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MULTI-UNIT OBJECT ALLOCATION PROBLEMS WITH MONEY FOR (NON)DECREASING INCREMENTAL VALUATIONS: IMPOSSIBILITY AND CHARACTERIZATION THEOREMS

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Multi-unit object allocation problems with money for (non)decreasing incremental valuations: Impossibility and characterization theorems*

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Abstract

We consider the problem of allocating multiple units of an object and collecting payments. Each agent can receive multiple units, and his (consumption) bundle is a pair consisting of the units he receives and his payment. An agent's preference over bundles may not be quasi-linear. A class of preferences is *rich* if it includes all quasi-linear preferences with constant incremental valuations. We show that for an odd number of units, if a class of preferences is rich and includes at least one preference exhibiting both decreasing incremental valuations and either positive or negative income effects, then no rule satisfies *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. In contrast, for an even number of units, the existence of a rule satisfying the four properties depends on the size of the income effects. We further show that if a rich class of preferences includes only preferences that exhibit nondecreasing incremental valuations, then the *generalized Vickrey rule* (Saitoh and Serizawa, 2008; Sakai, 2008) is the only rule satisfying the four properties. Our results suggest that (i) there a rule satisfying the four properties "almost" only when preferences exhibit nondecreasing incremental valuations, and (ii) it depends not only on the properties of preferences such as nondecreasing incremental valuations, but also on other characteristics of the environment such as the number of units.

JEL Classification Numbers. D44, D47, D71, D82

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1 Introduction

1.1 Purpose

The most important goal of many government auctions is to allocate public resources efficiently. These resources often consist of identical objects. Important examples are spectrum license auctions in some countries such as the Germany 3G auction and procurement auctions in some markets such as procurement of vaccines. In this paper, we focus on multi-unit object allocation problems where the objects are identical. Each agent can receive multiple units of the object, and his (*consumption*) *bundle* is a pair specifying the number of units he receives and his payment. Each agent has a (possibly) non-quasi-linear preference over bundles, which exhibits income effects or reflects non-linear borrowing costs.[1]

An *allocation* specifies a bundle for each agent. An (*allocation*) *rule* is a function from a set of preference profiles (a *domain*) to the set of allocations. An allocation is *efficient* for a given preference profile if no other allocation makes some agent better off without making any agent worse off, or decreasing the owner's revenue. A rule satisfies *efficiency* if it chooses an efficient allocation for each preference profile. It satisfies *individual rationality* if no agent is worse off than receiving nothing and making no payment. It satisfies *no subsidy for losers* if the payment of an agent who receives no object is nonnegative. This condition eliminates "fake" agents whose only interest is the participation subsidy. A rule satisfies *strategy-proofness* if no agent ever benefits from misrepresenting his preference. We regard these four properties as basic desiderata. Then, our goals are two-fold: (i) to identify domains on which there is a rule satisfying our desiderata, and (ii) to characterize the class of rules satisfying our desiderata when they exist.

1.2 Main results

A preference exhibits *nonincreasing* (resp. *nondecreasing*) *incremental valuations* if the incremental willingness to buy at each bundle is at least as large as (resp. at least as small as) the incremental willingness to sell at the bundle.[2] A preference exhibits *constant incremental valuations* if the incremental willingness to buy at each bundle coincides with the incremental willingness to sell at the bundle. Because of non-quasi-linearity, both the incremental willingness to buy and the incremental willingness to sell may vary depending on the bundle that we start from. The *domain with nonincreasing* (resp. *nondecreasing*) *incremental valuations* is the set of preference profiles at which the preference of each agent exhibits nonincreasing (resp. nondecreasing) incremental valuations. The *domain with constant incremental valuations* is the set of preference profiles at which the preference of each agent exhibits constant incremental valuations.

If a rule satisfies *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness* on a domain, then it also satisfies them on its subdomains. Thus, the smaller the

[1]Note that our setting is a private values model.

[2]A preference exhibiting decreasing or increasing incremental valuations can be defined analogously.

domain, the weaker the implications of the four properties. Thus, the existence of a rule satisfying the four properties becomes trivial on a small enough domain. To avoid such cases, we require a class of preferences to include an essential and small class of preferences. The quasi-linear domain with constant incremental valuations is contained both in the domains with nonincreasing and nondecreasing incremental valuations, and is single-dimensional. Moreover, on the quasi-linear domain with constant incremental valuations, the Vickrey rule satisfies the four properties (Vickrey, 1961; Holmström, 1979). We require a domain to include the quasi-linear domain with constant incremental valuations. We call the quasi-linear domain with constant incremental valuations a *minimal domain*. A class of preferences is *rich* if it includes all quasi-linear preferences that exhibit constant incremental valuations.

1.2.1 Nonincreasing incremental valuations

If a firm has a production technology that exhibits nonincreasing (resp. nondecreasing) returns to scale, its preference exhibits nonincreasing (resp. nondecreasing) incremental valuations. Nonincreasing returns to scale are a typical assumption in economic theory. Moreover, most of the literature on multi-unit object allocation problems assumes decreasing (or weaker nonincreasing) incremental valuations. Thus, we first investigate classes of preferences that exhibit decreasing incremental valuations.

A preference exhibits *positive* (resp. *negative*) *income effects* if the incremental willingness to buy increases (resp. decreases) as a payment decreases. Positive (resp. negative) income effects mean that the object is a normal (resp. inferior) good.

Typically borrowing costs progressively increase as borrowing increases. This factor causes preferences to exhibit positive income effects. The complements of the object also cause positive income effects. The less payments are, the more cash is available to buy the complements, which increases the demand for the object. For example, consider spectrum licenses for mobile phone. As more cash is available, mobile operators can afford to invest more for their services, which affects the profits from the licenses. Low frequency electromagnetic wave travels longer and enables mobile operators to save the cost of base stations. In contrast, high frequency electromagnetic wave transmits more information, and so is more suitable for latest services such as 5G services. Such features make mobile operators prefer a low frequency license more as the payment for the license increases. Accordingly, various factors cause income effects in different directions. Thus, we analyze both of positive and negative income effects.

Our results for preferences with nonincreasing incremental valuations depends on the number of units. Our first main result (Theorem 1) is as follows. *For an odd number of units, consider a class of preferences that is rich and includes at least one preference exhibiting both decreasing incremental valuations and either positive or negative income effects. Then, no rule satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness.*

The larger a domain is, the more preference profiles at which the four properties have to be satisfied, and the more difficult it is to satisfy the properties. This means that the implication of the properties gets stronger as a domain enlarges, and so an impossibility theorem on any domain carries over to any larger domain. Thus, Theorem 1 is a striking result as it shows adding at least one plausible preference to a small domain on which we find a positive result leads us to a negative result.

For preferences with decreasing incremental valuations, several agents typically win the object at an efficient allocation, and if an agent with a non-quasi-linear preference wins the object, then he may benefit from misrepresenting his preference so as to win one more or fewer unit, or it may be possible to Pareto improve the allocation by exchanging one unit between him and another winner.

The *inverse Vickrey rule* plays a key role in the proof of Theorem 1. It applies the inverse-demands of agents to the Vickrey rule (Vickrey, 1961) as their valuations. It is a new variant of the possible extensions of the Vickrey rule for quasi-linear preferences with nonincreasing incremental valuations to non-quasi-linear preferences with nonincreasing incremental valuations, and is different from the generalized Vickry rule, which has been studied in the literature (Saioh and Serizawa, 2008; Sakai, 2008; Malik and Mishra, 2021).

To illustrate the proof, consider the situation where there are two agents who have the same preferences that exhibit decreasing incremental valuations and positive income effects.[3] We emphasize that the assumption of an odd number of units plays an important role in the proof of Theorem 1. The main step of the proof is dedicated to showing that at the preference profile, the outcome of a rule satisfying the four properties coincides with that of the inverse Vickrey rule. If the number of units is odd, then the inverse Vickrey rule cannot treat the agents in a completely equal way. Thus, it treats an agent unfavorably, and when m is the number of units, an agent can receive $\frac{m-1}{2}$ units but the other agent receives $\frac{m+1}{2}$ units. We show that under the inverse Vickrey rule, the agent who receives $\frac{m-1}{2}$ units pays small payment compared to his inverse-demand of $\frac{m-1}{2}$ units. Then, positive income effects implies that he demands one more unit at his outcome bundle, and benefits from misrepresent his preference so as to win one more unit.[4]

We emphasize that the odd number of units is an indispensable assumption in Theorem 1. For an even number of units, the result depends on the size of the income effects of the preference added to the class of preferences.[5] We identify some positive value, and show that if the size of the income effects of a preference with decreasing incremental valuations and either positive or negative income effects is no greater than the value, then on the domain that contains our minimal domain and the preference, the inverse Vickrey rule satisfies the four properties (Propositions 1 (i) and 4 (i)). In contrast, if the size of the income effects is greater than the value, then no rule satisfies the properties (Propositions 1 (ii) and 4 (ii)).

[3]Similar discussion applies to the case where agents have the preferences with negative income effects.

[4]Recall the definition of positive income effects that the incremental willingness to buy increases as a payment decreases.

[5]The size of the income effects is the difference between the incremental willingness to buy at a pair of payments.

Since our minimal domain, that is, the quasi-linear domain with constant incremental valuations, contains only limited variations of preferences, it is an interesting question what if the variations of the minimal domain are expanded "slightly". Thus, we further investigate the class of rules satisfying the four properties by expanding the minimal domain slightly. In particular, we ask whether there is a rule satisfying the four properties on a domain that contains the expanded minimal domain and includes a non-quasi-linear preference with decreasing incremental valuations.

To clarify the precise meaning of "slightly", we introduce the following preferences. Given a positive number $\varepsilon > 0$, a preference exhibits *ε-nonincreasing incremental valuations* (resp. *ε-nondecreasing incremental valuations*) if it exhibits nonincreasing incremental valuations (resp. nondecreasing incremental valuations), and the absolute difference between the incremental willingness to buy and the incremental willingness to sell at each bundle is less than ε. The NI(ε)*-minimal domain* (resp. the ND(ε)*-minimal domain*) is the quasi-linear domain with ε-nonincreaing incremental valuations (ε-nondecreasing incremental valuations). When ε is sufficiently small, both the NI(ε)-minimal and the ND(ε)-minimal domains are slight expansions of the minimal domain in that both the domains include only quasi-linear preferences with almost constant incremental valuations. Moreover, the Vickrey rule still satisfies the properties on the those domains (Vickrey, 1961; Holmström, 1979). A class of preferences is *NI(ε)-rich* (resp. *ND(ε)-rich*) if it includes all quasi-linear preferences with ε-nonincreasing incremental valuations (resp. ε-nondecreasing incremental valuations).

We establish the following result. *Let* $\varepsilon > 0$, and *consider a class of preferences that is either* NI(ε)*-rich or* ND(ε)*-rich, and includes at least one preference that exhibits decreasing incremental valuations and either positive or negative income effects. Then, no rule satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness* (Proposition 2). The number $\varepsilon > 0$ can be arbitrarily small in Proposition 2. Note that Proposition 2 does not depend on the number of units. Thus, in contrast to Propositions 1 and 4, it states that in the case of an even number of units, a minimal increase of the variations of the domain leads to an impossibility theorem.

Although the conclusion of Proposition 2 is the same as that of Theorem 1, the intuition for the proof of Proposition 2 is different. To explain the intuition, consider a preference profile where agent 1 has a preference exhibiting decreasing incremental valuations and positive income effects, and agent 2 has a quasi-linear preference exhibiting ε-nonincreasing incremental valuations for a given $\varepsilon > 0$.[6] As in the proof of Theorem 1, we show that at the preference profile, the outcome of a rule satisfying the four properties coincides with that of the inverse Vickrey rule. Then, if agent 1 wins $m-1$ units, then his payment is slightly lower than when agent 2 has a quasi-linear preference with constant incremental valuations and agent 1 still wins $m-1$ units. Then, agent 1's positive income effects implies that agent 1's incremental willingness to buy of m units at his bundle exceeds agent 2's

[6]Similar discussions apply to the cases where agent 1 has a preference with negative income effects or agent 2 has a quasi-linear preference with ε-nondecreasing incremental valuations. For the detalies of the proof of Proposition 2, see Appendix B.4.

incremental willingness to sell of 1 unit at his bundle. Thus, a Pareto improvement is possible if agent 2 gives 1 unit to agent 1 in return for a compensation from agent 1. Note that the above intuition of the proof of Proposition 2 does not depend on the number of units unlike in the proof of Theorem 1.

1.2.2 Nondecreasing incremantal valuations

Nondecreasing returns to scale are also common in other industries. Such a technology causes preferences to exhibit nondecreasing incremental valuations.[7] Thus, we next consider classes of preferences that exhibit nondecreasing incremental valuations. The *generalized Vickrey rule* is a natural extension of the Vickrey rule (Saitoh and Serizawa 2008; Sakai, 2008). Our second main result (Theorem 2) is: *if a class of preferences is rich and includes only preferences that exhibit nondecreasing incremental valuations, then the generalized Vickrey rule is the only rule satisfying efficiency, individual rationality, no subsidy for losers, and strategy-proofness.*

To see the intuition of the proof of Theorem 2, consider a *bundling efficient allocation*. It is an allocation that bundles all the units into one package, and gives the package to a single agent with the highest valuation of it. For preferences that exhibit nondecreasing incremental valuations, a bundling efficient allocation is efficient. Thus, under an *efficient* rule for preferences that exhibit nondecreasing incremental valuations, the situation is close to the single-object environment, where the generalized Vickrey rule is the only rule satisfying the four properties (Saitoh and Serizawa, 2008; Sakai, 2008). This contrasts with the case of nonincreasing incremental valuations where several agents typically win the object at an efficient allocation, and we obtain impossibility theorems (Theorem 1 and Proposition 2).

Although nondecreasing incremental valuations together with *efficiency* make the situation close to the single-object environement, any domain of Theorem 2 includes preference profiles at which a bundling efficient allocation is not a unique efficient allocation. This complicates the proof of Theorem 2, and the result does not follow in a straightforward extension of the arguments in the previous literature (Saitoh and Serizawa, 2008; Sakai, 2008). In Section 5, we will discuss this point in detail.

1.3 Related literature

Most papers on multi-unit object allocation problems with money assume quasi-linear preferences with decreasing (or nonincreasing) incremental valuations (Perry and Reny, 2002, 2005; Ausubel, 2004; Milgrom, 2004; Krishna, 2009; Ausubel et al., 2014, etc).[8] This paper is different from this strand of research in covering not only preferences exhibiting

[7] For example, Baranov et al. (2017) argue that the vaccine industry is such an example because "new entry into the vaccine market may require making significant investments in R&D, performing clinical trials, obtaining regulatory approvals, and building production facilities" (Baranov et al., 2017).

[8] We also refer to Baranov et al. (2017), who remove the assumption of decreasing incremental valuations while maintaining quasi-linearity in the procurement auction model.

nonincreasing incremental valuations, but also those exhibiting nondecreasing incremental valuations in a non-quasi-linear environment.

In the literature on heterogeneous objects allocation problems with money (Kelso and Crawford, 1982; Gul and Stacchetti, 1999, 2000; Milgrom, 2000; Ausubel, 2006, etc.), substitutability among the objects is the key to several positive results such as the existence of an equilibrium allocation (Kelso and Crawford, 1982), revenue monotonicity of the Vickrey rule (Ausubel and Milgrom, 2002), etc. When preferences are quasi-linear and objects are identical, substitutability reduces to nonincreasing incremental valuations (Kelso and Crawford, 1982). Thus, our results (Theorems 1 and 2) are in contrast to the above literature in suggesting that in multi-unit object allocation problems with money, if preferences may not be quasi-linear, then the existence of a rule satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness* is guaranteed "almost" only when preferences exhibit nondecreasing incremental valuations.

The class of *efficient* and *strategy-proof* rules has been studied extensively. When a class of preferences includes only quasi-linear preferences and is sufficiently rich, the Vickrey rule is the only rule satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness* (Holmström 1979; Chew and Serizawa, 2007).

There is a small but expanding literature on object allocation problems with money that assume non-quasi-linear preferences. The papers in the literature mainly focuses on characterizations of rules satisfying *efficiency* and *strategy-proofness* together with the other desirable properties in various models. They are roughly classified into two categories: papers on unit-demand agents and those on multi-demand agents as ours.[9]

The minimum price Warlasian rule (Demange and Gale, 1985) plays a central role in in the first category. It is the only rule satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness* both on the classical domain (Morimoto and Serizawa, 2015) and on the common-tiered-object domain with positive income effects on which objects are ranked according to the common tiers (Zhou and Serizawa, 2018).

In the single-object environment, the generalized Vickrey rule coincides with the minimum price Walrasian rule, and is the only rule satisfying the four properties (Saitoh and Serizawa, 2008; Sakai, 2008). As already discussed, when preferences exhibit nondecreasing incremental valuations, it is efficient to bundle all the units into one package and to give the package to a single agent with the highest valuation of it. Thus, efficient allocations for preferences that exhibit nondecreasing incremental valuations are close to those in the single-object environment. Then, our characterization theorem (Theorem 2) may seem to be obtained as an application of the characterization for the single-object environment. However, we emphasize that our result is not a trivial extension of theirs since several

[9]Precisely, there is the third category of the papers on object allocation problems with money that assume non-quasi-linear preferences: the papers on hard budget constraint (Dobzinski et al., 2012; Lavi and May, 2012, etc). Papers in the first two categories including this paper assume continuity of a preference, and exclude any hard budget constraint. In contrast, papers in the third category assume quasi-linear preferences with hard budget constraint, and exclude any income effect in the feasible consumption set. Thus, the papers that belong to the third category are different from those in the first two categories.

agents may receive the object in our environment.[10]

In contrast, the papers in the latter category typically obtain impossibility theorems.

In the heterogeneous objects model, if a class of preferences includes all unit-demand preferences and includes at least one multi-demand preference, then no rule satisfies *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness* (Kazumura and Serizawa, 2016). We discuss in detail that the proof strategy of our impossibility theorem (Theorem 1) is different from Kazumura and Serizawa (2016)'s in Section 6.3.

In the heterogeneous objects model, a preference is *dichotomous* if it divides the set of packages of objects into the acceptable and the unacceptable sets, and packages in the former set has the same positive value, but those in the latter set are valueless. On the dichotomous domain, no rule satisfies *efficiency*, *individual rationality*, *no subsidy*, and *strategy-proofness* (Malik and Mishra, 2021). In contrast, on the dichotomous domain with nonnegative income effects, the generalized Vickrey rule is the only rule satisfying the same four properties (Malik and Mishra, 2021). Since Malik and Mishra (2021) treat heterogeneous objects, their results does not imply our results, and vice versa. In Section 6.2, we discuss in detail that our proof strategies are different from theirs.

All the above papers on non-quasi-linear preferences treat the different models from ours. Thus, the results in the above papers can not be applied to multi-unit auctions such as procurement auctions for vaccines and some spectrum auctions.[11] This paper contributes to the literature by treating the multi-unit object allocation problem with money, which is common in many important real-life auctions, but are not covered by the above papers.

The unique paper in the literature that considers the multi-unit object allocation problem with money as ours is Baisa (2020). He shows that on the domain with decreasing incremental valuations, positive income effects, and the single-crossing property, if preferences are of single-dimensional types, then there is a rule satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. He also shows that on the same domain, if preferences are of multi-dimensional types, then no rule satisfies the four properties. His results imply that for preferences exhibiting decreasing incremental valuations and positive income effects, there is a rule satisfying the four properties only when preferences are of single-dimensional types. Both of his results neither imply nor are implied by our results.

Compared to his results, our results contribute to the literature by covering a broad class of environments where preferences exhibit nonincreasing or nondecreasing incremental valuations, and does positive or negative income effects. Our results also contribute to the literature by showing that there is a rule satisfying the four properties if and "almost" only if preferences exhibit nondecreasing incremental valuations.[12] Thus, the two results draw

[10] Section 5 discusses this point in detail.

[11] Other examples are car license auctions in Singapore, emission auctions, government bond auctions, procurement auctions for electricity, etc.

[12] Baisa (2020) assumes that *all* preferences in a class of preferences exhibit decreasing incremental valuations. Thus, not only the "if" part, but also the "almost only if" part does not follow from his results because how the violation of nondecreasing incremental valuations of *a* preference affects the existence of a rule satisfying the four properties is beyond the scope of his results.

different insights into the existence of a rule satisfying the four properties, and our results complement his. We give a detailed discussion on the relationship between his results and ours in Section 6.1.

As long as preferences are quasi-linear, the existence of a rule satisfying the four properties does not depend on the characteristics of the environment such as the properties of valuations, the number of objects, etc. The prior literature has suggested that in non-quasi-linear environments, the existence of such a rule depends on the properties of preferences such as (i) the singe-dimensional types (Baisa, 2020), (ii) the dichotomous property with nonnegative income effects (Malik and Mishra, 2021), and (iii) the unit-demand property (Kazumura and Serizawa, 2016). Our results add other properties of preferences that guarantee the existence of such a rule to the literature, that is, (i) nondecreasing incremental valuations and (ii) the small size of the income effects.[13] Moreover, our results also contribute to the literature by suggesting that the existence of such a rule depends not only on the properties of preferences, but also on other characteristics of the environment such as the number of units.[14]

1.4 Organization

The remainder of the paper is organized as follows. Section 2 sets up the model. Section 3 introduces the generalized Vickrey rule and the inverse Vickrey rule. Section 4 provides the results for nonincreasing incremental valuations. Section 5 provides the results for nondecreasing incremental valuations. Section 6 discusses the relationship between our results and the related results obtained by other authors, and explains the difficulty of our proofs. Section 7 concludes. Most proofs are in Appendix, while missing ones can be found in the supplementary material.

2 The model and definitions

There are n agents and m units of an object, where $2 \leq n < \infty$ and $2 \leq m < \infty$. We denote the set of agents by $N \equiv \{1, \ldots, n\}$. Let $M \equiv \{0, \ldots, m\}$. Further, given $m' \in M$, let $M(m') \equiv \{0, \ldots, m'\}$.

Each agent $i \in N$ receives $x_i \in M$ units of the object. The amount of money paid by agent i is denoted by $t_i \in \mathbb{R}$. For each agent $i \in N$, his **consumption set** is $M \times \mathbb{R}$, and

[13] Baisa and Essig and Aberg (2021) assume the maximal size of the nonnegative income effects, and identify the worst case dead weight loss of the (indirect) Vickrey auction mechanism among all undominated strategies, which depends on the maximal size of the income effects. Because their result is concerned with the worst case efficiency loss among undominated strategies, it does not suggest that the existence of a rule satisfying *efficiency* and *strategy-proofness* together with the other desirable properties depends on the size of the income effects.

[14] Theorem 1 and Proposition 1 of Baisa (2020) also suggest that in the multi-unit object allocation problem with money, if agents have preferences with decreasing incremental valuations and positive income effects, then the existence of a rule satisfying the desirable properties of Vickrey rule depends on the number of units. Because his impossibility theorem (Proposition 1 of Baisa (2020)) imposes an additional property that he calls *strong monotonicity* together with the four properties, his results do not imply that the existence of a rule satisfying the four properties depends on the number of units.

his (**consumption**) **bundle** is a pair $z_i \equiv (x_i, t_i) \in M \times \mathbb{R}$. Let $\mathbf{0} \equiv (0, 0)$.

2.1 Preferences

Each agent $i \in N$ has a complete and transitive preference R_i over $M \times \mathbb{R}$. Let P_i and I_i be the strict and indifference relations associated with R_i, respectively. Preferences are privately known. Throughout this paper, we assume that a preference R_i satisfies the following four properties.

Money monotonicity. For each $x_i \in M$ and each pair $t_i, t'_i \in \mathbb{R}$ with $t_i < t'_i$, we have $(x_i, t_i)\ P_i\ (x_i, t'_i)$.

Object monotonicity. For each pair $x_i, x'_i \in M$ with $x_i > x'_i$ and each $t_i \in \mathbb{R}$, we have $(x_i, t_i)\ P_i\ (x'_i, t_i)$.

Possibility of compensation. For each $z_i \in M \times \mathbb{R}$ and each $x_i \in M$, there is a pair $t_i, t'_i \in \mathbb{R}$ such that $(x_i, t_i)\ R_i\ z_i$ and $z_i\ R_i\ (x_i, t'_i)$.

Continuity. For each $z_i \in M \times \mathbb{R}$, the upper contour set at z_i, $\{z'_i \in M \times \mathbb{R} : z'_i\ R_i\ z_i\}$, and the lower contour set at z_i, $\{z'_i \in M \times \mathbb{R} : z_i\ R_i\ z'_i\}$, are both closed.

A typical class of preferences satisfying the above four properties is denoted by $\mathcal{R}$. For each $i \in N$, each $R_i \in \mathcal{R}$, each $z_i \in M \times \mathbb{R}$, and each $x_i \in M$, the possibility of compensation and continuity together imply that there is a payment $V_i(x_i, z_i) \in \mathbb{R}$ such that $(x_i, V_i(x_i, z_i))\ I_i\ z_i$.[15] By money monotonicity, such a payment is unique. We call $V_i(x_i, z_i)$ the **valuation of** x_i **at** z_i **for** R_i. We define the **net valuation of** x_i **at** z_i **for** R_i as $v_i(x_i, z_i) \equiv V_i(x_i, z_i) - V_i(0, z_i)$. Note that for each $z_i \in M \times \mathbb{R}$, $v_i(0, z_i) = 0$. Moreover, by $V_i(0, \mathbf{0}) = 0$, for each $x_i \in M$, we have $v_i(x_i, \mathbf{0}) = V_i(x_i, \mathbf{0})$.

Remark 1. Let $R_i \in \mathcal{R}$. (i) Let $z_i, z'_i \in M \times \mathbb{R}$ be such that $z_i\ I_i\ z'_i$. For each $x_i \in M$, $V_i(x_i, z_i) = V_i(x_i, z'_i)$. (ii) Let $z_i \in M \times \mathbb{R}$ be such that $z_i\ I_i\ \mathbf{0}$. For each $x_i \in M$, $v_i(x_i, z_i) = v_i(x_i, \mathbf{0})$. (iii) For each $z_i \equiv (x_i, t_i) \in M \times \mathbb{R}$, $t_i = V_i(x_i, z_i)$.

A preference R_i is **quasi-linear** if for each pair (x_i, t_i), $(x'_i, t'_i) \in M \times \mathbb{R}$ and each $\delta \in \mathbb{R}$, $(x_i, t_i)\ I_i\ (x'_i, t'_i)$ implies $(x_i, t_i + \delta)\ I_i\ (x'_i, t'_i + \delta)$. Let $\mathcal{R}^Q$ denote the class of quasi-linear preferences.

Remark 2. Let $R_i \in \mathcal{R}^Q$. (i) For each $x_i \in M$, $v_i(x_i, \cdot)$ is independent of z_i, and we simply write $v_i(x_i)$ instead of $v_i(x_i, z_i)$. (ii) For each pair $(x_i, t_i), (x'_i, t'_i) \in M \times \mathbb{R}$, $(x_i, t_i)\ R_i\ (x'_i, t'_i)$ if and only if $v_i(x_i) - t_i \geq v_i(x'_i) - t'_i$.

[15] For the formal proof of the existence of such a payment, see Lemma 1 of Kazumura and Serizawa (2016).

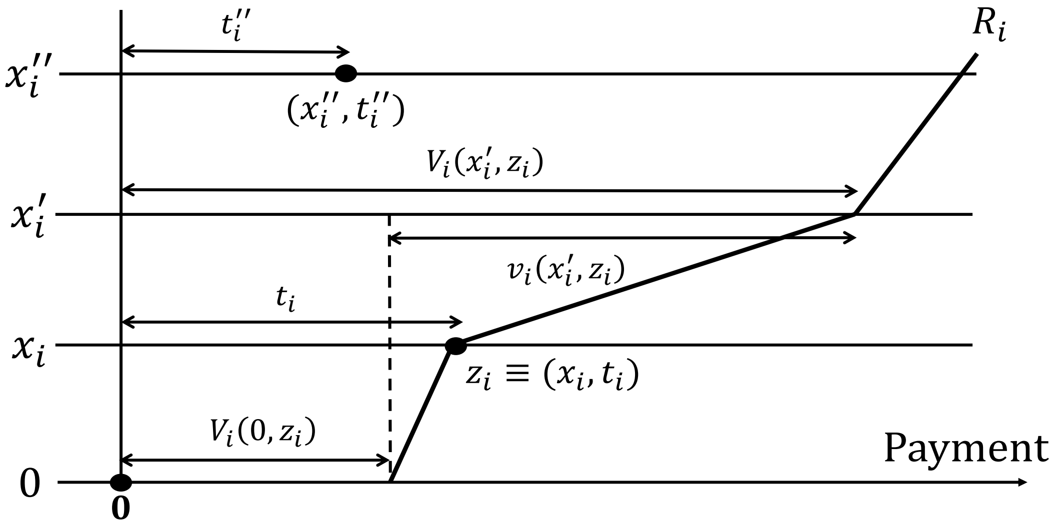

Figure 1: An illustration of the consumption set and indifference curves.

Figure 1 illustrates the consumption set and an indifference curve of a preference $R_i \in \mathcal{R}$. Each horizontal line corresponds to some consumption level of the object. The intersections of the horizontal lines and the vertical line are the points at which payments are zero. Each point on a horizontal line indicates the amount of money that he pays (if it is on the right side of the vertical line) or receives (if it is on the left side of the vertical line). A solid line is an indifference curve of his preference R_i. By money monotonicity, a bundle is more preferable as it goes to the left on a horizontal line. Thus, $(x_i'', t_i'')\ P_i\ z_i = (x_i, t_i)$.

2.2 Incremental valuations

In the multi-unit object allocation problem with money, the property of incremental valuations determines the characteristic of a preference. Given $R_i \in \mathcal{R}$, $z_i \in M \times \mathbb{R}$, and $x_i \in M \backslash \{0\}$, the **incremental valuation of x_i at z_i for R_i** is $v_i(x_i, z_i) - v_i(x_i - 1, z_i)$.

In words, the definition of nonincreasing (resp. nondecreasing) incremental valuations is that for each bundle z_i, the incremental valuation at z_i for R_i is nonincreasing (resp. nondecreasing) in the number of units, and that of constant incremental valuations is that for each bundle z_i, the incremental valuation at z_i for R_i is constant in the number of units. Our definitions of properties of incremental valuations are natural generalizations of the corresponding definitions for quasi-linear preferences.

Formally, a preference R_i exhibits **nonincreasing** (resp. **decreasing**) **incremental valuations** if for each $z_i \in M \times \mathbb{R}$ and each $x_i \in M \backslash \{0, m\}$,

$$v_i(x_i + 1, z_i) - v_i(x_i, z_i) \leq \text{(resp. } < \text{)}\ v_i(x_i, z_i) - v_i(x_i - 1, z_i).$$

A preference R_i exhibits **nondecreasing** (resp. **increasing**) **incremental valuations** if for each $z_i \in M \times \mathbb{R}$ and each $x_i \in M \backslash \{0, m\}$,

$$v_i(x_i + 1, z_i) - v_i(x_i, z_i) \geq \text{(resp. } > \text{)}\ v_i(x_i, z_i) - v_i(x_i - 1, z_i).$$

A preference R_i exhibits **constant incremental valuations** if for each $z_i \in M \times \mathbb{R}$ and each $x_i \in M \backslash \{0, m\}$,

$$v_i(x_i + 1, z_i) - v_i(x_i, z_i) = v_i(x_i, z_i) - v_i(x_i - 1, z_i).$$

Let $\mathcal{R}^{NI}$, $\mathcal{R}^{ND}$, and $\mathcal{R}^{C}$ denote the classes of preferences that exhibit nonincreasing, nondecreasing, and constant incremental valuations, respectively. Clearly, $\mathcal{R}^{NI} \cap \mathcal{R}^{ND} = \mathcal{R}^{C}$. Further, let $\mathcal{R}^{D}$ and $\mathcal{R}^{I}$ denote the classes of preferences that exhibit decreasing and increasing incremental valuations, respectively. Note that $\mathcal{R}^{D} \subsetneq \mathcal{R}^{NI}$, $\mathcal{R}^{D} \cap \mathcal{R}^{C} = \varnothing$, $\mathcal{R}^{I} \subsetneq \mathcal{R}^{ND}$, $\mathcal{R}^{I} \cap \mathcal{R}^{C} = \varnothing$, and $\mathcal{R}^{D} \cap \mathcal{R}^{I} = \varnothing$.

The next remark states that for a preference that exhibits nondecreasing incremental valuations, (i) the per-unit net valuation at each bundle is nondecreasing in the number of units, and (ii) if the per-unit net valuation at a bundle is not constant in the number of units, then it is strictly increasing.

Remark 3 (Nondecreasing per-unit net valuations). Let $m' \in M$ with $m' > 0$. Let $R_i \in \mathcal{R}^{ND}$ and $z_i \in M \times \mathbb{R}$. (i) For each $x_i \in M(m')$, $\frac{x_i}{m'} v_i(m', z_i) \geq v_i(x_i, z_i)$. (ii) If there is $x_i \in M(m') \backslash \{0, m'\}$ such that $\frac{x_i}{m'} v_i(m', z_i) > v_i(x_i, z_i)$, then for each $x'_i \in M(m') \backslash \{0, m'\}$, $\frac{x'_i}{m'} v_i(m', z_i) > v_i(x'_i, z_i)$.

Figure 2 below illustrates Remark 3 in the case where $m' = 3$ and $x_i = 2$.

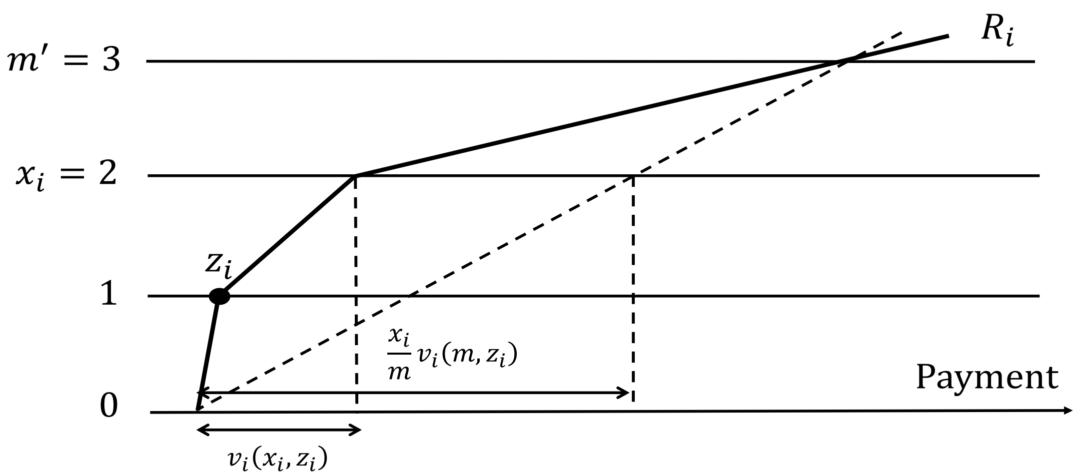

Figure 2: An illustration of Remark 3.

Although Remark 3 is graphically apparent from Figure 2, its formal proof can be found in the supplementary material.

In the classical pure exchange economy model, the convexity of a preference is a standard assumption. Recall that a preference is *convex* if the upper contour set at each bundle is a convex set (Mas-Colell et al., 1995). Because of the indivisibility of the object, we can not define a convex set in our model. However, a notion of convexity is extended to our model in a natural way if we focus on only feasible allocations. Formally, a set $L \subseteq M \times \mathbb{R}$

is *convex* if for each pair $z_i, z_i' \in L$ and each $\delta \in [0,1]$ with $\delta z_i + (1-\delta) z_i' \in M \times \mathbb{R}$, we have $\delta z_i + (1-\delta) z_i' \in L$. A set $L \subseteq M \times \mathbb{R}$ is *strictly convex* if for each distinct pair $z_i, z_i' \in L$ and each $\delta \in (0,1)$ with $\delta z_i + (1-\delta) z_i' \in M \times \mathbb{R}$, $\delta z_i + (1-\delta) z_i'$ belongs to the interior of L.[16]

The next remark states that (i) our definition of nonincreasing incremental valuations is equivalent to a convex preference, and (ii) nondecreasing incremental valuations are equivalent to a concave preference which means that the lower contour set at each bundle is a convex set.

Remark 4. Let $R_i \in \mathcal{R}$.
(i) $R_i \in \mathcal{R}^{NI}$ (resp. $R_i \in \mathcal{R}^{D}$) if and only if for each $z_i \in M \times \mathbb{R}$, the upper contour set at z_i, $\{z_i' \in M \times \mathbb{R} : z_i' \ R_i \ z_i\}$, is convex (resp. strictly convex).
(ii) $R_i \in \mathcal{R}^{ND}$ (resp. $R_i \in \mathcal{R}^{I}$) if and only if for each $z_i \in M \times \mathbb{R}$, the lower contour set at z_i, $\{z_i' \in M \times \mathbb{R} : z_i \ R_i \ z_i'\}$, is convex (resp. strictly convex).

2.3 Income effects

In this subsection, we introduce two classes of preferences which exhibit income effects.

Although our model does not take into account income explicitly, the zero payment can be regarded as the initial income. Then, the increase of the income by $\delta > 0$ induces the shift of the origin of the consumption space to the right by δ. If we fix the origin of the original consumption space, then this shift corresponds to the decrease of payments of all the bundles by δ. Then positive (resp. nonnegative) income effect requires that the increase of income (or equivalently, the decrease of a payment) by δ increase (resp. do not decrease) the incremental valuation.

Formally, a preference R_i exhibits the **positive** (resp. **nonnegative**) **income effect** if for each pair $(x_i, t_i), (x_i', t_i') \in M \times \mathbb{R}$ with $x_i > x_i'$ and $t_i > t_i'$, and each $\delta \in \mathbb{R}_{++}$, $(x_i, t_i) \ I_i \ (x_i', t_i')$ implies $(x_i, t_i - \delta) \ P_i \ (x_i', t_i' - \delta)$ (resp. $(x_i, t_i - \delta) \ R_i \ (x_i', t_i' - \delta)$).

Let $\mathcal{R}^{++}$ and $\mathcal{R}^{+}$ denote the classes of preferences that exhibit positive and nonnegative income effects, respectively. Note that $\mathcal{R}^{++} \subsetneq \mathcal{R}^{+}$, $\mathcal{R}^{++} \cap \mathcal{R}^{Q} = \varnothing$, and $\mathcal{R}^{Q} \subsetneq \mathcal{R}^{+}$.

Remark 5. Let $R_i \in \mathcal{R}^{++}$. (i) Let $x_i \in M \backslash \{m\}$ and $h^{+}(\cdot; x_i) : \mathbb{R} \to \mathbb{R}_{++}$ be such that for each $t_i \in \mathbb{R}$, $h^{+}(t_i; x_i) = V_i(x_i + 1, (x_i, t_i)) - t_i$. Then $h^{+}(\cdot; x_i)$ is strictly decreasing in t_i. (ii) Let $x_i \in M \backslash \{0\}$ and $h^{-}(\cdot; x_i) : \mathbb{R} \to \mathbb{R}_{++}$ be such that for each $t_i \in \mathbb{R}$, $h^{-}(t_i; x_i) = t_i - V_i(x_i - 1, (x_i, t_i))$. Then $h^{-}(\cdot; x_i)$ is strictly decreasing in t_i as well.

The proof of Remark 5 can be found in the supplementary material.

In contrast, negative (resp. nonpositive) income effect requires that the increase of income by δ decrease (resp. do not increase) the incremental valuation. A preference R_i exhibits the **negative** (resp. **nonpositive**) **income effect** if for each pair $(x_i, t_i), (x_i', t_i') \in M \times \mathbb{R}$ with $x_i > x_i'$ and $t_i > t_i'$, and each $\delta \in \mathbb{R}_{++}$, $(x_i, t_i) \ I_i \ (x_i', t_i')$ implies $(x_i', t_i' - \delta) \ P_i \ (x_i, t_i - \delta)$ (resp. $(x_i', t_i' - \delta) \ R_i \ (x_i, t_i - \delta)$).

[16] We endow $M \times \mathbb{R}$ with a distance function $d : (M \times \mathbb{R})^2 \to \mathbb{R}_{+}$ such that for each pair $(x_i, t_i), (x_i', t_i') \in M \times \mathbb{R}$, $d((x_i, t_i), (x_i', t_i')) = |x_i - x_i'| + |t_i - t_i'|$.

Let $\mathcal{R}^{--}$ and $\mathcal{R}^{-}$ denote the classes of preferences that exhibit negative and nonpositive income effects, respectively. Note that $\mathcal{R}^{--} \subsetneq \mathcal{R}^{-}$, $\mathcal{R}^{--} \cap \mathcal{R}^{Q} = \varnothing$, and $\mathcal{R}^{Q} \subsetneq \mathcal{R}^{-}$. Note also that $\mathcal{R}^{++} \cap \mathcal{R}^{--} = \varnothing$ and $\mathcal{R}^{+} \cap \mathcal{R}^{-} = \mathcal{R}^{Q}$.

Remark 6. Let $R_i \in \mathcal{R}^{--}$. (i) Let $x_i \in M \backslash \{m\}$ and $h^{+}(\cdot; x_i) : \mathbb{R} \to \mathbb{R}_{++}$ be such that for each $t_i \in \mathbb{R}$, $h^{+}(t_i; x_i) = V_i(x_i + 1, (x_i, t_i)) - t_i$. Then $h^{+}(\cdot; x_i)$ is strictly increasing in t_i. (ii) Let $x_i \in M \backslash \{0\}$ and $h^{-}(\cdot; x_i) : \mathbb{R} \to \mathbb{R}_{++}$ be such that for each $t_i \in \mathbb{R}$, $h^{-}(t_i; x_i) = t_i - V_i(x_i - 1, (x_i, t_i))$. Then $h^{-}(\cdot; x_i)$ is strictly increasing in t_i as well.

The proof of Remark 6 is symmetric to that of Remark 5, and we omit it.

2.4 Allocations and rules

Let $X \equiv \{(x_1, \ldots, x_n) \in M^n : 0 \leq \sum_{i \in N} x_i \leq m\}$. A (**feasible**) **allocation** is an n-tuple $z \equiv (z_1, \ldots, z_n) \equiv ((x_1, t_1), \ldots, (x_n, t_n)) \in (M \times \mathbb{R})^n$ such that $(x_1, \ldots, x_n) \in X$. Let Z denote the set of feasible allocations. We denote the object allocation and the payments at $z \in Z$ by $x \equiv (x_1, \ldots, x_n)$ and $t \equiv (t_1, \ldots, t_n)$, respectively. We write $z \equiv (x, t) \in Z$.

Given $N' \subseteq N$ and $m' \in M$, let

$$X(N', m') \equiv \left\{ x \in X : 0 \leq \sum_{i \in N'} x_i \leq m' \text{ and } x_i = 0 \text{ for each } i \in N \backslash N' \right\}$$

and $Z(N', m') \equiv \{z \equiv (x, t) \in Z : x \in X(N', m')\}$. These sets correspond to the sets of feasible object allocations and feasible allocations in the reduced economy where the set of agents is N' and there are m' units of the object, respectively.

We call $\mathcal{R}^n$ a **domain**. The partial list of domains to appear in this paper is as follows:

- The **quasi-linear domain**: $(\mathcal{R}^{Q})^n$.
- The **domain with nondecreasing incremental valuations**: $(\mathcal{R}^{ND})^n$.
- The **domain with nonincreasing incremental valuations**: $(\mathcal{R}^{NI})^n$.
- The **domain with constant incremental valuations**: $(\mathcal{R}^{C})^n$.

A **preference profile** is an n-tuple $R \equiv (R_1, \ldots, R_n) \in \mathcal{R}^n$. Given $R \in \mathcal{R}^n$ and $N' \subseteq N$, let $R_{N'} \equiv (R_i)_{i \in N'}$ and $R_{-N'} \equiv (R_i)_{i \in N \backslash N'}$. Specifically, for each distinct pair $i, j \in N$, we may write $R_{-i} \equiv (R_k)_{k \in N \backslash \{i\}}$ and $R_{-i,j} \equiv (R_k)_{k \in N \backslash \{i,j\}}$.

In this paper, we require a domain to be rich in the following sense. A class of preferences $\mathcal{R}$ is **rich** if $\mathcal{R} \supseteq \mathcal{R}^{C} \cap \mathcal{R}^{Q}$. We call the domain $(\mathcal{R}^{C} \cap \mathcal{R}^{Q})^n$ a **minimal domain**.

Our notion of richness is natural because a number of classes of preferences of interest are rich. Examples of a rich class of preferences include $\mathcal{R}^{C} \cap \mathcal{R}^{Q}$, $\mathcal{R}^{NI} \cap \mathcal{R}^{Q}$, $\mathcal{R}^{ND} \cap \mathcal{R}^{Q}$, $\mathcal{R}^{C} \cap \mathcal{R}^{+}$, $\mathcal{R}^{NI} \cap \mathcal{R}^{+}$, $\mathcal{R}^{ND} \cap \mathcal{R}^{+}$, $\mathcal{R}^{C} \cap \mathcal{R}^{-}$, $\mathcal{R}^{NI} \cap \mathcal{R}^{-}$, $\mathcal{R}^{ND} \cap \mathcal{R}^{-}$, etc.

An **allocation rule**, or simply a **rule**, on $\mathcal{R}^n$ is a function $f : \mathcal{R}^n \to Z$. With a slight abuse of notation, we may write $f \equiv (x, t)$, where $x : \mathcal{R}^n \to X$ and $t : \mathcal{R}^n \to \mathbb{R}^n$ are the object allocation and the payment rules associated with f, respectively. We denote agent

i's outcome bundle under a rule f at a preference profile R by $f_i(R) = (x_i(R), t_i(R))$, where $x_i(R)$ and $t_i(R)$ are the consumption level of the object and the payment made by agent i, respectively.

We now introduce the properties of a rule. The efficiency condition takes the preference of the owner of the object into account. We assume that he is only interested in his revenue. An allocation $z \equiv (x, t) \in Z$ is **(Pareto-)efficient** for a given preference profile $R \in \mathcal{R}^n$ if there is no other allocation $z' \equiv (x', t') \in Z$ such that (i) $z'_i \; R_i \; z_i$ for each $i \in N$, (ii) $\sum_{i \in N} t'_i \geq \sum_{i \in N} t_i$, and (iii) some agent has the strict relation in (i) or the inequality in (ii) is strict.

Note that if $R \in (\mathcal{R}^Q)^n$, then an allocation $z \equiv (x, t) \in Z$ is efficient for R if and only if $\sum_{i \in N} v_i(x_i) = \max_{x' \in X} \sum_{i \in N} v_i(x'_i)$. Remark 7 below generalizes this property to a non-quasi-linear preference profile. Since the (net) valuation depend on a bundle, an efficient allocation for a non-quasi-linear preference profile should depend on the bundles, unlike an efficient allocation for a quasi-linear preference profile.

Remark 7. Let $R \in \mathcal{R}^n$ and $z \equiv (x, t) \in Z$. Then z is efficient for R if and only if $\sum_{i \in N} v_i(x_i, z_i) = \max_{x' \in X} \sum_{i \in N} v_i(x'_i, z_i)$.

By Remark 7, we obtain another characterization of an efficient allocation for a preference profile at which the preference of each agent exhibits nonincreasing incremental valuations.

Remark 8. Let $N' \subseteq N$ and $m' \in M$. Let $R_{N'} \in (\mathcal{R}^{NI})^{|N'|}$ and $z \equiv (x, t) \in Z(N', m')$. Then $\sum_{i \in N'} v_i(x_i, z_i) = \max_{x' \in X(N', m')} \sum_{i \in N'} v_i(x'_i, z_i)$ if and only if $\sum_{i \in N'} x_i = m'$, and for each pair $i, j \in N'$ with $x_i < m'$ and $x_j > 0$, we have $V_i(x_i + 1, z_i) - t_i \leq t_j - V_j(x_j - 1, z_j)$. In particular, by Remark 7, z is efficient for R if and only if $\sum_{i \in N} x_i = m$, and for each pair $i, j \in N$ with $x_i < m$ and $x_j > 0$, we have $V_i(x_i + 1, z_i) - t_i \leq t_j - V_j(x_j - 1, z_j)$.

The first property states that a rule should select an efficient allocation.

Efficiency. For each $R \in \mathcal{R}^n$, $f(R)$ is efficient for R.

The second property is a participation constraint. It states that a rule never selects an allocation at which some agent is worse off than if he had received no object and paid nothing.

Individual rationality. For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R) \; R_i \; \mathbf{0}$.

The third and fouorth properties are both concerned with nonnegative payments.

No subsidy. For each $R \in \mathcal{R}^n$ and each $i \in N$, $t_i(R) \geq 0$.

No subsidy for losers. For each $R \in \mathcal{R}^n$ and each $i \in N$, if $x_i(R) = 0$, then $t_i(R) \geq 0$.

Clearly, *no subsidy* implies *no subsidy for losers*.

The last property is a dominant strategy incentive compatibility. It states that no agent ever benefits from misrepresenting his preference.

Strategy-proofness. For each $R \in \mathcal{R}^n$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $f_i(R)\ R_i\ f_i(R'_i, R_{-i})$.

3 The Vickrey rule and its extensions

In this section, we introduce the Vickrey rule for quasi-linear preferences (Vickrey, 1961) and its extensions for non-quasi-linear preferences.

Given $i \in N$, $R_{-i} \in \mathcal{R}^{n-1}$, and $x_i \in M$, define the maximum sum of net valuations at $\mathbf{0}$ for agents other than agent i, given that agent i has already obtained x_i units, as

$$\sigma_i(R_{-i}; x_i) \equiv \max_{x \in X(N \setminus \{i\}, m - x_i)} \sum_{j \in N \setminus \{i\}} v_j(x_j, \mathbf{0}).$$

Note that if $R_{-i} \in (\mathcal{R}^Q)^{n-1}$, then $\sigma_i(R_{-i}; x_i) = \max_{x \in X(N \setminus \{i\}, m - x_i)} \sum_{j \in N \setminus \{i\}} v_j(x_j)$.

3.1 The Vickrey rule

We first introduce the Vickrey rule for quasi-linear preferences. Given $\mathcal{R} \subseteq \mathcal{R}^Q$, a rule $f \equiv (x, t)$ on $\mathcal{R}^n$ is a **Vickrey rule** if the following two conditions hold:
(i) for each $R \in \mathcal{R}^n$,

$$x(R) \in \arg\max_{x \in X} \sum_{i \in N} v_i(x_i),$$

(ii) for each $R \in \mathcal{R}^n$ and $i \in N$,

$$t_i(R) = \sigma_i(R_{-i}; 0) - \sigma_i(R_{-i}; x_i(R)).$$

The condition (i) says that the object is allocated so as to maximize the sum of net valuations, and the condition (ii) says that each agent must pay the externality that he imposes on other agents.

3.2 The generalized Vickrey rule

We introduce an extension of the Vickrey rule to non-quasi-linear preferences introduced by Saitoh and Serizawa (2008) and Sakai (2008).

Remark 2 (i) states that the net valuations are independent of a bundle for quasi-linear preferences, and so we do not have to care about the bundles at which the net valuations are evaluated in the definition of the Vickrey rule. However, the net valuations may vary depending on a bundle for non-quasi-linear preferences, and we must specify the bundles at which the net valuations are evaluated in order to extend the Vickrey rule to a non-quasi-linear domain. The generalized Vickrey rule chooses the net valuations at $\mathbf{0}$, and apply those to the Vickrey rule.

A rule $f \equiv (x,t)$ on $\mathcal{R}^n$ is a **generalized Vickrey rule** if the following two conditions hold:

(i) for each $R \in \mathcal{R}^n$,

$$x(R) \in \arg\max_{x \in X} \sum_{i \in N} v_i(x_i, \mathbf{0}),$$

(ii) for each $R \in \mathcal{R}^n$ and each $i \in N$,

$$t_i(R) = \sigma_i(R_{-i}; 0) - \sigma_i(R_{-i}; x_i(R)).$$

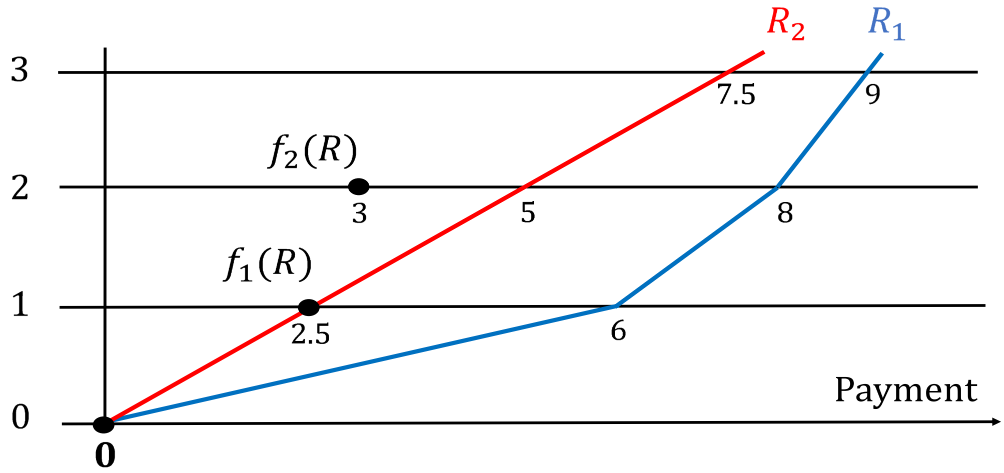

Figure 3: An illustration of Example 1.

The next example illustrates the generalized Vickrey rule.

Example 1 (The generalized Vickrey rule, Figure 3). Assume $n = 2$ and $m = 3$. Let $\mathcal{R} \supseteq \mathcal{R}^{NI} \cap \mathcal{R}^{+}$. Let $f \equiv (x,t)$ be a generalized Vickrey rule on $\mathcal{R}^2$. Let $R_1 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ be such that (i) $v_1(1,\mathbf{0}) = 6$, (ii) for each $t_1 \in [0,6]$, $v_1(2,(1,t_1)) - v_1(1,(1,t_1)) = -\frac{1}{3}t_1 + 4$, and (iii) for each $t_1 \in [0,8]$, $v_1(3,(2,t_1) - v_1(2,(2,t_1)) = -\frac{1}{4}t_1 + 3$. Let $R_2 \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that for each $x_2 \in M$, $v_2(x_2) = 2.5x_2$.

Note that $v_1(2,\mathbf{0}) - v_1(1,\mathbf{0}) = 2$ and $v_1(3,\mathbf{0}) - v_1(2,\mathbf{0}) = 1$. By the first condition (i) of the generalized Vickrey rule, $x(R) = (1,2)$. Further, by the second condition (ii) of the generalized Vickrey rule,

$$t_1(R) = v_2(3) - v_2(2) = 2.5 \quad \text{and} \quad t_2(R) = v_1(3,\mathbf{0}) - v_1(1,\mathbf{0}) = 3.$$

Thus, $f_1(R) = (1, 2.5)$ and $f_2(R) = (2,3)$.

Remark 9. Let $f \equiv (x,t)$ be a generalized Vickrey rule on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$ and $z \in Z$ be such that for each $i \in N$, $z_i \equiv (x_i(R), v_i(x_i(R),\mathbf{0}))$. Note that for each $i \in N$, by

$V_i(0, \mathbf{0}) = 0$, $v_i(x_i(R), \mathbf{0}) = V_i(x_i(R), \mathbf{0})$. Thus, for each $i \in N$, $z_i \ I_i \ \mathbf{0}$, and by Remark 1 (ii), $v_i(x_i(R), z_i) = v_i(x_i(R), \mathbf{0})$. Then by Remark 7, the first condition (i) of the generalized Vickrey rule implies that z is efficient for R.

Note that Remark 9 simply states that for each preference profile $R \in \mathcal{R}^n$, the allocation $(x_i(R), v_i(x_i(R), \mathbf{0}))_{i \in N}$ is efficient, and it does not imply that the generalized Vickrey rule satisfies *efficiency*.

3.3 The inverse Vickrey rule

In this subsection, we propose another extension of the Vickrey rule to non-quasi-linear preferences that exhibit nonincreasing incremental valuations.

Given $R_i \in \mathcal{R}$ and $x_i \in M$, the **inverse-demand set at** x_i **for** R_i is defined as $P(x_i; R_i) \equiv \{p \in \mathbb{R}_+ : (x_i, px_i) \ R_i \ (x'_i, px'_i)$ for each $x'_i \in M\}$. The **inverse-demand function for** R_i is a function $p(\cdot; R_i) : M \backslash \{0\} \to \mathbb{R}_+$ such that for each $x_i \in M \backslash \{0\}$, $p(x_i; R_i) = \sup P(x; R_i)$.[17]

The next remark states that if a quasi-linear preference R_i exhibits nonincreasing incremental valuations, then the inverse-demand $p(x_i; R_i)$ of x_i for R_i coincides with the incremental valuation $v_i(x_i) - v_i(x_i - 1)$ of x_i for R_i.

Remark 10. Let $R_i \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$. For each $x_i \in M \backslash \{0\}$, $p(x_i; R_i) = v_i(x_i) - v_i(x_i - 1)$.

Given a preference $R_i \in \mathcal{R}$ let $R_i^{inv} \in \mathcal{R}^Q$ be a quasi-linear preference such that for each $x_i \in M \backslash \{0\}$, $v_i^{inv}(x_i) - v_i^{inv}(x_i - 1) = p(x_i; R_i)$. That is, the incremental valuation of x_i for the new preference R_i^{inv} is equal to the inverse-demand $p(x_i; R_i)$ of x_i units for the original preference R_i. Note that by Remark 10, if $R_i \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$, then $R_i^{inv} = R_i$. Given $R \in \mathcal{R}$ and $i \in N$, let $R^{inv} \equiv (R_j^{inv})_{j \in N}$ and $R_{-i}^{inv} \equiv (R_j^{inv})_{j \in N \backslash \{i\}}$.

Now, we are ready to define another extension of the Vickrey rule to non-quasi-linear preferences exhibiting nonincreasing incremental valuations. For each preference profile $R \in (\mathcal{R}^{NI})^n$, the inverse Vickrey rule applies the transformed preference profile R^{inv} from R to the Vickrey rule.

Given $\mathcal{R} \subseteq \mathcal{R}^{NI}$, a rule $f \equiv (x, t)$ on $\mathcal{R}^n$ is an **inverse(-demand-based generalized) Vickrey rule** if for each $R \in \mathcal{R}^n$, the following two conditions hold:
(i) for each $R \in \mathcal{R}^n$,

$$x(R) \in \arg\max_{x \in X} \sum_{i \in N} v_i^{inv}(x_i),$$

(ii) for each $R \in \mathcal{R}^n$ and each $i \in N$,

$$t_i(R) = \sigma_i(R_{-i}^{inv}; 0) - \sigma_i(R_{-i}^{inv}; x_i(R)).$$

[17] Note that for each $R_i \in \mathcal{R}$, $\sup P(0; R_i) = \infty$. Thus. $p(0; R_i) \equiv \infty$. For each $x_i \in M \backslash \{0\}$, if $P(x_i; R_i) = \varnothing$, then set $p(x_i; R_i) \equiv v_i(x_i, \mathbf{0}) - v_i(x_i - 1, \mathbf{0})$.

By Remark 10, the inverse Vickrey rule coincides with the Vickrey rule on $\mathcal{R}^n \subseteq (\mathcal{R}^{NI} \cap \mathcal{R}^Q)^n$.[18] However, the inverse Vickrey rule is different from the generalized Vickrey rule, and the two rules typically produce the different outcomes for a preference profile $R \in (\mathcal{R}^{NI})^n$. The next example illustrates this point.

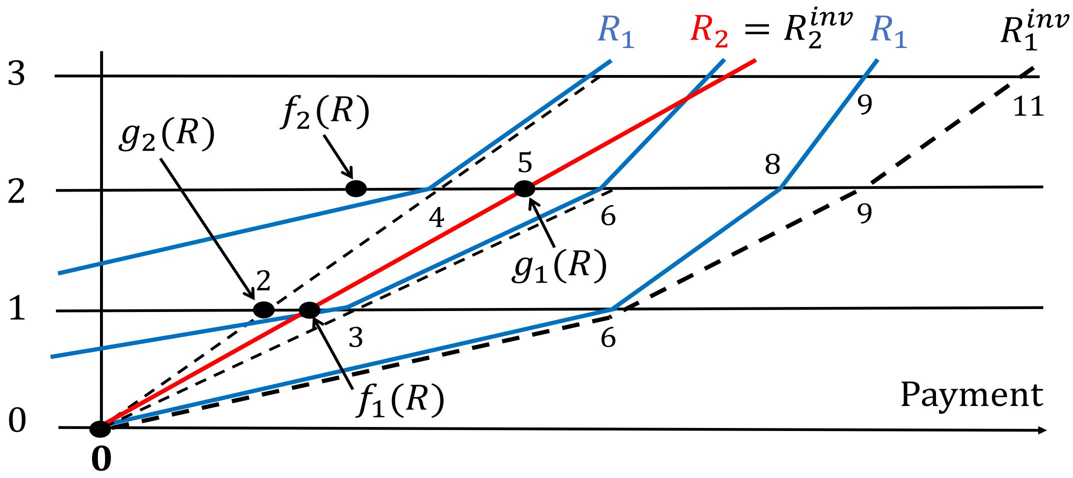

Figure 4: An illustration of Example 2.

Example 2 (The inverse Vickrey rule, Figure 4). Assume $n = 2$ and $m = 3$. Let $\mathcal{R}$ be such that $\mathcal{R}^{NI} \cap \mathcal{R}^+ \subseteq \mathcal{R} \subseteq \mathcal{R}^{NI}$. Let f be a generalized Vickrey rule on $\mathcal{R}^2$. Let $(R_1, R_2) \in \mathcal{R}^2$ be the same preference profile as in Example 1. Then, by Example 1, $f_1(R) = (1, 2.5)$ and $f_2(R) = (2, 3)$. Let $g \equiv (x, t)$ be an inverse Vickrey rule on $\mathcal{R}^2$.

Note that $p(1; R_1) = v_1(1, \mathbf{0}) = 6$. Then, we compute $p(2; R_1)$. Solving $v_1(2, (1, t_1)) - v_1(1, t_1)) = t_1$ for t_1, we have $t_1 = 3$. Then, we show that $p(2; R_1) = 3$.

First, we show that for each $p \in P(2; R_1)$, $p \leq 3$. Let $p \in P(2; R_1)$. By contradiction, suppose $p > 3$. Then,

$$V_1(2, (1, p)) - p = v_1(2, (1, p)) - v_1(1, (1, p)) = -\frac{p}{3} + 4 < p,$$

where the first equality follows from Remark 1 (iii) and the inequality from $p > 3$. This implies $V_1(2, (1, p)) < 2p$. Thus, $(1, p)\ P_1\ (2, 2p)$. However, this contradicts $p \in P(2; R_1)$.

Next, we show that $3 \in P(2; R_1)$. Since $t_1 = 3$ is the solution for the equation $v_1(2, (1, t_1)) - v_1(1, (1, t_1)) = t_1$,

$$V_1(2, (1, 3)) - 3 = v_1(2, (1, 3)) - v_1(1, (1, 3)) = 3,$$

or equivalently, $V_1(2, (1, 3)) = 6$. Thus, $(2, 6)\ I_1\ (1, 3)$. Moreover, by $V_1(1, \mathbf{0}) = v_1(1, \mathbf{0}) =$

[18] We can extend the definition of the inverse Vickrey to any domain $\mathcal{R}^n$, but if R_i does not exhibit nonincreasing incremental valuations for some $i \in N$, then it may not coincide with the Vickrey rule even when $R \in (\mathcal{R}^Q)^n$. In such a case, the inverse Vickrey rule can no longer be regarded as an extension of the Vickrey rule. Thus, we choose to define the inverse Vickrey rule only on $\mathcal{R}^n \subseteq (\mathcal{R}^{NI})^n$.

$6 > 3$, we have $(1,3)\ P_1\ \mathbf{0}$. Finally,

$$V_1(3,(2,6)) - 6 = v_1(3,(2,6)) - v_1(2,(2,6)) = \frac{3}{2} < 3,$$

or equivalently, $V_1(3,(2,6)) < 9$. This implies $(2,6)\ P_1\ (3,9)$.

Thus, we have established that for each $x_1 \in M$, $(2,6)\ R_1\ (x_1, 3x_1)$, that is, $3 \in P(2;R_1)$.[19] By $p \leq 3$ for each $p \in P(2;R_1)$, $p(2;R_1) = 3$.

Similarly, solving the equation $v_1(3,(2,t_1)) - v_1(2,(2,t_1)) = \frac{t_1}{2}$ for t_1, we get $t_1 = 4$, and we can show $p(3;R_1) = \frac{t_1}{2} = 2$. Thus, $v_1^{inv}(1) = 6$, $v_1^{inv}(2) - v_1^{inv}(1) = 3$, and $v_1^{inv}(3) - v_1^{inv}(2) = 2$. By Remark 10, $R_2^{inv} = R_2$. Thus, by the definition of the inverse Vickrey rule, $x(R) = (2,1)$, and

$$t_1(R) = v_2^{inv}(3) - v_2^{inv}(1) = 5 \quad \text{and} \quad t_2(R) = v_1^{inv}(3) - v_1^{inv}(2) = 2.$$

Then, $g_1(R) = (2,5)$ and $g(R) = (1,2)$. Note that the outcome of the inverse Vickrey rule for R is different from that of the generalized Vickrey rule for R.

4 Nonincreasing incremental valuations

The assumption of nonincreasing incremental valuations is standard in the literature on multi-unit object allocation problems, and is equivalent to a convex preference. Moreover, it is a natural assumption when firms have technologies that exhibit nonincreasing returns to scale. Thus, we first investigate the existence of a rule satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness* in such a standard situation.

4.1 An odd number of units

Our results depend on the number of units. First, we consider the case of an odd number of units.

The following theorem states that for an odd number of units, if a rich class of preference includes at least one preference that exhibits decreasing incremental valuations and either positive or negative income effects, then no rule satisfies *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*.

Theorem 1. *Assume m is odd. Let $R_0 \in \mathcal{R}^D \cap (\mathcal{R}^{++} \cup \mathcal{R}^{--})$. Let $\mathcal{R}$ be rich and $R_0 \in \mathcal{R}$. No rule on $\mathcal{R}^n$ satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness.*

Note that the Vickrey rule satisfies the four properties on our minimal domain $(\mathcal{R}^C \cap \mathcal{R}^Q)^n$ (Vickrey, 1961; Holmström, 1979). Thus, Theorem 1 states that adding at least one preference that exhibits decreasing incremental valuations and either positive or negative income

[19] By $(1,3)\ I_1\ (2,6)$, we also have $3 \in P(1;R_1)$.

effects to our minimal domain on which there is a rule satisfying the four properties leads to an impossibility theorem.

Intuitively, for preferences that exhibit decreasing incremental valuations, several agents typically win the object at an efficient allocation, and if an agent with a non-quasi-linear preference wins the object, then he may benefit from misrepresenting his preference so as to win one more or fewer unit, or may be possible to Pareto improve the allocation by exchanging one unit between him and another winner. Thus, it is difficult to guarantee the existence of a rule satisfying the four properties in the case of decreasing incremental valuations, and we have a negative result in Theorem 1.

To be more concrete, the inverse Vickrey rule plays an important role in the proof of Theorem 1. In order to explain the intuition of the proof, consider the case of two agents and the preference profile such that both the agents have the same preference that exhibits decreasing incremental valuations and positive income effects. In the proof, we show that at the preference profile, the outcome of a rule satisfying the four properties coincides with that of the inverse Vickrey rule. Then, we show that under the inverse Vickrey rule, an agent wins $\frac{m-1}{2}$ units, and he benefits from misrepresenting his preference so as to win one more unit.[20] The next example illustrates this point.

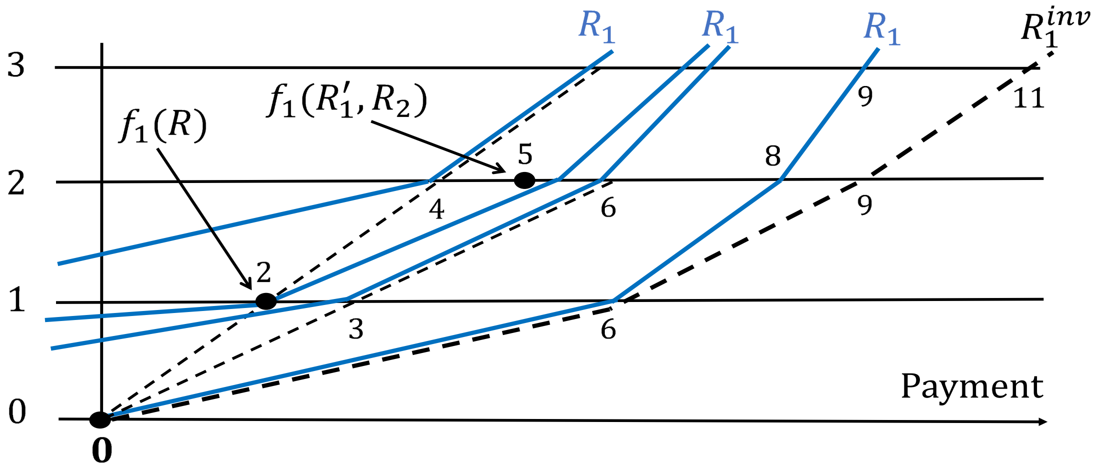

Figure 5: An illustration of Example 3.

Example 3 (Figure 5). Assume $n = 2$ and $m = 3$. Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ be the preference R_1 defined in Example 1. Let $\mathcal{R}$ ($\subseteq \mathcal{R}^{NI}$) be rich and $R_0 \in \mathcal{R}$. Let f be an inverse Vickrey rule on $\mathcal{R}^n$. Let $R \in \mathcal{R}^2$ be such that $R = (R_0, R_0)$. Recall that in Example 2, we observed that $v_0^{inv}(1) = 6$, $v_0^{inv}(2) - v_0^{inv}(1) = 3$, and $v_0^{inv}(3) - v_0^{inv}(2) = 2$. Then, by the definition

[20] When both the agents have the preference that exhibits negative income effects, the outcome of a rule satisfying the four properties does not necessarily coincide with that of the inverse Vickrey rule at the preference profile, and the proof strategy for the case of positive income effects does not work. In such a case, we solve the difficulty by identifying the *range of the payments* under a rule satisfying the properties. Then, we show that an agent receives at least $\frac{m+1}{2}$ units, and he gets better off by misrepresenting his preference so as to win one fewer unit. For the detailed discussion, see Appendix B.

of the inverse Vickrey rule, either $f(R) = ((1,2),(2,5))$ or $f(R) = ((2,5),(1,2))$. Without loss of generality, let $f(R) = ((1,2),(2,5))$.

Let $R_1' \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that for each $x_1 \in M$, $v_1'(x_1) = 4x_1$. By Remark 10, $R_1^{inv} = R_1$. Thus, by the definition of the inverse Vickrey rule, $f_1(R_1', R_2) = (2,5)$. Note that $t_1(R) = 2 < 3 = \min P(1; R_1)$.[21] Thus, agent 1's payment is smaller than his inverse demands of $\frac{m-1}{2} = 1$ unit. Then, by $2 < 3$ and $R_1 \in \mathcal{R}^{++}$, Remark 5 (i) implies

$$V_1(2,(1,2)) - 2 > V_1(2,(1,3)) - 3 = v_1(2,(1,3)) - v_1(1,(1,3)) = 3,$$

or equivalently, $V_1(x_1(R_1', R_2), f_1(R)) > 5 = t_1(R_1', R_2)$. This implies $f_1(R_1', R_2)\ P_1\ f_1(R)$. Thus, agent 1 benefits from misrepresenting his preference so as to win one more unit.

4.2 An even number of units

In contrast to the case of an odd number of units, when m is even, the results depend on the size of the income effects of a preference $R_0 \in \mathcal{R}^D \cap (\mathcal{R}^{++} \cup \mathcal{R}^{--})$. Throughout the subsection, we assume that m is even. In this subsection, we present the result only for the case of positive income effects. The result for the case of negative income effects can be found in Appendix C.

Given $R_i \in \mathcal{R}$ and $(x_i, t_i) \in (M\backslash\{m\}) \times \mathbb{R}$, the **willingness to buy at** (x_i, t_i) **for** R_i is $WB_i(x_i, t_i) \equiv V_i(x_i + 1, (x_i, t_i)) - t_i$. Given $R_i \in \mathcal{R}$ and $(x_i, t_i) \in (M\backslash\{0\}) \times \mathbb{R}$, the **willingness to sell at** (x_i, t_i) **for** R_i is $WS(x_i, t_i) \equiv t_i - V_i(x_i - 1, (x_i, t_i))$. Given $R_i \in \mathcal{R}$, a pair $t_i, t_i' \in \mathbb{R}$, and $x_i \in M\backslash\{m\}$, the **size of income effects of** x_i **between** t_i **and** t_i' **for** R_i is measured by $WB_i(x_i, t_i) - WB_i(x_i, t_i')$.

A preference $R_i \in \mathcal{R}^{NI} \cap \mathcal{R}^+$ has the **upper bound for the nonnegative income effects** if it satisfies the following inequality:

$$WB_i(\beta, t^*) - WB_i(\beta, \beta p(\beta; R_i)) \le WS_i(\beta, \beta p(\beta; R_i)) - WB_i(\beta, \beta p(\beta; R_i)), \tag{UB}$$

where $\beta \equiv \frac{m}{2}$ and $t^* \equiv \sum_{x=0}^{\beta-1} p(m - x; R_i)$.

Note that the bundle (β, t^*) is an outcome of the inverse Vickrey rule for the preference profile (R_i, R_i) in the case of two agents. The RHS of (UB) is the difference between the willingness to sell and the willingness to buy at the bundle $(\beta, \beta p(\beta; R_i))$ for R_i. Since R_i exhibits nonincreasing incremental valuations, this difference is nonnegative. The LHS of (UB) is the size of the income effects of β between t^* and $\beta p(\beta; R_i)$ for R_i. Thus, the inequality (UB) requires that the size of the income effects of x_i between t^* and $\beta p(\beta; R_i)$ be bounded by the difference between the willingness to sell and the willingness to buy at the bundle $(\beta, \beta p(\beta; R_i))$. We interpret it as the small size of the nonnegative income effects.

[21] To see that $3 = \min P(1; R_1)$, note first that by footnote 19, $3 \in P(1; R_1)$. Suppose by contradiction that there is $p \in P(1; R_1)$ such that $p < 3$. Then, $V_1(2,(1,p)) - p = v_1(2,(1,p)) - v_1(1,(1,p)) = -\frac{p}{3} + 4 > p$. This implies $V_1(2,(1,p)) > 2p$. Thus, $(2, 2p)\ P_1\ (1,p)$. This contradicts $p \in P(1; R_1)$. Thus, for each $p \in P(1; R_1)$, $p \ge 3$.

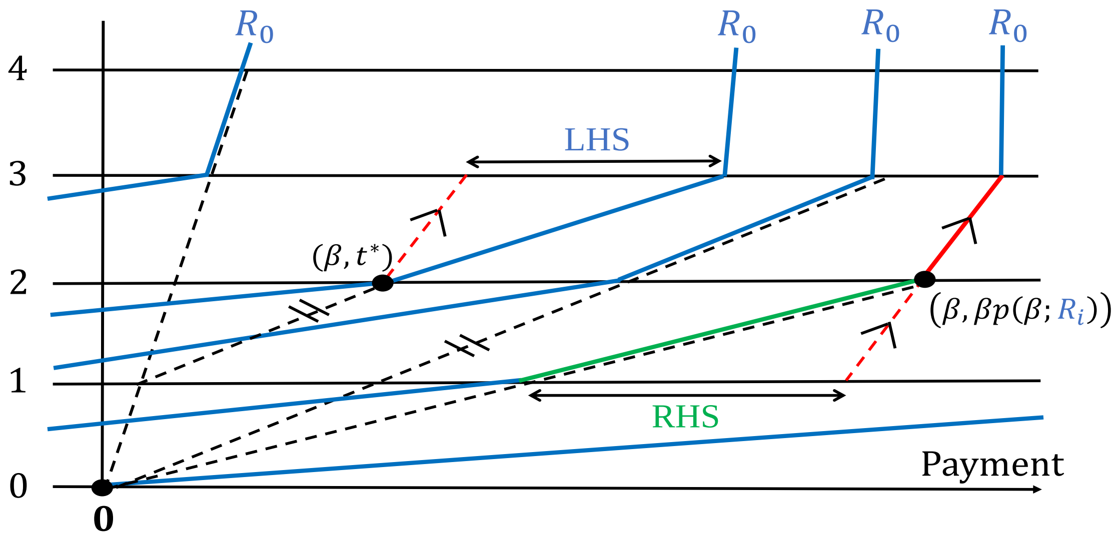

Figure 6: The upper bound for the nonnegative income effects.

Figure 6 is an illustration of a preference that has the upper bound for the nonnegative income effects. RHS (resp. LHS) in Figure 6 corresponds to the RHS (resp. the LHS) of the inequality (UB).

The next proposition states that when m is even, (i) if $n = 2$, and $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ has the upper bound for the nonnegative income effects, then on the domain $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^2$, the inverse Vickrey rule satisfies the four properties, and (ii) if R_0 does not have the upper bound for the nonnegative income effects, and a rich class of preferences includes R_0, then no rule satisfies the four properties.[22][23]

Proposition 1. *Assume m is even. Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$.*
(i) Assume $n = 2$. Assume R_0 has the upper bound for the nonnegative income effects. An inverse Vickrey rule on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^2$ satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness.
(ii) Assume R_0 does not have the upper bound for the nonnegative income effects. Let $\mathcal{R}$ be rich and $R_0 \in \mathcal{R}$. No rule on $\mathcal{R}^n$ satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness.

The proof of Proposition 1 can be found in the supplementary material.

Recall that we interpret the inequality (UB) as the small size of the nonnegative income effects. Thus, Proposition 1 states that if the size of the positive income effects of a preference added to $\mathcal{R}^C \cap \mathcal{R}^Q$ is sufficiently small (resp. large), then there exists (resp. does not exist) a rule satisfying the four properties. Note that Proposition 1 implies that the upper bound for the nonnegative income effects is necessary and sufficient for the existence of a rule satisfying the four properties on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^2$, where $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$.

[22] Given Proposition 1 (i), one might expect that when $n \geq 3$, the inverse Vickrey rule on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^n$ satisfies the four properties for some $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$. However, when $n \geq 3$ and $m = 6a - 2$ for some $a \in \mathbb{N}$ with $a \geq 2$, it violates *strategy-proofness* for *each* $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$. See Example 9 in the supplementary material.

[23] In Proposition 1 (i), we can further show that the inverse Vickrey rule is the only rule satisfying the four properties by applying Step 3 of the proof of Theorem 1.

Recall the intuition for the proof of Theorem 1 that under the inverse Vickrey rule, an agent with income effects benefits from misrepresenting his preference in the case of an odd number of units. The intuition for the proof of Proposition 1 (i) is that when the number of units is an even number, if the size of the positive income effects is small, then no agent benefits from misrepresenting his preference, and the inverse Vickrey rule satisfies the four properties. In contrast, the intuition for the proof of Proposition 1 (ii) is that if the size of the income effects is sufficiently large, then an agent with positive income effects benefits from misrepresenting his preference so as to win one more unit as in Theorem 1.

4.3 Expanding our minimal domain

In this section, to further explore the existence of a rule satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*, we expand our minimal domain. Then, we ask whether there is a rule satisfying the four properties on a domain that contains the expanded minimal domain and includes preferences with decreasing incremental valuations and either positive or negative income effects.

To slightly expand our minimal domain, we introduce preferences which exhibit almost constant incremental valuations. Given $\varepsilon > 0$, a preference R_i **exhibits ε-nonincreasing incremental valuations** if for each $z_i \in M \times \mathbb{R}$ and each $x_i \in M \setminus \{0, m\}$,

$$0 \leq v_i(x_i, z_i) - v_i(x_i - 1, z_i) - (v_i(x_i + 1, z_i) - v_i(x_i, z_i)) < \varepsilon.$$

Given $\varepsilon > 0$, a preference R_i **exhibits ε-nonincreasing incremental valuations** if for each $z_i \in M \times \mathbb{R}$ and each $x_i \in M \setminus \{0, m\}$,

$$0 \leq v_i(x_i + 1, z_i) - v_i(x_i, z_i) - (v_i(x_i, z_i) - v_i(x_i - 1, z_i)) < \varepsilon.$$

Given $\varepsilon > 0$, let $\mathcal{R}^{NI}(\varepsilon)$ and $\mathcal{R}^{ND}(\varepsilon)$ denote the classes of ε-nonincreasing and ε-nondecreasing incremental valuations, respectively. For each $\varepsilon > 0$, $\mathcal{R}^{NI}(\varepsilon) \subsetneq \mathcal{R}^{NI}$ and $\mathcal{R}^{ND}(\varepsilon) \subsetneq \mathcal{R}^{ND}$. For each $\varepsilon > 0$, it holds that $\mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^{ND}(\varepsilon) = \mathcal{R}^{C}$. Moreover, both $\mathcal{R}^{NI}(\varepsilon)$ and $\mathcal{R}^{ND}(\varepsilon)$ converges to $\mathcal{R}^{C}$ as ε goes to 0, that is, $\bigcap_{\varepsilon \in \mathbb{R}_{++}} \mathcal{R}^{NI}(\varepsilon) = \bigcap_{\varepsilon \in \mathbb{R}_{++}} \mathcal{R}^{ND}(\varepsilon) = \mathcal{R}^{C}$.

Given $\varepsilon > 0$, a class of preferences $\mathcal{R}$ is **NI(ε)-rich** (resp. **ND(ε)-rich**) if $\mathcal{R} \supseteq \mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^{Q}$ (resp. $\mathcal{R} \supseteq \mathcal{R}^{ND}(\varepsilon) \cap \mathcal{R}^{Q}$). We call the domain $(\mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^{Q})^n$ (resp. $(\mathcal{R}^{ND}(\varepsilon) \cap \mathcal{R}^{Q})^n$) the **NI($\varepsilon$)-minimal domain** (resp. the **ND(ε)-minimal domain**).

The next proposition states that for each $\varepsilon > 0$, if a class of preference is either NI(ε)-rich or ND(ε)-rich, and includes at least one preference that exhibits decreasing incremental valuations and either positive or negative income effects, then no rule satisfies the four properties on the domain *regardless of the number of units*.

Proposition 2. *Let $R_0 \in \mathcal{R}^{D} \cap (\mathcal{R}^{++} \cup \mathcal{R}^{--})$. Let $\varepsilon \in \mathbb{R}_{++}$. Let $\mathcal{R}$ be either NI(ε)-rich or ND(ε)-rich, and $R_0 \in \mathcal{R}$. No rule on $\mathcal{R}^n$ satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness.*

In Proposition 2, we allow $\varepsilon > 0$ to be arbitrarily small. Thus, even when both the NI(ε)-minimal and ND(ε)-minimal rich domains have almost the same variations of preferences as our minimal domain $(\mathcal{R}^C \cap \mathcal{R}^Q)^n$, for any $R_0 \in \mathcal{R}^D \cap (\mathcal{R}^{++} \cup \mathcal{R}^{--})$, both on $((\mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\})^n$ and on $((\mathcal{R}^{ND}(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\})^n$, no rule satisfies the four properties. This means that the positive result in the case of an even number of units (Proposition 1 (i)) is vulnerable to a minimal increase of the variations of preference.

We emphasize that for each $\varepsilon > 0$, both $\mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q$ and $\mathcal{R}^{ND}(\varepsilon) \cap \mathcal{R}^Q$ include only quasi-linear preferences, that is, $\mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q \subseteq \mathcal{R}^Q$ and $\mathcal{R}^{ND}(\varepsilon) \cap \mathcal{R}^Q \subseteq \mathcal{R}^Q$. Thus, both on $(\mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q)^n$ and on $(\mathcal{R}^{ND}(\varepsilon) \cap \mathcal{R}^Q)^n$, the Vickrey rule satisfies the four properties (Vickrey, 1961; Holmström, 1979). As with Theorem 1, Proposition 2 implies that adding at least one non-quasi-linear preference with decreasing incremental valuations to domains on which there is a rule satisfying the four properties leads to an impossibility theorem.

As in the proof of Theorem 1, the inverse Vickrey rule plays a key role in the proof of Proposition 2. In order to illustrate the proof of Proposition 2, for a given $\varepsilon > 0$, we here focus on the case of two agents and an NI(ε)-rich class of preferences that includes a preference exhibiting decreasing incremental valuations and positive income effects. Consider a preference profile such that agent 1 has a preference that exhibits positive income effects and agent 2 has a quasi-linear preference that exhibits ε-nonincreasing incremental valuations. As in the proof of Theorem 1, we show in the proof that the outcome of a rule satisfying the four properties coincides with that of the inverse Vickrey rule at the preference profile. Then, we show the outcome of the rule is not efficient. This contrasts with the proof of Theorem 1, where we show that an agent with non-quasi-linear preference benefits from misrepresenting his preference. To illustrate this point, we consider the following example.

Example 4. Let $n = 2$ and $m = 2$. Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ be such that (i) $v_0(1, \mathbf{0}) = 4$, and (ii) for each $t_0 \in [0, 4]$, $v_0(2, (1, t_0)) - v_0(1, (1, t_0)) = -\frac{1}{2}t_0 + 3$.[24] Let $\varepsilon = 0.05$. Let $\mathcal{R}$ $(\subseteq \mathcal{R}^{NI})$ be NI(ε)-rich and $R_0 \in \mathcal{R}$. Let f be an inverse Vickrey rule on $\mathcal{R}^n$. Let $R_2 \in \mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q$ be such that $v_2(1) = 2.01$ and $v_2(2) - v_2(1) = 1.97$. Let $R = (R_0, R_2) \in \mathcal{R}^2$. Note that $p(1; R_1) = v_1(1, \mathbf{0}) = 4$. By solving the equation $v_1(2, (1, t_1)) - v_1(1, (1, t_1)) = t_1$ for t_1, we get $t_1 = 2$. Then, we can show $p(2; R_1) = 2$ in the same way as in Example 2. Thus, $v_1^{inv}(1) = 4$ and $v_1^{inv}(2) - v_1^{inv}(1) = 2$. By Remark 10, $R_2^{inv} = R_2$. By the definition of the inverse Vickrey rule, $f(R) = ((1, 1.97), (1, 2))$. Then,

$$v_1(2, f_1(R)) - v_1(1, f_1(R)) - v_2(1) = (-0.985 + 3) - 2.01 = 2.015 - 2.01 > 0.$$

Thus, by $R \in (\mathcal{R}^{NI})^2$, Remark 8 implies $f(R)$ is not efficient for R.

In the above example, if a preference R_2 exhibits constant incremental valuations, and agent 1 wins $m - 1$ units under the inverse Vickrey rule, then agent 1's payment is in

[24] Note that R_0 has the upper bound for the nonnegative income effects. Thus, by Proposition 1 (i), the inverse Vickrey rule satisfies the four properties on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^2$.

$[2, 4]$, which is greater than his actual payment 1.97 in the above example. Thus, the above example illustrates that the ε-nonincreasing incremental valuations make agent 1's payment relatively small, and then positive income effects enable us to Pareto improve the outcome allocation of the inverse Vickrey rule. Although an outcome of a rule satisfying the four properties does not necessarily coincide with that of the inverse Vickrey rule if $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{--}$ or $\mathcal{R}$ is ND(ε)-rich (note that the inverse Vickrey rule is defined only for preferences with nonincreasing incremental valuations), in the proof of Proposition 2, we show that the Pareto improvement is possible when agents have the preferences as in the above example.

Corollary 1. *Let $R_0 \in \mathcal{R}^D \cap (\mathcal{R}^{++} \cup \mathcal{R}^{--})$.*
(i) Let $\mathcal{R}$ be such that $\mathcal{R}^{NI} \cap \mathcal{R}^Q \subseteq \mathcal{R} \subseteq \mathcal{R}^{NI}$. No rule on $(\mathcal{R} \cup \{R_0\})^n$ satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness.
(ii) Let $\mathcal{R}$ be such that $\mathcal{R}^{ND} \cap \mathcal{R}^Q \subseteq \mathcal{R} \subseteq \mathcal{R}^{ND}$. No rule on $(\mathcal{R} \cup \{R_0\})^n$ satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness.

Corollary 1 (i) and (ii) do not follow from the existing results such as Baisa (2020). Corollary 1 (i) implies that both on $(\mathcal{R}^{NI} \cap \mathcal{R}^{+})^n$ and on $(\mathcal{R}^{NI} \cap \mathcal{R}^{-})^n$, no rule satisfies the four properties. The impossibility on $(\mathcal{R}^{NI} \cap \mathcal{R}^{+})^n$ also follows from the result by Baisa (2020), but that on $(\mathcal{R}^{NI} \cap \mathcal{R}^{-})^n$ is a new result.

Corollary 1 has some implications. As we argued in Section 1.3, both the assumptions of nonincreasing incremental valuations and quasi-linear preferences are standard in the literature on multi-unit object allocation problems with money. Corollary 1 (i) implies that if we investigate the existence of a rule satisfying the four properties without quasi-linearity while keeping the assumption of nonincreasing incremental valuations, we may immediately encounter an impossibility result. Thus, the assumption of quasi-linear preferences is an important source of the existence of a rule satisfying the four properties. This may contrast with the results of Baisa (2020), which suggest that the deviation from the assumption of the single-dimensional types leads to an impossibility result in a non-quasi-linear environment with nonincreasing incremental valuations.[25]

The new entry into some industries such as vaccine and electricity industries requires a huge invest in new plants and equipment, and so new entrants typically have technologies that exhibit nondecreasing returns to scale. Thus, new entrants in such industries have preferences with nondecreasing incremental valuations. We will show in the next section that if agents have preferences with nondecreasing incremental valuations (with or without quasi-linearity), then the generalized Vickrey rule satisfies the four properties (Theorem 2 in Section 5). In contrast to new entrants, incumbents have already paid the large initial costs, and thus, may have technologies that exhibit decreasing returns to scale. Thus, it may be reasonable to assume that incumbents have preferences with decreasing incremental valuations. Corollary 1 (ii) suggests that in such an industry, if agents have non-quasi-linear preferences, then the existence of a rule satisfying the four properties is guaranteed only when there is no incumbent, that is, all the agents are new entrants.

[25]For the details of the results of Baisa (2020), see Section 6.1.

5 Nondecreasing incremental valuations

In this section, we turn to the situation where agents have preferences that exhibits nondecreasing incremental valuations. Note that nondecreasing incremental valuations correspond to the technology of a firm that exhibits nondecreasing returns to scale, which is a natural assumption in some industries of sufficiently large sunk costs.

The following theorem states that if a rich class of preferences includes only preferences that exhibit nondecreasing incremental valuations, then the generalized Vickrey rule is the only rule satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*.

Theorem 2. *Let $\mathcal{R}$ be rich and $\mathcal{R} \subseteq \mathcal{R}^{ND}$. A rule on $\mathcal{R}^n$ satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness if and only if it is a generalized Vickrey rule on $\mathcal{R}^n$.*

Since a domain in Theorem 2 might include non-quasi-linear preferences, the existence of a rule satisfying the four properties is not trivial. Indeed, Theorem 2 is not an immediate consequence of the existing results on the quasi-linear domain (Holmström, 1979; Chew and Serizawa, 2007).

The independence of properties in Theorem 2 is demonstrated in the examples below. In the examples, we fix a rich $\mathcal{R} \subseteq \mathcal{R}^{ND}$.

Example 5 (Dropping efficiency). Let f be the *no-trade rule* on $\mathcal{R}^n$ such that each agent receives $\mathbf{0}$ for each preference profile. Then f satisfies all the properties in Theorem 2 other than *efficiency*.

Example 6 (Dropping individual rationality). Let f be the generalized Vickrey rule with fixed and common entry fee $e > 0$ on $\mathcal{R}^n$. Then f satisfies all the properties in Theorem 2 other than *individual rationality*.

Example 7 (Dropping no subsidy for losers). Let f be the generalized Vickrey rule with fixed and common participation subsidy $s < 0$ on $\mathcal{R}^n$. Then f satisfies all the properties in Theorem 2 other than *no subsidy for losers*.

Example 8 (Dropping strategy-proofness). Let $f \equiv (x, t)$ be the *generalized pay-as-bid rule* on $\mathcal{R}^n$ such that for each preference profile $R \in \mathcal{R}^n$, (i) the object is allocated so as to maximize the sum of net valuations at $\mathbf{0}$, and (ii) each agent has to pay his net valuation of $x_i(R)$ at $\mathbf{0}$. By Remark 9, f satisfies *efficiency*. Further, it satisfies *individual rationality* and *no subsidy for losers*, but violates *strategy-proofness*.

The key intuition for Theorem 2 is that if preferences exhibit nondecreasing incremental valuations, then a *bundling efficient allocation* at which an agent who has the highest net valuation of m receives all the units is efficient, and the situation is close to the single-object environment.[26] This contrasts with the case of decreasing incremental valuations where

[26] Let $\overline{X} \equiv X \cap \{0, m\}^n$. An allocation $z \equiv (x, t) \in Z$ is a *bundling allocation* if $x \in \overline{X}$. Let $\overline{Z}$ denote the set of feasible bundling allocations. Given $R \in \mathcal{R}^n$, an allocation $z \in Z$ is *bundling efficient for* R if $z \in \overline{Z}$, and there is no other bundling allocation $z' \in \overline{Z}$ that Pareto dominates z for R.

several agents typically win the object at an efficient allocation. Here, the nondecreasing per-unit net valuations (Remark 3) play an important role as it make the above allocation efficient, and we repeatedly exploit the property in our proof.

As we stated in Section 1.3, our characterization theorem (Theorem 2) cannot be obtained simply by bundling all the units and applying the characterization of the generalized Vickrey rule in the single-object environment (Saitoh and Serizawa, 2008; Sakai, 2008) although a bundling efficient allocation is an efficient allocation. In what follows, we pursue the implications of their results in the single-object environment, and discuss that they do not lead us to a characterization of the class of rules satisfying the four properties, which is one of our goals in this paper.

First, note that the results by Saitoh and Serizawa (2008) and Sakai (2008) in the single-object environment imply that if a class of preferences is rich, then the bundling second-price rule is the only rule satisfying *bundling efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness* (see the bottom arrow in Figure 7 below).[27] Note also that when preferences exhibit nondecreasing incremental valuations, *bundling efficiency* implies *efficiency* (see Lemma 13 in Appendix D). Thus, their results imply that there is a rule satisfying our four properties.

However, the converse is not true, that is, *efficiency* does not necessarily imply *bundling efficiency*. To illustrate this point, consider a preference profile such that two agents have the same quasi-linear preference with constant incremental valuations, and their net valuations of m are the highest. For such a preference profile, by the first condition (i) of the generalized Vickrey rule, any object allocation maximizes the sum of valuations at $\mathbf{0}$ if the two agents share all the units, and so the generalized Vickrey rule violates *bundling efficiency* although it satisfies *efficiency* (by Theorem 2).[28] Thus, the class of rules satisfying *bundling efficiency* and the other three properties is a *proper subset* of the class of rules satisfying our four properties (see the right $\subsetneq$ relation in Figure 7).

In summary, the results by Saitoh and Serizawa (2008) and Sakai (2008) can be used (i) to show that there is a rule satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*, and (ii) to obtain a characterization of the class of rules satisfying an ad hoc property of *bundling efficiency* and the other three properties. However, it does not provide a characterization of the class of rules satisfying standard *efficiency* and the other three properties. Thus, the proof of our characterization theorem inevitably treats non-bundling efficient allocations and non-bundling generalized Vickrey rules, and requires a different proof strategy from applying the characterization of the generalized Vickrey rule in the single-object environment.

[27] A rule $f \equiv (x, t)$ on $\mathcal{R}^n$ is a *bundling second-price rule* if for each $R \in \mathcal{R}^n$, the following two conditions hold: (i) $x(R) \in \arg\max_{x \in \overline{X}} \sum_{i \in N} v_i(x_i, \mathbf{0})$, and (ii) for each $i \in N$, $t_i(R) = \max_{j \in N \setminus \{i\}} v_j(x_i(R), \mathbf{0})$. A rule f on $\mathcal{R}^n$ satisfies *bundling efficiency* if for each $R \in \mathcal{R}^n$, $f(R)$ is bundling efficient for R.

[28] Note that the generalized Vickrey rule violates *bundling efficiency*, that is, it produces a different outcome from the bundling second-price rule, only when the preferences of the agents whose net valuations of m at $\mathbf{0}$ are the highest exhibit constant incremental valuations at $\mathbf{0}$.

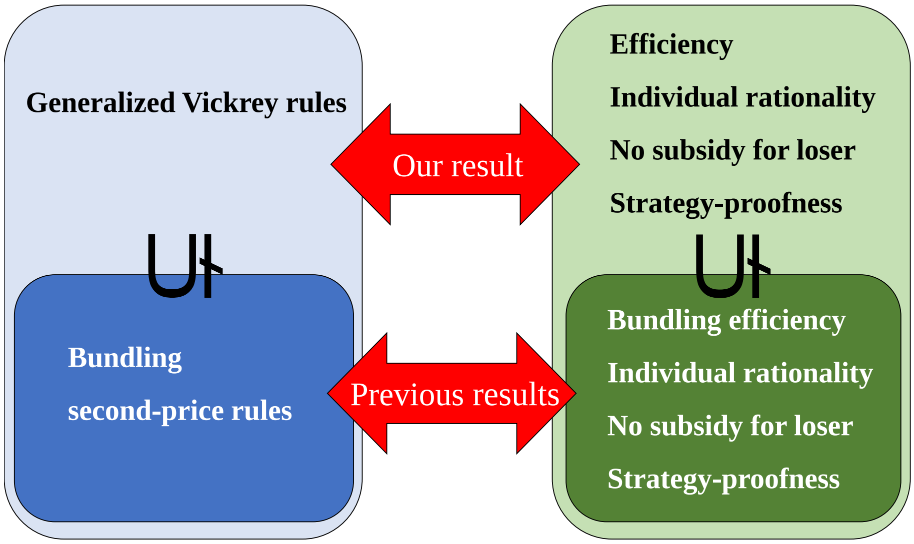

Figure 7: Our result (Theorem 2) and the previous results by Saitoh and Serizawa (2008) and Sakai (2008)

6 Discussion

6.1 Comparison to Baisa (2020)

We compare our results with the related results obtained by Baisa (2020). He considers the multi-unit object allocation problem with money as in this paper, and obtains the results for preferences exhibiting decreasing incremental valuations and positive income effects (or nonnegative income effects). He parameterizes preferences by type.

6.1.1 Positive results

First, we compare our characterization theorem (Theorem 2) with his positive results. His first two results (Theorems 1 and 2 of Baisa (2020)) are concerned with the existence of a rule satisfying *efficiency*, *individual rationality*, *no susbidy for losers*, and *strategy-proofness*. In Theorem 1 of Baisa (2020), he shows that for a case of two agents and an arbitrary number of units, if preferences are of single-dimensional types, then on the domain with decreasing incremental valuations, nonnegative income effects, and the single-crossing property, there is a rule satisfying the four properties.[29] He further shows that for a case of an arbitrary number of agents and two units of the object, under the assumption of single-dimensional types, there is a rule satisfying the four properties on the same domain as in his Theorem 1 (Theorem 2 of Baisa (2020)). Note that the rules in his positive results are equivalent neither to the generalized Vickrey rule nor to the inverse Vickrey rule.

Firstly and most importantly, his results are for preferences with decreasing incremental valuations, but our result is for preferences with nondecreasing incremental valuations. This

[29]A class of preferences exhibits *single-crossing property* if the incremental valuation at each bundle is increasing in a type.

means that the two results apply to different environments. Secondly, his results impose assumptions on the number of agents or units. Thirdly, they are for the case of preferences of single-dimensional types. Our result is free from those assumptions. However, his results involve technical discussions, such as a fixed-point argument, to show that the Vickrey rule can be generalized in a quite novel way so that it satisfies the desirable properties. In contrast, we characterize a natural generalization of the Vickrey rule by only elementary argument. Clearly, our result neither implies nor is implied by his results.

6.1.2 Negative result

Baisa (2020) also shows that if a class of preferences admits multi-dimensional types, then on the domain with decreasing incremental valuations, positive income effects, and the single-crossing property, no rule satisfies *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. (Theorem 3 of Baisa (2020)).[30] This result is different from ours (Theorem 1 and Proposition 2) in terms of domains. First, we emphasize that our domains in Theorem 1 and Proposition 2 neither contain nor are contained by the domain of his result. Thus, our results do not imply his result, and vice versa. Second, Baisa (2020) focuses on preferences with positive income effects, but we consider not only preferences with positive income effects, but also those with negative income effects.

6.1.3 Overall comparison

Baisa (2020) focuses on the case of decreasing incremental valuations and positive income effects. In contrast, our results cover the case of decreasing incremental valuations or nondecreasing incremental valuations, and that of positive or negative income effects. Thus, in principle, our results cover a broader class of environment than his results. However, none of his results follow from our results, and vice versa.

As a whole, the results of Baisa (2020) suggest that when preferences exhibit decreasing incremental valuations and positive income effects, the existence of a rule satisfying the four properties is guaranteed if and only if preferences are of single-dimensional types. In contrast, our results suggest that when preferences exhibit (either positive or negative) income effects, then the existence of such a rule is guaranteed if and "almost" only if preferences exhibit nondecreasing incremental valuations. Thus, our results complement the results of Baisa (2020) by drawing a different insight of the existence of a rule satisfying the four properties.

[30]To be precise, he shows that no rule satisfies *efficiency*, *individual rationality*, *no deficit, and strategy-proofness.*. A rule f on $\mathcal{R}^n$ satisfies *no deficit* if for each $R \in \mathcal{R}^n$, we have $\sum_{i \in N} t_i(R) \geq 0$. However, his proof can be directly applied to the proof of the impossibility theorem stated in the body.

6.2 Comparison to Malik and Mishra (2021)

Malik and Mishra (2021) consider the heterogeneous objects model, and study dichotomous preferences.[31] Compared to Malik and Mishra (2021), this paper contributes to the literature by treating the identical objects model, which applies to multi-unit auctions of practical importance. Below, we compare our results and proof strategies with theirs in detail.

6.2.1 Positive result

First, we compare our positive result (Theorem 2) with their positive result (Theorem 3 of Malik and Mishra (2021)). They show that on a domain that contains the quasi-linear dichotomous domain and is contained by the dichotomous domain with nonnegative income effects, the generalized Vickry rule is the only rule satisfying *efficiency*, *individual rationality*, *no subsidy*, and *strategy-proofness*.[32]

The key condition for their positive result is the assumption of nonnegative income effects. Indeed, the payments under the generalized Vickrey rule are quite small at some preference profile in their setting, and if agents have preferences with negative income effects, then the winners' willingness to sell of the objects that they obtain becomes smaller than the losers' willingness to buy of the objects, and a Pareto improvement among the winners and the losers is possible at an outcome of the generalized Vickrey rule. Thus, if preferences exhibit negative income effects, then the generalized Vickrey rule violates *efficiency*. Malik and Mishra (2021) observe that under the assumption of nonnegative income effects, such a Pareto improvement is impossible, and the generalized Vickrey rule satisfies *efficiency* together with the other desirable properties.

In contrast, the key intuition for our positive result is that preferences with nondecreasing incremental valuations (in particular, nondecreasing per-unit net valuations in Remark 3) and the implication of *efficiency* make the situation close to the single-object environment where we obtain a positive result (Saitoh and Serizawa, 2008; Sakai, 2008). In particular, unlike in Malik and Mishra (2021), our positive result holds without the assumption of nonnegative income effects. Thus, the proof strategy of our positive result is different from theirs.

6.2.2 Negative result

Next, we compare our negative results (Theorem 1 and Proposition 2) with their negative result (Theorem 4 of Malik and Mishra (2021)). They establish that if a class of preferences includes all quasi-linear dichotomous preferences and at least one non-quasi-linear and non-dichotomous preference (precisely, a heterogeneous demand preference with positive income

[31] A preference is *dichotomous* if the set of packages is divided into the acceptable set and the unacceptable set, and given a payment, each acceptable package has the same positive value, but each unacceptable one is valueless

[32] Note that the GVCG mechanism in Malik and Mishra (2021) is equivalent to the generalized Vickrey rule in this paper.

effects and decreasing incremental valuations), then no rule satisfies *efficiency*, *individual rationality*, *no subsidy*, and *strategy-proofness*. Note that our negative results and theirs have the same formats: both the results add at least one preference to a domain where there is a rule satisfying the desirable properties, and show that no rule satisfies the properties on the expanded domain.

Here, we discuss that our proof strategy is different from theirs.

First, the domain of their result includes quasi-linear preferences together with a preference with positive income effects. In contrast, the domains of our results may include a preference with negative income effects. Because of this difference, we cannot apply their proof strategy directly to our results.

Second, the main step of their proof is to show that at some preference profile, the outcome of a rule satisfying their four properties coincides with the outcome of a *unique* natural extension of the rule satisfying the properties on their minimal domain to non-quasi-linear domains: the generalized Vickrey rule. (see Step 1 of the proof of Theorem 4 of Malik and Mishra (2021)).[33] We cannot follow their proof strategy because on our minimal domain $(\mathcal{R}^C \cap \mathcal{R}^Q)^n$, we have *at least two* natural extensions of the rule satisfying the four properties to non-quasi-linear domains: the generalized Vickrey rule and the inverse Vickrey rule. In our view, there is no reason why we expect that one of the two rules is dominant in our model, and the outcome of a rule satisfying the four properties must be equivalent to that of the rule. In the proof, we show that at some preference profile, the outcome of a rule satisfying the four properties coincides not with that of the generalized Vickrey rule, but with that of the inverse Vickrey rule.

6.3 Comparison to Kazumura and Serizawa (2016)

Kazumura and Serizawa (2016) consider the heterogeneous objects model as in Malik and Mishra (2021), and establish that if the number of agents is greater than that of objects, and a class of preferences includes all unit-demand preferences and at least one multi-demand preference, then no rule satisfies *efficiency*, *individual rationality*, *no subsidy*, and *strategy-proofness*. Since on the unit-demand domain, the minimum price Walrasian rule satisfies the four properties (Demange and Gale, 1985; Morimoto and Serizawa, 2015), and they show that adding at least one preference to the domain leads to an impossibility theorem, their result has the same form as our negative results (Theorem 1 and Proposition 2). As with Malik and Mishra (2021), the contribution of this paper compared with their result is that our results apply to practically important environments to which their result does not apply, that is, multi-unit auctions.

In their proof, unit-demand preferences that exhibit negative income effects play an important role (see Step 1 of the proof of Theorem of Kazumura and Serizawa (2016) for the construction of preferences). For a preference profile where one agent has the multi-demand preference and other agents have preferences with negative income effects, they

[33] Note that in their setting with heterogeneous objects, it is not clear how we define the inverse Vickrey rule.

exploit negative income effects to show that the sum of the willingness to sell for the unit-demand agents is less than the willingness to buy for the multi-demand agent, and a Pareto improvement is possible (see Step 6 of the proof of their result).[34] In contrast, our proof makes use of the positive (resp. negative) income effects to show that an agent with a preference exhibiting positive (resp. negative) income effects benefits from misrepresenting his preference so that he wins one more (resp. fewer) unit. Thus, our proof strategy is different from theirs.[35]

6.4 Minimal domains

The larger the domain of rules, the stronger the implications of the properties of rules on the domain. Unless domains are rich enough, the implications of the properties are too weak to yield meaningful conclusions. Indeed, since the beginning of mechanism design theory and social choice theory, authors assume rich domains to establish characterization or impossibility theorems (e.g., Arrow, 1951; Hurwicz 1972; Holmström, 1979; Moulin 1980, etc.). However, if a domain is so large that it includes even non-natural preferences, the conclusions from properties on the domain can be applied only to limited situations. This motivates the concept of minimal domains. Many authors explicitly or implicitly assume that domains include their respective minimal domains.

Kazumura and Serizawa (2016) and Malik and Mishra (2021) are such examples in the literature on object allocation problems with money. The minimal domain of the first corresponds to the *unit-demand domain*, and that of the second to the *quasi-linear dichotomous domain*. In these papers, for a given package of objects, their respective minimal domains include a preference such that the valuation of the package is sufficiently large, but those for other packages are small. Then, the property of *efficiency* implies that an agent who has such a preference should get the given package. In a similar way, it is possible to construct a preference profile in their minimal domains such that the implications of the properties pin down the object allocation (see, e.g., Claim 5 in the proof of Theorem 4 of Malik and Mishra (2021)). In the proofs of both the papers, such a construction of preference profiles plays an essential role.

Remember that our minimal domain includes only quasi-linear preferences with constant incremental valuations.[36] This weak requirement of our minimal domain makes it possible to apply our results to various situations such as the case of nonincreasing incremental valuations and that of nondecreasing incremental valuations. At the same time, it also makes the property of *efficiency* unable to pin down the units an agent receives, and only implies that an agent receives all the units or nothing. Thus, the proof technique

[34] Precisely, when n is the number of agents and m is the number of heterogeneous objects, each agent $i \in \{2, \ldots, m\}$ has a preference with negative income effects, and each agent $i \in \{m+1, \ldots, n\}$ has a quasi-linear preference in their proof.

[35] The proof strategy of Proposition 2 is also different from their proof strategy because the proof of Proposition 2 does not rely on preferences with negative income effects, and applies not only to preferences with negative income effects, but also to those with positive income effects as illustrated in Example 4.

[36] Note that domains in Proposition 2 include a little bit more preferences.

as in Kazumura and Serizawa (2016) and Malik and Mishra (2021) does not work in our environment, and our proofs need to overcome such weak implications of properties. This point makes our proofs challenging.

7 Conclusion

We have considered the multi-unit object allocation problem with money. The distinguishing feature of our model is to allow agents to have preferences that may not be quasi-linear. Moreover, our results cover broad class of situations not only where preferences exhibit nonincreasing incremental valuation, but also where preferences exhibit nondecreasing incremental valuations. We have established that when preferences exhibit nonincreasing incremental valuations, the existence of a rule satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness* depends both on the number of units and on the size of the income effects. In contrast, when preferences exhibit nondecreasing incremental valuations, the generalized Vickrey rule is the only rule satisfying the four properties. Our results suggest that (i) the existence of a rule satisfying the four properties is guaranteed "almost" only when preferences exhibit nondecreasing incremental valuations, and (ii) the existence of such a rule depends not only on the properties of preferences such as nondecreasing incremental valuations and the size of the income effects, but also other characteristics of the environment such as the number of units.

Appendix

A Preliminaries

In this section, we provide some basic lemmas that will be used to prove the results.

The proofs of all the lemmas in this section are trivial, and we omit those.

The following lemma immediately follows from efficiency and object monotonicity.

Lemma 1 (No remaining object). *Let $R \in \mathcal{R}^n$ and $z \equiv (x, t) \in Z$ be efficient for R. Then $\sum_{i \in N} x_i = m$.*

The next lemma is immediate from *individual rationality* and *no subsidy for losers*.

Lemma 2 (Zero payment for losers). *Let $f \equiv (x, t)$ be a rule on $\mathcal{R}^n$ satisfying individual rationality and no subsidy for losers. Let $R \in \mathcal{R}^n$ and $i \in N$. If $x_i(R) = 0$, then $t_i(R) = 0$.*

The following lemma gives implications of *individual rationality*.

Lemma 3. *Let $f \equiv (x, t)$ be a rule on $\mathcal{R}^n$ satisfying individual rationality. Let $R \in \mathcal{R}^n$ and $i \in N$. We have (i) $V_i(0, f_i(R)) \leq 0$, (ii) $t_i(R) \leq V_i(x_i(R), \mathbf{0}) = v_i(x_i(R), \mathbf{0})$, and (iii) for each $x_i \in M$, $V_i(x_i, f_i(R)) \leq V_i(x_i, \mathbf{0}) = v_i(x_i, \mathbf{0})$.*

Let f be a rule on $\mathcal{R}^n$. Let $i \in N$. Given $R_{-i} \in \mathcal{R}^{n-1}$, agent i's **option set under f for R_{-i}** is defined by

$$o_i^f(R_{-i}) \equiv \{z_i \in M \times \mathbb{R} : \exists R_i \in \mathcal{R} \text{ s.t. } f_i(R_i, R_{-i}) = z_i\}.$$

Further, given $R_{-i} \in \mathcal{R}^{n-1}$, let $M_i^f(R_{-i}) \equiv \{x_i \in M : \exists R_i \in \mathcal{R} \text{ s.t. } x_i(R_i, R_{-i}) = x_i\}$.

Let f be a rule on $\mathcal{R}^n$ satisfying *strategy-proofness*. Let $i \in N$ and $R_{-i} \in \mathcal{R}^{n-1}$. Given $x_i \in M_i^f(R_{-i})$, let $t_i^f(R_{-i}; x_i) \in \mathbb{R}$ be a payment such that $(x_i, t_i^f(R_{-i}; x_i)) \in o_i^f(R_{-i})$. By *strategy-proofness*, such a payment is unique. Given $x_i \in M_i^f(R_{-i})$, let $z_i^f(R_{-i}; x_i) \equiv (x_i, t_i^f(R_{-i}; x_i))$. Then agent i's option set $o_i^f(R_{-i})$ under f for R_{-i} can be expressed as follows.

$$o_i^f(R_{-i}) = \{(x_i, t_i) \in M_i^f(R_{-i}) \times \mathbb{R} : t_i = t_i^f(R_{-i}; x_i)\} = \{z_i^f(R_{-i}; x_i) : x_i \in M_i^f(R_{-i})\}.$$

The following lemma is an immediate implication of *strategy-proofness*.

Lemma 4. *Let f be a rule on $\mathcal{R}^n$ satisfying strategy-proofness. Then, for each $R \in \mathcal{R}^n$, each $i \in N$, and each $x_i \in M_i^f(R_{-i})$, $f_i(R) \; R_i \; z_i^f(R_{-i}; x_i)$.*

B Proofs of the impossibility theorems

In this section, we provide the proofs of Theorem 1 and Proposition 2.

B.1 Preliminaries

Given $R_i \in \mathcal{R}$ and $x_i \in M \setminus \{0, m\}$, let

$$T(x_i; R_i) \equiv \Big\{ t_i \in [0, V_i(x_i, \mathbf{0})] : V_i(x_i + 1, (x_i, t_i)) - t_i = \frac{t_i}{x_i} \Big\}.$$

Thus, $T(x_i; R_i)$ is the set of payments in $[0, V_i(x_i, \mathbf{0})]$ at which the incremental valuation $V_i(x_i + 1, (x_i, t_i)) - t_i$ of $x_i + 1$ at (x_i, t_i) for R_i coincides with the per-unit payment $\frac{t_i}{x_i}$. Hereafter, we omit R_i if there is no risk of confusion, and simply write $T(x_i)$ instead of $T(x_i; R_i)$.

In this subsection, we investigate the properties of the set $T(x_i)$ for a preference $R_i \in \mathcal{R}^D$. The discussion in Sections B.1 and B.2 only relies on the incremental valuations being decreasing, and is valid for any $R_i \in \mathcal{R}^D$. In Sections B.3 and B.4, we require positive or negative income effects to derive a contradiction.

The next remark gives an alternative definition of $T(x_i)$.

Remark 11. Let $R_i \in \mathcal{R}$ and $x_i \in M \setminus \{0, m\}$. Let $t_i \in [0, V_i(x_i, \mathbf{0})]$. Then, $t_i \in T(x_i)$ if and only if

$$\frac{x_i + 1}{x_i} t_i - V_i\Big(x_i, \Big(x_i + 1, \frac{x_i + 1}{x_i} t_i\Big)\Big) = \frac{t_i}{x_i}.$$

Lemma 5. *Let $R_i \in \mathcal{R}^D$ and $x_i \in M \setminus \{0, m\}$. Then, $T(x_i) \neq \varnothing$.*

Proof. Let $h_i : \mathbb{R} \to \mathbb{R}$ be such that for each $t_i \in \mathbb{R}$, $h_i(t_i) = V_i(x_i+1, (x_i, t_i)) - t_i - \frac{t_i}{x_i}$. By object monotonicity, $(x_i + 1, 0)\ P_i\ (x_i, 0)$. Thus, $V_i(x_i + 1, (x_i, 0)) > 0$. Thus, $h_i(0) > 0$. Further,

$$\begin{aligned} V_i(x_i+1,(x_i,V_i(x_i,\mathbf{0}))) - V_i(x_i,\mathbf{0}) &= V_i(x_i+1,\mathbf{0}) - V_i(x_i,\mathbf{0}) && \text{(by Remark 1 (i))} \\ &< \frac{V_i(x_i,\mathbf{0})}{x_i}. && \text{(by } R_i \in \mathcal{R}^D\text{)} \end{aligned}$$

Thus, $h(V_i(x_i, \mathbf{0})) < 0$. By continuity of R_i, $h_i(\cdot)$ is a continuous function.[37] Thus, by the intermediate value theorem, there is $t_i \in (0, V_i(x_i, \mathbf{0}))$ such that $h_i(t_i) = 0$. Then, $t_i \in T(x_i)$. □

Lemma 6. *Let $R_i \in \mathcal{R}^D$ and $x_i \in M\backslash\{0, m\}$. Then, $\max T(x_i)$ exists.*

Proof. By Lemma 5, $T(x_i)$ is nonempty. Clearly, $V_i(x_i, \mathbf{0})$ is an upper bound of $T(x_i; R_i)$. Thus, it suffices to show that $T(x_i)$ is a closed set.

Let $(t^n)_{n\in\mathbb{N}}$ be a convergent sequence in $T(x_i)$. Let $t \in \mathbb{R}$ be such that $\lim_{n\to\infty} t^n = t$. For each $n \in \mathbb{N}$, $t^n \in [0, V_i(x_i, \mathbf{0})]$. Thus, $t \in [0, V_i(x_i, \mathbf{0})]$. Moreover, for each $n \in \mathbb{N}$, since $t^n \in T(x_i)$,

$$V_i(x_i + 1, (x_i, t^n)) - t^n = \frac{t^n}{x_i}.$$

By the continuity of the function $V_i(x_i + 1, (x_i, \cdot))$, taking the limit of the both sides yields

$$V_i(x_i + 1, (x_i, t)) - t = \frac{t}{x_i}.$$

Thus, $t \in T(x_i)$. □

Given $R_i \in \mathcal{R}^D$ and $x_i \in M\backslash\{0, m\}$, let

$$\bar{t}(x_i; R_i) \equiv \max T(x_i; R_i).$$

Moreover, let $\bar{t}(m; R_i) \equiv 0$, $\bar{t}(0; R_i) \equiv V_i(1, \mathbf{0})$, and $\bar{t}(-1; R_i) \equiv \infty$. We may omit R_i in $\bar{t}(\cdot; R_i)$ when it is clear.

Lemma 7. *Let $R_i \in \mathcal{R}^D$ and $x_i \in M\backslash\{0, m\}$. Then, $\bar{t}(x_i) < V_i(x_i, \mathbf{0})$.*

Proof. By contradiction, suppose $\bar{t}(x_i) \geq V_i(x_i, \mathbf{0})$. By $\bar{t}(x_i) \in T(x_i)$, $\bar{t}(x_i) = V_i(x_i, \mathbf{0})$. Then,

$$\begin{aligned} \frac{V_i(x_i,\mathbf{0})}{x_i} &= V_i(x_i+1,(x_i,V_i(x_i,\mathbf{0}))) - V_i(x_i,\mathbf{0}) && \text{(by } \bar{t}(x_i) \in T(x_i)\text{)} \\ &= V_i(x_i+1,\mathbf{0}) - V_i(x_i,\mathbf{0}), && \text{(by Remark 1 (i))} \end{aligned}$$

which contradicts $R_i \in \mathcal{R}^D$. □

[37]For the formal argument of this statement, see Lemma 1 of Kazumura and Serizawa (2016).

The next lemma states that if a payment t_i is greater than $\bar{t}(x_i)$, then the incremental valuation of x_i+1 at (x_i, t_i) is smaller than the per-unit payment at t_i.

Lemma 8. *Let $R_i \in \mathcal{R}^D$ and $x_i \in M \backslash \{0, m\}$. For each $t_i \in (\bar{t}(x_i), V_i(x_i, \mathbf{0})]$,*

$$V_i(x_i+1, (x_i, t_i)) - t_i < \frac{t_i}{x_i}.$$

Proof. Let $t_i \in (\bar{t}(x_i), V_i(x_i, \mathbf{0})]$. By contradiction, suppose

$$V_i(x_i+1, (x_i, t_i)) - t_i \geq \frac{t_i}{x_i}. \tag{1}$$

By $t_i > \bar{t}(x_i) = \max T(x_i)$, $t_i \notin T(x_i)$. Thus, by (1),

$$V_i(x_i+1, (x_i, t_i)) - t_i > \frac{t_i}{x_i}.$$

By Remark 1 (i) and $R_i \in \mathcal{R}^D$,

$$V_i(x_i+1, (x_i, V_i(x_i, \mathbf{0}))) - V_i(x_i, \mathbf{0}) = V_i(x_i+1, \mathbf{0}) - V_i(x_i, \mathbf{0}) < \frac{V_i(x_i, \mathbf{0})}{x_i}.$$

Thus, as in the proof of Lemma 5, the intermediate value theorem implies that there is $t_i' \in (t_i, V_i(x_i, \mathbf{0}))$ such that

$$V_i(x_i+1, (x_i, t_i')) - t_i' = \frac{t_i'}{x_i}.$$

Thus, by $t_i' \in [0, V_i(x_i, \mathbf{0})]$, $t_i' \in T(x_i)$. However, by $t_i' > t_i > \bar{t}(x_i)$, this contradicts the definition of $\bar{t}(x_i)$ that $\bar{t}(x_i) = \max T(x_i)$. □

The next lemma states that the per-unit payment $\frac{\bar{t}(x_i)}{x_i}$ is strictly decreasing in x_i.

Lemma 9. *Let $R_i \in \mathcal{R}^D$ and $x_i \in M \backslash \{0, m\}$. Then, $\frac{\bar{t}(x_i)}{x_i} > \frac{\bar{t}(x_i+1)}{x_i+1}$.*

Proof. By contradiction, suppose $\frac{\bar{t}(x_i)}{x_i} \leq \frac{\bar{t}(x_i+1)}{x_i+1}$. There are two cases.

Case 1. $\frac{\bar{t}(x_i)}{x_i} < \frac{\bar{t}(x_i+1)}{x_i+1}$.

We have

$$\begin{aligned} \frac{\bar{t}(x_i+1)}{x_i+1} &= V_i(x_i+2, (x_i+1, \bar{t}(x_i+1))) - \bar{t}(x_i+1) && (\text{by } \bar{t}(x_i+1) \in T(x_i+1)) \\ &< \bar{t}(x_i+1) - V_i(x_i, (x_i+1, \bar{t}(x_i+1))). && (\text{by } R_i \in \mathcal{R}^D) \end{aligned}$$

This implies

$$V_i(x_i, (x_i+1, \bar{t}(x_i+1))) < x_i \frac{\bar{t}(x_i+1)}{x_i+1}. \tag{1}$$

Thus,

$$V_i(x_i, (x_i+1, \bar{t}(x_i+1))) < x_i \frac{V_i(x_i+1, \mathbf{0})}{x_i+1} < V_i(x_i, \mathbf{0}), \tag{2}$$

where the first inequality follows from $\bar{t}(x_i+1) \in T(x_i+1)$, and the second one from $R_i \in \mathcal{R}^D$.

By $\bar{t}(x_i) \in T(x_i)$ and $\frac{\bar{t}(x_i)}{x_i} < \frac{\bar{t}(x_i+1)}{x_i+1}$,

$$V_i(x_i+1, (x_i, \bar{t}(x_i))) = \frac{x_i+1}{x_i} \bar{t}(x_i) < \bar{t}(x_i+1).$$

This implies $(x_i, \bar{t}(x_i))\ P_i\ (x_i+1, \bar{t}(x_i+1))$. Thus,

$$\bar{t}(x_i) < V_i(x_i, (x_i+1, \bar{t}(x_i+1))). \tag{3}$$

Let $t_i \equiv V_i(x_i, (x_i+1, \bar{t}(x_i+1)))$. By (2) and (3), $t_i \in (\bar{t}(x_i), V_i(x_i, \mathbf{0}))$. Moreover,

$$\begin{aligned}
& V_i(x_i+1, (x_i, t_i)) \\
&= V_i(x_i+1, (x_i+1, \bar{t}(x_i+1))) && \text{(by } (x_i, t_i)\ I_i\ (x_i+1, \bar{t}(x_i+1)) \text{ and Remark 1 (i))} \\
&= \bar{t}(x_i+1) && \text{(by Remark 1 (iii))} \\
&> (x_i+1)\frac{t_i}{x_i}. && \text{(by (1))}
\end{aligned}$$

Thus,

$$V_i(x_i+1, (x_i, t_i)) - t_i > \frac{t_i}{x_i}.$$

However, this contradicts Lemma 8.

Case 2. $\frac{\bar{t}(x_i)}{x_i} = \frac{\bar{t}(x_i+1)}{x_i+1}$.

By $\bar{t}(x_i) \in T(x_i)$,

$$V_i(x_i+1, (x_i, \bar{t}(x_i))) - \bar{t}(x_i) = \frac{\bar{t}(x_i)}{x_i}.$$

Similarly, by $\bar{t}(x_i+1) \in T(x_i+1)$,

$$V_i(x_i+2, (x_i+1, \bar{t}(x_i+1))) - \bar{t}(x_i+1) = \frac{\bar{t}(x_i+1)}{x_i+1}.$$

Thus, by $\frac{\bar{t}(x_i)}{x_i} = \frac{\bar{t}(x_i+1)}{x_i+1}$,

$$V_i(x_i+1, (x_i, \bar{t}(x_i))) - \bar{t}(x_i) = V_i(x_i+2, (x_i+1, \bar{t}(x_i+1))) - \bar{t}(x_i+1). \tag{4}$$

Further, by $\bar{t}(x_i) \in T(x_i)$ and $\frac{\bar{t}(x_i)}{x_i} = \frac{\bar{t}(x_i+1)}{x_i+1}$,

$$V_i(x_i+1, (x_i, \bar{t}(x_i))) = \frac{x_i+1}{x_i}\bar{t}(x_i) = \bar{t}(x_i+1). \tag{5}$$

Then,

$$
\begin{aligned}
&\bar{t}(x_i+1) - V_i(x_i,(x_i+1,\bar{t}(x_i+1))) \\
&= V_i(x_i+1,(x_i,\bar{t}(x_i))) - V_i\Big(x_i,\Big(x_i+1,V_i(x_i+1,(x_i,\bar{t}(x_i)))\Big)\Big) && \text{(by (5))} \\
&= V_i(x_i+1,(x_i,\bar{t}(x_i))) - V_i(x_i,(x_i,\bar{t}(x_i))) && \text{(by Remark 1 (i))} \\
&= V_i(x_i+1,(x_i,\bar{t}(x_i))) - \bar{t}(x_i) && \text{(by Remark 1 (iii))} \\
&= V_i(x_i+2,(x_i+1,\bar{t}(x_i+1))) - \bar{t}(x_i+1), && \text{(by (4))}
\end{aligned}
$$

which contradicts the assumption that $R_i \in \mathcal{R}^D$. □

By Lemma 9, the closed interval $[x_i\frac{\bar{t}(x_i+1)}{x_i+1}, \bar{t}(x_i)]$ is well-defined.

Lemma 10. *Let $R_i \in \mathcal{R}^D$ and $x_i \in M\backslash\{0,m\}$. Then, $\min\Big(T(x_i) \cap [x_i\frac{\bar{t}(x_i+1)}{x_i+1}, \bar{t}(x_i)]\Big)$ exists.*

Proof. By Lemma 9, $x_i\frac{\bar{t}(x_i+1)}{x_i+1} < \bar{t}(x_i)$. By $\bar{t}(x_i) \in T(x_i)$, $T(x_i) \cap [x_i\frac{\bar{t}(x_i+1)}{x_i+1}, \bar{t}(x_i)] \neq \varnothing$. Clearly, $x_i\frac{\bar{t}(x_i+1)}{x_i+1}$ is a lower bound of $T(x_i) \cap [x_i\frac{\bar{t}(x_i+1)}{x_i+1}, \bar{t}(x_i)]$. Thus, it suffices to show that the set is closed.

As shown in the proof of Lemma 6, $T(x_i)$ is closed. Thus, since the closed interval $[x_i\frac{\bar{t}(x_i+1)}{x_i+1}, \bar{t}(x_i)]$ is a closed set, $T(x_i) \cap [x_i\frac{\bar{t}(x_i+1)}{x_i+1}, \bar{t}(x_i)]$ is also a closed set. □

Given $R_i \in \mathcal{R}^D$ and $x_i \in M\backslash\{0,m\}$, let

$$\underline{t}(x_i;R_i) \equiv \min\Big(T(x_i;R_i) \cap \Big[x_i\frac{\bar{t}(x_i+1;R_i)}{x_i+1}, \bar{t}(x_i;R_i)\Big]\Big).$$

Moreover, let $\underline{t}(m;R_i) \equiv 0$, $\underline{t}(0;R_i) \equiv V_i(1,\mathbf{0})$, and $\underline{t}(-1;R_i) \equiv \infty$. Again, we may omit R_i in $\underline{t}(\cdot;R_i)$ when there is no risk of confusion. Clearly, for each $x_i \in M \cup \{-1\}$, $\underline{t}(x_i) \leq \bar{t}(x_i)$.

The next lemma states that the per-unit payment $\frac{\underline{t}(x_i)}{x_i}$ is greater than $\frac{\bar{t}(x_i+1)}{x_i+1}$.

Lemma 11. *Let $R_i \in \mathcal{R}^D$ and $x_i \in M\backslash\{0,m\}$. Then, $\underline{t}(x_i) > x_i\frac{\bar{t}(x_i+1)}{x_i+1}$.*

Proof. First, suppose $x_i < m-1$. By $\underline{t}(x_i) \in [x_i\frac{\bar{t}(x_i+1)}{x_i+1}, \bar{t}(x_i)]$, $\underline{t}(x_i) \geq x_i\frac{\bar{t}(x_i+1)}{x_i+1}$. By contradiction, suppose $\underline{t}(x_i) = x_i\frac{\bar{t}(x_i+1)}{x_i+1}$. Then, in the same way as in Case 2 of the proof of Lemma 9, we can show

$$V_i(x_i+2,(x_i,\underline{t}(x_i+1))) - \underline{t}(x_i+1) = \underline{t}(x_i+1) - V_i(x_i,(x_i+1,\underline{t}(x_i+1))),$$

which contradicts the assumption that $R_i \in \mathcal{R}^D$.

Next, suppose $x_i = m-1$. By $\underline{t}(m-1) \in T(m-1)$, $\underline{t}(m-1) \geq 0 = (m-1)\frac{\bar{t}(m)}{m}$. By contradiction, suppose that $\underline{t}(m-1) = 0$. Then, by object monotonicity, $(m,0) = (m,\underline{t}(m-1))\ P_i\ (m-1,\underline{t}(m-1))$. This implies

$$V_i(m,(m-1,\underline{t}(m-1))) > 0 = (m-1)\frac{\bar{t}(m)}{m}.$$

However, this contradicts $\underline{t}(m-1) \in T(m-1)$. $\square$

By Lemma 11, for each $x_i \in M \backslash \{0,1\}$, the interval $[\bar{t}(x_i), x_i \frac{\underline{t}(x_i-1)}{x_i-1})$ is well-defined.

Lemma 12. *Let $R_i \in \mathcal{R}^D$ and $x_i \in M \backslash \{0\}$.*
(i) If $x_i > 1$, then for each $t_i \in [\bar{t}(x_i), x_i \frac{\underline{t}(x_i-1)}{x_i-1})$,

$$t_i - V_i(x_i - 1, (x_i, t_i)) > \frac{t_i}{x_i}.$$

(ii) If $x_i = 1$, then for each $t_i \in [\bar{t}(1), \underline{t}(0))$,

$$t_i - V_i(x_i - 1, (x_i, t_i)) > \frac{t_i}{x_i}.$$

Proof. The proof has two parts.

Part 1. First, we show (i). Suppose $x_i > 1$. Let $t_i \in [\bar{t}(x_i), x_i \frac{\underline{t}(x_i-1)}{x_i-1})$. By contradiction, suppose

$$t_i - V_i(x_i - 1, (x_i, t_i)) \leq \frac{t_i}{x_i}.$$

First, we claim that

$$t_i - V_i(x_i - 1, (x_i, t_i)) < \frac{t_i}{x_i}. \tag{1}$$

By contradiction, suppose

$$t_i - V_i(x_i - 1, (x_i, t_i)) = \frac{t_i}{x_i}. \tag{2}$$

Let $t(x_i - 1) \equiv (x_i - 1)\frac{t_i}{x_i}$. Then,

$$\frac{x_i}{x_i - 1} t(x_i - 1) - V_i\Big(x_i - 1, \Big(x_i, \frac{x_i}{x_i - 1} t(x_i - 1)\Big)\Big) = \frac{t(x_i - 1)}{x_i - 1}.$$

By $t_i < \frac{x_i}{x_i-1}\underline{t}(x_i - 1)$,

$$t(x_i - 1) < \underline{t}(x_i - 1) \leq \bar{t}(x_i - 1) < V_i(x_i - 1, \mathbf{0}),$$

where the last inequality follows from Lemma 7. Thus, by Remark 11, $t(x_i - 1) \in T(x_i - 1)$.

By $\underline{t}(x_i - 1) \in T(x_i - 1)$,

$$V_i(x_i, (x_i - 1, \underline{t}(x_i - 1))) = x_i \frac{\underline{t}(x_i - 1)}{x_i - 1}.$$

Thus, by $t_i < x_i \frac{\underline{t}(x_i-1)}{x_i-1}$,

$$t_i < V_i(x_i, (x_i - 1, \underline{t}(x_i - 1))). \tag{3}$$

Let $t(x_i) \equiv V_i(x_i, (x_i - 1, \underline{t}(x_i - 1)))$. Then, by Remark 1 (i),

$$(x_i, t(x_i)) \; I_i \; (x_i - 1, \underline{t}(x_i - 1)). \tag{4}$$

Then, by (2),

$$\begin{aligned}
t(x_i-1) &= V_i(x_i-1,(x_i,t_i)) && \text{(by (2))}\\
&< V_i(x_i-1,(x_i,t(x_i))) && \text{(by (3))}\\
&= V_i(x_i-1,(x_i-1,\underline{t}(x_i-1))) && \text{(by (4) and Remark 1 (i))}\\
&= \underline{t}(x_i-1). && \text{(by Remark 1 (iii))}
\end{aligned}$$

By $\underline{t}(x_i-1) \leq \bar{t}(x_i-1)$, $t(x_i-1) < \bar{t}(x_i-1)$. Moreover, by $t_i \geq \bar{t}(x_i)$,

$$t(x_i-1) = (x_i-1)\frac{t_i}{x_i} \geq (x_i-1)\frac{\bar{t}(x_i)}{x_i}.$$

Thus, we obtain $t(x_i-1) \in [(x_i-1)\frac{\bar{t}(x_i)}{x_i}, \bar{t}(x_i-1)]$. By $t(x_i-1) \in T(x_i-1)$, we have $t(x_i-1) \in T(x_i-1) \cap [(x_i-1)\frac{\bar{t}(x_i)}{x_i}, \bar{t}(x_i-1)]$. However, by $t(x_i-1) < \underline{t}(x_i-1)$, this contradicts the definition of $\underline{t}(x_i-1)$ that $\underline{t}(x_i-1) = \min T(x_i-1) \cap [(x_i-1)\frac{\bar{t}(x_i)}{x_i}, \bar{t}(x_i-1)]$.

Thus, we obtain (1). This implies

$$(x_i-1)\frac{t_i}{x_i} < V_i(x_i-1,(x_i,t_i)).$$

Thus,

$$\left(x_i-1,(x_i-1)\frac{t_i}{x_i}\right)\ P_i\ (x_i-1,V_i(x_i-1,(x_i,t_i)))\ I_i\ (x_i,t_i).$$

This implies

$$V_i\Big(x_i,\Big(x_i-1,(x_i-1)\frac{t_i}{x_i}\Big)\Big) < t_i. \tag{5}$$

By $R_i \in \mathcal{R}^D$ and $\bar{t}(x_i) \in T(x_i)$,

$$\bar{t}(x_i) - V_i(x_i-1,(x_i,\bar{t}(x_i))) > V_i(x_i+1,(x_i,\bar{t}(x_i))) - \bar{t}(x_i) = \frac{\bar{t}(x_i)}{x_i},$$

or equivalently,

$$(x_i-1)\frac{\bar{t}(x_i)}{x_i} > V_i(x_i-1,(x_i,\bar{t}(x_i))).$$

This implies

$$(x_i,\bar{t}(x_i))\ I_i\ \Big(x_i-1,V_i(x_i-1,(x_i,\bar{t}(x_i)))\Big)\ P_i\ \Big(x_i-1,(x_i-1)\frac{\bar{t}(x_i)}{x_i}\Big).$$

Thus,

$$\bar{t}(x_i) < V_i\Big(x_i,\Big(x_i-1,(x_i-1)\frac{\bar{t}(x_i)}{x_i}\Big)\Big),$$

or equivalently,

$$V_i\Big(x_i,\Big(x_i-1,(x_i-1)\frac{\bar{t}(x_i)}{x_i}\Big)\Big) - (x_i-1)\frac{\bar{t}(x_i)}{x_i} > \frac{\bar{t}(x_i)}{x_i}. \tag{6}$$

Let $h_i : \mathbb{R} \to \mathbb{R}$ be such that for each $t \in \mathbb{R}$, $h_i(t) = V_i(x_i, (x_i - 1, t)) - \frac{t}{x_i - 1}$. By the continuity of R_i, h_i is continuous. Thus, by (5) and (6), the intermediate value theorem implies that there is $t'_i \in ((x_i - 1)\frac{\overline{t}(x_i)}{x_i}, (x_i - 1)\frac{t_i}{x_i})$ such that

$$V_i(x_i, (x_i - 1, t'_i)) - t'_i = \frac{t'_i}{x_i - 1}.$$

By $t_i < x_i \frac{\underline{t}(x_i - 1)}{x_i - 1}$ and Lemma 7, $t'_i < V_i(x_i - 1, \mathbf{0})$. Thus, $t'_i \in T(x_i - 1)$. By $t'_i < (x_i - 1)\frac{t_i}{x_i}$ and $t_i < x_i \frac{\underline{t}(x_i - 1)}{x_i - 1}$, we have $t'_i < \underline{t}(x_i - 1)$. By $\underline{t}(x_i - 1) \leq \overline{t}(x_i - 1)$, $t'_i < \overline{t}(x_i - 1)$. Thus, by $t'_i > (x_i - 1)\frac{\overline{t}(x_i)}{x_i}$, we have $t'_i \in [(x_i - 1)\frac{\overline{t}(x_i)}{x_i}, \underline{t}(x_i - 1)]$. By $t'_i \in T(x_i - 1)$, we get $t'_i \in T(x_i - 1) \cap [(x_i - 1)\frac{\overline{t}(x_i)}{x_i}, \underline{t}(x_i - 1)]$. However, by $t'_i < \underline{t}(x_i - 1)$, this contradicts the definition of $\underline{t}(x_i - 1)$ that $\underline{t}(x_i - 1) = \min\Big(T(x_i - 1) \cap [(x_i - 1)\frac{\overline{t}(x_i)}{x_i}, \overline{t}(x_i - 1)]\Big)$.

Part 2. Next, we show (ii). Suppose $x_i = 1$. Let $t_i \in [\overline{t}(x_i), \underline{t}(x_i - 1))$. Then, by $t_i < \underline{t}(0) = V_i(1, \mathbf{0})$, $(1, t_i)\ P_i\ \mathbf{0}$. This implies $V_i(0, (1, t_i)) < 0$. Thus,$t_i - V_i(0, (1, t_i)) > t_i$, as desired. □

Given $R_i \in \mathcal{R}^D$ and $x_i \in M \backslash \{0\}$, let $\overline{d}(x_i; R_i) \equiv \frac{\overline{t}(x_i)}{x_i}$ and $\underline{d}(x_i) \equiv \frac{\underline{t}(x_i)}{x_i}$. Further, let $\overline{d}(0; R_i) = \underline{d}(0; R_i) \equiv V_i(1, \mathbf{0})$ and $\overline{d}(-1; R_i) = \underline{d}(-1; R_i) \equiv -\infty$. We may omit R_i if it is obvious from the context. Then, for each $x_i \in M \cup \{-1\}$, $\underline{d}(x_i) \leq \overline{d}(x_i)$. Moreover, by Lemmas 7 and 11, for each $x_i \in M$, $\overline{d}(x_i) < \underline{d}(x_i - 1)$.

Here, we provide an interpretation of $\underline{d}(x_i)$ and $\overline{d}(x_i)$ for some class of preferences. To do so, we introduce the notion of single-intersection condition on a preference. In words, it states that for each $x_i \in M \backslash \{0, m\}$, the incremental valuation $V_i(x_i + 1, (x_i, t_i)) - t_i$ and the per-unit payment $\frac{t_i}{x_i}$ coincides only at once in $[0, V_i(x_i, \mathbf{0})]$. Formally, a preference $R_i \in \mathcal{R}$ satisfies the **single-intersection condition** if for each $x_i \in M \backslash \{0, m\}$, $|T(x_i)| = 1$.

Remark 12. Let $R_i \in \mathcal{R}^{NI} \cap \mathcal{R}^+$. Then, it satisfies the single-intersection condition.

Note that some preference $R_i \in \mathcal{R}^{NI} \cap \mathcal{R}^{--}$ that exhibits nonincreasing incremental valuations and negative income effects violates the single-crossing condition.

The next proposition gives an interpretation of $\underline{d}(x_i)$ and $\overline{d}(x_i)$. It states that if a preference $R_i \in \mathcal{R}^D$ satisfies the single-intersection condition, then both $\underline{d}(x_i)$ and $\overline{d}(x_i)$ are equivalent to the inverse-demand $p(x_i + 1; R_i)$ of $x_i + 1$ units.

Proposition 3. *Let $R_i \in \mathcal{R}^D$ satisfy the single-intersection condition. Then, for each $x_i \in M \backslash \{m\}$, $\underline{d}(x_i) = \overline{d}(x_i) = p(x_i + 1; R_i)$.*

The proof of Proposition 3 can be found in the supplementary material.

The single-intersection condition is indispensable for Proposition 3. Indeed, we can find a preference $R_i \in \mathcal{R}^D$ such that for some $x_i \in M \backslash \{0, m\}$, $|T(x_i)| \geq 2$, and neither $\underline{d}(x_i)$ nor $\overline{d}(x_i)$ coincides with $p(x_i + 1; R_i)$.

By Proposition 3 and Remark 12, we obtain the following.

Corollary 2. *Let $R_i \in \mathcal{R}^D \cap \mathcal{R}^+$. For each $x_i \in M \backslash \{m\}$, $\underline{d}(x_i) = \overline{d}(x_i) = p(x_i + 1; R_i)$.*

B.2 Proof of Theorem 1 and Proposition 2

We are now in a position to prove Theorem 1 and Proposition 2. Both the proofs have many parts in common, and we do not distinguish them for a while. Before providing the proofs, we invoke the following fact.

Fact 1. (Holmström, 1979) *Let $\mathcal{R}$ be such that $\mathcal{R} \cap \mathcal{R}^Q$ is convex.*[38] *Let f be a rule on $\mathcal{R}^n$ satisfying efficiency, individual rationality, no subsidy for losers, and strategy-proofness. Then, for each $i \in N$, each $R_{-i} \in (\mathcal{R} \cap \mathcal{R}^Q)^{n-1}$, and each $x_i \in M_i^f(R_{-i})$,*

$$t_i^f(R_{-i}; x_i) = \sigma_i(R_{-i}; 0) - \sigma_i(R_{-i}; x_i).$$

For each $\varepsilon > 0$, $\mathcal{R}^C \cap \mathcal{R}^Q$, $\mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q$, and $\mathcal{R}^{ND}(\varepsilon) \cap \mathcal{R}^Q$ are all convex. This allows us to use Fact 1 in the subsequent proof.

Let $R_0 \in \mathcal{R}^D \cap (\mathcal{R}^{++} \cup \mathcal{R}^{--})$ and $\varepsilon \in \mathbb{R}_{++}$. Let

$$\mathcal{R} \in \Big\{(\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}, (\mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\}, (\mathcal{R}^{ND}(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\}\Big\}.$$

Suppose that there is a rule f on $\mathcal{R}^n$ satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness.*[39]

Step 1. In what follows, for each $x_i \in M \cup \{-1\}$, we simply write $\overline{d}(x_i)$ and $\underline{d}(x_i)$ instead of $\overline{d}(x_i; R_0)$ and $\underline{d}(x_i; R_0)$, respectively. For each $x \in M \backslash \{m\}$, let

$$\mathcal{R}(x) \equiv \Big\{R_i \in \mathcal{R}^C \cap \mathcal{R}^Q : v_i(x'+1) - v_i(x') \in (\overline{d}(x), \underline{d}(x-1) \text{ for each } x' \in M\backslash\{m\}\Big\}.$$

By Lemma 11, we can choose $\delta > 0$ such that $\delta < \underline{d}(m-1)$, and it is sufficiently small positive number compared to R_0.[40] Let

$$\mathcal{R}(m) \equiv \Big\{R_i \in \mathcal{R}^C \cap \mathcal{R}^Q : v_i(x+1) - v_i(x) \in (\delta, \underline{d}(m-1)) \text{ for each } x \in M\backslash\{m\}\Big\}.$$

Note that $\bigcup_{x \in M} \mathcal{R}(x) \subsetneq \mathcal{R}^C \cap \mathcal{R}^Q \subsetneq \mathcal{R}$.

Let $R_1 \equiv R_0$. For each $i \in N\backslash\{1,2\}$, let $R_i \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that for each $x_i \in M$, $v_i(x_i) = \delta x_i$. Since δ is sufficiently small compared to R_1, *efficiency* implies that for each $R_2 \in \mathcal{R}$ and each $i \in N\backslash\{1,2\}$, $x_i(R) = 0$. For each $R_2 \in \bigcup_{x \in M}\mathcal{R}(x)$ with the constant incremental valuation v_2, since $v_2 > \delta$ and $R_{-1} \in (\mathcal{R}^{NI})^{n-1}$, Remark 8 implies that for

[38] A class of preferences $\mathcal{R} \subseteq \mathcal{R}^Q$ is *convex* if for each pair R_i, R_i' with valuations functions $v_i(\cdot), v_i'(\cdot)$ and each $\lambda \in [0,1]$, a preference R_i^λ with valuation function $v_i^\lambda(\cdot) = \lambda v_i(\cdot) + (1-\lambda)v_i'(\cdot)$ is in $\mathcal{R}$.

[39] Note that an impossibility theorem on a domain implies the impossibility theorem on any superdomain. Thus, to show Theorem 1, we only have to show the impossibility on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^n$. The parallel discussion applies to Proposition 2.

[40] For example, let $\delta > 0$ be such that

$$\delta < \min\{V_0(m, \mathbf{0}) - V_0(m-1, \mathbf{0}), V_0(m, (m-1, 0)), \underline{d}(m-1)\}.$$

Then the subsequent discussion is valid for such $\delta > 0$.

each $x_1 \in M$, $\sigma_1(R_{-1}; x_1) = (m - x_1)v_2$. Thus, for each $R_2 \in \bigcup_{x \in M} \mathcal{R}(x)$ with the constant incremental valuation v_2 and each $x_1 \in M_1^f(R_{-1})$, Fact 1 gives

$$t_1^f(R_{-1}; x_1) = v_2 x_1. \tag{1}$$

Step 2. Let $x_1 \in M$ and $R_2^{x_1} \in \mathcal{R}(x_1)$. Let $R^{x_1} \equiv (R_1, R_2^{x_1}, R_{-1,2})$. We show $x_1(R^{x_1}) = x_1$ and $x_2(R^{x_1}) = m - x_1$. Since, for each $i \in N \backslash \{1, 2\}$, $x_i(R) = 0$, by Lemma 1 we only have to show $x_1(R^{x_1}) = x_1$. Let $v_2^{x_1}$ be the constant incremental valuation associated with $R_2^{x_1}$.

Case 1. $1 \leq x_1 \leq m$.

We show $x_1(R^{x_1}) = x_1$. Suppose by contradiction that $x_i(R^{x_1}) \neq x_1$.

Suppose first $x_1(R^{x_1}) > x_1$. Then, by *efficiency* and $R^{x_1} \in (\mathcal{R}^{NI})^n$, Remark 8 implies

$$t_1(R^{x_1}) - V_1(x_1(R^{x_1}) - 1, f_1(R^{x_1})) \geq v_2^{x_1}.$$

By $x_1(R^{x_1}) > x_1$ and $R_1 \in \mathcal{R}^D$, we have

$$t_1(R^{x_1}) - V_1(x_1, f_1(R^{x_1})) \geq (x_1(R^{x_1}) - x_1)v_2^{x_1}.$$

Then, by (1),

$$V_1(x_1, f_1(R^{x_1})) \leq x_1 v_2^{x_1}.$$

This implies $f_1(R^{x_1})\ R_1\ (x_1, x_1 v_2^{x_1})$.

By $x_1 < x_1(R^{x_1})$, $x_1 < m$. Thus, by $x_1 \geq 1$, $x_1 \in M \backslash \{0, m\}$. By $R_2^{x_1} \in \mathcal{R}(x_1)$, Lemma 8 implies

$$V_1(x_1 + 1, (x_1, x_1 v_2^{x_1})) - x_1 v_2^{x_1} < \frac{x_1 v_2^{x_1}}{x_1} = v_2^{x_1}.$$

By $x_1(R^{x_1}) > x_1$ and $R_1 \in \mathcal{R}^D$,

$$V_1(x_1(R^{x_1}), (x_1, x_1 v_2^{x_1})) - x_1 v_2^{x_1} < (x_1(R^{x_1}) - x_1)v_2^{x_1} = t_1(R^{x_1}) - x_1 v_2^{x_1},$$

where the equality follows from (1). This implies

$$V_1(x_1(R^{x_1}), (x_1, x_1 v_2^{x_1})) < t_1(R^{x_1}).$$

Thus, $(x_1, x_1 v_2^{x_1})\ P_1\ f_1(R^{x_1})$. However, this contradicts $f_1(R^{x_1})\ R_1\ (x_1, x_1 v_2^{x_1})$.

Suppose instead $x_1(R^{x_1}) < x_1$. By *efficiency* and $R_1 \in \mathcal{R}^D$, Remark 8 implies

$$V_1(x_1(R^{x_1}) + 1, f_1(R^{x_1})) - t_1(R^{x_1}) \leq v_2^{x_1}.$$

By $x_1 > x_1(R^{x_1})$, $R_1 \in \mathcal{R}^D$ implies

$$V_1(x_1, f_1(R^{x_1})) - t_1(R^{x_1}) \leq (x_1 - x_1(R^{x_1}))v_2^{x_1} = x_1 v_2^{x_1} - t_1(R^{x_1}),$$

where the equality follows from (1). Thus,

$$V_1(x_1, f_1(R^{x_1})) \leq x_1 v_2^{x_1},$$

which implies $f_1(R^{x_1})\ R_1\ (x_1, v_2^{x_1} x_1)$.

By $x_1 > x_1(R^{x_1})$, $x_1 > 0$. Thus, by $R_1 \in \mathcal{R}^D$ and $R_2^{x_1} \in \mathcal{R}(x_1)$, Lemma 12 implies

$$v_2^{x_1} x_1 - V_1(x_1 - 1, (x_1, x_1 v_2^{x_1})) > \frac{x_1 v_2^{x_1}}{x_1} = v_2^{x_1}.$$

By $x_1 > x_1(R^{x_1})$, $R_1 \in \mathcal{R}^D$ gives

$$v_2^{x_1} x_1 - V_1(x_1(R^{x_1}), (x_1, x_1 v_2^{x_1})) > (x_1 - x_1(R^{x_1}))v_2^{x_1} = v_2^{x_1} x_1 - t_1(R^{x_1}),$$

where the equality follows from (1). Thus,

$$t_1(R^{x_1}) > V_1(x_1(R^{x_1}), (x_1, x_1 v_2^{x_1})).$$

This implies $(x_1, x_1 v_2^{x_1})\ P_1\ f_1(R^{x_1})$. However, this contradicts $f_1(R^{x_1})\ R_1\ (x_1, x_1 v_2^{x_1})$.

Case 2. $x_1 = 0$.

We show $x_1(R^0) = 0$. Suppose by contradiction that $x_1(R^0) > 0$. Then,

$$V_1(x_1(R^0), \mathbf{0}) \leq x_1(R^0) V_1(1, \mathbf{0}) < x_1(R^0) v_2^0, \tag{2}$$

where the first inequality follows from $R_1 \in \mathcal{R}^D$, and the second one from $R_2^0 \in \mathcal{R}(0)$. Then, by (2) and (1),

$$\mathbf{0}\ I_1\ (x_1(R^0), V_1(x_1(R^0), \mathbf{0}))\ P_1\ (x_1(R^0), x_1(R^0) v_2^0) = f_1(R^0).$$

However, this contradicts *individual rationality*.

Step 3. Note that Step 2 implies that for each $x_2 \in M$ and each $R_2^{m-x_2} \in \mathcal{R}(m - x_2)$, $x_2(R_2^{m-x_2}, R_{-2}) = x_2$. Thus, $M_2^f(R_{-2}) = M$, and the domain of the function $t_2^f(R_{-2}; \cdot)$ is M. In this step, we show that for each $x_2 \in M \backslash \{m\}$,

$$\underline{d}(m - x_2 - 1) \leq t_2^f(R_{-2}; x_2 + 1) - t_2^f(R_{-2}; x_2) \leq \overline{d}(m - x_2 - 1).$$

Let $x_2 \in M \backslash \{m\}$. Suppose by contradiction that

$$t_2^f(R_{-2}; x_2+1) - t_2^f(R_{-2}; x_2) \notin [\underline{d}(m-x_2-1), \overline{d}(m-x_2-1)].$$

There are two cases.

Case 1. $t_2^f(R_{-2}; x_2+1) - t_2^f(R_{-2}; x_2) < \underline{d}(m-x_2-1)$.

Let $R_2^{m-x_2} \in \mathcal{R}(m-x_2)$ be such that

$$t_2^f(R_{-2}; x_2+1) - t_2^f(R_{-2}; x_2) < v_2^{m-x_2} < \underline{d}(m-x_2-1),$$

where $v_2^{m-x_2}$ is a constant incremental valuation associated with $R_2^{m-x_2}$. By Step 2, we have $x_2(R^{m-x_2}) = x_2$. By $v_2^{m-x_2} > t_2^f(R_{-2}; x_2+1) - t_2^f(R_{-2}; x_2)$,

$$(x_2+1)v_2^{m-x_2} - t_2^f(R_{-2}; x_2+1) > x_2 v_2^{m-x_2} - t_2^f(R_{-2}; x_2).$$

Thus, $z_2^f(R_{-2}; x_2+1)\ P_2^{m-x_2}\ z_2^f(R_{-2}; x_2) = f_2(R^{m-x_2})$, which contradicts Lemma 4.

Case 2. $t_2^f(R_{-2}; x_2+1) - t_2^f(R_{-2}; x_2) > \overline{d}(m-x_2-1)$.

Let $R_2^{m-x_2-1} \in \mathcal{R}(m-x_2-1)$ be such that

$$\overline{d}(m-x_2-1) < v_2^{m-x_2-1} < t_2^f(R_{-2}; x_2+1) - t_2^f(R_{-2}; x_2),$$

where $v_2^{m-x_2-1}$ is a constant incremental valuation associated with $R_2^{m-x_2-1}$. By Step 2, we have $x_2(R^{m-x_2-1}) = x_2+1$. By $v_2^{m-x_2-1} < t_2^f(R_{-2}; x_2+1) - t_2^f(R_{-2}; x_2)$,

$$x_2 v_2^{m-x_2-1} - t_2^f(R_{-2}; x_2) > (x_2+1)v_2^{m-x_2-1} - t_2^f(R_{-2}; x_2+1).$$

Thus, $z_2^f(R_{-2}; x_2)\ P_2^{m-x_2-1}\ z_2^f(R_{-2}; x_2+1) = f_2(R^{m-x_2-1})$, which contradicts Lemma 4.

B.3 Proof of Theorem 1

We complete the proof of Theorem 1. Suppose m is odd and $\mathcal{R} = (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$.

Let $R_2 \equiv R_0$. Let $\alpha \equiv \frac{m-1}{2}$. Note that as m is odd, $\alpha \in M$. There are two cases.

Case 1. $R_0 \in \mathcal{R}^{++}$.

By $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Remark 12 implies that for each $x \in M$, $\underline{d}(x) = \overline{d}(x) \equiv d(x)$. Let $x_2 \in M$ be such that $0 < x_2 \leq \alpha$. We show that $z_2^f(R_{-2}; x_2+1)\ P_2\ z_2^f(R_{-2}; x_2)$. We have

$$t_2^f(R_{-2}; x_2) = \sum_{x=0}^{x_2-1} d(m-x-1) \leq x_2 d(m-x_2) < x_2 d(x_2) = \bar{t}(x_2),$$

where the first equality follows from Step 3 and Lemma 2, the first inequality follows from $R_1 \in \mathcal{R}^D$ and Lemma 11, the second inequality comes from $R_1 \in \mathcal{R}^D$, $x_2 \leq \alpha$, and Lemma 11, and the last equality follows from $0 < x_2 < m$. Thus, by $R_2 \in \mathcal{R}^{++}$ and Remark 5 (i),

$$V_2(x_2+1, z_2^f(R_{-2}; x_2)) - t_2^f(R_{-2}; x_2) > V_2(x_2+1, (x_2, \bar{t}(x_2))) - \bar{t}(x_2). \tag{3}$$

We also have

$$\begin{aligned}
t_2^f(R_{-2}; x_2+1) - t_2^f(R_{-2}; x_2) &= d(m - x_2 - 1) && \text{(by Step 3)} \\
&\leq d(x_2) && \text{(by Lemma 11)} \\
&= \frac{\bar{t}(x_2)}{x_2} \\
&= V_2(x_2+1, (x_2, \bar{t}(x_2))) - \bar{t}(x_2). && \text{(by } \bar{t}(x_2) \in T(x_2))
\end{aligned}$$

This, together with (3), implies

$$V_2(x_2+1, z_2^f(R_{-2}; x_2)) - t_2^f(R_{-2}; x_2) > t_2^f(R_{-2}; x_2+1) - t_2^f(R_{-2}; x_2),$$

or equivalently, $V_2(x_2+1, z_2^f(R_{-2}; x_2)) > t_2^f(R_{-2}; x_2+1)$. Thus, $z_2^f(R_{-2}; x_2+1)\ P_2\ z_2^f(R_{-2}; x_2)$.

We then show that $z_2^f(R_{-2}; 1)\ P_2\ z_2^f(R_{-2}; 0)$. We have

$$t_2^f(R_{-2}; 1) = d(m-1) < \frac{V_2(m-1, \mathbf{0})}{m-1} < V_2(1, \mathbf{0}),$$

where the equality follows from Step 3 and Lemma 2, the first inequality from $R_2 \in \mathcal{R}^D$ and Lemma 7, and the last one from $R_2 \in \mathcal{R}^D$. Thus, $z_2^f(R_{-2}; 1)\ P_2\ \mathbf{0} = z_2^f(R_{-2}; 0)$.

We have established that for each $x_2 \in M$ with $x_2 < \alpha + 1$,

$$z_2^f(R_{-2}; x_2+1)\ P_2\ z_2^f(R_{-2}; x_2).$$

By Lemma 4, $x_2(R) \geq \alpha + 1$. Thus,

$$x_1(R) \leq m - x_2(R) \leq m - \alpha - 1 = \alpha.$$

Since both agents 1 and 2 have the same preferences R_0, the name of agents does not matter in the above discussion. Thus, a symmetric argument implies that for each $x_1 \in M$ with $x_1 < \alpha + 1$,

$$z_1^f(R_{-1}; x_1+1)\ P_1\ z_1^f(R_{-1}; x_1).$$

In particular, by $x_1(R) \leq \alpha$, we have

$$z_1^f(R_{-1}; x_1(R)+1)\ P_1\ z_1^f(R_{-1}; x_1(R)) = f_1(R).$$

However, this contradicts Lemma 4.

CASE 2. $R_0 \in \mathcal{R}^{--}$.

We show that $z_2^f(R_{-2};\alpha)\ P_2\ z_2^f(R_{-2};\alpha+1)$. We divide the argument into two subcases.

CASE 2-1. $t_2^f(R_{-2};\alpha+1) \geq (\alpha+1)\underline{d}(\alpha)$.

We have
$$t_2^f(R_{-2};\alpha) \leq \sum_{x=0}^{\alpha-1} \overline{d}(m-x-1) \leq \alpha\overline{d}(m-\alpha) < \alpha\underline{d}(\alpha),$$
where the first inequality follows from Step 3 and Lemma 2, the second one from $R_2 \in \mathcal{R}^D$ and Lemma 11, and the last one from $R_2 \in \mathcal{R}^D$, $2\alpha < m$, and Lemma 11. Note that by $0 < \alpha < m$, $\alpha\underline{d}(\alpha) = \underline{t}(\alpha)$. Thus, $t_2^f(R_{-2};\alpha) < \underline{t}(\alpha)$. This implies

$$z_2^f(R_{-2};\alpha)\ P_2\ (\alpha, \underline{t}(\alpha)). \tag{4}$$

By $\underline{t}(\alpha) \in T(\alpha)$,
$$V_2(\alpha+1, (\alpha, \underline{t}(\alpha))) = (\alpha+1)\underline{d}(\alpha).$$
This implies
$$(\alpha, \underline{t}(\alpha))\ I_2\ (\alpha+1, (\alpha+1)\underline{d}(\alpha)). \tag{5}$$
Further, by $t_2^f(R_{-2};\alpha+1) \geq (\alpha+1)\underline{d}(\alpha)$,

$$(\alpha+1, (\alpha+1)\underline{d}(\alpha))\ R_2\ z_2^f(R_{-2};\alpha+1). \tag{6}$$

Combining (4), (5), and (6), we get

$$z_2^f(R_{-2};\alpha)\ P_2\ z_2^f(R_{-2};\alpha+1),$$

as desired.

CASE 2-2. $t_2^f(R_{-2};\alpha+1) < (\alpha+1)\underline{d}(\alpha)$.

By $\underline{t}(\alpha) \in T(\alpha)$,
$$V_2(\alpha+1, (\alpha, \underline{t}(\alpha))) = (\alpha+1)\underline{d}(\alpha).$$
This implies $(\alpha+1, (\alpha+1)\underline{d}(\alpha))\ I_2\ (\alpha, \underline{t}(\alpha))$. Thus, by Remark 1 (i) and (iii),

$$V_2(\alpha, (\alpha+1, (\alpha+1)\underline{d}(\alpha))) = V_2(\alpha, (\alpha, \underline{t}(\alpha))) = \underline{t}(\alpha). \tag{7}$$

We have

$$\begin{aligned}
& t_2^f(R_{-2};\alpha+1) - V_2(\alpha, z_2^f(R_{-2};\alpha+1)) \\
&< (\alpha+1)\underline{d}(\alpha) - V_2(\alpha,(\alpha+1,(\alpha+1)\underline{d}(\alpha))) && \text{(by Remark 6 (ii))} \\
&= (\alpha+1)\underline{d}(\alpha) - \underline{t}(\alpha) && \text{(by (7))} \\
&= \underline{d}(\alpha) \\
&= \underline{d}(m-\alpha-1) && \text{(by } 2\alpha = m-1\text{)} \\
&\leq t_2^f(R_{-2};\alpha+1) - t_2^f(R_{-2};\alpha), && \text{(by Step 3)}
\end{aligned}$$

or equivalently,

$$V_2(\alpha, z_2^f(R_{-2};\alpha+1)) > t_2^f(R_{-2};\alpha).$$

This implies $z_2^f(R_{-2};\alpha)\ P_2\ z_2^f(R_{-2};\alpha+1)$.

Then, we show for each $x_2 \in M$ with $\alpha < x_2 < m$, $z_2^f(R_{-2};x_2)\ P_2\ z_2^f(R_{-2};x_2+1)$. Let $x_2 \in M$ be such that $\alpha < x_2 < m$. By contradiction, suppose $z_2^f(R_{-2};x_2{+}1)\ R_2\ z_2^f(R_{-2};x_2)$. This implies

$$t_2^f(R_{-2};x_2+1) - V_2(x_2, z_2^f(R_{-2};x_2+1)) \geq t_2^f(R_{-2};x_2+1) - t_2^f(R_{-2};x_2). \tag{8}$$

We have

$$\begin{aligned}
& t_2^f(R_{-2};x_2+1) - V_2(\alpha, z_2^f(R_{-2};x_2+1)) \\
&> (x_2+1-\alpha)\Big(t_2^f(R_{-2};x_2+1) - V_2(x_2; z_2^f(R_{-2};x_2+1))\Big) && \text{(by } R_2 \in \mathcal{R}^D\text{)} \\
&\geq (x_2+1-\alpha)(t_2^f(R_{-2};x_2+1) - t_2^f(R_{-2};x_2)) && \text{(by (8))} \\
&\geq t_2^f(R_{-2};x_2+1) - t_2^f(R_{-2};x_2) + (x_2-\alpha)\underline{d}(m-x_2-1) && \text{(by Step 3)} \\
&> t_2^f(R_{-2};x_2+1) - t_2^f(R_{-2};x_2) + \sum_{x=\alpha}^{x_2-1} \overline{d}(m-x-1) && \text{(by Lemma 11)} \\
&\geq t_2^f(R_{-2};x_2+1) - t_2^f(R_{-2};\alpha), && \text{(by Step 3)}
\end{aligned}$$

or equivalently,

$$V_2(\alpha, z_2^f(R_{-2};x_2+1)) < t_2^f(R_{-2};\alpha).$$

This implies $z_2^f(R_{-2};x_2+1)\ P_2\ z_2^f(R_{-2};\alpha)$.

By $z_2^f(R_{-2};\alpha+1)\ P_2\ z_2^f(R_{-2};\alpha)$,

$$V_2(\alpha+1, z_2^f(R_{-2};\alpha)) - t_2^f(R_{-2};\alpha) < t_2^f(R_{-2};\alpha+1) - t_2^f(R_{-2};\alpha). \tag{9}$$

Then,

$$
\begin{aligned}
&V_2(x_2+1, z_2^f(R_{-2};\alpha)) - t_2^f(R_{-2};\alpha) \\
&< (x_2+1-\alpha)\Big(V_2(\alpha+1, z_2^f(R_{-2};\alpha)) - t_2^f(R_{-2};\alpha)\Big) && \text{(by } R_2 \in \mathcal{R}^D\text{)} \\
&\leq t_2^f(R_{-2};\alpha+1) - t_2^f(R_{-2};\alpha) + (x_2-\alpha)\overline{d}(m-\alpha-1) && \text{(by Step 3)} \\
&< t_2^f(R_{-2};\alpha+1) - t_2^f(R_{-2};\alpha) + \sum_{x=\alpha+1}^{x_2} \underline{d}(m-x-1) && \text{(by Lemmas 7 and 11)} \\
&\leq t_2^f(R_{-2};x_2+1) - t_2^f(R_{-2};\alpha), && \text{(by Step 3)}
\end{aligned}
$$

or equivalently,

$$V_2(x_2+1, z_2^f(R_{-2};\alpha)) < t_2^f(R_{-2};x_2+1).$$

Thus, $z_2^f(R_{-2};\alpha)\ P_2\ z_2^f(R_{-2};x_2+1)$, which contradicts $z_2^f(R_{-2};x_2+1)\ P_2\ z_2^f(R_{-2};\alpha)$.

We have established that for each $x_2 \in M$ with $\alpha \leq x_2 < m$,

$$z_2^f(R_{-2};x_2)\ P_2\ z_2^f(R_{-2};x_2+1).$$

Thus, by Lemma 4, $x_2(R) \leq \alpha$. By $x_i(R) = 0$ for each $i \in N\backslash\{1,2\}$, Lemma 1 implies

$$x_1(R) = m - x_2(R) \geq m - \alpha = \alpha + 1.$$

By a symmetric argument, we can show that for each $x_1 \in M$ with $\alpha \leq x_1 < m$,

$$z_1^f(R_{-1};x_1)\ P_1\ z_1^f(R_{-1};x_1+1).$$

Thus, by $x_1(R) \geq \alpha+1$,

$$z_1^f(R_{-1};x_1(R)-1)\ P_1\ z_1^f(R_{-1};x_1(R)) = f_1(R),$$

which contradicts Lemma 4. ■

B.4 Proof of Proposition 2

Next, we complete the proof of Proposition 2. We divide the argument into two cases.

Case 1. $R_0 \in \mathcal{R}^{++}$

By $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Remark 12 implies $\underline{d}(x) = \overline{d}(x) \equiv d(x)$ for each $x \in M$. Suppose that $\mathcal{R} = (\mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\}$. The proof of the other case is similar, and we relegate it to the supplementary material.

Recall that $\delta < d(m-1)$. Let $0 < \varepsilon_1 < \min\{(m-1)\varepsilon, \frac{d(m-1)-\delta}{2}\}$. Then

$$\begin{aligned} d(m-1) &= V_1(m,(m-1,\bar{t}(m-1))) - \bar{t}(m-1) && (\text{by } \bar{t}(m-1) \in T(m-1)) \\ &< V_1(m,(m-1,\bar{t}(m-1)-\varepsilon_1)) - (\bar{t}(m-1)-\varepsilon_1) \end{aligned}$$

Thus,

$$\varepsilon_2 \equiv V_1(m,(m-1,\bar{t}(m-1)-\varepsilon_1)) - (\bar{t}(m-1)-\varepsilon_1) - d(m-1) > 0.$$

Let $\varepsilon_3 > 0$ be such that $\varepsilon_1 + m\varepsilon_3 < \varepsilon$ and

$$\varepsilon_3 < \min\Big\{\frac{d(m-1)-\delta}{2}, \varepsilon_2\Big\}.$$

Let $R_2 \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ be such that $v_2(1) = d(m-1) + \varepsilon_3$, and for each $x_2 \in M\backslash\{0,m\}$,

$$v_2(x_2+1) - v_2(x_2) = d(m-1) - \frac{\varepsilon_1 + \varepsilon_3}{m-1}.$$

Note that by $\varepsilon_1 + m\varepsilon_3 < \varepsilon$, $R_2 \in \mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q$. By $\delta < d(m-1)$, $v_2(1) > \delta$. Further, by $\varepsilon_1, \varepsilon_3 < \frac{d(m-1)-\delta}{2}$, $v_2(x_2+1) - v_2(x_2) > \delta$ for each $x_2 \in M\backslash\{0,m\}$. Thus, by $R_{-1} \in (\mathcal{R}^{NI})^{n-1}$, Remark 8 implies $\sigma_1(R_{-1};x_1) = v_2(m-x_1)$ for each $x_1 \in M$. Then by Fact 1, for each $x_1 \in M_1^f(R_{-1})$,

$$t_1^f(R_{-1};x_1) = v_2(m) - v_2(m-x_1). \tag{10}$$

By Step 3 and Lemma 2, $t_2^f(R_{-2};1) = d(m-1)$. Thus, $v_2(1) - t_2^f(R_{-2};1) = (d(m-1)+\varepsilon_3) - d(m-1) = \varepsilon_3 > 0$. For each $x_2 \in M\backslash\{0,1\}$,

$$\begin{aligned} v_2(x_2) - v_2(1) &= (x_2-1)\Big(d(m-1) - \frac{\varepsilon_1+\varepsilon_3}{m-1}\Big) \\ &< (x_2-1)d(m-1) && (\text{by } \varepsilon_1, \varepsilon_3 > 0) \\ &< \sum_{x=2}^{x_2} d(m-x) && (\text{by Lemma 11}) \\ &= \sum_{x=2}^{x_2} (t_2^f(R_{-2};x) - t_2^f(R_{-2};x-1)) && (\text{by Step 3}) \\ &= t_2^f(R_{-2};x_2) - t_2^f(R_{-2};1), \end{aligned}$$

or equivalently, $v_2(1) - t_2^f(R_{-2};1) > v_2(x_2) - t_2^f(R_{-2};x_2)$. Hence, for each $x_2 \in M\backslash\{1\}$, $z_2^f(R_{-2};1)\ P_2\ z_2^f(R_{-2};x_2)$. By Lemma 4, we obtain $f_2(R) = z_2^f(R_{-2};1)$. By $x_i(R) = 0$ for each $i \in N\backslash\{1,2\}$, Lemma 1 implies $x_1(R) = m-1$. By (10), $t_1(R) = \bar{t}(m-1) - (\varepsilon_1 + \varepsilon_3)$.

Therefore,

$$\begin{aligned}
&V_1(x_1(R)+1, f_1(R)) - t_1(R) - (v_2(x_2(R)) - v_2(x_2(R)-1)) \\
&= V_1(m, (m-1, \bar{t}(m-1) - (\varepsilon_1+\varepsilon_3))) - (\bar{t}(m-1) - (\varepsilon_1+\varepsilon_3)) - (d(m-1)+\varepsilon_3) \\
&> V_1(m, (m-1, \bar{t}(m-1) - \varepsilon_1)) - (\bar{t}(m-1) - \varepsilon_1) - (d(m-1)+\varepsilon_3) \\
&= \varepsilon_2 - \varepsilon_3 \\
&> 0,
\end{aligned}$$

where the first inequality follows from $R_1 \in \mathcal{R}^{++}$, $\varepsilon_3 > 0$, and Remark 5 (i), the second equality follows from the definition of ε_2, and the second inequality follows from $\varepsilon_3 < \varepsilon_2$. By $R \in (\mathcal{R}^{NI})^n$ and Remark 8, this contradicts *efficiency*.

CASE 2. $R_0 \in \mathcal{R}^{--}$.

Suppose $\mathcal{R} = (\mathcal{R}^{ND}(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\}$. Again, the the proof of the other case is similar, and we relegate it to the supplementary material.

By Lemmas 7 and 11, $\underline{d}(m-2) > \bar{d}(m-1)$. Let $\varepsilon_1 > 0$ be such that $\varepsilon_1 < \min\{\underline{d}(m-2) - \bar{d}(m-1), \varepsilon\}$. Then,

$$\begin{aligned}
\bar{d}(m-1) &= V_1(m, (m-1, \bar{t}(m-1))) - \bar{t}(m-1) && (\text{by } \bar{t}(m-1) \in T(m-1)) \\
&< V_1(m, (m-1, \bar{t}(m-1)+\varepsilon_1)) - (\bar{t}(m-1)+\varepsilon_1) && (\text{by Remark 6 (i)})
\end{aligned}$$

Thus,

$$\varepsilon_2 \equiv V_1(m, (m-1, \bar{t}(m-1)+\varepsilon_1)) - (\bar{t}(m-1)+\varepsilon_1) - \bar{d}(m-1) > 0.$$

Let $\varepsilon_3 > 0$ be such that $m\varepsilon_3 < \varepsilon_1$ and

$$\varepsilon_3 < \min\{\underline{d}(m-2) - \bar{d}(m-1) - \varepsilon_1, \varepsilon_2\}. \tag{11}$$

Let $R_2 \in \mathcal{R}^{ND} \cap \mathcal{R}^Q$ be such that $v_2(1) = \bar{d}(m-1) + \varepsilon_3$, and for each $x_2 \in M \backslash \{0, m\}$,

$$v_2(x_2+1) - v_2(x_2) = \bar{d}(m-1) + \frac{\varepsilon_1+\varepsilon_3}{m-1}.$$

By $m\varepsilon_3 < \varepsilon_1 < \varepsilon$, $R_2 \in \mathcal{R}^{ND}(\varepsilon) \cap \mathcal{R}^Q$. By $\delta < \bar{d}(m-1)$, $v_2(x_2+1) - v_2(x_2) > \delta$ for each $x_2 \in M \backslash \{m\}$. Thus, for each $x_1 \in M$, $\sigma_1(R_{-1}; x_1) = v_2(m - x_1)$. By Fact 1, for each $x_1 \in M_1^f(R_{-1})$,

$$t_1^f(R_{-1}; x_1) = v_2(m) - v_2(m - x_1). \tag{12}$$

We have

$$v_2(1) - t_2^f(R_{-2}; 1) \geq \bar{d}(m-1) + \varepsilon_3 - \bar{d}(m-1) = \varepsilon_3 > 0,$$

where the first inequality follows from Step 3 and Lemma 2. Further, for each $x_2 \in M \backslash \{0, 1\}$,

$$\begin{aligned}
v_2(x_2) - v_2(1) &= (x_2 - 1)\Big(\overline{d}(m-1) + \frac{\varepsilon_1 + \varepsilon_3}{m-1}\Big) \\
&< (x_2 - 1)(\overline{d}(m-1) + \underline{d}(m-2) - \overline{d}(m-1)) && \text{(by (11))} \\
&= (x_2 - 1)\underline{d}(m-2) \\
&\leq \sum_{x=1}^{x_2-1} \underline{d}(m-x-1) && \text{(by Lemmas 7 and 11)} \\
&\leq \sum_{x=1}^{x_2-1} (t_2^f(R_{-2}; x_2+1) - t_2^f(R_{-2}; x_2)) && \text{(by Step 3)} \\
&= t_2^f(R_{-2}; x_2) - t_2^f(R_{-2}; 1),
\end{aligned}$$

or equivalently, $v_2(1) - t_2^f(R_{-2}; 1) > v_2(x_2) - t_2^f(R_{-2}; 1)$. Thus, for each $x_2 \in M \backslash \{1\}$, we have $z_2^f(R_{-2}; 1)\ P_2\ z_2^f(R_{-2}; x_2)$. By Lemma 4, $x_2(R) = 1$. By $x_i(R) = 0$ for each $i \in N \backslash \{1, 2\}$, $x_1(R) = m-1$ by Lemma 1. By (12), $t_1(R) = \overline{t}(m-1) + \varepsilon_1 + \varepsilon_3$. Then,

$$\begin{aligned}
&V_1(x_1(R)+1, f_1(R)) - t_1(R) - (v_2(x_2(R)) - v_2(x_2(R)-1)) \\
&= V_1(m, (m-1, \overline{t}(m-1) + \varepsilon_1 + \varepsilon_3)) - (\overline{t}(m-1) + \varepsilon_1 + \varepsilon_3) - (\overline{d}(m-1) + \varepsilon_3) \\
&> v_1(m-1, (\overline{t}(m-1) + \varepsilon_1)) - (\overline{t}(m-1) + \varepsilon_1) - (\overline{d}(m-1) + \varepsilon_3) && \text{(by Remark 6 (i))} \\
&= \varepsilon_2 - \varepsilon_3 && \text{(by } \varepsilon_3 < \varepsilon_2\text{)} \\
&> 0,
\end{aligned}$$

which contradicts *efficiency* by Remark 7. ■

C Even number of units and negative income effects

In this section, we provide the result for the case of an even number of units and negative income effects. Throughout the section, assume that m is even.

First, we introduce the condition that is an analogue of the upper bound for nonnegative income effects.. A preference $R_i \in \mathcal{R}^{NI} \cap \mathcal{R}^{-}$ has the **upper bound for the nonpositive income effects** if it satisfies

$$WB_i(\beta, \beta p(\beta+1; R_i)) - WB_i(\beta, t^*) \leq WS_i(\beta, t^*) - WB_i(\beta, t^*),$$

where $\beta \equiv \frac{m}{2}$ and $t^* \equiv \sum_{x=0}^{\beta-1} p(m-x; R_i)$.

The RHS of the above inequality is the difference between the willingness to sell and the willingness to buy at the outcome bundle of the inverse Vickrey rule for the preference profile (R_i, R_i). By $R_i \in \mathcal{R}^{NI}$, it is nonnegative. The LHS is the size of the income effect of β between $\beta p(\beta+1; R_i)$ and t^*. Similarly to the upper bound for the nonnegative income effects, we interpret the upper bound for the nonpositive income effects as the small size of the nonpositive income effects.

The next proposition states that under the single-intersection condition, (i) when $n = 2$ and m is even, if $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{--}$ has the upper bound for the nonpositive income effects, then on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^2$, the inverse Vickrey rule satisfies the four properties, and (ii) if $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{--}$ does not have the upper bound for the nonpositive income effects, $\mathcal{R}$ is rich, and $R_0 \in \mathcal{R}$, then no rule on $\mathcal{R}^n$ satisfies the four properties.[41]

Proposition 4. *Assume m is even, Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{--}$ satisfy the single-intersection condition.*
(i) Assume $n = 2$. Assume R_0 has the upper bound for the nonpositive income effects. An inverse Vickrey rule on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^2$ satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness.
(ii) Assume R_0 does not have the upper bound for the nonpositive income effects. Let $\mathcal{R}$ be rich and $R_0 \in \mathcal{R}$. No rule on $\mathcal{R}^n$ satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness.

The proof of Proposition 4 can be found in the supplementary material.

The assumption of the single-intersection condition is indispensable for Proposition 4. Indeed, without the single-intersection condition, the upper bound for the nonpositive income effects is no longer a necessary and sufficient condition for the existence of a rule satisfying the four properties on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^2$, where $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{--}$. Further, even if there is a rule satisfying the four properties on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^2$ for some $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{--}$ that violates the single-intersection condition, it does not necessarily coincide with the inverse Vickrey rule.[42]

[41] As with Proposition 1 (i), by applying Step 3 of the proof of Theorem 1, we can show that the inverse Vickrey rule is the only rule satisfying the four properties in Proposition 4 (i). See also footnote 23.

[42] Without the single-intersection condition, we obtain a necessary and sufficient condition for the existence of a rule satisfying the four properties on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^2$, but is complicated in general. To be more precise, the next conditions is a necessary and sufficient condition of a preference $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{--}$ for the existence of a rule satisfying the four properties on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^2$: For each $i \in \{1, 2\}$, there is a sequence $(\pi_i(x))_{x \in M\setminus\{0\}}$ satisfying the following properties.
(i) For each $x \in M\setminus\{0\}$, $\pi_i(x) \in [\underline{d}(x-1), \overline{d}(x-1)]$.
(ii) We have

$$t^* - V_0(\beta - 1, (\beta, t^*)) \leq \pi_i(\beta + 1),$$

where $\beta \equiv \frac{m}{2}$ and $t^* \equiv \sum_{x=0}^{\beta-1} \pi_i(m - x)$.
(iii) For each $d \in \mathbb{R}_+$ with $d \in (\underline{d}(x), \overline{d}(x))$ for some $x \in M\setminus\{0, m\}$, there is $x' \in M\setminus\{0\}$ such that

$$\pi_i(x') \leq d \leq dx' - V_0(x' - 1, (x', dx')),$$

and if $x' < m$, then

$$V_0(x' + 1, (x', dx')) - dx' \leq d \leq \pi_i(x' + 1).$$

Note that if R_0 satisfies the single-intersection condition, then (iii) is vacuously true, and (ii) is equivalent to the upper bound for the nonpositive income effects.

Under the above conditions, given $i \in \{1, 2\}$ and a preference $R_i \in (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$, let $\hat{R}_i \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ be a quasi-linear preference with nonincreasing incremental valuations such that if $R_i \neq R_0$, then $\hat{R}_i = R_i$, and if $R_i = R_0$, then for each $x_i \in M\setminus\{0\}$, $\hat{v}_i(x_i) - \hat{v}_i(x_i - 1) = \pi_i(x_i)$, where $\pi_i(\cdot)$ is the sequence satisfying the the above conditions. Then, the following rule f on $((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^2$ is the only rule satisfying the four properties: for each $R \in \mathcal{R}^2$, if (I) $R_i = R_0$ and $R_j \neq R_0$ for some pair $i, j \in N$ and $v_j \in (\underline{d}(x), \overline{d}(x))$ for some $x \in M\setminus\{0, m\}$, where v_j is the constant incremental valuation associated with R_j, then $f_i(R) = (x', v_j x')$ and $f_j(R) = (m - x', \sum_{y=0}^{m-x'-1} \pi_i(m - y))$ for some $x' \in M\setminus\{0\}$

D Proof of the characterization theorem

In this section, we provide the proof of Theorem 2.

D.1 Preliminary

We first show some preliminary results related to nondecreasing incremental valuations.

The following lemma says that bundling is one way to allocate the object efficiently.

Lemma 13 (Optimality of bundling). *Let $N' \subseteq N$ and $m' \in M$ be such that $m' > 0$. Then for each $R_{N'} \in (\mathcal{R}^{ND})^{|N'|}$ and each $z \in Z(N', m')$, we have $\max_{i \in N'} v_i(m', z_i) = \max_{x \in X(N', m')} \sum_{j \in N'} v_j(x_j, z_j)$.*

Proof. For each $x \in X(N', m')$,

$$\max_{i \in N'} v_i(m', z_i) \geq \sum_{j \in N'} \frac{x_j}{m'} \max_{i \in N'} v_i(m', z_i) \geq \sum_{j \in N'} \frac{x_j}{m'} v_j(m', z_j) \geq \sum_{j \in N'} v_j(x_j, z_j),$$

where the first inequality follows from $x \in X(N', m')$, and the last one from $R_{N'} \in (\mathcal{R}^{ND})^{|N'|}$ and Remark 3 (i). $\square$

The next lemma states that under an efficient allocation (resp. the outcome of the generalized Vickrey rule), if an agent's net valuation of m at his bundle (resp. $\mathbf{0}$) is not the highest one, then he can receive no object.

Lemma 14. *Let $R \in (\mathcal{R}^{ND})^n$ and $i \in N$. (i) Let $z \equiv (x, t) \in Z$ be efficient for R. If $v_i(m, z_i) < \max_{j \in N} v_j(m, z_j)$, then $x_i = 0$. (ii) Let $g(R) \equiv (x(R), t(R))$ be an outcome of the generalized Vickrey rule for R. If $v_i(m, \mathbf{0}) < \max_{j \in N} v_j(m, \mathbf{0})$, then $x_i(R) = 0$.*

Proof. (i) Suppose $v_i(m, z_i) < \max_{j \in N} v_j(m, z_j)$ and $x_i > 0$. Then,

$$\max_{j \in N} v_j(m, z_j) = \sum_{k \in N} \frac{x_k}{m} \max_{j \in N} v_j(m, z_j) > \sum_{k \in N} \frac{x_k}{m} v_k(m, z_k) \geq \sum_{k \in N} v_k(x_k, z_k),$$

where the equality follows from Lemma 1, the first inequality follows from $x_i > 0$ and $\max_{j \in N} v_j(m, z_j) > v_i(m, z_i)$, and the second one follows from $R \in (\mathcal{R}^{ND})^n$ and Remark 3 (i). By Remark 7, this contradicts efficiency.

(ii) Next, let $g(R) \equiv (x(R), t(R))$ be an outcome of the generalized Vickrey rule for R. By Remark 9, $z \equiv (z_j)_{j \in N} \equiv (x_j(R), v_j(x_j(R), \mathbf{0}))_{j \in N} = (x_j(R), V_j(x_j(R), \mathbf{0}))_{j \in N}$ is efficient for R. For each $j \in N$, by $z_j \; I_j \; \mathbf{0}$, Remark 1 (ii) gives $v_j(m, z_j) = v_j(m, \mathbf{0})$. Thus, we can show Lemma 14 (ii) in the same way as Lemma 14 (i) by using the efficient

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satisfying the inequalities in (iii) of the above condition and $\sum_{y=0}^{-1} \pi_i(m-y) \equiv 0$, and (II) otherwise, $f(R)$ is an outcome of the Vickrey rule for $\hat{R}$. If R_0 satisfies the single-intersection condition, then $\hat{R}_i = R_0^{inv}$ and the case (I) does not occur. Thus, in such a case, the rule f coincides with the inverse Vickrey rule. The proof of the above facts are available upon request.

allocation z for R. $\square$

Given $x \in X$, let $N^+(x) \equiv \{i \in N : x_i > 0\}$.

Note that Lemma 14 implies that if there are at least two agents who receive the object under an efficient allocation (resp. the outcome of the generalized Vickrey rule), then their net valuations of m at their bundles (resp. $\mathbf{0}$) must coincide. The following lemma further says that in such a case, the indifference curves of agents who receive the object through the bundles are flat.

Lemma 15 (Flat indifference curves). *Let $R \in (\mathcal{R}^{ND})^n$. (i) Let $z \equiv (x,t) \in Z$ be efficient for R. If $|N^+(x)| \geq 2$, then for each $i \in N^+(x)$ and each $x_i' \in M$, we have $v_i(x_i', z_i) = \frac{x_i'}{m} v_i(m, z_i)$. (ii) Let $g(R) \equiv (x(R), t(R))$ be an outcome of the generalized Vickrey rule for R. If $|N^+(x(R))| \geq 2$, then for each $i \in N^+(x(R))$ and each $x_i \in M$, we have $v_i(x_i, \mathbf{0}) = \frac{x_i}{m} v_i(m, \mathbf{0})$.*

Proof. (i) Suppose there is $x_i' \in X \backslash \{0, m\}$ such that $\frac{x_i'}{m} v_i(m, z_i) \neq v_i(x_i', z_i)$. By $R_i \in \mathcal{R}^{ND}$ and Remark 3 (i), $\frac{x_i'}{m} v_i(m, z_i) > v_i(x_i', z_i)$. By $|N^+(x)| \geq 2$ and $i \in N^+(x)$, $x_i \in M \backslash \{0, m\}$. Thus, by $R_i \in \mathcal{R}^{ND}$, Remark 3 (ii) gives $\frac{x_i}{m} v_i(m, z_i) > v_i(x_i, z_i)$. Then

$$v_i(m, z_i) = \sum_{j \in N} \frac{x_j}{m} v_i(m, x_i) = \sum_{j \in N} \frac{x_j}{m} v_j(m, z_j) > \sum_{j \in N} v_j(x_j, z_j),$$

where the first equality follows from Lemma 1, the second one, from $R \in (\mathcal{R}^{ND})^n$ and Lemma 14 (i), and the inequality follows from $x_i \in M \backslash \{0, m\}$, $\frac{x_i}{m} v_i(m, z_i) > v_i(x_i, z_i)$, $R_{-i} \in (\mathcal{R}^{ND})^{n-1}$, and Remark 3 (i). By Remark 7, this contradicts efficiency.

(ii) We can show Lemma 15 (ii) similarly to Lemma 15 (i), but by using Lemma 14 (ii) instead of Lemma 14 (i) and the efficient allocation $(x_j(R), v_j(x_j(R), \mathbf{0}))_{j \in N}$ for R instead of z. $\square$

The following proposition identifies the form of the payments under the generalized Vickrey rule for preferences with nondecreasing incremental valuations.

Proposition 5 (The generalized Vickrey rule payments). *Let $\mathcal{R} \subseteq \mathcal{R}^{ND}$. Let $g \equiv (x, t)$ be a generalized Vickrey rule on $\mathcal{R}^n$. Let $R \in (\mathcal{R}^{ND})^n$. Then for each $i \in N$, $t_i(R) = \frac{x_i(R)}{m} \max_{j \in N \backslash \{i\}} v_j(m, \mathbf{0})$. Moreover, if $|N^+(x(R))| \geq 2$, then $t_i(R) = v_i(x_i(R), \mathbf{0})$ for each $i \in N$.*

Proof. By the definition of the generalized Vickrey rule, for each $i \in N$, if $x_i(R) = 0$, then $t_i(R) = 0$. Thus, we only have to consider an agent $i \in N^+(x(R))$.

First, suppose $|N^+(x(R))| = 1$. Note that by $|N^+(x(R))| = 1$ and Remark 9, Lemma 1 implies $x_i(R) = m$. By $R_{-i} \in (\mathcal{R}^{ND})^{n-1}$ and Lemma 13, $t_i(R) = \sigma_i(R_{-i}; 0) = \max_{j \in N \backslash \{i\}} v_j(m, \mathbf{0})$.

Suppose instead $|N^+(x(R))| \geq 2$. For each $j \in N^+(x(R))$, each $x_j \in M$, and each $k \in N$,

$$v_j(x_j, \mathbf{0}) = \frac{x_j}{m} v_j(m, \mathbf{0}) \geq \frac{x_j}{m} v_k(m, \mathbf{0}) \geq v_k(x_j, \mathbf{0}), \tag{1}$$

where the equality follows from $R \in (\mathcal{R}^{ND})^n$ and Lemma 15 (ii), the first inequality follows from $R \in (\mathcal{R}^{ND})^n$ and Lemma 14 (ii), and the second one comes from $R_k \in \mathcal{R}^{ND}$ and Remark 3 (i). By $|N^+(x(R))| \geq 2$, there is $j \in N^+(x(R)) \backslash \{i\}$. By (1), for each $x_i \in M$,

$$v_i(x_i, \mathbf{0}) = v_j(x_i, \mathbf{0}) = \max_{k \in N \backslash \{i\}} v_k(x_i, \mathbf{0}). \tag{2}$$

Then

$$\begin{aligned}
t_i(R) &= \sigma_i(R_{-i}; 0) - \sigma_i(R_{-i}; x_i(R)) \\
&= \max_{k \in N \backslash \{i\}} v_k(m, \mathbf{0}) - \max_{k \in N \backslash \{i\}} v_k(m - x_i(R), \mathbf{0}) && \text{(by Lemma 13)} \\
&= v_i(m, \mathbf{0}) - v_i(m - x_i(R), \mathbf{0}) && \text{(by (2))} \\
&= \frac{x_i(R)}{m} v_i(m, \mathbf{0}) && \text{(by Lemma 15 (ii))} \\
&= \frac{x_i(R)}{m} \max_{k \in N \backslash \{i\}} v_k(m, \mathbf{0}). && \text{(by (2))}
\end{aligned}$$

Further, by $R \in (\mathcal{R}^{ND})^n$ and Lemma 15 (ii), $t_i(R) = \frac{x_i(R)}{m} v_i(m, \mathbf{0}) = v_i(x_i(R), \mathbf{0})$. □

D.2 Proof of the "if" part

Let $\mathcal{R}$ be a class of preferences such that $\mathcal{R}^C \cap \mathcal{R}^Q \subseteq \mathcal{R} \subseteq \mathcal{R}^{ND}$ and $g \equiv (x, t)$ be a generalized Vickrey rule on $\mathcal{R}^n$. Since *individual rationality* and *no subsidy for losers* are immediate from the definition of the generalized Vickrey rule, we omit the proofs.

Efficiency. Let $R \in \mathcal{R}^n$. We show $g(R)$ is efficient for R.

Suppose $|N^+(x(R))| = 1$. By Remark 9 and Lemma 1, $x_i(R) = m$ for $i \in N^+(x(R))$. By $\mathcal{R} \subseteq \mathcal{R}^{ND}$, Proposition 5 gives $g_i(R) = (m, \max_{j \in N \backslash \{i\}} v_j(m, \mathbf{0}))$ and $g_j(R) = \mathbf{0}$ for each $j \in N \backslash \{i\}$. Thus, $t_i(R) = \max_{j \in N \backslash \{i\}} v_j(m, g_j(R))$. By Remark 1 (iii) and Lemma 3 (i),

$$v_i(m, g_i(R)) = \max_{j \in N \backslash \{i\}} v_j(m, g_j(R)) - V_i(0, g_i(R)) \geq \max_{j \in N \backslash \{i\}} v_j(m, g_j(R)). \tag{1}$$

For each $x \in X$,

$$v_i(m, g_i(R)) \geq \sum_{j \in N} \frac{x_j}{m} v_i(m, g_i(R)) \geq \sum_{j \in N} \frac{x_j}{m} v_j(m, g_j(R)) \geq \sum_{j \in N} v_j(x_j, g_j(R)),$$

where the second inequality follows from (1), and the last one follows from $R \in (\mathcal{R}^{ND})^n$ and Remark 3 (i). By Remark 7, $g(R)$ is efficient for R.

Suppose instead $|N^+(x(R))| \geq 2$. By $\mathcal{R} \subseteq \mathcal{R}^{ND}$ and Proposition 5, $t_i(R) = v_i(x_i(R), \mathbf{0})$ for each $i \in N$. Thus, Remark 9 implies that $g(R)$ is efficient for R.

Strategy-proofness. Let $R \in \mathcal{R}^n$, $i \in N$, and $R'_i \in \mathcal{R}$. By $\mathcal{R} \subseteq \mathcal{R}^{ND}$, Proposition 5 implies $g_i(R'_i, R_{-i}) = (x_i, \frac{x_i}{m} \max_{j \in N \backslash \{i\}} v_j(m, \mathbf{0}))$, where $x_i \equiv x_i(R'_i, R_{-i})$. We show that

$g_i(R)\ R_i\ (x_i, \frac{x_i}{m} \max_{j \in N\setminus\{i\}} v_j(m, \mathbf{0}))$.

Suppose first $x_i(R) = m$. Let $s_i \equiv V_i(0, g_i(R))$. By Lemma 3 (i), $s_i \le 0$. Then

$$V_i(x_i, g_i(R)) = v_i(x_i, g_i(R)) + s_i \le \frac{x_i}{m}\big(v_i(m, g_i(R)) + s_i\big) = \frac{x_i}{m} \max_{j \in N\setminus\{i\}} v_j(m, \mathbf{0}),$$

where the inequality follows from $R_i \in \mathcal{R}^{ND}$, Remark 3 (i), and $s_i \le 0$, and the last equality follows from Remark 1 (iii), $\mathcal{R} \subseteq \mathcal{R}^{ND}$, and Proposition 5.

Next, suppose $x_i(R) < m$. By Lemma 1, there is $j \in N^+(x(R))\setminus\{i\}$. By $R \in (\mathcal{R}^{ND})^n$ and Lemma 14 (ii), $v_j(m, \mathbf{0}) = \max_{k \in N} v_k(m, \mathbf{0})$. Then

$$v_i(m, \mathbf{0}) \le v_j(m, \mathbf{0}) = \max_{k \in N\setminus\{i\}} v_k(m, \mathbf{0}). \tag{2}$$

We have

$$V_i(x_i, g_i(R)) \le v_i(x_i, \mathbf{0}) \le \frac{x_i}{m} v_i(m, \mathbf{0}) \le \frac{x_i}{m} \max_{k \in N\setminus\{i\}} v_k(m, \mathbf{0}),$$

where the first inequality follows from Lemma 3 (iii), the second one follows from $R_i \in \mathcal{R}^{ND}$ and Remark 3 (i), and the last one, from (2).

In either case, we obtain $g_i(R)\ R_i\ (x_i, \frac{x_i}{m} \max_{j \in N\setminus\{i\}} v_j(m, \mathbf{0}))$. ■

D.3 Proof of the "only if" part

In this section, we provide the proof of the "only if" part. Throughout the subsection, we fix a class of preferences $\mathcal{R}$ such that $\mathcal{R}^C \cap \mathcal{R}^Q \subseteq \mathcal{R} \subseteq \mathcal{R}^{ND}$ and a rule f on $\mathcal{R}^n$ satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*.

We first set up the following lemma.

Lemma 16. *Let $R \in \mathcal{R}^n$ and $i \in N$. If $v_i(m, \mathbf{0}) < \max_{j \in N} v_j(m, \mathbf{0})$, then $x_i(R) = 0$.*

Proof. The proof is in two steps.

STEP 1. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $R_i \in \mathcal{R}^C \cap \mathcal{R}^Q$ and $v_i(m) < \max_{j \in N} v_j(m, \mathbf{0})$. We show $x_i(R) = 0$. Suppose by contradiction that $x_i(R) > 0$.

Let $j \in \arg\max_{k \in N} v_k(m, \mathbf{0})$. We show $x_j(R) > 0$. Suppose $x_j(R) = 0$. By Lemma 2, $f_j(R) = \mathbf{0}$. By $v_i(m) < V_j(m, \mathbf{0})$ and $R \in (\mathcal{R}^{ND})^n$, Lemma 14 (i) implies $x_i(R) = 0$, a contradiction. Thus, $x_j(R) > 0$.

Next, we show $t_j(R) \ge v_i(x_j(R))$. Suppose $t_j(R) < v_i(x_j(R))$. Let $R'_j \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that $t_j(R) < v'_j(x_j(R)) < v_i(x_j(R))$. By $R'_j \in \mathcal{R}^C$, $v'_j(x_j(R)) < v_i(x_j(R))$, and $R_i \in \mathcal{R}^C$,

$$v'_j(m) = \frac{m}{x_j(R)} v'_j(x_j(R)) < \frac{m}{x_j(R)} v_i(x_j(R)) = v_i(m).$$

By $R \in (\mathcal{R}^{ND})^n$, Lemmas 2 and 14 (i) together imply $f_j(R'_j, R_{-j}) = \mathbf{0}$. By $t_j(R) < v'_j(x_j(R))$, $f_j(R)\ P'_j\ \mathbf{0} = f_j(R'_j, R_{-j})$, contradicting *strategy-proofness*. Thus, $t_j(R) \ge v_i(x_j(R))$.

Note that by *individual rationality*, $f_j(R)\ R_j\ \mathbf{0}$. If $f_j(R)\ I_j\ \mathbf{0}$, then by Remark 1 (ii),

$$v_j(m, f_j(R)) = v_j(m, \mathbf{0}) > v_i(m).$$

Instead, if $f_j(R)\ P_j\ \mathbf{0}$, then $V_j(0, f_j(R)) < 0$, and so

$$v_j(m, f_j(R)) = \frac{m}{x_j(R)}\Big(t_j(R) - V_j(0, f_j(R))\Big) > \frac{m}{x_j(R)} v_i(x_j(R)) = v_i(m),$$

where the first equality follows from $R \in (\mathcal{R}^{ND})^n$, Lemma 15 (i), and Remark 1 (iii), the inequality follows from $t_j(R) \geq v_i(x_j(R))$ and $V_j(0, f_j(R)) < 0$, and the last equality follows from $R_i \in \mathcal{R}^C$. In either case, by $R \in (\mathcal{R}^{ND})^n$, Lemma 14 (i) implies $x_i(R) = 0$. But this contradicts $x_i(R) > 0$.

Step 2. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $v_i(m, \mathbf{0}) < \max_{j \in N} v_j(m, \mathbf{0})$. Suppose $x_i(R) > 0$. Let $R'_i \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that $v_i(m, \mathbf{0}) < v'_i(m) < \max_{j \in N} v_j(m, \mathbf{0})$. By Step 1, $x_i(R'_i, R_{-i}) = 0$. By Lemma 2, $f_i(R'_i, R_{-i}) = \mathbf{0}$. Then,

$$t_i(R) \leq v_i(x_i(R), \mathbf{0}) \leq \frac{x_i(R)}{m} v_i(m, \mathbf{0}) < \frac{x_i(R)}{m} v'_i(m) = v'_i(x_i(R)),$$

where the first inequality follows from Lemma 3 (ii), the second one follows from $R_i \in \mathcal{R}^{ND}$ and Remark 3 (i), the third one follows from $v_i(m, \mathbf{0}) < v'_i(m)$ and $x_i(R) > 0$, and the equality follows from $R'_i \in \mathcal{R}^C$. Thus, $f_i(R)\ P'_i\ \mathbf{0} = f_i(R'_i, R_{-i})$, contradicting *strategy-proofness*. $\square$

We now show that f is a generalized Vickrey rule on $\mathcal{R}^n$.

Step 1. We first show that the payments under f coincide with those of the generalized Vickrey rule. Let $R \in \mathcal{R}^n$ and $i \in N$. Note that by $\mathcal{R} \subseteq \mathcal{R}^{ND}$ and Proposition 5, we only have to show that $t_i(R) = \frac{x_i(R)}{m} \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0})$. Suppose not. By Lemma 2, we must have $x_i(R) > 0$. We divide the argument into two cases.

Case 1. $t_i(R) > \frac{x_i(R)}{m} \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0})$.

We have

$$v_i(m, \mathbf{0}) \geq \frac{m}{x_i(R)} v_i(x_i(R), \mathbf{0}) \geq \frac{m}{x_i(R)} t_i(R) > \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0}),$$

where the first inequality follows from $R_i \in \mathcal{R}^{ND}$ and Remark 3 (i), the second one follows from Lemma 3 (ii), and the last one comes from $t_i(R) > \frac{x_i(R)}{m} \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0})$. Thus, by Lemma 16, $x_j(R) = 0$ for each $j \in N \setminus \{i\}$. By Lemma 1, $x_i(R) = m$. Thus, by $t_i(R) > \frac{x_i(R)}{m} \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0})$, $t_i(R) > \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0})$. Let $R'_i \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that $\max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0}) < v'_i(m) < t_i(R)$. Again, by Lemmas 1 and 16, $x_i(R'_i, R_{-i}) =$

m. Thus, by Lemma 3 (ii), $t_i(R'_i, R_{-i}) \leq v'_i(m) < t_i(R)$, which implies $f_i(R'_i, R_{-i}) \ P_i \ f_i(R)$. However, this contradicts *strategy-proofness*.

CASE 2. $t_i(R) < \frac{x_i(R)}{m} \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0})$.

Let $R'_i \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that $t_i(R) < v'_i(x_i(R)) < \frac{x_i(R)}{m} \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0})$. By $R'_i \in \mathcal{R}^C$ and $v'_i(x_i(R)) < \frac{x_i(R)}{m} \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0})$,

$$v'_i(m) = \frac{m}{x_i(R)} v'_i(x_i(R)) < \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0}).$$

Thus, Lemmas 2 and 16 together imply $f_i(R'_i, R_{-i}) = \mathbf{0}$. However, by $t_i(R) < v'_i(x_i(R))$, $f_i(R) \ P'_i \ \mathbf{0} = f_i(R'_i, R_{-i})$, contradicting *strategy-proofness*.

STEP 2. Let $R \in \mathcal{R}^n$. We show $\sum_{i \in N} v_i(x_i(R), \mathbf{0}) = \max_{x \in X} \sum_{i \in N} v_i(x_i, \mathbf{0})$.

Suppose $|N^+(x(R))| = 1$. Let $i \in N^+(x(R))$. By $|N^+(x(R))| = 1$ and Lemma 1, $x_i(R) = m$. Thus, $x_j(R) = 0$ for each $j \in N \setminus \{i\}$. Then

$$\sum_{j \in N} v_j(x_j(R), \mathbf{0}) = v_i(m, \mathbf{0}) = \max_{j \in N} v_j(m, \mathbf{0}) = \max_{x \in X} \sum_{j \in N} v_j(x_j, \mathbf{0}),$$

where the second equality follows from Lemma 16, and the last one comes from $R \in (\mathcal{R}^{ND})^n$ and Lemma 13.

Next, suppose $|N^+(x(R))| \geq 2$. We show $f_i(R) \ I_i \ \mathbf{0}$ for each $i \in N$. By *individual rationality*, $f_i(R) \ R_i \ \mathbf{0}$ for each $i \in N$. Suppose there is $i \in N$ such that $f_i(R) \ P_i \ \mathbf{0}$. Then $t_i(R) < V_i(x_i(R), \mathbf{0}) = v_i(x_i(R), \mathbf{0})$. Thus, by $R_i \in \mathcal{R}^{ND}$, Remark 3 (i), and Step 1,

$$\frac{x_i(R)}{m} v_i(m, \mathbf{0}) \geq v_i(x_i(R), \mathbf{0}) > t_i(R) = \frac{x_i(R)}{m} \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0}).$$

This implies $v_i(m, \mathbf{0}) > \max_{j \in N \setminus \{i\}} v_j(m, \mathbf{0})$. By Lemma 16, $x_j(R) = 0$ for each $j \in N \setminus \{i\}$. But this contradicts $|N^+(x(R))| \geq 2$.

Thus, $f_i(R) \ I_i \ \mathbf{0}$ for each $i \in N$. By Remark 1 (ii), $v_i(x_i, f_i(R)) = v_i(x_i, \mathbf{0})$ for each $i \in N$ and $x_i \in M$. Thus, we have

$$\sum_{i \in N} v_i(x_i(R), \mathbf{0}) = \sum_{i \in N} v_i(x_i(R), f_i(R)) = \max_{x \in X} \sum_{i \in N} v_i(x_i, f_i(R)) = \max_{x \in X} \sum_{i \in N} v_i(x_i, \mathbf{0}),$$

where the second equality follows from Remark 7 and *efficiency*. ■

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