 1.3.2, our characterization theorem (Theorem 1) cannot be obtained simply by bundling all the units and applying the characterization of the generalized Vickrey rule in the single-object environment (Saitoh and Serizawa, 2008; Sakai, 2008) although preferences exhibit nondecreasing marginal valuations.

To illustrate this point, consider a preference profile such that two agents have the same quasi-linear preference with constant marginal valuations, and their net valuations of m are the highest. For such a profile, any allocation is efficient if the two agents share all the units, and so an efficient allocation does not necessarily bundle all the units. By the first condition (i) of the generalized Vickrey rule, all units are allocated so as to maximize the sum of net valuations at $\mathbf{0}$. For the above profile, any allocation also maximizes the sum of valuations at $\mathbf{0}$ if the two agents share all the units, and so the generalized Vickrey rule does not necessarily bundle all the units either. Thus, the proof of our characterization theorem inevitably treats non-bundling efficient allocations and non-bundling generalized Vickrey rules, and requires different logics from applying the characterization of the generalized Vickrey rule in the single-object environment.

6.2.2 Minimally rich domain

The larger the domain of rules, the stronger the implications of the properties of rules on the domain. Unless domains include a sufficient variety of preferences, that is, domains are rich enough, the implications of the properties are too weak to yield meaningful conclusions. Indeed, since the beginning of mechanism design theory and social choice theory, authors assume rich domains to establish characterization or impossibility theorems (e.g., Arrow, 1951; Hurwicz 1972; Holmström, 1979; Moulin 1980, etc.). However, if a domain is so large that it includes even non-natural preferences, the conclusions from properties on the domain can be applied only to limited situations. This motivates the concept of minimal richness of domains. Many authors explicitly or implicitly assume minimal richness.

Chew and Serizawa (2007) and Malik and Mishra (2019) are such examples in the auction literature. The minimally rich domain of the first corresponds to the *additive quasi-linear domain*;[15] and that of the second to the *quasi-linear dichotomous domain*.[16] In these papers, for a given bundle (or an object), their respective minimally rich domains include a preference such that the valuation for the bundle is sufficiently large but those for other bundles are small. Then, the property of *efficiency* implies that an agent who

[15] A preference is *additive* if the valuation at each bundle is an additive function.

[16] A preference is *dichotomous* if it divides the set of bundles into the acceptable and the unacceptable sets, and given a payment, it assigns a positive value to any bundle in the acceptable set while any bundle in the unacceptable set is valueless.

has such a preference should get the given bundle. In a similar way, it is possible to construct a preference profile such that the implications of the properties pin down the outcome of the constructed profile.

Remember that our minimally rich domain includes only quasi-linear preferences with constant marginal valuations.[17] This weak requirement of the minimally rich domain makes it possible to apply our results to various situations. At the same time, it also makes the property of *efficiency* unable to pin down the units an agent receives, and only implies that an agent receives all the units or nothing. Our proofs need to overcome such weak implications of properties. This point makes our proofs challenging.

7 Conclusion

We have considered the multi-unit object allocation problem with money. The distinguishing feature of our model is to allow agents to have general preferences that may not be quasi-linear. Our results give a new insight into the design of an efficient multi-unit auction mechanism in non-quasi-linear environments: it is possible only when agents have nondecreasing marginal valuations.

Although we regard *efficiency* as the goal of an auction in this paper, another important goal is revenue maximization. Recently, Kazumura et al. (2020b) establish that the minimum price Warlasian rule maximizes the ex-post revenue among the class of rules satisfying *strategy-proofness*, *individual rationality*, and other mild properties in the setting of unit-demand and (possibly) non-quasi-linear preferences. We leave the ex-post revenue maximization problem in our setting for future research.

Appendix

A Preliminaries

In this section, we provide some basic lemmas that will be used to prove the theorems. The proofs of all the lemmas in this section are trivial, and we omit those.

The following lemma immediately follows from efficiency and object monotonicity.

Lemma 1 (No remaining object). *Let $R \in \mathcal{R}^n$ and $z \equiv (x, t) \in Z$ be efficient for R. Then $\sum_{i \in N} x_i = m$.*

The next lemma is immediate from *individual rationality* and *no subsidy for losers*.

[17] Note that domains in Proposition 1 include a little bit more preferences.

Lemma 2 (Zero payment for losers). *Let $f \equiv (x,t)$ be a rule on $\mathcal{R}^n$ satisfying individual rationality and no subsidy for losers. Let $R \in \mathcal{R}^n$ and $i \in N$. If $x_i(R) = 0$, then $t_i(R) = 0$.*

The following lemma gives useful implications of *individual rationality*.

Lemma 3. *Let $f \equiv (x,t)$ be a rule on $\mathcal{R}^n$ satisfying individual rationality. Let $R \in \mathcal{R}^n$ and $i \in N$. We have (i) $V_i(0, f_i(R)) \leq 0$, (ii) $t_i(R) \leq V_i(x_i(R), \mathbf{0}) = v_i(x_i(R), \mathbf{0})$, and (iii) for each $x_i \in M$, $V_i(x_i, g_i(R)) \leq V_i(x_i, \mathbf{0}) = v_i(x_i, \mathbf{0})$.*

Let f be a rule on $\mathcal{R}^n$. Let $i \in N$. Given $R_{-i} \in \mathcal{R}^{n-1}$, agent i's **option set under f for R_{-i}** is defined by

$$o_i^f(R_{-i}) \equiv \{z_i \in M \times \mathbb{R} : \exists R_i \in \mathcal{R} \text{ s.t. } f_i(R_i, R_{-i}) = z_i\}.$$

Further, given $R_{-i} \in \mathcal{R}^{n-1}$, let $M_i^f(R_{-i}) \equiv \{x_i \in M : \exists R_i \in \mathcal{R} \text{ s.t. } x_i(R_i, R_{-i}) = x_i\}$.

Let f be a rule on $\mathcal{R}^n$ satisfying *strategy-proofness*. Let $i \in N$ and $R_{-i} \in \mathcal{R}^{n-1}$. Given $x_i \in M_i^f(R_{-i})$, let $t_i^f(R_{-i}; x_i) \in \mathbb{R}$ be a payment such that $(x_i, t_i^f(R_{-i}; x_i)) \in o_i^f(R_{-i})$. By *strategy-proofness*, such a payment must be unique. Further, given $x_i \in M_i^f(R_{-i})$, let $z_i^f(R_{-i}; x_i) \equiv (x_i, t_i^f(R_{-i}; x_i))$. Then agent i's option set $o_i^f(R_{-i})$ under f for R_{-i} can be expressed as follows:

$$o_i^f(R_{-i}) = \{(x_i, t_i) \in M_i^f(R_{-i}) \times \mathbb{R} : t_i = t_i^f(R_{-i}; x_i)\} = \{z_i^f(R_{-i}; x_i) : x_i \in M_i^f(R_{-i})\}.$$

The following lemma is an immediate implication of *strategy-proofness*.

Lemma 4. *Let f be a rule on $\mathcal{R}^n$ satisfying strategy-proofness. Then, for each $i \in N$, each $R_{-i} \in \mathcal{R}^{n-1}$, and each $x_i \in M_i^f(R_{-i})$, $f_i(R)\ R_i\ z_i^f(R_{-i}; x_i)$.*

B Proof of the characterization theorem

In this section, we provide the proof of Theorem 1.

B.1 Preliminary

We first show some preliminary results related to nondecreasing marginal valuations.

The following lemma says that bundling is one way to allocate the object efficiently.

Lemma 5 (Optimality of bundling). *Let $N' \subseteq N$ and $m' \in M$ be such that $m' > 0$. Then for each $R_{N'} \in (\mathcal{R}^{ND})^{|N'|}$ and each $z \in Z(N', m')$, we have $\max_{i \in N'} v_i(m', z_i) = \max_{x \in X(N', m')} \sum_{j \in N'} v_j(x_j, z_j)$.*

Proof. For each $x \in X(N', m')$,

$$\max_{i \in N'} v_i(m', z_i) \geq \sum_{j \in N'} \frac{x_j}{m'} \max_{i \in N'} v_i(m', z_i) \geq \sum_{j \in N'} \frac{x_j}{m'} v_j(m', z_i) \geq \sum_{j \in N'} v_j(x_j, z_j),$$

where the first inequality follows from $x \in X(N', m')$, and the last one from $R_{N'} \in (\mathcal{R}^{ND})^{|N'|}$ and Remark 3 (i). □

The next lemma states that under an efficient allocation (resp. the outcome of the generalized Vickrey rule), if an agent's net valuation of m at his bundle (resp. $\mathbf{0}$) is not the highest one, then he can receive no object.

Lemma 6. *Let $R \in (\mathcal{R}^{ND})^n$ and $i \in N$. (i) Let $z \equiv (x, t) \in Z$ be efficient for R. If $v_i(m, z_i) < \max_{j \in N} v_j(m, z_j)$, then $x_i = 0$. (ii) Let $g(R) \equiv (x(R), t(R))$ be an outcome of the generalized Vickrey rule for R. If $v_i(m, \mathbf{0}) < \max_{j \in N} v_j(m, \mathbf{0})$, then $x_i(R) = 0$.*

Proof. (i) Suppose $v_i(m, z_i) < \max_{j \in N} v_j(m, z_j)$ and $x_i > 0$. Then,

$$\max_{j \in N} v_j(m, z_j) = \sum_{k \in N} \frac{x_k}{m} \max_{j \in N} v_j(m, z_j) > \sum_{k \in N} \frac{x_k}{m} v_k(m, z_k) \geq \sum_{k \in N} v_k(x_k, z_k),$$

where the equality follows from Lemma 1, the first inequality follows from $x_i > 0$ and $\max_{j \in N} v_j(m, z_j) > v_i(m, z_i)$, and the second one follows from $R \in (\mathcal{R}^{ND})^n$ and Remark 3 (i). By Remark 4, this contradicts efficiency.

(ii) Next, let $g(R) \equiv (x(R), t(R))$ be an outcome of the generalized Vickrey rule for R. By Remark 6, $z \equiv (z_j)_{j \in N} \equiv (x_j(R), v_j(x_j(R), \mathbf{0}))_{j \in N} = (x_j(R), V_j(x_j(R), \mathbf{0}))_{j \in N}$ is efficient for R. For each $j \in N$, by $z_j \; I_j \; \mathbf{0}$, Remark 1 (ii) gives $v_j(m, z_j) = v_j(m, \mathbf{0})$. Thus, we can show Lemma 6 (ii) in the same way as Lemma 6 (i) by using the efficient allocation z for R. □

Given $x \in X$, let $N^+(x) \equiv \{i \in N : x_i > 0\}$.

Note that Lemma 6 implies that if there are at least two agents who receive the object under an efficient allocation (resp. the outcome of the generalized Vickrey rule), then their net valuations of m at their bundles (resp. $\mathbf{0}$) must coincide. The following lemma further says that in such a case, the indifference curves of agents who receive the object through the bundles are flat.

Lemma 7 (Flat indifference curves). *Let $R \in (\mathcal{R}^{ND})^n$. (i) Let $z \equiv (x, t) \in Z$ be efficient for R. If $|N^+(x)| \geq 2$, then for each $i \in N^+(x)$ and each $x'_i \in M$, we have $v_i(x'_i, z_i) = \frac{x'_i}{m} v_i(m, z_i)$. (ii) Let $g(R) \equiv (x(R), t(R))$ be an outcome of the generalized Vickrey rule for R. If $|N^+(x(R))| \geq 2$, then for each $i \in N^+(x(R))$ and each $x_i \in M$, we have $v_i(x_i, \mathbf{0}) = \frac{x_i}{m} v_i(m, \mathbf{0})$.*

Proof. (i) Suppose there is $x_i' \in X\backslash\{0, m\}$ such that $\frac{x_i'}{m}v_i(m, z_i) \neq v_i(x_i', z_i)$. By $R_i \in \mathcal{R}^{ND}$ and Remark 3 (i), $\frac{x_i'}{m}v_i(m, z_i) > v_i(x_i', z_i)$. By $|N^+(x)| \geq 2$ and $i \in N^+(x)$, $x_i \in M\backslash\{0, m\}$. Thus, by $R_i \in \mathcal{R}^{ND}$, Remark 3 (ii) gives $\frac{x_i}{m}v_i(m, z_i) > v_i(x_i, z_i)$. Then

$$v_i(m, z_i) = \sum_{j \in N} \frac{x_j}{m} v_i(m, x_i) = \sum_{j \in N} \frac{x_j}{m} v_j(m, z_j) > \sum_{j \in N} v_j(x_j, z_j),$$

where the first equality follows from Lemma 1, the second one, from $R \in (\mathcal{R}^{ND})^n$ and Lemma 6 (i), and the inequality follows from $x_i \in M\backslash\{0, m\}$, $\frac{x_i}{m}v_i(m, z_i) > v_i(x_i, z_i)$, $R_{-i} \in (\mathcal{R}^{ND})^{n-1}$, and Remark 3 (i). By Remark 4, this contradicts efficiency.

(ii) We can show Lemma 7 (ii) similarly to Lemma 7 (i), but by using Lemma 6 (ii) instead of Lemma 6 (i) and the efficient allocation $(x_j(R), v_j(x_j(R), \mathbf{0}))_{j \in N}$ for R instead of z. □

The following proposition identifies the form of the payments under the generalized Vickrey rule for preferences with nondecreasing marginal valuations.

Proposition 2 (The generalized Vickrey rule payments). *Let $\mathcal{R} \subseteq \mathcal{R}^{ND}$. Let $g \equiv (x, t)$ be a generalized Vickrey rule on $\mathcal{R}^n$. Let $R \in (\mathcal{R}^{ND})^n$. Then for each $i \in N$, $t_i(R) = \frac{x_i(R)}{m} \max_{j \in N\backslash\{i\}} v_j(m, \mathbf{0})$. Moreover, if $|N^+(x(R))| \geq 2$, then $t_i(R) = v_i(x_i(R), \mathbf{0})$ for each $i \in N$.*

Proof. By the definition of the generalized Vickrey rule, for each $i \in N$, if $x_i(R) = 0$, then $t_i(R) = 0$. Thus, we only have to consider an agent $i \in N^+(x(R))$.

First, suppose $|N^+(x(R))| = 1$. Note that by $|N^+(x(R))| = 1$ and Remark 6, Lemma 1 implies $x_i(R) = m$. By $R_{-i} \in (\mathcal{R}^{ND})^{n-1}$ and Lemma 5, $t_i(R) = \sigma_i(R_{-i}; 0) = \max_{j \in N\backslash\{i\}} v_j(m, \mathbf{0})$.

Suppose instead $|N^+(x(R))| \geq 2$. For each $j \in N^+(x(R))$, each $x_j \in M$, and each $k \in N$,

$$v_j(x_j, \mathbf{0}) = \frac{x_j}{m} v_j(m, \mathbf{0}) \geq \frac{x_j}{m} v_k(m, \mathbf{0}) \geq v_k(x_j, \mathbf{0}), \tag{1}$$

where the equality follows from $R \in (\mathcal{R}^{ND})^n$ and Lemma 7 (ii), the first inequality follows from $R \in (\mathcal{R}^{ND})^n$ and Lemma 6 (ii), and the second one comes from $R_k \in \mathcal{R}^{ND}$ and Remark 3 (i). By $|N^+(x(R))| \geq 2$, there is $j \in N^+(x(R))\backslash\{i\}$. By (1), for each $x_i \in M$,

$$v_i(x_i, \mathbf{0}) = v_j(x_i, \mathbf{0}) = \max_{k \in N\backslash\{i\}} v_k(x_i, \mathbf{0}). \tag{2}$$

Then

$$\begin{aligned} t_i(R) &= \sigma_i(R_{-i}; 0) - \sigma_i(R_{-i}; x_i(R)) \\ &= \max_{k \in N \setminus \{i\}} v_k(m, \mathbf{0}) - \max_{k \in N \setminus \{i\}} v_k(m - x_i(R), \mathbf{0}) && \text{(by Lemma 5)} \\ &= v_i(m, \mathbf{0}) - v_i(m - x_i(R), \mathbf{0}) && \text{(by (2))} \\ &= \frac{x_i(R)}{m} v_i(m, \mathbf{0}) && \text{(by Lemma 7 (ii))} \\ &= \frac{x_i(R)}{m} \max_{k \in N \setminus \{i\}} v_k(m, \mathbf{0}). && \text{(by (2))} \end{aligned}$$

Further, by $R \in (\mathcal{R}^{ND})^n$ and Lemma 7 (ii), $t_i(R) = \frac{x_i(R)}{m} v_i(m, \mathbf{0}) = v_i(x_i(R), \mathbf{0})$. □

B.2 Proof of the "if" part

Let $\mathcal{R}$ be a class of preferences such that $\mathcal{R}^C \cap \mathcal{R}^Q \subseteq \mathcal{R} \subseteq \mathcal{R}^{ND}$ and $g \equiv (x, t)$ be a generalized Vickrey rule on $\mathcal{R}^n$. Since *individual rationality* and *no subsidy for losers* are immediate from the definition of the generalized Vickrey rule, we omit the proofs.

Efficiency. Let $R \in \mathcal{R}^n$. We show $g(R)$ is efficient for R.

Suppose $|N^+(x(R))| = 1$. By Remark 6 and Lemma 1, $x_i(R) = m$ for $i \in N^+(x(R))$. By $\mathcal{R} \subseteq \mathcal{R}^{ND}$, Proposition 2 gives $g_i(R) = (m, \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0}))$ and $g_j(R) = \mathbf{0}$ for each $j \in N \setminus \{i\}$. By Remark 1 (ii), $t_i(R) = \max_{j \in N \setminus \{i\}} v_j(m, g_j(R))$. Then by Remark 1 (iii) and Lemma 3 (i),

$$v_i(m, g_i(R)) = \max_{j \in N \setminus \{i\}} v_j(m, g_j(R)) - V_i(0, g_i(R)) \geq \max_{j \in N \setminus \{i\}} v_j(m, g_j(R)). \tag{1}$$

For each $x \in X$,

$$v_i(m, g_i(R)) \geq \sum_{j \in N} \frac{x_j}{m} v_i(m, g_i(R)) \geq \sum_{j \in N} \frac{x_j}{m} v_j(m, g_j(R)) \geq \sum_{j \in N} v_j(x_j, g_j(R)),$$

where the second inequality follows from (1), and the last one follows from $R \in (\mathcal{R}^{ND})^n$ and Remark 3 (i). By Remark 4, $g(R)$ is efficient for R.

Suppose instead $|N^+(x(R))| \geq 2$. By $\mathcal{R} \subseteq \mathcal{R}^{ND}$ and Proposition 2, $t_i(R) = v_i(x_i(R), \mathbf{0})$ for each $i \in N$. Thus, Remark 6 implies that $g(R)$ is efficient for R.

Strategy-proofness. Let $R \in \mathcal{R}^n$, $i \in N$, and $R'_i \in \mathcal{R}$. By $\mathcal{R} \subseteq \mathcal{R}^{ND}$, Proposition 2 implies $g_i(R'_i, R_{-i}) = (x_i, \frac{x_i}{m} \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0}))$, where $x_i \equiv x_i(R'_i, R_{-i})$. We show that $g_i(R) \; R_i \; (x_i, \frac{x_i}{m} \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0}))$.

Suppose first $x_i(R) = m$. Let $s_i \equiv V_i(0, g_i(R))$. By Lemma 3 (i), $s_i \leq 0$. Then

$$V_i(x_i, g_i(R)) = v_i(x_i, g_i(R)) + s_i \leq \frac{x_i}{m}\big(v_i(m, g_i(R)) + s_i\big) = \frac{x_i}{m} \max_{j \in N \setminus \{i\}} V_j(m, \mathbf{0}),$$

where the inequality follows from $R_i \in \mathcal{R}^{ND}$, Remark 3 (i), and $s_i \leq 0$, and the last equality follows from Remark 1 (iii), $\mathcal{R} \subseteq \mathcal{R}^{ND}$, and Proposition 2.

Next, suppose $x_i(R) < m$. By Lemma 1, there is $j \in N^+(x(R)) \setminus \{i\}$. By $R \in (\mathcal{R}^{ND})^n$ and Lemma 6 (ii), $v_j(m, \mathbf{0}) = \max_{k \in N} v_k(m, \mathbf{0})$. Then

$$v_i(m, \mathbf{0}) \leq v_j(m, \mathbf{0}) = \max_{k \in N \setminus \{i\}} v_k(m, \mathbf{0}). \tag{2}$$

We have

$$V_i(x_i, g_i(R)) \leq v_i(x_i, \mathbf{0}) \leq \frac{x_i}{m} v_i(m, \mathbf{0}) \leq \frac{x_i}{m} \max_{k \in N \setminus \{i\}} v_k(m, \mathbf{0}),$$

where the first inequality follows from Lemma 3 (iii), the second one follows from $R_i \in \mathcal{R}^{ND}$ and Remark 3 (i), and the last one, from (2).

In either case, we obtain $g_i(R)\ R_i\ (x_i, \frac{x_i}{m} \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0}))$. ■

B.3 Proof of the "only if" part

In this section, we provide the proof of the "only if" part. Throughout the subsection, we fix a class of preferences $\mathcal{R}$ such that $\mathcal{R}^C \cap \mathcal{R}^Q \subseteq \mathcal{R} \subseteq \mathcal{R}^{ND}$ and a rule f on $\mathcal{R}^n$ satisfying *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers*.

We first set up the following lemma.

Lemma 8. *Let $R \in \mathcal{R}^n$ and $i \in N$. If $v_i(m, \mathbf{0}) < \max_{j \in N} v_j(m, \mathbf{0})$, then $x_i(R) = 0$.*

Proof. The proof is in two steps.

Step 1. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $R_i \in \mathcal{R}^C \cap \mathcal{R}^Q$ and $v_i(m) < \max_{j \in N} v_j(m, \mathbf{0})$. We show $x_i(R) = 0$. Suppose by contradiction that $x_i(R) > 0$.

Let $j \in \arg\max_{k \in N} v_k(m, \mathbf{0})$. We show $x_j(R) > 0$. Suppose $x_j(R) = 0$. By Lemma 2, $f_j(R) = \mathbf{0}$. By $v_i(m) < V_j(m, \mathbf{0})$ and $R \in (\mathcal{R}^{ND})^n$, Lemma 6 (i) implies $x_i(R) = 0$, a contradiction. Thus, $x_j(R) > 0$.

Next, we show $t_j(R) \geq v_i(x_j(R))$. Suppose $t_j(R) < v_i(x_j(R))$. Let $R'_j \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that $t_j(R) < v'_j(x_j(R)) < v_i(x_j(R))$. By $R'_j \in \mathcal{R}^C$, $v'_j(x_j(R)) < v_i(x_j(R))$, and $R_i \in \mathcal{R}^C$,

$$v'_j(m) = \frac{m}{x_j(R)} v'_j(x_j(R)) < \frac{m}{x_j(R)} v_i(x_j(R)) = v_i(m).$$

By $R \in (\mathcal{R}^{ND})^n$, Lemmas 2 and 6 (i) together imply $f_j(R'_j, R_{-j}) = \mathbf{0}$. By $t_j(R) < v'_j(x_j(R))$, $f_j(R)\ P'_j\ \mathbf{0} = f_j(R'_j, R_{-j})$, contradicting *strategy-proofness*. Thus, $t_j(R) \geq v_i(x_j(R))$.

Note that by *individual rationality*, $f_j(R)\ R_j\ \mathbf{0}$. If $f_j(R)\ I_j\ \mathbf{0}$, then by Remark 1 (ii),

$$v_j(m, f_j(R)) = v_j(m, \mathbf{0}) > v_i(m).$$

Instead, if $f_j(R)\ P_j\ \mathbf{0}$, then $V_j(0, f_j(R)) < 0$, and so

$$v_j(m, f_j(R)) = \frac{m}{x_j(R)}\Big(t_j(R) - V_j(0, f_j(R))\Big) > \frac{m}{x_j(R)} v_i(x_j(R)) = v_i(m),$$

where the first equality follows from $R \in (\mathcal{R}^{ND})^n$, Lemma 7 (i), and Remark 1 (iii), the inequality follows from $t_j(R) \geq v_i(x_j(R))$ and $V_j(0, f_j(R)) < 0$, and the last equality follows from $R_i \in \mathcal{R}^C$. In either case, by $R \in (\mathcal{R}^{ND})^n$, Lemma 6 (i) implies $x_i(R) = 0$. But this contradicts $x_i(R) > 0$.

Step 2. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $v_i(m, \mathbf{0}) < \max_{j \in N} v_j(m, \mathbf{0})$. Suppose $x_i(R) > 0$. Let $R'_i \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that $v_i(m, \mathbf{0}) < v'_i(m) < \max_{j \in N} v_j(m, \mathbf{0})$. By Step 1, $x_i(R'_i, R_{-i}) = 0$. By Lemma 2, $f_i(R'_i, R_{-i}) = \mathbf{0}$. Then,

$$t_i(R) \leq v_i(x_i(R), \mathbf{0}) \leq \frac{x_i(R)}{m} v_i(m, \mathbf{0}) < \frac{x_i(R)}{m} v'_i(m) = v'_i(x_i(R)),$$

where the first inequality follows from Lemma 3 (ii), the second one follows from $R_i \in \mathcal{R}^{ND}$ and Remark 3 (i), the third one follows from $v_i(m, \mathbf{0}) < v'_i(m)$ and $x_i(R) > 0$, and the equality follows from $R'_i \in \mathcal{R}^C$. Thus, $f_i(R)\ P'_i\ \mathbf{0} = f_i(R'_i, R_{-i})$, contradicting *strategy-proofness*. □

We now show that f is a generalized Vickrey rule on $\mathcal{R}^n$.

Step 1. We first show that the payments under f coincide with those of the generalized Vickrey rule. Let $R \in \mathcal{R}^n$ and $i \in N$. Note that by $\mathcal{R} \subseteq \mathcal{R}^{ND}$ and Proposition 2, we only have to show that $t_i(R) = \frac{x_i(R)}{m} \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0})$. Suppose not. By Lemma 2, we must have $x_i(R) > 0$. We divide the argument into two cases.

Case 1. $t_i(R) > \frac{x_i(R)}{m} \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0})$.

We have

$$v_i(m, \mathbf{0}) \geq \frac{m}{x_i(R)} v_i(x_i(R), \mathbf{0}) \geq \frac{m}{x_i(R)} t_i(R) > \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0}),$$

where the first inequality follows from $R_i \in \mathcal{R}^{ND}$ and Remark 3 (i), the second one follows from Lemma 3 (ii), and the last one comes from $t_i(R) > \frac{x_i(R)}{m} \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0})$.

Thus, by Lemma 8, $x_j(R) = 0$ for each $j \in N \backslash \{i\}$. By Lemma 1, $x_i(R) = m$. Thus, by $t_i(R) > \frac{x_i(R)}{m} \max_{j \in N \backslash \{i\}} v_j(m, \mathbf{0})$, $t_i(R) > \max_{j \in N \backslash \{i\}} v_j(m, \mathbf{0})$. Let $R'_i \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that $\max_{j \in N \backslash \{i\}} v_j(m, \mathbf{0}) < v'_i(m) < t_i(R)$. Again, by Lemmas 1 and 8, $x_i(R'_i, R_{-i}) = m$. Thus, by Lemma 3 (ii), $t_i(R'_i, R_{-i}) \leq v'_i(m) < t_i(R)$, which implies $f_i(R'_i, R_{-i})\ P_i\ f_i(R)$. However, this contradicts *strategy-proofness*.

Case 2. $t_i(R) < \frac{x_i(R)}{m} \max_{j \in N \backslash \{i\}} v_j(m, \mathbf{0})$.

Let $R'_i \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that $t_i(R) < v'_i(x_i(R)) < \frac{x_i(R)}{m} \max_{j \in N \backslash \{i\}} v_j(m, \mathbf{0})$. By $R'_i \in \mathcal{R}^C$ and $v'_i(x_i(R)) < \frac{x_i(R)}{m} \max_{j \in N \backslash \{i\}} v_j(m, \mathbf{0})$,

$$v'_i(m) = \frac{m}{x_i(R)} v'_i(x_i(R)) < \max_{j \in N \backslash \{i\}} v_j(m, \mathbf{0}).$$

Thus, Lemmas 2 and 8 together imply $f_i(R'_i, R_{-i}) = \mathbf{0}$. However, by $t_i(R) < v'_i(x_i(R))$, $f_i(R)\ P'_i\ \mathbf{0} = f_i(R'_i, R_{-i})$, contradicting *strategy-proofness*.

Step 2. Let $R \in \mathcal{R}^n$. We show $\sum_{i \in N} v_i(x_i(R), \mathbf{0}) = \max_{x \in X} \sum_{i \in N} v_i(x_i, \mathbf{0})$.

Suppose $|N^+(x(R))| = 1$. Let $i \in N^+(x(R))$. By $|N^+(x(R))| = 1$ and Lemma 1, $x_i(R) = m$. Thus, $x_j(R) = 0$ for each $j \in N \backslash \{i\}$. Then

$$\sum_{j \in N} v_j(x_j(R), \mathbf{0}) = v_i(m, \mathbf{0}) = \max_{j \in N} v_j(m, \mathbf{0}) = \max_{x \in X} \sum_{j \in N} v_j(x_j, \mathbf{0}),$$

where the second equality follows from Lemma 8, and the last one comes from $R \in (\mathcal{R}^{ND})^n$ and Lemma 5.

Next, suppose $|N^+(x(R))| \geq 2$. We show $f_i(R)\ I_i\ \mathbf{0}$ for each $i \in N$. By *individual rationality*, $f_i(R)\ R_i\ \mathbf{0}$ for each $i \in N$. Suppose there is $i \in N$ such that $f_i(R)\ P_i\ \mathbf{0}$. Then $t_i(R) < V_i(x_i(R), \mathbf{0}) = v_i(x_i(R), \mathbf{0})$. Thus, by $R_i \in \mathcal{R}^{ND}$, Remark 3 (i), and Step 1,

$$\frac{x_i(R)}{m} v_i(m, \mathbf{0}) \geq v_i(x_i(R), \mathbf{0}) > t_i(R) = \frac{x_i(R)}{m} \max_{j \in N \backslash \{i\}} v_j(m, \mathbf{0}).$$

This implies $v_i(m, \mathbf{0}) > \max_{j \in N \backslash \{i\}} v_j(m, \mathbf{0})$. By Lemma 8, $x_j(R) = 0$ for each $j \in N \backslash \{i\}$. But this contradicts $|N^+(x(R))| \geq 2$.

Thus, $f_i(R)\ I_i\ \mathbf{0}$ for each $i \in N$. By Remark 1 (ii), $v_i(x_i, f_i(R)) = v_i(x_i, \mathbf{0})$ for each $i \in N$ and $x_i \in M$. Thus, we have

$$\sum_{i \in N} v_i(x_i(R), \mathbf{0}) = \sum_{i \in N} v_i(x_i(R), f_i(R)) = \max_{x \in X} \sum_{i \in N} v_i(x_i, f_i(R)) = \max_{x \in X} \sum_{i \in N} v_i(x_i, \mathbf{0}),$$

where the second equality follows from Remark 4 and *efficiency*. ■

C Proofs of the impossibility theorems

In this section, we provide the proofs of Theorem 2 and Proposition 1.

C.1 Preliminary

We first show the following lemma, which states the existence and uniqueness of some payments.

Lemma 9. *Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$. For each $x \in M \backslash \{0, m\}$, there is a unique payment $t^*(x) \in (0, V_0(x, \mathbf{0}))$ such that*

$$V_0(x+1, (x, t^*(x))) - t^*(x) = \frac{t^*(x)}{x}.$$

Proof. Let $x \in M \backslash \{0, m\}$. Let $h : \mathbb{R}_+ \to \mathbb{R}$ be such that for each $t \in \mathbb{R}_+$, $h(t) = V_0(x+1, (x, t)) - t - \frac{t}{x}$. By object monotonicity, $(x+1, 0)\ P_0\ (x, 0)$. This implies $h(0) = V_0(x+1, (x, 0)) > 0$. By Remark 1 (i), $R_0 \in \mathcal{R}^D$, and $x \in M \backslash \{0\}$,

$$h(V_0(x, \mathbf{0})) = \Big(V_0(x+1, \mathbf{0}) - V_0(x, \mathbf{0})\Big) - \frac{V_0(x, \mathbf{0})}{x} < 0.$$

By continuity of R_0, the function $h^+(t) \equiv V_0(x+1, (x, t)) - t$ is continuous on $[0, V_0(x, \mathbf{0})]$.[18] Thus, $h(\cdot)$ is continuous on $[0, V_0(x, \mathbf{0})]$ as well. Then, by the intermediate value theorem, there is a payment $t^*(x) \in (0, V_0(x, \mathbf{0}))$ such that $h(t^*(x)) = 0$. Thus,

$$V_0(x+1, (x, t^*(x))) - t^*(x) = \frac{t^*(x)}{x}.$$

By Remark 7 (i), $h(\cdot)$ is strictly decreasing. Thus, such a payment $t^*(x)$ is unique. □

Given $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, by Lemma 9, we can define a function $t^*(\cdot; R_0)$ from $M \backslash \{0, m\}$ to $\mathbb{R}_{++}$ such that for each $x \in M \backslash \{0, m\}$, $t^*(x; R_0)$ is in $(0, V_0(x, \mathbf{0}))$ and satisfies the equation in Lemma 9. We simply write $t^*(\cdot)$ instead of $t^*(\cdot; R_0)$ unless there is no risk of confusion.

Lemma 10. *Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ and $x \in M \backslash \{0, m\}$. (i) For each $t \in \mathbb{R}_+$,*

$$t < t^*(x) \text{ if and only if } V_0(x+1, (x, t)) - t > \frac{t}{x}.$$

(ii) For each $t \in \mathbb{R}_+$,

$$t < \frac{x+1}{x} t^*(x) \text{ if and only if } t - V_0(x, (x+1, t)) > \frac{t}{x+1}.$$

[18] For the formal proof of this statement, see Lemma 1 of Kazumura and Serizawa (2016).

Proof. Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ and $x \in M\backslash\{0, m\}$.

(i) Let $h : \mathbb{R}_+ \to \mathbb{R}$ be such that for each $t \in \mathbb{R}_+$, $h(t) = V_0(x+1,(x,t)) - t - \frac{t}{x}$. By Remark 7 (i), $h(\cdot)$ is strictly decreasing. By Lemma 9, $h(t^*(x)) = 0$. Thus, for each $t \in \mathbb{R}_+$, $t < t^*(x)$ is equivalent to $h(t) > h(t^*(x)) = 0$, which is also equivalent to $V_0(x+1,(x,t)) - t > \frac{t}{x}$.

(ii) Let $g : \mathbb{R}_+ \to \mathbb{R}$ be such that $g(t) = t - V_0(x,(x+1,t)) - \frac{t}{x+1}$. We first claim that $g(\frac{x+1}{x}t^*(x)) = 0$. We have

$$\begin{aligned} V_0\Big(x,(x+1,\frac{x+1}{x}t^*(x))\Big) &= V_0\Big(x,(x+1,V_i(x+1,(x,t^*(x))))\Big) \\ &= V_0(x,(x,t^*(x))) \\ &= t^*(x), \end{aligned} \tag{1}$$

where the first equality follows from Lemma 9, the second equality follows from Remark 1 (i), and the last equality follows from Remark 1 (iii). Thus,

$$\begin{aligned} g\Big(\frac{x+1}{x}t^*(x)\Big) &= \frac{x+1}{x}t^*(x) - V_0\Big(x,(x+1,\frac{x+1}{x}t^*(x))\Big) - \frac{1}{x+1}\frac{x+1}{x}t^*(x) \\ &= \frac{x+1}{x}t^*(x) - t^*(x) - \frac{t^*(x)}{x} && \text{(by (1))} \\ &= 0. \end{aligned}$$

Since $g(\cdot)$ is strictly decreasing by Remark 7 (ii), for each $t \in \mathbb{R}_+$, $t < \frac{x+1}{x}t^*(x)$ is equivalent to $g(t) > g(\frac{x+1}{x}t^*(x)) = 0$, which is equivalent to $t - V_0(x,(x+1,t)) > \frac{t}{x+1}$. □

The following lemma shows that the per-unit payments $\frac{t^*(x)}{x}$ specified in Lemma 9 are strictly decreasing in the number of units.

Lemma 11. *For each* $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$,

$$V_0(1,\mathbf{0}) > t^*(1) > \frac{t^*(2)}{2} > \cdots > \frac{t^*(m-1)}{m-1}.$$

Proof. By $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Lemma 9 gives $t^*(1) < V_0(1,\mathbf{0})$. Suppose that there is $x \in M\backslash\{0, m-1, m\}$ such that $\frac{t^*(x)}{x} \leq \frac{t^*(x+1)}{x+1}$. By $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ and Lemma 9, $\frac{x+1}{x}t^*(x) = V_0(x+1,(x,t^*(x)))$. Thus,

$$\Big(x+1,\frac{x+1}{x}t^*(x)\Big) = \Big(x+1,V_0(x+1,(x,t^*(x)))\Big)\ I_0\ (x,t^*(x)).$$

Thus, by Remark 1 (i),

$$V_0\Big(x+2,(x+1,\frac{x+1}{x}t^*(x))\Big) = V_0(x+2,(x,t^*(x))). \tag{1}$$

Then

$$
\begin{aligned}
&V_0\Big(x+2,(x+1,\frac{x+1}{x}t^*(x))\Big)-\frac{x+1}{x}t^*(x) \\
&= V_0(x+2,(x,t^*(x))) - V_0(x+1,(x,t^*(x))) && \text{(by (1) and Lemma 9)} \\
&< V_0(x+1,(x,t^*(x))) - t^*(x) && \text{(by } R_0 \in \mathcal{R}^D \text{ and Remark 1 (iii))} \\
&= \frac{t^*(x)}{x} && \text{(by Lemma 9)} \\
&\leq \frac{t^*(x+1)}{x+1} && \text{(by assumption)} \\
&= V_0(x+2,(x+1,t^*(x+1))) - t^*(x+1). && \text{(by Lemma 9)}
\end{aligned}
$$

However, by $R_0 \in \mathcal{R}^{++}$ and $\frac{x+1}{x}t^*(x) \leq t^*(x+1)$, this contradicts Remark 7 (i). □

Figure 3 below illustrates the payments $t^*(\cdot)$ described in Lemmas 9, 10, and 11.

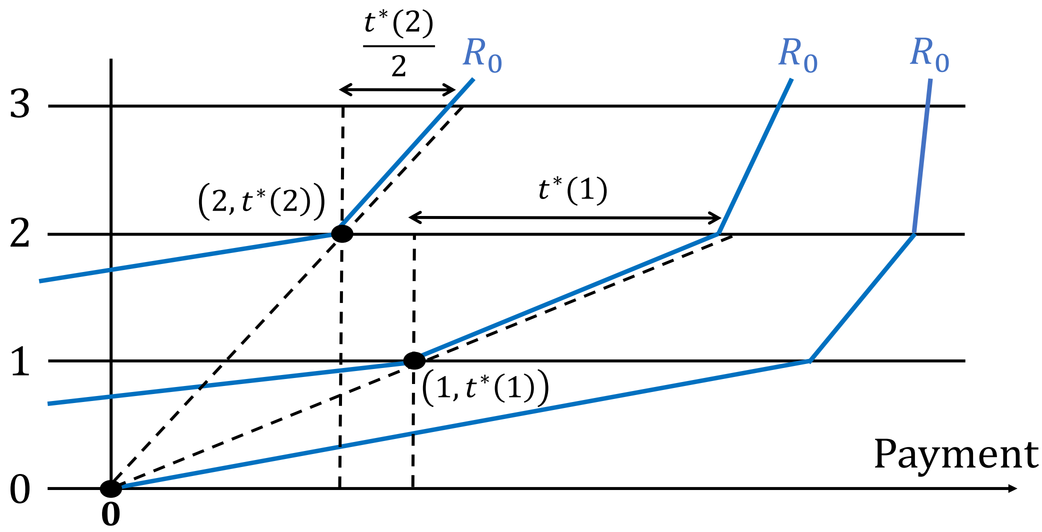

Figure 3: An illustration of $t^*(\cdot)$.

Here, we provide an interpretation of the per-unit payments $\frac{t^*(x)}{x}$ specified in Lemma 9. Let $R_i \in \mathcal{R}$ and $x_i \in M$. Then, the **inverse-demand set at** x_i **for** R_i is defined as $P(x_i; R_i) \equiv \{p \in \mathbb{R}_+ : (x_i, px_i)\ R_i\ (x_i', px_i') \text{ for each } x_i' \in M\}$. The **inverse-demand function for** R_i is a function $p(\cdot; R_i) : M \to \mathbb{R}_+ \cup \{\infty\}$ such that for each $x_i \in M$, $p(x_i; R_i) = \inf P(x; R_i)$.[19]

The next proposition identifies the inverse-demand sets of a preference $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$.

Proposition 3 (The inverse-demand sets). *Let* $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$. *Then (i)* $P(0; R_0) = [V_0(1, \mathbf{0}), \infty)$, *(ii)* $P(1; R_0) = [t^*(1), V_0(1, \mathbf{0})]$, *(iii)* $P(x; R_0) = [\frac{t^*(x)}{x}, \frac{t^*(x-1)}{x-1}]$ *for each* $x \in M \backslash \{0, 1, m\}$, *and (iv)* $P(m; R_0) = [0, \frac{t^*(m-1)}{m-1}]$.

[19]If $P(x_i; R_i) = \varnothing$, then set $p(x_i; R_i) \equiv \infty$.

The proof of Proposition 3 can be found in the supplementary material.

The following corollary of Proposition 3 states that the per-unit payment $\frac{t^*(x)}{x}$ specified in Lemma 9 is exactly the image of the inverse-demand function $p(x; R_0)$ for each $x \in M\backslash\{0, m\}$.

Corollary 1 (The inverse-demand function). *Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$. We have (i) $p(0; R_0) = V_0(1, \mathbf{0})$, (ii) $p(x; R_0) = \frac{t^*(x)}{x}$ for each $x \in M\backslash\{0, m\}$, and, (iii) $p(m; R_0) = 0$.*

Corollary 1 gives the redefinition of the inverse-demand function of a preference $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, and hereafter we will repeatedly use the implication of Corollary 1 without referring to it.

C.2 Proofs of Theorem 2 and Proposition 1

We are now in a position to prove Theorem 2 and Proposition 1. Both the proofs have many parts in common, and we do not distinguish them for a while. Before providing the proofs, we invoke the following fact.

Fact 1 (Holmström, 1979). *Let $\mathcal{R}$ be such that $\mathcal{R} \cap \mathcal{R}^Q$ is convex.*[20] *Let f be a rule on $\mathcal{R}^n$ satisfying efficiency, strategy-proofness, individual rationality, and no subsidy for losers. Then, for each $i \in N$, each $R_{-i} \in (\mathcal{R} \cap \mathcal{R}^Q)^{n-1}$, and each $x_i \in M_i^f(R_{-i})$,*

$$t_i^f(R_{-i}; x_i) = \sigma_i(R_{-i}; 0) - \sigma_i(R_{-i}; x_i).$$

Note that $\mathcal{R}^C \cap \mathcal{R}^Q$ and $\mathcal{R}^C(\varepsilon) \cap \mathcal{R}^Q$ are both convex for each $\varepsilon > 0$. This allows us to use Fact 1 in the subsequent proof.

Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ and $\varepsilon \in \mathbb{R}_{++}$. Let

$$\mathcal{R} \in \Big\{(\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}, (\mathcal{R}^C(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\}\Big\}.$$

Suppose that there is a rule f on $\mathcal{R}^n$ satisfying *efficiency*, *strategy-proofness*, *individual rationality, and no subsidy for losers.*[21]

Step 1. For notational convenience, we expand the domain of $p(\cdot; R_0)$ from M to $M \cup \{-1\}$, and define $p(-1; R_0) \equiv \infty$. By $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Lemma 11 implies that $p(x-1; R_0) > p(x; R_0)$ for each $x \in M$. For each $x \in M\backslash\{m\}$, let

$$\mathcal{R}(x) \equiv \{R_i \in \mathcal{R}^C \cap \mathcal{R}^Q : v_i(1) \in (p(x; R_0), p(x-1; R_0))\}.$$

[20] A class of preferences $\mathcal{R} \subseteq \mathcal{R}^Q$ is *convex* if for each pair R_i, R_i' with valuations functions $v_i(\cdot), v_i'(\cdot)$ and each $\lambda \in [0, 1]$, a preference R_i^λ with valuation function $v_i^\lambda(\cdot) = \lambda v_i(\cdot) + (1-\lambda)v_i'(\cdot)$ is in $\mathcal{R}$.

[21] Note that an impossibility theorem on a domain implies the impossibility theorem on any superdomain. Thus, to show Theorem 2, we only have to show the impossibility on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^n$. The parallel discussion applies to Proposition 1.

Let $\delta > 0$ be such that $\delta < \min\{p(m-1;R_0), V_0(m,\mathbf{0}) - V_0(m-1,\mathbf{0})\}$.[22] Let

$$\mathcal{R}(m) \equiv \{R_i \in \mathcal{R}^C \cap \mathcal{R}^Q : v_i(1) \in (\delta, p(m-1;R_0))\}.$$

Note that $\bigcup_{x \in M} \mathcal{R}(x) \subsetneq \mathcal{R}^C \cap \mathcal{R}^Q \subsetneq \mathcal{R}$.

Let $R_1 \equiv R_0$. For each $i \in N \backslash \{1,2\}$, let $R_i \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that $v_i(x_i) = \delta x_i$ for each $x_i \in M$. Then, *efficiency*, in conjunction with *individual rationality*, $\delta < V_1(m,\mathbf{0}) - V_1(m-1,\mathbf{0})$, and $R_1 \in \mathcal{R}^D \cap \mathcal{R}^D$, implies that for each $R_2 \in \mathcal{R}$ and each $i \in N \backslash \{1,2\}$, $x_i(R) = 0$.[23] Further, for each $R_2 \in \bigcup_{x \in M} \mathcal{R}(x)$, since $v_2 > \delta$ and $R_{-1} \in (\mathcal{R}^{NI})^{n-1}$, Remark 5 implies $\sigma_1(R_{-1};x_1) = (m-x_1)v_2$ for each $x_1 \in M$, where v_2 is a constant marginal valuation associated with R_2. Thus, for each $R_2 \in \bigcup_{x \in M} \mathcal{R}(x)$ with constant marginal valuation v_2 and each $x_1 \in M_1^f(R_{-1})$, Fact 1 gives

$$t_1^f(R_{-1};x_1) = v_2 x_1. \tag{1}$$

STEP 2. Let $x_1 \in M$ and $R_2^{x_1} \in \mathcal{R}(x_1)$. Let $R^{x_1} \equiv (R_1, R_2^{x_1}, R_{-1,2})$. We show $x_1(R^{x_1}) = x_1$ and $x_2(R^{x_1}) = m - x_1$. By Lemma 1 and $x_i(R^{x_1}) = 0$ for each $i \in N \backslash \{1,2\}$, we only have to show $x_1(R^{x_1}) = x_1$. Let $v_2^{x_1}$ be a constant marginal valuation associated with $R_2^{x_1}$.

CASE 1. $1 \leq x_1 \leq m$.

We show $x_1(R^{x_1}) = x_1$. Suppose by contradiction that $x_i(R^{x_1}) \neq x_1$.

Suppose first $x_1(R^{x_1}) > x_1$. Then,

$$t_1(R^{x_1}) = v_2^{x_1} x_1(R^{x_1}) > p(x_1;R_1)x_1(R^{x_1}) \geq p(x_1(R^{x_1})-1;R_1)x_1(R^{x_1}), \tag{2}$$

where the equality follows from (1), the first inequality follows from $R_2^{x_1} \in \mathcal{R}(x_1)$, and the second follows from $R_1 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, $x_1(R^{x_1}) > x_1$, and Lemma 11. Note that by $0 < x_1(R^{x_1}) - 1 < m$, $p(x_1(R^{x_1})-1;R_1)x_1(R^{x_1}) = \frac{x_1(R^{x_1})}{x_1(R^{x_1})-1} t^*(x_1(R^{x_1})-1)$. Thus, by $R_1 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Lemma 10 (ii), (2), and (1), we have

$$t_1(R^{x_1}) - V_1(x_1(R^{x_1})-1, f_1(R^{x_1})) < \frac{t_1(R^{x_1})}{x_1(R^{x_1})} = \frac{v_2^{x_1} x_1(R^{x_1})}{x_1(R_1^x)} = v_2^{x_1}.$$

[22] By Lemma 9, $p(m-1;R_0) = \frac{t^*(m-1)}{m-1} > 0$. Moreover, by object monotonicity, $V_0(m,\mathbf{0}) > V_0(m-1,\mathbf{0})$. Thus, we can pick such δ.

[23] The formal argument is as follows. Suppose $x_i(R) > 0$ for some $i \in N \backslash \{1,2\}$. Then, $x_1(R) < m$, and

$$V_1(x_1(R)+1, f_1(R)) - t_1(R) \geq V_1(x_1(R)+1,\mathbf{0}) - V_1(x_1(R),\mathbf{0}) > \delta,$$

where the first inequality follows from $R_1 \in \mathcal{R}^{++}$, Lemma 3 (ii), and Remarks 7 (i) and 1 (i), and the second one follows from $R_1 \in \mathcal{R}^D$ and $\delta < V_1(m,\mathbf{0}) - V_1(m-1,\mathbf{0})$. However, this contradicts Remark 5.

By $R^{x_1} \in (\mathcal{R}^{NI})^n$, this contradicts Remark 5.

Suppose $0 < x_1(R^{x_1}) < x_1$. Then

$$t_1(R^{x_1}) = v_2^{x_1} x_1(R^{x_1}) < p(x_1 - 1; R_1) x_1(R^{x_1}) \leq p(x_1(R^{x_1}); R_1) x_1(R^{x_1}) = t^*(x_1(R^{x_1})),$$

where the first equality follows from (1), the first inequality follows from $R_2^{x_1} \in \mathcal{R}(x_1)$, the second follows from $R_1 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, $x_1 > x_1(R^{x_1})$, and Lemma 11, and the last equality follows from $0 < x_1(R^{x_1}) < m$. Thus, by $R_1 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Lemma 10 (i), and (1),

$$V_1(x_1(R^{x_1}) + 1, f_1(R^{x_1})) - t_1(R^{x_1}) > \frac{t_1(R^{x_1})}{x_1(R^{x_1})} = \frac{v_2^{x_1} x_1(R^{x_1})}{x_1(R^{x_1})} = v_2^{x_1}. \tag{3}$$

By $R^{x_1} \in (\mathcal{R}^{NI})^n$, this contradicts Remark 5.

Suppose finally $x_1(R^{x_1}) = 0$. Then $t_1(R^{x_1}) = 0$ by Lemma 2. By $R_2^{x_1} \in \mathcal{R}(x_1)$, $R_1 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, and Lemma 11, $v_2^{x_1} < p(x_1 - 1; R_1) \leq p(0; R_1) = V_1(1, \mathbf{0})$. However, by $f_1(R^{x_1}) = \mathbf{0}$ and $R^{x_1} \in (\mathcal{R}^{NI})^n$, this contradicts Remark 5.

Case 2. $x_1 = 0$.

We show $x_1(R^0) = 0$. Suppose by contradiction that $x_1(R^0) > 0$. Then

$$V_1(x_1(R^0), \mathbf{0}) \leq x_1(R^0) V_1(1, \mathbf{0}) = x_1(R^0) p(0; R_1) < x_1(R^0) v_2^0, \tag{4}$$

where the first inequality follows from $R_1 \in \mathcal{R}^D$, and the last one follows from $x_1(R^0) > 0$ and $R_2^0 \in \mathcal{R}(0)$. By (4) and (1),

$$\mathbf{0} \ I_1 \ (x_1(R^0), V_1(x_1(R^0), \mathbf{0})) \ P_1 \ (x_1(R^0), x_1(R^0) v_2^0) = f_1(R^0).$$

However, this contradicts *individual rationality*.

Step 3. Note that Step 2 implies that for each $x_2 \in M$ and each $R_2^{m-x_2} \in \mathcal{R}(m - x_2)$, $x_2(R_2^{m-x_2}, R_{-2}) = x_2$. Thus, $M_2^f(R_{-2}) = M$, and the domain of the function $t_2^f(R_{-2}; \cdot)$ is M. In this step, we show that for each $x_2 \in M$,

$$t_2^f(R_{-2}; x_2) = \sum_{x=0}^{x_2} p(m - x; R_1).$$

By Lemma 2, $t_2^f(R_{-2}; 0) = 0 = p(m; R_1)$. Thus, it suffices to show that for each $x_2 \in M \backslash \{m\}$, $t_2^f(R_{-2}; x_2 + 1) - t_2^f(R_{-2}; x_2) = p(m - x_2 - 1; R_1)$. Let $x_2 \in M \backslash \{m\}$. Suppose by contradiction that $t_2^f(R_{-2}; x_2 + 1) - t_2^f(R_{-2}; x_2) \neq p(m - x_2 - 1; R_1)$.

Case 1. $t_2^f(R_{-2}; x_2 + 1) - t_2^f(R_{-2}; x_2) < p(m - x_2 - 1; R_1)$.

Let $R_2^{m-x_2} \in \mathcal{R}(m - x_2)$ be such that

$$t_2^f(R_{-2}; x_2 + 1) - t_2^f(R_{-2}; x_2) < v_2^{m-x_2} < p(m - x_2 - 1; R_1),$$

where $v_2^{m-x_2}$ is a constant marginal valuation associated with $R_2^{m-x_2}$. By Step 2, we have $x_2(R^{m-x_2}) = x_2$. By $v_2^{m-x_2} > t_2^f(R_{-2}; x_2 + 1) - t_2^f(R_{-2}; x_2)$,

$$(x_2 + 1)v_2^{m-x_2} - t_2^f(R_{-2}; x_2 + 1) > x_2 v_2^{m-x_2} - t_2^f(R_{-2}; x_2).$$

Thus, $z_2^f(R_{-2}; x_2 + 1) \ P_2^{m-x_2} \ z_2^f(R_{-2}; x_2) = f_2(R^{m-x_2})$, which contradicts Lemma 4.

Case 2. $t_2^f(R_{-2}; x_2 + 1) - t_2^f(R_{-2}; x_2) > p(m - x_2 - 1; R_1)$.

Let $R_2^{m-x_2-1} \in \mathcal{R}(m - x_2 - 1)$ be such that

$$p(m - x_2 - 1; R_1) < v_2^{m-x_2-1} < t_2^f(R_{-2}; x_2 + 1) - t_2^f(R_{-2}; x_2),$$

where $v_2^{m-x_2-1}$ is a constant marginal valuation associated with $R_2^{m-x_2-1}$. By Step 2, we have $x_2(R^{m-x_2-1}) = x_2 + 1$. By $v_2^{m-x_2-1} < t_2^f(R_{-2}; x_2 + 1) - t_2^f(R_{-2}; x_2)$,

$$x_2 v_2^{m-x_2-1} - t_2^f(R_{-2}; x_2) > (x_2 + 1)v_2^{m-x_2-1} - t_2^f(R_{-2}; x_2 + 1).$$

Thus, $z_2^f(R_{-2}; x_2) \ P_2^{m-x_2-1} \ z_2^f(R_{-2}; x_2 + 1) = f_2(R^{m-x_2-1})$, which contradicts Lemma 4.

C.2.1 Proof of Theorem 2

We complete the proof of Theorem 2. Suppose m is odd and $\mathcal{R} = (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$.

Let $R_2 \equiv R_0$. Let $\alpha \equiv \frac{m-1}{2}$. Note that as m is odd, $\alpha \in M$. Let $x_2 \in M$ be such that $0 < x_2 \leq \alpha$. We show that $z_2^f(R_{-2}; x_2 + 1) \ P_2 \ z_2^f(R_{-2}; x_2)$. We have

$$t_2^f(R_{-2}; x_2) = \sum_{x=1}^{x_2} p(m - x; R_1) \leq p(m - x_2; R_1)x_2 < p(x_2; R_1)x_2 = t^*(x_2; R_1),$$

where the first equality follows from Step 3, the first inequality follows from $R_1 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ and Lemma 11, the second inequality comes from $R_1 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, $x_2 \leq \alpha$, and Lemma 11, and the last equality follows from $0 < x_2 < m$. Thus, by $R_2 \in \mathcal{R}^{++}$ and Remark 7 (i),

$$V_2(x_2 + 1, z_2^f(R_{-2}; x_2)) - t_2^f(R_{-2}; x_2) > V_2(x_2 + 1, (x_2, t^*(x_2; R_1))) - t^*(x_2; R_1). \quad (5)$$

We also have

$$\begin{aligned} t_2^f(R_{-2}; x_2+1) - t_2^f(R_{-2}; x_2) &= p(m - x_2 - 1; R_1) \\ &\leq p(x_2; R_1) \\ &= \frac{t^*(x_2; R_1)}{x_2} \\ &= V_2(x_2+1, (x_2, t^*(x_2; R_1))) - t^*(x_2; R_1), \end{aligned}$$

where the first equality follows from Step 3, the inequality follows from $R_1 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, $x_2 \leq \alpha$, and Lemma 11, the second equality follows from $0 < x_2 < m$, and the last one, from $R_2 = R_1 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ and Lemma 9. This, together with (5), implies

$$V_2(x_2+1, z_2^f(R_{-2}; x_2)) - t_2^f(R_{-2}; x_2) > t_2^f(R_{-2}; x_2+1) - t_2^f(R_{-2}; x_2),$$

or $V_2(x_2+1, z_2^f(R_{-2}; x_2)) > t_2^f(R_{-2}; x_2+1)$. Thus, $z_2^f(R_{-2}; x_2+1) \; P_2 \; z_2^f(R_{-2}; x_2)$.

We then show that $z_2^f(R_{-2}; 1) \; P_2 \; z_2^f(R_{-2}; 0)$. By Step 3, $R_2 = R_1 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, and Lemma 11, $t_2^f(R_{-2}; 1) = p(m-1; R_1) < V_2(1, \mathbf{0})$. Thus, $z_2^f(R_{-2}; 1) \; P_2 \; \mathbf{0} = z_2^f(R_{-2}; 0)$.

We have established $z_2^f(R_{-2}; \alpha+1) \; P_2 \; z_2^f(R_{-2}; x_2)$ for each $x_2 \in M$ with $x_2 < \alpha + 1$. By Lemma 4, $x_2(R) \geq \alpha + 1$. Since both agents 1 and 2 have the same preferences R_0, the name of agents does not matter in the above discussion. Thus, a symmetric argument implies $x_1(R) \geq \alpha + 1$. Therefore,

$$x_1(R) + x_2(R) \geq 2(\alpha + 1) = m + 1.$$

However, this contradicts feasibility. ∎

C.2.2 Proof of Proposition 1

Next, we complete the proof of Proposition 1. Suppose $\mathcal{R} = (\mathcal{R}^C(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\}$.

Recall that $\delta < p(m-1; R_1)$. Let $0 < \varepsilon_1 < \min\{\frac{\varepsilon}{2}, \frac{p(m-1;R_1)-\delta}{2}\}$. Then

$$\begin{aligned} & V_1(m, (m-1, t^*(m-1) - \varepsilon_1)) - (t^*(m-1) - \varepsilon_1) \\ & > V_1(m, (m-1, t^*(m-1))) - t^*(m-1) && \text{(by Remark 7 (i))} \\ & = \frac{t^*(m-1)}{m-1} && \text{(by Lemma 9)} \\ & = p(m-1; R_1). \end{aligned}$$

Thus,

$$\varepsilon_2 \equiv V_1(m, (m-1, t^*(m-1) - \varepsilon_1)) - (t^*(m-1) - \varepsilon_1) - p(m-1; R_1) > 0.$$

Let $\varepsilon_3 > 0$ be such that

$$\varepsilon_3 < \min\Big\{\frac{\varepsilon}{2}, \frac{p(m-1;R_1)-\delta}{2}, \varepsilon_2\Big\}.$$

Let $R_2 \in \mathcal{R}^C(\varepsilon) \cap \mathcal{R}^Q$ be such that $v_2(1) = p(m-1;R_1)+\varepsilon_3$, and for each $x_2 \in M\backslash\{0,m\}$,

$$v_2(x_2+1) - v_2(x_2) = p(m-1;R_1) - \frac{\varepsilon_1+\varepsilon_3}{m-1}.$$

Note that by $\varepsilon_1, \varepsilon_3 < \frac{\varepsilon}{2}$, R_2 is ε-perturbation of $R_2' \in \mathcal{R}^C \cap \mathcal{R}^Q$ whose constant marginal valuation is $p(m-1;R_1)$. Note also that $R_2 \in \mathcal{R}^{NI}$. By $\delta < p(m-1;R_1)$, $v_2(1) > \delta$. Further, by $\varepsilon_1, \varepsilon_3 < \frac{p(m-1;R_1)-\delta}{2}$, $v_2(x_2+1) - v_2(x_2) > \delta$ for each $x_2 \in M\backslash\{0,m\}$. Thus, by $R_{-1} \in (\mathcal{R}^{NI})^{n-1}$, Remark 5 implies $\sigma_1(R_{-1};x_1) = v_2(m-x_1)$ for each $x_1 \in M$. Then by Fact 1, for each $x_1 \in M_1^f(R_{-1})$,

$$t_1^f(R_{-1};x_1) = v_2(m) - v_2(m-x_1). \tag{6}$$

By Step 3, $v_2(1) - t_2^f(R_{-2};1) = (p(m-1;R_1)+\varepsilon_3) - p(m-1;R_1) = \varepsilon_3 > 0$. For each $x_2 \in M\backslash\{0,1\}$,

$$\begin{aligned}
v_2(x_2) - v_2(1) &= (x_2-1)\Big(p(m-1;R_1) - \frac{\varepsilon_1+\varepsilon_3}{m-1}\Big) \\
&< (x_2-1)p(m-1;R_1) && \text{(by } \varepsilon_1, \varepsilon_3 > 0) \\
&< \sum_{x=2}^{x_2} p(m-x;R_1) && \text{(by Lemma 11)} \\
&= t_2^f(R_{-2};x_2) - t_2^f(R_{-2};1), && \text{(by Step 3)}
\end{aligned}$$

or $v_2(1) - t_2^f(R_{-2};1) > v_2(x_2) - t_2^f(R_{-2};x_2)$. Hence, $z_2^f(R_{-2};1)\ P_2\ z_2^f(R_{-2};x_2)$ for each $x_2 \in M\backslash\{1\}$. By Lemma 4, we obtain $f_2(R) = z_2^f(R_{-2};1)$. By Lemma 1, $x_1(R) = m-1$. By (6), $t_1(R) = t^*(m-1) - (\varepsilon_1+\varepsilon_3)$. Therefore,

$$\begin{aligned}
&V_1(x_1(R)+1, f_1(R)) - t_1(R) - (v_2(x_2(R)) - v_2(x_2(R)-1)) \\
&= V_1(m, (m-1, t^*(m-1) - (\varepsilon_1+\varepsilon_3))) - (t^*(m-1) - (\varepsilon_1+\varepsilon_3)) - (p(m-1;R_1)+\varepsilon_3) \\
&> V_1(m, (m-1, t^*(m-1) - \varepsilon_1)) - (t^*(m-1) - \varepsilon_1) - (p(m-1;R_1)+\varepsilon_3) \\
&= \varepsilon_2 - \varepsilon_3 \\
&> 0,
\end{aligned}$$

where the first inequality follows from $R_1 \in \mathcal{R}^{++}$, $\varepsilon_3 > 0$, and Remark 7 (i), the second equality follows from the definition of ε_2, and the second inequality follows from $\varepsilon_2 < \varepsilon_3$. By $R \in (\mathcal{R}^{NI})^n$, this contradicts Remark 5. ■

D Even number of units of the object

In this section, we provide the results in the case of even number of units omitted in the main text.

Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ and $\mathcal{R} \equiv (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$. We will assume that R_0 satisfies the following condition:

$$V_0(\beta + 1, (\beta, t^*)) - t^* \leq p(\beta - 1; R_0), \tag{1}$$

where $\beta \equiv \frac{m}{2}$ and $t^* \equiv \sum_{x=1}^{\beta} p(m - x; R_0)$.

Given a preference $R_i \in \mathcal{R}^{NI} \cap \mathcal{R}^+$, let $R_i^{inv} \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ be such that for each $x_i \in M \backslash \{m\}$, $v_i^{inv}(x_i + 1) - v_i^{inv}(x_i) = p(x_i; R_i)$.[24] Note that if $R_i \in \mathcal{R}^C \cap \mathcal{R}^Q$, then $R_i^{inv} = R_i$. Given a preference profile $R \in (\mathcal{R}^{NI} \cap \mathcal{R}^+)^n$, let $R^{inv} \equiv (R_i^{inv})_{i \in N}$. Further, given $i \in N$ and $R_{-i} \in (\mathcal{R}^{NI} \cap \mathcal{R}^+)^{n-1}$, let $R_{-i}^{inv} \equiv (R_j^{inv})_{j \in N \backslash \{i\}}$.

Definition 8. Let $\mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^+$. A rule f on $\mathcal{R}^n$ is an **inverse-demand-based generalized Vickrey rule** if the following two conditions hold:
(i) for each $R \in \mathcal{R}^n$,

$$x(R) \in \arg\max_{x \in X} \sum_{i \in N} v_i^{inv}(x_i),$$

(ii) for each $R \in \mathcal{R}^n$ and each $i \in N$,

$$t_i(R) = \sigma_i(R_{-i}^{inv}; 0) - \sigma_i(R_{-i}^{inv}; x_i(R)).$$

Note that given a preference profile $R \in \mathcal{R}^n$, the inverse-demand-based generalized Vickrey rule produces an outcome of the Vickrey rule for the preference profile R^{inv} induced by the original preference profile R.

Assume $n = 2$. Then we obtain the following positive result.[25]

Proposition 4. *Assume $n = 2$ and m is even. Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ satisfy (1). Let $\mathcal{R} \equiv (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$. An inverse-demand-based-generalized Vickrey rule on $\mathcal{R}^2$ satisfies efficiency, strategy-proofness, individual rationality, and no subsidy for losers.*

The proof of Proposition 4 can be found in the supplementary material.

[24] By identifying some payments that have the similar properties to those in Lemmas 9 and 11, we can show $P(x_i; R_i) \neq \varnothing$ for each $x_i \in M \backslash \{m\}$. Moreover, we have $p(x_i; R_i) \geq v_i(x_i + 1) - v_i(x_i) > 0$ and $p(x_i; R_i) \geq p(x_i + 1; R_i)$ for each $x_i \in M \backslash \{m\}$. Thus, R_i^{inv} is well-defined.

[25] One might expect that when $n \geq 3$, the inverse-demand-based generalized Vickrey rule on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^n$ satisfies the four properties for some $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$. However, when $n \geq 3$ and $m = 6a - 2$ for some $a \in \mathbb{N}$, it violates *strategy-proofness* regardless of the choice of $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$. The supplementary material contains the example that demonstrates this fact.

If $m = 2$, the condition (1) holds for any $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$.[26] Thus, Proposition 4 implies that for any $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, the inverse-demand based generalized Vickrey rule on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^2$ satisfies the four properties if $n = m = 2$.

The next proposition states that when $m \geq 4$, the condition (1) is a necessary condition for the existence of a rule satisfying the desirable properties.

Proposition 5. *Assume $m \geq 4$ is even. Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ violate (1). Let $\mathcal{R}$ be a class of preferences satisfying $\mathcal{R} \supseteq (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$. No rule on $\mathcal{R}^n$ satisfies efficiency, strategy-proofness, individual rationality, and no subsidy for losers.*

The proof of Proposition 5 is almost same as that of Theorem 2, and we relegate it to the supplementary material.

Propositions 4 and 5 together imply that when $n = 2$ and m is even, (1) is a necessary and sufficient condition for the existence of a rule on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^2$ satisfying the desirable properties.

Corollary 2 (Necessary and sufficient condition). *Assume $n = 2$ and m is even. Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$. Let $\mathcal{R} \equiv (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$. There is a rule on $\mathcal{R}^2$ satisfying efficiency, strategy-proofness, individual rationality, and no subsidy for losers if and only if R_0 satisfies (1).*

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[26] To see this, let $m = 2$ and $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$. Note that $t^* = p(1; R_0) = t^*(1)$. Thus, by Lemmas 9 and 11,

$$V_0(2, (1, t^*)) - t^* = t^*(1) < V_0(1, \mathbf{0}) = p(0; R_0).$$

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