Supplementary material for "Multi-unit object allocation problems with money for (non)decreasing incremental valuations: Impossibility and characterization theorem"

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In this supplementary material, we provide the proofs and the example omitted in the main text (Shinozaki et al., 2022).

1 Proofs of Remarks

In this section, we give the proofs of some remarks in the main text.

Remark 3 (Nondecreasing per-unit net valuations). Let $m' \in M$ with $m' > 0$. Let $R_i \in \mathcal{R}^{ND}$ and $z_i \in M \times \mathbb{R}$. (i) For each $x_i \in M(m')$, $\frac{x_i}{m'} v_i(m', z_i) \geq v_i(x_i, z_i)$. (ii) If there is $x_i \in M(m')\backslash\{0, m'\}$ such that $\frac{x_i}{m'} v_i(m', z_i) > v_i(x_i, z_i)$, then for each $x_i' \in M(m')\backslash\{0, m'\}$, $\frac{x_i'}{m'} v_i(m', z_i) > v_i(x_i', z_i)$.

Proof. (i) Let $x_i \in M(m')$. Then

$$\begin{aligned} m' v_i(x_i, z_i) &= m' \Bigg(\sum_{x=0}^{x_i - 1} (v_i(x+1, z_i) - v_i(x, z_i)) \Bigg) \\ &\leq x_i \Bigg(\sum_{x=0}^{m'-1} (v_i(x+1, z_i) - v_i(x, z_i)) \Bigg) && \text{(by } R_i \in \mathcal{R}^{ND}) \\ &= x_i v_i(m', z_i). \end{aligned}$$

(ii) Suppose there is $x_i \in M(m')\backslash\{0, m'\}$ such that $\frac{x_i}{m'} v_i(m', z_i) > v_i(x_i, z_i)$. Then there is $x \in M(m')\backslash\{0, m'\}$ such that $v_i(x+1, z_i) - v_i(x, z_i) > v_i(x, z_i) - v_i(x-1, z_i)$. Let $x_i' \in M(m')\backslash\{0, m'\}$. Then the inequality above holds strictly for x_i'. □

Remark 5. Let $R_i \in \mathcal{R}^{++}$. (i) Let $x_i \in M\backslash\{m\}$ and $h^+(\cdot; x_i) : \mathbb{R} \to \mathbb{R}_{++}$ be such that for each $t_i \in \mathbb{R}$, $h^+(t_i; x_i) = V_i(x_i + 1, (x_i, t_i)) - t_i$. Then $h^+(\cdot; x_i)$ is strictly decreasing in t_i. (ii) Let $x_i \in M\backslash\{0\}$ and $h^-(\cdot; x_i) : \mathbb{R} \to \mathbb{R}_{++}$ be such that for each $t_i \in \mathbb{R}$, $h^-(t_i; x_i) = t_i - V_i(x_i - 1, (x_i, t_i))$. Then $h^-(\cdot; x_i)$ is strictly decreasing in t_i as well.

Proof. (i) Let $t_i, t_i' \in \mathbb{R}$ be such that $t_i' < t_i$. Note that $(x_i+1, V_i(x_i+1,(x_i,t_i)))\ I_i\ (x_i,t_i)$. Thus, by $t_i - t_i' > 0$, $R_i \in \mathcal{R}^{++}$ implies

$$(x_i+1, V_i(x_i+1,(x_i,t_i))-(t_i-t_i'))\ P_i\ (x_i, t_i-(t_i-t_i')) = (x_i,t_i')\ I_i\ (x_i+1, V_i(x_i+1,(x_i,t_i'))).$$

Thus, $V_i(x_i+1,(x_i,t_i))-(t_i-t_i') < V_i(x_i+1,(x_i,t_i'))$, or equivalently, we have $h^+(t_i;x_i) = V_i(x_i+1,(x_i,t_i)) - t_i < V_i(x_i+1,(x_i,t_i')) - t_i' = h^+(t_i';x_i)$.

(ii) We can show (ii) in the symmetric way, and we omit the proof. □

2 Proof of Proposition 3

In this section, we give the proof of Proposition 3.

Proposition 3. *Let $R_i \in \mathcal{R}^D$ satisfy the single-intersection condition. Then, for each $x_i \in M\backslash\{m\}$, $\underline{d}(x_i) = \overline{d}(x_i) = p(x_i+1;R_i)$.*

Proof. By $|T(x_i)| = 1$ for each $x_i \in M$, $\underline{d}(x_i) = \overline{d}(x_i) \equiv d(x_i)$ for each $x_i \in M$. Let $x_i \in M\backslash\{m\}$. The proof has two steps.

Step 1. Let $p \in P(x_i+1;R_i)$. We show that $p \le d(x_i)$. Suppose by contradiction that $p > d(x_i)$. We consider the following two cases.

Case 1. $x_i > 0$.

Now, we claim that $px_i < V_i(x_i,\mathbf{0})$. Suppose by contradiction that $px_i \ge V_i(x_i,\mathbf{0})$. Then,

$$V_i(x_i+1,\mathbf{0}) < \frac{x_i+1}{x_i} V_i(x_i,\mathbf{0}) \le p(x_i+1),$$

where the first inequality follows from $R_i \in \mathcal{R}^D$, and the second one from $px_i \ge V_i(x_i,\mathbf{0})$. Thus, $\mathbf{0}\ P_i\ (x_i+1, p(x_i+1))$. However, this contradicts $p \in P(x_i+1;R_i)$.

Thus, $px_i < V_i(x_i,\mathbf{0})$. By $p > d(x_i)$, $px_i \in (\bar{t}(x_i), V_i(x_i,\mathbf{0}))$. Thus, by $R_i \in \mathcal{R}^D$, Lemma 8 implies

$$V_i(x_i+1,(x_i,px_i)) - px_i < \frac{px_i}{x_i} = p,$$

or equivalently, $V_i(x_i+1,(x_i,px_i)) < p(x_i+1)$. This implies $(x_i,px_i)\ P_i\ (x_i+1,p(x_i+1))$. However, this contradicts $p \in P(x_i+1;R_i)$.

Case 2. $x_i = 0$.

Note that $d(0) = V_i(1,\mathbf{0})$. Thus, by $p > d(0)$, $p > V_i(1,\mathbf{0})$. This implies $\mathbf{0}\ P_i\ (1,p)$. However, this contradicts $p \in P(1;R_i)$.

Step 2. We show that $d(x_i) \in P(x_i+1;R_i)$. Note that by Step 1, this implies $d(x_i) = p(x_i+1;R_i)$.

By $d(x_i) \in T(x_i)$,

$$V_i(x_i + 1, (x_i, d(x_i)x_i)) - d(x_i)x_i = d(x_i), \tag{1}$$

or equivalently,

$$V_i(x_i + 1, (x_i, d(x_i)x_i)) = d(x_i)(x_i + 1).$$

This implies $(x_i + 1, d(x_i)(x_i + 1)) \ I_i \ (x_i, d(x_i)x_i)$.

Let $x \in M$ be such that $x > x_i + 1$. Then, by $R_i \in \mathcal{R}^D$ and (1),

$$V_i(x, (x_i, d(x_i)(x_i))) - d(x_i)x_i \leq (x - x_i)\Big(V_i(x_i + 1, (x_i, d(x_i)x_i)) - d(x_i)x_i\Big) = d(x_i)(x - x_i),$$

or equivalently,

$$V_i(x, (x_i, d(x_i)x_i)) \leq d(x_i)x.$$

This implies $(x_i, d(x_i)x_i) \ R_i \ (x, d(x_i)x)$.

Let $x \in M$ be such that $x < x_i$. Then, $x_i > 0$. By $R_i \in \mathcal{R}^D$ and $d(x_i)x_i = \bar{t}(x_i)$, Lemma 12 implies

$$d(x_i)x_i - V_i(x_i - 1, (x_i, d(x_i)x_i)) > \frac{d(x_i)x_i}{x_i} = d(x_i).$$

Thus, by $R_i \in \mathcal{R}^D$,

$$d(x_i)x_i - V_i(x, (x_i, d(x_i)x_i)) \geq (x_i - x)\Big(d(x_i)x_i - V_i(x_i - 1, (x_i, d(x_i)x_i))\Big) > d(x_i)(x_i - x),$$

or equivalently,

$$V_i(x, (x_i, d(x_i)x_i)) < d(x_i)x.$$

This implies $(x_i, d(x_i)x_i) \ P_i \ (x, d(x_i)x)$.

Thus, we have established that for each $x \in M$, $(x_i, d(x_i)x_i) \ R_i \ (x, d(x)x)$. This implies $d(x_i) \in P(x_i + 1; R_i)$. □

As a corollary of Proposition 3 and Lemmas 7 and 11, we obtain the following

Corollary 3. *Let $R_i \in \mathcal{R}^D$ satisfies the single-intersection condition. Then, $R_i^{inv} \in \mathcal{R}^D \cap \mathcal{R}^Q$.*

3 Proofs of Propositions 1 and 4

In this section, we provide the proofs of Propositions 1 and 4.

3.1 Preliminaries

In this subsection, we show some preliminary results.

Lemma 17. *Assume m is even.*

(i) Let $R_i \in \mathcal{R}^D \cap \mathcal{R}^{++}$. Then, it has the upper bound for the nonnegative income effects if and only if we have $V_i(\beta+1,(\beta,t^*)) - t^* \leq \overline{d}(\beta-1;R_i)$, where $\beta \equiv \frac{m}{2}$ and $t^* \equiv \sum_{x=0}^{\beta-1} p(m-x;R_i)$.

(ii) Let $R_i \in \mathcal{R}^D \cap \mathcal{R}^{--}$ satisfy the single-intersection condition. Then, it has the upper bound for the nonpositive income effects if and only if we have $t^* - V_i(\beta-1,(\beta,t^*)) \geq \overline{d}(\beta;R_i)$, where $\beta \equiv \frac{m}{2}$ and $t^* \equiv \sum_{x=0}^{\beta-1} p(m-x;R_i)$.

Proof. (i) Note that the second terms of both the sides of the inequality in the upper bound for the nonnegative income effects are the same. Thus, the upper bound for the nonnegative income effects is equivalent to the following inequality:

$$WB_i(\beta,t^*) \leq WS_i(\beta,\beta p(\beta;R_i)).$$

By Remark 1 (iii), we have

$$WB_i(\beta,t^*) = V_i(\beta+1,(\beta,t^*)) - t^*.$$

By $R_i \in \mathcal{R}^D \cap \mathcal{R}^{++}$ and Corollary 2, $\overline{d}(\beta-1;R_i) = p(\beta;R_i)$. Then, we have

$$\begin{aligned}
&WS_i(\beta,\beta p(\beta;R_i)) \\
&= \beta p(\beta;R_i) - V_i(\beta-1,(\beta,\beta p(\beta;R_i))) && \text{(by Remark 1 (iii))} \\
&= \beta\overline{d}(\beta-1;R_i) - V_i(\beta-1,(\beta,\beta\overline{d}(\beta-1;R_i))) && \text{(by } p(\beta;R_i) = \overline{d}(\beta-1;R_i)\text{)} \\
&= \overline{d}(\beta-1;R_i). && \text{(by Remark 11)}
\end{aligned}$$

Thus, R_i has the upper bound for the nonnegative income effects if and only if

$$V_i(\beta+1,(\beta,t^*)) - t^* \leq \overline{d}(\beta-1;R_i).$$

(ii) We can show (ii) in a similar way to (i), and omit the proof. □

In order to prove Propositions 1 (i) and 4 (i), we show the following proposition. It states that if a preference $R_0 \in \mathcal{R}^D$ satisfies the single-intersection condition, then the inverse Vickrey rule (i) satisfies *efficiency*, *individual rationality*, and *no subsidy for losers*, and (ii) satisfies *strategy-proofness* at any preference profile $R \in ((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^2 \backslash \{(R_0,R_0)\}$.

Proposition 6. *Assume $n=2$ and m is even. Let $R_0 \in \mathcal{R}^D$ satisfy the single-intersection condition. Let $\mathcal{R} \equiv (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$. Let $f \equiv (x,t)$ be an inverse Vickrey rule on $\mathcal{R}^2$.*
(i) f satisfies efficiency, individual rationality, and no subsidy for losers.
(ii) For each $R \in \mathcal{R}^2$, each $i \in N$ and each $R'_i \in \mathcal{R}$, if $R \neq (R_0,R_0)$, then $f_i(R)\ R_i\ f_i(R'_i,R_j)$.

Suppose $n=2$ and m is even. Let $f \equiv (x,t)$ be an inverse Vickrey rule on $\mathcal{R}^2$. By the single-intersection condition, $\underline{d}(x;R_0) = \overline{d}(x;R_0) \equiv d(x;R_0)$ for each $x \in M \cup \{-1\}$. In what follows, we may omit R_0.

First, we show the following lemma.

Lemma 18. *Let $R \in \mathcal{R}^2$ and $i, j \in N$ be a pair such that $R_i = R_0$ and $R_j \in \mathcal{R}^C \cap \mathcal{R}^Q$. Let $v_j > 0$ be a constant incremental valuation associated with R_j. (i) If $x_i(R) > 0$, then $t_i(R) - V_i(x_i(R)-1, f_i(R)) \geq v_j$. (ii) If $x_i(R) < m$, then $V_i(x_i(R)+1, f_i(R)) - t_i(R) \leq v_j$.*

Proof. By Corollary 3 and Remark 10, $R^{inv} \in (\mathcal{R}^{NI})^n$. Thus, by the definition of the inverse Vickrey rule and Remark 8, $p(x_i(R)+1; R_i) \leq v_j \leq p(x_i(R); R_i)$ and $t_i(R) = v_j x_i(R)$, where $p(m+1; R_i) \equiv 0$. Then, by Proposition 3, $v_j \in [d(x_i(R)), d(x_i(R))-1)]$.

(i) Suppose $x_i(R) = 1$. Then $v_j \leq d(x_i(R)-1) = V_i(1, \mathbf{0})$. Then, we have $f_i(R) = (1, v_j)\ R_i\ (1, V_i(1, \mathbf{0}))\ I_i\ \mathbf{0}$. This implies $V_i(0, f_i(R)) \leq 0$. Thus, by $t_i(R) = v_j$, $t_i(R) - V_i(0, f_i(R)) \geq v_j$.

Suppose instead $x_i(R) > 1$. Note that $v_j \in [d(x_i(R)), d(x_i(R)-1)]$. Suppose $v_j = d(x_i(R)-1)$. By $(x_i(R)-1)v_j \in T(x_i(R)-1)$,

$$V_i(x_i(R), (x_i(R)-1, (x_i(R)-1)v_j)) = x_i(R)v_j,$$

which implies $f_i(R) = (x_i(R), x_i(R)v_j)\ I_i\ (x_i(R)-1, (x_i(R)-1)v_j)$. This implies

$$(x_i(R)-1)v_j = V_i(x_i(R)-1, f_i(R)).$$

Thus, by $t_i(R) = v_j x_i(R)$,

$$t_i(R) - V_i(x_i(R)-1, f_i(R)) = v_j.$$

Instead, if $v_j \in [d(x_i(R)), d(x_i(R)-1))$, then by $R_i \in \mathcal{R}^D$ and Lemma 12 (i),

$$t_i(R) - V_i(x_i(R)-1, f_i(R)) > \frac{t_i(R)}{x_i(R)} = v_j.$$

(ii) Suppose $x_i(R) = 0$. By $v_j \geq d(x_i(R)) = V_i(1, \mathbf{0})$ and $f_i(R) = \mathbf{0}$, we have $V_i(1, f_i(R)) - t_i(R) \leq v_j$.

Suppose $0 < x_i(R) < m$. First, we claim that $v_j \leq \frac{V_i(x_i(R), \mathbf{0})}{x_i(R)}$. If $x_i(R) = 1$, then $v_j \leq d(x_i(R)-1) = V_i(1, \mathbf{0})$. Instead, suppose $x_i(R) > 1$. Then, by $(x_i(R)-1)d(x_i(R)-1) \in T(x_i(R)-1)$,

$$V_i\Big(x_i(R), (x_i(R)-1, d(x_i(R)-1)(x_i(R)-1))\Big) = d(x_i(R)-1)x_i(R). \tag{1}$$

Then,

$$(x_i(R), v_j x_i(R))\ R_i\ (x_i(R), d(x_i(R)-1)x_i(R))\ I_i\ (x_i(R)-1, d(x_i(R)-1)(x_i(R)-1))\ P_i\ \mathbf{0},$$

where the first relation follows from $v_j \leq d(x_i(R)-1)$, the second one from (1), and the last one from $R_i \in \mathcal{R}^D$ and Lemma 7. Thus, $v_j x_i(R) < V_i(x_i(R), \mathbf{0})$, or equivalently, $v_j < \frac{V_i(x_i(R), \mathbf{0})}{x_i(R)}$.

Thus, in either case, we obtain $v_j \leq \frac{V_i(x_i(R), \mathbf{0})}{x_i(R)}$. By $v_j \geq d(x_i(R))$, $v_j \in [d(x_i(R)), \frac{V_i(x_i(R), \mathbf{0})}{x_i(R)}]$.

If $v_j = d(x_i(R))$, then by $x_i(R)v_j \in T(x_i(R))$,

$$V_i(x_i(R)+1, f_i(R)) - t_i(R) = \frac{t_i(R)}{x_i(R)} = v_j.$$

Instead, if $v_j \in (d(x_i(R)), \frac{V_i(x_i(R),\mathbf{0})}{x_i(R)}]$, then by $R_i \in \mathcal{R}^D$ and Lemma 8,

$$V_i(x_i(R)+1, f_i(R)) - t_i(R) < \frac{t_i(R)}{x_i(R)} = v_j,$$

as desired. □

We now proceed to the proof of Proposition 6. Let $\beta \equiv \frac{m}{2}$ and $t^* \equiv \sum_{x=0}^{\beta-1} p(m-x; R_0)$. Note that by Proposition 3, $t^* = \sum_{x=1}^{\beta} d(m-x)$. The proof has two parts.

Part 1. First, we show (i). Since *no subsidy for losers* is immediate from the definition of the rule, we here show the other two properties.

Individual rationality. Let $R \in \mathcal{R}^2$ and $i, j \in N$ be a distinct pair. If $R_i \in \mathcal{R}^C \cap \mathcal{R}^Q$, then by Remark 10, $R_i^{inv} = R_i$. Since $f_i(R)$ is an outcome of the Vickrey rule for R^{inv}, $f_i(R)\ R_i\ \mathbf{0}$ by *individual rationality* of the Vickrey rule. Thus, we assume $R_i = R_0$. We consider the following two cases.

Case 1. $R_j = R_0$.

By Corollary 3, $R_0^{inv} \in \mathcal{R}^D$. Thus, by the definition of the inverse Vickrey rule and Remark 8, $x_i(R) = \beta$ and

$$t_i(R) = \sum_{x=1}^{m} p(x; R_0) - \sum_{x=1}^{\beta} p(x; R_0) = \sum_{x=\beta+1}^{m} p(x; R_0) = \sum_{x=0}^{\beta-1} p(m-x; R_0) = t^*.$$

By $t^* = \sum_{x=1}^{\beta} d(m-x)$, $t_i(R) = \sum_{x=1}^{\beta} d(m-x)$. Then,

$$t_i(R) = \sum_{x=1}^{\beta} d(m-x) \leq \beta d(\beta) < V_i(\beta, \mathbf{0}),$$

where the first inequality follows from $R_0 \in \mathcal{R}^D$, $\beta = \frac{m}{2}$, and Lemma 11, and the second inequality follows from $R_i = R_0 \in \mathcal{R}^D$ and Lemma 7. Thus, $f_i(R)\ P_i\ \mathbf{0}$.

Case 2. $R_j \in \mathcal{R}^C \cap \mathcal{R}^Q$.

By Remark 10, $R_j^{inv} = R_j$. Thus, by the definition of the inverse Vickrey rule, $t_i(R) = v_j x_i(R)$, where $v_j > 0$ is a constant incremental valuation associated with R_j. If $x_i(R) > 0$, then

$$
\begin{aligned}
t_i(R) - V_i(0, f_i(R)) &= \sum_{x=1}^{x_i(R)} \Big(V_i(x, f_i(R)) - V_i(x-1, f_i(R)) \Big) && \text{(by Remark 1 (iii))} \\
&\geq x_i(R) \Big(V_i(x_i(R), f_i(R)) - V_i(x_i(R)-1, f_i(R)) \Big) && \text{(by } R_i \in \mathcal{R}^D\text{)} \\
&\geq x_i(R) v_j && \text{(by Lemma 18 (i))} \\
&= t_i(R),
\end{aligned}
$$

or equivalently, $V_i(0, f_i(R)) \leq 0$. Thus, $f_i(R)\ R_i\ \mathbf{0}$. In contrast, if $x_i(R) = 0$, then we have $f_i(R) = \mathbf{0}$.

EFFICIENCY. Let $R \in \mathcal{R}^2$. Since $f(R)$ is an outcome of the Vickrey rule for R^{inv}, *efficiency* of the Vickrey rule and Lemma 1 together imply $x_1(R) + x_2(R) = m$.

If $R \in (\mathcal{R}^C \cap \mathcal{R}^Q)^2$, then by Remark 10, $R = R^{inv}$. Thus, by *efficiency* of the Vickrey rule, $f(R)$ is efficient for R. Thus, we consider the following two cases.

CASE 1. $R_1 = R_2 = R_0$.

By Corollary 3, $R_1^{inv}, R_2^{inv} \in \mathcal{R}^D$. Thus, by the definition of the inverse Vickrey rule and Remark 8, $f_1(R) = f_2(R) = (\beta, t^*)$. By $f_1(R) = f_2(R)$ and Remark 1 (iii), $R_1 = R_2 \in \mathcal{R}^D$ implies $V_i(x_i(R)+1, f_i(R)) - t_i(R) < t_j(R) - V_j(x_j(R)-1, f_j(R))$ for each pair $i, j \in N$. Thus, by $R \in (\mathcal{R}^{NI})^2$, $x_1(R) + x_2(R) = m$, and Remark 8, $f(R)$ is efficient for R.

CASE 2. $R_i = R_0$ and $R_j \in \mathcal{R}^C \cap \mathcal{R}^Q$ for some pair $i, j \in N$.

Without loss of generality, let $i = 1$ and $j = 2$. By the definition of the inverse Vickrey rule, $t_1(R) = v_2 x_1(R)$, where $v_2 > 0$ is a constant incremental valuation associated with R_2.

If $x_1(R) = 0$, then by Lemma 18 (ii), $v_2 \geq V_1(1, f_1(R)) - t_1(R)$. If $0 < x_1(R) < m$, then by Lemma 18 (i) and (ii),

$$
V_1(x_1(R)+1, f_1(R)) - t_1(R) \leq v_2 \leq t_1(R) - V_1(x_1(R)-1, f_1(R)).
$$

Finally, if $x_i(R) = m$, then by Lemma 18 (i), $t_1(R) - V_1(m-1, f_1(R)) \geq v_2$. In either case, by $R \in (\mathcal{R}^{NI})^2$ and $x_1(R) + x_2(R) = m$, Remark 8 implies that $f(R)$ is efficient for R.

PART 2. We show (ii). Let $R \in \mathcal{R}^2$, $i, j \in N$ be a distinct pair, and $R'_i \in \mathcal{R}$ be such that $R \neq (R_0, R_0)$. If $R_i \in \mathcal{R}^C \cap \mathcal{R}^Q$, then by Remark 10, $R_i = R_i^{inv}$. Then, since $f(R)$ and $f(R'_i, R_j)$ are the outcomes of the Vickrey rule for (R_i, R_j^{inv}) and $(R_i'^{inv}, R_j^{inv})$, respectively, *strategy-proofness* of the Vickrey rule gives $f_i(R)\ R_i\ f_i(R'_i, R_j)$. Thus, we assume $R_i = R_0$. By $R \neq (R_0, R_0)$, $R_j \in \mathcal{R}^C \cap \mathcal{R}^Q$.

Let $x_i \equiv x_i(R'_i, R_{-i})$. We show $f_i(R)\ R_i\ (x_i, t_i(R'_i, R_j))$. By the definition of the inverse

Vickrey rule, $t_i(R) = v_j x_i(R)$ and $t_i(R'_i, R_j) = v_j x_i$, where $v_j > 0$ is a constant incremental valuations associated with R_j. If $x_i > x_i(R)$, then $x_i(R) < m$. Then,

$$\begin{aligned}
&V_i(x_i, f_i(R)) - t_i(R) \\
&= \sum_{x=x_i(R)}^{x_i-1} \Big(V_i(x+1, f_i(R)) - V_i(x, f_i(R))\Big) && \text{(by Remark 1 (iii))} \\
&\leq (x_i - x_i(R))\Big(V_i(x_i(R)+1, f_i(R)) - V_i(x_i(R), f_i(R))\Big) && \text{(by } R_i \in \mathcal{R}^D\text{)} \\
&\leq (x_i - x_i(R))v_j && \text{(by Lemma 18 (ii))} \\
&= t_i(R'_i, R_j) - t_i(R),
\end{aligned}$$

or equivalently, $V_i(x_i, f_i(R)) \leq t_i(R'_i, R_j)$. Thus, $f_i(R)\ R_i\ (x_i, t_i(R'_i, R_j))$.

The other case can be treated symmetrically, and we omit the proof. ■

3.2 Proof of Proposition 1

In this section, we provide the proof of Proposition 1.

Proposition 1. *Assume m is even. Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$.*
(i) Assume $n = 2$. Assume R_0 has the upper bound for the nonnegative income effects. An inverse Vickrey rule on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^2$ satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness.
(ii) Assume R_0 does not have the upper bound for the nonnegative income effects. Let $\mathcal{R}$ be rich and $R_0 \in \mathcal{R}$. No rule on $\mathcal{R}^n$ satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness.

Suppose m is even. By $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Remark 12 implies $\underline{d}(x_i; R_0) = \overline{d}(x; R_0) \equiv d(x; R_0)$ for each $x \in M$. In the following, we may omit R_0 in $d(\cdot; R_0)$.

3.2.1 Proof of Proposition 1 (i)

Now, we complete the proof of Proposition 1 (i). Suppose $n = 2$ and $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ has the upper bound for the nonnegative income effects. Then, by Lemma 17 (i),

$$V_0(\beta + 1, (\beta, t^*)) - t^* \leq d(\beta - 1), \tag{1}$$

where $\beta \equiv \frac{m}{2}$ and $t^* \equiv \sum_{x=0}^{\beta-1} p(m - x; R_0)$.

Let $\mathcal{R} \equiv (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$. Let f be an inverse Vickrey rule on $\mathcal{R}^2$. By Proposition 6, it suffices to show that for each $R \in \mathcal{R}^2$, each $i \in N$, and each $R'_i \in \mathcal{R}$, if $R = (R_0, R_0)$, then $f_i(R)\ R_i\ f_i(R'_i, R_{-i})$. Let $R \equiv (R_0, R_0)$. Let $i \in N$ and $R'_i \in \mathcal{R}$. Let $x_i \equiv x_i(R'_i, R_{-i})$.

If $x_i = 0$, then by Proposition 6 (i), $f_i(R)\ R_i\ \mathbf{0} = f_i(R'_i, R_{-i})$. Thus, suppose $x_i > 0$.

By Corollary 3 and Remark 12, $R^{inv} \in (\mathcal{R}^D)^2$. Thus, by the definition of the inverse Vickrey rule and Remark 8, $f_i(R) = (\beta, t^*)$ and $t_i(R'_i, R_{-i}) = \sum_{x=0}^{x_i-1} p(m - x; R_0)$. By

Proposition 3 and Remark 12, $t_i(R) = \sum_{x=1}^{\beta} d(m-x)$ and $t_i(R'_i, R_{-i}) = \sum_{x=1}^{x_i} d(m-x)$. If $x_i = x_i(R)$, then $f_i(R) = f_i(R'_i, R_{-i})$. Thus, suppose $x_i \neq x_i(R)$.

First, suppose $x_i > x_i(R)$. Then,

$$\begin{aligned}
t_i(R'_i, R_{-i}) - t_i(R) &= \sum_{x=1}^{x_i} d(m-x) - \sum_{x=1}^{\beta} d(m-x) \\
&= \sum_{x=\beta+1}^{x_i} d(m-x) \\
&\geq (x_i - \beta) d(m - \beta - 1) && \text{(by Lemmas 7 and 11)} \\
&= (x_i - \beta) d(\beta - 1) && \text{(by } 2\beta = m\text{)} \\
&\geq (x_i - \beta)\Big(V_i(\beta + 1, f_i(R)) - t_i(R)\Big) && \text{(by (1))} \\
&\geq V_i(x_i, f_i(R)) - t_i(R), && \text{(by } R_i \in \mathcal{R}^D\text{)}
\end{aligned}$$

or equivalently, $t_i(R'_i, R_{-i}) \geq V_i(x_i, f_i(R))$. This implies $f_i(R)\ R_i\ (x_i, t_i(R'_i, R_{-i}))$.

Suppose instead $x_i < x_i(R)$. Note that by $\beta = x_i(R) > x_i > 0$, $\beta - 1 > 0$. We have

$$t_i(R) = \sum_{x=1}^{\beta} d(m-x) < \beta d(m - \beta - 1) = \beta d(\beta - 1),$$

where the inequality follows from $R_i \in \mathcal{R}^D$ and Lemma 11, and the last equality from $2\beta = m$. Thus, by $R_i \in \mathcal{R}^{++}$, Remark 5 (ii) implies

$$t_i(R) - V_i(\beta - 1, f_i(R)) > \beta d(\beta - 1) - V_i\Big(\beta - 1, (\beta, \beta d(\beta - 1))\Big). \tag{2}$$

Note that by $0 < \beta - 1 < m$, $(\beta - 1)d(\beta - 1) \in T(\beta - 1)$. Thus,

$$V_i\Big(\beta, (\beta - 1, (\beta - 1)d(\beta - 1))\Big) = \beta d(\beta - 1).$$

This implies $(\beta, \beta d(\beta - 1))\ I_i\ (\beta - 1, (\beta - 1)d(\beta - 1))$. Thus, by Remark 1 (i) and (iii),

$$V_i\Big(\beta - 1, (\beta, \beta d(\beta - 1))\Big) = V_i\Big(\beta - 1, (\beta - 1, (\beta - 1)d(\beta - 1))\Big) = (\beta - 1)d(\beta - 1). \tag{3}$$

By (2) and (3),

$$t_i(R) - V_i(\beta - 1, f_i(R)) > \beta d(\beta - 1) - (\beta - 1)d(\beta - 1) = d(\beta - 1). \tag{4}$$

Then,

$$
\begin{aligned}
t_i(R) - t_i(R_i', R_{-i}) &= \sum_{x=1}^{\beta} d(m-x) - \sum_{x=1}^{x_i} d(m-x) \\
&= \sum_{x=x_i+1}^{\beta} d(m-x) \\
&< (\beta - x_i) d(m-\beta-1) && \text{(by Lemma 11)} \\
&= (\beta - x_i) d(\beta - 1) && \text{(by } 2\beta = m\text{)} \\
&< (\beta - x_i)\Big(t_i(R) - V_i(\beta-1, f_i(R))\Big) && \text{(by (4))} \\
&\leq t_i(R) - V_i(x_i, f_i(R)), && \text{(by } R_i \in \mathcal{R}^D\text{)}
\end{aligned}
$$

or equivalently, $V_i(x_i, f_i(R)) < t_i(R_i', R_{-i})$. This implies $f_i(R)\ P_i\ (x_i, t_i(R_i', R_{-i}))$. ■

3.2.2 Proof of Proposition 1 (ii)

In this subsection, we provide the proof of Proposition 1 (ii). Suppose $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ does not have the upper bound for the nonnegatinve income effects. By Lemma 17 (i),

$$
V_0(\beta+1, (\beta, t^*)) - t^* > d(\beta - 1), \tag{5}
$$

where $\beta \equiv \frac{m}{2}$ and $t^* \equiv \sum_{x=0}^{\beta-1} p(m-x; R_0)$.

Suppose by contradiction that there is a rule f on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^n$ satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. Note that Steps 1 to 3 in the proof of Theorem 1 only depends on the assumption of decreasing incremental valuations, and so the discussion is valid here as well. Thus, we hereafter take over the results and the notations in Steps 1 to 3 in the proof of Theorem 1.

As in the proof of Theorem 1, let $R_2 \equiv R_0$. First, we show $z_2^f(R_{-2}; \beta+1)\ P_2\ z_1^f(R_{-2}; \beta)$. Note that by Proposition 3 and Remark 12, $t^* = \sum_{x=1}^{\beta} d(m-x)$. Thus, by Step 3 and Lemma 2, $z_2^f(R_{-2}; \beta) = (\beta, t^*)$. Also,

$$
\begin{aligned}
t_2^f(R_{-2}; \beta+1) - t_2^f(R_{-2}; \beta) &= d(m-\beta-1) && \text{(by Step 3)} \\
&= d(\beta-1) && \text{(by } 2\beta = m\text{)} \\
&< V_2(\beta+1, z_2^f(R_{-2}; \beta)) - t_2^f(R_{-2}; \beta). && \text{(by (5))}
\end{aligned}
$$

Thus, $t_2^f(R_{-2}; \beta+1) < V_2(\beta+1, z_2^f(R_{-2}; \beta))$. This implies $z_2^f(R_{-2}; \beta+1)\ P_2\ z_2^f(R_{-2}; \beta)$.

By the same argument as in the proof of Theorem 1, we can show that for each $x_2 \in M$ with $x_2 < \beta$, $z_2^f(R_{-2}; x_2+1)\ P_2\ z_2^f(R_{-2}; x_2)$. Thus, for each $x_2 \in M$ with $x_2 \leq \beta$, we have $z_2^f(R_{-2}; \beta+1)\ P_2\ z_2^f(R_{-2}; x_2)$. By Lemma 4, $x_2(R) \geq \beta+1$. Thus, by $x_1(R) + x_2(R) \leq m$ and $2\beta = m$,

$$
x_1(R) \leq m - x_2(R) \leq m - \beta - 1 = \beta - 1.
$$

Since both agents 1 and 2 have the same preferences R_0, a symmetric argument implies that for for each $x_1 \in M$ with $x_1 \leq \beta$, $z_1^f(R_{-1}; x_1 + 1)\ P_1\ z_1^f(R_{-1}; x_1)$. Thus, by $x_1(R) \leq \beta - 1$,

$$z_1^f(R_{-1}; x_1(R) + 1)\ P_1\ z_1^f(R_{-1}; x_1(R)) = f_1(R).$$

However, this contradicts Lemma 4. ∎

3.3 Proof of Proposition 4

In this section, we prove Proposition 4.

Proposition 4. *Assume m is even. Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{--}$ satisfy the single-intersection condition.*
(i) Assume $n = 2$. Assume R_0 has the upper bound for the nonpositive income effects. An inverse Vickrey rule on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^2$ satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness.
(ii) Assume R_0 does not have the upper bound for the nonpositive income effects. Let $\mathcal{R}$ be rich and $R_0 \in \mathcal{R}$. No rule on $\mathcal{R}^n$ satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness.

Suppose m is even, and R_0 satisfies the single-intersection condition. Then, $\underline{d}(x; R_0) = \overline{d}(x; R_0) \equiv d(x; R_0)$ for each $x \in M$. In the following, we may omit R_0 in $d(\cdot; R_0)$.

3.3.1 Proof of Proposition 4 (i)

In this subsection, we prove Proposition 4 (i). Suppose $n = 2$, and $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{--}$ has the upper bound for the nonpositive income effects. By Lemma 17 (ii),

$$t^* - V_0(\beta - 1, (\beta, t^*)) \geq d(\beta), \tag{1}$$

where $\beta \equiv \frac{m}{2}$ and $t^* \equiv \sum_{x=0}^{\beta-1} p(m - x; R_0)$.

Let $\mathcal{R} \equiv (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$. Let f be an inverse Vickrey rule on $\mathcal{R}^2$. By Proposition 6, it suffices to show that for each $R \in \mathcal{R}^2$, each $i \in N$, and each $R_i' \in \mathcal{R}$ with $R = (R_0, R_0)$, $f_i(R)\ R_i\ f_i(R_i', R_{-i})$. Let $R \equiv (R_0, R_0)$, $i \in N$, and $R_i' \in \mathcal{R}$. Further, let $x_i \equiv x_i(R_i', R_{-i})$.

If $x_i = 0$, then by Proposition 6 (i), $f_i(R)\ R_i\ \mathbf{0} = f_i(R_i', R_{-i})$. Thus, suppose $x_i > 0$.

By Corollary 3, $R_0^{inv} \in \mathcal{R}^D$. Thus, by the definition of the inverse Vickrey rule and Remark 8, $f_i(R) = (\beta, t^*)$ and $t_i(R_i', R_j) = \sum_{x=0}^{x_i - 1} p(m - x; R_0)$. By Proposition 3, $t_i(R) = \sum_{x=1}^{\beta} d(m - x)$ and $t_i(R_i', R_{-i}) = \sum_{x=1}^{x_i} d(m - x)$. If $x_i = x_i(R)$, then $f_i(R) = f_i(R_i', R_{-i})$. Thus, suppose $x_i \neq x_i(R)$.

Suppose $x_i < x_i(R)$. Then,

$$\begin{aligned}
t_i(R) - t_i(R'_i, R_{-i}) &= \sum_{x=1}^{\beta} d(m-x) - \sum_{x=1}^{x_i} d(m-x) \\
&= \sum_{x=x_i+1}^{\beta} d(m-x) \\
&\leq (\beta - x_i) d(m-\beta) && \text{(by Lemma 11)} \\
&= (\beta - x_i) d(\beta) && \text{(by } 2\beta = m\text{)} \\
&\leq (\beta - x_i)\Big(t_i(R) - V_i(\beta - 1, f_i(R))\Big) && \text{(by (1))} \\
&\leq t_i(R) - V_i(x_i, f_i(R)), && \text{(by } R_i \in \mathcal{R}^D\text{)}
\end{aligned}$$

or equivalently, $t_i(R'_i, R_{-i}) \geq V_i(x_i, f_i(R))$. This implies $f_i(R)\ R_i\ (x_i, t_i(R'_i, R_{-i}))$.

Next, suppose $x_i > x_i(R)$. Then,

$$t_i(R) = \sum_{x=1}^{\beta} d(m-x) \leq \beta d(m-\beta) = \beta d(\beta),$$

where the inequality follows from $R_0 \in \mathcal{R}^D$ and Lemma 11, and the second equality from $2\beta = m$. Thus, by $R_i \in \mathcal{R}^{--}$, Remark 6 (i) implies

$$V_i(\beta + 1, f_i(R)) - t_i(R) \leq V_i(\beta + 1, (\beta, \beta d(\beta))) - \beta d(\beta) = d(\beta), \tag{2}$$

where the equality follows from $\beta d(\beta) \in T(\beta)$. Then,

$$\begin{aligned}
t_i(R'_i, R_{-i}) - t_i(R) &= \sum_{x=1}^{x_i} d(m-x) - \sum_{x=1}^{\beta} d(m-x) \\
&= \sum_{x=\beta+1}^{x_i} d(m-x) \\
&> (x_i - \beta) d(m-\beta) && \text{(by Lemmas 7 and 11)} \\
&= (x_i - \beta) d(\beta) && \text{(by } 2\beta = m\text{)} \\
&\geq (x_i - \beta)\Big(V_i(\beta + 1, f_i(R)) - t_i(R)\Big) && \text{(by (2))} \\
&\geq V_i(x_i, f_i(R)) - t_i(R), && \text{(by } R_i \in \mathcal{R}^D\text{)}
\end{aligned}$$

or equivalently, $t_i(R'_i, R_{-i}) > V_i(x_i, f_i(R))$. Thus, $f_i(R)\ P_i\ (x_i, t_i(R'_i, R_{-i}))$. ■

3.3.2 Proof of the Proposition 4 (ii)

In this subsection, we prove Proposition 4 (ii). Suppose $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{--}$ does not have the upper bound for the nonpositive income effects. Then, by Lemma 17 (ii),

$$t^* - V_0(\beta - 1, (\beta, t^*)) < d(\beta), \tag{3}$$

where $\beta \equiv \frac{m}{2}$ and $t^* \equiv \sum_{x=0}^{\beta-1} p(m - x; R_0)$.

Suppose by contradiction that there is a rule f on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^n$ satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. By the same reason as in the proof of Proposition 1 (ii), we can take over the results and the notations in Steps 1 to 3 in the proof of Theorem 1.

Let $R_2 \equiv R_0$. First, we show $z_2^f(R_{-2}; \beta)\ P_2\ z_1^f(R_{-2}; \beta - 1)$. By Proposition 3, $t^* = \sum_{x=1}^{\beta} d(m - x)$. Thus, by Step 3 and Lemma 2, $z_2^f(R_{-2}; \beta) = (\beta, t^*)$. Then,

$$\begin{aligned} t_2^f(R_{-2}; \beta) - t_2^f(R_{-2}; \beta - 1) &= d(m - \beta) && \text{(by Step 3)} \\ &= d(\beta) && \text{(by } 2\beta = m) \\ &> t_2^f(R_{-2}; \beta) - V_2(\beta - 1, z_2^f(R_{-2}; \beta)). && \text{(by (3))} \end{aligned}$$

Thus, $t_2^f(R_{-2}; \beta - 1) < V_2(\beta - 1, z_2^f(R_{-2}; \beta))$. This implies $z_2^f(R_{-2}; \beta - 1)\ P_2\ z_2^f(R_{-2}; \beta)$.

By the same argument as in the proof of Theorem 1, we can show that for each $x_2 \in M$ with $\beta \leq x_2 < m$, $z_2^f(R_{-2}; x_2)\ P_2\ z_2^f(R_{-2}; x_2 + 1)$. Thus, $z_2^f(R_{-2}; \beta - 1)\ P_2\ z_2^f(R_{-2}; x_2)$ for each $x_2 \in M$ with $x_2 \geq \beta$. By Lemma 4, $x_2(R) \leq \beta - 1$. By Lemma 1 and $x_i(R) = 0$ for each $i \in N \backslash \{1, 2\}$, $x_1(R) = m - x_2(R)$. Thus, by $x_2(R) \leq \beta - 1$ and $2\beta = m$,

$$x_1(R) = m - x_2(R) \geq m - \beta + 1 = \beta + 1.$$

By a symmetric argument, we can show that for each for each $x_1 \in M$ with $x_1 \geq \beta$, $z_1^f(R_{-1}; x_1 - 1)\ P_1\ z_1^f(R_{-1}; x_1)$. By $x_1(R) \geq \beta + 1$,

$$z_1^f(R_{-1}; x_1(R) - 1)\ P_1\ z_1^f(R_{-1}; x_1(R)) = f_1(R),$$

which contradicts Lemma 4. ■

4 Remaining parts of the proof of Proposition 2

In this section, we prove the remaining parts of Proposition 2. We divide the argument into two cases.

Case 1. $R_0 \in \mathcal{R}^{++}$.

In Shinozaki et al. (2022), we have already considered the case where $\mathcal{R} = (\mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\}$. Thus, suppose $\mathcal{R} = (\mathcal{R}^{ND}(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\}$.

By $R_0 \in \mathcal{R}^D$, Lemma 11 implies $d(m-1) < d(m-2)$. Let $\varepsilon_1 > 0$ be such that $\varepsilon_1 < \min\{d(m-2) - d(m-1), \varepsilon\}$. We have

$$\begin{aligned} d(m-1) &= md(m-1) - V_1(m-1,(m, md(m-1))) && \text{(by Remark 11)} \\ &> md(m-1) + \varepsilon_1 - V_1(m-1,(m, md(m-1)+\varepsilon_1)). && \text{(by Remark 5 (ii))} \end{aligned}$$

Then,

$$\varepsilon_2 \equiv d(m-1) - (md(m-1)+\varepsilon_1) + V_1(m-1, md(m-1)+\varepsilon_1)) > 0.$$

Note that $\delta < d(m-1)$. Let $\varepsilon_3 > 0$ be such that $\varepsilon_1 + m\varepsilon_3 < \varepsilon$, $\varepsilon_3 < \min\{d(m-1)-\delta, \varepsilon_2\}$, and

$$\varepsilon_1 + \varepsilon_3 < d(m-2) - d(m-1). \tag{1}$$

Let $R_2 \in \mathcal{R}^{ND} \cap \mathcal{R}^Q$ be such that $v_2(1) = d(m-1) - \varepsilon_3$, and for each $x_2 \in M \backslash \{0, m\}$,

$$v_2(x_2+1) - v_2(x_2) = d(m-1) + \frac{\varepsilon_1 + \varepsilon_3}{m-1}.$$

By $\varepsilon_1 + m\varepsilon_3 < \varepsilon$, we have $R_2 \in \mathcal{R}^{ND}(\varepsilon) \cap \mathcal{R}^Q$. By $\varepsilon_3 < d(m-1)-\delta$, $v_2(1) > \delta$. Moreover, $v_2(x_2+1) - v_2(x_2) > \delta$ for each $x_2 \in M\backslash\{0,m\}$. Thus, for each $x_1 \in M$, $\sigma_1(R_{-1}; x_1) = v_2(m - x_1)$. By Fact 1, for each $x_1 \in M_1^f(R_{-1})$,

$$t_1^f(R_{-1}; x_1) = v_2(m) - v_2(m - x_1). \tag{2}$$

By Step 3 and Lemma 2, $t_2^f(R_{-2}; 1) = d(m-1)$. Thus, $v_2(1) - t_2^f(R_{-2}; 1) = d(m-1) - \varepsilon_3 - d(m-1) = -\varepsilon_3 < 0$. Further, for each $x_2 \in M\backslash\{0,1\}$,

$$\begin{aligned} v_2(x_2) - v_2(1) &= (x_2 - 1)\Big(d(m-1) + \frac{\varepsilon_1 + \varepsilon_3}{m-1}\Big) \\ &< (x_2-1)(d(m-1) + d(m-2) - d(m-1)) && \text{(by (1))} \\ &= (x_2 - 1)d(m-2) \\ &\leq \sum_{x=1}^{x_2-1} d(m-x-1) && \text{(by Lemma 11)} \\ &= \sum_{x=1}^{x_2-1} (t_2^f(R_{-2}; x+1) - t_2^f(R_{-2}; x)) && \text{(by Step 3)} \\ &= t_2^f(R_{-2}; x_2) - t_2^f(R_{-2}; 1), \end{aligned}$$

or equivalently, $v_2(1) - t_1^f(R_{-2}; 1) > v_2(x_2) - t_2^f(R_{-2}; x_2)$. Thus, by Lemma 2, for each $x_2 \in M\backslash\{0\}$, $z_2^f(R_{-2}; 0) = \mathbf{0} \; P_2 \; z_2^f(R_{-2}; x_2)$. By Lemma 4, $x_2(R) = 0$. Since $x_i(R) = 0$ for each $i \in N\backslash\{1,2\}$, Lemma 1 implies $x_1(R) = m$. Let $x \in X$ be such that $x_1 = m-1$ and

$x_2 = 1$. Then,

$$\begin{aligned}
&v_2(x_2) - v_2(x_2(R)) - (t_1(R) - V_1(x_1, f_1(R))) \\
&= d(m-1) - \varepsilon_3 - (md(m-1) + \varepsilon_1) + V_1(m-1, (m, md(m-1) + \varepsilon_1)) && \text{(by (2))} \\
&= \varepsilon_2 - \varepsilon_3 \\
&> 0. && \text{(by } \varepsilon_3 < \varepsilon_2\text{)}
\end{aligned}$$

By Remark 7, this contradicts *efficiency*.

CASE 2. $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{--}$.

We have already considered the case where $\mathcal{R} = (\mathcal{R}^{ND}(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\}$ in Shinozaki et al. (2022). Thus, suppose $\mathcal{R} = (\mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\}$.

Note that $\delta < \underline{d}(m-1)$. Let $\varepsilon_1 > 0$ be such that $\varepsilon_1 < \min\{\underline{d}(m-1) - \delta, \varepsilon\}$. Then,

$$\begin{aligned}
\underline{d}(m-1) &= m\underline{d}(m-1) - V_1(m-1, (m, m\underline{d}(m-1))) && \text{(by Remark 11)} \\
&> m\underline{d}(m-1) - \varepsilon_1 - V_1(m-1, (m, m\underline{d}(m-1) - \varepsilon_1)). && \text{(by Remark 6 (ii))}
\end{aligned}$$

Thus,

$$\varepsilon_2 \equiv \underline{d}(m-1) - (m\underline{d}(m-1) - \varepsilon_1) + V_1(m-1, (m, m\underline{d}(m-1) - \varepsilon_1)) > 0.$$

Let $\varepsilon_3 > 0$ be such that $m\varepsilon_3 < \varepsilon_1$ and $\varepsilon_1 + \varepsilon_3 < \underline{d}(m-1) - \delta$.

Let $R_2 \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ be such that $v_2(1) = \underline{d}(m-1) - \varepsilon_3$, and for each $x_2 \in M\backslash\{0, m\}$,

$$v_2(x_2 + 1) - v_2(x_2) = \underline{d}(m-1) - \frac{\varepsilon_1 + \varepsilon_3}{m-1}.$$

By $m\varepsilon_3 < \varepsilon_1$, $R_2 \in \mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q$. By $\varepsilon_1 + \varepsilon_3 < \underline{d}(m-1) - \delta$, $v_2(x_2+1) - v_2(x_2) > \delta$ for each $x_2 \in M\backslash\{m\}$. Thus, for each $x_1 \in M$, $\sigma_1(R_{-2}; x_1) = v_2(m - x_1)$. By Fact 1, for each $x_1 \in M_1^f(R_{-1})$,

$$t_1^f(R_{-1}; x_1) = v_2(m) - v_2(m - x_1). \tag{3}$$

For each $x_2 \in M\backslash\{0\}$,

$$\begin{aligned}
v_2(x_2) &= x_2\underline{d}(m-1) - \varepsilon_3 - (x_2-1)\frac{\varepsilon_1+\varepsilon_3}{m-1} \\
&< x_2\underline{d}(m-1) && \text{(by } \varepsilon_1, \varepsilon_3 > 0\text{)} \\
&\leq \underline{d}(m-1) + (x_2-1)\overline{d}(m-1) && \text{(by } \underline{d}(m-1) \leq \overline{d}(m-1)\text{)} \\
&\leq \sum_{x=0}^{x_2-1} \underline{d}(m-x-1) && \text{(by Lemmas 7 and 11)} \\
&\leq \sum_{x=0}^{x_2-1} (t_2^f(R_{-2}; x+1) - t_2^f(R_{-2}; x)) && \text{(by Step 3)} \\
&= t_2^f(R_{-2}; x_2) - t_2^f(R_{-2}; 0),
\end{aligned}$$

or equivalently, $v_2(0) - t_2^f(R_{-2}; 0) > v_2(x_2) - t_2^f(R_{-2}; x_2)$. Thus, by Lemma 4, $x_2(R) = 0$. By $x_i(R) = 0$ for each $i \in N\backslash\{1,2\}$, Lemma 1 gives $x_1(R) = m$. By (3), $t_1(R) = m\underline{d}(m-1) - \varepsilon_1 - 2\varepsilon_3$. Then,

$$\begin{aligned}
&v_2(x_2(R)+1) - v_2(x_2(R)) - (t_1(R) - V_1(x_1(R)-1, f_1(R))) \\
&= \underline{d}(m-1) - \varepsilon_3 - (m\underline{d}(m-1) - \varepsilon_1 - 2\varepsilon_3) + V_1(m-1, (m, m\underline{d}(m-1) - \varepsilon_1 - 2\varepsilon_3)) \\
&> \underline{d}(m-1) - \varepsilon_3 - (m\underline{d}(m-1) - \varepsilon_1) + V_1(m-1, (m, m\underline{d}(m-1) - \varepsilon_1)) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Remark 6 (ii))} \\
&= \varepsilon_2 - \varepsilon_3 \\
&> 0. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by } \varepsilon_3 < \varepsilon_2\text{)}
\end{aligned}$$

By $R \in (\mathcal{R}^{NI})^n$ and Remark 8, this contradicts *efficiency*. ■

5 Even number of units and more than two agents

In this section, we give an example which demonstrates that when $n \geq 3$ and $m = 6a-2$ for some $a \in \mathbb{N}$ with $a \geq 2$, for any $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, the inverse Vickrey rule violates *strategy-proofness* on $\mathcal{R}^n \equiv ((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^n$.[1]

Example 9. Let $n \geq 3$ and $m = 6a-2$, where $a \in \mathbb{N}$ and $a \geq 2$. Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$. By $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Remark 12 implies that $\underline{d}(x; R_0) = \overline{d}(x; R_0) \equiv d(x; R_0)$ for each $x \in M$. Hereafter, we may omit R_0 in $d(\cdot; R_0)$.

Let $\mathcal{R} \equiv (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$. Let $f \equiv (x, t)$ be an inverse Vickrey rule on $\mathcal{R}^n$. For each $i \in \{1,2,3\}$, let $R_i = R_0$. By $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Lemma 11 and Corollary 2 imply $p(m; R_0) = d(m-1) > 0$. Thus, we can choose $\delta > 0$ such that $\delta < p(m; R_0)$. For each $j \in N\backslash\{1,2,3\}$, let $R_j \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that for each $x_j \in M$, $v_j(x_j) = \delta x_j$.

Note that by Corollary 3 and Remark 12, $R_0^{inv} \in \mathcal{R}^D$. Thus, by the definition of the

[1] Parallel discussion applies to any preference $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{--}$ satisfying the single-intersection condition.

inverse Vickrey rule, $R^{inv} \in (\mathcal{R}^{NI})^n$, $\delta < p(m; R_0)$, and Remark 8, $x_i(R) = x_j(R) = 2a-1$ and $x_k(R) = 2a$ for some distinct triple $i, j, k \in \{1,2,3\}$, $x_l(R) = 0$ for each $l \in N \backslash \{1,2,3\}$,

$$t_i(R) = t_j(R) = p(2a; R_0) + 2 \sum_{x=2a+1}^{3a-1} p(x; R_0), \quad t_k(R) = 2 \sum_{x=2a}^{3a-1} p(x; R_0),$$

and $t_l(R) = 0$ for each $l \in N \backslash \{1,2,3\}$. Without loss of generality, let $i = 1$, $j = 2$, and $k = 3$.

By $R_0 \in \mathcal{R}^D$, Lemma 11 implies $d(2a-2) < d(2a-1)$. Thus, we can choose $R'_1 \in \mathcal{R}^C \cap \mathcal{R}^Q$ such that $d(2a-2) < v'_1 < d(2a-1)$, where v'_1 is a constant incremental valuation associated with R'_1. By $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Corollary 2 implies $p(2a-1; R_0) < v'_1 < p(2a; R_0)$. Thus, by the definition of the inverse Vickrey rule, $R^{inv} \in (\mathcal{R}^{NI})^n$, and Remark 8, $f_1(R'_1, R_{-1}) = (2a, 2\sum_{x=2a+1}^{3a-1} p(x; R_0))$.

We have

$$t_1(R) = d(2a-1) + 2 \sum_{x=2a}^{3a-2} d(x) < (2a-1)d(2a-1).$$

where the equality follows from $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ and Corollary 2, and the inequality from $R_0 \in \mathcal{R}^D$ and Lemma 11. Thus,

$$\begin{aligned}
& V_1(2a, f_1(R)) - t_1(R) \\
&> V_1(2a, (2a-1, (2a-1)d(2a-1)) - (2a-1)d(2a-1) && \text{(by Remark 5 (i))} \\
&= d(2a-1) && \text{(by } (2a-1)d(2a-1) \in T(2a-1)) \\
&= p(2a; R_0) && \text{(by Corollary 2)} \\
&= t_1(R'_1, R_{-1}) - t_1(R),
\end{aligned}$$

or equivalently, $V_1(2a, f_1(R)) > t_1(R'_1, R_{-1})$. This implies $f_1(R'_1, R_{-1}) \; P_1 \; f_1(R)$. Thus, f violates *strategy-proofness*.

References

[1] Shinozaki, H., T. Kazumura, and S. Serizawa (2022), "Multi-unit object allocation problems with money for (non)decreasing incremental valuations: Impossibility and characterization theorems." Working paper.