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MULTI-OBJECT AUCTION DESIGN BEYOND QUASI-LINEARITY: LEADING EXAMPLES

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Multi-object Auction Design Beyond Quasi-linearity: Leading Examples*

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Abstract

In multi-object auction models with unitary demand agents, if agents' utility functions satisfy quasi-linearity, the exact ascending auction of Demange et al. (1986), the sealed-bid Vickrey auction, as well as the approximate ascending auction of Demange et al. (1986) are fundamental to the auction theory and known to identify the minimum price equilibrium (MPE). In particular, these auctions exhibit elegant efficiency and incentive-compatibility. We *exemplify* that these auctions fail to identify the MPEs and are substantially inefficient and manipulatable without assuming the coincidence between price increment and valuation unit or without assuming the quasi-linearity of utility functions. The implications of our negative results for multi-object auction models with agents with multi-unit demand, and matching with contracts models are discussed as well.

Keywords: Multi-object auction, minimum price equilibrium, examples, the coincidence assumption, quasi-linearity

JEL Classification: C78, D44, D74

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1 Introduction

The most successful examples of auction theory are spectrum license auctions organized by governments in OECD countries that auction off several licenses simultaneously, often generating enormous revenues. As a case in point, in the 2000 British 3G spectrum license auctions, the revenue generated from selling five licenses amounted to 2.5% of the UK's GNP. These auctions play important roles in the overall development of the mobile phone industry. In these auctions, it is widely observed that winning bids involve copious sum of money and the price increment is set sufficiently large in advance (to have a reasonable stopping time) (Klemperer, 2004). The huge winning bids end up violating the *quasi-linearity assumption* on preferences, i.e., an agent's benefit from the auctioned object can be presented by her valuation of that object, independent of the associated payment. The sizable increment leads to the violation of *the coincidence assumption* that the price increment coincides with the valuation unit, which further restricts the quasi-linearity of preferences. These two assumptions are common in auction theory but remain undesirable in many real-life applications. This paper analyzes the ramifications of the violations of both assumptions on the performance of auctions that are of utmost significance in theory, but generally studied in the quasi-linear settings.

In particular, we focus on auctions with government-concerned and socially desirable properties, *efficiency* and *strategy-proofness*. *Efficiency* requires that objects should be given to those who value them the most. However, agents' preference information of objects is only privately known in many cases. Thus, it is indispensable for the auction to directly or indirectly extract true information from agents to attain efficiency. *Strategy-proofness* requires agents to have no incentive to manipulate information in the auction in the sense that revealing true information is a dominant strategy for each agent.

We study the multi-object auction model where agents have unit demand, i.e., each agent gets at most one object, and also have the *classical* utility functions that satisfy the standard monotonicity and continuity, taking the quasi-linear utility functions as special cases. In such settings, there is a *minimum price equilibrium* (MPE). The *MPE mechanism*, a direct mechanism to select an MPE allocation for each utility profile, is the only mechanism that satisfies efficiency, strategy-proofness, individual rationality, and no subsidy when agents have either quasi-linear utility functions (Holmstrom, 1979) or classical utility functions (Saitoh and Serizawa, 2008; Morimoto and Serizawa, 2015). For this reason, any auction that duplicates the outcome of the MPE mechanism is efficient and is imbued with a nice incentive property, thus eliciting particular attention by auction theorists.

There are three well-known auctions that find the MPEs in the quasi-linear settings. The first one is proposed by Demange et al. (1986), the exact ascending (EA) auction. The EA auction generates the MPE price by further imposing the coincidence assumption. The second one is the sealed-bid Vickrey auction, whose outcome coincides with the MPE (Leonard, 1983). Meanwhile, the third one is provided by Demange et al. (1986), the approximate ascending (AA) auction. The AA auction originates from the salary adjustment process in Crawford and Knoer (1981) and Kelso and Crawford (1982). The AA auction generates a price that then coordinate-wise approximates the MPE price. In these auctions, objects can be heterogenous.[1] As explicated later, these auctions are fundamental to auction theory and provide the essential ideas of several auction designs in more general environments.

In practice, variants of the aforementioned auctions are also utilized. For example, the multi-unit demand variant of the EA auction runs in power purchase in New Jersey, U.S.A. (Chapter 7, Milgrom, 2004). Facebook uses the sealed-bid Vickrey auction to sell its online ads slots (Varian and Harris, 2014). It is also noteworthy that the simultaneous ascending auction adopted by the U.S. Federal Communications Commission (FCC) for selling spectrum licences is similar to the multi-unit demand variant of the AA auction (Chapter 7, Milgrom, 2004).

We study the performance of three aforementioned auctions by dropping the coincidence assumption and quasi-linearity assumption. In practice, the coincidence assumption may be unsuitable for auctions with sizeable increments. The willingness of an agent to pay is the primitive data independent of auctions, which can take arbitrary values, whereas the price increment is set in advance. The coincidence assumption is not realistic except for technical reasons. The quasi-linear assumption is dubious for the aforementioned large-scale auction. Notably the coincidence assumption is a further restriction in the quasi-linear setting, and a violation of quasi-linearity assumption leads to a violation of the coincidence assumption.

To evaluate the performance of three aforementioned auctions, in terms of their associated (direct) mechanisms, apart from the standard notions of efficiency, approximate efficiency, strategy-proofness, and approximate strategy-proofness, we formulate two additional concepts of *absolute inefficiency* and *absolute manipulability.* These are the properties of the mechanism. The former states that it

[1] When applied to the case of homogenous objects, the sealed-bid Vickrey auction is sometimes referred to as the Vickrey auction for k objects where k objects are assigned to the agents whose values are among the kth highest and they just pay the $(k+1)th$ highest valuation. On the other hand, the EA auction is sometimes called the ascending auction for k objects where information revelation of all bidders follows the same fashion as the ascending auction for one object.

is impossible for the mechanism to achieve approximate efficiency to any degree. The latter states that it is impossible for the mechanism to achieve any degree of approximate incentive compatibility.

When objects are homogenous, the three aforementioned auctions perform well on the classical domain. The intuition is as follows. The key to determine the MPE is an agent's generalized valuation, a natural generalization of valuations for quasi-linear preferences, defined as the maximum willingness to pay for the object relative to the status of getting nothing and paying nothing. The MPE price is the same across all the objects, which is equal to the $(k+1)th$-highest generalized valuation if there are k objects to be auctioned and those objects are assigned to agents whose generalized valuation is among the $(k+1)th$-highest generalized valuation. However, the generalized valuation does not include sufficient information to determine the MPE when objects are heterogenous and agents have classical utility functions. Our main results stated below exhibit such abnormalities.

- In the quasi-linear settings, we consider a "perturbation" of the quasi-linear domain with valuations that are mutiples of natural numbers, by allowing rational numbers in the neighborhood of the natural numbers. This, in turn, breaches the coincidence assumption. We demonstrate that the EA auction always overshoots the MPE prices by an arbitrarily large distance in the perturbed quasi-linear domain, leading to absolute inefficiency and absolute manipulability of the mechanism associated with EA auction. If such a perturbation allows the agents to have valuations that are multples of irrational numbers, we further contend that it could be the case that the EA auction always largely overshoots the MPE for any positive increment. Furthermore, the continuous-time clock auction for selling one object can not be extended to multi-object cases with classical utility functions.

- When agents have classical utility functions, the generalized sealed-bid Vickrey auction is defined by replacing the valuations in the formula of the Vickrey allocation with the generalized valuations. Both the price and the assignment generated by this auction are different from MPE price and assignment. Its associated mechanism, the generalized Vickrey mechanism is absolutely inefficient and absolutely manipulable.

- When agents have classical utility functions, the AA auction overshoots and undershoots the MPE prices by an arbitrarily large distance if the increment is fixed in advance. It never approximates the MPE and leads to absolute manipulability of the mechanism associated with the AA auction. By contrast, for a fixed classical utility profile, a decrease in increment makes the outcome of AA auction approximate the MPE.

In the following, we contrast our negative results with the auction designs in various settings. First, auctions that target MPEs in quasi-linear settings with unitary demand agents, as proposed by Mishra and Parkes (2009) and Andersson and Erlanson (2013), are premised on the coincidence assumption. Therefore, our negative results of EA auctions can be carried over to their models without the coincidence assumption by constructing similar examples.

Second, consider multi-object auction models with agents with multi-unit demand quasi-linear utility functions. In such models, a Vickrey allocation is not an MPE allocation; however the Vickrey mechanism remains strategy-proof and efficient. When agents have classical utility functions, our negative result of the generalized sealed-bid Vickrey auction defined via generalized valuations can be carried over.[2]

If the utility functions of agents further satisfy certain substitutable properties, the MPE is well-defined, and the MPE mechanism is efficient, albeit not strategy-proof. Under the coincidence assumption, Gul and Stacchetti (2000), Ausubel (2006), and Sun and Yang (2009) propose auctions that identify the MPEs. When applying their auctions to our model, those auctions are essentially the same as the EA auction; thus our negative results of the EA auction hold for their auctions.

Finally, we consider the matching with contracts models with transfers (Hatfield and Milgrom, 2005). With regard to the one-to-one setting, if agents are buyers and each object is owned by one seller whose utility hinges on only the transfer, then those models coincide with ours, and their buyer-optimal outcomes coincide with the MPEs. If transfers are discretized and both sellers and buyers have strict preferences over contracts, the cumulative offer process of buyers coincides with the AA auction and finds the buyer-sided optimal outcome (Echenique, 2012). Consequently, our negative results of the AA auction imply: The one-sided optimal outcome with continuous transfers cannot be approximated by the one-sided optimal outcome derived from discretized transfers and induced strict preferences via variants of the cumulative offer process.[3]

Our paper is also related to the literature of efficient and incentive compatible auction designs when agents have classical utility functions. When objects are homogenous, Saitoh and Serizawa (2008) characterize the efficient and strategy-proof direct mechanisms. On the other hand, when objects are heterogenous, Morimoto

[2] A similar conclusion is reached by Malik and Mishra (2020).

[3] Such an insight holds even for the trading network model with continuous transfers, see, Fleiner et al. (2019) and Schlegel (2022).

and Serizawa (2015), Zhou and Serizawa (2018), Baisa (2020), Kazumura et al. (2020), and Malik and Mishra (2021) characterize the efficient, strategy-proof, and fair direct mechanisms. Prices are continuous variables in all these auction models. Hatfield et al. (2021) characterize the cumulative offers process that can be regarded as the auctions with discretiezed payments. Apart from the analysis of properties of direct mechanisms studied by those papers, their indirect implementations through various auction formats also bear significance for practical purpose.

The novelty of this paper is to exemplify that the myriad well-known auctions listed above are neither efficient nor incentive-compatible even when applied to simple non-quasi-linear settings where agents have unit demand. Our key message is to note the challenges and inspire the development of novel analytical techniques for the auction design by dropping the coincidence assumption along with the quasi-linearity assumption.

The remainder of this paper is organized in the following manner: We define the model and MPEs in Section 2. In Section 3, we expound on the notions for efficiency, inefficiency, strategy-proofness, and manipulability. In Section 4 we define auctions for MPEs. We review the existing results of auctions for MPEs on the quasi-linear domain in Section 5. In Section 6, we present our main findings and study the performance of auctions in Section 5 by dropping the coincidence assumption and quasi-linearity assumption. Finally, we conclude the study in Section 7.

2 The model and minimum price equilibrium

There is a finite set of agents N and a finite set of objects M. For the purpose of this paper, we ignore the trivial analysis of $|N| = 1$ or $|M| = 1$ and assume $|N|, |M| \geq 2$.[4] Not receiving an object is called receiving the *null* object, which is denoted by 0. Let $L \equiv M \cup \{0\}$. Each agent has unit demand: She either receives an object or the null object.

Agents have preferences on the consumption set $L \times \mathbb{R}$.[5] We abuse language and identify a preference of agent i with her utility representation u_i.

Definition 1: A utility function $u_i : L \times \mathbb{R} \to \mathbb{R}$ is *classical* if:

[4] Here $| \cdot |$ is the cardinality of set $\cdot$.

[5] Following the convention, $\mathbb{R}$ is the set of reals, $\mathbb{Q}$ is the set of rational numbers, $\mathbb{Z}$ is the set of integers, and $\mathbb{N} = \{0, 1, \cdots\}$ is the set of natural numbers. Let $\mathbb{R}_+$, $\mathbb{Q}_+$, and $\mathbb{N}_+$ be the sets of non-negative reals, non-negative rational numbers, and positive integers, respectively. For a positive real $d > 0$, $d \cdot \mathbb{Z} = \{\cdots -d, 0, d, \cdots\}$ and $d \cdot \mathbb{T}$ where $\mathbb{T} \in \{\mathbb{Q}, \mathbb{Q}_+, \mathbb{N}, \mathbb{N}_+\}$ can be similarly defined.

(i) For each $l \in L$, $u_i(l, \cdot)$ is continuous and strictly decreasing in $\mathbb{R}$.
(ii) For each pair $l, l' \in L$, each $t \in \mathbb{R}$, there is $t' \in \mathbb{R}$ such that $u_i(l, t) = u_i(l', t')$.

Let $\mathcal{U}$ be the set of classical utility functions and $\mathcal{U}^N$ be the *classical domain*. Let $u \equiv (u_i)_{i \in N} \in \mathcal{U}^N$ be a profile of utility functions, i.e., the utility profile.

Definition 2: A utility function $u_i \in \mathcal{U}$ is *quasi-linear* if there is a valuation function $v_i : L \to \mathbb{R}_+$ such that for each $(l, p_l) \in L \times \mathbb{R}$, $u_i(l, p_l) = v_i(l) - p_l$.

Each quasi-linear utility function u_i can be represented by a valuation function v_i. We assume, without loss of generality (w.l.o.g.), that for each $i \in N$, $v_i(0) = 0$. Let $\mathcal{U}^{QL}$ be the set of quasi-linear utility functions and $(\mathcal{U}^{QL})^N$ be the *quasi-linear domain*. Notice that $\mathcal{U}^{QL} \subsetneq \mathcal{U}$.

For each agent $i \in N$, let $x_i \in L$ be her assigned object. An assignment $x \equiv (x_i)_{i \in N} \in L^N$ is a list of individually assigned objects such that except for the null object, no two agents obtain the same object, i.e., if $x_i \neq 0$ and $i \neq j$, $x_i \neq x_j$. Let X be the set of assignments.

For each $l \in L$, let $p_l \in \mathbb{R}_+$ denote the price of object l and $p = (p_l)_{l \in L} \in \mathbb{R}^L_+$ be a price. Agent $i's$ demand set at price p is defined as $D_i(p) \equiv \{l \in L : u_i(l, p_l) \geq u_i(l', p_{l'}), \forall l' \in L\}$. An object in agent $i's$ demand set maximizes her welfare at the given price. We assume, w.l.o.g., that the price of the null object always remains zero and the reserve prices of all the objects are zero.

Definition 3: A pair $(x, p) \in X \times \mathbb{R}^L_+$ is a (Walrasian) *equilibrium* if:
(i) For each $i \in N$, $x_i \in D_i(p)$.
(ii) For each $l \in M$, if $p_l > 0$, there is $i \in N$ such that $x_i = l$.

Definition 3(i) states that each agent receives an object in her demand set. Definition 3(ii) specifies that an object with a positive price must be assigned. Equivalently, the price of an unassigned object is its reserve price fixed at zero.

For each utility profile from the classical domain, there is an equilibrium. In particular, the set of equilibrium prices is a complete lattice (Demange and Gale, 1985). Therefore, there is a *minimum price equilibrium (MPE)* whose price is *unique* and *coordinate-wise minimum* among all equilibrium prices. For each utility profile $u \in \mathcal{U}^N$, let $p^{\min}(u)$ be the associated MPE price. Notably, the associated MPE price is unique for each given utility profile, but the corresponding assignment may not be unique since indifference in preferences is allowed. Moreover, each agent is welfare-equivalent across all the MPEs.

The equilibrium prices and the MPE price indeed can be characterized via the interactions between demand and supply.

Definition 4: (i) A non-empty set of objects $M' \subseteq M$ is *overdemanded at price* p if $|\{i \in N : D_i(p) \subseteq M'\}| > |M'|$.

(ii) A non-empty set of objects $M' \subseteq M$ is (*weakly) underdemanded at price* p if $[\forall x \in M', p_x > 0] \Rightarrow |\{i \in N : D_i(p) \cap M' \neq \emptyset\}| (\leq) < |M'|$.

The following characterizations of the equilibrium prices and MPE price hold.

Fact 1 (Mishra and Talman, 2010; Morimoto and Serizawa, 2015): Let $u \in \mathcal{U}^N$.
(i) A price p is an equilibrium price for u if and only if no set of objects is overdemanded and no set of objects is underdemanded at p for u.
(ii) A price p is an MPE price for u if and only if no set of objects is overdemanded and no set of objects is weakly underdemanded at p for u.

Fact 1(ii) introduces the following property of MPEs.

Fact 2 (**Demand connectedness**) (Morimoto and Serizawa, 2015): Let $u \in \mathcal{U}^N$ and $(x, p^{\min})$ be an MPE for u. For each $l \in M$ such that $p_l^{\min} > 0$, there is a sequence $\{i_k\}_{k=1}^{\Lambda}$ of Λ distinct agents such that
(i) $x_{i_1} = 0$ or $p_{x_{i_1}}^{\min} = 0$,
(ii) $x_{i_\Lambda} = l$,
(iii) for each $k \in \{2, \cdots, K-1\}$, $x_{i_k} \in M$ and $p_{x_{i_k}}^{\min} > 0$, and
(iv) for each $k \in \{1, \cdots, K-1\}$, $\{x_{i_k}, x_{i_{k+1}}\} \subseteq D_{i_k}(p^{\min})$.

Fact 2 states that for each object with a positive price, it will be connected to the object with zero price alternatively via agents' demands.

Facts 1 and 2 are vital in designing auctions that implement the MPEs on the quasi-linear domain and understanding the strategy-proofness of the MPE mechanisms (Mishra and Talman, 2010; Morimoto and Serizawa, 2015).

A weaker notion of equilibrium, the *approximate equilibrium*, called the "$\varepsilon-$equilibrium," is studied as well. The $\varepsilon-$equilibrium tackles the case where equilibrium prices are restricted to be discrete, i.e., multiples of some grid say ε.

For some $\varepsilon > 0$, agent i's $\varepsilon-$*demand set* at $p \in \mathbb{R}_+^L$ is given by:

$$D_i^{\varepsilon}(p) \equiv \{l \in L : u_i(l, p_l) \geq u_i(0,0), \text{ and } \forall l' \in M, u_i(l, p_l) \geq u_i(l', p_{l'} + \varepsilon)\}.$$

An object in $D_i^{\varepsilon}(p)$ approximately maximizes agent i's welfare at price p. When $\varepsilon = 0$, an $\varepsilon-$demand set is reduced to a demand set.

Definition 5: Let some $\varepsilon > 0$ be given. A pair $(x, p) \in X \times (\varepsilon \cdot \mathbb{N})^L$ is an $\varepsilon-$*equilibrium* if:
(i) For each $i \in N$, $x_i \in D_i^{\varepsilon}(p)$.
(ii) For each $l \in M$, if $p_l > 0$, there is $i \in N$ such that $x_i = l$.

Definition 5(i) is parallel to Definition 3(i) in the approximate sense and Definition 5(ii) is the same as Definition 3(ii).

When prices are discrete, an equilibrium compatible with discrete prices may not exist; refer to Example 1 below for more details. The $\varepsilon-$equilibrium is useful in the auction model where ε is set to be the bid increment, as discussed later.

3 Notions of (in)efficiency, and (non-)manipulability

An allocation $z \equiv (x_i, t_i)_{i\in N} \in (L \times \mathbb{R})^N$ such that $(x_i)_{i\in N} \in X$ denotes a list of individually assigned objects, paired with the associated transfers. Let Z be the set of allocations. A *(direct) mechanism* f is defined as a function from $\mathcal{U}^N$ to Z that maps to each utility profile u an allocation z. For each agent $i \in N$, let $x_i(u)$ be the object assigned and $t_i(u)$ represent the associated transfer specified by mechanism f, and let $f_i(u) = (x_i(u), t_i(u))$. Given a utility profile $u \in \mathcal{U}^N$, $(f(\cdot), u)$ forms a revelation game: agents report their utility functions and the outcome of their reports is selected by $f(\cdot)$.

We now let $\mathcal{D} \subseteq \mathcal{U}$ and define the (in)efficiency and (non-)manipulability of mechanisms on domain $\mathcal{D}^N$.

We first introduce efficiency. Given an allocation $z \in (L \times \mathbb{R})^N$, let $Rev(z) \equiv \sum_{i\in N} t_i$ be the revenue generated by z. An allocation $z \in Z$ is *efficient* for $u \in \mathcal{D}^N$ if there is no $z' \in Z$ such that (i) for each $i \in N$, $u_i(z'_i) \geq u_i(z_i)$ with at least one strict inequality, and (ii) $Rev(z) \leq Rev(z')$.

Efficiency: A mechanism f is *efficient* on domain $\mathcal{D}^N$ if for each $u \in \mathcal{D}^N$, $f(u)$ is efficient for u.[6]

Due to practical considerations, achieving efficiency is sometimes demanding. Instead, approximate efficiency is often taken into consideration. Given $r \in \mathbb{R}_+$, an allocation $z \in Z$ is $r-$*efficient* for $u \in \mathcal{D}^N$ if there is no $z' \in Z$ such that (i) for each $i \in N$, $u_i(z'_i) > u_i(z_i)$, and (ii) $Rev(z) + r \cdot |N| \leq Rev(z')$.[7]

r-**efficiency**: Given $r \in \mathbb{R}_+$, a mechanism f is $r-$*efficient* on domain $\mathcal{D}^N$ if for each $u \in \mathcal{D}^N$, $f(u)$ is $r-$efficient for u.

In the case of $r = 0$, $r-$efficiency coincides with efficiency. For a small $r > 0$, $r-$efficiency is "approximately efficient."

Absolute inefficiency: A mechanism f is *absolutely inefficient* on domain $\mathcal{D}^N$ if there is no $r \in \mathbb{R}_+$ such that f is $r-$efficient on domain $\mathcal{D}^N$.

[6] A mechanism f on $(\mathcal{U}^{QL})^N$ is efficient if and only if for each $u \in (\mathcal{U}^{QL})^N$, $x(u) \in \underset{x\in X}{\arg\max} \sum_{i\in N} v_i(x_i)$ (See Zhou and Serizawa (footnote 29, 2018) for a complete proof of this statement).

[7] Our notion is in the spirit of the definition of the approximate core allocation in classical general equilibrium theory, see, e.g., Hildenbrand et al. (1973).

Absolute inefficiency specifies that it is impossible for a mechanism to achieve approximate efficiency to any degree.

Thereafter, we introduce the incentive notions. A mechanism f is *manipulable* on domain $\mathcal{D}^N$ if there are $u \in \mathcal{D}^N$, $i \in N$, and $u'_i \in \mathcal{D}$ such that $u_i(f_i(u'_i, u_{-i})) > u_i(f_i(u))$. If a mechanism is immune to manipulability, it is strategy-proof.

Strategy-proofness: A mechanism f is *strategy-proof* on domain $\mathcal{D}^N$ if it is not manipulable on domain $\mathcal{D}^N$.

Strategy-proofness implies that in the revelation game $(f(\cdot), u)$, truthfully reporting her utility function is a dominant strategy for each agent.

Next, we define a weaker notion of strategy-proofness. Given $r \in \mathbb{R}_+$, "r-manipulability" states that an agent benefits more than r from misrepresenting her utility function, with respect to the payment, paired with the assigned object under truth telling. Given $r \in \mathbb{R}_+$, a mechanism f on domain $\mathcal{D}^N$ is *r-manipulable* if there are $u \in \mathcal{D}^N$, $i \in N$, and $u'_i \in \mathcal{D}$ such that $u_i(f_i(u'_i, u_{-i})) > u_i(x_i(u), p_i(u)-r)$. If a mechanism is immune to r-manipulability, it is r-strategy-proof.

$r-$**strategy-proofness:**[8] Given $r \in \mathbb{R}_+$, a mechanism f is *$r-$strategy-proof* on domain $\mathcal{D}^N$ if it is not *r-manipulable* on domain $\mathcal{D}^N$.

If $r = 0$, $r-$strategy-proofness coincides with strategy-proofness. For a small $r > 0$, $r-$strategy-proofness is approximately strategy-proof: Each agent has only a small incentive to manipulate.

Absolute manipulability: A mechanism f is *absolutely manipulable* on domain $\mathcal{D}^N$ if there is no $r \in \mathbb{R}_+$ such that f is $r-$strategy-proof on domain $\mathcal{D}^N$.

Absolute manipulability points out that it is impossible for a mechanism to achieve approximate strategy-proofness to any degree.

An *MPE mechanism* is a function that maps to each utility profile an MPE allocation[9] for that profile. On the quasi-linear domain, the MPE mechanism is equivalent to the Vickrey mechanism defined in Section 4 (Leonard, 1983). More specifically, the MPE/Vickery mechanism is the unique mechanism satisfying *efficiency*, *strategy-proofness*, *individual rationality*, as well as *no subsidy* (Holmstrom, 1979). This characterization of the MPE mechanism holds on the non-quasi-linear domains (Morimoto and Serizawa, 2015; Zhou and Serizawa, 2018).

[8]Our notion is in the spirit of limiting incentive compatibility of Robert and Postlewaite (1976).

[9]An MPE allocation is an allocation compatible with the MPE where each agent gets an object assigned by the MPE and pays its MPE price.

4 Auctions for MPEs

We introduce three well-known auctions in this section. The first auction is the exact ascending auction introduced by Demange et al. (1986). This dynamic procedure finds an MPE price by recursively increasing the prices of objects in a minimally overdemanded set identified at the current price. The minimal overdemanded set is a particular type of overdemanded set such that none of its non-empty proper subsets is overdemanded.[10]

Definition 6: *The exact ascending (EA) auction* is defined as follows:

Let $d > 0$ be the increment. Starting from reserve prices, each agent reports her demand set at the current price. If there is a set of objects that are minimally overdemanded, then the prices of those objects are increased by d; otherwise, stop at the current price.

The assignment of the EA auction is not explicitly specified, but implicitly given in Theorem 2 in Demange et al. (1986). Put succinctly, each agent is assigned an object from her demand set at the termination price. Among all these assignments, find the assignment such that objects with positive prices are assigned.

The second auction is known as the sealed-bid Vickrey auction (Leonard, 1983).[11] In this auction, the auctioneer asks each agent to report her maximum willingness to pay for each object relative to the status of getting the null object and paying nothing, which is formulated as follows, to facilitate our discussion on both the quasi-linear domain and the classical domain. For each $i \in N$, each $u_i \in \mathcal{U}$, and each $l \in L$, let $V_i(l) \in \mathbb{R}$ be the maximum willingness to pay for object l, called the *generalized valuation* of object l, such that $u_i(l, V_i(l)) = u_i(0,0)$. In the quasi-linear settings, the generalized valuation $V_i(\cdot)$ is simply the valuation $v_i(\cdot)$.

Definition 7: *The generalized sealed-bid Vickrey auction* is defined in the following manner: Each agent reports $V_i(\cdot)$. Then calculate $(x^V, p^V) \in X \times \mathbb{R}^L$ as follows:

(i) $x^V \in \arg\max_{x \in X} \sum_{i \in N} V_i(x_i)$, and

(ii) $p_0^V = 0$, if $l \in M$ is unassigned, $p_l^V = 0$, and if $l \in M$ is assigned to agent i, $p_l^V = \max_{x \in X} \sum_{j \in N \setminus \{i\}} V_j(x_j) - \sum_{j \in N \setminus \{i\}} V_j(x_j^V)$.

[10] A non-empty set of objects $M' \subseteq M$ is minimally overdemanded at p if (i) $|\{i \in N : D_i(p) \subseteq M'\}| > |M'|$ and (ii) there is no non-empty set $M'' \subsetneq M'$ such that $|\{i \in N : D_i(p) \subseteq M', D_i(p) \cap M'' \neq \emptyset\}| > |M''|$.

[11] The closed-form expression of the sealed-bid Vickrey auction is essentially introduced by Vickrey (1961), Clarke (1971), and Groves (1977).

If $u \in (\mathcal{U}^{QL})^N$, i.e., for each $i \in N$, $V_i(\cdot) = v_i(\cdot)$, we simply call the generalized sealed-bid Vickrey auction the *sealed-bid Vickrey auction.* For each utility profile $u \in \mathcal{U}^N$, let $p^V(u)$ be the associated generalized sealed-bid Vickrey price. By further assigning zero prices to unassigned objects, it extends the well-known Vickrey payment. We do so just for the sake of convenience when comparing it to the MPE price, see, e.g., Fact 5 and Proposition 5.

The third auction is also proposed by Demange (1986), known as the approximate ascending auction. The following definition comes from Demange (1986).[12]

Definition 8: *The approximate ascending (AA) auction* is defined as follows: Let $d > 0$ be the increment. Initially, all the agents are uncommitted and stand in a queue. These uncommitted agents are called one by one to bid. When agent i is called, she is presented with the following three options.
Option 1 is to bid on an unassigned object l. This option commits agent i to object l at its reserve price.
Option 2 is to bid on an object l that is tentatively assigned to some other agent j at price p_l. This option increases the price of object l by d, commits agent i to object l at price $p_l + d$ and drives agent j back into the queue of uncommitted agents.
Option 3 is to drop out by bidding on the null object.

The auction terminates when all uncommitted agents drop out and each committed agent buys her assigned object at its current price.

We finally define the mechanisms associated with the aforementioned three auctions on the domain $\mathcal{D}^N \subseteq \mathcal{U}^N$. An allocation is specified by some auction in such a manner that each agent pays the price of her assigned object upon the termination of the auction. The *EA mechanism* is a function that maps to each utility profile from D^N an allocation specified by the EA auction for that utility profile. The *generalized Vickrey mechanism* is a function maps to each utility profile from $\mathcal{D}^N$ an allocation specified by the generalized sealed-bid auction for that utility profile. If it is defined on the quasi-linear domain, it is called the Vickrey mechanism. The *AA mechanism* is a function that maps to each utility profile from $\mathcal{D}^N$ an allocation specified by the AA auction for that utility profile.

[12] In Page 867 of Demange et al. (1986), they wrote "[...]If one wishes to structure this procedure the auctioneer could call on the uncommitted bidders, say in alphabetical order, requiring them to choose one of the three alternatives listed above[...]" Our formulation of the AA auction is congruent with its original definition in Demange et al. (1986), but utilizes a more general way of dealing with the uncommitted agents, taking the alphabetical order as a special case.

5 Implementability of MPEs on the quasi-linear domain

In this section, we review the current results of the EA auction, the sealed-bid auction, as well as the AA auction on the quasi-linear domain.[13]

To begin with, we review the results of the EA auction. Demange et al. (1986) assume that agents' valuations are multiples of the increment $d > 0$, i.e., they focus on the class of quasi-linear utility functions $\mathcal{U}^{QLd}_{\mathbb{N}} \equiv \{u_i \in \mathcal{U}^{QL} : \forall l \in M, v_i(l) \in d \cdot \mathbb{N}\}$ and the EA auction operates on domain $(\mathcal{U}^{QLd}_{\mathbb{N}})^N$. They obtain the following results.

Fact 3 (Demange et al. 1986): Let $d > 0$ and $u \in (\mathcal{U}^{QLd}_{\mathbb{N}})^N$. Then, the EA auction with increment d finds an MPE price for u in a finite number of rounds.

The EA mechanism has nice efficiency and incentive properties.

Fact 4 (Leonard, 1983): Let $d > 0$. The EA mechanism with increment d on $(\mathcal{U}^{QLd}_{\mathbb{N}})^N$ coincides with the MPE mechanism on $(\mathcal{U}^{QLd}_{\mathbb{N}})^N$. It is efficient and strategy-proof.

The coincidence assumption is essential to establish Facts 3 and 4. We will elaborate further on this point in Section 6.1.

Remark 1: Facts 3 and 4 can be established on the classical domain that satisfies the "generalized coincidence assumption." Given an increment $d > 0$, let $\mathcal{U}^d \equiv \{u_i \in \mathcal{U} : \forall l, l' \in L, \forall k \in d \cdot \mathbb{Z}, \exists k' \in d \cdot \mathbb{Z} \text{ s.t. } u_i(l,k) = u_i(l',k')\}$ be the set of classical utility functions that exhibits quasi-linearity over $L \times (d \cdot \mathbb{Z})$. Then $\mathcal{U}^d \backslash \mathcal{U}^{QL} \neq \emptyset$ and $\mathcal{U}^{QLd}_{\mathbb{N}} \subsetneq \mathcal{U}^d$.[14] The *generalized coincidence assumption* says that the domain is $(\mathcal{U}^d)^N$. Facts 3 and 4 hold on $(\mathcal{U}^d)^N$ since their original proofs also work on $(\mathcal{U}^d)^N$. A violation of the generalized coincidence assumption must lead to a violation of the coincidence assumption.

Second, we overview the results of the sealed-bid Vickrey auction. The MPE allocation coincides with the sealed-bid Vickrey allocation for each quasi-linear utility profile (Leonard 1983). Thus, we have the following result.

Fact 5 (Leonard, 1983): Let $u \in (\mathcal{U}^{QL})^N$, $(x^*, p^{\min}(u))$ be an MPE, and $p^V(u)$ be the sealed-bid Vickrey price. Then,
(i) $x^* \in \underset{x \in X}{\arg\max} \sum_{i \in N} v_i(x_i)$.

[13] Facts 3 to 7 are established on the quasi-linear domain where each agent has non-negative valuations of objects. It is not difficult to ascertain that Facts 3 to 7 hold even when agents have negative valuations of some objects.

[14] It is not difficult to construct a utility function satisfying quasi-linearity only on $L \times (d \cdot \mathbb{Z})$, but violating it on $L \times (\mathbb{R} \backslash (d \cdot \mathbb{Z}))$.

(ii) For each $l \in M$, $p_l^{\min}(u) = p_l^V(u)$.

Therefore, on the quasi-linear domain, an MPE can be implemented via the sealed-bid Vickrey auction and an equivalence exists between the MPE mechanism and the Vickrey mechanism. The efficiency and strategy-proofness of the Vickrey mechanism on the quasi-linear domain comes from Holmstrom (1979). In Section 6.2, we demonstrate that none of these results hold on the classical domain.

Finally, we overview the results of AA auction. Some factors might affect the AA auction's outcome. First, it is possible to form the bidding queue in myriad ways. The initial order when agents stand in a queue has $|N|!$ variants. Several possibilities open up when an agent is replaced and is driven back into the queue of uncommitted agents, there are also several possibilities. That agent could be placed first in the queue, second in the queue, or so forth. Moreover, different agents may be treated differently as well. This implies that the number of variants of queues of uncommitted agents is much greater than $|N|!$. Definitely, the outcomes of the AA auction are predicated on how the bidding queue is formed. Second, when an agent is called to bid, her demand set *at the price that agent faces* may contain several objects, and the agent is supposed to bid on one of them. Each uncommitted agent could bid arbitrarily on any object in her demand set. Therefore, the outcome of the AA auction relies on agents' bidding choices even if the utility profile is fixed. As shown below, the results of the AA auction and the AA mechanism are robust to these factors.

Fact 6: Let $u \in (\mathcal{U}^{QL})^N$, $d > 0$, and the bidding queue q be arbitrarily given.
(i)[15] (Roughtgarden, 2014) The AA auction with increment d and bidding queue q finds a $d-$equilibrium in a finite number of rounds.
(ii) (**Deviation bound**) (Demange et al. 2016) Let $p(u)$ be the price generated by the AA auction in (i). For each $l \in M$, $\left|p_l(u) - p_l^{\min}(u)\right| \leq d \cdot \min\{|M|, |N|\}$.

Fact 6(i) states that the outcome of AA auction is an approximate equilibrium by setting $\varepsilon = d$, i.e., $d-$equilibrium. Fact 6(ii) is the most interesting property of the AA auction. It mentions that for any quasi-linear utility profile, the deviation $\left|p_l(u) - p_l^{\min}(u)\right|$ is bounded by $d \cdot \min\{|M|, |N|\}$, which is independent of the bidding queues and agents' choices from their demand sets. Thus, as d goes to zero, the outcome of the AA auction converges to an MPE. Unlike Fact 3, Fact 6 enables agents' valuations to be arbitrary non-negative reals and does not necessitate the coincidence assumption.

The AA mechanism is neither efficient nor strategy-proof, but it does manage to achieve some degrees of approximate efficiency and approximate strategy-

[15] Step 1 in the proof of Proposition 9 gives an independent proof of Fact 6(i).

proofnes.

Fact 7: Let $d > 0$, $k = 2 \cdot \min\{|M|, |N|\}$, and a bidding queue q be arbitrarily given. The AA mechanism with the increment d and the bidding queue q on $(\mathcal{U}^{QL})^N$ is $d-$efficient and $k \cdot d-$strategy-proof.

Indeed, $d-$equilibrium is $d-$efficient,[16] which ensures the $d-$efficiency of the AA mechanism. The $k \cdot d-$strategy-proofness of AA auction is shown by Roughgarden (2014). Fact 7 implies that the AA auction works well on the quasi-linear domain even in the absence of the coincidence assumption. Nevertheless, it fails to work on the classical domain. We discuss this point in Section 6.2.

6 Implementability of MPE on the classical domain

To emphasize the feature of our model with heterogenous objects, we discuss the case of homogenous objects where agents have classical utility functions. Homogeneity implies that for each payment, an agent's utility hinges only on whether she is assigned an object or the null; for this reason, each object gives her the same utility level. In such cases, the MPE price is the same across all the objects, represented by a single number, and the MPE assignment is determined by the ranking of generalized valuations. Put differently, the MPE assigns objects to agents whose generalized valuations are among $|M|\, th$-highest, and have them pay a price equal to the $(|M| + 1)th$-highest generalized valuation, whereas the remaining agents get the null and pay nothing.[17] When it comes to homogeneous objects, the MPE allocation is exactly the generalized sealed-bid Vickrey allocation and the generalized Vickrey mechanism is the MPE mechanism, which, in turn, satisfies efficiency and strategy-proofness (Saitoh and Serizawa, 2008).

However, when objects are heterogenous and agents have classical utility functions, the information of the generalized valuation is insufficient to determine the MPE. In such an environment, for each payment, an agent's utility depends on

[16] We demonstrate that $d-$equilibrium is $d-$efficient. By contradiction, suppose that there is a $d-$equilibrium (x, p), which is not $d-$efficient. Let $z \in Z$ be the allocation compatible with (x, p), i.e., for each $i \in N$, $t_i = p_{x_i}$. Thus there is $z' \in Z$ such that (i) for each $i \in N$, $u_i(z'_i) > u_i(z_i)$, and (ii) $\sum_{i \in N} t_i + d \cdot |N| \leq \sum_{i \in N} t'_i$. By (ii), there is $i \in N$ such that $z'_i = (x'_i, t'_i)$ and $t'_i \geq p_{x'_i} + d$. If not, i.e., for each $i \in N$, $t'_i < p_{x'_i} + d$, then $\sum_{i \in N} t'_i < \sum_{i \in N} p_{x'_i} + d \cdot |N| \leq \sum_{x \in M} p_x + d \cdot |N| = \sum_{i \in N} t_i + d \cdot |N|$, contradicting (ii). Thus by (i), $u_i(x'_i, p_{x'_i} + d) \geq u_i(z'_i) > u_i(z_i)$, contradicting $x_i \in D^d_i(p)$.

[17] To be precise, if $|M| \geq |N|$ or the $(|M| + 1)th-$highest generalized valuation is non-positive, the MPE price is zero. Otherwise, the MPE price is the $(|M| + 1)th-$highest generalized valuation.

which object is assigned to her, and not merely on whether she is assigned an object or the null. Moreover, an agent's preference over object also depends on payments. For example, at the payment t, an agent may prefer object a to b, but at t', e.g., $t' > t$, the opposite may occur, i.e., the agent prefers b to a. Notably, such a complexity never occurs in the case of homogenous objects or in the quasi-linear settings.

In the following, we demonstrate that the above complexity results in the poor performance of the auctions studied in Section 5.

6.1 The exact ascending auction

Since the coincidence assumption requires the quasi-linearity, a violation of quasi-linearity assumption leads to a violation of the coincidence assumption. We further argue that if we arbitrarily select a utility profile from $(\mathcal{U})^N$, in most cases, it does not come from $(\mathcal{U}^d)^N$ defined in Remark 1 and so the generalized coincidence assumption, as well as the coincidence assumption, is violated. Let $\mathcal{U}^d_* \equiv \{u_i \in \mathcal{U} : \forall l, l' \in L, \forall k, k' \in d \cdot \mathbb{Z},\ u_i(l,k) \neq u_i(l',k')\}$ be the set of utility functions that represents the strict preferences over $L \times (d \cdot \mathbb{Z})$. It is noteworthy that $\mathcal{U}^d_*$ is indeed open and dense in $\mathcal{U}$, and so $\mathcal{U}^d \subseteq \mathcal{U} \diagdown \mathcal{U}^d_*$ is non-generic in $\mathcal{U}$. This implies that if we arbitrarily select a utility profile from $(\mathcal{U})^N$, in most cases it will come from $(\mathcal{U}^d_*)^N$, instead of $(\mathcal{U}^d)^N$, and so the argument holds.

In the following, we examine the performance of EA auction by dropping the coincidence assumption, but still keep quasi-linearity. We show that the EA auction and EA mechanism perform poorly without the coincidence assumption.

First, we demonstrate that even a small "perturbation" of $\mathcal{U}^{QLd}_{\mathbb{N}}$ may lead to the break-down of the EA auction. We fix an increment $d > 0$. For agent i, let $\mathcal{U}^{QLd}_{\mathbb{Q}_+} \equiv \{u_i \in \mathcal{U}^{QL} : \forall l \in M, v_i(l) \in d \cdot \mathbb{Q}_+\}$. Given $\delta > 0$, let $\mathcal{U}^{QLd}_{\mathbb{N}}(\mathbb{Q}_+, \delta) \equiv \{u_i \in \mathcal{U}^{QLd}_{\mathbb{Q}_+} : \exists \overline{u}_i \in \mathcal{U}^{QLd}_{\mathbb{N}}$ such that $\forall a \in M, |v_i(a) - \overline{v}_i(a)| \leq d\delta\}$. It is easily seen that $\mathcal{U}^{QLd}_{\mathbb{N}} \subsetneq \mathcal{U}^{QLd}_{\mathbb{N}}(\mathbb{Q}_+, \delta) \subseteq \mathcal{U}^{QLd}_{\mathbb{Q}_+}$, that all the three sets are countable, and that when $\delta \to 0$, $\mathcal{U}^{QLd}_{\mathbb{N}}(\mathbb{Q}_+, \delta) \to \mathcal{U}^{QLd}_{\mathbb{N}}$.

Example 1 shows that for an arbitrarily small $\delta > 0$, the EA auction does not work.

Example 1: Let $d > 0$, $r > 0$, $0 < \delta < 1$, and $t \in \mathbb{N}_+$ be such that $(t-1) \cdot d > r$. Let $M = \{a, b\}$ and $N = \{1, 2\}$. Let $u \in \mathcal{U}^{QLd}_{\mathbb{N}}(\mathbb{Q}_+, \delta)$ be represented by a valuation profile $(v_1(\cdot), v_2(\cdot))$ such that:
(1) $v_1(a) = (t + \alpha_1) \cdot d$ and $v_1(b) = (t + \beta_1) \cdot d$.
(2) $v_2(a) = (t + \alpha_2) \cdot d$ and $v_2(b) = (t + \beta_2) \cdot d$.
(3) $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Q}_+$, $0 < \alpha_1 < \beta_1 < 1$, $0 < \alpha_2 < \beta_2 < 1$, and $\beta_1 - \alpha_1 < \beta_2 - \alpha_2$.

(4) $\beta_1 < \delta$ and $\beta_2 < \delta$.[18]
We remark that the following analysis depends only on (1), (2), and (3), and that (4) is just used to specify where $(v_1(\cdot), v_2(\cdot))$ comes from. There are several ways to perturb valuation profiles in $\mathcal{U}_{\mathbb{N}}^{QLd}$ to get $(v_1(\cdot), v_2(\cdot))$, by varying (4).

Note that

$$\begin{aligned} v_1(0) &= v_2(0) = 0 \underset{(3)}{<} (\beta_1 - \alpha_1) \cdot d \underset{(1)}{=} v_1(b) - v_1(a) \\ &\qquad \underset{(3)}{<} v_2(b) - v_2(a) \underset{(2)}{=} (\beta_2 - \alpha_2) \cdot d \underset{(3)}{<} d. \end{aligned} \tag{$*$}$$

The MPE $(x, p^{\min}(u))$ is: $x = (x_1, x_2) = (a, b)$ and $p^{\min}(u) = (p_0^{\min}, p_a^{\min}, p_b^{\min}) = (0, 0, (\beta_1 - \alpha_1) \cdot d)$.

We confirm that $(x, p^{\min}(u))$ is an MPE. For agent 1,

$$v_1(a) - p_a^{\min} = v_1(b) - p_b^{\min} = (t + \alpha_1) \cdot d > 0 = v_1(0) - p_0^{\min}$$

and so $x_1 = a \in D_1(p^{\min}(u)) = \{a, b\}$. For agent 2,

$$v_2(b) - p_b^{\min} = (t + \beta_2 - \beta_1 + \alpha_1) \cdot d \underset{(3)}{>} (t + \alpha_2) \cdot d = v_2(a) - p_a^{\min} > 0 = v_2(0) - p_0^{\min}$$

and so $x_2 = b \in D_2(p^{\min}(u)) = \{b\}$. Thus Definition 3(i) holds and Definition 3(ii) holds vacuously. Thus $(x, p^{\min}(u))$ is an equilibrium.

We use Fact 1(ii) to show that $(x, p^{\min}(u))$ is an MPE. Note that $\{i \in N : D_i(p^{\min}(u)) \subseteq \{a\}\} = \emptyset$, $\big|\{i \in N : D_i(p^{\min}(u)) \subseteq \{b\}\big| = |\{2\}| = 1 = |\{b\}|$, and $\big|\{i \in N : D_i(p^{\min}(u)) \subseteq \{a, b\}\big| = 2 = |\{a, b\}|$. Thus no set of objects is overdemanded at $p^{\min}(u)$. Since $p_b^{\min} > 0$, $\big|\{i \in N : D_i(p^{\min}(u)) \cap \{b\} \neq \emptyset\}\big| = |\{1, 2\}| = 2 > 1 = |\{b\}|$. Thus no set of object is weakly underdemanded, as desired. Since $x_2 = b \in D_1(p^{\min}(u)) = \{a, b\}$ and $x_1 = a$ with $p_a^{\min} = 0$, b is connected via agent 1's demand set. Therefore, Fact 2 is also illustrated.

We now illustrate how the EA auction proceeds. The auction starts from round 0 with the initial price $p^0 = (0, 0, 0)$. At p^0, by $(*)$, both agents prefer b to a and 0, i.e., $D_1(p^0) = D_2(p^0) = \{b\}$. Therefore, only b is minimally overdemanded at p^0 and so, its price p_b^0 is raised by d while the price of a remains unchanged. Therefore, we proceed to round 1 with $p^1 = (0, 0, d)$. At p^1, by $(*)$, both agents prefer a to b and 0, i.e., $D_1(p^1) = D_2(p^1) = \{a\}$. Therefore, only a is minimally overdemanded at p^1 and so, its price p_a^1 is raised by d while the price of b does not change. Therefore, we proceed to round 2 with $p^2 = (0, d, d)$ and so on.

[18] When δ is given, there is $k \in \mathbb{N}_+$ such that $10^{-k} \leq \delta$. For example, we can set $\alpha_1 = 0.1/10^k$, $\beta_1 = 0.5/10^k$, $\alpha_2 = 0.2/10^k$, and $\beta_2 = 0.8/10^k$. Then (1) to (4) hold.

At round $2t+1$ with price $p^{2t+1} = (0, t \cdot d, (t+1) \cdot d)$, by $(*)$, both agents prefer a to b and 0, i.e., $D_1(p^{2t+1}) = D_2(p^{2t+1}) = \{a\}$. Therefore, only a is minimally overdemanded at p^{2t+1} and so, its price p_a^{2t+1} is raised by d while the price of b does not change. The auction proceeds to round $2t+2$ with $p^{2t+2} = (0, (t+1) \cdot d, (t+1) \cdot d)$. At price p^{2t+2}, no agent is willing to obtain either a or b, and they prefer to drop the auction, i.e., $D_1(p^{2t+2}) = D_2(p^{2t+2}) = \{0\}$. At price p^{2t+2}, there are no sets of objects that are minimally overdemanded. Therefore, the auction terminates at p^{2t+2}, with an allocation that each agent receives the null object and pays nothing.

In the above case, note that $p_a^{2t+2} - p_a^{\min} = (t+1) \cdot d > r$ and $p_b^{2t+2} - p_b^{\min} = (t + 1 - (\beta_1 - \alpha_1)) \cdot d > t \cdot d > r$. $\triangle$

In Example 1, since r can be arbitrarily large, the EA auction may overshoot the MPE price by a substantial margin. Example 1 can be easily generalized to cases of more agents and objects. Proposition 1 summarizes this overshooting result.

Proposition 1 (**Substantial overshooting**): For each $d > 0$, each arbitrarily large $r > 0$, and each arbitrarily small $\delta > 0$, there is $u \in (\mathcal{U}_{\mathbb{N}}^{QLd}(\mathbb{Q}_+, \delta))^N$ such that the EA auction with increment d generates a price p with $p_l > p_l^{\min}(u) + r$ for each $l \in M$.

It is easy to confirm that no equilibrium is compatible with discrete prices in $(d \cdot \mathbb{N})^L$ in Example 1. One may wonder whether this nonexistence result is a key factor that results in the large overshooting of the EA auction. Thus, we consider approximate equilibria where the deviation is set to be the increment, i.e., $\varepsilon = d$. The prices compatible with $d-$equilibria for the utility profile u in Example 1 are such that $p = (0, k \cdot d, k \cdot d)$ for $k = 0, 1, \ldots, t$, or $p' = (0, k' \cdot d, (k'+1) \cdot d)$ for $k' = 0, 1, \ldots, t-1$. However, in Example 1, the outcome price is not among them, and moreover, the outcome assignment is not a $d-$equilibrium assignment either. Thus, the EA auction does not find a $d-$equilibrium.

In the following, we argue that the EA auction largely overshoot even the $d-$equilibrium price that is "closest" to the MPE price. Formally, given a utility profile $u \in \mathcal{U}^N$, a $d-$equilibrium price $p^A(u)$ is called *a closest $d-$equilibrium price* to $p^{\min}(u)$ if there is no other $d-$equilibrium price p' for that utility profile such that for each $l \in M$, $\left|p_l^A(u) - p_l^{\min}(u)\right| \geq \left|p'_l - p_l^{\min}(u)\right|$ with at least one strict inequality. The price $p^A(u)$ approximates $p^{\min}(u)$ among all $1-$equilibrium prices. In Example 1, if $\beta_1 - \alpha_1 < 0.5$, $p^0 = (0, 0, 0)$ is a closest $d-$equilibrium price to $p^{\min}(u)$ among all $d-$equilibrium prices.[19] The insight of large overshooting from

[19]If $\beta_1 - \alpha_1 > 0.5$, $p^1 = (0, 0, d)$ is a closest $d-$equilibrium price to $p^{\min}(u)$ among all $d-$equilibrium prices. If $\beta_1 - \alpha_1 = 0.5$, both p^0 and p^1 are closest $d-$equilibrium prices to

Example 1 extends to a closest $d-$equilibrium price.

Proposition 2: For each $d > 0$, each arbitrarily large $r > 0$, and each arbitrarily small $\delta > 0$, there is $u \in (\mathcal{U}_{\mathbb{N}}^{QLd}(\mathbb{Q}_+, \delta))^N$ such that the EA auction with increment d generates a price p with $p_l \geq p_l^A(u) + r$ for each $l \in M$.

Propositions 1 and 2 demonstrate that the EA auction completely breaks down on $(\mathcal{U}_{\mathbb{N}}^{QLd}(\mathbb{Q}_+, \delta))^N$ even when δ is sufficiently small. Notably, the smaller δ is, the smaller $\mathcal{U}_{\mathbb{N}}^{QLd}(\mathbb{Q}_+, \delta)\backslash\mathcal{U}_{\mathbb{N}}^{QLd}$ will be. Thus, the operation of EA auction to get the MPE is rather demanding.

Corollary 1: Let $d > 0$, an arbitrarily small $\delta > 0$ be given, an arbitrarily large $r > 0$ be given, and $\mathcal{U}_{\mathbb{N}}^{QLd}(\mathbb{Q}_+, \delta) \subseteq \overline{\mathcal{U}} \subseteq \mathcal{U}$. There is $u \in \overline{\mathcal{U}}^N$ such that the EA auction with increment d generates a price p with $p_l \geq p_l^{\min}(u) + r$ and $p_l \geq p_l^A(u) + r$ for each $l \in M$.

Proposition 1 perturbs $\mathcal{U}_{\mathbb{N}}^{QLd}$ only by allowing agents' valuations to take positive *rational* numbers in the neighborhood of natural numbers. It concludes that there is *no* common increment such that for each $u \in \mathcal{U}_{\mathbb{N}}^{QLd}(\mathbb{Q}_+, \delta)$, the EA auction finds the associated MPE price $p^{\min}(u)$. However, for each $u \in \mathcal{U}_{\mathbb{N}}^{QLd}(\mathbb{Q}_+, \delta)$, there is an increment such that the EA auction finds the associated MPE price $p^{\min}(u)$.[20]

On the other hand, if we perturb $\mathcal{U}_{\mathbb{N}}^{QLd}$ by allowing agents' valuations to take positive *irrational* numbers in the neighborhood of natural numbers as well, we can conclude a stronger result as in illustrated by Example 2 below: For some valuation profile, *regardless of increments* the EA auction will always overshoot the MPE price and a closest $d-$equilibrium price.

Given $\delta > 0$, let $\mathcal{U}_{\mathbb{N}}^{QLd}(\mathbb{R}_+, \delta) \equiv \{u_i \in \mathcal{U}^{QL} : \exists \overline{u}_i \in \mathcal{U}_{\mathbb{N}}^{QLd}$ such that $\forall a \in M, |v_i(a) - \overline{v}_i(a)| \leq d\delta\}$. It is easily seen that $\mathcal{U}_{\mathbb{N}}^{QLd} \subsetneq \mathcal{U}_{\mathbb{N}}^{QLd}(\mathbb{R}_+, \delta) \subseteq \mathcal{U}^{QLd}$, and that when $\delta \to 0$, $\mathcal{U}_{\mathbb{N}}^{QLd}(\mathbb{R}_+, \delta) \to \mathcal{U}_{\mathbb{N}}^{QLd}$.

Example 2: Let $\overline{d} > 0$, $r > 0$, $0 < \delta < 1$, and $\lambda_1, \lambda_2, \lambda_3 \in \overline{d} \cdot \mathbb{N}_+$ be such that $r + 1 < \lambda_1 < \lambda_2 < \lambda_3$ be given. Let $M = \{a, b, c\}$ and $N = \{1, 2, 3\}$. Let $u \in (\mathcal{U}_{\mathbb{N}}^{QL\overline{d}}(\mathbb{R}_+, \delta))^3$ be represented by a valuation profile $(v_1(\cdot), v_2(\cdot), v_3(\cdot))$ such that:

$p^{\min}(u)$ among all $d-$equilibrium prices. It can be easily verified that Proposition 2 holds even when there are multiple closest $d-$equilibrium prices.

[20] To see this point, assume, w.l.o.g, that for each $i \in N$, there is an object $l \in M$ such that $v_i(l) > 0$, and let M_i' be the collection of such objects. For each $i \in N$ and each $l \in M_i'$, let $v_i(l) \in d \cdot \mathbb{Q}_+$ be such that $v_i(l) = d \cdot (p_i^l/q_i^l)$ where $p_i^l, q_i^l \in \mathbb{N}_+$ and p_i^l/q_i^l is irreducible, and let $d_i = d \cdot (\Pi_{l \in M'} \frac{1}{q_l^i})$. Then, if $d' = \Pi_{i \in N} d_i$ is the increment in the auction, agents' valuations are integer multiples of d'. For each $u \in \mathcal{U}_{\mathbb{Q}_+}^{QLd}$, an increment d' defined as above guarantees that the EA auction finds an MPE for u.

(1) $v_1(a) = \lambda_1$, $v_1(b) = \lambda_1 + \alpha$, and $v_1(c) = \lambda_1 + \beta$
(2) $v_2(a) = \lambda_2$, $v_2(b) = \lambda_2 + \alpha$, and $v_2(c) = \lambda_2 + \gamma$
(3) $v_3(a) = \lambda_3 - 1$, $v_3(b) = \lambda_3 + \alpha$, and $v_3(c) = \lambda_3 + \beta$.
(4) $\alpha, \beta, \gamma \in \mathbb{R}_+ \backslash \mathbb{Q}_+$, $\frac{\beta}{\alpha} \in \mathbb{R}_+ \backslash \mathbb{Q}_+$ and $\alpha < \gamma < \beta < 1$.
(5) $\beta < \overline{d}\delta$.[21]
Similarly to Example 1, we remark that the following analysis relies only on (1), (2), (3), and (4), and that (5) is just used to specify where $(v_1(\cdot), v_2(\cdot), v_3(\cdot))$ comes from, and one may verify that by varying (5), there are several ways to perturb valuation profiles in $\mathcal{U}_{\mathbb{N}}^{QLd}$ to get $(v_1(\cdot), v_2(\cdot), v_3(\cdot))$.

The MPE price for u specified above is $p^{\min}(u) = (p_0^{\min}, p_a^{\min}, p_b^{\min}, p_c^{\min}) = (0, 0, \alpha, \beta)$.

Let $m_l = 0$ if $l = a$, $m_l = \alpha$ if $l = b$, and $m_l = \beta$ if $l = c$. We show that for each increment $d > 0$, the EA auction finds a price $p \equiv (0, p_a, p_b, p_c)$ such that $p_l \geq \lambda_1 + m_l$ for each $l \in M$.

We proceed by contradiction. Suppose that for some increment $d > 0$, the EA auction finds a price p such that there is a non-empty set $M' \subseteq M$ such that each $l \in M'$, $p_l < \lambda_1 + m_l$ and for each $l \in M \backslash M'$, $p_l \geq \lambda_1 + m_l$.

We begin by proving Claims 1 and 2 below.

Claim 1: (a) There is no set of objects that is overdemanded at $(0, p_a, p_b, p_c)$.
(b) For each $i \in N$, each $l \in M$, $v_i(l) - (\lambda_1 + m_l) \geq 0$.
(c) For each $i \in N$, $D_i(p) \subseteq M$.

Part (a): If there is a set of objects that is overdemanded at $(0, p_a, p_b, p_c)$, then it is easy to verify that set contains a minimally overdemanded set at $(0, p_a, p_b, p_c)$, contradicting the termination of the EA auction.
Part (b): For agent 1, for each $l \in M$, $v_1(l) = \lambda_1 + m_l$. For agent 2, if $l \in \{a, b\}$, $v_2(l) - (\lambda_1 + m_l) = \lambda_2 - \lambda_1 > 0$. If $l = c$, $v_2(c) - (\lambda_1 + m_l) = \lambda_2 - \lambda_1 + \gamma - \beta$. Since $\lambda_1, \lambda_2 \in \mathbb{N}_+$, $\lambda_2 > \lambda_1$, and (4), it holds $\lambda_2 - \lambda_1 + \gamma - \beta \geq 0$ and so $v_2(c) - (\lambda_1 + m_l) \geq 0$. For agent 3, if $l \in \{b, c\}$, $v_3(l) - (\lambda_1 + m_l) = \lambda_3 - \lambda_1 \geq 0$. If $l = a$, $v_3(a) - (\lambda_1 + m_l) = \lambda_3 - 1 - \lambda_1$. Since $\lambda_1, \lambda_3 \in \mathbb{N}_+$ and $\lambda_3 > \lambda_1$, it holds that $\lambda_3 - 1 - \lambda_1 \geq 0$ and so $v_3(a) - (\lambda_1 + m_l) \geq 0$. Thus, (b) holds.
Part (c): For each $i \in N$, each $l \in D_i(p)$, and each $l' \in M'$,

$$v_i(l) - p_l \geq v_i(l') - p_{l'} \underset{l' \in M'}{>} v_i(l') - (\lambda_1 + m_{l'}) \underset{\text{Claim 1(b)}}{\geq} 0,$$

so $0 \notin D_i(p)$ and $D_i(p) \subseteq M$.

Claim 2: (a) $p_b = p_a + \alpha$, and (b) $p_c = p_b + \beta - \alpha$.

[21] When δ is given, there is $k \in \mathbb{N}_+$ such that $10^{-k} \leq d\delta$. For example, we can set $\alpha = (\sqrt{5} - \sqrt{3})/10^{k+1}$, $\beta = (\sqrt{11} - \sqrt{3})/10^{k+1}$, and $\gamma = (\sqrt{7} - \sqrt{3})/10^{k+1}$. Then (1) to (4) hold.

Part (a): Suppose that $p_b \neq p_a + \alpha$. We consider two cases, i.e., $p_b > p_a + \alpha$ (Case 1), or $p_b < p_a + \alpha$ (Case 2), and derive a contradiction in each case.

Case 1: $p_b > p_a + \alpha$

For each $i \in \{1, 2\}$,

$$v_i(a) - p_a \underset{(1)\&(2)}{=} v_i(b) - \alpha - p_a \underset{\text{Case 1}}{>} v_i(b) - p_b,$$

so $b \notin D_i(p)$. Thus, by Claim 1(c), for each $i \in \{1, 2\}$, $D_i(p) \subseteq M\backslash\{b\} = \{a, c\}$. If $b \notin D_3(p)$, then by Claim 1(c) $D_3(p) \subseteq M\backslash\{b\} = \{a, c\}$, and so $\{a, c\}$ is overdemanded, contradicting Claim 1(a). Thus $b \in D_3(p)$. Since $b \in D_3(p)$ implies $v_3(b) - p_b \geq v_3(c) - p_c$, by (3), we have $p_c \geq p_b + \beta - \alpha$. Thus by $p_b > p_a + \alpha$, we have $p_c > p_a + \beta$. Therefore, for each $i \in \{1, 2\}$,

$$v_i(a) - p_a \underset{(1),(2)\&(4)}{\geq} v_i(c) - \beta - p_a \underset{p_c > p_a + \beta}{>} v_i(c) - p_c,$$

so $c \notin D_i(p)$. Thus for each $i \in \{1, 2\}$, by Claim 1(c) and $D_i(p) \subseteq \{a, c\}$, we have $D_1(p) = D_2(p) = \{a\}$, contradicting Claim 1(a).

Case 2: $p_b < p_a + \alpha$

For each $i \in \{1, 2\}$,

$$v_i(b) - p_b \underset{(1)\&(2)}{=} v_i(a) + \alpha - p_b \underset{\text{Case 2}}{>} v_i(a) - p_a,$$

so $a \notin D_i(p)$. Thus for each $i \in \{1, 2\}$, by Claim 1(c), $D_i(p) \subseteq M\backslash\{a\} = \{b, c\}$. If $a \notin D_3(p)$, then by Claim 1(c), $D_3(p) \subseteq M\backslash\{a\} = \{b, c\}$, and so $\{b, c\}$ is overdemanded, contradicting Claim 1(a). Thus $a \in D_3(p)$. Since $a \in D_3(p)$ implies $v_3(a) - p_a \geq v_3(c) - p_c$, by (3), we have $p_c \geq p_a + \beta + 1$. Thus by $p_b < p_a + \alpha$, we have $p_c > p_b - \alpha + \beta + 1 > p_b + \beta - \alpha$. Note that by (1), $v_1(b) = v_1(c) + \alpha - \beta$, and by (2) and $\gamma < \beta$, $v_2(b) = v_2(c) + \alpha - \gamma > v_2(c) + \alpha - \beta$. Thus, for each $i \in \{1, 2\}$,

$$v_i(b) - p_b \geq v_i(c) - (\beta - \alpha) - p_b \underset{p_c > p_b + \beta - \alpha}{>} v_i(c) - p_c,$$

so $c \notin D_i(p)$. Thus, for each $i \in \{1, 2\}$, by Claim 1(c) and $D_i(p) \subseteq \{a, c\}$, we have $D_1(p) = D_2(p) = \{a\}$, contradicting Claim 1(a).

Thus, $p_b = p_a + \alpha$.

Part (b): Suppose that $p_c \neq p_b + \beta - \alpha$. We consider two cases, i.e., $p_c > p_b + \beta - \alpha$ (Case 1), or $p_c < p_b + \beta - \alpha$ (Case 2), and derive a contradiction in each case.

Case 1: $p_c > p_b + \beta - \alpha$

For each $i \in \{1, 3\}$,

$$v_i(b) - p_b \underset{(1)\&(3)}{=} v_i(c) + \alpha - \beta - p_b \underset{\text{Case 1}}{>} v_i(c) - p_c,$$

so $c \notin D_i(p)$. Thus, by Claim 1(c), for each $i \in \{1,3\}$, $D_i(p) \subseteq M \backslash \{c\} = \{a,b\}$. Note that

$$v_2(b) - p_b \underset{(2)}{=} v_2(c) + \alpha - \gamma - p_b \underset{\beta>\gamma}{>} v_2(c) + \alpha - \beta - p_b \underset{\text{Case 1}}{>} v_2(c) - p_c$$

so $c \notin D_2(p)$. Thus by Claim 1(c), $D_2(p) \subseteq M\backslash\{c\} = \{a,b\}$. Thus for each $i \in \{1,2,3\}$, $D_i(p) \subseteq \{a,b\}$ and $\{a,b\}$ is overdemanded, contradicting Claim 1(a).

Case 2: $p_c < p_b + \beta - \alpha$

Note that by Claim 2(a), i.e., $p_b = p_a + \alpha$, (1) implies $v_1(b) - p_b = v_1(a) - p_a$, while (3) implies $v_3(b) - p_b = v_3(a) + 1 - p_a > v_3(a) - p_a$. Thus for each $i \in \{1,3\}$,

$$v_i(c) - p_c \underset{(1)\&(3)}{=} v_i(b) - \alpha + \beta - p_c \underset{\text{Case 2}}{>} v_i(b) - p_b \geq v_i(a) - p_a$$

so $a, b \notin D_i(p)$. Thus for each $i \in \{1,3\}$, by Claim 1(c), $D_i(p) = \{c\}$. Thus $D_1(p) = D_3(p) = \{c\}$ and $\{c\}$ is overdemanded, contradicting Claim 1(a).

Thus $p_c = p_b + \beta - \alpha$. Thus Claim 2 holds.

Since the EA auction stops at $(0, p_a, p_b, p_c)$, there are $k_a, k_b, k_c \in \mathbb{N}$ such that $p_a = k_a d$, $p_b = k_b d$, and $p_c = k_c d$. By Claim 2 and (4), we have $p_b - p_a = (k_b - k_a)d = \alpha > 0$ and $p_c - p_b = (k_c - k_b)d = \beta - \alpha > 0$. Therefore,

$$\frac{k_c - k_b}{k_b - k_a} = \frac{p_c - p_b}{p_b - p_a} = \frac{\beta - \alpha}{\alpha}.$$

Note that by (4), $\frac{\beta-\alpha}{\alpha}$ is an irrational number. However, by $k_a, k_b, k_c \in \mathbb{N}$, $(k_c - k_b)/(k_b - k_a)$ is a rational number. This is a contradiction.

Thus we conclude that for each increment $d > 0$, the EA auction finds a price $(0, p_a, p_b, p_c)$ such that $p_a \geq \lambda_1$, $p_b \geq \lambda_1 + \alpha$, and $p_c \geq \lambda_1 + \beta$.

By (4) and $\lambda_1 > r + 1$, for each $l \in M$, $p_l > p_l^{\min}(u) + r$. $\triangle$

Example 2 can be easily generalized to cases of more agents and objects. Example 2 also works to demonstrate that for any positive increment d, the EA auction is always largely overshooting a closest $d-$equilibrium price, which can be skipped to avoid the redundancy. Proposition 3 summarizes the insights of Example 2.

Proposition 3: Let $d > 0$. For an arbitrarily large $r > 0$ and an arbitrarily small $\delta > 0$, there is $u \in (\mathcal{U}_{\mathbb{N}}^{QLd}(\mathbb{R}_+, \delta))^N$ such that for any positive increment, the EA auction generates a price p with $p_l \geq p_l^{\min}(u) + r$ and $p_l \geq p_l^A(u) + r$ for each $l \in M$.

Proposition 3 strengthens the results in Propositions 1 and 2, by further enlarging the range of agents' valuations slightly.

In the following, Example 1 is used to study the $r-$efficiency and $r-$strategy-proofness of the EA mechanism.

Example 3: Consider the economy in Example 1. We first consider the efficiency issue. Let $z_1 = (a, (t-1) \cdot d)$ and $z_2 = (b, (t-1) \cdot d)$. For each $i \in \{1, 2\}$, $u_i(z_i) > u_i(0, 0) = 0$, and $Rev(z) = (t + t - 2) \cdot d > 2r > 0$. Thus, the EA mechanism fails to find an $r-$efficient allocation for u.

Next, we consider the incentive issue. Let $u'_2 \in \mathcal{U}^{QL}$ be denoted by $v'_2(\cdot)$ such that $v'_2(a) = (t + \beta_2) \cdot d$ and $v'_2(b) = (t + \alpha_2) \cdot d$. Then, agent 2 with u'_2 demands only object a at the initial price $p^0 = (0, 0, 0)$, i.e., $D'_2(p^0) = \{a\}$ while as analyzed in Example 1, $D_1(p^0) = \{b\}$. Since no set of objects is minimally overdemanded, the EA auction concludes at p^0, and agents 1 and 2 receive objects b and a with no payment. Since $t-1 > r > 0$, it holds that $u_2(a, 0) = (t+\alpha_2) \cdot d > u_2(0, -r) = v_2(0) + r = r$. As a result, agent 2 benefits more than r from misreporting $v'_2(\cdot)$ when her true valuation function is $v_2(\cdot)$. Therefore, the EA mechanism with increment d is $r-$manipulable. $\triangle$

Example 3 can easily be generalized to cases of more agents and objects, which indicates the following result.

Proposition 4: Let $d > 0$, an arbitrarily small $\delta > 0$ be given, and $\mathcal{U}_{\mathbb{N}}^{QLd}(\mathbb{Q}_+, \delta) \subseteq \overline{\mathcal{U}} \subseteq \mathcal{U}^{QL}$. The EA mechanism with increment d is absolutely inefficient and absolutely manipulable on $\overline{\mathcal{U}}^N$.

By Proposition 4, we have the following result.

Corollary 2: Let $d > 0$, an arbitrarily small $\delta > 0$ be given, and $\mathcal{U}_{\mathbb{N}}^{QLd}(\mathbb{Q}_+, \delta) \subseteq \overline{\mathcal{U}} \subseteq \mathcal{U}$. The EA mechanism with increment d is absolutely inefficient and absolutely manipulable on $\overline{\mathcal{U}}^N$.

We end this section by discussing the predicament of extending the continuous-time clock auction for one object with quasi-linear utility functions to our model for heterogeneous objects with classical utility functions.

Continuous-time clock auction*:* The continuous-time clock auction, originated from the Japanese simultaneous-bidding auction (Cassady, 1967), is a particular type of the English auction for selling only one object. The price increases continuously at some rate kept by a clock, and agents drop out at some point. The auction stops a price when all agents drop out except for one and the remaining agent wins the object and pays that stopping price. This auction duplicates the MPE mechanism for one object.

We begin by discussing its extension to multiple heterogenous objects. Let $M = \{a, b\}$ and $N = \{1, 2, 3\}$. Let $u \in (\mathcal{U}^{QL})^3$ be such that $v_1(a) = 3$, $v_1(b) = 1$,

$v_2(a) = 1$, $v_2(b) = 3$, and $v_3(a) = v_3(b) = 2$. In such a case, the MPE $(x, p^{\min}(u))$ is $x = (x_1, x_2, x_3) = (a, b, 0)$ and $p^{\min}(u) = (p_0^{\min}, p_a^{\min}, p_b^{\min}) = (0, 2, 2)$.

One possible extension of the continuous-time clock auction is that each agent chooses an object from her demand at the current price, and bids on it. The prices of minimally overdemanded objects are increased continuously at the same rate. At $p^0 = (0, 0, 0)$, $D_1(p^0) = \{a\}$, $D_2(p^0) = \{b\}$, and $D_3(p^0) = \{a, b\}$. Thus at p^0, agent 1 bids on a, agent 2 bids on b, and agent 3 bids on a or b. Whenever the price of a (or b) is higher than b (or a), agents 3 demands and bids on b (or a). Therefore, beginning from $(0, 0, 0)$, if the price increases continuously, agents 3's bid needs to move between a and b continuously until the price reaches $p^{\min}(u) = (0, 2, 2)$. Such a bidding behavior is physically impossible. Furthermore, the associated price path to $p^{\min}(u)$ is not well-defined.

Another possible extension is that each agent reports her demand set at the current price, and the prices of minimally overdemanded objects are increased continuously at the same rate. Indeed, this is *the continuous variant of the EA auction*. It works well on the quasi-linear domain.[22] However, such an auction is fraught with problems on the classical domain. Consider a classical utility function $u_3' \in \mathcal{U}$ of agent 3 such that (i) $u_3'(0, 0) = u_3'(a, 1) = u_3'(b, 2)$, and (ii) $u_3'(a, t) = u_3'(b, 2t)$ for each $t \in [0, 1]$. Let $u' = (u_1, u_2, u_3')$. Then the MPE price is $p^{\min}(u') = (0, 1, 2)$. Note that for $p_a \in [0, 1]$, $D_3(p) = \{a\}$ if $p_b > 2p_a$, $D_3(p) = \{b\}$ if $p_b < 2p_a$, $D_3(p) = \{a, b\}$ if $p_b = 2p_a$. Thus, starting from $(0, 0, 0)$, if the prices of minimally overdemanded objects increase continuously with the same rate, agents 3 needs to move between $\{a, b\}$ and $\{b\}$ continuously, which is physically impossible. Note that in such a case, the price path to $p^{\min}(u') = (0, 1, 2)$ is not well-defined either.

Morimoto and Serizawa (Proposition 1, 2015) show that if the price path generated by the aforementioned continuous variant of the EA auction is well-defined, then such an auction converges to the MPE price with multiple objects in a finite time on the classical domain. However, their result does not guarantee the existence of such a price path. Our discussion demonstrates that the price path of the EA auction's continuous variant may not be well-defined for some classical utility profiles. In particular, it is both theoretically and practically impossible to use such an auction on the classical domain as an alternative option to approximate or duplicate the MPE mechanism.

[22] If the prices increase continuously along the path $p(t) = (0, t, t)$ where $t \in [0, 2]$, agents 1, 2 and 3 keep reporting $D_1(p(t)) = \{a\}$, $D_2(p(t)) = \{b\}$, and $D_3(p(t)) = \{a, b\}$, respectively. Finally, the price reaches $p^{\min}(u) = (0, 2, 2)$.

6.2 The generalized sealed-bid Vickrey auction and approximate ascending auction

In contrast to the EA auction, the operations of the sealed-bid Vickrey auction and the AA auction are not predicated on the coincidence assumption and work well on the quasi-linear domain. However, in this section, we show that the nice properties of those two auctions elucidated in Section 5 do not hold on the classical domain.

6.2.1 The generalized sealed-bid Vickrey auction

Example 4 below highlights that both the assignments and prices are different between the MPE and generalized sealed-bid Vickrey on the classical domain.

Example 4: Let $M = \{a, b\}$ and $N = \{1, 2, 3\}$. Let $r > 0$ and $u \in \mathcal{U}^3$ be such that:
(1) $V_1(a) = r$ and $V_1(b) = 2r$.
(2) $V_2(a) = 20r$, $V_2(b) = 10r$, $u_2(a, r) = u_2(b, 0)$, and $u_2(a, 3r) = u_2(b, 2r)$.
(3) $V_3(a) = 10r$, $V_3(b) = 30r$, $u_3(a, 3r) = u_3(b, -2r)$, and $u_3(a, 8r) = u_3(b, 2r)$.

The generalized sealed-bid Vickrey outcome $(x^V, p^V(u))$ is $x^V = (x_1^V, x_2^V, x_3^V) = (0, a, b)$ and $p^V(u) = (p_0^V, p_a^V, p_b^V) = (0, r, 2r)$. In this a case, the generalized sealed-bid Vickrey assignment is unique.

We show how to calculate $(x^V, p^V(u))$. It is not hard to see that assigning a to agent 2 and b to agent 3 while agent 1 gets the null object maximizes the sum of generalized valuations $V_1(\cdot) + V_2(\cdot) + V_3(\cdot)$, which is equal to $50r$. Thus x^V satisfies Definition 7(i). In the absence of agent 2, assigning b to agent 3 and assigning a to agent 1 maximizes the sum of generalized valuations $V_1(\cdot) + V_3(\cdot)$, which is equal to $31r$. Therefore, by Definition 7(ii), agent 2 gets a and pays $V_1(a) + V_3(b) - (V_1(x_1^V) + V_3(x_3^V)) = r$. Without agent 3, assigning a to agent 2 and assigning b to agent 1 maximizes the sum of generalized valuations $V_1(\cdot) + V_2(\cdot)$, which is equal to $22r$. Therefore, by Definition 7(ii), agent 2 gets a and pays $V_1(b) + V_2(a) - (V_1(x_1^V) + V_2(x_2^V)) = 2r$.

The MPE $(x, p^{\min}(u))$ is $x = (x_1, x_2, x_3) = (0, b, a)$ and $p^{\min}(u) = (p_0^{\min}, p_a^{\min}, p_b^{\min}) = (0, 3r, 2r)$. In this case, the MPE assignment is unique as well.

Thus, $p^{\min}(u) \neq p^V(u)$ and $x \neq x^V$. △

Since Example 4 can be easily generalized to cases of more agents and objects, we have Proposition 5.

Proposition 5: There is $u \in \mathcal{U}^N$ such that (i) for each $l \in M$, $p_l^{\min}(u) \neq p_l^V(u)$, and (ii) there is no intersection between the set of the MPE assignments and the set of the generalized sealed-bid Vickrey assignments for u.

We discuss the intuition behind Proposition 5 via Example 4. By its definition, the generalized sealed-bid Vickrey price is determined only by $\{V_i(\cdot)\}_{i\in N}$. However, the information in $\{V_i(\cdot)\}_{i\in N}$ is not sufficient to determine equilibrium prices. Indeed, at the generalized sealed-bid Vickrey price $p^V(u) = (0, r, 2r)$, since $u_i(a, r) > u_i(b, 2r) > u_i(0, 0)$ for $i = 2, 3$, agents 2 and 3 both demand only a, and so $p^V(u)$ is not an equilibrium price. Put differently, the determination of the generalized sealed-bid Vickrey price and equilibrium price may require different parts of preference information in $\{u_i(\cdot)\}_{i\in N}$, which may not convey in $\{V_i(\cdot)\}_{i\in N}$. Therefore, the different usage of preference information results in various outcomes. Moreover the larger deviation of utility profiles from quasi-linearity may lead to larger differences in such outcomes. Indeed, that the generalized sealed-bid Vickrey price is not an equilibrium price; it is rather a robust property on the classical domain although the generalized sealed-bid Vickrey prices could be equilibrium prices for some utility profiles.[23]

In contrast to the classical domain, on the quasi-linear domain, since for each agent i, $v_i(\cdot)$ contains all the information of her preference $u_i(\cdot)$, the generalized sealed-bid Vickrey price is an equilibrium price as well. For instance, consider a utility profile $u' \in (\mathcal{U}^{QL})^3$ in Example 4 that for each $i \in \{1, 2, 3\}$, $v_i'(\cdot) = V_i(\cdot)$. Then, $p^V(u') = p^V(u) = (0, r, 2r)$, and $x'^V = x^V = (0, a, b)$. Since $u_1'(0, 0) = u_1'(a, r) = u_1'(b, 2r)$, $u_2'(a, r) > u_2'(b, 2r) > u_2'(0, 0)$, and $u_3'(b, 2r) > u_2'(a, r) > u_2'(0, 0)$, each agent demands her assignment at x'^V. Thus, $(x'^V, p^V(u'))$ is an equilibrium for u'.

Next, we show the inefficiency and manipulability of the generalized Vickrey mechanism.

Example 5: Consider the economy in Example 4. Let $z_1 = (0, -0.5r)$, $z_2 = (b, -0.5r)$, and $z_3 = (a, 7.5r)$. For each $i \in \{1, 2, 3\}$, $u_i(z_i) > u_i(x_i^V, p^V_{x_i^V})$, and $Rev(z) = 6.5r > p_0^V + p_a^V + p_b^V + 3r = 6r$. Thus (x^V, p^V) is not $r-$efficient for u.

Let $u_3' \in \mathcal{U}$ be such that $V_3'(a) = 35r$, $V_3'(b) = 8r$, and $u_3'(a, 10r) = u_3'(b, 4r)$.

[23] Fix $r > 0$ and u_1 as shown in Example 4. Let $\mathcal{U}_2(20r; 10r) \equiv \{u_2(\cdot,\cdot) \in \mathcal{U} : V_2(a) = 20r, V_2(b) = 10r\}$, and $\mathcal{U}_3(10r; 30r) \equiv \{u_3(\cdot,\cdot) \in \mathcal{U} : V_3(a) = 10r, V_3(b) = 30r\}$. Any pair $(u_2, u_3) \in \mathcal{U}_2(20r; 10r) \times \mathcal{U}_3(10r; 30r)$ of utility functions of agents 2 and 3, together with u_1, have the generalized sealed-bid Vickrey price $p^V = (0, r, 2r)$ and the associated assignment $x^V = (0, a, b)$.

However, for any utility profile $u' \in \{u_1\} \times \mathcal{U}_2(20r; 10r) \times \mathcal{U}_3(10r; 30r)$ such that $u_2'(a, r) < u_2'(b, 2r)$ or $u_3'(b, 2r) < u_3'(a, r)$, (x^V, p^V) is not an equilibrium. Thus, for these utility profiles, the generalized sealed-bid Vickrey outcome is not an equilibrium.

For any utility profile $u' \in \{u_1\} \times \mathcal{U}_2(20r; 10r) \times \mathcal{U}_3(10r; 30r)$ such that $u_2'(a, r) \geq u_2'(b, 2r)$ and $u_3'(b, 2r) \geq u_3'(a, r)$, (x^V, p^V) is an equilibrium. Thus, for these utility profiles, the generalized sealed-bid Vickrey outcome is an equilibrium.

Agent 3 obtains $(a, 12r)$ under the generalized sealed-bid Vickrey auction for the utility profile (u_1, u_2, u_3'). Since $u_3'(b, 2r) > u_3'(b, 4r) = u_3'(a, 10r) > u_3'(a, 12r)$, agent 3 benefits from reporting u_3 when her true utility function is u_3'. Thus, it can be inferred that the generalized Vickrey mechanism is $r-$manipulable. $\triangle$

Since Example 5 can be easily generalized to cases of more agents and objects, we have Proposition 6.

Proposition 6: The generalized Vickrey mechanism is absolutely inefficient and absolutely manipulable on $\mathcal{U}^N$.

Morimoto and Serizawa (2015) characterize the MPE mechanism via efficiency, strategy-proofness, and some mild axioms on the classical domain with some mild restriction. In their Section 6.2, an example is cited to show that the MPE mechanism is different from the generalized Vickrey mechanism, which, coupled with their characterization of the MPE mechanism, indicates that the generalized Vickrey mechanism is neither efficient nor strategy-proof. Proposition 5 is congruent with their result, but Proposition 6 complements their result by arguing that when the deviation of utility profiles from quasi-linearity is large, the generalized Vickrey mechanism cannot achieve even approximate efficiency and approximate strategy-proofness to any degree.

6.2.2 The approximate ascending auction

Next, we study whether the AA auction works as predicted by Facts 6 and 7 when agents have the classical utility functions.

In an economy with two objects and three agents, Example 6 below demonstrates that the AA auction substantially undershoots the MPE price even if only one agent has the classical utility function.

Example 6: Let $M = \{a, b\}$ and $N = \{1, 2, 3\}$. The bidding queue is in the order of agent 1, 2 and 3, without any specification about the manner in which uncommitted agents are driven back into the queue in the auction. Let $d > 0$ be the increment and $r > 0$ be a large number. Let $t \in \mathbb{N}_+$ be such that $t \cdot d > r$ and $u \in \mathcal{U}^3$ be such that:

(1) Let $u_1 \in \mathcal{U}^{QL}$ be such that $u_1(a, p_a) = td - p_a$ and $u_1(b, p_b) = 3td - p_b$.

(2) Let $u_2 \in \mathcal{U}$ be such that

$$u_2(a, 0) > u_2(b, d) > u_2(a, 0.5d) = u_2(b, 2td) > 0 = u_2(0, 0).^{24}$$

[24] For example, u_2 can take the form of $u_2(a, p_a) = 20td - (4t - 1.8)p_a$ and $u_2(b, p_b) = (20t + 0.9)d - p_b$.

(3) Let $u_3 \in \mathcal{U}^{QL}$ be such that $u_3(a, p_a) = 0.5d - p_a$ and $u_3(b, p_b) = 0.6d - p_b$.
The MPE $(x, p^{\min}(u))$ is $x = (x_1, x_2, x_3) = (b, a, 0)$ and $p^{\min}(u) = (p_0^{\min}, p_a^{\min}, p_b^{\min}) = (0, 0.5d, 2td)$.

We illustrate how the AA auction proceeds. AA auction begins from the initial price $\widehat{p}^0 = (0, 0, 0)$. At $\widehat{p}^0$, agent 1 bids first. Agent 1 only demands b at p^0 since $u_1(b, 0) = 3td > u_1(a, 0) = td > u_1(0, 0) = v_1(0) = 0$. Thus $D_1(p^0) = \{b\}$, she bids on b and is tentatively assigned $(b, 0)$. The market price generated in round 0 is $p^0 = (0, 0, 0)$ and auction proceeds to round 1. Then agent 2 is called. By the definition of the AA auction, if agent 2 bids b, the price of b that she faces is updated, i.e., $\widehat{p}_b^1 = p_b^0 + d = d$ while the price of a that she faces remains the same, i.e., $\widehat{p}_a^1 = p_a^0 = 0$. Agent 2 only demands a at $\widehat{p}^1 = (0, 0, d)$ since $u_2(a, 0) > u_2(b, d) > u_2(0, 0)$. Thus $D_2(\widehat{p}^1) = \{a\}$, agent 2 bids on a and is tentatively assigned $(a, 0)$. Then, the market price generated in round 1 is $p^1 = (0, 0, 0)$ and the auction proceeds to round 2. Then agent 3 is called. If agent 3 bids a or b, the prices of a and b that she faces are updated, i.e., $\widehat{p}_a^2 = p_a^1 + d = d$ and $\widehat{p}_b^2 = p_b^1 + d = d$. Since agent 3 only demands the null object at $\widehat{p}^2 = (0, d, d)$, i.e., $D_3(\widehat{p}^2) = \{0\}$, agent 3 drops out. Since there is no uncommitted agent in the queue, the auction terminates at round 2 with an unchanged market price $p^2 = p^1 = (0, 0, 0)$ and agents 1, 2, and 3 get $(b, 0)$, $(a, 0)$ and $(0, 0)$, respectively. Notice that $p_b^2 = 0$ and $p_b^{\min} - p_b^2 = 2td > r$. Thus, the AA auction generates zero revenue and p_b is smaller than $p_b^{\min}$ by an arbitrarily large amount r. $\triangle$

Next, we use Example 7 to demonstrate that the AA auction may substantially overshoot the MPE price.

Example 7: Let $M = \{a, b\}$ and $N = \{1, 2, 3\}$. The biding queue is such that the initial order of agents is 1, 2 and 3, and when driven back, the uncommitted agent is placed last in the queue. Let $d > 0$ be the increment and $r > 0$ be a large number. Let $t \in \mathbb{N}_+$ be such that $t = 3k$ for some odd number $k \in \mathbb{N}_+$ and $(t - 2) \cdot d > r$. Let $u = (u_1, u_2, u_3) \in \mathcal{U}^{QL} \times \mathcal{U} \times \mathcal{U}$ satisfy the following:

$$
\begin{aligned}
u_1(a, p_a) &= -p_a, \text{ and } u_1(b, p_b) = (t + 0.1)d - p_b, \\
u_2(b, (t - 0.3)d) &> u_2(a, 0) \text{ and} \\
u_2(a, 0.5d) &= u_2(b, (t + 0.1)d) > u_2(0, 0) = u_2(a, (t - 0.5)d) = u_2(b, (t + 0.5)d), \\
u_3(b, (t - 0.4)d) &> u_3(a, 0) > u_3(0, -r) \text{ and} \\
u_3(a, 0.6d) &= u_3(b, td) > u_3(0, 0) = u_3(a, (t - 0.4)d) = u_3(b, (t + 0.4)d).
\end{aligned}
$$

The MPE $(x, p^{\min}(u))$ is $x = (x_1, x_2, x_3) = (0, b, a)$ and $p^{\min}(u) = (p_0^{\min}, p_a^{\min}, p_b^{\min}) = (0, 0.5d, (t + 0.1)d)$.

We illustrate how the AA auction proceeds. AA auction begins from the initial price $\widehat{p}^0 = (0,0,0)$. At $\widehat{p}^0$, agent 1 bids first. Since agent 1 only demands b at $\widehat{p}^0$, i.e., $D_1(\widehat{p}^0) = \{b\}$, she bids on b and is tentatively assigned $(b,0)$. The market price generated in round 0 is $p^0 = (0,0,0)$ and auction proceeds to round 1. Then agent 2 is called. By the definition of the AA auction, if agent 2 bids b, the price of b that she faces is updated, i.e., $\widehat{p}^1_b = p^0_b + d = d$ while the price of a that she faces remains unchanged, i.e., $\widehat{p}^1_a = p^0_a = 0$. Since agent 2 only demands b at $\widehat{p}^1 = (0,0,d)$, i.e., $D_2(\widehat{p}^1) = \{b\}$, agent 2 bids on b and is tentatively assigned (b,d). Then, agent 1 is placed last in the queue behind agent 3. Then, the market price generated in round 1 is $p^1 = (0,0,d)$ and the auction proceeds to round 2. Then agent 3 is called. By the definition of the AA auction, if agent 3 bids b, the price of b that she faces is updated, i.e., $\widehat{p}^2_b = p^1_b + d = 2d$ while the price of a that she faces remains the same, i.e., $\widehat{p}^2_a = p^1_a = 0$. Since agent 3 only demands b at $\widehat{p}^2 = (0,0,2d)$, i.e., $D_3(\widehat{p}^2) = \{b\}$, agent 3 bids on b and is tentatively assigned $(b,2d)$. Then, agent 2 is placed last in the queue behind agent 1. Then, the market price generated in round 2 is $p^2 = (0,0,2d)$ and the auction proceeds to round 3. Then agent 1 is called and so on.

Since $u_1(b,(t-1)d) > u_1(a,0)$, $u_2(b,(t-0.3)d) > u_2(a,0)$, and $u_3(b,(t-0.4)d) > u_3(a,0)$, three agents will just compete for b, which makes the auction proceed to round $t-1$. The market price generated in round $t-1$ is $p^{t-1} = (0,0,(t-1)d)$ and since $t = 3k$ for some odd number $k \in \mathbb{N}_+$, agent 3 gets $(b,(t-1)d)$; in the queue, agent 2 is behind agent 1. In round t, agent 1 is called. If agent 1 bids b, the price of b that she faces is updated, i.e., $\widehat{p}^t_b = p^{t-1}_b + d = td$ while the price of a that she faces remains unchanged, i.e., $\widehat{p}^t_a = p^{t-1}_a = 0$. Since agent 1 only demands b at $\widehat{p}^t = (0,0,td)$, i.e., $D_1(\widehat{p}^t) = \{b\}$, agent 1 bids on b and is tentatively assigned (b,td). Then, agent 3 is placed last in the queue behind agent 2. Then, the market price generated in round t is $p^t = (0,0,td)$ and the auction proceeds to round $t+1$.

In round $t+1$, agent 2 is called. If agent 2 bids b, the price of b that she faces is updated, i.e., $\widehat{p}^{t+1}_b = p^t_b + d = (t+1)d$ while the price of a that she faces remains unchanged, i.e., $\widehat{p}^{t+1}_a = p^t_a = 0$. Since agent 2 only demands a at $\widehat{p}^{t+1} = (0,0,(t+1)d)$, i.e., $D_2(\widehat{p}^{t+1}) = \{a\}$, agent 2 bids on a and is tentatively assigned $(a,0)$. The market price generated in round $t+1$ is $p^{t+1} = (0,0,td)$ and the auction then proceeds to round $t+2$. Agent 3 is called. If agent 3 bids b, the price of b that she faces is updated, i.e., $\widehat{p}^{t+2}_b = p^{t+1}_b + d = (t+1)d$ and if she bids a, the price of a that she faces is updated either, i.e., $\widehat{p}^{t+2}_a = p^{t+1}_a + d = d$. Since agent 3 only demands a at $\widehat{p}^{t+2} = (0,d,(t+1)d)$, i.e., $D_3(\widehat{p}^{t+2}) = \{a\}$, agent 3 bids on a and is tentatively assigned (a,d). Then, agent 2 is placed in the queue.

Agents 2 and 3 will recursively bid on a.

The market price generated by round $2t$ is $(0,(t-1)d,td)$. Since t is an odd number, agent 2 is tentatively assigned $(a,(t-1)d)$ while agent 1 is tentatively assigned (b,td). Agent 3 is in the queue. The auction proceeds to round $2t+1$ and agent 3 is called. If agent 3 bids b, the price of b that she faces is updated, i.e., $\widehat{p}_b^{2t+1} = p_b^{2t} + d = (t+1)d$ and if she bids a, the price of a that she faces is updated either, i.e., $\widehat{p}_a^{2t+1} = p_a^{2t} + d = td$. Since agent 3 only demands the null object at $\widehat{p}^{2t+1} = (0,td,(t+1)d)$, i.e., $D_3(\widehat{p}^{2t+1}) = \{0\}$, agent 3 drops out. The auction terminates at round $2t+1$ and the market price generated in round $2t+1$ is $p^{2t+1} = (0,(t-1)d,td)$ where agent 1 is assigned (b,td), agent 2 is assigned $(a,(t-1)d)$, and agent 3 drops out.

Note that $p_a - p_a^{\min} = (t-1.5)d > r$ can be arbitrarily large. $\triangle$

Examples 6 and 7 can be easily generalized to cases with more agents and objects and with an arbitrary bidding queue. Thus, we have Proposition 7.

Proposition 7: Let $d > 0$ be the increment and a bidding queue be arbitrarily given in the AA auction. Then, for an arbitrarily large $r > 0$, the following results hold.
(i) (**Substantial undershooting**) For each $a \in M$, there is $u \in \mathcal{U}^N$ such that AA auction generates price $p = (0,\ldots,0)$ with $p_a < p_a^{\min} - r$.
(ii) (**Substantial overshooting**) For each $a \in M$, there is $u' \in \mathcal{U}^N$ such that the AA auction generates price p with $p_a > p_a^{\min} + r$.

As seen in the case of the sealed-bid Vickrey auction, when the deviation of utility profiles from quasi-linearity is large, the outcome of the AA auction may not even approximate the MPE. Notably, by Proposition 7(i), it is possible for the AA auction to generate zero revenue even if some agent has a very high willingness to pay for some object.

In the following, let $p^d(u;q)$ be the price generated by the AA auction with increment d and a bidding queue q for $u \in \mathcal{U}^N$. By Proposition 7, we have the following result.

Corollary 3: There are no increment $d > 0$ and no bidding queue q such that for each $u \in \mathcal{U}^N$ and each $l \in M$, $\left|p_l^d(u;q) - p_l^{\min}(u)\right| \leq d \cdot \min\{|M|, |N|\}$.

Corollary 3 implies there is *no* increment across all the classical utility profiles such that the AA auction neither substantially overshoots nor undershoots the MPE price. In other words, Fact 6(ii) does not hold on the classical domain.

Finally, we study the incentive of AA auction.

Example 8: Consider the economy in Example 7 with an replacement of agent 3's utility function. Let $u_3' \in \mathcal{U}^{QL}$ be such that $v_3'(a) = 1.5d$ and $v_3'(b) = 2d$. Rounds

0 and 1 are the same in Example 7. Recall that the market price generated in round 1 is $p^1 = (0, 0, d)$ where agent 2 is tentatively assigned b and 1 stands behind 3 in the queue of uncommitted agents. The auction then proceeds to round 2 and agent 3 is called. If agent 3 bids b, the price of b that she faces is updated, i.e., $\widehat{p}_b^2 = p_b^1 + d = 2d$ whereas the price of a that she faces remains unchanged, i.e., $\widehat{p}_a^2 = p_a^1 = 0$. Since $v_3'(a) - \widehat{p}_a^2 = 1.5d > v_3'(b) - \widehat{p}_b^2 = v_3'(0) = 0$, agent 3 only demands a at $\widehat{p}^2 = (0, 0, 2d)$, i.e., $D_3(\widehat{p}^2) = \{a\}$ so she bids a and is tentatively assigned $(a, 0)$. The market price generated in round 2 is $p^2 = (0, 0, d)$. The auction proceeds to round 3 and agent 1 is called. If agent 1 bids b, the price of b that she faces is also updated, i.e., $\widehat{p}_b^3 = p_b^2 + d = 2d$ and if she bids a, the price of a that she faces is updated either, i.e., $\widehat{p}_a^3 = p_a^2 + d = d$. Since agent 1 only demands b at $\widehat{p}^3 = (0, d, 2d)$, i.e., $D_1(\widehat{p}^3) = \{b\}$, agent 1 bids b and is tentatively assigned $(b, 2d)$. Then, agent 2 is driven to the queue.

In the later round of the auction, agents 1 and 2 will compete for b. The market price generated by round $t + 1$ is $(0, 0, td)$ where agent 2 is tentatively assigned (b, td), agent 3 is tentatively assigned $(a, 0)$, and agent 1 is in the queue. The auction proceeds to round $t + 2$ and agent 1 is called. If agent 1 bids b, the price of b that she faces is updated as well, i.e., $\widehat{p}_b^{t+2} = p_b^{t+1} + d = (t+1)d$ and if she bids a, the price of a that she faces is updated as well, i.e., $\widehat{p}_a^{t+2} = p_a^{t+1} + d = d$. Since agent 1 only demands the null object at $\widehat{p}^{t+2} = (0, d, (t+1)d)$, i.e., $D_1(\widehat{p}^{t+2}) = \{0\}$, she drops out. The auction terminates at price $(0, 0, td)$ where agent 1 drops out, agent 2 gets (b, t), and agent 3 gets $(a, 0)$.

Recall that in Example 7, agent 3 obtains $(0, 0)$ for (u_1, u_2, u_3). Since $r > 0$, and $u_3(a, 0) > u_3(0, -r) > u_3(0, 0)$, when agent 3's true utility function is u_3, she has the incentive to misreport u_3'. $\triangle$

The insight of Example 8 can be extended to show the following result.

Proposition 8: For each $d > 0$, the AA mechanism with increment d and with an arbitrary bidding queue q is absolutely manipulable on $\mathcal{U}^N$.

Proposition 8 demonstrates that $k \cdot d-$strategy-proofness of Fact 7 does not hold on the classical domain. However, even if the AA auction may substantially undershoot or overshoot on the classical domain, it can still find an approximate equilibrium, i.e., $d-$equilibrium.[25] In other words, the AA mechanism still could achieve some degree of approximate efficiency. One may consider this point to be a merit of the AA auction in comparison to the EA auction.

In both Propositions 7 and 8, the increment is given regardless of utility profiles and the properties of AA auction and mechanisms are derived when agents have

[25] The statement is implied by Step 1 in the proof of Proposition 9.

the flexibility to alter heir utility functions. In the following, we show that in case of a fixed classical utility profile, if the increment is sufficiently small, the outcome price of an AA auction will be sufficiently close to the MPE price.

Proposition 9: Let $u \in \mathcal{U}^N$ be given. Let $\{d_n\}$ be a decreasing sequence such that for each $n \in \mathbb{N}_+$, $d_n > 0$ and $\lim_{n\to\infty} d_n = 0$. Let p^{d_n} be the price generated by the AA auction with increment d_n and with an arbitrary bidding queue. Then, $\lim_{n\to\infty} p^{d_n} = p^{\min}(u)$.

The proof of Proposition 9 is relegated to Appendix A. Proposition 9 implies that when a utility profile is fixed, the AA auction works well in approximating the MPE mechanism for a sufficiently small increment. Even when agents have classical utility functions, the AA auction is still a good candidate to obtain the MPE if we carefully choose the increment, so practitioners should limit themselves to auctions of this form and its variant such as the cumulative offer procedure mentioned in the introduction.

7 Conclusion

In this paper, we study multi-object auction models with unitary demand agents whose utility functions may not be quasi-linear. We contend that the exact ascending auction of Demange et al. (1986), the sealed-bid Vickrey auction, and the approximate ascending auction of Demange et al. (1986) finding the minimum price equilibrium in the quasi-linear settings, cannot identify or even approximate the MPEs and fail to be efficient and incentive-compatible. Our results allude to the challenges of efficient and incentive-compatible auction design when agents have utility functions without assuming quasi-linearity, and inspire development of novel analytical techniques in the future research.

Appendix A: Proof of Proposition 9

Let (x^{d_n}, p^{d_n}) be the outcome generated by the AA auction with increment d_n and with an arbitrarily given bidding queue. The proof comprises five steps.

Step 1: For each $d_n > 0$, (x^{d_n}, p^{d_n}) is a d_n−equilibrium.

Consider an agent i who drops out. Agent i bids 0 when she faces a price lower than or equal to $(0, (p_l^{d_n} + d_n)_{l\in M})$. Thus, $0 \in D_i^{d_n}(p^{d_n})$. Consider an agent i who obtains $x_i^{d_n} \in M$. She bids on $x_i^{d_n}$ when the price of $x_i^{d_n}$ that she faces is $p^{d_n}_{x_i^{d_n}}$ and the price of any other object $l \in M$ that she faces is less than or equal to $p_l^{d_n} + d_n$. Thus, $x_i^{d_n} \in D_i^{d_n}(p^{d_n})$. Thus, Definition 5(i) holds.

In the AA auction, whenever an object is bided on by some agent, it will keep getting assigned till the end. Thus, Definition 5(ii) holds.

Step 2: There is a convergent subsequence $\{p^{d'_n}\}$ in $\{p^{d_n}\}$ whose assignments remain the same.

For each $l \in M$ and each $n \in \mathbb{N}_+$, $0 \leq p_l^{d_n} \leq \max_{i \in N} V_i(l) + 2d_n$. Thus, $\{p^{d_n}\}$ contains a convergent subsequence $\{p^{d''_n}\}$. Since agents and objects are both finite, $\{p^{d''_n}\}$ contains a subsequence $\{p^{d'_n}\}$ whose assignments remain the same.

Step 3: $(x, p) \equiv \lim_{n\to\infty}(x^{d'_n}, p^{d'_n})$ is an equilibrium.

By Step 2, for each $n \in \mathbb{N}_+$, $x = x^{d'_n}$. By Definition 5(ii), Definition 3(ii) holds. Thus, we show Definition 3(i). For each $n \in \mathbb{N}_+$ and each $i \in N$, by Step 2, $x_i \in D_i^{d_n}(p^{d_n})$ and moreover $x_i \in D_i^{d_n}(p^{d_n})$ implies that for each $y \in M$, $u_i(x_i, p_{x_i}^{d_n}) \geq u_i(y, p_y^{d_n} + d_n)$ and $u_i(x_i, p_{x_i}^{d_n}) \geq u_i(0, 0)$. Thus, for each $y \in M$, $\lim_{n\to\infty} p^{d'_n} = p$ implies $u_i(x_i, p_{x_i}) \geq u_i(0, 0)$ and $u_i(x_i, p_{x_i}) \geq u_i(y, p_y)$. Thus $x_i \in D_i(p)$.

Step 4: Let (x, p) be the equilibrium at Step 3. Then $p = p^{\min}$.

Suppose $p \neq p^{\min}$. Since (x, p) is an equilibrium, by Fact 1, there is a weakly underdemanded set $M' \subseteq M$ at p, that is, for each $l \in M'$, $p_l > 0$ and $|\{i \in N : D_i(p) \cap M' \neq \emptyset\}| = |M'|$. Let $N' \equiv \{i \in N' : x_i \in M'\}$. Then, $N' = \{i \in N : D_i(p) \cap M' \neq \emptyset\}$ and for each $i \in N \backslash N'$ and each $l \in M'$, $u_i(x_i, p_{x_i}) > u_i(l, p_l)$. Thus, since for each $l \in M'$, $p_l > 0$, Definition 1(i) implies that there is $\delta > 0$ such that for each $l \in M'$, $p_l - \delta > 0$, and for each $i \in N \backslash N'$,

$$u_i(x_i, p_{x_i} + \delta) > u_i(l, p_l - 2\delta). \tag{a}$$

and since $\lim_{n\to\infty} d'_n = 0$ and $\lim_{n\to\infty} p^{d'_n} = p$, for some $d''_n \in \{d'_n\}$,
(b) $p_{x_i}^{d''_n} \leq p_{x_i} + d''_n$ and $d''_n \leq \delta$, and
(c) $p_l^{d''_n} \geq p_l - \delta > 0$ for each $l \in M'$.

Thus for each $i \in N \backslash N'$ and each $l \in M'$,

$$\begin{aligned} u_i(x_i, p_{x_i}^{d''_n}) &\underset{(b)}{\geq} u_i(x_i, p_{x_i} + d''_n) \\ &\underset{(b)}{\geq} u_i(x_i, p_{x_i} + \delta) \underset{(a)}{>} u_i(l, p_l - 2\delta) \underset{(c)}{\geq} u_i(l, p_l^{d''_n} - \delta) \underset{(b)}{\geq} u_i(l, p_l^{d''_n} - d''_n). \end{aligned}$$

Thus, no agent in $N \backslash N'$ bids an objects in M' when the price of $l \in M'$ reaches $p_l^{d''_n} - d''_n$. In contrast, by (c), l is assigned to some $i \in N'$ at price $p_l^{d''_n} - d''_n$. Thus, by $|N'| = |M'|$, the price of any object $l \in M'$ cannot be increased to $p_l^{d''_n}$, contradicting that $p^{d''_n}$ is the outcome price of auction with increment d''_n.

Step 5: $\lim_{n\to\infty} p^{d_n} = p^{\min}(u)$.

Recall that a bounded sequence converges if and only if any of its convergent subsequence has the same limit. Hence, we prove that any convergent subsequence in $\{p^{d_n}\}$ has the same limit. Let $\{p^{d'_n}\}$ and $\{p^{d''_n}\}$ be two convergent subsequences in $\{p^{d_n}\}$ such that $\lim_{n\to\infty} p^{d'_n} = p'$ and $\lim_{n\to\infty} p^{d''_n} = p''$. In the following, we show $p' = p''$. Then Step 5 holds, as desired.

Analogous to Step 2, there is a subsequence of $\{p^{d'_n}\}$ converging to p' whose assignments remain unchanged, say x'. By Step 3, (x', p') is an equilibrium and moreover, by Step 4, $p' = p^{\min}$. Similarly, there is a subsequence of $\{p^{d''_n}\}$ converging to p'' whose assignments remains unchanged, say x''. By Step 3, (x'', p'') is an equilibrium and moreover, by Step 4, $p'' = p^{\min}$. Since the MPE price is unique, $p' = p''$.

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