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SIGNALING UNDER DOUBLE-CROSSING PREFERENCES

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Abstract. This paper provides a general analysis of signaling under double-crossing preferences with a continuum of types. There are natural economic environments where indifference curves of two types cross twice, so that the celebrated single-crossing property fails to hold. Equilibrium exhibits a particular form of pooling: there is a threshold type below which types choose actions that are fully revealing and above which they choose actions that are clustered in possibly non-monotonic ways, with a gap separating these two sets of types. We also provide an algorithm to establish equilibrium existence by construction under mild conditions.

Keywords. single-crossing property; double-crossing property; counter-signaling; pairwise pooling; mass pooling

JEL Classification. D82; I21

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1. Introduction

There are a few assumptions in economics that have earned gold standard status. The single-crossing property, also known as the Spence-Mirrlees condition, which is routinely assumed in signaling (Spence, 1973) and screening (Mirrlees, 1971) models, is one of them. In the context of the classic education signaling model of Spence (1973), for example, the single-crossing property states that an indifference curve of a higher type (in the space of education level and wages) crosses that of a lower type once and only once. This assumption captures the idea that the marginal cost of education is relatively cheaper for more able workers—as a result more able workers find it profitable to signal their ability through investing in education while less able workers do not choose to mimic—thus making it possible to separate the two types by observing their education choices. Many insights we learn from various analyses of signaling behavior, such as corporate financing decisions (Leland and Pyle, 1977), advertising (Milgrom and Roberts, 1986), or even biological signals (Grafen, 1990), are rooted in this property.

While the single-crossing property has been widely accepted and used, economists do not always think of it as an accurate reflection of reality; it is rather a convenient assumption for analytical clarity and tractability. Although this property can be a good local approximation for some range of signaling levels, Mailath (1987, p. 1355) notes that "in many applications, it is difficult, if not impossible, to verify that the single crossing condition is satisfied for all [signaling and reputation levels]." Moreover, as Hörner (2008) remarks in an encyclopedic article on singaling and screening, "Little is known about equilibria when single-crossing fails, as may occur in applications." There is no guarantee that any insight gained from the class of models characterized by the single-crossing property can be extended straightforwardly to a model with wider scope.

The possibility that the single-crossing property may fail to hold in some environments has been acknowledged in the literature, and there are sporadic and independent attempts to look into this situation in the analysis of signaling (Feltovich et al., 2002; Araujo et al., 2007; Daley and Green, 2014; Bobtcheff and Levy, 2017; Chen et al., 2020; Frankel and Kartik, 2019).[1] Much of this literature considers either a small number of discrete types or

[1] There are also some attempts to relax the single-crossing property in the analysis of screening. See Smart (2000), Araujo and Moreira (2010) and Schottmüller (2015). Matthews and Moore (1987) introduce double-crossing utility curves in a multi-dimensional screening problem, but their focus and formulation are different from ours, which relies on double-crossing indifference curves.

some specific payoff functions (or both). In this paper, we provide an analysis of a standard signaling model with a continuum of types, except that the usual single-crossing property is replaced by a *double-crossing property*—indifference curves of two types cross twice in the relevant space. To the best of our knowledge, this is the first general analysis of signaling under double-crossing preferences. The paper intends to make three contributions.

First, we show in Section 3 via examples that there are many situations of economic interest that exhibit the double-crossing property. One factor which potentially breaks the single-crossing property is that gains from signaling are typically not unbounded; beyond some level the gains diminish as an agent invests more in signaling. Moreover, higher, more productive, types may reach this point of diminishing returns at lower signaling levels than do lower types. Thus, the benefit-cost ratio of signaling is greater initially for higher types than for lower types, but the comparison is reversed past some signaling level, resulting in the double-crossing property. We provide several examples to capture this principle and show that the single-crossing property can be easily turned into the double-crossing property with minor modifications of the underlying specification.

Second, we provide a characterization of equilibria in Section 4. We introduce Low types Separate High types Pairwise-Pool (LSHPP) equilibrium, and show that any D1 equilibrium under the double-crossing property is LSHPP. In such an equilibrium, there is a threshold type above which two distinct types may pair up to choose the same signaling action, or two distinct intervals of types pair up, with pairs that are farther apart choosing lower actions than pairs that are closer to one another. Our notion of LSHPP is a generalized version of Low types Separate High types Pool (LSHP) introduced by Kartik (2009). An important difference from Kartik's (2009) model (and also from Bernheim (1994)) is that we do not impose an exogenous bound on the signaling space. Instead, "pairwise-pooling" is the result of endogenous constraints induced by the double-crossing property.

Finally, in Section 5, we provide an algorithm to find an LSHPP equilibrium and establish its existence by construction. Pairwise-pooling is related to a phenomenon known as "counter-signaling," where low and high types pool by refraining from costly signaling while intermediate types separate from those types by signaling (Feltovich et al., 2002; Araujo et al., 2007; Chung and Eso, 2013). When types are continuously distributed, however, establishing a counter-signaling equilibrium is not straightforward, and our understanding of counter-signaling has been limited to specific contexts.[2] Our equilibrium

[2] Araujo et al. (2007) provide a form of counter-signaling with a continuum of types, but their analysis relies on the assumption that an agent's two-dimensional type can be identified up to a linear combination.

construction generalizes the notion of counter-signaling to that of pairwise-pooling and enables us to establish its existence under mild conditions, suggesting that counter-signaling is not a pathological outcome that can occur only under a stringent set of circumstances.

2. Model

We consider a standard signaling model, except that the usual single-crossing property is replaced by a double-crossing property, which we will define more precisely below. An agent, characterized by his type $\theta \in [\underline{\theta}, \overline{\theta}]$, chooses a publicly observable action (signaling level) $a \in \mathbb{R}_+$. The type of an agent is his private information. The payoff to an agent is $u(a, t, \theta)$, where t is the market's perception of his type, or his "reputation," i.e., $t = \mathbb{E}[\theta \mid a]$. We assume that the agent benefits from a higher reputation.

Assumption 1. *$u : \mathbb{R}_+ \times [\underline{\theta}, \overline{\theta}]^2 \to \mathbb{R}$ is twice continuously differentiable, and is strictly increasing in t.*

In the subsequent analysis, we make heavy use of the marginal rate of substitution between signaling action a and reputation t, defined as

$$m(a, t, \theta) := -\frac{u_a(a, t, \theta)}{u_t(a, t, \theta)}.$$

It measures the increase in reputation that is needed to compensate an increase in signaling level. Loosely speaking, signaling is relatively cheap when the marginal rate of substitution is low. If we let $t = \phi(a, u, \theta)$ represent the indifference curve for type θ at utility level u in the (a, t)-space, then the marginal rate of substitution gives the slope of indifference curves. Specifically, $\phi_a(a, u, \theta) = m(a, \phi(a, u, \theta), \theta)$.

Preferences satisfy the single-crossing property if whenever a lower type θ'' is indifferent between a higher signaling action a_1 to a lower signaling action a_2, a higher type θ' strictly prefers the higher action a_1. This is equivalent to requiring that $m(a, t, \theta') < m(a, t, \theta'')$ for any $\theta' > \theta''$ and any (a, t). It implies that an indifference curve of a higher type crosses that of a lower type once and from above. We often refer to this case as the "standard setup."

We relax the standard setup to allow for "double-crossing preferences." Our focus is to study situations in which the single-crossing property holds when the signaling level is low, but fails when the signaling level is high.

Definition 1 (Double-crossing property). *For any $\theta' > \theta''$, there exists a continuous function $D(\cdot;\theta',\theta'') : [\underline{\theta},\overline{\theta}] \to \mathbb{R}_+$ such that*

(a) if $a < a_0 \leq D(t_0;\theta',\theta'')$, then

$$u(a,t,\theta'') \leq u(a_0,t_0,\theta'') \implies u(a,t,\theta') < u(a_0,t_0,\theta');$$

(b) if $a > a_0 \geq D(t_0;\theta',\theta'')$, then

$$u(a,t,\theta'') \leq u(a_0,t_0,\theta'') \implies u(a,t,\theta') < u(a_0,t_0,\theta').$$

The locus of points $\{(a,t) : a = D(t;\theta',\theta'')\}$ partitions the (a,t)-space into two regions. For signaling actions to the left of the "dividing line" $D(\cdot;\theta',\theta'')$, the standard single-crossing property holds for types θ' and θ''. To the right of the dividing line, the *reverse single-crossing property* holds: whenever the lower type θ'' is indifferent between a higher action a_1 and a lower action a_2, the higher type θ' strictly prefers the lower action. Note that the double-crossing property does not impose any specific restrictions on the rankings between actions on opposite sides of the dividing line. It also does not require $D(t;\theta',\theta'')$ to be monotone in t.

Assumption 2. *$u(\cdot)$ satisfies the double-crossing property.*

For $\theta' > \theta''$, $m(a,t,\theta') - m(a,t,\theta'')$ is negative in the standard sepup. Assumption 2, on the other hand, implies that this difference is single-crossing from below, with crossing point at $a = D(t;\theta',\theta'')$. But the latter condition alone does not imply Assumption 2. Suppose type θ'' is indifferent between (a_1,t_1) and (a_2,t_2). Parts (a) and (b) of Definition 1 together suggests that $D(t_2;\theta',\theta'') \leq a_2 < a_1 \leq D(t_1;\theta',\theta'')$ would lead to a contradiction. To avoid this situation, if (a_1,t_1) is to the left of the dividing line $D(\cdot;\theta',\theta'')$, then any combination (a_2,t_2) on the indifference curve of type θ'' passing through (a_1,t_1) with $a_2 < a_1$ must remain on the left of the dividing line. Likewise, if (a_2,t_2) is to the right of the dividing line $D(\cdot;\theta',\theta'')$, then any combination (a_1,t_1) on the indifference curve passing through (a_2,t_2) with $a_1 > a_2$ must remain on the right of the dividing line.

Formally, suppose type θ'' attains utility level u_0 at (a_0,t_0). We require that the difference in marginal rate of substitution between two types is single-crossing from below *along an indifference curve* of one type (say, the lower type): for $\theta' > \theta''$,

$$m(a,\phi(a,u_0,\theta''),\theta') - m(a,\phi(a,u_0,\theta''),\theta'') \begin{cases} \leq 0 & \text{if } a \leq a_0 \leq D(t_0;\theta',\theta''), \\ \geq 0 & \text{if } a \geq a_0 \geq D(t_0;\theta',\theta''); \end{cases} \tag{1}$$

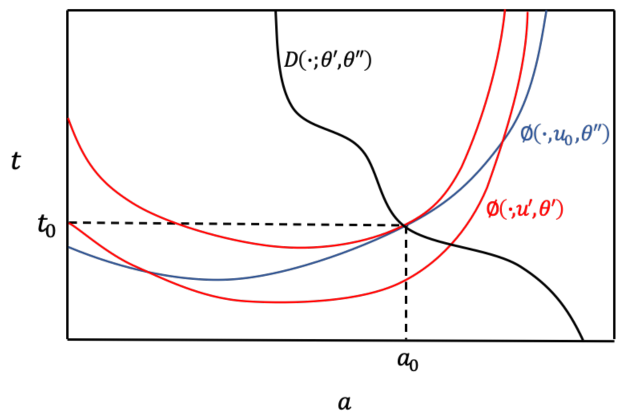

Figure 1. *Double-crossing property. The indifference curve of a higher type θ' crosses that of a lower type θ'' from above to the left of the dividing line $D(\cdot;\theta',\theta'')$, and crosses it again from below to the right of the dividing line. Along the dividing line, higher types have more convex indifference curves.*

with strict inequality except when $a = a_0 = D(t_0;\theta',\theta'')$. It is clear that Assumption 2 is satisfied if and only if there exists $D(\cdot;\theta',\theta'')$ such that (1) holds; so (1) can be adopted as an alternative definition of the double-crossing property.[3]

In Figure 1, we show the indifference curves of types θ'' and θ'' in the (a,t)-space. To the left of the dividing line $D(\cdot;\theta',\theta'')$, the indifference curve of the higher type θ' must cross $\phi(\cdot,u_0,\theta'')$ from above. To the right, they must cross $\phi(\cdot,u_0,\theta'')$ from below. At the boundary, the indifference curves of the two types are tangent to each other, with the high type having indifference curves that are "more convex." Any indifference curve can cross the dividing line $D(\cdot;\theta',\theta'')$ only once.[4]

Assumptions 1 and 2 are sufficient for an analysis of signaling under double-crossing preferences when there are only two types. To allow for a general analysis with multiple types, we need to make assumptions about how the dividing line $D(\cdot;\theta',\theta'')$ shifts with respect to θ' and θ''.[5]

Assumption 3. *For any t, $D(t;\theta',\theta'')$ strictly decreases in θ' and in θ''.*

[3] For completeness, we provide a proof of this claim in Online Appendix C.

[4] Otherwise, for $a < a_0$, it is possible to have $a_0 \le D(t_0;\theta',\theta'')$ and $a > D(\phi(a,u_0,\theta'');\theta',\theta'')$. This would lead to a contradiction under condition (1).

[5] With three types, for example, there would be three dividing lines (one for each pair of types) and six possible rankings of these dividing lines for each value of t. Any analysis will become unmanageable without further restrictions as the number of types increases.

The dividing line $D(\cdot;\theta',\theta'')$ is defined for $\theta' > \theta''$. We will extend the domain of D to allow for $\theta' \geq \theta''$ by defining, for any t,

$$D(t;\theta,\theta) := \lim_{\theta''\to\theta^-} D(t;\theta,\theta'') = \lim_{\theta'\to\theta^+} D(t;\theta',\theta).$$

Assumption 3 implies that $D(t;\theta',\theta'')$ is monotone in θ''; so the limit is well defined.

Definition 2. *(a,t) is in the SC-domain of type θ if it belongs to the set $SC(\theta) := \{(a,t) : a < D(t;\theta,\theta)\}$; and it is in the RSC-domain of type θ if it belongs to $RSC(\theta) := \{(a,t) : a > D(t;\theta,\theta)\}$.*

Assumption 3 implies that for any $\theta' > \theta''$, $SC(\theta') \subset SC(\theta'')$ and $RSC(\theta') \supset RSC(\theta'')$. When (a,t) is in the SC-domain of type θ, among any two types lower than θ, the higher type has a smaller marginal rate of substitution at this point than the lower type. This follows because $a < D(t;\theta,\theta) < D(t;\theta',\theta'')$ for any $\theta \geq \theta' > \theta''$. When (a,t) is in the RSC-domain of type θ, among any two types higher than θ, the higher type has a larger marginal rate of substitution than the lower type. When (a,t) is on the boundary of the SC-domain and RSC-domain of type θ, this type has a lower marginal rate of substitution than any other type. In other words,

$$a = D(t;\theta,\theta) \iff \theta = \operatorname*{argmin}_{\theta'} m(a,t,\theta'). \tag{2}$$

Assumption 3 is not easy to interpret in terms of preferences. The following result is useful for relating it to the marginal rate of substitution.

Lemma 1. *Suppose preferences satisfy the double-crossing property. Then Assumption 3 holds if and only if $m(a,t,\theta)$ is strictly quasi-convex in θ.*

Proof. Take any (a,t) such that $a = D(t;\theta',\theta'')$. Suppose $D(t;\theta',\cdot)$ is decreasing. For $\theta_1 \in (\theta'',\theta')$, $a > D(t;\theta',\theta_1)$ implies $m(a,t,\theta_1) < m(a,t,\theta')$. For $\theta_2 < \theta''$, $a < D(t;\theta',\theta_2)$ implies $m(a,t,\theta'') = m(a,t,\theta') < m(a,t,\theta_2)$. If $D(t;\cdot,\theta'')$ is decreasing, then for $\theta_3 > \theta'$, $a > D(t;\theta_3,\theta'')$ implies $m(a,t,\theta') = m(a,t,\theta'') < m(a,t,\theta_3)$. Thus Assumption 3 implies that $m(a,t,\cdot)$ is quasi-convex. If $a > D(t;\underline{\theta},\underline{\theta}) > D(t;\theta',\theta'')$, then $m(a,t,\theta') > m(a,t,\theta'')$ for any $\theta' > \theta''$; so $m(a,t,\cdot)$ is strictly increasing. If $a < D(t;\overline{\theta},\overline{\theta}) < D(t;\theta',\theta'')$, then $m(a,t,\theta') < m(a,t,\theta'')$ for any $\theta' > \theta''$; so $m(a,t,\cdot)$ is strictly decreasing. In the latter two cases, monotone functions are quasi-convex.

Conversely, suppose $m(a,t,\theta)$ is quasi-convex. Take any (a,t) such that $a = D(t;\theta',\theta'')$. For $\theta_1 \in (\theta'',\theta')$, $m(a,t,\theta_1) < m(a,t,\theta')$ implies $a > D(t;\theta',\theta_1)$. For $\theta_2 < \theta''$, $m(a,t,\theta_2) > m(a,t,\theta')$ implies $a < D(t;\theta',\theta_2)$. This shows that $D(t;\theta',\cdot)$ is decreasing. A similar argument establishes that $D(t;\cdot,\theta'')$ is decreasing. ∎

Given this result, an alternative way to state Definition 2 is that (a,t) belongs to the SC-domain of type θ if $m(a,t,\cdot)$ is locally decreasing at θ, and it belongs to the RSC-domain of type θ if $m(a,t,\cdot)$ is locally increasing at θ. In the standard setup, marginal rate of substitution strictly decreases in type, reflecting the assumption that higher types have lower signaling costs. The double-crossing property with Assumption 3 is relevant for situations in which the marginal costs of signaling are lowest for intermediate types.

Finally, the probability distribution over types is given by a continuous function F with full support. Signaling models typically exhibit a plethora of equilibria, and we adopt the D1 criterion (Cho and Kreps, 1987) to restrict off-equilibrium beliefs. Under the D1 criterion, the standard setup predicts the least-cost separating equilibrium, which is distribution-free. This is not the case for our model, where D1 equilibria often entail some pooling. As a consequence, the distribution of types has a nontrivial impact on the equilibrium allocation.

3. Examples

While our specification is a natural way to define double-crossing preferences, the assumptions we adopt do impose economically meaningful restrictions on preferences, which may or may not be reasonable depending on the context of application. Specifically, Assumption 2 implies that indifference curves of higher types are more convex than those of lower types. In the standard setup, the relevant issues is which type has a higher marginal rate of substitution. Under double-crossing preferences, the issue is of higher order: we need to determine how the slope of marginal rates of substitution is related to agent type, for which there appears to be no *a priori* obvious specification.

To better motivate the modeling choices we make and to demonstrate the relevance of our analysis, we provide four examples of economic applications that give rise to double-crossing preferences. We argue through these examples that, despite its pervasive use in signaling models, the single-crossing property is not as innocuous as it is generally believed, and there are many situations of economic interest that are better characterized by the double-crossing property. In the process, we attempt to justify our assumptions by showing that higher types naturally have more convex indifference curves in many economic settings, and by explaining the logic behind such preferences.

3.1. Signaling with news

Several works have pointed out that the single-crossing property fails in signaling models with additional information sources such as news or "grades" (Feltovich et al., 2002; Araujo et al., 2007; Daley and Green, 2014). For illustration, we use a very simple formulation of additional information; the literature has developed more complicated models.

Consider an environment where there are two sources of information: a signaling action and a test outcome. The test outcome is binary, either pass or fail, and the agent passes the test with probability $\beta_0 + \beta\theta$ (where $\beta > 0$). If the agent passes the test, he will be promoted and earn λV. If he fails, he will be fired and his outside payoff depends on his reputation. Let the outside payoff be $\lambda t < \lambda V$. The agent's utility is:

$$u(a,t,\theta) = (\beta_0 + \beta\theta)\lambda V + [1-(\beta_0+\beta\theta)]\lambda t - \left(\frac{\gamma a}{\theta} + \frac{a^2}{2}\right),$$

where the last term in parentheses represents the cost of signaling, and $\gamma > 0$ is a cost parameter. The marginal rate of substitution is

$$m(a,t,\theta) = \frac{\gamma + a\theta}{\lambda\theta[1-(\beta_0+\beta\theta)]}.$$

For $\theta' > \theta''$, $m(a,t,\theta')/m(a,t,\theta'')$ increases in a. This shows that $m(a,t,\theta') - m(a,t,\theta'')$ is single-crossing from below. Since $m(a,t,\theta)$ is independent of t, this suffices for Assumption 2 to hold. Assumption 3 also holds because $m(a,t,\theta)$ is quasi-convex in θ.[6]

In this class of models, the single-crossing property breaks down because higher types have less incentive to engage in costly signaling, knowing that their type will be partially revealed by exogenous news anyway. Because of this, the marginal gain from signaling is not necessarily higher for higher types. As Feltovich et al. (2002) illustrate, this type of model often leads to a phenomenon known as "counter-signaling," in which higher types refrain from costly signaling.[7] We will later show that the possibility of counter-signaling is a common feature of equilibrium under double-crossing preferences.

[6] We provide details of the relevant calculations for the examples in this section in Online Appendix D.

[7] Araujo et al. (2007) also show the possibility of counter-signaling, but the underlying logic is different. In Araujo et al. (2007), the agent's unknown attributes are two dimensional, but there is a public observable interview result which reveals this information up to some linear combination. Frankel and Kartik (2019) discuss how two-dimensional types may lead to failure of the single-crossing property. See also Ball (2020).

3.2. Reputation enhances the chances of success

In many facets of life a person's chances of success depend not only on his true ability, but on other people's perception of his ability as well. Take the case of a startup entrepreneur. His reputation in the market affects the availability of initial funding and the capacity to attract talents to work in his firm. These factors, together with his true entrepreneurial ability, determine the performance of his business and its chances of reaching the next milestone (such as developing a prototype product, or attracting the next round of funding) over the course of the project. In this example signaling incentive comes from the fact that reputation matters for improving performance.

Suppose the performance of a startup entrepreneur is $\theta + \beta t + \varepsilon$, where $\beta > 0$ is a weight that determines the importance of reputation relative to true ability. The term ε summarizes the random factors that may affect performance, and its distribution is given by $G(\cdot)$ with a corresponding log-concave density $g(\cdot)$. The startup business can reach the next milestone if its performance exceeds some exogenous threshold K, and the value of reaching the milestone is V. Let a represent the level of signaling activity he chooses to establish his reputation. The payoff to the entrepreneur is

$$u(a,t,\theta) = V\,(1 - G(K - \theta - \beta t)) - \left(\frac{\gamma a}{\theta} + \frac{a^2}{2}\right),$$

where $\gamma > 0$ is a signaling cost parameter. This gives

$$m(a,t,\theta) = \frac{\gamma + \theta a}{\theta \beta V g(K - \theta - \beta t)}.$$

One can verify that $m(a,t,\theta)$ is quasi-convex in θ. Moreover, for $\theta' > \theta''$, if $\phi(\cdot, u_0, \theta'')$ is an indifference curve of type θ'' at some utility level u_0, then the ratio,

$$\frac{m(a,\phi(a,u_0,\theta''),\theta')}{m(a,\phi(a,u_0,\theta''),\theta'')} = \left[\frac{\theta''(\gamma + \theta' a)}{\theta'(\gamma + \theta'' a)}\right]\left[\frac{g(K - \theta'' - \beta\phi(a,u_0,\theta''))}{g(K - \theta' - \beta\phi(a,u_0,\theta''))}\right],$$

strictly increases in a by log-concavity of $g(\cdot)$. Thus, condition (1) holds and the double-crossing property is satisfied.

In this example, the payoff from signaling to build a reputation is bounded from above by V. Moreover, log-concavity of $g(\cdot)$ implies that the density function is unimodal. This means that a higher reputation does not significantly improve the chances of success for very low types or very high types. The marginal increase in probability of reaching the target K is greatest for intermediate types, and they tend to have the greatest incentives to invest in signaling.[8]

3.3. Risky experimentation

This example is adapted from our previous work (Chen et al., 2020), extended to incorporate a continuum of types. The key question is whether an agent with superior ability, modeled here by a higher Poisson arrival rate of success, will signal his type by staying with a risky project for a longer duration (because his expected reward from success is higher), or he will signal by quitting early (because he learns more quickly that the risky project is not promising). Bobtcheff and Levy (2017) explore related incentives.

Suppose that an agent engages in risky experimentation with a hidden state of nature. If the state is good, success arrives stochastically with Poisson rate θ; if the state is bad, success never arrives. The prior probability that the state is good is π. Neither the agent nor the market knows the state, but the arrival rate θ is the agent's private information.

The model is an optimal stopping problem with reputation concerns. If the agent achieves success at some random time, he receives a payoff of V. If he abandons the project at time a, the outside-option payoff depends on his reputation at the time of termination, which we capture by $R(a,t)$. Here we measure reputation t by the market's interim belief about the agent's type, i.e., the belief derived from observation on the agent's choice of a and its consistency with the equilibrium strategies.[9] For further details, we encourage the reader to refer to Chen et al. (2020).

The utility function of type θ can be written as

$$u(a,t,\theta) = \int_0^a e^{-\rho\tau}\pi g(a;\theta)V\,\mathrm{d}a + e^{-\rho a}(1-\pi G(a;\theta))R(a,t),$$

where ρ is the discount rate, and $G(\cdot;\theta)$ and $g(\cdot;\theta)$ represent the exponential distribution and density with rate θ. It follows from this that

$$m(a,t,\theta) = -\frac{\tilde{g}(a;\theta)[V-R(a,t)]-\rho R(a,t)+R_a(a,t)}{R_t(a,t)},$$

[8] In different contexts, this non-monotonicity of the effect of investment to improve the chances of success has been exploited in models of hiring standards (Coate and Loury, 1993) and contest selection (Morgan et al., 2018).

[9] In this setting, the equilibrium reputation depends on: (1) inference based on the agent's choice and its consistency with the equilibrium strategies; and (2) observation about the timing of success. The interim belief t only captures (1) but not (2), i.e., the fact that the agent abandons the project before success arrives.

where $\tilde{g}(a;\theta)$ is the unconditional hazard rate of success, which is given by

$$\tilde{g}(a;\theta) = \frac{\pi g(a;\theta)}{1-\pi+\pi G(a;\theta)} = \frac{\pi\theta e^{-\theta a}}{1-\pi e^{-\theta a}}.$$

The marginal rate of substitution depends on agent type only through the hazard rate $\tilde{g}(\cdot;\theta)$. For $\theta' > \theta''$, the difference, $m(a,t,\theta') - m(a,t,\theta'')$, has the same sign as $\tilde{g}(a,\theta'') - \tilde{g}(a,\theta')$, and is single-crossing from below in a. Note also that $\tilde{g}(a;\theta)$ is quasi-concave in θ, meaning that $m(a,t,\theta)$ is quasi-convex in θ. Therefore, Assumptions 2 and 3 hold.

The reason why the double-crossing property emerges in this model is intuitive. Higher types are more likely to achieve success if the state is good. This implies that they have more incentive to persist with the risky project compared to lower types at early stages, when the difference in their beliefs about the state is relatively small. As the game progresses, higher types become pessimistic more quickly than lower types do, because they learn faster that their project is not promising. Past some point, therefore, they become more reluctant to persist with the project. This structure suggests that signaling by persisting with the risky project is relatively more attractive for higher types than for lower types when a is small, but the comparison flips when a is large.

3.4. Productive signaling

Many signaling models assume away any positive benefit of signaling activity in order to isolate its role in conveying hidden information. While this assumption may appear innocuous, once we admit the possibility that signals can be directly productive, details of the model specification can have substantial impact and yield qualitatively different predictions for signaling outcomes.

Assume that education is directly productive in addition to serving as a signal about private information. Specifically, let $s = a\theta$ represent an agent's skill, which depend both on his natural ability θ and on the level of education a. The labor-market benefit from having skill s and reputation t is $\beta s + t$, and the cost of acquiring skill through education is $C(a,\theta) = \gamma_0 a + \gamma(a\theta)^2$. This cost function is unconventional because $C_{a\theta} > 0$, indicating that high-ability agents have higher marginal cost of investing in education—say, due to opportunity cost reasons. However, we may also express the cost of acquiring skill as a function of the target skill level, and write $\tilde{C}(s,\theta) = \gamma_0 s/\theta + \gamma s^2$. This formulation shows that the total cost of reaching skill level s, as well as the marginal cost of increasing skill, is lower for higher types. In this example, the utility function has the form:

$$u(a,t,\theta) = \beta a\theta - \gamma_0 a - \gamma(a\theta)^2 + t,$$

and the marginal rate of substitution is

$$m(a,t,\theta) = 2\gamma a\theta^2 - \beta\theta + \gamma_0.$$

The marginal rate of substitution is obviously quasi-convex, and $m(a,t,\theta') - m(a,t,\theta'')$ is single-crossing from below in a. Thus, both Assumptions 2 and 3 are satisfied. What appears to be a minor—and not totally unreasonable—modification in specification converts the standard setup into a model that exhibits the double-crossing property.

4. Characterization

This section provides a characterization of perfect Bayesian equilibria that survive the D1 criterion. Let $S : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$ denote the sender's strategy, and let $T : [\underline{\theta}, \overline{\theta}] \to [\underline{\theta}, \overline{\theta}]$ be the equilibrium reputation. Let $Q(a) := \{\theta : S(\theta) = a\}$ denote the set of types who choose a in equilibrium. We refer to $Q(a)$ as a *pooling set* if it is not a singleton.

4.1. Full separation

Consider a fully separating strategy $s^*(\cdot)$ for some interval of types, where $T(\theta) = \theta$ in this interval. Incentive compatibility requires type θ to have no incentive to mimic adjacent types:

$$u(s^*(\theta), \theta, \theta) \geq u(s^*(\theta + \epsilon), \theta + \epsilon, \theta).$$

In the limit, this condition can be written as

$$s^{*\prime}(\theta) = \frac{1}{m(s(\theta), \theta, \theta)}. \tag{3}$$

An equilibrium is fully separating if the whole type space $[\underline{\theta}, \overline{\theta}]$ is separating. In this case, the initial condition must satisfy

$$s^*(\underline{\theta}) = \underline{a}^* := \underset{a}{\operatorname{argmax}}\, u(a, \underline{\theta}, \underline{\theta}), \tag{4}$$

if such an action exists.[10] If indifference curves are single-crossing, the solution to the differential equation (3) with initial condition (4) constitutes a fully separating equilibrium

[10] Obviously, no equilibrium exists if there is no such $\underline{a}^*$. We later assume that a unique optimal action exists for each type.

(Mailath, 1987). This solution is also known as the least cost separating equilibrium, or the "Riley outcome" (Riley, 1979).

In our model, there is a dividing line $D(\cdot;\cdot,\cdot)$ which separates the (a,t)-space into two distinct domains. No fully separating solution can extend beyond the dividing line.

Proposition 1. *There is no fully separating equilibrium if there exists $\theta' < \overline{\theta}$ such that $s^*(\theta') = D(\theta';\theta',\theta')$.*

Proof. Assume that $(\underline{a}^*, \underline{\theta})$ is in the SC-domain of type $\underline{\theta}$; the case where it is in the RSC-domain can be proved similarly. Let θ be a type that is slightly above θ', such that $s^*(\theta') = D(\theta';\theta',\theta')$. Recall from (2) that, at $(s^*(\theta'),\theta')$, type θ' has the lowest marginal rate of substitution. Moreover, by the double-crossing property, an indifference curve of the higher type θ that passes through $(s^*(\theta'),\theta')$ stays strictly above that of type θ' for all $a' > s^*(\theta')$. Therefore, if type θ' is indifferent between (a',t') and $(s^*(\theta'),\theta')$, type θ must strictly prefer (a',t'). This shows that $s^*(\cdot)$ cannot extend beyond the dividing line.

The remaining possibility is that $s^*(\cdot)$ jumps at some $\theta \leq \theta'$. Let $s^*(\theta^-)$ denote the left limit and $s^*(\theta^+)$ the right limit at θ. Since $(s^*(\theta),\theta)$ is in the SC-domain of type θ, $s^*(\cdot)$ is positively sloped, and signaling must be costly at that point. This means that $s^*(\cdot)$ cannot jump up because type θ would strictly prefer $(s^*(\theta^-),\theta)$ to $(s^*(\theta^+),\theta)$ if $s^*(\theta^+) > s^*(\theta^-)$. Observe also that $s^*(\cdot)$ cannot jump down either, because if type θ were indifferent between $(s^*(\theta^-),\theta)$ and $(s^*(\theta^+),\theta)$ for any $s^*(\theta^-) > s^*(\theta^+)$, types below θ would strictly prefer $(s^*(\theta^+),\theta)$ and have an incentive to deviate. ∎

If $(s^*(\theta),\theta)$ either belongs to $SC(\theta)$ for all θ, or belongs to $RSC(\theta)$ for all θ, the model reduces to the standard setup. For double-crossing preferences to have any bite, therefore, we need to look at the situation where $s^*(\cdot)$ hits the boundary before it reaches the highest type $\overline{\theta}$. The remainder of the paper deals with this situation.

4.2. Pooling equilibria under D1

Under double-crossing preferences, some form of pooling can survive the D1 criterion. This is a crucial difference from the standard setup, which generally predicts full separation when the D1 criterion is applied. For any (a,t) and any set of types $Q(a)$, let

$$\theta_{\max}(a,t;Q(a)) := \operatorname*{argmax}_{\theta\in\Theta}\; m(a,t,Q(a)),$$
$$\theta_{\min}(a,t;Q(a)) := \operatorname*{argmin}_{\theta\in\Theta}\; m(a,t,Q(a)).$$

We write $\theta_{\min}(a,t)$ for short when $Q(a)=[\underline{\theta},\overline{\theta}]$. Consider a pooling set $Q(a)$ of types who choose a in equilibrium. Let $t=\mathbb{E}[\theta \mid \theta\in Q(a)]$ be the reputation corresponding to action a. Suppose further that there is an open neighborhood $N_\epsilon(a)=(a-\epsilon,a+\epsilon)$ such that no other type chooses $a'\in N_\epsilon(a)\setminus\{a\}$. Then, under D1, a slight upward deviation from (a,t) to the off-equilibrium action a' is attributed to type $\theta_{\min}(a,t;Q(a))$, while a slight downward deviation is attributed to $\theta_{\max}(a,t;Q(a))$. To satisfy D1, we need to make sure that the equilibrium reputation is greater than these off-equilibrium beliefs, i.e.,

$$t \geq \max\{\theta_{\max}(a,t;Q(a)),\theta_{\min}(a,t;Q(a))\}\,. \tag{5}$$

If $m(a,t,\theta)$ is monotone in θ for a given (a,t), then $\theta_{\max}(a,t;Q(a))$ and $\theta_{\min}(a,t;Q(a))$ must be at the extremal points of $Q(a)$. Since $t\in(\min Q(a),\max Q(a))$, (5) cannot be satisfied for any pooling set $Q(a)$. This is why no pooling equilibrium can survive D1 in the standard setup. Under double-crossing preferences, on the other hand, $m(a,t,\theta)$ may not be monotone in θ for some (a,t), thereby leaving some room for pooling equilibria.

4.3. Low types separate high types pairwise-pool

Below, we show that equilibrium under double-crossing preferences exhibits a particular form of pooling, which can be seen as a generalized version of LSHP (Low types Separate High types Pool) equilibrium introduced by Kartik (2009).

Definition 3. *A sender's strategy is LSHPP (Low types Separate High types Pairwise-Pool) if there is some $\theta_0\in[\underline{\theta},\overline{\theta}]$ such that:*

- *(a) $S(\theta)=s^*(\theta)$ for $\theta\in[\underline{\theta},\theta_0)$;*
- *(b) $S(\theta)$ is discontinuous only at $\theta=\theta_0$, with an upward (resp. downward) jump if $s^*(\cdot)$ is increasing (resp. decreasing) on $[\underline{\theta},\theta_0)$.*
- *(c) $S(\theta)$ is weakly quasi-concave for $\theta\in[\theta_0,\overline{\theta}]$, with $S(\theta_0)=S(\overline{\theta})$.*

An equilibrium is an *LSHPP equilibrium* if the sender's strategy is LSHPP; the reason why we call it *pairwise-pooling* is due to the way we construct an equilibrium and will become clear later. Our notion of LSHPP equilibrium includes full separation ($\theta_0 = \overline{\theta}$), full pooling ($\theta_0 = \underline{\theta}$ and $S(\cdot)$ is constant for $\theta \in [\underline{\theta}, \overline{\theta}]$), and LSHP equilibrium ($\theta_0 > \underline{\theta}$ and $S(\cdot)$ is constant for $\theta \in [\theta_0, \overline{\theta}]$) as special cases. An important feature of LSHPP strategy is that it can have at most one "gap" (i.e., discontinuity) at θ_0.

Part (c) of Definition 3 describes what happens above the gap (i.e., among types above θ_0). Quasi-concavity of $S(\cdot)$ with $S(\theta_0) = S(\overline{\theta})$ implies that for any action $a \geq S(\theta_0)$ chosen in equilibrium, $Q(a)$ must be a pooling set (except possibly for $a = \max_\theta S(\theta)$, where $Q(a)$ may be a singleton or a pooling set). See Figure 2 for an illustration. An LSHPP equilibrium exhibits counter-signaling whenever $S(\cdot)$ is not constant above the gap. In Figure 2, the highest type $\overline{\theta}$ chooses a signaling action lower than that chosen by any other type in $(\theta_0, \overline{\theta})$. The highest equilibrium signaling action is chosen by some intermediate types.[11] This suggests that counter-signaling that has been discussed in various contexts is a consequence that pertains to double-crossing preferences. Also note that pairwise-pooling among types in $[\theta_0, \overline{\theta}]$ and full separation among types in $[\underline{\theta}, \theta_0)$ implies that the difference between $T(\theta_0)$ and $T(\theta_0^-)$ (i.e., the left-limit of $T(\cdot)$ at θ_0) must be bounded away from zero. Because the utility function is continuous, the discontinuity of $T(\cdot)$ accounts for the gap in $S(\cdot)$ at θ_0.

The next statement is one of the main results of this paper.

Theorem 1. *Any D1 equilibrium is LSHPP if Assumptions 1 to 3 are satisfied.*

4.4. A sketch of proof

The proof of Theorem 1 is lengthy, and we relegate it to Appendix A. Here, we provide the key steps and a heuristic argument to illustrate the underlying intuition of our characterization. Since the properties of the fully separating region are tightly pinned down by the differential equation (3) and the initial condition (4), we focus on restrictions on equilibrium pooling patterns to see what can happen above the gap. The following result is useful to narrow down possible forms of pooling.

Lemma 2. *Suppose there is an interval (θ'', θ') such that $S(\theta)$ is continuous and strictly monotone, and $Q(S(\theta))$ is a pooling set for some θ in this interval. Then, there exists $p(\cdot)$*

[11] Pairwise-pooling does not exclude the possibility that all types above θ_0 pool at the same action, in which case there is no counter-signaling. Also, in our case of a continuum of types, the highest type $\overline{\theta}$ pools with a lower type θ_0, rather than with the *lowest* type $\underline{\theta}$ (except in the special case of $\theta_0 = \underline{\theta}$). This is different from the model of Feltovich et al. (2002) with three discrete types, in which the highest type pools with the lowest type.

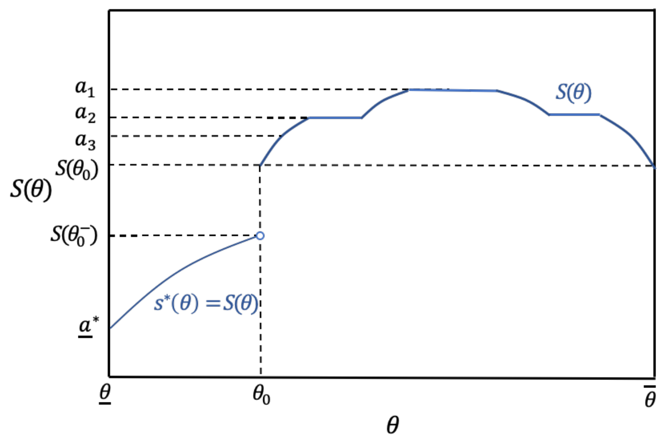

Figure 2. *LSHPP strategy. Below the gap, $S(\cdot)$ coincide with the least cost separating strategy $s^*(\cdot)$. Above the gap, $S(\cdot)$ is quasi-concave. There is mass pooling at a_1 and at a_2, and pairwise pooling in the neighborhood of a_3.*

such that, for all $\theta \in (\theta'', \theta')$, (a) $Q(S(\theta)) = \{\theta\} \cup \{p(\theta)\}$; and (b) $m(S(\theta), T(\theta), \theta) = m(S(\theta), T(\theta), p(\theta))$.

Proof. If there is pooling only at some points in the interval, $T(\cdot)$ must be discontinuous. This necessarily violates incentive compatibility when $S(\cdot)$ is continuous. So there must be pooling over the entire interval. Suppose some type $\theta \in (\theta', \theta'')$ chooses a pooling action a_p, and $Q(a_p)$ contains more than two types. By Lemma 1, we can find a type $\theta_1 \in Q(a_p)$ such that $m(a_p, t_p, \theta) \neq m(a_p, t_p, \theta_1)$ (where $t_p = T(\theta)$). Thus type θ_1 has an incentive to deviate to an action either slightly above or slightly below a_p. This means that $Q(S(\theta))$ can contain only two types, θ and $p(\theta)$. The fact that $m(S(\theta), T(\theta), \theta) = m(S(\theta), T(\theta), p(\theta))$ follows immediately. ■

In our model two different types of pooling can emerge in equilibrium. First, it is possible to have pooling in the usual sense, where a positive measure of types choose the same action. We refer to this pattern of pooling as *mass pooling*. Lemma 2 shows that there can be a different kind of pooling, which we call *atomless pooling*, where exactly two types paired together for each action level, and the pooling set $Q(a)$ has measure zero. For example, in Figure 2, the pooling set $Q(a_3)$ contains exactly two types, and $S(\cdot)$ is locally increasing at one of these types and locally decreasing at the other type. Under atomless pooling the marginal rate of substitution at $(S(\theta), T(\theta))$ must be the same for the paired types. One implication is clear: $(S(\theta), T(\theta))$ belongs to the SC-domain of type θ if and only if it belongs to the RSC-domain of type $p(\theta)$.

When there is mass pooling, the pooling set may be either connected or disconnected. In Figure 2, $Q(a_1)$ is a connected pooling set, while $Q(a_2)$ is disconnected. It is straightforward to deal with connected pooling sets, because it must be an interval. The case of disconnected pooling sets is more complicated, as they potentially admit infinitely many different forms.

Lemma 3. *Suppose there is pooling at (a_p, t_p) such that the pooling set $Q(a_p)$ is disconnected.*

(a) $Q(a_p) = Q_L(a_p) \cup Q_R(a_p)$*, where* $Q_L(a_p)$ *and* $Q_R(a_p)$ *are two disjoint intervals, with* $(a_p, t_p) \in SC(\theta)$ *for* $\theta \in Q_L(a_p)$ *and* $(a_t, t_p) \in RSC(\theta)$ *for* $\theta \in Q_R(a_p)$*.*

(b) $S(\theta) \geq a_p$ *for all* $\theta \in [\min Q(a_p), \max Q(a_p)]$*.*

(c) $S(\theta)$ *is continuous for all* $\theta \in [\min Q(a_p), \max Q(a_p)]$*.*

Consider the pooling set $Q(a_2)$ in Figure 2. This pooling set is the union of two disjoint intervals. If t_2 is the reputation corresponding to action a_2, Lemma 3 shows that (a_2, t_2) is in the SC-domain of all types in the left interval, and is in the RSC-domain of all types in the right interval. All types between $\min Q(a_2)$ and $\max Q(a_2)$ choose actions that are weakly higher than a_2. These two properties implies that $S(\cdot)$ is weakly quasi-concave on $[\min Q(a_2), \max Q(a_2)]$. Quasi-concavity of $S(\cdot)$ in turn implies that a pooling set $Q(a_1)$ can be a connected set only if $a_1 = \max_\theta S(\theta)$.

The formal proof of Lemma 3 is part of the proof of Theorem 1 in Appendix A. Suppose there is pooling at (a_p, t_p). Let $\underline{\theta}_p := \min Q(a_p)$ and $\overline{\theta}_p := \max Q(a_p)$. Suppose further that $Q(a_p)$ is disconnected. Then we can define an open set,

$$J(a_p) := \{\theta : \theta \notin Q(a_p),\ \theta \in (\underline{\theta}_p, \overline{\theta}_p)\},$$

to be the set of types in $(\underline{\theta}_p, \overline{\theta}_p)$ that do not choose a_p. Let $\underline{\theta}_j := \inf J(a_p)$ and $\overline{\theta}_j := \sup J(a_p)$. For any given θ', also define

$$M_h(a, t, \theta') := \{\theta : m(a, t, \theta) \geq m(a, t, \theta')\},$$
$$M_\ell(a, t, \theta') := \{\theta : m(a, t, \theta) < m(a, t, \theta')\}.$$

In words, $M_\ell(a, t, \theta')$ is the set of types whose marginal rate of substitution at (a, t) is lower than that of type θ'. By quasi-convexity of $m(a, t, \cdot)$, $M_\ell(a, t, \theta')$ must be an interval and it must contain $\theta_{\min}(a, t)$. See Figure 3 for an illustration.

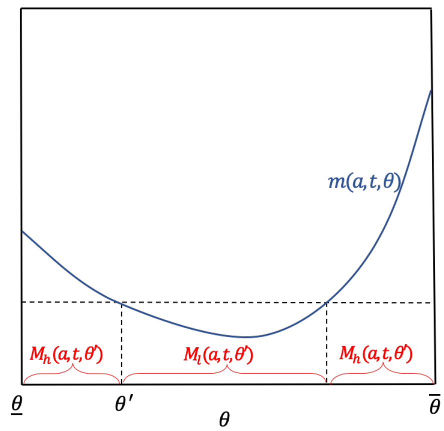

Figure 3. *The marginal rate of substitution is quasi-convex in θ. The set $M_\ell(a,t,\theta')$ is an interval.*

Consider two on-path choices (a_1,t_1) and (a_2,t_2), where $a_1 > a_2$. When a_1 and a_2 are arbitrarily close to each other, preference ranking between these two choices depends only on the marginal rate of substitution. If a type θ' is indifferent between the two choices, then no type in $M_h(a_1,t_1,\theta')$ would choose (a_1,t_1), and no type in $M_\ell(a_2,t_2,\theta')$ would choose (a_2,t_2).

Suppose that $S(\cdot)$ is continuous on $[\theta_0,\overline{\theta}]$, and there is a disconnected pooling set $H(a_p)$ in the interior of this interval. Given that $S(\cdot)$ is continuous, there must be a path $S(\cdot)$ converging to a_p as θ approaches $\underline{\theta}_p$ from below. Lemma 2 suggests that there must be a paired type $p(\cdot)$ and another path $S(p(\cdot))$ converging to a_p as θ approaches $\overline{\theta}_p$ from above. Since $m(a_p,t_p,\underline{\theta}_p) = m(a_p,t_p,\overline{\theta}_p)$ by Lemma 2, we have

$$M_\ell(a_p,t_p,\underline{\theta}_p) = M_\ell(a_p,t_p,\overline{\theta}_p) = (\underline{\theta}_p,\overline{\theta}_p).$$

This means that we must have $S(\theta) < a_p = S(\underline{\theta}_p)$ for $\theta \in (\underline{\theta}_p - \epsilon, \underline{\theta}_p)$, because otherwise types $\underline{\theta}_p$ and $\overline{\theta}_p$ cannot choose (a_p,t_p), a contradiction. Similarly, we must have $S(p(\theta)) < a_p$ for $\theta \in (\overline{\theta}_p, \overline{\theta}_p + \epsilon)$. This implies that when $S(\cdot)$ approaches a pooling action a_p from outside the interval $[\underline{\theta}_p,\overline{\theta}_p]$, it must be increasing on the left and decreasing on the right.

If the pooling set $Q(a_p)$ is disconnected and $S(\cdot)$ is continuous, there must be two more paths, $S(\cdot)$ and $S(p(\cdot))$, converging to a_p as θ approaches $\underline{\theta}_j$ and as $p(\theta)$ approaches $\overline{\theta}_j$. Again, we must have $m(a_p,t_p,\underline{\theta}_j) = m(a_p,t_p,\overline{\theta}_j)$ and hence

$$M_h(a_p,t_p,\underline{\theta}_j) = M_h(a_p,t_p,\overline{\theta}_j) = [\underline{\theta},\underline{\theta}_j] \cup [\overline{\theta}_j,\overline{\theta}].$$

Then, for $\theta \in (\underline{\theta}_j, \underline{\theta}_j + \epsilon)$, we must have $S(\theta) > a_p$ because otherwise no type in $[\underline{\theta}, \underline{\theta}_j] \cup [\overline{\theta}_j, \overline{\theta}]$ would choose (a_p, t_p), a contradiction. We conclude that when $S(\cdot)$ approaches a pooling action a_p from inside the interval $[\underline{\theta}_p, \overline{\theta}_p]$, it must be increasing on the left and decreasing on the right.

The above argument shows that types outside $[\underline{\theta}_p, \overline{\theta}_p]$ must take actions lower than a_p while types inside $(\underline{\theta}_j, \overline{\theta}_j)$ must take actions higher than a_p. This means that $S(\cdot)$ is weakly quasi-concave, with $Q(a_p) = [\underline{\theta}_p, \underline{\theta}_j] \cup [\overline{\theta}_j, \overline{\theta}_p]$. Since $S(\cdot)$ is locally increasing at the endpoints of one of these intervals and locally decreasing at the endpoints of the other interval, (a_p, t_p) is in the SC-domain for types in the former set and is in the RSC-domain for types in the latter set. The quasi-concavity of $S(\cdot)$ stems from the fact that $m(a, t, \cdot)$ decreases at first and then increases, so that middle types tend to have more incentive to choose higher actions. In other words, quasi-convexity of $m(a, t, \cdot)$ strongly suggests quasi-concavity of $S(\cdot)$ above the gap.

5. Existence

This section provides an algorithm to find an LSHPP equilibrium and exploits this algorithm to establish equilibrium existence by construction. To this end, we need to add more structure to the model.

Assumption 4. *For any θ, $u(\cdot, \theta, \theta)$ is quasi-concave, with a unique optimal action $a^*(\theta)$ such that $(a^*(\theta), \theta) \in SC(\theta)$.*

Assumption 5. $F : [\underline{\theta}, \overline{\theta}] \to [0, 1]$ *is continuously differentiable and strictly increasing.*

Assumption 4 allows the possibility that signaling is always costly (as is often assumed in standard signaling models), in which case $u(\cdot, \theta, \theta)$ is strictly decreasing, and the optimal action is $a^*(\theta) = 0$ for any type θ. It excludes the possibility that $u(\cdot, \theta, \theta)$ is strictly increasing, in which case an optimal action does not exist. When $(a^*(\theta), \theta) \in SC(\theta)$, together with quasi-concavity of $u(\cdot, \theta, \theta)$, the marginal rate of substitution is positive (signaling is locally costly) at any point (a, t) on the border or in the RSC-domain of type θ. Assumption 5 is a purely technical condition to ensure that the density function of types, denoted $f(\cdot)$, is well defined and positive everywhere.

The equilibrium signaling pattern for $\theta < \theta_0$ is pinned down by the least-cost separating solution $S(\theta) = s^*(\theta)$ and $T(\theta) = \theta$. Above the gap, there are three objects that need to be determined. Let $\theta_* \in \operatorname{argmax}_{\theta \in [\theta_0, \overline{\theta}]} S(\theta)$. Let $\sigma : [\theta_0, \theta_*] \to \mathbb{R}_+$ represent the signaling action taken by type θ, and $\tau : [\theta_0, \theta_*] \to [\theta_0, \overline{\theta}]$ represent the reputation of type θ. Also, let the (decreasing) function $p : [\theta_0, \theta_*] \to [\theta_*, \overline{\theta}]$ represent the type that is "paired with" type θ in choosing the same signaling action: this function means that each type in $[\theta_0, \theta_*]$ has a counterpart in $[\theta_*, \overline{\theta}]$, thereby giving rise to the term *pairwise-pooling*. Once we pin down these three functions, we can determine:

$$\begin{cases} S(\theta) = \sigma(\theta) \text{ and } T(\theta) = \tau(\theta) & \text{if } \theta \in [\theta_0, \theta_*], \\ S(\theta) = \sigma(p(\theta)) \text{ and } T(\theta) = \tau(p(\theta)) & \text{if } \theta \in (\theta_*, \overline{\theta}]. \end{cases}$$

These objects are defined this way because Lemmas 2 and 3 require that any pooling action is chosen either by exactly two types, or by two intervals of types.[12] When there is atomless pooling, $\sigma(\cdot)$ and $\tau(\cdot)$ are strictly increasing; when there is mass pooling, $\sigma(\cdot)$ and $\tau(\cdot)$ are locally flat.

Perfect Bayesian equilibrium requires a set of equilibrium conditions to be satisfied above the gap, and an indifference condition for type θ_0.

Bayes' rule. The equilibrium belief $\tau(\cdot)$ must be consistent with equilibrium strategies and Bayes' rule on the path of play. The consistency of beliefs requires that for any interval $[\theta_E, \theta_B) \subseteq [\theta_0, \theta_*)$,

$$\int_{\theta_E}^{\theta_B} \tau(\theta)\,\mathrm{d}F(\theta) + \int_{p(\theta_B)}^{p(\theta_E)} \tau(\theta)\,\mathrm{d}F(\theta) = \int_{\theta_E}^{\theta_B} \theta\,\mathrm{d}F(\theta) + \int_{p(\theta_B)}^{p(\theta_E)} \theta\,\mathrm{d}F(\theta).$$

In the limit as $\theta_B \to \theta_E$, we have

$$\tau(\theta_E) = \frac{f(\theta_E)}{f(\theta_E) + f(p(\theta_E))\,|p'(\theta_E)|}\theta_E + \frac{f(p(\theta_E))\,|p'(\theta_E)|}{f(\theta_E) + f(p(\theta_E))\,|p'(\theta_E)|}p(\theta_E),$$

which gives us a "pointwise" belief. This pointwise belief is relevant when the pooling set has measure zero, as in atomless pooling. It is often more convenient to solve for $p'(\theta)$ and write

$$p'(\theta) = \frac{f(\theta)}{f(p(\theta))}\frac{\theta - \tau(\theta)}{p(\theta) - \tau(\theta)}. \tag{6}$$

If there is mass pooling for $[\theta_E, \theta_B)$, then for all $\theta' \in [\theta_E, \theta_B)$, $\tau(\theta')$ is constant and we have

$$\tau(\theta') = \frac{\int_{\theta_E}^{\theta_B} \theta\,\mathrm{d}F(\theta) + \int_{p(\theta_B)}^{p(\theta_E)} \theta\,\mathrm{d}F(\theta)}{F(\theta_B) - F(\theta_E) + F(p(\theta_B)) - F(p(\theta_E))},$$

[12] If the pooling set is connected, we can arbitrarily partition it into two intervals.

Incentive compatibility. In equilibrium, no type has an incentive to mimic adjacent types. The incentive constraint for separation is

$$u(\sigma(\theta), \tau(\theta), \theta) \geq u(\sigma(\theta+\epsilon), \tau(\theta+\epsilon), \theta),$$

for $\theta \in [\theta_0, \theta_*)$.[13] In the limit, we obtain

$$\sigma'(\theta) = \frac{\tau'(\theta)}{m(\sigma(\theta), \tau(\theta), \theta)}, \tag{7}$$

Note that $\sigma'(\theta) > 0$ if and only if $\tau'(\theta) \neq 0$. This corresponds to atomless pooling. In the case of mass pooling, we have $\sigma'(\theta) = \tau'(\theta) = 0$.

Pairwise matching. When there is atomless pooling, incentive compatibility must be satisfied for both θ and $p(\theta)$. This boils down to the restriction (Lemma 2) that the two paired types must have the same marginal rate of substitution:

$$m(\sigma(\theta), \tau(\theta), p(\theta)) - m(\sigma(\theta), \tau(\theta), \theta) = 0.$$

For ease of notation, we sometimes use $m(\cdot)$ to represent the marginal rate of substitution evaluated at $(\sigma(\theta), \tau(\theta), \theta)$ and $\hat{m}(\cdot)$ to represent the value evaluated at $(\sigma(\theta), \tau(\theta), p(\theta))$. Taking derivative with respect to θ then gives

$$[\hat{m}_a(\cdot) - m_a(\cdot)]\,\sigma'(\cdot) + [\hat{m}_t(\cdot) - m_t(\cdot)]\,\tau'(\cdot) = m_\theta(\cdot) - \hat{m}_\theta(\cdot) p'(\cdot). \tag{8}$$

The left-hand side is strictly positive by condition (1).[14] Furthermore, $m_\theta(\cdot) < 0$ and $\hat{m}_\theta(\cdot) > 0$ because $(\sigma(\theta), \tau(\theta))$ is in the SC-domain of type θ and in the RSC-domain of type $p(\theta)$.

Indifference at the gap. The conditions mentioned above allow us to obtain a candidate equilibrium strategy above some threshold θ_0. Below the threshold, there must be full separation, i.e., $S(\theta) = s^*(\theta)$. To pin down an equilibrium for the whole type space, type θ_0 must be indifferent between choosing $s^*(\theta_0)$ and jumping to $\sigma(\theta_0)$ if there is an interior solution (i.e., if $\theta_0 > \underline{\theta}$). Since θ_0 depends on the choice of θ_*, we define

$$\Delta_u(\theta_*) := u(s^*(\theta_0), \theta_0, \theta_0) - u(\sigma(\theta_0), \tau(\theta_0), \theta_0), \tag{9}$$

[13] The incentive constraint for type θ_* is slightly irregular, as he can mimic either type $\theta - \epsilon$ or type $p(\theta - \epsilon)$. Since $S(\cdot)$ attains local maximum at θ_*, incentive compatibility requires $\sigma'(\theta_*) = 0$.

[14] Substituting (7) into the left-hand side shows that it has the same sign as $\hat{m}_a(\cdot) - m_a(\cdot) + \hat{m}(\cdot)(\hat{m}_t(\cdot) - m_t(\cdot))$. Under atomless pooling, types θ and $p(\theta)$ have the same marginal rate of substitution at $(\sigma(\theta), \tau(\theta))$. Letting $a = \sigma(\theta)$ and $\epsilon > 0$, condition (1) implies $m(a+\epsilon, \phi(a+\epsilon, u, \theta), p(\theta)) > m(a+\epsilon, \phi(a+\epsilon, u, \theta), u), \theta)$. Taking the limit gives $\hat{m}_a(\cdot) + \hat{m}_t(\cdot)\phi_a(\cdot) > m_a(\cdot) + m_t(\cdot)\phi_a(\cdot)$. The conclusion follows since $\hat{m}(\cdot) = \phi_a(\cdot)$.

where θ_0 is taken as an implicit function of θ_*.[15] Equilibrium requires $\Delta_u(\theta_*) \leq 0$, with strict inequality only if $\theta_0 = \underline{\theta}$.

5.1. Atomless pooling

Our characterization establishes that there can be two different types of pooling, which we call atomless pooling and mass pooling. The question is when each type of pooling obtains and how the transition between them occurs. We start with the case of atomless pooling.

If we begin with the initial condition $\sigma(\theta_B) = a_B$, $\tau(\theta_B) = t_B$, $p(\theta_B) = \hat{\theta}_B$, we can summarize the initial state by a 4-tuple, $\mathbf{c}_B = (\theta_B, \hat{\theta}_B, a_B, t_B)$. For this to be a legitimate initial state, we require

$$t_B \in (\theta_B, \hat{\theta}_B) \quad \text{and} \quad m(a_B, t_B, \theta_B) = m(a_B, t_B, \hat{\theta}_B). \tag{10}$$

Suppose there is a well defined solution to the differential equations (6), (7), and (8) for $\theta \in [\theta_E, \theta_B]$. We can then obtain the end state summarized by the 4-tuple, $\mathbf{c}_E = (\theta_E, p(\theta_E), \sigma(\theta_E), \tau(\theta_E))$. Obviously the end state will depend on the initial state and on the value of θ_E at which we choose to evaluate the solution functions, we denote this mapping by $\mathbf{c}_E = Z_A(\theta_E; \mathbf{c}_B)$. By construction, if the initial state $\mathbf{c}_B$ satisfies condition (10), then the output $\mathbf{c}_E$ of this mapping also satisfies (10).

The main constraint for pairwise matching is that $\sigma(\cdot)$ must be strictly increasing on (θ_E, θ_B), reflecting the requirement that $S(\cdot)$ is quasi-concave above the gap. Combining (7) and (8), this restriction can be expressed as

$$m_\theta(\cdot) - \hat{m}_\theta(\cdot) p'(\cdot) > 0, \tag{11}$$

which is a necessary condition for atomless pooling to be supported in equilibrium. Therefore, once $m_\theta(\cdot) - \hat{m}_\theta(\cdot) p'(\cdot)$ turns from positive to zero, the solution to the differential equation cannot be extended further back. Let

$$\chi_A(\mathbf{c}_B) = \left\{ \theta_E : \text{constraint (11) holds for all } \theta \in (\theta_E, \theta_B] \text{ and } p(\theta_E) \leq \overline{\theta} \right\}.$$

For any $\mathbf{c}_B$ satisfying (10) and any $\theta_E \in \chi_A(\mathbf{c}_B)$, the mapping $Z_A(\theta_E; \mathbf{c}_B)$ is well defined and produces a valid solution satisfying the monotonicity requirement on the domain $(\theta_E, \theta_B]$.

[15] For any given θ_*, there may be multiple solutions that satisfy the equilibrium conditions, so that the mapping from θ_* to θ_0 is in general a correspondence. For the purpose of showing equilibrium existence, we pick a particular solution even when there are others. We let θ_0 denote this particular solution produced by our algorithm and hence take it as a function of θ_*.

5.2. Mass pooling

Begin with an initial condition, summarized by $\mathbf{c}_B = (\theta_B, \hat{\theta}_B, a_B, t_B)$, that satisfies (10). To construct an equilibrium in which all types in $[\theta_E, \theta_B] \cup [\hat{\theta}_B, \hat{\theta}_E]$ pool to choose (a_B, t_B), the equilibrium conditions require:

$$m(a_B, t_B, \hat{\theta}_E) - m(a_B, t_B, \theta_E) = 0, \tag{12}$$

$$\mathbb{E}\left[\theta \mid \theta \in [\theta_E, \theta_B] \cup [\hat{\theta}_B, \hat{\theta}_E]\right] - t_B = 0. \tag{13}$$

Let $\psi(\cdot; a_B, t_B)$ represent the implicit function that gives the $\hat{\theta}_E$ satisfying (12) for each θ_E. Similarly, let $\eta(\cdot; \theta_B, \hat{\theta}_B, t_B)$ give the $\hat{\theta}_E$ satisfying (13) for each θ_E. Both functions are defined on the domain $[b, \theta_B]$, such that b solves $\eta(b; \theta_B, \hat{\theta}_B, t_B) = \overline{\theta}$. If no such b exists, we set $b = \underline{\theta}$. Whenever $\psi(\theta_E)$ is undefined for $\theta_E \in [b, \theta_B]$, we set $\psi(\theta_E) = \overline{\theta}$. According to this extended definition, $\psi(b; a_B, t_B) = \overline{\theta}$ if and only if $m(a_B, t_B, b) \geq m(a_B, t_B, \overline{\theta})$.

A solution to the equation system (12) and (13) exists if there is a θ_E such that $\psi(\theta_E) = \eta(\theta_E)$. By implicit differentiation, the slopes of these functions are:

$$\psi'(\theta; a_B, t_B) = \frac{m_\theta(a_B, t_B, \theta)}{m_\theta(a_B, t_B, \psi(\theta))},$$

$$\eta'(\theta; \theta_B, \hat{\theta}_B, t_B) = \frac{f(\theta)}{f(\eta(\theta))} \frac{\theta - t_B}{\eta(\theta) - t_B}.$$

Both functions are decreasing for any $\mathbf{c}_B$ satisfying (10). The condition that $m_\theta(\cdot) - \hat{m}_\theta(\cdot) p'(\cdot)$ is non-negative corresponds to $\psi'(\cdot) \geq \eta'(\cdot)$. To satisfy the conditions for mass pooling at (a_B, t_B), θ_E and $\hat{\theta}_E$ have to satisfy (12) and (13). Further, for any interior crossing point (i.e., $\theta_E > b$), we require that $\psi'(\theta_E) \geq \eta'(\theta_E)$. This would allow the end point of mass pooling θ_E to serve as an initial starting point for atomless pooling immediately to the left of θ_E.

Figure 4 illustrates this situation where we fix (a_B, t_B) in the background. At (a_B, t_B), $\eta'(\theta_B) = \psi'(\theta_B)$. For θ slightly smaller than θ_B, we have $\psi'(\cdot) < \eta'(\cdot)$, and it is hence not feasible to extend atomless pooling any further. We instead have mass pooling starting from θ_B. In general, the two-equation system may produce multiple termination points (as in the left panel), in which case we may terminate mass pooling at any one of them, subject to the constraint that $\phi'(\theta_E) \geq \eta'(\theta_E)$ at the termination point.

To summarize, we let

$$\chi_M(\mathbf{c}_B) = \left\{\theta_E : \psi(\theta_E) = \eta(\theta_E) \text{ and } \phi'(\theta_E) \geq \eta'(\theta_E), \text{ or } \theta_E = b \text{ and } \psi(b) \geq \eta(b)\right\}.$$

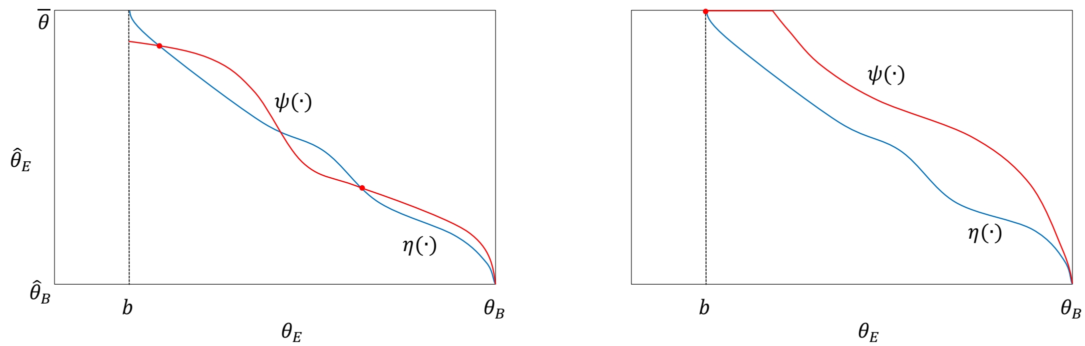

Figure 4. *The figures show $\psi(\cdot)$ and $\eta(\cdot)$ with (a_B, t_B) fixed. In the left panel, mass pooling starts from θ_B with two possible termination points marked by the red dots. In the right panel, $\psi(\cdot)$ is consistently above $\eta(\cdot)$, and mass pooling continues all the way to $\overline{\theta}$.*

Given an initial state $\mathbf{c}_B$, and for any $\theta_E \in \chi_M(\mathbf{c}_B)$, we can obtain an end state $\mathbf{c}_E = (\theta_E, \psi(\theta_E), a_B, t_B)$. We denote this mapping by $\mathbf{c}_E = Z_M(\theta_E; \mathbf{c}_B)$. By construction, the output of this mapping satisfies (10) except possibly at $\theta_E = b$. But in this case, the pairwise-pooling region is $[b, \overline{\theta}]$, and $m(a_E, t_E, b) \geq m(a_E, t_E, \overline{\theta})$ ensures that there is no incentive for downward deviation below a_E.

5.3. Algorithm and equilibrium existence

If $S(\cdot)$ attains a maximum at a unique θ_*, there is atomless pooling in a neighborhood of θ_*. In this neighborhood, $(\sigma(\theta), \tau(\theta))$ is in the SC-domain of type θ and in the RSC-domian of type $p(\theta)$. This means that $(\sigma(\theta_*), \theta_*)$ must be on the boundary of the SC-domain and RSC-domain of type θ_*. Therefore, a boundary condition that satisfies (10) in the limit is:

$$\sigma(\theta_*) = D(\theta_*; \theta_*, \theta_*), \quad \tau(\theta_*) = \theta_*, \quad p(\theta_*) = \theta_*.$$

If there is mass pooling in the neighborhood of θ_*, using this boundary condition ensures that the off-equilibrium belief for an upward deviation above $\sigma(\theta_*)$ is θ_*, which does not exceed the equilibrium belief θ_*.

For any given θ_*, we go through the following iterative procedure to ensure that the equilibrium conditions for pairwise-pooling are satisfied:

1. Initialize $k = 1$. Set $\mathbf{c}_k = (\theta_*, \theta_*, D(\theta_*; \theta_*, \theta_*), \theta_*)$, and set $\theta_{Bk} = \theta_*$. If $\inf \chi_P(\mathbf{c}_k) < \theta_*$, go to step 2; otherwise go to step 3.
2. Let $\theta_E = \inf \chi_P(\mathbf{c}_k)$. Construct the atomless-pooling solution for $\theta \in (\theta_E; \theta_{Bk}]$. If $p(\theta_E) = \overline{\theta}$, stop. Otherwise, let $\mathbf{c}_{k+1} = Z_A(\theta_E, \mathbf{c}_k)$ and $\theta_{Bk+1} = \theta_E$, increment k and

go to step 3.

3. Let $\theta_E = \max \chi_M(\mathbf{c}_k)$. Construct the mass-pooling solution for $\theta \in (\theta_E; \theta_{Bk}]$. If $\theta_E = b$, stop. Otherwise, let $\mathbf{c}_{k+1} = Z_M(\theta_E, \mathbf{c}_k)$ and $\theta_{Bk+1} = \theta_E$, increment k and go to step 2.

Once θ_* is fixed, this algorithm yields a well defined θ_E such that $p(\theta_E) = \overline{\theta}$ at the end of the procedure, along with $\sigma(\theta)$, $\tau(\theta)$, and $p(\theta)$ for $\theta \in [\theta_E, \theta_*]$. By construction, these objects satisfy Bayes' rule, incentive compatibility, and pairwise matching. Let $\zeta : [\underline{\theta}, \overline{\theta}] \to [\underline{\theta}, \overline{\theta}]$ denote this mapping, where $\zeta(\theta_*)$ is the θ_E obtained at the end of the procedure starting from θ_*. If we let θ_0 equal $\zeta(\theta_*)$, then $\Delta_u(\theta_*)$ defined in (9) is a well defined function of θ_*.

Give this construction, we can establish existence of an LSHPP equilibrium. The proof of Theorem 2 is relegated to Appendix B. In the proof, we first show that $\zeta(\cdot)$ is continuous with respect to boundary condition θ_*. This implies that $\Delta_u(\cdot)$ is also continuous with respect to θ_*, which ensures existence of $\theta_0 = \zeta(\theta_*)$ such that $\Delta_u(\theta_*) \leq 0$ (with strict inequality only if $\theta_0 = \underline{\theta}$). By construction, the candidate solution so obtained satisfies all the local incentive compatibility constraints. In the final step, we check global incentive compatibility to make sure that the candidate solution constitutes an equilibrium.

The highest action chosen above the gap is $\sigma(\theta_*) = D(\theta_*; \theta_*, \theta_*)$. Since type θ_* has the lowest marginal rate of substitution among all types at that point, upward deviation above $\sigma(\theta_*)$ is attributed to type θ_* under D1, meaning that the equilibrium reputation for action $\sigma(\theta_*)$ is the same as the off-equilibrium reputation for an action above $\sigma(\theta_*)$. If there is a type that prefers higher action to lower action (signaling is not costly) at that point, such a deviation may be profitable. Assumption 4 guarantees that $m(D(\theta_*; \theta_*, \theta_*), \theta_*, \theta_*) > 0$, and hence $m(D(\theta_*; \theta_*, \theta_*), \theta_*, \theta) > 0$ for all θ. Since signaling is locally costly, no type has an incentive to deviate to an off-equilibrium action higher than $\sigma(\theta_*)$.

Theorem 2. *An LSHPP equilibrium exists if Assumptions 1 to 5 are satisfied.*

6. Discussion

6.1. Multiplicity of equilibria

We establish equilibrium existence by construction. The algorithm in Section 5 consistently picks the infimum from the feasible set $\chi_A(\mathbf{c}_k)$, or the maximum from the feasible set

$\chi_M(\mathbf{c}_k)$, at each round k. It is possible to obtain other candidate solutions that may also satisfy all the equilibrium restrictions if we adopt a different algorithm.

To see the possibility of multiple equilibria, it is easiest to consider the case of full pooling equilibria. Let μ represent the unconditional mean of θ. Suppose all types pool at (a_p, μ). To prevent off-equilibrium deviation, D1 requires that condition (5) hold. In the context of full pooling, this requirement is equivalent to

$$D(\mu; \overline{\theta}, \underline{\theta}) \geq a_p \geq D(\mu; \mu, \mu).$$

Of course, we also require that type $\underline{\theta}$ should have no incentive to deviate to his optimal action:

$$u(a_p, \mu, \underline{\theta}) \geq u(\underline{a}^*, \underline{\theta}, \underline{\theta}).$$

When both of these conditions hold, there is a full pooling equilibrium, but these two conditions do not pin down a unique value of a_p.

For an explicit numerical example, consider the "signaling with news" application of Section 3.1. We choose parameters so that

$$u(a, t, \theta) = \lambda(\theta + (1-\theta)t) - \left(\frac{a}{\theta} + \frac{a^2}{2}\right),$$

and let θ be uniformly distributed on $[0.1, 0.5]$. The mean of θ is $\mu = 0.3$, and

$$m(a, t, \theta) = \frac{1 + a\theta}{\lambda\theta(1-\theta)}.$$

Let a_P represent a pooling action in a full pooling equilibrium. To prevent downward deviation requires $m(a_P, t, 0.1) \geq m(a_P, t, 0.5)$. Since the marginal rate of substitution does not depend on t, this requirement reduces to $a_P \leq 8$. To prevent upward deviation requires $\theta_{\min}(a_P, t) \leq 0.3$, which reduces to $a_P \geq 40/9$. Furthermore, $u(a_P, 0.3, 0.1) \geq u(0, 0.1, 0.1)$ for any $a_P \leq 8$ if $\lambda \geq 5600/9$. We can conclude that for $\lambda \geq 5600/9$, any action $a_P \in [40/9, 8]$ can constitute part of a full pooling equilibrium.

6.2. Comparative statics

To further illustrate the properties of equilibria under double-crossing preferences, we continue to work with the same example as in the previous subsection and examine how equilibrium varies with changes in some key parameters of the model. Although comparative statics is cumbersome when there are multiple equilibria, this exercise still allows us to elucidate some general tendencies and important insights.

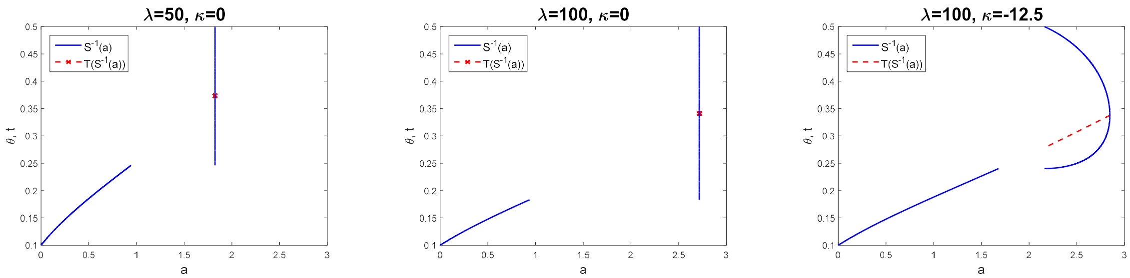

Figure 5. *Equilibrium actions for different returns to signaling (parameter λ) and different type distributions (parameter b). Larger values of λ corresponds to larger returns to signaling. The type distribution has density $f(\theta) = 2.5 + \kappa(\theta - 0.3)$ on the support $[0.1, 0.5]$, and κ is the slope of the density. The red line shows that locus of $(S(\theta), T(\theta))$ in the (a, t)-space above the gap.*

The first observation is that we have less separation and more pooling as the returns to signaling become larger (i.e., θ_0 decreases towards $\underline{\theta}$ as λ increases). As λ gets larger, higher types need to take even higher actions to separate because lower types now have more incentive to mimic. The equilibrium action taken by higher types cannot be unbounded, however, because of the double-crossing property: as the equilibrium action increases, it will inevitably enter the RSC-domain where it is *more costly* for higher types to choose higher actions. As we have seen above, we can always construct a fully pooling equilibrium when $\lambda \geq 5600/9$. In short, the double-crossing property imposes an endogenous upper bound for actions, which must bind at some point as λ increases. Figure 5 further illustrates this tendency: when λ increases from 50 to 100, where the range of the fully separating region shrinks (i.e., θ_0 decreases), with an increase in equilibrium actions for all types. This is different from the standard setup, where an increase in the returns to signaling only stretches out equilibrium actions but yields no qualitative impact on the form of equilibrium.

In the left and middle panels of Figure 5, $S(\cdot)$ is flat above the gap (i.e., LSHP equilibrium); this example thus shows that counter-signaling is not a necessary consequence of the double-crossing property. To construct an equilibrium with atomless pooling (and counter-signaling), we manipulate the type distribution by letting $f(\theta) = 2.5 + \kappa(\theta - 0.3)$ for $\theta \in [0.1, 0.5]$. Atomless pooling is more likely to emerge as the slope parameter κ becomes smaller. Figure 5 shows that for $\lambda = 100$, the equilibrium is LSHP when $b = 0$ (uniform distribution) but exhibits atomless pooling when $\kappa = -12.5$.[16] To see why, recall

[16] For intermediate values of κ, we can find equilibria in which both atomless and mass pooling coexist.

that $\eta(\cdot)$ solves

$$\mathbb{E}\big[\theta \mid \theta \in [\theta_E, \theta_B] \cup [\hat{\theta}_B, \eta(\theta_E)]\big] - t_B = 0,$$

for given t_B, θ_B and $\hat{\theta}_B$. For a given θ_E, $\eta(\theta_E)$ must go up to keep the mean constant at t_B as the distribution becomes more skewed to the right. Since $\psi(\cdot)$ is independent of the type distribution, a decrease in κ makes atomless pooling more likely to emerge.

6.3. Other variants of double-crossing preferences

The structure of preferences in our model is essentially determined by Assumptions 2 and 3. Our Assumption 2 (A2) requires that, for $\theta' > \theta''$, the difference $m(a, \phi(a, u_0, \theta''), \theta') - m(a, \phi(a, u_0, \theta''), \theta'')$ be single-crossing from below in a. One may imagine an alternative assumption (A2′) which requires the same difference be single-crossing *from above*. Assumption 3 (A3) in our specification requires $D(t; \theta', \theta'')$ to be decreasing in θ' and θ''. An alternative assumption (A3′) may require that it is *increasing* in these two arguments. Assumptions 2 and 3 capture independent aspects of double-crossing preferences and can be altered separately, leading to four different specifications. We argue through the examples in Section 3 that our specification is the most natural and useful one for applied economic analysis. It turns out that it is also much more tractable than the alternatives.

To make the exposition simple, assume for the moment that signaling is always costly (i.e., $u_a(\cdot) < 0$). We will return to this latter assumption at the end of this discussion.

Consider first a model in which we assume (A3′) while maintaining (A2). These two assumptions imply that $m(a, t, \cdot)$ is *quasi-concave*. Since middle types have the highest marginal signaling costs, equilibrium cannot exhibit counter-signaling. Indeed, local incentive compatibility requires that if two types θ' and θ'' pool at some action a_p, then $S(\theta) \leq a_p$ for $\theta \in (\theta'', \theta')$—a result opposite to Lemma 3(b). This may seem to suggest that pairwise-pooling would take the form in which $S(\cdot)$ is quasi-convex above the gap. However, a model with (A3′) and (A2) is not well-behaved because, unlike our model, local incentive compatibility does not imply global incentive compatibility. If type θ' chooses (a', t') and type θ'' chooses (a'', t'') (with $a' > a''$), it is possible that (a', t') lies to the right of the dividing line $D(\cdot; \theta', \theta'')$ while (a'', t'') lies to the left of it, even though both points are to the left of $D(\cdot; \theta', \theta')$. The concept of "SC-domain," as defined in Definition 2, loses its force under (A3′) and (A2). There is no tractable way of ensuring the types θ' and θ'' have no incentive to mimic each other, even though local incentive compatibility constraints are satisfied at those two points.

In a model with (A2′) and (A3′), $m(a,t,\cdot)$ is quasi-convex (contrary to Lemma 1, which relies on (A2)). Because the reverse single-crossing property prevails at low levels of signaling action, generally such a model cannot support an LSHPP equilibrium. In principle, it may support an equilibrium in which types lower than some θ_0 pairwise-pool, while types higher than θ_0 separate with actions in the "SC-domain" of type θ_0. However, as in the previous case, when two types θ' and θ'' (with $\theta' > \theta'' \geq \theta_0$) choose two actions (a', t') and (a'', t'') both in $SC(\theta_0)$, these two points may lie on opposite sides of the dividing line $D(\cdot; \theta', \theta'')$, making global incentive compatibility difficult to ascertain.

Finally, suppose we impose (A2′) and (A3). In this case, $m(a,t,\cdot)$ is quasi-concave, so $S(\cdot)$ would be quasi-convex when there is pairwise-pooling. If there is an equilibrium in which low types pairwise-pool while high types separate, the main problem is that it may not satisfy the D1 criterion. Specifically, suppose types in $[\theta'', \theta']$ pool to choose the lowest on-path action-reputation pair (a_p, t_p), such that $a_p = D(t_p; \theta_*, \theta_*)$. Because type θ_* has the steepest indifference curve, D1 requires $t_p > \theta_*$ to prevent a small downward deviation. Moreover, under (A2′), since large downward deviations would occur in the RSC-domain of type θ_* (where signaling is more costly for higher types), D1 may attribute large downward deviations to types even higher than θ_*, making the conditions for equilibrium existence more stringent and difficult to specify in a tractable way.

We will now provide a brief discussion of the situation when signaling is not always costly. Of course, signaling cannot be always beneficial because the optimal signaling action would then be unbounded. Therefore, we consider an alternative assumption (A4′) to replace Assumption 4. It requires that, for all θ, $u(\cdot, \theta, \theta)$ is quasi-concave and $(a^*(\theta), \theta)$ belongs to $RSC(\theta)$, Suppose we maintain (A3′) and (A4′), together with Assumptions 1, 2, and 5. There is an equilibrium which is the "mirror image" of our LSHPP equilibrium. In such an equilibrium, types $\theta < \theta_0$ separate with $S(\theta) = s^*(\theta)$ and $s^*(\underline{\theta}) = \underline{a}^* \in RSC(\underline{\theta})$. Under (A4′), because separation occurs in the RSC-domain of type θ_0 and signaling is locally beneficial, $s^*(\cdot)$ is strictly decreasing—signaling occurs by choosing inefficiently low levels of action. The equilibrium signaling action jumps down from $s^*(\theta_0)$ to $S(\theta_0)$, and pairwise-pooling occurs for types above θ_0. Under (A3′) and (A4′), $m(a,t,\cdot)$ is *quasi-concave and negative,* meaning that middle types have the most incentive to signal by choosing inefficiently low actions. Thus, $S(\cdot)$ is quasi-convex below the gap. Finally, this combination of assumptions ensures that local incentive compatibility implies global incentive compatibility. It is also straightforward to verify that such a "mirror image" equilibrium survives the D1 refinement.

7. Conclusion

Despite its widespread use in economic analysis, the single-crossing property imposes strong restrictions on the structure of preferences, and its validity and robustness are not necessarily always evident in economic applications. Because many insights about signaling behavior we learn from standard models depend on this property, it is important to extend the scope of analysis to circumstances that are not constrained by the single-crossing property. We take a step in this direction by providing a formal framework to capture double-crossing preferences in signaling models. Our characterization shows that equilibrium under double-crossing preferences exhibits a particular form of pooling at the higher end of types, which we label as pairwise-pooling. Pairwise-pooling generalizes a phenomenon known as counter-signaling in the literature: double-crossing preferences often induce middle types to invest more in signaling whereas higher types are content with pooling with lower types. Our model identifies the assumptions on preferences that tend to produce pairwise-pooling, as well as the constraints that affect the form it takes (i.e., atomless or mass pooling). We provide a simple algorithm to find an LSHPP equilibrium and show that it exists under fairly weak conditions.

From the theoretical point of view, it is perhaps not so controversial to say that the single-crossing property may fail in some situations. The problem is rather that this can happen in many different ways. Section 6.3 touches upon this issue, but even that does not exhaust all the possible ways through which the single-crossing property breaks down. Although we argue that our framework covers a broad range of economically relevant situations, and this framework turns out to be relatively tractable, it does not by any means exclude other variations of non-single-crossing preferences. We hope to see more work along these lines, in order to gain a more comprehensive understanding of signaling behavior that goes beyond the single-crossing property.

Appendix

A. Proof of Theorem 1

A.1. Preliminaries

Denote the set of types that choose action a in equilibrium by $Q(a) = \{\theta : S(\theta) = a\}$. If there is some action a_p such that $Q(a_p)$ is not a singleton, we refer to a_p as a pooling action and to $Q(a_p)$ as a pooling set. We assume that a pooling set is closed. Recall that we define $\overline{\theta}_p := \max Q(a_p)$ and $\underline{\theta}_p := \min Q(a_p)$.

To apply D1, it is crucial whether actions slightly above or below a pooling action are chosen in equilibrium. Consider some pooling action a_p. We say that actions below a_p are *on-path* if there exists a small $\epsilon > 0$ such that $Q(a) \neq \emptyset$ for all $a \in (a_p - \epsilon, a_p)$; otherwise, actions below a_p are *off-path*. Similarly, actions above a_p are on-path if there exists a small $\epsilon > 0$ such that $Q(a) \neq \emptyset$ for all $a \in (a_p, a_p + \epsilon)$; otherwise, actions above a_p are off-path.

For any θ, we use $S(\theta^-)$ and $T(\theta^-)$ to denote the left limit and $S(\theta^+)$ and $T(\theta^+)$ to denote the right limit at θ. If there exists a sequence $\theta^n \to \theta'$ for some θ' such that $S(\theta^n) \to a_p$, with either $S(\theta^n) > a_p$ or $S(\theta^n) < a_p$ for all n, we call θ' a *limit type*.

The following lemma is a crucial property which we exploit repeatedly. To this end, it is convenient to define $q(a,t,\theta)$ such that $m(a,t,q(a,t,\theta)) = m(a,t,\theta)$, with $q(a,t,\theta) = \theta$ if $\theta = \theta_{\min}(a,t)$. This mapping gives a counterpart type that has the same marginal rate of substitution at (a,t). If no such counterpart type exists, let $q(a,t,\theta) = \overline{\theta}$ if $\theta < \theta_{\min}(a,t)$ and $q(a,t,\theta) = \underline{\theta}$ if $\theta > \theta_{\min}(a,t)$.

Lemma 4. *Consider two choices (a_1,t_1) and (a_2,t_2) where $a_1 > a_2$, and some type θ'.*

(a) Suppose $\theta' < \theta_{\min}(a_1,t_1)$ and $u(a_1,t_1,\theta') \geq u(a_2,t_2,\theta')$. Then, there exists some $\delta(a_2) \geq 0$ such that $u(a_1,t_1,\theta) > u(a_2,t_2,\theta)$ for all $\theta \in (\theta', q(a_1,t_1,\theta') + \delta(a_2))$.

(b) Suppose $\theta' > \theta_{\min}(a_2,t_2)$ and $u(a_1,t_1,\theta') \geq u(a_2,t_2,\theta')$. Then, there exists some $\delta(a_1) \geq 0$ such that $u(a_1,t_1,\theta) > u(a_2,t_2,\theta)$ for all $\theta \in (q(a_2,t_2,\theta') - \delta(a_1), \theta')$.

(c) $\delta(a_i) \to 0$ for $i = 1,2$ as $a_2 - a_1 \to 0$.

Proof. (a) If $\theta' < \theta_{\min}(a_1,t_1)$, $m(a_1,t_1,\theta') > m(a_1,t_1,\theta)$ for all $\theta \in (\theta', q(a_1,t_1,\theta'))$. Moreover, the indifference curve of type θ' that passes through (a_1,t_1) stays strictly below the indifference curve of any type $\theta \in (\theta', q(a_1,t_1,\theta')]$ to the left of a_1. If a_2 is bounded from a_1 and is lower than a_1, type $q(a_1, t_1, \theta')$ strictly prefers (a_1, t_1) to (a_2, t_2). By continuity, we can find a type slightly above $q(a_1, t_1, \theta')$ who strictly prefers (a_1, t_1).

(b) If $\theta' > \theta_{\min}(a_2, t_2)$, $m(a_2, t_2, \theta') > m(a_2, t_2, \theta)$ for all $\theta \in (q(a_2, t_2, \theta'), \theta')$. Moreover, the indifference curve of type θ' that passes through (a_2, t_2) stay strictly above the indifference curve of any type $\theta \in [q(a_2, t_2, \theta'), \theta')$ to the right of a_2. If a_1 is bounded from a_2 and is higher than a_2, type $q(a_2, t_2, \theta')$ strictly prefers (a_1, t_1) to (a_2, t_2). By continuity, we can find a type slightly below $q(a_2, t_2, \theta')$ who strictly prefers (a_1, t_1).

(c) If a_2 is arbitrarily close to a_1, preferences depend only on the marginal rate of substitution at (a_1, t_1). If type θ' is indifferent between (a_1, t_1) and (a_2, t_2), type $q(a_1, t_1, \theta')$ is also indifferent. ∎

Lemma 4 simply states that a type with a lower marginal rate of substitution has more incentive to choose a higher action. In particular, when a_1 and a_2 are arbitrarily close to each other, preference ranking between (a_1, t_1) and (a_2, t_2) depends only on the marginal rate of substitution at that point.

A.2. Connected pooling sets

There are two forms of pooling, depending on whether a pooling set is connected or not. We start with connected pooling sets.

Lemma 5. *In any D1 equilibrium, if there is a connected pooling set at (a_p, t_p), it is in the SC-domain for type $\underline{\theta}_p$ and in the RSC-domain for type $\overline{\theta}_p$.*

Proof. Suppose $Q(a_p)$ is connected and actions below and above a_p are off-path. In this case, D1 requires that

$$m(a_p, t_p, \underline{\theta}_p) \geq m(a_p, t_p, \overline{\theta}_p) > m(a_p, t_p, \theta_{\min}(a_p, t_p)).$$

This is possible only if $m(a_p, t_p, \cdot)$ is decreasing at $\underline{\theta}_p$ (in the SC-domain) and increasing at $\overline{\theta}_p$ (in the RSC-domain).

If $Q(a_p)$ is connected, and there is a path $S(\cdot)$ that converges to a_p, we must have two limit types $\underline{\theta}_p$ and $\overline{\theta}_p$. Suppose to the contrary that there is only one limit type, either $\underline{\theta}_p$ or $\overline{\theta}_p$. Then, we have either $T(\underline{\theta}_p{}^-) = \underline{\theta}_p$ or $T(\overline{\theta}_p{}^+) = \overline{\theta}_p$, but since $t_p \in (\underline{\theta}_p, \overline{\theta}_p)$, it necessarily violates incentive compatibility. This means that there must be some $p(\cdot)$ and $\epsilon > 0$ such that $S(\theta) = S(p(\theta))$ for all $\theta \in (\underline{\theta}_p - \epsilon, \underline{\theta}_p)$. Lemma 2 then ensures

$m(a_p, t_p, \underline{\theta}_p) = m(a_p, t_p, \overline{\theta}_p)$, which in turn implies that (a_p, t_p) is in the SC-domain of type $\underline{\theta}_p$ and in the RSC-domain of type $\overline{\theta}_p$. ■

A.3. Disconnected pooling sets

The case of disconnected pooling sets is much more complicated. We begin with a preliminary result.

Lemma 6. *If actions above a_p are on-path with limit type $\theta_1 \in [\underline{\theta}_p, \overline{\theta}_p]$, no type between θ_1 and $q(a_p, t_p, \theta_1)$ chooses a_p. If actions below a_p are on-path with limit type θ_1, only types between θ_1 and $q(a_p, t_p, \theta_1)$ may choose a_p.*

Proof. When there is a continuous path $S(\cdot)$ to a_p, preferences are determined entirely by the marginal rate of substitution at (a_p, t_p). Since all types between θ_1 and $q(a_p, t_p, \theta_1)$ have a lower marginal rate of substitution than type θ_1, they strictly prefer an action slightly above a_p. For the second statement, let $\theta_1 < \theta_{\min}(a_p, t_p)$ (the opposite case follows by the same argument). Then, all types below θ_1 and above $q(a_p, t_p, \theta_1)$ have a higher marginal rate of substitution than type θ_1; they strictly prefer an action slightly below a_p. ■

Consider a disconnected pooling set $Q(a_p)$. Recall that we define $J(a_p) = \{\theta : \theta \notin Q(a_p),\ \theta \in (\underline{\theta}_p, \overline{\theta}_p)\}$, and let $\underline{\theta}_j = \inf J(a_p)$ and $\overline{\theta}_j = \sup J(a_p)$. If there is a disconnected pooling set at (a_p, t_p), Lemma 3 in the text states that:

(a) $Q(a_p) = Q_L(a_p) \cup Q_R(a_p)$, where $Q_L(a_p)$ and $Q_R(a_p)$ are two disjoint intervals, and (a_p, t_p) is in the SC-domain for all types in $Q_L(a_p)$ and in the RSC-domain for all types in $Q_R(a_p)$.

(b) $S(\theta) \geq a_p$ for all $\theta \in [\underline{\theta}_p, \overline{\theta}_p]$.

(c) $S(\theta)$ is continuous for all $\theta \in [\underline{\theta}_p, \overline{\theta}_p]$.

Proof of Lemma 3. Part (b). Suppose otherwise, i.e., there is some type $\theta_1 \in (\underline{\theta}_p, \overline{\theta}_p)$ who chooses (a_1, t_1) with $a_1 < a_p$.

If $m(a_p, t_p, \underline{\theta}_p) \geq m(a_p, t_p, \overline{\theta}_p)$, then $q(a_p, t_p, \underline{\theta}_p) > \overline{\theta}_p$. By Lemma 4(a), whenever type $\underline{\theta}_p$ weakly prefers (a_p, t_p) to (a_1, t_1), type θ_1 would strictly prefer (a_p, t_p).

If $m(a_p, t_p, \underline{\theta}_p) < m(a_p, t_p, \overline{\theta}_p)$, some type θ_1 above $q(a_p, t_p, \underline{\theta}_p)$ may choose $a_1 < a_p$. Let $a_1 \to a_p$. In this case, all types above θ_1, including type $\overline{\theta}_p$, strictly prefer (a_1, t_1) to

(a_p, t_p), a contradiction. This means that a_1 must be bounded away from a_p. If actions below a_p are off-path, and $m(a_p, t_p, \underline{\theta}_p) < m(a_p, t_p, \overline{\theta}_p)$, then the highest type would have an incentive to deviate downward under D1. If actions below a_p are on-path with limit type θ_2, Lemma 4(a) requires that $\theta_2 > \theta_{\min}(a_p, t_p)$. But Lemma 6 implies that no type above θ_2 can choose a_p, so we must have $\theta_2 = \overline{\theta}_p$. Since $t_p \in (\underline{\theta}_p, \overline{\theta}_p)$, this type must pool with some type $\theta_3 < \theta_{\min}(a_p, t_p)$. Then, Lemma 6 again implies that no type below θ_3 can choose a_p, and so we must have $\theta_3 = \underline{\theta}_p$. This contradicts Lemma 2, which requires that $m(a_p, t_p, \underline{\theta}_p) = m(a_p, t_p, \overline{\theta}_p)$.

Part (c). Suppose $S(\cdot)$ is discontinuous on $[\underline{\theta}_p, \overline{\theta}_p]$. There are two cases, one in which $S(\cdot)$ jumps up and the other in which it jumps down.

Case 1. Suppose that $S(\cdot)$ jumps up at some $\theta_1 \in (\underline{\theta}_p, \overline{\theta}_p)$. Let $(S(\theta_1^+), T(\theta_1^+)) = (a_1, t_1)$. Also, let θ_2 be the type such that $S(\theta_2) = a_p$ and $S(\theta) > a_p$ for all $\theta \in (\theta_1, \theta_2)$. By part (b), only types in (θ_1, θ_2) can choose a_1. We further argue that there cannot be pooling at (a_1, t_1).

Suppose that there is some pooling at (a_1, t_1) instead. Then, actions below a_1 must be off-path. To see this, suppose that there is some type θ' who chooses an action slightly below a_1. By continuity, type θ' must weakly prefer (a_1, t_1) to (a_p, t_p). If $\theta' \leq \theta_{\min}(a_1, t_1)$, no type below θ' chooses a_1 by Lemma 6, so it must be that $\theta' < \theta_1$. Note also that since $S(\cdot)$ jumps up at θ_1, θ' must be bounded away from θ_1. Since $t_1 > \theta_1$, there must be another type $\theta_2 > \theta_1$ who pools with type θ', with $m(a_1, t_1, \theta') = m(a_1, t_1, \theta_2)$. However, if type θ' weakly prefers (a_1, t_1) to (a_p, t_p), all types below θ_2 strictly prefer (a_1, t_1), a contradiction. If $\theta' > \theta_{\min}(a_1, t_1)$, again no type above θ' can choose a_1 and there must be another limit type θ_2 who pools with θ'. We can derive a contradiction by applying the same argument.

Moreover, if there is some pooling at (a_1, t_1), and if actions above a_1 are off-path, D1 requires $t_1 \geq \theta_{\min}(a_1, t_1)$, which in turn implies $\max Q(a_1) > \theta_{\min}(a_1, t_1)$. If actions above a_1 are on-path with some limit type θ', Lemma 6 suggests that no type between θ' and $q(a_1, t_1, \theta_1)$ can choose a_1 and hence $\max Q(a_1) > \theta_{\min}(a_1, t_1)$. In either case, (a_1, t_1) must be in the RSC-domain for type $\max Q(a_1)$. This means that a type slightly above $\max Q(a_1)$ must choose some (a'', t'') such that $a'' < a_1$. Since we have shown that actions below a_1 are off-path, D1 requires $m(a_1, t_1, \theta_1) \geq m(a_1, t_1, \max Q(a_1))$ to prevent a downward deviation from a_1. But then this implies that type θ_1 must strictly prefer (a'', t'') to (a_1, t_1), a contradiction. This shows that there cannot be any pooling at (a_1, t_1).

This argument establishes that $S(\cdot)$ must be fully separating in a right neighborhood of θ_1. As long as $S(\cdot)$ is continuous, there cannot be any pooling along the way because otherwise $T(\cdot)$ would be discontinuous. If there is a jump, the same argument as above applies, and there cannot be any pooling immediately after the jump. But if $S(\cdot)$ is fully separating after the jump, $T(\cdot)$ must be continuous and incentive compatibility cannot be satisfied. This shows that $S(\theta)$ must be fully separating for all $\theta \in (\theta_1, \theta_2)$

Because $S(\cdot)$ is fully separating for all $\theta \in (\theta_1, \theta_2)$, there are two possibilities. One possibility is that $S(\cdot)$ jumps up at θ_1, strictly increases on (θ_1, θ_2), and jumps down to a_p at θ_2. If $S(\cdot)$ is increasing, however, $(S(\theta_2^-), T(\theta_2^-))$ must be in the SC-domain of type θ_2. Then, $S(\cdot)$ cannot jump down to a_p because a type slightly above θ_2 would have an incentive to deviate to $S(\theta_2^-)$.

The only remaining possibility is that $S(\cdot)$ jumps up at θ_1 and strictly decreases for $\theta \in (\theta_1, \theta_2)$. Since $S(\cdot)$ decreases while $T(\cdot)$ increases, the indifference curve of type θ_1 must be downward sloping at (a_1, t_1), and $(a_1, t_1) \in RSC(\theta_1)$. Now consider an action a' slightly above a_1 with corresponding reputation t'. Suppose there is some type $\theta' > \theta_2$ who chooses (a', t'). Then, by continuity, type θ' must weakly prefer (a_1, t_1) to (a_p, t_p). However, since $(a_1, t_1) \in RSC(\theta)$ for all $\theta \geq \theta_1$, all types in $[\theta_1, \theta')$, including type θ_2, must strictly prefer (a_1, t_1) to (a_p, t_p), a contradiction. If any type below θ_1 chooses a', then $t' < \theta_1 = t_1$, which cannot be incentive compatible. We thus conclude that actions slightly above a_1 are off-path. Given this, if there is a deviation to a', D1 assigns belief $t' = \theta_1$. But then type θ_1 would have an incentive to deviate upward, because $t' = t_1$ and $a' > a_1$, and the indifference curve of θ_1 is downward-sloping at that point.

Case 2. Now suppose that $S(\cdot)$ jumps down at some $\theta_2 \in (\underline{\theta}_p, \overline{\theta}_p)$. As above, let $(S(\theta_2^-), T(\theta_2^-)) = (a_2, t_2)$, and let θ_1 be the type such that $S(\theta_1) = a_p$ and $S(\theta) > a_p$ for all $\theta \in (\theta_1, \theta_2)$. Case 1 establishes that $S(\cdot)$ cannot jump up, so $S(\cdot)$ must be continuous at θ_1. Lemma 6 requires that $\theta_2 \geq q(a_p, t_p, \theta_1)$. Note also that for this to be an equilibrium, type θ_2 weakly prefers (a_2, t_2) to (a_p, t_p), while type θ_1 weakly prefers (a_p, t_p) to (a_2, t_2). However, $\theta_2 \geq q(a_p, t_p, \theta_1)$ implies that $\theta_2 > \theta_{\min}(a_p, t_p)$. This would lead to a contradiction, because Lemma 4(b) requires that whenever type θ_2 weakly prefers (a_2, t_2) to (a_p, t_p), type θ_1 must strictly prefer (a_2, t_2).

Part (a). By part (b) and part (c), if $J(a_p)$ is not empty, we must have $S'(\underline{\theta}_j) > 0$ and $S'(\overline{\theta}_j) < 0$. Therefore, there exists $\epsilon > 0$ such that for any $a \in (a_p, a_p + \epsilon)$, $Q(a)$ is a pooling set. Using part (b) again, all types in $[\min Q(a), \max Q(a)]$ choose actions higher than or equal to a and cannot choose a_p. This establishes that $Q(a_p) = Q_L(a_p) \cup H_R(a_p)$, with $Q_L(a_p) = [\underline{\theta}_p, \overline{\theta}_j]$ and $Q_R(a_p) = [\overline{\theta}_j, \overline{\theta}_p]$. Further, $S'(\underline{\theta}_j) > 0$ implies that $(a_p, t_p) \in SC(\underline{\theta}_j)$. By Assumption 3, (a_p, t_p) is in the SC-domain for all types in $Q_L(a_p)$. Similarly, $S'(\overline{\theta}_j) < 0$ implies that $(a_p, t_p) \in RSC(\overline{\theta}_j)$. So it must be in the RSC-domain for all types in $Q_R(a_p)$. ■

Lemma 3 also implies that for any disconnected pooling set $Q(a_p)$, there are two limit types that approaches a_p from inside the interval $(\underline{\theta}_p, \overline{\theta}_p)$, given by $\underline{\theta}_j$ and $\overline{\theta}_j$, with $\underline{\theta}_j > \theta_{\min}(a_p, t_p) > \overline{\theta}_j$ such that $S(\underline{\theta}_j{}^+) = S(\overline{\theta}_j{}^-) = a_p$. Moreover, $m(a_p, t_p, \underline{\theta}_j) = m(a_p, t_p, \overline{\theta}_j)$. Both $\underline{\theta}_j$ and $\overline{\theta}_j$ belong to $Q(a_p)$, and types in $J(a_p)$ choose actions higher than a_p.

To obtain an LSHPP strategy, we need to ensure that $\max Q(a_p) = \overline{\theta}$ for some a_p.

Lemma 7. *If actions below any pooling action a_p are off-path, then $Q(a_p)$ must include $\overline{\theta}$.*

Proof. Lemma 3(a) and Lemma 5 establish that, regardless of whether $Q(a_p)$ is connected or not, $(a_p, t_p) \in RSC(\overline{\theta}_p)$. Suppose $\overline{\theta}_p < \overline{\theta}$, and let $(S(\overline{\theta}_p{}^+), T(\overline{\theta}_p{}^+)) = (a_1, t_1)$. The reverse single-crossing property implies $a_1 < a_p$. By continuity, type $\overline{\theta}_p$ must be indifferent between (a_p, t_p) and (a_1, t_1). But to prevent downward deviation from a_p under D1 requires $m(a_p, t_p, \underline{\theta}_p) \geq m(a_p, t_p, \overline{\theta}_p)$, which means that $a_p \leq D(t_p; \overline{\theta}_p, \underline{\theta}_p)$. By the double-crossing property, whenever the higher type $\overline{\theta}_p$ is indifferent between (a_p, t_p) and (a_1, t_1), the lower type $\underline{\theta}_p$ strictly prefers the lower action (a_1, t_1), a contradiction. ■

A.4. Below the gap

The argument thus far characterizes equilibrium above the gap. Below we deal with the situation below the gap.

Lemma 8. *Suppose $S(\theta) = s^*(\theta)$ for $\theta \in (\theta_1, \theta_2)$. If $S(\cdot)$ jumps at θ_2,*

(a) $Q(S(\theta_2^+))$ is not a singleton.

(b) If $m(s^*(\theta_2^-), \theta_2, \theta_2) > 0$, then $S(\theta_2^+) > s^*(\theta_2^-)$; if $m(s^*(\theta_2^-), \theta_2, \theta_2) < 0$, then $S(\theta_2^+) < s^*(\theta_2^-)$.

Proof. (a) If $(s^*(\theta), \theta) \in SC(\theta)$, incentive compatibility requires $s^{*'}(\theta) > 0$. Since $T(\theta) = \theta$ under full separation, any incentive compatible solution must have $m(s^*(\theta), \theta, \theta) > 0$. Suppose there is a jump at θ_2 to $S(\theta_2^+)$ which is larger than $S(\theta_2^-)$. If $Q(S(\theta_2^+))$ is also a singleton, this cannot be incentive compatible because $T(\theta_2^+) = T(\theta_2^-) = \theta_2$ while $S(\theta_2^+) > S(\theta_2^-)$. The same argument follows for the case where $(s^*(\theta), \theta) \in RSC(\theta)$ and $s^{*'}(\theta) < 0$.

(b) Suppose $m(s^*(\theta_2^-), \theta_2, \theta_2) > 0$. In this case $s^*(\cdot)$ must be upward sloping at θ_2, and a type sightly below θ_2 must have a higher marginal rate of substitution. By the double-crossing property, their indifference curves never cross to the left of $s^*(\theta_2^-)$. Therefore, if there exists $S(\theta_2^+)$ such that type θ_2 is indifferent between $(s^*(\theta_2^-), \theta_2)$ and $(S(\theta_2^+), T(\theta_2^+))$, a type slightly below θ_2 strictly prefers $(S(\theta_2^+), T(\theta_2^+))$ and has an incentive to deviate. The other case can be proved similarly. ■

Suppose there is some pooling at (a_0, t_0) where $a_0 = \min\{a : Q(a) \text{ is not a sigleton}\}$. Lemma 8 suggests that once an equilibrium starts from a fully separating segment $s^*(\cdot)$, there are only two possibilities. First, there could be some θ_0 such that $s^*(\theta_0) = a_0$. Second, $s^*(\cdot)$ could jump at θ_0 to $S(\theta_0^+) = a_0$. The first possibility is ruled out because $T(\theta_0^-) = \theta_0$ while $t_0 > \theta_0$. This means that if there is an equilibrium in which full separation and pooling coexist, there must be a jump at the point of transition between separation and pooling.

Part (b) of the lemma states that if there is a jump when $s^*(\cdot)$ is upward sloping, the jump must also be an upward jump. In other words, any jump must be in the same direction as $s^*(\cdot)$ points to.

A.5. Equilibrium characterization

With all the preceding lemmas, we are now ready to complete the proof that any D1 equilibrium must be LSHPP.

First, if there is no fully separating region, i.e., $\theta_0 = \underline{\theta}$, then $Q(S(\underline{\theta}))$ must be a pooling set. Let $a_0 = S(\underline{\theta})$ and $t_0 = T(\underline{\theta})$. By Lemma 3, no type in $[\theta_0, \max Q(a_0)]$ can choose an action lower than a_0. If there is any type who chooses an action slightly lower than a_0, it must be types above $\max Q(a_0)$, but this cannot be incentive compatible because $t_0 < \max Q(a_0)$. Since actions below a_0 are off-path, Lemma 7 suggests $\overline{\theta} \in Q(a_0)$. Other properties of LSHPP equilibrium follow directly from the lemmas, suggesting that if a_0 is in a pooling set, the equilibrium must LSHPP.

Now suppose that $S(\underline{\theta})$ is a singleton and $T(\underline{\theta}) = \underline{\theta}$, in which case $S(\cdot) = s^*(\cdot)$ in a right neighborhood of $\underline{\theta}$. Lemma 8 implies that $S(\cdot) = s^*(\cdot)$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$ (the case of a fully separating equilibrium), or $S(\cdot)$ must jump at some point. Note that the fully separating equilibrium is a special case of LSHPP equilibrium. If $S(\cdot)$ jumps at θ_0 to $a_0 = S(\theta_0^+)$, we can apply the same argument as above to show that actions slightly below a_0 are off-path. This in turn implies that the equilibrium must be LSHPP.

B. Proof of Theorem 2

The existence proof consists of three parts. Let $\zeta(\theta_*)$ represent the θ_E obtained at the end of our algorithm starting from an initial state θ_*. We first establish continuity of $\zeta(\cdot)$, which in turn implies that $\Delta_u(\cdot)$ is also continuous. The second part establishes the existence of θ_* such that $\Delta_u(\theta_*) \leq 0$ (with strict inequality only if $\zeta(\theta_*) = \underline{\theta}$). The candidate solution obtained from such θ_* satisfies all the local incentive compatibility constraints. In the final step, we show that the candidate solution satisfies global incentive compatibility and constitutes an equilibrium.

B.1. Continuity

Under our algorithm, the solution switches from atomless pooling to mass pooling when $m_\theta(\cdot) - \hat{m}_\theta(\cdot)p'(\cdot)$ switches from positive to negative, and it switches back from mass pooling to atomless pooling as soon as $m(\cdot) - \hat{m}(\cdot)$ turns from positive to zero. We can rewrite equation (8) as

$$[\hat{m}_a(\cdot) - m_a(\cdot)]\,\sigma' + [\hat{m}_t(\cdot) - m_t(\cdot)]\,\tau' = \max\left\{\left[m_\theta(\cdot) - \hat{m}_\theta(\cdot)p'\right]\mathbb{I}\left(m(\cdot) = \hat{m}(\cdot)\right), 0\right\},$$

which incorporates both atomless pooling and mass pooling. Let $x = (p, \sigma, \tau)$. For ease of notation, we write

$$\begin{aligned}\tilde{h}(\theta, x) &:= m_\theta(\sigma, \tau, \theta) - \hat{m}_\theta(\sigma, \tau, p)\frac{f(\theta)}{f(p)}\frac{\theta - \tau}{p - \tau},\\ h(\theta, x) &:= \max\left\{\tilde{h}(\theta, x)\,\mathbb{I}\left(\Delta_m(\theta, x) = 0\right), 0\right\}.\end{aligned}$$

where $\Delta_m(\theta, x) := m(\sigma, \tau, \theta) - \hat{m}(\sigma, \tau, p)$. Together with (6) and (7), we obtain a system of differential equations of the form $x' = H(\theta, x)$, where

$$\begin{cases} p' = \frac{f(\theta)}{f(p)}\frac{\theta - \tau}{p - \tau}, \\ \sigma' = \frac{h(\theta, x)}{\hat{m}_a(\sigma,\tau,p) - m_a(\sigma,\tau,\theta) + m(\sigma,\tau,\theta)[\hat{m}_t(\sigma,\tau,p) - m_t(\sigma,\tau,\theta)]}, \\ \tau' = \frac{m(\sigma,\tau,\theta)h(\theta,x)}{\hat{m}_a(\sigma,\tau,p) - m_a(\sigma,\tau,\theta) + m(\sigma,\tau,\theta)[\hat{m}_t(\sigma,\tau,p) - m_t(\sigma,\tau,\theta)]}. \end{cases}$$

We solve this system backwards from $\mathbf{c}_1 = (\theta_*, x_*(\theta_*))$, where $x_*(\theta_*) = (\theta_*, D(\theta_*; \theta_*, \theta_*), \theta_*)$.

The initial value problem we consider is as follows:

$$\begin{cases} x' = H(\theta, x), \\ x(\theta_B) = x_B := (p_B, \tau_B, \sigma_B), \end{cases}$$

where (θ_B, x_B) is an arbitrary initial state. Let $y(\cdot; \theta_B, x_B)$ denote the solution to this problem. By standard argument, $y(\cdot; \theta_B, x_B)$ is continuous with respect to the initial state in a neighborhood of (θ_B, x_B) if $h(\cdot,\cdot)$ is locally Lipschitz at (θ_B, x_B).[17]

Suppose first that there is either mass pooling or atomless pooling in a neighborhood of (θ_B, x_B). If there is mass pooling, we have $h(\cdot,\cdot) = 0$; if there is atomless pooling, we have $h(\cdot,\cdot) = \tilde{h}(\cdot,\cdot)$. In either case, $h(\cdot,\cdot)$ is locally Lipschitz at (θ_B, x_B).

If there is a transition from atomless pooling to mass pooling at (θ_B, x_B), we have both $\tilde{h}(\theta_B, x_B) = 0$ and $\Delta_m(\theta_B, x_B) = 0$ by construction. In this case, $h(\cdot,\cdot)$ is still locally Lipschitz at (θ_B, x_B).

When there is a transition from mass pooling to atomless pooling, the indicator function turns from 0 to 1, and $h(\cdot,\cdot)$ is discontinuous at (θ_B, x_B) if $\tilde{h}(\theta_B, x_B) > 0$. To deal with this case, consider an initial state (θ_B, x_B) such that

$$\Delta_m(\theta_B, x_B) = 0, \qquad \tilde{h}(\theta_B, x_B) > 0,$$

which represents a point of transition from mass pooling to atomless pooling.[18] Pick an arbitrary state x from a set $X(\theta_B)$ such that

$$X(\theta_B) := \{x : \Delta_m(\theta_B, x) = 0\}.$$

By this definition, there is mass pooling in a neighborhood of (θ_B, x) if $x \in X(\theta_B)$. Define

$$\theta_T(x) := \max\{\theta : \Delta_m(\theta, y(\theta; \theta_B, x)) = 0\} < \theta_B,$$

for $x \in X(\theta_B)$ if it exists, and let $N_\delta(x_B) := \{x : \|x - x_B\| < \delta\}$.

Lemma 9. *For any $\epsilon > 0$, there is a δ such that $\theta_T(x)$ exists and $\theta_B - \theta_T(x) < \epsilon$ for $x \in N_\delta(x_B) \cap X(\theta_B)$.*

[17] The system is well defined except at θ_* where $p(\theta_*) = \tau(\theta_*) = \theta_*$ is imposed by construction. In this case, however, we can show $p'(\theta_*) = -1$ and $\tau'(\theta_*) = \sigma'(\theta_*) = 0$ for any θ_*. See Online Appendix E.

[18] We can have a (non-generic) case with $\tilde{h}(\theta_B, x_B) = 0$ even when there is a transition from mass pooling to atomless pooling. This occurs if $\psi(\cdot)$ and $\eta(\cdot)$ are tangent to each other at θ_B (and possibly over some interval that contains θ_B). We can disregard this possibility because $h(\cdot,\cdot)$ is continuous in this case.

Proof. We write $\psi(\cdot;x)$ and $\eta(\cdot;x)$ to denote their dependence on x. Recall that $\Delta_m(\theta_B, x) > 0$ is equivalent to $\psi(\theta_B;x) > \eta(\theta_B;x)$, so that we consider a change in x which makes $\psi(\cdot;x)$ go above $\eta(\cdot;x)$ evaluated at θ_B. Note also that $h(\theta_B, x_B) > 0$ is equivalent to $\psi'(\theta_B;x_B) > \eta'(\theta_B;x_B)$ and therefore that $\psi(\cdot;x_B) < \eta(\cdot;x_B)$ in a left neighborhood of θ_B. Then, since both $\psi(\cdot;x)$ and $\eta(\cdot;x)$ are continuous in x, for any $\epsilon > 0$, we can find a $\delta > 0$ such that $\psi(\theta_B - \epsilon;x) < \eta(\theta_B - \epsilon;x)$ and $\psi(\theta_B;x) > \eta(\theta_B;x)$ for $x \in N_\delta(x_B) \cap X(\theta_B)$. By continuity of $\psi(\cdot)$ and $\eta(\cdot)$, $\theta_T(x)$ must lie in $(\theta_B - \epsilon, \theta_B)$. ■

The lemma shows that $\theta_T(x)$ converges to θ_B as x gets arbitrarily close to x_B. Therefore, the solution induced from (θ_B, x) also converges pointwise to the solution induced from (θ_B, x_B) as $x \to x_B$.

This completes the proof that the solution from our algorithm is continuous with respect to the initial state. Suppose that (θ_B, x_B) represents the first point of transition from mass pooling to atomless pooling, so that continuity up to that point is ensured. This means that $x = y(\theta_B; \theta_*, x_*(\theta_*))$ is continuous in θ_*. Since $y(\cdot;\theta_B, x)$ is also continuous in x, we can ensure that the mapping $\zeta(\cdot)$ consistently produces a θ_0 which varies continuously with θ_*.

B.2. Indifference at the gap

Recall that $\Delta_u(\cdot)$ is defined as

$$\Delta_u(\theta_*) = u(s^*(\zeta(\theta_*)), \zeta(\theta_*), \zeta(\theta_*)) - u(\sigma(\zeta(\theta_*);\theta_*), \tau(\zeta(\theta_*);\theta_*), \zeta(\theta_*)),$$

where for clarity we use $(\sigma(\cdot;\theta_*), \tau(\cdot;\theta_*))$ to indicate the action-reputation pair induced from boundary type θ_*. Since $\zeta(\cdot)$ is continuous, $\Delta_u(\cdot)$ is also continuous.

Define z to be the boundary type such that $\zeta(z) = \underline{\theta}$; such type exists dues to continuity of $\zeta(\cdot)$. If $\Delta_u(\underline{z}) \leq 0$, then $(\sigma(\cdot;z), \tau(\cdot;z))$ with $\theta_0 = \underline{\theta}$ constitute a candidate solution.

Now suppose $\Delta_u(z) > 0$. Note that $\zeta(\overline{\theta}) = \overline{\theta}$, and therefore $\sigma(\overline{\theta};\overline{\theta}) = D(\overline{\theta};\overline{\theta},\overline{\theta}) < s^*(\overline{\theta})$ (otherwise we can have a fully separating equilibrium) and $\tau(\overline{\theta};\overline{\theta}) = \overline{\theta}$. Assumption 4 implies that $m(D(\overline{\theta};\overline{\theta},\overline{\theta}),\overline{\theta},\overline{\theta}) > 0$, and hence $((\sigma(\overline{\theta};\overline{\theta}), \tau(\overline{\theta};\overline{\theta}))$ is strictly preferred to $(s^*(\overline{\theta}),\overline{\theta})$. We therefore have $\Delta_u(\overline{\theta}) < 0$. It then follows that there exists $\theta_* \in (z, \overline{\theta})$ such that $\Delta_u(\theta_*) = 0$. For such θ_*, the solution $(\sigma(\cdot;\theta_*), \tau(\cdot;\theta_*))$ with $\theta_0 = \zeta(\theta_*)$ constitute a candidate solution.

B.3. Global incentive compatibility

Actions in $[\underline{a}^*, s^*(\theta_0)) \cup [\sigma(\theta_0), \sigma(\theta_*)]$ are on-path actions chosen by some types in equilibrium. We first show that no type has an incentive to deviate to any on-path action.

Since $(S(\theta), T(\theta)) \in SC(\theta)$ for all $\theta \in [\underline{\theta}, \theta_*)$, local incentive compatibility implies global incentive compatibility for types in $[\underline{\theta}, \theta_*]$. Consider next types in $(\theta_*, \overline{\theta}]$. Any action-reputation pair $(\sigma(\theta), \tau(\theta))$ belongs to $RSC(p(\theta))$ for all $p(\theta) \in (\theta_*, \overline{\theta}]$. According to the reverse single-crossing property, local incentive compatibility implies global incentive compatibility for action-reputation pairs in the RSC-domain, meaning that any type $p(\theta) \in (\theta_*, \overline{\theta}]$ has no incentive to deviate to any action $a \in [\sigma(\theta_0), \sigma(\theta_*)]$. For deviation to $(a, t) = (s^*(\theta'), \theta')$ for some $\theta' < \theta_0$, our earlier argument establishes that, for type $\theta \in [\theta_0, \theta_*]$,

$$u(a, t, \theta) < u(\sigma(\theta), \tau(\theta), \theta)).$$

But $m(\sigma(\theta), \tau(\theta), \theta) = m(\sigma(\theta), \tau(\theta), p(\theta))$ and the fact that the higher type $p(\theta)$ has more convex indifference curve than that of the lower type θ along the dividing line $D(\cdot; p(\theta), \theta)$ means that the indifference curve of type $p(\theta)$ passing through $(\sigma(\theta), \tau(\theta))$ is everywhere above the indifference curve of type θ. Therefore, the above inequality implies

$$u(a, t, p(\theta)) < u(\sigma(\theta), \tau(\theta), p(\theta)).$$

In other words type $p(\theta) \in (\theta_*, \overline{\theta}]$ has no incentive to deviate to (a, t).

The remaining issue is deviation to some off-path action. For the following argument, we refer to the indifference curve of type θ that passes through his equilibrium choice $(S(\theta), T(\theta))$ as the *equilibrium indifference curve* of type θ for brevity.

Case 1: Deviation to $a < \underline{a}^*$. This case is relevant only when $\underline{a}^* > 0$, which implies $m(\underline{a}^*, \underline{\theta}, \underline{\theta}) = 0$. We have already shown that all types above $\underline{\theta}$ strictly prefer their equilibrium choice to $(\underline{a}^*, \underline{\theta})$. So the equilibrium indifference curve of any type θ can cross that of type $\underline{\theta}$ at a point to the right of $\underline{a}^*$ and from above. By condition (1) and Assumption 4, this implies that, for all $a < \underline{a}^*$,

$$m(a, \phi(a, \underline{u}, \underline{\theta}), \theta) < m(a, \phi(a, \underline{u}, \underline{\theta}), \underline{\theta}) < 0,$$

where $\underline{u}$ is the equilibrium utility of type $\underline{\theta}$. This means that any deviation to $a < \underline{a}^*$ is attributed to the lowest type under D1. Furthermore, since the marginal rate of substitution is negative, if type θ prefers his equilibrium choice to $(\underline{a}^*, \underline{\theta})$, he will prefer his equilibrium choice to $(a, \underline{\theta})$ for $a < \underline{a}^*$.

Case 2: Deviation to $a > \sigma(\theta_*)$. At $(\sigma(\theta_*), \theta_*)$, all types above θ_* have a higher marginal rate of substitution and moreover that their equilibrium indifference curves stay strictly above that of type θ_* for all $a > \sigma(\theta_*)$. This means that the belief assigned to any deviation to an action higher than $\sigma(\theta_*)$ must be lower than θ_*. Since $\sigma(\theta_*) = D(\theta_*; \theta_*, \theta_*)$ by construction, Assumption 4 implies that, for all θ,

$$m(\sigma(\theta_*), \theta_*, \theta) \geq m(\sigma(\theta_*), \theta_*, \theta_*) > 0,$$

and $m(a, \theta_*, \theta) > 0$ for $a > \sigma(\theta_*)$. Thus no type can benefit from deviating to an action higher than $\sigma(\theta_*)$.

Case 3: Deviation to $a \in [s^*(\theta_0), \sigma(\theta_0))$. Global incentive compatibility for on-path actions means that the equilibrium indifference curve of any type (other than type θ_0) is strictly above the points $(s^*(\theta_0), \theta_0)$ and $(\sigma(\theta_0), \tau(\theta_0))$. For a type $\theta \in [\underline{\theta}, \theta_*]$, both points are in $SC(\theta)$, and therefore his equilibrium indifference curve must be entirely above that of type θ_0 for all $a \in [s^*(\theta_0), \sigma(\theta_0)]$. For a type $p(\theta) \in (\theta_*, \overline{\theta}]$, his equilibrium indifference curve is entirely above that of type $\theta \in [\theta_0, \theta_*]$, and is therefore also above that of type θ_0 for all $a \in [s^*(\theta_0), \sigma(\theta_0)]$. This means that any deviation to an action between $s^*(\theta_0)$ and $\sigma(\theta_0)$ is attributed to type θ_0 under D1. Moreover, incentive compatibility in the SC-domain implies that $s^*(\cdot)$ is increasing at θ_0. Together with the quasi-concavity of $u(\cdot, \theta_0, \theta_0)$, this implies that $m(a, \theta_0, \theta_0) > 0$ for all $a \in (s^*(\theta_0), \sigma(\theta_0))$. Thus type θ_0 has no incentive to deviate to such a for no gain in reputation. It follows that no other type has an incentive to deviate to such a either.

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Online Appendix to
"Signaling under Double-Crossing Preferences"

(Not for publication)

C. Double-Crossing Property and Marginal Rate of Substitution

Lemma C.1. *If preferences satisfy the double-crossing property, then for $\theta' > \theta''$,*

$$m(a,t,\theta') - m(a,t,\theta'') \begin{cases} \leq 0 & \text{if } a \leq D(t;\theta',\theta''), \\ \geq 0 & \text{if } a \geq D(t;\theta',\theta''). \end{cases}$$

Proof. Let u'' and u' be the utility levels of types θ'' and θ', respectively, at (a_1, t_1). For $a_2 < a_1 \leq D(t_1;\theta',\theta'')$, part (a) of Definition 1 requires that $t_2 = \phi(a_2, u'', \theta'')$ implies $t_2 < \phi(a_2, u', \theta')$. Take the limit as a_2 approaches a_1 from below, we obtain $\phi_a(a_1, u', \theta') \leq \phi_a(a_1, u'', \theta'')$, which implies that $m(a_1, t_1, \theta') \leq m(a_1, t_1, \theta'')$, with equality only if $a_1 = D(t_1;\theta',\theta'')$.

If $a_1 > a_2 \geq D(t_2, \theta', \theta'')$, we let u'' and u' represent the utility levels of the corresponding types at (a_2, t_2). Part (b) of the definition requires $t_1 = \phi(a_1, u'', \theta'') > \phi(a_1, u', \theta')$. Take the limit as a_1 approaches a_2 from above, we obtain $\phi_a(a_2, u'', \theta'') \leq \phi_a(a_2, u', \theta')$, which implies that $m(a_2, t_2, \theta'') \leq m(a_2, t_2, \theta')$, with equality only if $a_2 = D(t_2;\theta',\theta'')$. ■

Lemma C.2. *Preferences satisfy the double-crossing property if and only if, for $\theta' > \theta''$, there exists $D(\cdot;\theta',\theta'')$ such that*

$$m(a, \phi(a, u_0, \theta''), \theta') - m(a, \phi(a, u_0, \theta''), \theta'') \begin{cases} \leq 0 & \text{if } a \leq a_0 \leq D(t_0;\theta',\theta''), \\ \geq 0 & \text{if } a \geq a_0 \geq D(t_0;\theta',\theta''); \end{cases}$$

with strict inequality except when $a = a_0 = D(t_0;\theta',\theta'')$.

Proof. Suppose preferences satisfy the double-crossing property. If type θ'' is indifferent between (a_0, t_0) and $(a, \phi(a, u_0, \theta''))$, parts (a) and (b) of Definition 1 together imply that $a < a_0 \leq D(t_0;\theta',\theta'')$ and $a \geq D(\phi(a, u_0, \theta'');\theta',\theta'')$ would lead to a contradiction. Therefore, $a < a_0 \leq D(t_0;\theta',\theta'')$ implies $a < D(\phi(a, u_0, \theta'');\theta',\theta'')$. By Lemma C.1, we have $m(a,\phi(a,u_0,\theta''),\theta') - m(a,\phi(a,u_0,\theta''),\theta'') \leq 0$, with equality only if $a_2 = a_1 = D(t_1;\theta',\theta'')$. Similarly, $a > a_0 \geq D(t_0;\theta',\theta'')$ implies $a > D(\phi(a,u_0,\theta'');\theta',\theta'')$. By Lemma C.1, we have $m(a,\phi(a,u_0,\theta''),\theta') - m(a,\phi(a,u_0,\theta''),\theta'') \geq 0$, with equality only if $a_1 = a_2 = D(t_2;\theta',\theta'')$.

For sufficiency, let u' represent the utility level of type θ' at (a_0,t_0). If $a < a_0 \leq D(t_0;\theta',\theta'')$, we have $m(a,\phi(a,u_0,\theta''),\theta') > m(a,\phi(a,u_0,\theta''),\theta'')$ and $\phi(a,u',\theta') = \phi(a,u_0,\theta'')$. By the comparison theorem of differential equations, we have $\phi(a,u',\theta') > \phi(a,u_0,\theta'')$. Because $u(\cdot)$ strictly increases in t, this in turn implies that the higher type θ' strictly prefers (a_0,t_0) to $(a,\phi(a,u_0,\theta''))$. If $a > a'' \geq D(t'';\theta',\theta'')$, the comparison theorem of differential equations implies that $\phi(a,u',\theta') < \phi(a,u_0,\theta'')$. This in turn implies that the higher type strictly prefers $(a,\phi(a,u_0,\theta''))$ to (a_0,t_0). ■

D. Verifying the Assumptions in the Examples

D.1. Signaling with news

The marginal rate of substitution in this example is

$$m(a,t,\theta) = \frac{\gamma + a\theta}{\lambda\theta[1-(\beta_0+\beta\theta)]}.$$

For $\theta' > \theta''$,

$$\frac{m(a,t,\theta')}{m(a,t,\theta'')} = \left(\frac{\gamma + a\theta'}{\gamma + a\theta''}\right)\left(\frac{\theta'[1-(\beta_0+\beta\theta'')]}{\theta''[1-(\beta_0+\beta\theta')]}\right),$$

strictly increases in a. Therefore $m(a,t,\theta') - m(a,t,\theta'')$ is single-crossing from below in a. Furthermore,

$$m_\theta(a,t,\theta) = \frac{\beta a\theta^2 - (1-(\beta_0+2\beta\theta))\gamma}{\lambda\theta^2[1-(\beta_0+\beta\theta)]^2}.$$

At $m_\theta(a,t,\theta) = 0$, the second derivative is

$$m_{\theta\theta}(a,t,\theta) = \frac{2\beta a\theta + 2\beta\gamma}{\lambda\theta^2[1-(\beta_0+\beta\theta)]^2} > 0.$$

Thus, $m_\theta(a,t,\theta)$ is single-crossing from below, meaning that $m(a,t,\theta)$ is quasi-convex.

D.2. Reputation enhances the chances of success

The marginal rate of substitution is

$$m(a,t,\theta) = \frac{\gamma + \theta a}{\theta\beta V g(K-\theta-\beta t)}.$$

We have

$$\frac{\partial \log m(a,t,\theta)}{\partial \theta} = -\frac{\gamma}{\theta(\gamma+\theta a)} + \frac{g'(K-\theta-\beta t)}{g(K-\theta-\beta t)}.$$

Since $g'(\cdot)/g(\cdot)$ increases in θ, the above expression increases in θ. Thus, $m(a,t,\theta)$ is quasi-convex. Moreover,

$$\frac{m(a,\phi(a,u_0,\theta''),\theta')}{m(a,\phi(a,u_0,\theta''),\theta'')} = \left[\frac{\theta''(\gamma+\theta' a)}{\theta'(\gamma+\theta'' a)}\right]\left[\frac{g(K-\theta''-\beta\phi(a,u_0,\theta''))}{g(K-\theta'-\beta\phi(a,u_0,\theta''))}\right].$$

For $\theta' > \theta''$, the first bracketed term increases in a, and the second bracketed term also increases in a by log-concavity of $g(\cdot)$. This shows that $m(a,\phi(a,u_0,\theta''),\theta')$ crosses $m(a,\phi(a,u_0,\theta''),\theta'')$ from below.

D.3. Risky experimentation

The marginal rate of substitution is:

$$m(a,t,\theta) = -\frac{\tilde{g}(a;\theta)[V-R(a,t)] - \rho R(a,t) + R_a(a,t)}{R_t(a,t)},$$

where

$$\tilde{g}(a;\theta) = \frac{\pi\theta e^{-\theta a}}{1-\pi e^{-\theta a}}.$$

For $\theta' > \theta''$, $m(a,t,\theta') - m(a,t,\theta'')$ has the same sign as:

$$\frac{\tilde{g}(a;\theta'')}{\tilde{g}(a;\theta')} - 1 = \frac{\theta''}{\theta'} e^{(\theta'-\theta'')a} \frac{1-\pi e^{-\theta' a}}{1-\pi e^{-\theta'' a}} - 1$$

When $\tilde{g}(a;\theta'') = \tilde{g}(a;\theta')$, the derivative of the above with respect to a is equal to

$$\frac{\theta'' e^{(\theta'-\theta'')a}}{\theta'(1-\pi e^{-\theta'' a})}\left[(\theta'-\theta'')(1-\pi e^{-\theta' a}) + \theta'\pi e^{-\theta' a} - \frac{\theta''\pi e^{-\theta'' a}(1-\pi e^{-\theta' a})}{1-\pi e^{-\theta'' a}}\right]$$

$$= \frac{\theta'' e^{(\theta'-\theta'')a}}{\theta'(1-\pi e^{-\theta'' a})}(\theta'-\theta'')(1-\pi e^{-\theta' a}) > 0.$$

This shows that $m(a,t,\theta') - m(a,t,\theta'')$ is single-crossing from below. Moreover,

$$\tilde{g}_\theta(a;\theta) = \frac{\pi e^{-\theta a}}{(1+\pi e^{-\theta a})^2}(1 - a\theta - \pi e^{-\theta a}).$$

At $\tilde{g}_\theta(a;\theta) = 0$, we have

$$\tilde{g}_{\theta\theta}(a;\theta) = \frac{\pi e^{-\theta a}}{(1+\pi e^{-\theta a})^2}(-a + \pi a e^{-\theta a}) = -\frac{\pi a^2 \theta e^{-\theta a}}{(1+\pi e^{-\theta a})^2} < 0.$$

Therefore, $\tilde{g}_\theta(a;\theta)$ is single-crossing from above, which implies that $m_\theta(a,t,\theta)$ is single-crossing from below. Hence $m(a,t,\theta)$ is quasi-convex.

E. The solution at the boundary

The solution of our model is characterized by the system of differential equations $x' = H(\theta, x)$ where $x = (p, \sigma, \tau)$. Observe that the differential equations are not well defined at θ_* since $p(\theta_*) = \tau(\theta_*) = \theta_*$ is imposed by construction. Below, we argue that $p'(\theta_*) = -1$ and $\sigma'(\theta_*) = \tau'(\theta_*) = 0$ hold for any θ_*, so that the system always produces a well behaved solution.

To this end, we apply l'Hopital's rule to obtain

$$p'(\theta_*) = \frac{1 - \tau'(\theta_*)}{p'(\theta_*) - \tau'(\theta_*)}.$$

Solving this for $p'(\theta_*)$ yields

$$p'(\theta_*) = \frac{\tau'(\theta_*) \pm \sqrt{\tau'(\theta_*)^2 + 4(1 - \tau'(\theta_*))}}{2} = \frac{\tau'(\theta_*) \pm (\tau'(\theta_*) - 2)}{2}.$$

Since $p'(\cdot)$ must be negative, we must have $p'(\theta_*) = \tau'(\theta_*) - 1$. This immediately implies that if there is mass pooling in a neighborhood of θ_*, we have $p'(\theta_*) = -1$.

Now suppose that there is atomless pooling in a neighborhood of θ_*. Note that the local incentive compatibility constraint for type θ_* is slightly irregular, as he may mimic either type $\theta_* - \epsilon$ or type $p(\theta_* - \epsilon)$. The conditions for this can be written as

$$u(\sigma(\theta_*), \tau(\theta_*), \theta_*) \geq u(\sigma(\theta_* - \epsilon), \tau(\theta_* - \epsilon), \theta_*),$$
$$u(\sigma(\theta_*), \tau(\theta_*), \theta_*) \geq u(\sigma(p(\theta_* - \epsilon)), \tau(p(\theta_* - \epsilon)), \theta_*),$$

where $(\sigma(\cdot), \tau(\cdot)) = (\sigma(p(\cdot)), \tau(p(\cdot)))$ by definition. In the limit, we must have

$$\sigma'(\theta_*) = \frac{\tau'(\theta_*)}{m(\sigma(\theta_*), \tau(\theta_*), \theta_*)} = \frac{\tau'(\theta_*) p'(\theta_*)}{m(\sigma(\theta_*), \tau(\theta_*), \theta_*)}.$$

Since $p'(\theta_*) = \tau'(\theta_*) - 1$, the only consistent solution is $\sigma'(\theta_*) = \tau'(\theta_*) = 0$ and $p'(\theta_*) = -1$ for any θ_*, regardless of whether there is atomless pooling or mass pooling in a neighborhood of θ_*.