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STRATEGY-PROOF MECHANISM DESIGN WITH NON-QUASILINEAR PREFERENCES: EX-POST REVENUE MAXIMIZATION FOR AN ARBITRARY NUMBER OF OBJECTS

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Strategy-proof mechanism design with non-quasilinear preferences: Ex-post revenue maximization for an arbitrary number of objects*

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Abstract

We consider the multi-object allocation problem with monetary transfers where each agent obtains at most one object (unit-demand). We focus on allocation rules satisfying *individual rationality*, *non-wastefulness*, *equal treatment of equals*, and *strategy-proofness*. Extending the result of Kazumura et al. (2020B), we show that for an arbitrary number of agents and objects, the minimum price Walrasian is the unique ex-post revenue maximizing rule among the rules satisfying *no subsidy* in addition to the four properties, and that no subsidy in this result can be replaced by no bankruptcy on the positive income effect domain.

JEL classification: D82, D47, D63.

Keywords: Multi-object allocation problem, Strategy-proofness, Ex-post revenue maximization, Minimum price Walrasian rule, Non-quasi-linear preference, Equal treatment of equals, Non-wastefulness.

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1 Introduction

This is to extend Kazumura et al. (2020B) to a general case of an arbitrary number of agents and objects. They consider the multi-object allocation problem with monetary transfers where each agent obtains at most one object (unit-demand). A (*consumption*) *bundle* is a pair of object and payment. Each agent has a continuous preference relation over bundles satisfying the possibility of compensation, money monotonicity, and object desirability. Such preferences are called *classical*. The *classical domain* is the class of all classical preferences.

An (*allocation*) *rule*, or simply *rule* chooses, for each preference profile, the object each agent receives and how much each agent pays. We focus on the following four properties of rules. A rule is *desirable* if it satisfies those properties.

(i) *Individual rationality* requires that each agent's bundle is at least as good as receiving nothing with no payment. Without this condition, agents does not participate the rule voluntarily. (ii) *Non-wastefulness* means that no agent prefers his own bundle to unassigned object with no payment. This condition is the property of weak efficiency. (iii) *Equal treatment of equals* requires that if there are two agents with same preferences, then their bundles are indifference for their preferences. This is the weak condition of fairness. (iv) *Strategy-proofness* is the incentive compatible condition, which means that no agent has incentive to misreport his preference.

In multi-object allocation problem, for each preference profile, *Walrasian equilibrium* exists (Alkan and Gale, 1990), and Demange and Gale (1985) show that the set of Walrasian prices has a lattice structure; that is, there is the minimum price Walrasian equilibrium for each preference profile. The *minimum price Walrasian rule* is desirable (Demange and Gale, 1985), and it satisfies *efficiency* and *no subsidy* (no-negative monetary transfer).

A subdomain of the classical domain is *rich* if for each object, there is a preference in the subdomain which demands only the object for some price and demands no object for some other price.[1] Kazumura et al. (2020B) illustrate that various domains including the quasi-linear domain are rich, and show that in the case where the number of agents is greater than the number of objects, the minimum price Walrasian rule is the unique ex-post revenue maximizing rule among the desirable rules satisfying no subsidy on a rich domain. We extend their result to a general case of an arbitrary number of agents and objects.

This article is organized as follows. Section 2 introduces the model and basic concepts. Section 3 checks the properties of minimum price Walrasian rules. Our results are in Section 4. Section 5 provides proofs. Section 6 concludes.

[1] See Subsection 4.1 for the formal definition of richness.

2 The model

Let $N = \{1, 2, \cdots, n\}$ be the set of agents and $M = \{1, 2, \cdots, m\}$ be the set of different objects. Not consuming an object in M is called consuming the "null object". Let $L \equiv M \cup \{0\}$, where 0 denotes the null object. Each agent consumes at most one object. A typical (consumption) bundle for agent i is a pair $z_i = (x_i, t_i) \in L \times \mathbb{R}$: agent i receives object x_i and pays t_i.

Each agent has a complete and transitive preference relation R_i over $L \times \mathbb{R}$. Let I_i and P_i be the indifference relation and strict preference relation associated with R_i. A typical class of preferences is denoted by $\mathcal{R}$. We call $\mathcal{R}^n$ a **domain**. $\mathcal{R}$ is **classical** if it satisfies the following assumptions:

1. *Continuity:* For each $z_i \in L \times \mathbb{R}$, the sets $\{z_i' \in L \times \mathbb{R} : z_i' \, R_i \, z_i\}$ and $\{z_i' \in L \times \mathbb{R} : z_i \, R_i \, z_i'\}$ are closed.

2. *Possibility of compensation:* For each pair $a, b \in L$ and each $t \in \mathbb{R}$, there exist $t', t'' \in \mathbb{R}$ such that $(a, t) \; R_i \; (b, t')$ and $(b, t') \; R_i \; (a, t'')$.

3. *Money monotonicity:* For each $a \in L$ and each pair $t, t' \in \mathbb{R}$, if $t < t'$, then $(a, t) \; P_i \; (a, t')$.

4. *Object desirability:* For each $a \in M$ and each $t \in \mathbb{R}$, $(a, t) \; P_i \; (0, t)$.

Let $\mathcal{R}^C$ be the set of classical preferences. We assume that $\mathcal{R} \subseteq \mathcal{R}^C$. A preference profile is a list of preferences $R \equiv (R_1, \cdots, R_n)$. Given $i \in N$ and $N' \subseteq N$, let $R_{-i} \equiv (R_j)_{j \neq i}$ and $R_{-N'} \equiv (R_j)_{j \in N \setminus N'}$.

A (feasible) **object allocation** is an n-tuple $x = (x_1, x_2, \ldots, x_n) \in L^n$ such that for each pair $i, j \in N$, if $x_i = x_j$, then $x_i = x_j = 0$. Let A be the set of all object allocations. An **allocation** is a pair of an object allocation and a vector of payments, $z = ((x_1, x_2, \ldots, x_n), (t_1, t_2, \ldots, t_n)) \in A \times \mathbb{R}^n$. Given $z \in A \times \mathbb{R}^n$ and $i \in N$, $z_i = (x_i, t_i) \in L \times \mathbb{Z}$ denotes the bundle of agent i.

An **(allocation) rule** associates an allocation to each preference profile. Formally, a **rule** is a mapping $f = (x, t) \colon \mathcal{R}^n \to A \times \mathbb{R}^n$. Given a rule f and a preference profile $R \in \mathcal{R}^n$, agent i's assignment under f at R is denoted by $f_i(R)$. Moreover, we write $f_i(R) \equiv (x_i(R), t_i(R)) \in L \times \mathbb{R}$, where $x_i(R)$ denotes i's object assignment and $t_i(R)$ denotes his payment. We define $f(R) \equiv (f_1(R), \cdots, f_n(R))$.

Definition 1. An allocation rule $f = (x, t)$ is **desirable** if it satisfies the following axioms:

- *Individual rationality:* For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R) \; R_i \; (0, 0)$.

- *Non-wastefulness:* For each $R \in \mathcal{R}$, each $i \in N$ and each $a \in M$, if $x_i(R) \neq a$ and $(a, 0) \, P_i \, f_i(R)$, then there is $j \neq i$ such that $x_j(R) = a$.

- *Equal treatment of equals:* For each $R \in \mathcal{R}^n$ and each $i, j \in N$ with $R_i = R_j$, $f_i(R) \, I_i \, f_j(R)$.
- *Strategy-proofness:* For each $R \in \mathcal{R}^n$, each $i \in N$ and each $R_i' \in \mathcal{R}$, $f_i(R) \; R_i \; f_i(R_i', R_{-i})$.

Non-wastefulness means that no agent prefers unassigned object with no payment to his own bundle. This condition is a weak condition of efficiency.

We say the allocation rule satisfies **no-wastage** if for each $R \in \mathcal{R}^n$ and each $a \in M$, there is $i \in N$ such that $x_i(R) = a$. Note that when $n < m$, no rule satisfies no-wastage. Kazumura et al. (2020B) define desirability by (i) individual rationality, (ii) *no-wastage*, (iii) equal treatment of equals, and (iv) strategy-proofness, which is different from our definition. However, since they assume that the number of agents is greater than the number of objects, no-wastage implies no-wastefulness. Thus, it is natural to relax no-wastage to non-wastefulness.

Remark 1. (i) If $n < m$, no rule satisfies no-wastage. (ii) If $n \geq m$ and f satisfies no-wastage, then it is non-wasteful.

Definition 2. An allocation rule $f = (x, t)$ satisfies **no subsidy** if for each $R \in \mathcal{R}^n$ and each $i \in N$, $t_i(R) \geq 0$.

Let $p = (p_1, p_2, \cdots, p_m) \in \mathbb{R}^m_+$ be a price vector. We assume that the price of null object is equal to zero; that is, $p_0 = 0$. Given $i \in N$, $R_i \in \mathcal{R}$, and $p \in \mathbb{R}^m_+$, let $D(R_i, p) \equiv \{a \in L : \forall b \in L, \; (a, p_a) \, I_i \, (b, p_b)\}$ denote the demand set of agent i with R_i at p.

Next, we define the concept of Walrasian equilibrium. It is a pair of a price vector and an allocation such that each agent an object he demands and pays its price, and the price of an unassigned object is zero.

Definition 3. Given $R \in \mathcal{R}^n$, a pair $((x, t), p) \in (A \times \mathbb{R}^n) \times \mathbb{R}^m_+$ is a **Walrasian equilibrium** for R if
WE-i: for each $i \in N$, $x_i \in D(R_i, p)$ and $t_i = p_{x_i}$; and
WE-ii: for each $a \in M \setminus \{x_j\}_{j \in N}$, $p_a = 0$.

Given $R \in \mathcal{R}^n$, let $W(R)$ be the set of Walrasian equilibria for R, and define

$$Z(R) \equiv \{z \in A \times \mathbb{R}^n : \exists p \in \mathbb{R}^m_+ \text{ s.t. } (z, p) \in W(R)\}$$

and

$$P(R) \equiv \left\{p \in \mathbb{R}^m_+ : \exists z \in A \times \mathbb{R}^n \text{ s.t. } (z, p) \in W(R)\right\}.$$

3 Facts

Fact 1 (Alkan and Gale, 1990)**.** For each $R \in \mathcal{R}^n$, there is a Walrasian equilibrium; that is, $W(R) \neq \emptyset$.

Fact 2 (Demange and Gale, 1985)**.** For each $R \in \mathcal{R}^n$, there is $p \in \mathbb{R}^m_+$ such that for each $p' \in P(R)$, $p \leq p'$. [2]

Given $R \in \mathcal{R}^n$, we denote the minimum Walrasian price for R by $p^{min}(R)$ and define

$$Z^{min}(R) \equiv \{z \in A \times \mathbb{R}^n : (z, p^{min}(R)) \in W(R)\}.$$

We say an allocation rule f is a **minimum price Walrasian rule** if for each $R \in \mathcal{R}^n$, $f(R) \in Z^{min}(R)$.

Fact 3 (Demange and Gale, 1985)**.** The minimum price Walrasian rule f is strategy-proof.[3]

By the definition of Walrasian equilibrium, the minimum price Walrasian rule satisfies individual rationality, non-wastefulness, and equal treatment for equals, and so it is desirable; moreover, it satisfies no subsidy.

Fact 4 (Demange and Gale, 1985)**.** The minimum price Walrasian rule f is desirable and satisfies no subsidy.

4 Results

Definition 4. A rule $f = (x, t)$ **revenue dominates** $g = (x', t')$ such that for each $R \in \mathcal{R}^n$, $\sum_{i \in N} t_i(R) \geq \sum_{i \in N} t'_i(R)$.

A rule is **ex-post revenue optimal** among a class of rules if it is in a class of rule and revenue dominates each rules in the class. We explore ex-post revenue optimal rules among the class of desirable satisfying no subsidy on rich domains in Subsection 4.1, and among the class of desirable satisfying no bankruptcy on the positive income effect domain in Subsection 4.2.

[2] $p \leq p'$ means that $p_a \leq p'_a$ for each $a \in M$.

[3] Precisely, they show that the minimum price Walrasian rule is *group strategy-proofness*; we say a rule f is *group strategy-proof* if for each $R \in \mathcal{R}^n$ and each $N' \subseteq N$, there is no $R'_{N'} \in \mathcal{R}^{|N'|}$ such that for each $i \in N$, $f_i(R'_{N'}, R_{-N'})\, P_i\, f_i(R)$.

4.1 The result on Rich domains

We define the **richness** of a domain.

Definition 5. A domain $\mathcal{R}$ is **rich** if for each $a \in M$ and each $p \in \mathbb{R}^m_+$ with $p_a > 0$ and $p_b = 0$ for each $b \in M \setminus \{a\}$ and for each $p' > p$, [4] there is $R_i \in \mathcal{R}$ such that

$$D(R_i, p) = \{a\} \text{ and } D(R_i, p') = \{0\}.$$

Fact 5 (Theorem 1 in Kazumura et al., 2020B)**.** Let $\mathcal{R}$ be rich and $n > m$. The minimum price Walrasian rule is the unique ex-post revenue optimal rule among the class of desirable rules satisfying no subsidy.

Fact 6 (Morimoto and Serizawa, 2015)**.** Let $R \in \mathcal{R}^n$ and $p = p^{min}(R)$. Then, (i) if $n > m$, then for each $a \in M$, $p_a > 0$, and (ii) if $n \leq m$, then there is $a \in M$ such that $p_a = 0$.

In Kazumura et al. (2020B), the proof of Fact 5 depends on Fact 6 (i). However, when the number of agents is less or equal to the number of objects, we cannot use their method directly.

Theorem 1 shows that for an arbitrary number of agents and objects, the minimum price Walrasian rule is revenue optimal among the same class on a rich domain.

Theorem 1. Let $\mathcal{R}$ be rich. The minimum price Walrasian rule is the unique ex-post revenue optimal rule among the class of desirable rules satisfying no subsidy.

4.2 The result on the positive income effect domain

We define the positive income effect domain.

Definition 6. A preference R_i satisfies **positive income effect** if for each $a, b \in L$ and each $t, t' \in \mathbb{R}$ with $t < t'$ and $(b, t')\, I_i\, (a, t)$, for each $\delta > 0$,

$$(b, t' - \delta)\, P_i\, (a, t - \delta).$$

Let $\mathcal{R}^{++}$ be the set of positive income effect preferences.

Definition 7. A rule f satisfies **no bankruptcy** if there is $l \leq 0$ such that for each $R \in \mathcal{R}^n$, $\sum_{i \in N} t_i(R) \geq l$.

Fact 7 (Kazumura et al., 2020B)**.** Let $\mathcal{R} \supseteq \mathcal{R}^{++}$ and $n > m$. The minimum price Walrasian rule is the unique ex-post revenue optimal rule among the class of desirable rules satisfying no bankruptcy.

Similarly to Fact 7, but for an arbitrary number of agents and objects, we have:

[4] $p' > p$ means that $p'_a > p_a$ for each $a \in M$.

Theorem 2. Let $\mathcal{R} \supseteq \mathcal{R}^{++}$. The minimum price Walrasian rule is the unique ex-post revenue optimal rule among the class of desirable rules satisfying no bankruptcy.

Since the classical domain is a rich domain including the positive income effect domain, we have Corollary 1:

Corollary 1. The minimum price Walrasian rule is the unique ex-post revenue optimal rule among the class of desirable rules satisfying no bankruptcy on the classical domain.

4.3 Efficiency

Finally, we discuss the property of **efficiency**.

Definition 8. A rule $f = (x, t)$ satisfies **efficiency** if for each $R \in \mathcal{R}^n$, there is no allocation $z \in A \times \mathbb{R}^n$ such that (i) for each $i \in N$, $z_i \, R_i \, f_i(R)$, (ii) for some $j \in N$, $z_j \, P_j \, f_j(R)$ and (iii) $\sum_{k \in N} t_k \geq \sum_{k \in N} t_k(R)$.

Since ex-post revenue optimal rule is the minimum price Walrasian rule and it is an efficient rule (Morimoto and Serizawa, 2015), ex-post revenue optimal rule is efficient (Kazumura et al., 2020B). By the results of Kazumura et al. (2020B) and ours, we have:

Corollary 2. Let $\mathcal{R}$ be rich. For each $n, m \in \mathbb{N}$, if f is ex-post revenue optimal among desirable rules satisfying no subsidy, then f is efficient.

Corollary 3. Let $\mathcal{R} \supseteq \mathcal{R}^{++}$. For each $n, m \in \mathbb{N}$, if f is ex-post revenue optimal among desirable rules satisfying no bankruptcy, then f is efficient.

5 Proofs

Given $i \in N$, $R_i \in \mathcal{R}$, $a \in L$ and $(b, t) \in L \times \mathbb{R}_+$, the **compensated valuation** $V^{R_i}(a; (b, t))$ of a from (b, t) for R_i is the value such that $(a, V^{R_i}(a; (b, t))) \, I_i \, (b, t)$.

Fact 8. Let f satisfies individual rationality. Let $R \in \mathcal{R}^n$, $i \in N$ and $z \in Z(R)$. If $z_i \, P_i \, f_i(R)$, then $x_i \neq 0$.

Fact 9 (Lemma 1 in Kazumura et al., 2020B)**.** Let $g = (x', t')$ be a minimum price Walrasian rule. Then, for each rule f and each $R \in \mathcal{R}^n$,

$$[\forall i \in N, \ f_i(R) \, R_i \, g_i(R)] \Rightarrow \sum_{i \in N} t'_i(R) \geq \sum_{i \in N} t_i(R).$$

5.1 Proof of Theorem 1

Throughout this subsection, we assume that $\mathcal{R}$ is rich. Our proof employes many results of Kazumura et al. (2020B). We omit the proofs of such results, and write only the proofs of our new results.

We introduce the concept of favoring preferences. Given $a \in M$, let $\mathcal{R}^a \equiv \{R_i \in \mathcal{R} : \forall b \in M \setminus \{a\}, (a,0)\, P_i\, (b,0)\}$. By the richness of domains, for each $a \in M$, $\mathcal{R}^a \neq \emptyset$. We say a preference $R_i \in \mathcal{R}^a$ is **a-favoring**.

Definition 9. Given $(a,t) \in M \times \mathbb{R}_+$, $R_i' \in \mathcal{R}^a$ is **(a,t)-favoring** if for each $b \in M \setminus \{a\}$, $V^{R_i'}(b;(a,t)) < 0$.

Note that for each $R \in \mathcal{R}^n$, each $i \in N$ and each $z \in Z(R)$, if $x_i \neq 0$, then there is a z_i-favoring preference $R_i' \in \mathcal{R}$.

Remark 2. Let f satisfy no subsidy. Let $R \in \mathcal{R}^n$, $i \in N$, and $z \in Z(R)$ be such that $x_i \neq 0$ and R_i is z_i-favoring. If $x_i(R) \neq x_i$, then $z_i\, P_i\, f_i(R)$.

Proof. Let $x_i(R) \neq x_i$. Suppose $f_i(R)\, R_i\, z_i$. Then, $t_i(R) \leq V^{R_i}(x_i(R); z_i)$. Since R_i is z_i-favoring, $V^{R_i}(x_i(R); z_i) < 0$. Thus, $t_i(R) < 0$, which contradicts no subsidy. ■

We say a preference R_i is **$(a,t)^{\varepsilon}$-favoring** if at price $p \in \mathbb{R}^m_+$ such that $p_a = t$ and $p_b = 0$ for each $b \neq a$, agent i demands only object a, but when all objects' prices slightly increase, then he depends nothing.

Definition 10. Given $(a,t) \in M \times \mathbb{R}_+$ and $\varepsilon_i > 0$, R_i is **$(a,t)^{\varepsilon_i}$-favoring** if R_i is (a,t)-favoring and

$$V^{R_i}(a;(0,0)) < t + \varepsilon_i \text{ and } V^{R_i}(b;(0,0)) < \varepsilon_i \text{ for each } b \in M \setminus \{a\}.$$

Given $a \in M$ and a preference $R_i \in \mathcal{R}^a$, let $t^*(R_i,a) \equiv \min_{b \in M \setminus \{a\}}\{V^{R_i}(a;(b,0))\}$.

Remark 3. Let $a \in M$ and $R_i \in \mathcal{R}^a$. (i) R_i is (a,t)-favoring preference if and only if $t^*(R_i,a) > t$. (ii) For each $b \in L \setminus \{a\}$ and each $t \in [0, t^*(R_i,a))$, $V^{R_i}(b;(a,t)) < 0$.

Note that even if all object price is zero except object a, agent i with preference $R_i \in \mathcal{R}^a$ demands only object a if object a's price is less than $t^*(R_i,a)$. Given $a \in M$ and $R_i \in \mathcal{R}^a$, $t^*(R_i,a) > 0$ and we call $t^*(R_i,a)$ the **supremum a-favoring payment for R_i**.

Remark 4. Let $(a,t) \in M \times \mathbb{R}_+$ and $R_i \in \mathcal{R}$ be (a,t)-favoring preference. Then, $t^*(R_i,a) > t$.

Remark 5. Let $(a,t) \in M \times \mathbb{R}_+$, $\varepsilon > 0$ and $R_i \in \mathcal{R}$ be $(a,t)^{\varepsilon}$-favoring preference. Then, $0 < t^*(R_i,a) - t < \varepsilon$.

Facts 10 and 11 do not depend on the numbers of agents and objects.

Fact 10 (Lemma 2 in Kazumura et al., 2020B)**.** Let $\mathcal{R}$ be rich. Then, for each $(a,t) \in M \times \mathbb{R}_+$ and each $\varepsilon > 0$, there is $R_i \in \mathcal{R}$ such that it is $(a,t)^{\varepsilon}$-favoring.

Fact 11 (Lemma 3 in Kazumura et al., 2020B)**.** Let f be desirable and satisfy no subsidy. For each $R \in \mathcal{R}^n$, each $i \in N$, and each $t \geq 0$, if there is $j \neq i$ such that R_j is $(x_i(R), t)$-favoring, then $t_i(R) > t$.

Lemma 1. Let f satisfy individual rationality and strategy-proofness. Let $R \in \mathcal{R}^n$ and $z \in Z(R)$. Assume that there is $i \in N$ such that $z_i \, P_i \, f_i(R)$. Then, (i) there are $\varepsilon_i \in (0, V^{R_i}(x_i; f_i(R)) - t_i)$ and $z_i^{\varepsilon_i}$-favoring preference $R_i' \in \mathcal{R}$, and (ii) $x_i(R_i', R_{-i}) \neq x_i$.

Proof. (i) By $z_i \, P_i \, f_i(R)$, $t_i < V^{R_i}(x_i; f_j(R))$. Thus, there is $\varepsilon_i \in (0, V^{R_i}(x_i; f_i(R)) - t_i)$. Moreover, by $z_i \, P_i \, f_i(R)$, $z \in Z(R)$ and Fact 8, $x_i \neq 0$, Thus by Fact 10, there is $z_i^{\varepsilon_i}$-favoring preference $R_i' \in \mathcal{R}$.

(ii) Suppose $x_i(R_i', R_{-i}) = x_i$. Then, by individual rationality, $t_i(R_i') \leq V^{R_i'}(x_i; (0,0))$, and by (i), $V^{R_i'}(x_i; (0,0)) < V^{R_i}(x_i; f_i(R))$. Thus, $t_i(R_i') < V^{R_i}(x_i; f_j(R))$, and hence $f_i(R_i', R_{-i}) \, P_i \, f_i(R)$. This contradicts strategy-proofness. Thus, $x_i(R_i', R_{-i}) \neq x_i$. ■

Lemma 2. Let f be desirable and satisfy no subsidy. Let $R \in \mathcal{R}^n$ and $z \in Z(R)$. Let $N' \subseteq N$ and $R_{N'}' \in \mathcal{R}^{|N'|}$ be such that for each $i \in N'$, $x_i \neq 0$ and R_i' is z_i-favoring. Then,

(i) for each $i \in N'$, there is $j \in N$ such that $x_j(R_{N'}', R_{-N'}) = x_i$ and

(ii) if $i \neq j$, then $t_j(R_{N'}', R_{-N'}) \geq t^*(R_i', x_i) > t_i$.

Proof. (i) Let $i \in N'$. Then $x_i \neq 0$. Suppose that for each $j \in N$, $x_j(R_{N'}', R_{-N'}) \neq x_i$. Since $x_i(R_{N'}', R_{-N'}) \neq x_i$ and R_i' is z_i-favoring, by Remark 2,

$$(x_i, 0) \underset{\text{by } 0 \leq t_i}{R_i'} z_i \underset{\text{by Remark 2}}{P_i'} f_i(R_{N'}', R_{-N'}),$$

which contradicts non-wastefulness.

(ii) Let $i \in N'$ and $j \in N$ be such that $i \neq j$ and $x_j(R_{N'}', R_{-N'}) = x_i$. Since R_i' is z_i-favoring, by Remark 3 (ii), for each $t \in [0, t^*(R_i', x_i))$, R_i' is (x_i, t)-favoring. Thus by Fact 11, $t_j(R) \geq t^*(R_i', x_i)$, and by Remark 4, $t_i < t^*(R_i', x_i)$. Thus, $t_j(R_{N'}', R_{-N'}) \geq t^*(R_i', x_i) > t_i$. ■

Proposition 1. Let $\mathcal{R}$ be rich. Let f be desirable and satisfy no subsidy. For each $R \in \mathcal{R}^n$, each $z \in Z^{min}(R)$ and each $i \in N$, $f_i(R) \, R_i \, z_i$.

Proof. Let $R \in \mathcal{R}^n$, $p = p^{min}(R)$ and $z \in Z^{min}(R)$. Let

$$\underline{p} \equiv \begin{cases} \min\{p_a \in \mathbb{R} : a \in M \text{ and } p_a > 0\} & \text{if } \exists a \in M \text{ such that } p_a > 0 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that there is $i \in N$ such that $z_i \, P_i \, f_i(R)$. Without loss of generality, let $i \equiv 1$.

Claim. For each $k \geq 0$, there are sets $N(k)$ and $N(k+1)$ of distinct agents such that $N(k+1) \supseteq N(k)$, $|N(k)| = k$, $|N(k+1)| = k+1$, say $N(k) = \{1, 2, \ldots, k\}$, $N(k+1) = \{1, 2, \ldots, k+1\}$, and $(\varepsilon_j)_{j \in N(k+1)} \in \mathbb{R}^{k+1}_{++}$, $R^{(k)} \equiv (R'_{N(k)}, R_{-N(k)}) \in \mathcal{R}^n$ and $R^{(k+1)} \equiv (R'_{N(k+1)}, R_{-N(k+1)}) \in \mathcal{R}^n$ such that
(i-a) $z_{k+1} \, P_{k+1} \, f_{k+1}(R^{(k)})$ and
(i-b) $x_{k+1} \neq 0$,
(ii-a) $\varepsilon_1 < \min(\{\underline{p}, V^{R_1}(x_1; f_1(R)) - t_1\} \setminus \{0\})$ and R'_1 is $z_1^{\varepsilon_1}$-favoring,
(ii-b) for each $j \in N(k+1) \setminus \{1\}$, $\varepsilon_j < \min\{t^*(R'_{j-1}, x_{j-1}) - t_{j-1}, V^{R_j}(x_j; f_j(R^{(j-1)})) - t_j\}$ and R'_j is $z_j^{\varepsilon_j}$-favoring,
(ii-c) for each $j \in N(k)$, $\varepsilon_{k+1} < t^*(R'_j, x_j)$,
(iii) $x_{k+1}(R^{(k+1)}) \neq x_{k+1}$ and $z_{k+1} \, P'_{k+1} \, f_{k+1}(R^{(k+1)})$,
(iv) $x_{k+1}(R^{(k+1)}) \notin \{x_l\}_{l \in N(k+1)}$, and
(v) there is $j \in N \setminus N(k+1)$ such that $x_j \in \{x_l\}_{l \in N(k+1)}$ and $z_j \, P_j \, f_j(R^{(k+1)})$.

We prove Claim by induction on k.

Base Case. Let $k = 0$.

(i) By assumption, $z_1 \, P_1 \, f_1(R)$. Thus, (i-a) holds. By (i-a), $z \in Z(R)$ and Fact 8, $x_1 \neq 0$. Hence, (i-b) holds.

(ii) By $z_1 \, P_1 \, f_1(R)$, $t_1 < V^{R_1}(x_1; f_1(R)))$. Thus, there is $\varepsilon_1 > 0$ such that $\varepsilon_1 < \min(\{\underline{p}, V^{R_1}(x_1; f_1(R)) - t_1\} \setminus \{0\})$. By (i-b), $x_1 \neq 0$. Thus, by Fact 10, there is $z_1^{\varepsilon_1}$-favoring preference $R'_1 \in \mathcal{R}$. Hence, (ii-a) holds. By $k = 0$, (ii-b) and (ii-c) hold vacantly.

(iii) By (i-a), (ii-a) and Lemma 1 (ii), $x_1(R^{(1)}) \neq x_1$. Thus since R'_1 is $z_1^{\varepsilon_1}$-favoring, by Remark 2, $z_1 \, P'_1 \, f_1(R^{(1)})$.

(iv) By $k = 0$, (iv) directly follows from (iii).

(v) By $x_1(R^{(1)}) \neq x_1$ and Lemma 2 (i), there is $j \in N \setminus \{1\}$ such that $x_j(R^{(1)}) = x_1$. Without loss of generality, let $j \equiv 2$. We show that $z_2 \, P_2 \, f_2(R^{(1)})$. By Lemma 2 (ii) and $(z, p) \in W(R)$, $t_2(R^{(1)}) > t_1 = p_{x_1}$. Thus,

$$z_2 \underset{\text{by } (z,p) \in W(R)}{R_2} (x_1, p_{x_1}) \underset{\text{by } t_2(R^{(1)}) > p_{x_1}}{P_2} (x_1, t_2(R^{(1)})) \underset{\text{by } x_2(R^{(1)}) = x_1}{=} f_2(R^{(1)}).$$

Thus, $z_2 \, P_2 \, f_2(R^{(1)})$.

Inductive Hypothesis. Let $k \geq 1$. There are sets $N(k-1)$ and $N(k)$ of distinct agents such that $N(k) \supseteq N(k-1)$, $|N(k-1)| = k-1$, $|N(k)| = k$, say $N(k-1) = \{1, 2, \ldots, k-1\}$, $N(k) = \{1, 2, \ldots, k\}$, and $(\varepsilon_j)_{j \in N(k)} \in \mathbb{R}^k_{++}$, $R^{(k-1)} \equiv (R'_{N(k-1)}, R_{-N(k-1)}) \in \mathcal{R}^n$ and $R^{(k)} \equiv (R'_{N(k)}, R_{-N(k)}) \in \mathcal{R}^n$ such that
(i-a-k) $z_k \, P_k \, f_k(R^{(k-1)})$ and
(i-b-k) $x_k \neq 0$,
(ii-a-k) $\varepsilon_1 < \min(\{\underline{p}, V^{R_1}(x_1; f_1(R)) - t_1\} \setminus \{0\})$ and R'_1 is $z_1^{\varepsilon_1}$-favoring,
(ii-b-k) for each $j \in N(k)$, $\varepsilon_j < \min\{t^*(R'_{j-1}, x_{j-1}) - t_{j-1}, V^{R_j}(x_j; f_j(R^{(j-1)})) - t_j\}$ and R'_j is $z_j^{\varepsilon_j}$-favoring,

(ii-c-k) for each $j \in N(k-1)$, $\varepsilon_k < t^*(R'_j, x_j)$,
(iii-k) $x_k(R^{(k)}) \neq x_k$ and $z_k \, P'_k \, f_k(R^{(k)})$,
(iv-k) $x_k(R^{(k)}) \notin \{x_l\}_{l \in N(k)}$, and
(v-k) there is $j \in N \setminus N(k)$ such that $x_j \in \{x_l\}_{l \in N(k)}$ and $z_j \, P_j \, f_j(R^{(k)})$.

Inductive Step.

(i) By (iv-k), there is $j \in N \setminus N(k)$ such that $z_j \, P_j \, f_j(R^{(k)})$. Without loss of generality, let $j = k+1$. Then, (i-a) holds. By (i-a), $z \in Z(R)$ and Fact 8, $x_{k+1} \neq 0$. Thus, (i-b) holds.

(ii) The hypothesis (ii-a-k) is equivalent to (ii-a).

Next we show (ii-b). By (i-a), $t_{k+1} < V^{R_{k+1}}(x_{k+1}; f_{k+1}(R^{(k)}))$. By Remark 5 and the hypothesis (ii-b-k), $t^*(R'_k, x_k) - t_k > 0$. Thus, there is $\varepsilon_{k+1} > 0$ such that $\varepsilon_{k+1} < \min\{t^*(R'_k, x_k) - t_k, V^{R_{k+1}}(x_{k+1}; f_{k+1}(R^{(k)})) - t_{k+1}\}$. By (i-b), $x_{k+1} \neq 0$. Thus, by Fact 10, there is $z_{k+1}^{\varepsilon_{k+1}}$-favoring preference $R'_{k+1} \in \mathcal{R}$. Hence by (ii-b-k), (ii-b) holds.

Note that $\varepsilon_{k+1} < t^*(R'_k, x_k) - t_k \leq t^*(R'_k, x_k)$, and so $\varepsilon_{k+1} < t^*(R'_k, x_k)$. By Remark 5, $\varepsilon_{k+1} < t^*(R'_k, x_k) - t_k < \varepsilon_k$, and so by (ii-c-k), for each $j \in N(k) \setminus \{k\}$, $\varepsilon_{k+1} < \varepsilon_k < t^*(R'_j, x_j)$. Thus, for each $j \in N(k)$, $\varepsilon_{k+1} < t^*(R'_j, x_j)$. Hence (ii-c) holds.

(iii) By (i-a), (ii-a) and Lemma 1 (ii), $x_{k+1}(R^{(k+1)}) \neq x_{k+1}$. Thus since R'_{k+1} is z_{k+1}-favoring, by Remark 2, $z_{k+1} \, P'_{k+1} \, f_{k+1}(R^{(k+1)})$.

(iv) Suppose that $x_{k+1}(R^{(k+1)}) \in \{x_l\}_{l \in N(k+1)}$. By (iii), since $x_{k+1}(R^{(k+1)}) \neq x_{k+1}$, there is $j \in N(k+1) \setminus \{k+1\}$ such that $x_{k+1}(R^{(k+1)}) = x_j$. By $x_{k+1}(R^{(k+1)}) \neq x_{k+1}$ and (ii-b), $V^{R'_{k+1}}(x_{k+1}(R^{(k+1)}); (0,0)) < \varepsilon_{k+1}$. By individual rationality, $t_{k+1}(R^{(k+1)}) \leq V^{R'_{k+1}}(x_{k+1}(R^{(k+1)}); (0,0))$, and so $t_{k+1}(R^{(k+1)}) < \varepsilon_{k+1}$. By (ii-c), $\varepsilon_{k+1} < t^*(R'_j, x_j)$, and so $t_{k+1}(R^{(k+1)}) < t^*(R'_j, x_j)$. However, by Lemma 2 (ii), $t_{k+1}(R^{(k+1)}) \geq t^*(R'_j, x_j)$. This is a contradiction. Thus, $x_{k+1}(R^{(k+1)}) \notin \{x_l\}_{l \in N(k+1)}$.

(v) By (ii-a), (ii-b) and Lemma 2 (i), for each $i \in N(k+1)$, there is $j \in N$ such that $x_j(R^{(k+1)}) = x_i$. By (iv), since $x_{k+1}(R^{(k+1)}) \notin \{x_l\}_{l \in N(k+1)}$, there is $j \in N \setminus N(k+1)$ such that $x_j(R^{(k+1)}) \in \{x_l\}_{l \in N(k+1)}$.

By Lemma 2 (ii) and $(z, p) \in W(R)$, $t_j(R^{(k+1)}) > t_j = p_{x_j(R^{(k+1)})}$. Thus,

$$
\begin{aligned}
z_j \quad & R_j \quad (x_j(R^{(k+1)}), p_{x_j(R^{(k+1)})}) && \text{by } (z,p) \in W(R) \\
& P_j \quad (x_j(R^{(k+1)}), t_j(R^{(k+1)})) && \text{by } t_j(R^{(k+1)}) > p_{x_j(R^{(k+1)})} \\
& = \quad f_j(R^{(k+1)}).
\end{aligned}
$$

Hence $z_j \, P_j \, f_j(R^{(k+1)})$. The proof of Claim is completed.

By the above Claim, we derive a contradiction. For $k = n-1$, by (i-b), for each $i \in N$, $x_i \neq 0$. If $n > m$, it is impossible. Thus, we assume that $n \leq m$.

By (ii-a), (ii-b) and Lemma 2 (i), for each $i \in N$, there is $j \in N$ such that $x_j(R^{(n)}) = x_i$. However, by Claim (iv), since $x_n(R^{(n)}) \notin \{x_i\}_{i \in N}$, this is also impossible. ■

Proof of Theorem 1. By Proposition 1 and Fact 9, a minimum price Walrasian rule is ex-post revenue optimal in the class of desirable rules satisfying no subsidy. The uniqueness of the ex-post revenue optimal rule directly follows from the proof of Theorem 1 in Kazumura et al. (2020B). ■

5.2 Proof of Theorem 2

Throughout this subsection, we assume that $\mathcal{R} \supseteq \mathcal{R}^{++}$ and $l \leq 0$ is the associated lower bound of no bankruptcy.

Given $a \in M$ and $\delta > 0$, define $\mathcal{R}^a(\delta) \equiv \{R_i \in \mathcal{R} : \forall b \in M \setminus \{a\},\ (a, 0)\, P_i\, (b, -\delta)\}$. Note that for each $a \in M$ and each $\delta > 0$, there is $R_i \in \mathcal{R}^{++}$ such that $R_i \in \mathcal{R}^a(\delta)$. Thus, for each $a \in M$ and each $\delta > 0$, $\mathcal{R}^a(\delta) \neq \emptyset$.

Definition 11. Given $(a, t) \in M \times \mathbb{R}_+$ and $\delta > 0$, $R'_i \in \mathcal{R}^a(\delta)$ is **(a, t)-favoring with subsidy** δ if for each $b \in M \setminus \{a\}$, $V^{R'_i}(b; (a, t)) < -\delta$.

Note that for each $R \in \mathcal{R}^n$, each $z \in Z(R)$ and each $\delta > 0$, if $x_i \neq 0$, by $\mathcal{R} \supseteq \mathcal{R}^{++}$, there is a z_i-favoring preference R'_i with subsidy δ.

Given $a \in M$, $\delta > 0$ and $R_i \in \mathcal{R}^a(\delta)$, let $t^*(R_i, a, \delta) \equiv \min_{b \in M \setminus \{a\}} \{V^{R_i}(a; (b, -\delta))\}$.

Remark 6. Let $a \in M$, $\delta > 0$ and $R_i \in \mathcal{R}^a(\delta)$. (i) R_i is (a, t)-favoring with subsidy δ if and only if $t^*(R_i, a, \delta) > t$. (ii) For each $b \in L \setminus \{a\}$ and each $t \in [0, t^*(R_i, a, \delta))$, $V^{R_i}(b; (a, t)) < -\delta$.

For each $a \in M$, each $\delta > 0$ and each $R_i \in \mathcal{R}^a(\delta)$, $t^*(R_i, a, \delta) > 0$.

Given $(a, t) \in M \times \mathbb{R}_+$, $\varepsilon > 0$ and $\delta > 0$, let $\mathcal{R}((a, t), \varepsilon, \delta)$ be the set of $(a, t)^{\varepsilon}$-favoring preferences in $\mathcal{R}^a(\delta)$.

Fact 12. For each $(a, t) \in M \times \mathbb{R}_+$, each $\varepsilon > 0$ and each $\delta > 0$, $\mathcal{R}^{++} \cap \mathcal{R}((a, t), \varepsilon, \delta) \neq \emptyset$.

Remark 7. Let $(a, t) \in M \times \mathbb{R}_+$, $\varepsilon > 0$ and $R'_i \in \mathcal{R}^{++} \cap \mathcal{R}((a, t), \varepsilon, \delta)$. Then, (i) $t^*(R_i, a, \delta) > t$ and (ii) $0 < t^*(R_i, a, \delta) - t < \varepsilon$.

Given $R \in \mathcal{R}^n$ and $l \leq 0$, let $\delta(R, l) \equiv n(\max_{k \in N} \max_{b \in M} V^{R_k}(b; (0, 0))) - l$.

Fact 13 (Lemma 4 in Kazumura et al., 2020B)**.** Let f be desirable and satisfy no bankruptcy. For each $R \in \mathcal{R}^n$, each $i \in N$, and each $(a, t) \in M \times \mathbb{R}_+$, if there is $j \neq i$ such that for each $a \in L \setminus \{x_i(R)\}$, $V^{R_j}(a; f_i(R)) < -\delta(R, l)$, then $t_i(R) > t$.

Lemma 3. Let f satisfy individual rationality and no bankruptcy. For each $R \in \mathcal{R}^n$ and $i \in N$, $t_i(R) \geq -\delta(R, l)$.

Proof. Suppose that there is $i \in N$ such that $t_i(R) < -\delta(R,l)$. Note that by individual rationality, for each $j \in N \setminus \{i\}$, $t_j(R) \leq \max_{b \in M} V^{R_j}(b;(0,0))$. Thus,

$$\begin{aligned}\sum_{k\in N} t_k(R) &\leq \sum_{j\in N\setminus\{i\}} \max_{b\in M} V^{R_j}(b;(0,0)) - \delta(R,l)\\ &\leq (n-1)\max_{k\in N}\max_{b\in M} V^{R_k}(b;(0,0)) - \delta(R,l)\\ &= (n-1)\max_{k\in N}\max_{b\in M} V^{R_k}(b;(0,0)) - n(\max_{k\in N}\max_{b\in M} V^{R_k}(b;(0,0))) + l\\ &< l\end{aligned}$$

which contradicts no bankruptcy. ■

Remark 8. Let f satisfy individual rationality and no bankruptcy. Let $R \in \mathcal{R}^n$, $i \in N$, and $z \in Z(R)$ be such that $x_i \neq 0$ and $R_i \in \mathcal{R}^{++} \cap \mathcal{R}(z_i, \varepsilon, \delta(R,l))$. If $x_i(R) \neq x_i$, then $z_i \, P_i \, f_i(R)$.

Proof. Let $x_i(R) \neq x_i$. Suppose $f_i(R) \, R_i \, z_i$. Then, $t_i(R) \leq V^{R_i}(x_i(R); z_i)$. Since $R_i \in \mathcal{R}(z_i, \varepsilon, \delta(R,l))$, $V^{R_i}(x_i(R); z_i) < -\delta(R,l)$. Thus, $t_i(R) < -\delta(R,l)$, which contradicts Lemma 3. ■

Lemma 4. Let f satisfy individual rationality and strategy-proofness. Let $R \in \mathcal{R}^n$ and $z \in Z(R)$. Assume that there is $i \in N$ such that $z_i \, P_i \, f_i(R)$. Then, (i) there are $\varepsilon_i > 0$ and $R'_i \in \mathcal{R}$ such that $\varepsilon_i < V^{R_i}(x_i; f_i(R)) - t_i$ and $R'_i \in \mathcal{R}^{++} \cap \mathcal{R}(z_i, \varepsilon_i, \delta(R,l))$. (ii) $x_i(R'_i, R_{-i}) \neq x_i$.

Proof. (i) By $z_i \, P_i \, f_i(R)$, $t_i < V^{R_i}(x_i; f_j(R))$. Thus, there is $\varepsilon_i \in (0, V^{R_i}(x_i; f_i(R)) - t_i)$. Moreover, by $z_i \, P_i \, f_i(R)$ $z \in Z(R)$ and Fact 8, $x_i \neq 0$. Thus, by Fact 12, there is $R'_i \in \mathcal{R}^{++} \cap \mathcal{R}(z_i, \varepsilon, \delta(R,l))$

(ii) Directly follows from the proof of Lemma 1 (ii). ■

Lemma 5. Let f be desirable and satisfy no bankruptcy. Let $R \in \mathcal{R}^n$ and $z \in Z(R)$. Let $N' \subseteq N$ and $R'_{N'} \in \mathcal{R}^{|N'|}$ be such that for each $i \in N'$, $x_i \neq 0$ and $R'_i \in \mathcal{R}^{++} \cap \mathcal{R}(z_i, \varepsilon, \delta(R,l))$. Then,

(i) for each $i \in N'$, there is $j \in N$ such that $x_j(R'_{N'}, R_{-N'}) = x_i$ and

(ii) if $i \neq j$, then $t_j(R'_{N'}, R_{-N'}) \geq t^*(R'_i, x_i, \delta(R,l)) > t_i$

Proof. (i) Let $i \in N'$. Suppose that for each $j \in N$, $x_j(R'_{N'}, R_{-N'}) \neq x_i$. Since $x_i(R'_{N'}, R_{-N'}) \neq x_i$ and R'_i is z_i-favoring, by Remark 8,

$$(x_i, 0) \underset{\text{by } 0 \leq t_i}{R'_i} z_i \underset{\text{by Remark 8}}{P'_i} f_i(R'_{N'}, R_{-N'}),$$

which contradicts non-wastefulness.

(ii) Let $i \in N'$ and $j \in N$ be such that $i \neq j$ and $x_j(R'_{N'}, R_{-N'}) = x_i$. Since R'_i is z_i-favoring, by Remark 6 (ii), for each $t \in [0, t^*(R'_i, x_i))$, R'_i is (x_i, t)-favoring. Thus by Fact 13, $t_j(R) \geq t^*(R'_i, x_i, \delta(R,l))$. By Remark 7 (i), $t_i < t^*(R'_i, x_i, \delta(R,l))$. Thus, $t_j(R'_{N'}, R_{-N'}) \geq t^*(R'_i, x_i, \delta(R,l)) > t_i$. ■

Proposition 2. Let $\mathcal{R} \supseteq \mathcal{R}^{++}$. Let f be desirable and satisfy no bankruptcy. For each $R \in \mathcal{R}^n$, each $z \in Z^{min}(R)$ and each $i \in N$, $f_i(R)\, R_i\, z_i$.

Proof. Let $R \in \mathcal{R}^n$, $p = p^{min}(R)$ and $z \in Z^{min}(R)$. Let

$$\underline{p} \equiv \begin{cases} \min\{p_a \in \mathbb{R} : a \in M \text{ and } p_a > 0\} & \text{if } \exists a \in M \text{ such that } p_a > 0 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that there is $i \in N$ such that $z_i\, P_i\, f_i(R)$. Without loss of generality, let $i \equiv 1$.

Claim. For each $k \geq 0$, there are sets $N(k)$ and $N(k+1)$ of distinct agents such that $N(k+1) \supseteq N(k)$, $|N(k)| = k$, $|N(k+1)| = k+1$, say $N(k) = \{1, 2, \ldots, k\}$, $N(k+1) = \{1, 2, \ldots, k+1\}$, and $(\varepsilon_j)_{j \in N(k+1)} \in \mathbb{R}^{k+1}_{++}$, $R^{(k)} \equiv (R'_{N(k)}, R_{-N(k)}) \in \mathcal{R}^n$ and $R^{(k+1)} \equiv (R'_{N(k+1)}, R_{-N(k+1)}) \in \mathcal{R}^n$ such that
(i-a) $z_{k+1}\, P_{k+1}\, f_{k+1}(R^{(k)})$ and
(i-b) $x_{k+1} \neq 0$,
(ii-a) $\varepsilon_1 < \min(\{\underline{p}, t^*(R'_1, x_1, \delta(R, l)) - t_1\} \setminus \{0\})$ and $R'_1 \in \mathcal{R}^{++} \cap \mathcal{R}(z_1, \varepsilon_1, \delta(R, l))$,
(ii-b) for each $j \in N(k+1)\setminus\{1\}$, $\varepsilon_j < \min\{t^*(R'_{j-1}, x_{j-1}, \delta(R, l)) - t_{j-1}, V^{R_j}(x_j; f_j(R^{(j-1)})) - t_j\}$ and $R'_j \in \mathcal{R}^{++} \cap \mathcal{R}(z_j, \varepsilon_j, \delta(R, l))$,
(ii-c) for each $j \in N(k)$, $\varepsilon_{k+1} < t^*(R'_j, x_j, \delta(R, l))$,
(iii) $x_{k+1}(R^{(k+1)}) \neq x_{k+1}$ and $z_{k+1}\, P'_{k+1}\, f_{k+1}(R^{(k+1)})$,
(iv) $x_{k+1}(R^{(k+1)}) \notin \{x_l\}_{l \in N(k+1)}$, and
(v) there is $j \in N \setminus N(k+1)$ such that $x_j \in \{x_l\}_{l \in N(k+1)}$ and $z_j\, P_j\, f_j(R^{(k+1)})$

To replace Lemma 1 with Lemma 4, Lemma 2 with Lemma 5, Remark 5 with Remark 7, and Fact 10 with Fact 12, by the same logic in Proposition 1, we can prove the above claim. ■

Proof of Theorem 2. The same logic in Theorem 1. ■

6 Concluding remarks

By extending the results of Kazumura et al. (2020B), we showed that for an arbitrary numbers of agents and objects, the minimum price Walrasian rule is the unique ex-post revenue maximizing rule on rich domains among desirable rules, and that no subsidy in this result can be replaced by no bankruptcy on the positive income effect domain. There is the literature on auction with non-quasi-linear preferences. We conclude by referring them and mentioning relating research in future.

Most of the literature on auction with non-quasi-linear preferences focus on rather efficiency than revenue maximization. Saitoh and Serizawa (2008) and Sakai (2008) show that in the cases of homogeneous objects and unit-demand preferences, the generalized

Vickrey rule is the only rule satisfying strategy-proofness, efficiency, individual rationality, and no subsidy. Morimoto and Serizawa (2015) extend these results to the case of heterogeneous objects by maintaining unit-demand preferences, and show that the minimum price Walrasian rule is the only rule satisfying the same four properties on classical domain. These works assume that the number of agents is greater than objects. Thus, it is an open question whether these results hold for an arbitrary numbers of agents and objects.

Zhou and Serizawa (2018) also maintain unit-demand preferences, but study the special class of preferences, *the common-tiered domains*. It says that objects are partitioned into several tiers, and if objects are equally priced, agents prefer an object in the higher tier to one in the lower. They show that the minimum price Walrasian rule is the only rule satisfying same four properties on the common-tiered domains. It is an open question whether their results also hold on the common-tiered domains for an arbitrary numbers of agents and objects.

There is also the literature on auction with non-quasi-linear preferences admitting multi-demand in various settings. Kazumura and Serizawa (2016) study classes of preferences that include unit-demand preferences and additionally includes at least one multi-demand preference, and show that no rule satisfies the four properties on such a domain. Malik and Mishra (2019) study the special classes of preferences, "dichotomous" domains. A preference is *dichotomous* if there is a set of objects such that the valuations of its supersets are constant and the valuations of other sets are zero. A *dichotomous domain* includes all such dichotomous preferences for a given set of objects. They show that no rule satisfies the four properties on a dichotomous domain, but that the generalized Vickrey rule is the only rule satisfying the four properties on a class of dichotomous preferences exhibiting positive income effects.

Baisa (2020) assumes that objects are homogeneous and shows that on the class of preferences exhibiting decreasing marginal valuations, positive income effect, and single-crossing property, if the preferences are parametrized by one dimensional types, there is a rule satisfying the above four properties, but that if types are multi-dimensional, no rule satisfies these properties. Shinozaki et al. (2020) also assume the homogeneity of objects, and show that on the class of preferences includes sufficiently various preferences exhibiting non-decreasing marginal valuations (*minimal richness*), the generalized Vickrey rule is the only rule satisfying the four properties, but that no rule satisfies these properties on the class of preferences that additionally includes at least one preference exhibiting decreasing marginal valuations.

These different results in various settings of multi-demand suggest that analyzing revenue maximization rules in multi-demand settings would be technically challenging. However, such research is important in practical applications. Recently, Kazumura et al. (2020A) develop methods to analyze strategy-proof rules in general settings including multi-demand cases. We believe that such methods would be useful to analyze revenue maximization rules in multi-demand settings.

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