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MULTI-OBJECT AUCTION DESIGN BEYOND QUASI-LINEARITY: LEADING EXAMPLES

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Multi-object Auction Design Beyond Quasi-linearity: Leading Examples*

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Abstract

In multi-object auction models with unitary demand agents, if agents' utility functions satisfy quasi-linearity, three auction formats, sealed-bid auction, exact ascending auction, and approximate ascending auction, are known to identify the minimum price equilibrium (MPE), and exhibit elegant efficiency and incentive-compatibility. These auctions are conjured to preserve their properties beyond quasi-linearity. Nevertheless, we exemplify that with general utility functions, these auctions fail to identify the MPEs and are substantially inefficient and manipulatable. The implications of our negative results for multi-object auction models with agents with multi-unit demand, and matching with contracts models are also discussed.

Keywords: Multi-object auction, minimum price equilibrium, examples, quasi-linear utility functions, general utility functions

JEL Classification: C78, D44, D74

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1 Introduction

Governments in many countries conduct auctions to allocate social resources. The most important and innovative examples are spectrum license auctions in many countries including most of the OECD. They auction off several licenses simultaneously and often generate enormous revenue. For instance, in the 2000 British 3G spectrum license auctions, the revenue from selling five licenses amounted to 2.5% of the UK's GNP (Binmore and Klemperer, 2002). Those auctions also have considerable impacts on industry development, and so it is vitally important to design governments' auctions to achieve their goals. However, such large-scale auctions cause violations of quasi-linearity and small (or continuous) price increment. These are the basic assumptions of auction theory on which auction designs are based. We analyze the effects of these violations on the performance of well-known auctions and the implications for auction design.

Efficiency is a central goal of government auctions. It requires that objects be given to those who value them most. However, in many cases, information such as preferences over objects is only privately known by agents. Thus, to attain efficiency, it is indispensable for auction mechanisms to directly or indirectly extract true information from agents. *Strategy-proofness* is such a requirement for direct mechanisms, and it gives agents no incentive to manipulate information in the sense that revealing true information is a dominant strategy for each agent.

We focus on the setting in which each agent has unitary demand. In this setting, there is a *minimum price equilibrium* (MPE) whose price (vector) is a unique and coordinate-wise minimum among all equilibrium prices. The *MPE mechanism*, the direct mechanism choosing an MPE allocation for each profile of agents' preferences, is the only mechanism that satisfies efficiency, strategy-proofness and other essential requirements such as individual rationality and no subsidy (Holmstrom, 1979; Morimoto and Serizawa, 2015). Moreover, the MPE mechanism maximizes revenue among mechanisms satisfying strategy-proofness, individual rationality, no subsidy and equal treatment of equals for each preference profile (Kazumura et al., 2020). Thus, if an auction mechanism duplicates the outcome of the MPE mechanism, it achieves important goals for governments.

In quasi-linear environments where agents' preferences are quasi-linear, there are three representative formats of auctions that find the MPEs. These auctions are fundamental to the auction theory literature and provide the basic ideas of many auction designs in an even more general environment, e.g., agents have multi-unit demand (Gul and Stacchetti, 2000; Ausubel, 2006; Sun and Yang, 2009). The first one is a sealed-bid auction, called the "Vickrey mechanism." The MPE coincides with the Vickrey allocation under quasi-linearity (Leonard, 1983). Demange et al. (1986) provide another two auction formats, exact ascending

(EA) auction and approximate ascending (AA) auction. For the EA auction, they assume that the price increment coincides with the unit of agents' valuations, for example, agents' valuations and price increments are both integers and show that under such an assumption, the EA auction generates an MPE. For the AA auction, originating from the salary adjustment process in Crawford and Knoer (1981), they assume that the increment is small and find that the AA auction generates a price that deviates the MPE price coordinate-wise within some bounds. In other words, the AA auction approximately duplicates the MPE mechanism.

However, the assumption of quasi-linearity is dubious for the government auctions cited above. A preference satisfies quasi-linearity with respect to a good if the expenditure on the good is so much smaller than an agent's income or wealth that its income effect is negligible. However, in many government auctions, bidders pay enormous amounts and face financial constraints. These factors make their preferences far from quasi-linear.

The assumption that the increment is small and coincides with the valuation unit is also unreasonable. Agent valuations are primitive data independent of auctions and can be any real numbers or any amounts measured by monetary units. In contrast, price increments are set in advance, as a part of auction design. The speed of auctions crucially depends on the price increments, and in practice, they are set large enough that the auction concludes within a reasonable time. Typically, the price increments are much larger than the unit of agents' valuations.

Many auction mechanisms in the literature and practice are based on the **implicit conjecture** that even without quasi-linearity and the coincidence of increment and valuation units, the above three auction formats duplicate or approximate the MPE mechanism, and preserve efficiency and strategy-proofness. *However, we demonstrate that such a conjecture is far from validated.*

First, to analyze the Vickrey mechanism, we generalize agents' valuations over objects for non-quasi-linear environments. We define the *generalized valuation* of an object as an agent's willingness to pay for the object from the point where she obtains no object and pays nothing. Then, we define the *generalized Vickrey mechanism* by replacing the valuations in the formula of the Vickrey allocation with the generalized valuations. We show that *in non-quasi-linear environments, the generalized Vickrey mechanism fails to find the MPE, is far from efficient, and provides agents with strong incentives to manipulate.*

Second, we analyze whether an EA auction will work without the coincidence of increment and valuations' unit. We exemplify that *even in quasi-linear environments, for any price increment larger than the units of agents' valuations, the EA auction overshoots the MPE prices by an arbitrarily large distance, generates allocations that are far from efficient, and provides agents with strong incentives to manipulate.* The auction mechanisms targeting MPEs in quasi-linear environments, as proposed by, e.g., Mishra and Parkes (2009), Andersson and Erlanson (2013), and Liu and Bagh (2019), are all based on the coincidence of increment and valuation units. Our negative results extend to their auction mechanisms by constructing similar examples.

Third, we analyze whether an AA auction will work without quasi-linearity and the coincidence of increment and valuation units. We exemplify that *in non-quasi-linear environments, the AA auction overshoots and undershoots the MPE prices by an arbitrarily large distance.* Thus, in contrast to the quasi-linear environments, the AA auction fails to approximate the MPE mechanism. We also exemplify that *the AA auction generates allocations that are far from efficient and provides agents with strong incentives to manipulate.*

In addition, we also discuss the implications of our negative results for multi-object auction models with agents with multi-unit demand and matching with contracts models at the end of this paper.

The remainder is organized as follows: Section 2 defines the model and MPEs. Section 3 defines conditions for inefficiency, manipulability, efficiency and strategy-proofness. Section 4 reviews the auctions for MPEs in the quasi-linear environment. Section 5 exemplifies that without quasi-linearity and the coincidence of increment and valuation units, the auctions shown in Section 4 fail to identify the MPEs and are substantially inefficient and manipulatable. Section 6 concludes the paper and discusses the implications of the results obtained in Section 5.

2 The model and minimum price equilibrium

There is a finite set of agents N and a finite set of heterogeneous objects M. Not receiving an object is called receiving the dummy, denoted by 0. Let $L \equiv M \cup \{0\}$. Each agent either receives an object or the dummy.

Agents have preferences on the consumption space $L \times \mathbb{R}$. We abuse language and identify a preference of agent i with her utility representation u_i.

Definition 1: A utility function $u_i : L \times \mathbb{R} \to \mathbb{R}$ is *general* if:
(i) For each $l \in L$, $u_i(l, \cdot)$ is continuous and strictly decreasing in $\mathbb{R}$.
(ii) For each pair $l, l' \in L$, each $t \in \mathbb{R}$, there is $t' \in \mathbb{R}$ such that $u_i(l, t) = u_i(l', t')$.

Let $\mathcal{U}$ be the set of *general* utility functions and $u \equiv (u_i)_{i \in N} \in \mathcal{U}^n$ be a utility profile.

Definition 2: A utility function $u_i \in \mathcal{U}$ is *quasi-linear* if there is a valuation function $v_i : L \to \mathbb{R}$ such that for each $(l, p_l) \in L \times \mathbb{R}$, $u_i(l, p_l) = v_i(l) - p_l$.

Each quasi-linear utility function u_i can be represented by a valuation function v_i. We assume, w.o.l.g., that for each $i \in N$, $v_i(0) = 0$. Let $\mathcal{U}^{QL}$ be the set of quasi-linear utility functions. Notice that $\mathcal{U}^{QL} \subsetneq \mathcal{U}$.

For each agent $i \in N$, let $x_i \in L$ be her assigned object. An assignment $x \equiv (x_i)_{i \in N} \in L^N$ is a list of individually assigned objects such that except for the dummy, no two agents obtain the same object, i.e., if $x_i \neq 0$ and $i \neq j$, $x_i \neq x_j$. Let X be the set of assignments.

Agent $i's$ demand set at price $p \in \mathbb{R}^L_+$ is defined as $D_i(p) \equiv \{l \in L : u_i(l, p_l) \geq u_i(l', p_{l'}), \forall l' \in L\}$. We assume, w.o.l.g., that the price of the dummy is zero and the reserve prices of all the objects are zero.

Definition 3: A pair $(x, p) \in X \times \mathbb{R}^L_+$ is an *equilibrium* if:
(i) For each $i \in N$, $x_i \in D_i(p)$.
(ii) For each $l \in M$, if $p_l > 0$, there is $i \in N$ such that $x_i = l$.

When agents have general utility functions, there is an equilibrium, and the set of equilibrium prices is a complete lattice (Demange and Gale, 1985). Thus, there is a **minimum price equilibrium** (**MPE**) whose price is *unique* and *coordinate-wise minimum* among all equilibrium prices. For each utility profile $u \in \mathcal{U}^n$, let $p^{\min}(u)$ be the MPE price. Although the MPE price is unique, the corresponding assignment may not be unique since indifference is allowed. However, for each agent, her assignment at any MPE is welfare-equivalent: given a utility profile u, if $(x, p^{\min}(u))$ and $(x', p^{\min}(u))$ are two MPEs, then $u_i(x_i, p^{\min}_{x_i}) = u_i(x'_i, p^{\min}_{x'_i})$.

3 Efficiency and incentive-compatibility

This section studies the properties of (direct) mechanisms, which are shown to be cornerstones of practical auction design.

An allocation $z \equiv (x_i, t_i)_{i \in N} \in X \times \mathbb{R}^N$ is a list of individually assigned objects, paired with the corresponding payments. Let Z be the set of allocations. A *mechanism* f is defined as a function from $\mathcal{U}^n$ to Z that maps to each utility profile u an allocation z. For each agent $i \in N$, let $x_i(u)$ be the object assigned and $t_i(u)$ be the associated payment specified by mechanism f, and let $f_i(u) = (x_i(u), t_i(u))$.

First, we introduce efficiency.

Efficiency: An allocation $z \in Z$ is *efficient* for $u \in \mathcal{U}^n$ if there is no $z' \in Z$ such that.
(i) for each $i \in N$, $u_i(z'_i) \geq u_i(z_i)$ with at least one strict inequality,
(ii)$\sum_{i \in N} t_i \leq \sum_{i \in N} t'_i$.
A mechanism f is *efficient* if for each $u \in \mathcal{U}^n$, $f(u)$ is efficient for u.

A mechanism f on $(\mathcal{U}^{QL})^n$ is efficient if and only if for each $u \in (\mathcal{U}^{QL})^n$, $x(u) \in \arg\max_{x \in X} \sum_{i \in N} v_i(x_i)$.[1] However, this result typically does not hold for general utility functions (Zhou and Serizawa, 2018).

[1]See footnote 29 in Zhou and Serizawa (2018) for a complete proof of this statement.

We introduce variants of efficiency. It allows the allocation to be inefficient but per capita inefficiency to be bounded.

r-**efficiency**:[2] Given $r \in \mathbb{R}_+$, an allocation $z \in Z$ is $r-$*inefficient* for $u \in \mathcal{U}^n$ if there is $z' \in Z$ such that
(i) for each $i \in N$, $u_i(z_i') > u_i(z_i)$, and
(ii) $\sum_{i \in N} t_i + r \cdot |N| \leq \sum_{i \in N} t_i'$,
and z are $r-$*efficient* if no such z' exists. A mechanism f is $r-$*inefficient* if there is $u \in \mathcal{U}^n$ such that $f(u)$ is $r-$inefficient for u, and f is $r-$*efficient* if for each $u \in \mathcal{U}^n$, $f(u)$ is $r-$efficient for u.

In the case of $r = 0$, $r-$efficiency coincides with efficiency. For a small $r > 0$, $r-$efficiency is "approximate efficiency." However, given a large $r > 0$, $r-$efficiency is a rather weak requirement.

Next, we define the incentive properties of mechanisms. "Manipulability" states that an agent benefits from misrepresenting her utility function, i.e., a mechanism f on $\mathcal{U}^n$ is *manipulable* if there are a utility function profile $u \in \mathcal{U}^n$, an agent $i \in N$ and $u_i' \in \mathcal{U}$ such that $u_i(f_i(u_i', u_{-i})) > u_i(f_i(u))$. *Strategy-proofness* states that a mechanism is not manipulable.

Strategy-proofness: For each $u_i \in \mathcal{U}$, each $i \in N$, and each $u_i' \in \mathcal{U}$, $u_i(f_i(u)) \geq u_i(f_i(u_i', u_{-i}))$.

We also introduce variants of manipulability and strategy-proofness. Given $r \in \mathbb{R}_+$, "r-manipulability" states that an agent benefits more than r from misrepresenting her utility function, in terms of the payment, paired with the object under truth telling.

r-**manipulability:** Given $r \in \mathbb{R}_+$, a mechanism f on $\mathcal{U}^n$ is r-*manipulable* if there are a utility function profile $u \in \mathcal{U}^n$, an agent $i \in N$ and $u_i' \in \mathcal{U}$ such that $u_i(f_i(u_i', u_{-i})) > u_i(x_i(u), p_i(u) - r)$.
$r-$**strategy-proofness**: Given $r \in \mathbb{R}_+$, a mechanism f on $\mathcal{U}^n$ is $r-$*strategy-proof* if f on $\mathcal{U}^n$ is not r-*manipulable.*

If $r = 0$, $r-$strategy-proofness is exactly strategy-proofness. For a small $r > 0$, $r-$strategy-proofness is "approximate strategy-proofness," that is, each agent has only a small incentive to manipulate. However, given a large $r > 0$, $r-$manipulability states that an agent may have a strong incentive to manipulate and thus implies that the mechanism cannot be implemented.

An *MPE mechanism* is a function that maps to each utility profile an MPE. When agents have quasi-linear utility functions, the MPE mechanism is equivalent to the Vickrey mechanism, which associates with each quasi-linear utility profile

[2] Our definition is in the spirit of that of the approximate core in classical general equilibrium theory, when the size of the coalition is the set of all agents; see, e.g., Hildenbrand et al. (1973).

a Vickrey assignment and payment (Leonard, 1983). The MPE (Vickrey) mechanism is the unique mechanism satisfying *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy* (Holmstrom, 1979). Such a characterization of MPE mechanism holds for the general utility functions (Morimoto and Serizawa, 2015).

4 Auctions for MPEs with quasi-linear utility functions

In this section, we introduce three well-known auction formats designed to implement MPEs under various assumptions in the quasi-linear environment.

4.1 The sealed-bid auction

In the quasi-linear environment, the MPE allocation coincides with the Vickrey allocation (Leonard 1983). Thus, the MPE can be characterized by a closed-form expression. Formally, we have the following results.

Fact 1: Let $u \in (\mathcal{U}^{QL})^n$ and $(x^*, p^{\min}(u))$ be an MPE. Then,
(i) $x^* \in \underset{x \in X}{\arg\max} \sum_{i \in N} v_i(x_i)$.
(ii) If $l \in M$ is unassigned, $p_l^{\min}(u) = 0$, and if $l \in M$ is assigned to agent i, $p_l^{\min}(u) = \max_{x \in X} \sum_{j \in N \setminus \{i\}} v_j(x_j) - \sum_{j \in N \setminus \{i\}} v_j(x_j^*)$.

Fact 1 indicates that a sealed-bid auction can be used to implement an MPE (Leonard, 1983). In the auction, each bidder reports all her valuations of the objects to the auctioneer. Then, the auctioneer specifies the assignment and price to bidders in the formula presented in Fact 1.

The mechanism induced by this sealed-bid auction is exactly the MPE mechanism, and so it is efficient and strategy-proof.

In Section 5.1, we show that the sealed-bid auction or its generalized variant fails to find the MPE and to be efficient and strategy-proof.

4.2 The exact ascending auction

Demange et al. (1986) provide an exact ascending auction that finds an MPE price. The idea of this auction is to iteratively eliminate the set of objects that are "overdemanded" by raising their prices.

Definition 4: A nonempty set of objects $M' \subseteq M$ is *overdemanded at* p if

$$|\{i \in N : D_i(p) \subseteq M'\}| > |M'| .$$

A set of objects M' is *minimally overdemanded* at p if M' is overdemanded at p and no proper subset of M' is overdemanded at p.

Now, we define the exact ascending auction.

The exact ascending (EA) auction: Let $d > 0$ be the increment.

Starting with reserve prices, each agent reports her demand set at the current price. If there is a set of objects that are minimally overdemanded, then the auctioneer raises the prices of those objects by d. Otherwise, the auctioneer stops the auction at the current price.

Demange et al. (1986) assume that agents' valuations are discrete and multiples of the increment, i.e., *the increment coincides with unit of valuation unit.* (Hereafter, **coincidence assumption**) In other words, they focus on the class of utility functions, $\mathcal{U}^{QLd} \equiv \{u_i \in \mathcal{U}^{QL} : v_i(L) \subseteq d \cdot \mathbb{N}\}$. They obtain the following result for utility profiles in this class.

Fact 2: Let $d > 0$ and $u \in (\mathcal{U}^{QLd})^n$. Then, the EA auction with increment d finds an MPE price for u in a finite number of steps.

The **EA mechanism with increment** d is a function that maps to each utility profile the outcome of the EA auction with increment d for that profile.

Fact 3: Let $d > 0$. The EA mechanism with increment d on $(\mathcal{U}^{QLd})^n$ coincides with the MPE mechanisms on $(\mathcal{U}^{QLd})^n$, and so it satisfies efficiency and strategy-proofness.

Compared with sealed-bid auctions, one merit of EA auctions is that the price formation process is transparent to all bidders, i.e., all bidders can see how prices change in the auction and where the prices stop changing. Researchers design different formats of auctions to improve the speed of finding an MPE in the quasi-linear environment, see, e.g., Mishra and Parkes (2009) and Andersson and Erlanson (2013). They explicitly assume that agents' valuations and increment\decrement are both integers.

Note that the coincidence assumption is crucial to find the MPE price in above-mentioned works. We discuss these points in Section 5.2.

4.3 The approximate ascending auction

Demange et al. (1986) propose another type of auction, the approximate ascending auction, that finds an "approximate" equilibrium price coordinate-wise close to the MPE price. This auction is introduced by Crawford and Knoer (1981).

The approximate ascending (AA) auction: Let $d > 0$ be the increment. Initially, all the agents are uncommitted and stand in a queue. Agents are called one by one to bid. When agent i is called, she has the following three options.

Option 1 is to bid on an unassigned object l. This option commits agent i to object l at its reserve price.

Option 2 is to bid on an object l that is tentatively assigned to some other agent j at price p_l. This option increases the price of object l by d, commits agent i to

object l at price $p_l + d$ and drives agent j back into the queue of uncommitted agents.
Option 3 is to drop out by bidding on the dummy.

The auction terminates when all uncommitted agents drop out.

Note that *the bidding queue* can be formed in many ways. In essence, it contains two parts. The first part is the initial order when agents stand in a queue. The permutations of agents make $n!$ variants. The second part concerns treating agents when driven back into the queue of uncommitted agents. A natural rule is to place an agent at the end of the queue when driven back. Other rules include placing the driven back agent first in the queue, second in the queue, and so forth. Moreover, different agents may be treated differently. Thus, the number of variants of queues is much more than $n!$. The outcomes of the AA auction also depend on how we form the bidding queue.

Additionally, note that when a bidder is called, she is supposed to bid on an object from her demand set at the price she faces. Her demand set may contain several objects. In that case, she needs to choose one object among them on which to bid. The outcomes of the AA auction depend on such choices of bidders even if the utility function profile is fixed.

It is known that when prices vary discretely and agents' valuations of objects are arbitrary real numbers, an equilibrium may not exist. Moreover, the outcome of the AA auction may not be an equilibrium even if it exists. Thus, we introduce a concept of approximate equilibrium, the "$\varepsilon-$equilibrium."

Given $\varepsilon \geq 0$, agent i's $\varepsilon-$*demand set* at p is given by:

$$D_i^{\varepsilon}(p) \equiv \{l \in L : u_i(l, p_l) \geq u_i(0, 0), \text{ and } \forall l' \in M, u_i(l, p_l) \geq u_i(l', p_{l'} + \varepsilon)\}.$$

An object in $D_i^{\varepsilon}(p)$ approximately maximizes agent i's welfare at price p for small ε, and when $\varepsilon = 0$, $D_i^{\varepsilon}(p) = D_i(p)$.

Definition 5: A pair $(x, p) \in X \times \varepsilon \cdot \mathbb{N}_+^L$ is an $\varepsilon-$*equilibrium* if:
(i) For each $i \in N$, $x_i \in D_i^{\varepsilon}(p)$.
(ii) For each $l \in M$, if $p_l > 0$, there is $i \in N$ such that $x_i = l$.

Definition 5(i) states that each agent receives an object in her $\varepsilon-$demand set. Definition 5(ii) coincides with Definition 3(ii). When $\varepsilon = 0$, an $\varepsilon-$equilibrium is an equilibrium. In Definition 5, $p \in \varepsilon \cdot \mathbb{N}_+^L$ and so prices vary discretely for $\varepsilon > 0$. In general, any $\varepsilon-$equilibrium is $\varepsilon-$efficient.[3]

In the following, we set $\varepsilon = d$, where d represents the increment, and replace the $\varepsilon-$equilibrium with the $d-$equilibrium.

[3]Suppose that there is an $\varepsilon-$equilibrium that is not $\varepsilon-$efficient. Thus there is $z' \in Z$ such that (i) for each $i \in N$, $u_i(z_i') > u_i(z_i)$, and (ii) $\sum_{i \in N} t_i + \varepsilon \cdot |N| \leq \sum_{i \in N} t_i'$. By (ii), there is $i \in N$ such that $t_{x_i'}' \geq p_{x_i'} + \varepsilon$. Thus by (i), $u_i(x_i', p_{x_i'} + \varepsilon) \geq u_i(z_i') > u_i(z_i)$, contradicting $x_i \in D_i^{\varepsilon}(p)$.

Fact 4: Let $u \in (\mathcal{U}^{QL})^n$, $d > 0$, and the bidding queue q be given. Suppose that each uncommitted agent arbitrarily bids on an object from her demand set at the price proposed to her.
(i) The AA auction with increment d and bidding queue q finds an $d-$equilibrium in finitely many steps.
(ii) (**Deviation bound**) Let $p(u)$ be the price generated by the AA auction in (i). For each $l \in M$, $\left|p_l(u) - p_l^{\min}(u)\right| \leq d \cdot \min\{|M|, |N|\}$.

In contrast to Fact 2, Fact 3 allows agents' valuations to be *arbitrary real numbers* and hence can have different measurement units from the increment.

As noted above, the outcomes of the AA auction depend on the queueing rules and bidders' choices from their demand sets even if the utility function profile is fixed. Fact 4 (ii) states that **for any quasi-linear utility profile**, the deviation $\left|p_l(u) - p_l^{\min}(u)\right|$ is bounded by $d \cdot \min\{|M|, |N|\}$ regardless of queueing rules and bidders' choices from their demand sets. Thus, as d goes to zero, the outcome of the AA auction with increment d converges to an MPE price.

The **AA mechanism with increment** d **and bidding queue** q is a function that maps to each utility profile the outcome of the AA auction with increment d and bidding queue q for that profile. The AA mechanism is neither efficient nor strategy-proof. However, Fact 5 holds.

Fact 5 (Roughgarden, 2014): Let $d > 0$, $k = 2\min\{|M|, |N|\}$, and a bidding queue q be arbitrarily given. The AA mechanism with increment d and bidding queue q on $(\mathcal{U}^{QL})^n$ is $d-$efficient and $k \cdot d-$strategy-proof.

Fact 5 implies that the AA auction works well for quasi-linear utility functions even without the assumption of the same measurement unit. Nevertheless, it fails to work without assuming quasi-linearity. We discuss this point in Section 5.3.

5 Implementability of MPE for general utility functions

5.1 The sealed-bid auction

First, we investigate the possibility of generalizing the sealed-bid auction of Fact 1 to implement MPEs for general utility functions. The formula in Fact 1 is based on valuations. Thus, to conduct the sealed-bid auction using this formula for general utility function profiles, the valuations need to be generalized. A natural generalization of the valuation of an object is the agent's willingness to pay for that object from the dummy with no payment. Formally, for each $i \in N$, each $u_i \in \mathcal{U}$, and each $l \in L$, let $V_i(l) \in \mathbb{R}$ be the *generalized valuation* of an object l such that $u_i(l, V_i(l)) = u_i(0, 0)$. For a quasi-linear utility function, for each $l \in L$,

$V_i(l) = v_i(l)$, an $V_i(0) = v_i(0) = 0$. "Generalized Vickrey allocations" employ generalized valuations to reformulate Fact 1.

Definition 6: A pair $(x^V, p^V) \in X \times \mathbb{R}_+^L$ is a *generalized Vickrey allocation* if
(i) $x^V \in \underset{x \in X}{\arg\max} \sum_{i \in N} V_i(x_i)$,
(ii) if $l \in M$ is unassigned, $p_l^V = 0$ and if $l \in M$ is assigned to agent i, $p_l^V = \max_{x \in X} \sum_{j \in N \setminus \{i\}} V_j(x_j) - \sum_{j \in N \setminus \{i\}} V_j(x_j^V)$.

Let $p^V(u)$ be the generalized Vickrey payment for utility profile u.

Proposition 1: There is $u \in \mathcal{U}^n$ such that (i) the MPE price $p^{\min}(u)$ is different from the generalized Vickrey payment $p^V(u)$, and (ii) no MPE assignment coincides with the generalized Vickrey assignment $x^V(u)$.

We demonstrate Proposition 1 by Example 1 below. This 2-object and 3-agent example can be easily generalized to the cases of more agents and objects.

Example 1: Let $M = \{a, b\}$ and $N = \{1, 2, 3\}$. Let $r > 0$ and $u \in \mathcal{U}^3$ be such that:
(1) $V_1(a) = r$ and $V_1(b) = 2r$.
(2) $V_2(a) = 20r$, $V_2(b) = 10r$, $u_2(a, r) = u_2(b, 0) = u_2(0, -3r)$, and $u_2(0, -1) = u_2(a, 3r) = u_2(b, 2r) = u_2(0, -r)$.
(3) $V_3(a) = 10r$, $V_3(b) = 30r$, $u_3(a, 3r) = u_3(b, -2r)$, and $u_3(a, 8r) = u_3(b, 2r)$.

By the generalized valuations in (1), (2), and (3), $p^V(u) = (p_0^V, p_a^V, p_b^V) = (0, r, 2r)$, and there is a unique generalized Vickrey assignment $x^V = (x_1^V, x_2^V, x_3^V) = (0, a, b)$.

By the generalized valuations in (1) and utility functions in (2) and (3), $p^{\min}(u) = (p_0^{\min}, p_a^{\min}, p_b^{\min}) = (0, 3r, 2r)$ and $x = (x_1, x_2, x_3) = (0, b, a)$ is the unique MPE assignment. Thus, $p^{\min}(u) \neq p^V(u)$ and $x \neq x^V$. △

Proposition 1 states that the sealed-bid auction for the generalized Vickrey allocation may not be an MPE, and in some cases, it is even not an equilibrium; see, e.g., (x^V, p^V) in Example 1.

In the following, we discuss $r-$efficiency and $r-$strategy-proofness of the generalized Vickrey mechanism.

Given $r > 0$, consider the economy in Example 1. Let $z_1 = (0, -0.5r)$, $z_2 = (b, -0.5r)$, and $z_3 = (a, 7.5r)$. For each $i \in \{1, 2, 3\}$, $u_i(z_i) > u_i(x_i^V, p_{x_i^V}^V)$, and $7.5r - 0.5r - 0.5r = 6.5r > 2r + r + 3r$. Thus (x^V, p^V) is not $r-$efficient for u, and so the generalized Vickrey mechanism is not $r-$efficient.

Suppose that agent 3 has a utility function u_3' such that $V_3'(a) = 35r$, $V_3'(b) = 8r$, and $u_3'(a, 10r) = u_3'(b, 4r)$. For the utility profile (u_1, u_2, u_3'), agent 3 obtains $(a, 12r)$ under the generalized Vickrey allocation. Since $u_3'(b, 2r) > u_3'(b, 4r) =$

$u_3'(a, 10r) > u_3'(a, 12r)$, agent 3 benefits from reporting u_3 when her true utility function is u_3'. Thus, the generalized Vickrey mechanism is $r-$manipulable. Therefore, the following holds.

Proposition 2: For an arbitrarily large $r > 0$, the generalized Vickrey mechanism is $r-$inefficient and $r-$manipulable on $\mathcal{U}^n$.

5.2 The exact ascending auction

In this subsection, we study whether the EA auction works without the coincidence assumption but in the quasi-linear environment.

Example 2 below illustrates the large overshooting of the EA auction without the coincidence assumption even if utility functions are quasi-linear.

Example 2: Let $d > 0$ be an increment and $r > d$ be a large number. Let $t \in \mathbb{N}_{++}$ be such that $t \cdot d > r > (t-1) \cdot d$. Let $M = \{a, b\}$ and $N = \{1, 2\}$. Let $u \in (\mathcal{U}^{QL})^2$ be represented by a valuation profile $(v_1(\cdot), v_2(\cdot))$ such that:
(1) $v_1(a) = (t + 0.1) \cdot d$ and $v_1(b) = (t + 0.5) \cdot d$.
(2) $v_2(a) = (t + 0.2) \cdot d$ and $v_2(b) = (t + 0.8) \cdot d$.

Note that

$$v_1(b) - v_1(a) = 0.4d \text{ and } v_2(b) - v_2(a) = 0.6d, \tag{$*$}$$

Note further that $p^{\min}(u) = (p_0^{\min}, p_a^{\min}, p_b^{\min}) = (0, 0, 0.4d)$ and the MPE assignment is $x^{\min} = (a, b)$.

We illustrate how the EA auction proceeds. The auction starts from round 0 with the initial price $p^0 = (0, 0, 0)$. By $(*)$, $D_1(p^0) = D_2(p^0) = \{b\}$. Since only b is overdemanded at p^0, its price p_b^0 is raised by d. Let $p^1 = (0, 0, d)$. By $(*)$, $D_1(p^1) = D_2(p^1) = \{a\}$. Since only a is overdemanded at p^1, its price p_a^1 is raised by d. Let $p^2 = (0, d, d)$. The EA auction repeats this process until prices exceed agents' valuations, i.e., until $p^{2t+2} = (0, (t + 1) \cdot d, (t + 1) \cdot d)$. However, $D_1(p^{2t+2}) = D_2(p^{2t+2}) = \{0\}$. Thus, both bidders simultaneously drop out, and the auction terminates at p^{2t+2}. That is, the outcome of the EA auction is that the price is p^{2t+2} and each agent receives the dummy and pays nothing. $\triangle$

Example 2 demonstrates that the EA auction may fail to find an MPE price. Since r is arbitrarily large, the auction may substantially overshoot an MPE price. This 2-object and 2-agent example can be easily generalized to the cases of more agents and objects. Thus, Proposition 3 holds.

Proposition 3: (**Large overshooting**) For each $d > 0$ and arbitrarily large $r > 0$, there is $u \in (\mathcal{U}^{QL})^n$ such that the EA auction with increment d generates a price p with $p_l > p_l^{\min}(u) + r$ for each $l \in M$.

Since prices vary only discretely with increment $d > 0$, no equilibrium exists for the utility profile u in Example 2. The nonexistence of equilibrium may be a fundamental factor of the above problems. Thus, we consider approximate equilibria and focus on $d-$equilibria. The $d-$equilibria for the utility profile u are prices p such that $p = (0, k \cdot d, k \cdot d)$ for $k = 0, 1, \ldots, t$. However, the outcome price of the EA auction for u in Example 2 is not among them. Thus, the EA auction may fail to find even an $d-$equilibrium price. Moreover, the assignment in the EA auction outcome for u is not a $d-$equilibrium assignment. Thus, the EA auction may also fail to find a $d-$equilibrium assignment.

Note that in Example 2, $p^0 = (0, 0, 0)$ is the closest $d-$equilibrium price to $p^{\min}(u)$ among all $d-$equilibrium prices, and so p^0 is an approximate MPE price. Formally, an $d-$equilibrium price p^A is called **the closest $d-$equilibrium price to** $p^{\min}(u)$ if there is no other $d-$equilibrium price p' such that for each $l \in M$, $\left|p_l^A - p_l\right| \geq |p_l' - p_l|$ with at least one strict inequality. The price p^A approximates the MPE price among $d-$equilibrium prices.

Based on the insight from Example 2, we generalize it to the case with more agents and objects and obtain the following result.

Proposition 4: For each $d > 0$ and arbitrarily large $r > 0$, there is $u \in (\mathcal{U}^{QL})^n$ such that the EA auction with increment d generates a price p with $p_l \geq p_l^A + r$ for each $l \in M$.

Finally, we study $r-$efficiency and $r-$strategy-proofness of the EA mechanism.

Given $r > 0$, consider the economy in Example 2. Let $z_1 = (a, td)$ and $z_2 = (b, td)$. For each $i \in \{1, 2\}$, $u_i(z_i) > u_i(0, 0) = 0$, and by $t \cdot d > r$, it holds that $t \cdot d + t \cdot d > 0 + 2r$. Thus, the EA mechanism with increment d fails to find an $r-$efficient allocation for u.

Let $u_2' \in \mathcal{U}^{QL}$ be represented by a valuation function $v_2'(\cdot)$ such that $v_2'(a) = (t + 0.3) \cdot d$ and $v_1'(b) = (t + 0.1) \cdot d$. Then, agent 2 with u_2' demands only object a at the initial price $p^0 = (0, 0, 0)$. In this case, the EA auction concludes instantly, and agents 1 and 2 receive objects b and a with no payment. Since $td > r$, it holds that $u_2(a, 0) = (t + 0.2) \cdot d > u_2(0, -r) = r$. Thus, agent 2 benefit more than r from misreporting u_2' when her true utility function is u_2. Thus the EA mechanism with increment d is $r-$manipulable.

The above discussion can be easily generalized to the cases of more agents and objects, which indicates the following result.

Proposition 5: For each $d > 0$ and an arbitrarily large $r > 0$, the EA mechanism with increment d is $r-$inefficient and $r-$manipulable on $(\mathcal{U}^{QL})^n$.

Propositions 3, 4 and 5 imply that the EA auction fails to work without the coincidence assumption even in the quasi-linear environment. Note that in Example 2, if one of two agents is given a priority to obtain object b at p^0 at the beginning of the auction, each agent obtains some object at a low price. This outcome is close to the MPE allocation and makes both agents much better off than the outcome of the EA auction. This raises the question of whether the above negative results on the EA auction might be due to the lack of a priority rule that should be applied when several agents demand the same object. We discuss this point in Subsection 5.3 since the AA auction has such a priority rule.

Mishra and Parkes (2009), Andersson and Erlanson (2013), and Liu and Bagh (2019), among others, propose auctions targeting MPEs in a quasi-linear environment with the coincidence assumption. By constructing similar examples, the above negative results extend to their auctions.

5.3 The approximate ascending auction

In this subsection, we study whether the AA auction works as predicted by Facts 4 and 5 when agents have general utility functions. In the following, we set $\varepsilon = d$, where d represents the increment in the auction.

First, we show that for **a given general utility profile**, if the increment is sufficiently small, the outcome price of an AA auction will be sufficiently close to the MPE price.

Proposition 6: Let $u \in \mathcal{U}^n$ be given. Suppose that each uncommitted agent arbitrarily bids on an object from her demand set at the price proposed to her and the bidding queue is arbitrarily given. Let $\{d_n\}$ be a decreasing sequence such that for each $n \in \mathbb{N}$, $d_n > 0$ and $\lim_{n\to\infty} d_n = 0$. Let p^{d_n} be the price generated by the AA auction with increment d_n. Then, $\lim_{n\to\infty} p^{d_n} = p^{\min}(u)$.

The proof of Proposition 6 is relegated to the Appendix. Proposition 6 implies that at the limit, i.e., as d goes to zero, the induced mechanism is indeed the MPE mechanism. This proposition assumes that when the utility profile $u \in \mathcal{U}^n$ is fixed, the AA auction works well for a sufficiently small increment d. Note that how fast auctions conclude depends on d.

However, it is common in practice for d to be set large enough in advance so that the auction can terminate reasonably fast, without knowing the utility profile u. *Thus, it is necessary to analyze whether the AA auction works for any utility profile when* d *is fixed.*

In an economy with two objects and three agents, Example 3 below illustrates that the AA auction substantially undershoots the MPE price for any bidding queue if only one agent has the non-quasi-linear utility function. The insight can be easily generalized to the cases of more agents and objects.

Example 3: Let $M = \{a, b\}$ and $N = \{1, 2, 3\}$. The bidding queue is in the order of agent 1, 2 and 3, without any specification about how uncommitted agents are driven back into the queue in the auction. Let $d > 0$ be an increment and $r > 0$ be a large number. Let $t \in \mathbb{N}_{++}$ be such that $t \cdot d > r$.

(1) Let $u_1 \in \mathcal{U}^{QL}$ be such that $u_1(a, p_a) = td - p_a$ and $u_1(b, p_b) = 3td - p_b$.

(2) Let $u_2 \in \mathcal{U}$ be such that[4]

$$u_2(a, 0) > u_2(b, d) > u_2(a, 0.5d) = u_2(b, 2td) > 0.$$

(3) Let $u_3 \in \mathcal{U}^{QL}$ be such that $u_3(a, p_a) = 0.5d - p_a$ and $u_3(b, p_b) = 0.6d - p_b$.

For this profile u, $p^{\min} = (0, 0.5d, 2td)$, and $x = (x_1, x_2, x_3) = (b, a, 0)$ is a unique MPE assignment.

We illustrate how the AA auction proceeds. The auction starts in round 0 with the initial price $p^0 = (0, 0, 0)$. By $D_1(p^0) = \{b\}$, agent 1 bids on b and is tentatively assigned $(b, 0)$. Then, agent 2 is called. If she bids on b, the price of b that agent 2 faces is d. Thus, let $p^1 = (0, 0, d)$. By $D_2(p^1) = \{a\}$, agent 2 bids on a and is tentatively assigned $(a, 0)$.

Then, agent 3 is called. If agent 3 bids on a, the price of a that she faces is d. If agent 3 bids on b, the price of b that she faces is also d. Thus, let $p^2 = (0, d, d)$. By $D_3(p^2) = \{0\}$, agent 3 drops out by biding 0 and is assigned $(0, 0)$.

There is no uncommitted agent in the queue, and so the auction terminates at a price $p = (0, 0, 0)$ and an allocation where agents 1, 2 and 3 get $(b, 0)$, $(a, 0)$ and $(0, 0)$, respectively. Note that $p_b^{\min} - p_b = 2td > r$. Thus, the outcome price of the AA auction can be smaller than $p_b^{\min}$ by an arbitrarily large amount r. △

Next, we provide Example 4 to show that in an economy with two objects and three agents, the AA auction may substantially overshoot the MPE price.

Example 4:[5] Let $M = \{a, b\}$ and $N = \{1, 2, 3\}$. The bidding queue is such that the initial order of agents is 1, 2 and 3, and when driven back, the uncommitted agent is placed last in the queue. Let $d > 0$ be an increment and $r > 0$ be a large number. Let $t \in \mathbb{N}_{++}$ be such that $t = 3k$ for some odd number $k \in \mathbb{N}_{++}$ and $(t - 2) \cdot d > r$. Let $u = (u_1, u_2, u_3) \in \mathcal{U}^{QL} \times \mathcal{U} \times \mathcal{U}$ satisfy the following[6]:

$$\begin{aligned}
u_1(a, p_a) &= -p_a, \text{ and } u_1(b, p_b) = (t + 0.1)d - p_b, \\
u_2(b, (t - 0.3)d) &> u_2(a, 0) \text{ and} \\
u_2(a, 0.5d) &= u_2(b, (t + 0.1)d) > u_2(0, 0) = u_2(a, (t - 0.5)d) = u_2(b, (t + 0.5)d), \\
u_3(b, (t - 0.4)d) &> u_3(a, 0) \text{ and} \\
u_3(a, 0.6d) &= u_3(b, td) > u_3(0, 0) = u_3(a, (t - 0.4)d) = u_3(b, (t + 0.4)d).
\end{aligned}$$

[4]For example, we can have that $u_2(a, p_a) = 20td - (20t - 1.8)p_a$ and $u_2(b, p_b) = (12t + 0.9)d - p_b$.

[5]Liu and Bagh (2019) provide a hybrid format of the EA and AA auctions that finds an MPE for quasi-linear utility functions. When applying their auction to the economy in Example 4, overshooting also occurs.

[6]The conditions for u_2 and u_3 are satisfied by the following utility functions: $u_2(0, p_0) = -p_0$, $u_2(a, p_a) = (t - 0.5) \cdot d - p_a$, and $u_2(b, p_b) = (2.5t - 2.5) \cdot [(t + 0.5) \cdot d - p_b]$; $u_3(0, p_0) = -p_0$, $u_3(a, p_a) = (t - 0.4) \cdot d - p_a$, and $u_3(b, p_b) = (2.5t - 2.5) \cdot [(t + 0.4) \cdot d - p_b]$.

For this profile u, the MPE price is $p^{\min} = (0, 0.5d, (t + 0.1) \cdot d)$ and the unique MPE assignment is $x^{\min} = (x_1^{\min}, x_2^{\min}, x_3^{\min}) = (0, b, a)$.

We illustrate how the AA auction proceeds. The auction starts from round 0 with the initial price $p^0 = (0, 0, 0)$. By $D_1(p^0) = \{b\}$, agent 1 bids on b and is tentatively assigned $(b, 0)$. Then agent 2 is called. If she bids on b, the price of b that agent 2 faces is d. Thus, let $p^1 = (0, 0, d)$. By $D_2(p^1) = \{b\}$, agent 2 bids on b and is tentatively assigned (b, d). Then, agent 1 is placed last in the queue, and agent 3 is called. If agent 3 bids on b, the price of b that agent 3 faces is $2d$. Thus, let $p^2 = (0, 0, 2d)$. By $D_3(p^2) = \{b\}$, agent 3 bids on b and is tentatively assigned $(b, 2d)$. Then, agent 2 is placed last in the queue, and agent 1 is called. Note that similarly the three agents bid only on b until round t and $p^t = (0, 0, td)$. Then, by $t = 3k$, agent 1 is tentatively assigned (b, td).

At round $t+1$, since agent 2 is first in the queue, she is called. If she bids on b, the price of b that agent 2 faces is $(t+1) \cdot d$. Thus, let $p^{t+1} = (0, 0, (t+1) \cdot d)$. By $D_2(p^{t+1}) = \{a\}$, agent 2 bids on a and is tentatively assigned $(a, 0)$. Then agent 3 is called. Let $p^{t+2} = (0, d, (t + 1) \cdot d)$. By $D_3(p^{t+2}) = \{a\}$, agent 3 bids on a and is tentatively assigned (a, d). Note that agents 2 and 3 bid only on a until round $2t$. In round $2t$ with price $p^{2t} = (0, (t - 1) \cdot d, (t + 1) \cdot d)$, since t is an odd number, agent 2 is called, bids on a and is tentatively assigned $(a, (t-1) \cdot d)$. Let $p^{2t+1} = (0, td, (t + 1) \cdot d)$. Then agent 1 is called. By $D_3(p^{2t+1}) = \{0\}$, agent 3 drops out by biding 0 and is assigned $(0, 0)$.

Since there is no uncommitted agent in the queue, the auction terminates at price $p = (0, (t - 1) \cdot d, t \cdot d)$ and assignment $x = (x_1, x_2, x_3) = (b, a, 0)$. Note that $p_a - p_a^{\min} = (t - 1.5) \cdot d > r$ can be arbitrarily large. △

The insight of Example 4 can be easily generalized to the cases with more agents and objects with an arbitrary bidding queue. We summarize the insights of Examples 3 and 4 by Proposition 7.

Proposition 7: Let an increment $d > 0$ and a bidding queue be given in the AA auction. Suppose that each uncommitted agent bids on an object from her demand set at the price proposed to her. Then, the following results hold for arbitrarily large $r > 0$.
(i) (**Substantial undershooting**) There is $u \in \mathcal{U}^n$ such that the AA auction generates the price p with $p_a < p_a^{\min} - r$ for each $a \in M$.
(ii) (**Substantial overshooting**) For each $a \in M$, there is $u \in \mathcal{U}^n$ such that the AA auction generates the price p with $p_a > p_a^{\min} + r$.

In contrast to Fact 4(ii), Proposition 7 implies that there is **no increment across all the general utility profiles** such that the AA auction neither substantially overshoots nor undershoots the MPE price.

Corollary 1: Let $p^d(u; q)$ be the price generated by the AA auction with increment

d and a bidding queueing q for $u \in \mathcal{U}^n$. Then, there are no $d > 0$ and a bidding queue q such that for each $u \in \mathcal{U}^n$ and each $l \in M$, $\left|p_l^d(u) - p_l^{\min}(u)\right| \leq d$.

Finally, we discuss the incentive issue of AA auctions. Let $u_3' \in \mathcal{U}^{QL}$ be such that $v_3'(a) = 1.5d$ and $v_3'(b) = 2d$ in Example 4. Then when agent 3 is called in round 2 with price $p^2 = (0, 0, 2d)$, $v_3'(b) - v_3'(a) < p_b^2 - p_a^2 = 2d$, and so $D_3(p^2) = \{a\}$. Then, agent 3 bids on a and is tentatively assigned $(a, 0)$. Next, agent 1 is called and faces price $p^3 = (0, d, 2d)$. By $D_1(p^3) = \{b\}$, agent 1 bids on b and is tentatively assigned $(b, 2d)$. Then, agent 2 is called and faces price $p^4 = (0, d, 3d)$. Agents 1 and 2 bid only on b until round $t + 1$. In round $t + 1$ with the price $p^{t+1} = (0, d, td)$, since t is an odd number, agent 2 is called. Since $D_2(p^{t+1}) = \{b\}$, agent 2 bids on b and is tentatively assigned (b, td). In round $t + 2$, agent 1 faces a price $p^{t+2} = (0, d, (t + 1)d)$, since $D_1(p^{t+2}) = \{0\}$, agent 1 drops out by biding 0 and is assigned $(0, 0)$. Since there is no uncommitted agent in the queue, the auction terminates, and generates a price $p' = (0, 0, td)$ and an assignment $x' = (x_1', x_2', x_3') = (0, b, a)$ where agent 3 obtains $(a, 0)$.

Recall that in Example 4, under the utility profile (u_1, u_2, u_3), agent 3 obtains $(0, 0)$. Since $t = 3k \geq 3$ and $u_3(a, 0) > u_3(a, (t - 0.4)t) = u_3(0, 0)$, when agent 3's true utility function is u_3, she has the incentive to misreport u_3'.

The insight of Example 4 and above discussion can be extended to show the following result.

Proposition 8: For each $d > 0$ and arbitrarily large $r > 0$, the AA mechanism with increment d and an arbitrary bidding queue q is $r-$manipulable $\mathcal{U}^n$.

6 Concluding remarks

We conclude by discussing the implications of our negative insights. First, in the one-object auction model, existing work also studies a different auction format, called a "clock auction." It is the continuous variant of the ascending price auction. The price rises continuously at a rate kept by a clock, and bidders drop out at some point in time. The auction stops when there is only one bidder remaining and generates a price at which the winning bidder obtains the object and pays that price. Since it is a dominant strategy for each bidder to drop out at her valuation, this auction is simple and expeditious but also duplicates the MPE mechanism for one object. In Appendix B, we demonstrate the impossibility of extending such a clock auction to multi-object cases with general utility functions.

Second, consider multi-object auction models with agents who have multi-unit demand quasi-linear utility functions. In such models, the Vickrey payment is not an MPE price, but the Vickrey mechanism is strategy-proof and efficient. However, our result in Subsection 5.1 indicates that when agents have multi-unit demand general utility functions, the generalized Vickrey mechanism that is defined in parallel to ours is far from efficient and incentive-compatible.

If agents' multi-unit demand quasi-linear utility functions further satisfy certain substitutable and complementary properties, the MPE is well defined, and the MPE mechanism is efficient, although not strategy-proof. Assuming the coincidence of the increment and the unit of agents' valuations, Gul and Stacchetti (2000), Ausubel (2006), and Sun and Yang (2009) propose auctions that identify the MPEs. Note that unit-demand quasi-linear utility functions are special cases of their utility functions. When applied to the unit-demand settings, the auctions in those three papers are essentially the same as the EA auction. Thus, our results in Subsection 5.2 also imply that for any price increment larger than the unit of agents' valuations, the auctions proposed by those papers also overshoot the MPE prices by an arbitrarily large distance, generate allocations far from efficient outcomes and provide agents with strong incentives to manipulate.

Third, we examine the matching with contracts models with transfers. In the one-to-one setting, if agents are buyers and each object is owned by one seller whose utility only depends on the transfer, then those models coincide with ours, and their buyer-optimal outcomes coincide with the MPE. If transfers are discretized and sellers and buyers have strict preferences over contracts, the cumulative offer process of buyers coincides with the AA auction and finds the buyer-sided optimal outcome. Thus, our results in Subsection 5.3 also imply that in non-quasi-linear environments, the cumulative offer process overshoots or undershoots the MPE prices by an arbitrarily large distance, which generates an outcome far from efficiency and provides agents with strong incentives to manipulate. This insight extends to the many-to-one matching with contracts models and trading network models for general utility functions; see, e.g., Fleiner et al. (2019) and Schlegel (2020): the one-sided optimal outcome with continuous transfers cannot be approximated by the one-sided optimal outcome obtained with discretized transfers and induced strict preferences through variants of the cumulative offer process.

Overall, our results inspire new techniques to study efficient and incentive-compatible auction design when agents have general utility functions.

Appendix A: Proof of Proposition 6

First, we introduce the following characterization.

A set $M' \subseteq M$ of objects is *weakly underdemanded* at p if $[\forall x \in M', p_x > 0] \Rightarrow |\{i \in N : D_i(p) \cap M' \neq \emptyset\}| \leq |M'|$.

Lemma (Morimoto and Serizawa, 2015): p is an MPE price if and only if no set of objects is overdemanded and no set of objects is weakly underdemanded at p.

Let (x^{d_n}, p^{d_n}) be the assignment and price generated by the AA auction with increment d_n. The proof consists of five steps.

Step 1: For each $d_n > 0$, (x^{d_n}, p^{d_n}) is an d_n-equilibrium.

Consider an agent i who drops out. She bids 0 either at the price lower than $\overline{p}^{d_n}$ or at the price $\overline{p}^{d_n} \equiv (p_1^{d_n} + d_n, \dots, p_1^{d_n} + d_n)$. Thus, $0 \in D_i^{d_n}(p^{d_n})$. Consider an agent i who obtains $x_i^{d_n} \in M$. She bids object $x_i^{d_n}$ just at the price $p^{d_n}_{x_i^{d_n}}$, where the price of other object l is equal or less than $p_l^{d_n} + d_n$. Thus, $x_i^{d_n} \in D_i^{d_n}(p^{d_n})$. Thus, Definition 5(i) holds.

In the AA auction, whenever an object is bided on by some agent, it will keep assigned till the end. Thus, Definition 5(ii) holds.

Step 2: There is a convergent subsequence $\{p^{d'_n}\}$ in $\{p^{d_n}\}$ whose assignments remain the same.

For each $l \in M$ and each n, $0 \leq p_l^{d_n} \leq \max_{i\in N} V_i(l)+2d_n$. Thus, $\{p^{d_n}\}$ contains a convergent subsequence $\{p^{d'''_n}\}$. Since agents and objects are both finite, $\{p^{d'''_n}\}$ contains a subsequence $\{p^{d'_n}\}$ whose assignments remain the same.

Step 3: $(x, p) \equiv \lim_{n\to\infty}(x^{d'_n}, p^{d'_n})$ is an equilibrium.

By Definition 5(ii), Definition 3(ii) holds. Thus, we show Definition 3(i). For each n and each $i \in N$, $x_i \in D^{d_n}(p^{d_n})$ implies that for each $y \in M$, $u_i(x_i, p_{x_i}^{d_n}) \geq u_i(y, p_y^{d_n} + d_n)$ and $u_i(x_i, p_{x_i}^{d_n}) \geq u_i(0, 0)$. Thus, for each $y \in M$, $\lim_{n\to\infty} p^{d'_n} = p$ implies $u_i(x_i, p_{x_i}) \geq u_i(0, 0)$ and $u_i(x_i, p_{x_i}) \geq u_i(y, p_y)$. Thus $x_i \in D_i(p)$.

Step 4: Let (x, p) be the equilibrium at Step 3. Then $p = p^{\min}$.

Suppose $p \neq p^{\min}$. Since (x, p) is an equilibrium, by Lemma, there is a weakly underdemanded set $M' \subseteq M$ at p, that is, for each $l \in M'$, $p_l > 0$ and $|\{i \in N : D_i(p) \cap M' \neq \emptyset\}| = |M'|$. Let $N' \equiv \{i \in N' : x_i \in M'\}$. Then, $N' = \{i \in N : D_i(p) \cap M' \neq \emptyset\}$ and for each $i \in N\backslash N'$ and each $l \in M'$, $u_i(x_i, p_{x_i}) > u_i(l, p_l)$. Thus, since for each $l \in M'$, $p_l > 0$, Definition 1(i) implies that there is $\delta > 0$ such that for each $l \in M'$, $p_l - \delta > 0$, and for each $i \in N\backslash N'$,

$$u_i(x_i, p_{x_i} + \delta) > u_i(l, p_l - 2\delta). \tag{a}$$

Since $\lim_{n\to\infty} d'_n = 0$ and $\lim_{n\to\infty} p^{d'_n} = p$, for some $d''_n \in \{d'_n\}$,

(b) $d''_n \leq \delta$ and

(c) $p_l^{d''_n} \geq p_l - \delta > 0$ for each $l \in M'$.

Thus for each $i \in N\backslash N'$ and each $l \in M'$,

$$\begin{aligned} u_i(x_i, p_{x_i}^{d''_n}) &\underset{\text{Step 1}}{\geq} u_i(x_i, p_{x_i} + d''_n) \\ &\underset{\text{(b)}}{\geq} u_i(x_i, p_{x_i} + \delta) \underset{\text{(a)}}{>} u_i(l, p_l - 2\delta) \underset{\text{(c)}}{\geq} u_i(l, p_l^{d''_n} - \delta) \underset{\text{(b)}}{\geq} u_i(l, p_l^{d''_n} - d''_n). \end{aligned}$$

Thus, no agent in $N\backslash N'$ bids an objects in M' when the price of $l \in M'$ reaches $p_l^{d''_n} - d''_n$. In contrast, by (c), l is assigned to some $i \in N'$ at price $p_l^{d''_n} - d''_n$. Thus, by $|N'| = |M'|$, the price of any object $l \in M'$ cannot be increased to $p_l^{d''_n}$, contradicting that $p^{d''_n}$ is the outcome of auction with increment d''_n.

Step 5: $\lim_{n\to\infty} p^{d_n} = p^{\min}(u)$.

Recall that a bounded sequence converges if and only if any of its convergent subsequence has the same limit. Thus, we prove that any convergent subsequence in $\{p^{d_n}\}$ has the same limit. Let $\{p^{d'_n}\}$ and $\{p^{d''_n}\}$ be two convergent subsequences in $\{p^{d_n}\}$ such that $\lim_{n\to\infty} p^{d'_n} = p'$ and $\lim_{n\to\infty} p^{d''_n} = p''$.

Analogous to Step 2, there is a subsequence of $\{p^{d'_n}\}$ converging to p' whose assignments remain the same, say x'. Similarly, there is a subsequence of $\{p^{d''_n}\}$ converging to p'' whose assignments remains the same, say x''. Analogous to Steps 3 and 4, we can show $p' = p^{\min}$ and $p'' = p^{\min}$. Thus, since the MPE price is unique, $p' = p''$.

Appendix B: Clock auction for multiple objects with non-quasi-linearity

Let $M = \{a, b\}$ and $N = \{1, 2, 3\}$. Let $u \in (\mathcal{U}^{QL})^3$ be such that $v_1(a) = 3$, $v_1(b) = 1$, $v_2(a) = 1$, $v_2(b) = 3$, $v_3(a) = v_2(b) = 2$. Then $p^{\min}(u) = (0, 2, 2)$ and $x^{\min}(u) = (a, b, 0)$.

One possible extension of the clock auction is that each agent chooses an object from her demand set given a price, and bids on it. The prices of overdemanded objects are increased. At $p^0 = (0, 0, 0)$, $D_1(p(0)) = \{a\}$, $D_2(p(0)) = \{b\}$, and $D_3(p(0)) = \{a, b\}$. Thus agent 1 bids on a, agent 2 bids on b, and agent 3 bids on a or b. When the price of a or b, say a, increases higher than b, agents 3 bids on b. Thus, starting from $(0, 0, 0)$, if the price is raised continuously, agents 3 needs to move between a and b continuously until the price reaches $p^{\min}(u) = (0, 2, 2)$. Such a bidding behavior is physically impossible, and moreover, the price path in the auction is not well-defined.

Another possible extension is that each agent reports her demand set at a given price, and the prices of minimally overdemanded objects are increased. Indeed, this is the continuous variant of the EA auction. In this extended clock auction, if the prices increase continuously along the path $p(t) = (t, t)$ where $t \in [0, 2]$, agents 1, 2 and 3 keep reporting $D_1(p(t)) = \{a\}$, $D_2(p(t)) = \{b\}$, and $D_3(p(t)) = \{a, b\}$, respectively. Finally, the price reaches $p^{\min}(u) = (0, 2, 2)$.

However, consider a general utility function of agent 3, $u'_3 \in \mathcal{U}$ such that (i) $u'_3(0, 0) = u'_3(a, 1) = u'_3(b, 2)$, and (ii) $u'_3(a, t) = u'_3(b, 2t)$ for each $t \in [0, 1]$. Let $u' = (u_1, u_2, u'_3)$. Then, $p^{\min}(u') = (0, 1, 2)$ and $x^{\min}(u') = (a, b, 0)$. Note that for $p_a \in [0, 1]$, $D_3(p) = \{a\}$ if $p_b > 2p_a$, $D_3(p) = \{b\}$ if $p_b < 2p_a$, $D_3(p(0)) = \{a, b\}$ if $p_b = 2p_a$. Thus, starting from $(0, 0, 0)$, if the prices of minimally overdemanded objects increase continuously with the same rate, agents 3 needs to move between $\{a, b\}$ and $\{b\}$ continuously, which is physically impossible. Note that in such a case, the price path to $p^{\min}(u) = (1, 2)$ is not well-defined either.

Overall, it is both theoretically and practically impossible to extend the clock auction for multiple objects with general utility profiles to duplicate or approximate the MPE mechanism.

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