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A CHARACTERIZATION OF THE VICKERY RULE IN SLOT ALLOCATION PROBLEMS

Yu Zhou
Youngsub Chun
Shigehiro Serizawa

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The Institute of Social and Economic Research
Osaka University
6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan

A characterization of the Vickery rule in slot allocation problems*

Yu Zhou† Youngsub Chun‡ Shigehiro Serizawa§

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Abstract

We study the slot allocation problem where agents have quasi-linear single-peaked preferences over slots and identify the rules satisfying efficiency, strategy-proofness, and individual rationality. Since the quasi-linear single-peaked domain is not connected, the famous characterization of the Vickrey rule in terms of the three properties in Holmström (1979) cannot be applied. However, we are able to establish that *on the quasi-linear single-peaked domain, the Vickrey rule is still the only rule satisfying efficiency, strategy-proofness, and individual rationality.*

Keywords: Slot allocation problem, single-peakedness, efficiency, strategy-proofness, individual rationality, Vickrey rule

JEL Classification: C78, D44, D71

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†Graduate School of Economics, Kyoto University, Japan. E-mail: zhouyu_0105@hotmail.com

‡Department of Economics, Seoul National University, Republic of Korea. E-mail: ychun@snu.ac.kr

§Institute of Social and Economic Research, Osaka University, Japan. E-mail: serizawa@iser.osaka-u.ac.jp

1 Introduction

We study the slot allocation problem of deciding how to assign slots to agents and how much each agent should pay. We focus on the case where agents have quasi-linear single-peaked preferences over slots, represented by the single-peaked valuations of slots. We allow the possibility that some agent cannot get a slot, due to the facility capacity. The allocation of time slots in the gym or golf games matches our settings. For example, each gym member or golfer may have her most preferred starting time, and the preferred starting time may be different across gym members or golfers. Moreover, the capacity of gym or gold club may not be sufficient enough to accommodate all members.

We try to identify the rule satisfying *efficiency*, *strategy-proofness*, and *individual rationality*. An *allocation* specifies how slots are assigned to agents and how much each agent should pay. A *rule* is a mapping from the domain, i.e., the class of valuation profiles, to the set of allocations. An allocation is *efficient* if the assignment of slots maximizes the sum of all agents' valuations. *Efficiency* describes the property of a rule that for each valuation profile, the rule always selects an efficient allocation. *Strategy-proofness* states that for each agent and each valuation profile, revealing one's true valuation is a weakly dominant strategy. *Individual rationality* states that for each agent and each valuation profile, everyone should not be worse off than receiving no slot and paying nothing. This property guarantees the agents' voluntary participation.

On the quasi-linear domain, the Vickrey rule, originated by Vickrey (1961), Clarke (1971) and Groves (1973), satisfies the above-mentioned three properties. Holmström (1979) establishes a stronger result: on the smoothly connected domain, the Vickrey rule is the only rule satisfying those three properties. Since the quasi-linear domain is smoothly connected, the Holmström's characterization holds.

On the other hand, we are interested in the quasi-linear single-peaked domain, i.e., the class of single-peaked valuation profiles, which is a proper subdomain of the quasi-linear domain. The smaller a domain is, the weaker the properties of rules are, such as *efficiency* and *strategy-proofness*. Thus, smaller domains imply the possibility of better rules tailored to our problems. Although the Vickrey rule is *efficient*, *strategy-proof*, and *individually rational* in the quasi-linear single-peaked domain, it is not clear whether it is the only rule satisfying the three properties. In fact, since the quasi-linear single-peaked domain is not smoothly connected, the Holmström's technique cannot be applied in our setting to establish the characterization.

In this paper, we show that the Holmström's result still holds on our domain: *on the quasi-linear single-peaked domain, the Vickrey rule is the only rule satisfying* *efficiency, strategy-proofness, and individual rationality.*

On the quasi-linear domain with the identical objects, Ashlagi and Serizawa (2012) characterize the Vickrey rule in terms of strategy-proofness, anonymity in welfare, and individual rationality. In the task-assignment settings where individual rationality may not be respected, Yengin (2012) characterizes the class of egalitarian-equivalent Groves mechanisms and studies different fairness properties of the Groves mechanisms. In the queueing setting,[1] Chun et al. (2014) characterize the VCG mechanisms in terms of queue efficiency, strategy-proofness, and egalitarian equivalence. Kayı and Ramaekers (2010) and Chun et al. (2019) focus on the symmetrically balanced Vickrey mechanisms and characterize it by outcome efficiency, and strategy-proofness, together with some fairness axioms. Yengin and Chun (2020) characterize some subfamilies of the VCG mechanisms in terms of no-envy or solidarity properties.

On the non-quasi-linear domain, the minimum price (MP) rule is a natural extension of the Vickrey rule, which associates each preference profile to a MP equilibrium. Notice that on the quasi-linear domain, the MP equilibrium coincides with the Vickrey allocation (Leonard, 1983). However, on the non-quasi-linear domain, the MP equilibrium price in general has no closed-form expression as the Vickrey payment. In the settings of identical objects, Saitoh and Serizawa (2008) and Sakai (2008) characterize the MP rule by Pareto-efficiency, strategy-proofness, individual rationality, and non-negative payments. In the setting of heterogenous objects, Morimoto and Serizawa (2015) characterize the MP rule by Pareto-efficiency, strategy-proofness, individual rationality, and no subsidy for losers. In the generalized queuing settings, i.e., agents have the non-quasi-linear common-object-ranking preferences, Zhou and Serizawa (2018) characterize the MP rule by using the same axioms as Morimoto and Serizawa (2015).[2]

It is worth mentioning that Schummer and Vohra (2013), Hougaard et al. (2014), and Chun and Park (2017) also study the slot allocation problems. Schummer and Vohra (2013) study the reassignment of landing slots without transfers and focus on the mechanisms with incentive and property rights properties. They provide a new reassignment mechanism that respects these properties, and discuss the pros and cons between their mechanism and the reassignment algorithm currently used by the Federal Aviation Administration. Hougaard et al. (2014) consider the problem of assigning agents to slots on a line where each agent prefers to be served as close as possible to her target. They consider the aggregate gap minimizing methods that minimize the total gap between targets and assigned slots, in terms of both deterministic and probabilistic assignments. Chun and

[1] Chun (2016) gives a detailed survey of the queueing problem.

[2] In this paper, we do not impose no subsidy or the non-negative payment as an axiom. Instead, we assume that the payment in an allocation is always non-negative.

Park (2017) consider a problem of assigning slots to a group of agents with the identical unit cost. They form such a problem in the setting of bipartite graph, and present a simple way of identifying all efficient assignments. In addition, they also introduce the leximin and the leximax rules and discuss their properties.

The remainder is organized as follows: Section 2 defines the model. Section 3 defines the Vickrey rule. Section 4 shows the characterization result, together with its proof. Section 5 concludes.

2 The model

There is a non-empty finite set of agents N and a non-empty finite set of slots M. Not getting a slot is called receiving the dummy, denoted by 0. Let L be the set of slots, together with the dummy, i.e., $L \equiv M \cup \{0\}$. Each agent either receives a slot or the dummy. We denote the slot that agent $i \in N$ receives by $x_i \in L$. We denote the amount that agent i pays by $t_i \in \mathbb{R}$. A generic **bundle** for agent i is a pair $z_i \equiv (x_i, t_i) \in L \times \mathbb{R}$.

Each agent has a **quasi-linear** preference over $L \times \mathbb{R}$: There is a valuation function $v_i : L \to \mathbb{R}_+$ such that (i) $v_i(0) = 0$, (ii) for each $m \in M$, $v_i(m) > 0$, and (iii) for each pair $(m,t), (m',t') \in L \times \mathbb{R}$, $(m,t)\, R_i\, (m',t')$ if and only if $v_i(m) - t \geq v_i(m') - t'$. Let V_i be the class of agent i's valuation functions. Let $V \equiv V_1 \times \cdots \times V_n$ be the **quasi-linear domain**. A valuation profile is an element $v \equiv (v_i)_{i \in N} \in V$. Given $v \in V$ and $N' \subseteq N$, let $v_{N'} \equiv (v_i)_{i \in N'}$ and $v_{-N'} \equiv (v_i)_{i \in N \setminus N'}$.

Let m^* be the cardinality of M. Let $\pi \equiv (\pi(1), \ldots, \pi(m^*))$ be a permutation of slots in M where $\pi(1)$ denotes the first slot, $\pi(2)$ denotes the second slot, and so forth. For each pair $m, m' \in M$, $m >_\pi m'$ means that slot m has a higher rank than slot m' according to π.

Definition 1: A valuation function $v_i \in V_i$ is **single-peaked according to** π if there is $k \in M$ such that
(i) for each pair $m, m' \in M$ such that $k >_\pi m >_\pi m'$, $v_i(k) > v_i(m) > v_i(m')$.
(ii) for each pair $m, m' \in M$ such that $m' >_\pi m >_\pi k$, $v_i(k) > v_i(m) > v_i(m')$.

Let $V_i^{SP}(\pi)$ be the class of agent i's valuation functions satisfying single-peakedness according to π. Let $V^{SP}(\pi) \equiv V_1^{SP}(\pi) \times \cdots \times V_n^{SP}(\pi)$ be the **quasi-linear single-peaked domain according to** π. It is easily seen that $V^{SP}(\pi) \subsetneq V$. We assume that the slot permutation π is exogenously given and commonly known by all agents, and agents' valuation functions are privately known by themselves.

An **assignment** is an n-tuple $(x_1, \ldots, x_n) \in L^n$ such that for each pair $i, j \in N$, if $x_i \neq 0$ and $i \neq j$, then $x_i \neq x_j$, i.e., each slot is assigned to at most one agent. We denote the set of assignments by X. An **allocation** is an n-tuple $z \equiv (z_1, \ldots, z_n) \equiv$

$((x_1, t_1), \dots, (x_n, t_n)) \in [L \times \mathbb{R}_+]^n$ such that $(x_1, \dots, x_n) \in X$. In an allocation, the payment for each slot is non-negative. We denote the set of allocations by Z. Given $z \in Z$, we denote its assignment and payment (components) at z by $x \equiv (x_1, \dots, x_n)$ and $t \equiv (t_1, \dots, t_n)$, respectively.

A **rule** on V is a mapping f from V to Z. Given a rule f and $v \in V$, we denote the bundle assigned to agent i by $f_i(v) \equiv (x_i(v), t_i(v))$, where $x_i(v)$ denotes the assigned object and $t_i(v)$ the associated payment. We write

$$f(v) \equiv (f_i(v))_{i \in N},\ x(v) \equiv (x_i(v))_{i \in N}, \text{ and } t(v) \equiv (t_i(v))_{i \in N}.$$

We focus on the following three properties of a rule, "efficiency," "strategy-proofness," and "individual rationality."

First, we define the notion of efficiency. An allocation $z \in Z$ for $v \in V$ is efficient if its assignment maximizes the sum of agents' valuations, i.e., $x \in \arg\max_{x' \in X} \sum_{i \in N} v_i(x'_i)$.

Efficiency of a rule states that for each valuation profile, the rule chooses an efficient allocation.

Efficiency: For each $v \in V$, $x(v)$ is efficient for v.

Remark 1: An allocation $z \in Z$ for $v \in \mathcal{R}^n$ is Pareto-efficient if there is no allocation $z' \equiv (x', t') \in [L \times \mathbb{R}]^n$ such that (i) for each $i \in N$, $v_i(x'_i) - t'_i \geq v_i(x_i) - t_i$ with at least one strict relation, and (ii) $\sum_{i \in N} t_i \leq \sum_{i \in N} t'_i$.[3] A rule is *Pareto-efficient* if for each valuation profile, it selects a Pareto-efficient allocation. On the quasi-linear domain, efficiency is independent of payments. A rule is efficient if and only if it is Pareto-efficient.[4] This result does not hold if agents have non-quasi-linear preferences.

[3]There are two equivalent ways to define the Pareto-efficient allocation: (1) An allocation $z \in Z$ for $v \in \mathcal{R}^n$ is Pareto-efficient if there is no allocation $z' = (x', t') \in [L \times \mathbb{R}]^n$ such that (i) for each $i \in N$, $v_i(x'_i) - t'_i \geq v_i(x_i) - t_i$ with at least strict inequality, and (ii) $\sum_{i \in N} t_i = \sum_{i \in N} t'_i$; (2) An allocation $z \in Z$ for $v \in \mathcal{R}^n$ is Pareto-efficient if there is no allocation $z' = (x', t') \in [L \times \mathbb{R}]^n$ such that (i) for each $i \in N$, $v_i(x'_i) - t'_i = v_i(x_i) - t_i$, and (ii) $\sum_{i \in N} t_i < \sum_{i \in N} t'_i$.

[4]Let f be a rule on V. First, we show that if f is efficient, then f is Pareto-efficient. By contradiction, suppose that f is not Pareto-efficient. By (1) in footnote 3, there is a valuation profile v and an allocation $z' \in Z$ such that (i) for each $i \in N$, $v_i(x'_i) - t'_i \geq v_i(x_i(v)) - t_i(v)$, (ii) there is $j \in N$ such that $v_j(x'_j) - t'_j > v_j(x_j(v)) - t_j(v)$, and (iii) $\sum_{i \in N} t'_j = \sum_{i \in N} t_i(v)$. If we sum up (i) side-by-side for all the agents, then by (ii) and (iii), $\sum_{i \in N} v_i(x'_i) > \sum_{i \in N} v_i(x_i(v))$, contradicting that f is efficient.

Second, we show that if f is Pareto-efficient, then f is efficient. We prove the contra-positive argument: If f is not efficient, then f is not Pareto-efficient. If f is not efficient, then there is a valuation profile $v \in V$ and an assignment $x^* \in X$ such that $\sum_{i \in N} v_i(x^*_i) > \sum_{i \in N} v_i(x_i(v))$. Let $z' \equiv (x', t') \in Z$ be such that for each $i \in N$, $x'_i \equiv x^*_i$ and $t'_i \equiv v_i(x^*_i) - v_i(x_i(v)) + t_i(v)$. Then, for each $i \in N$, $v_i(x^*_i) - t'_i = v_i(x_i(v)) - t_i(v)$. Note that $\sum_{i \in N} t'_i = \sum_{i \in N} t_i(v) + \sum_{i \in N} v_i(x^*_i) -$

Second, we define the incentive property of a rule. *Strategy-proofness* states that no agent ever benefits from misrepresenting her valuation function.

Strategy-proofness: For each $v \in V$, each $i \in N$, and each $v_i' \in V_i$, $v_i(x_i(v)) - t_i(v) \geq v_i(x_i(v_i', v_{-i})) - t_i(v_i', v_{-i})$.

Individual rationality states that no agent is ever assigned a bundle that makes her worse off than she would be if she had received the dummy and paid nothing.

Individual rationality: For each $v \in V$ and each $i \in N$, $v_i(x_i(v)) - t_i(v) \geq 0$.

3 The Vickrey rule

In this section, we define the "Vickrey rule," and show the famous characterization of Vickrey rule in terms of efficiency, strategy-proofness, and individual rationality by Holmström (1979). In addition, we argue that Holmström's result cannot be applied to the quasi-linear single-peaked domain.

Definition 2: A rule f on V is called a **Vickrey rule** if
(i) for each $v \in V$, $x(v) \in \arg\max_{x' \in X} \sum_{i \in N} v_i(x_i')$.
(ii) for each $i \in N$, $t_i(v) \equiv \sigma_{-i}(v) - \sigma_{-i}^*(v)$ where $\sigma_{-i}(v) \equiv \max_{x \in X} \sum_{j \in N \setminus \{i\}} v_j(x_j)$ and $\sigma_{-i}^*(v) \equiv \sum_{j \in N \setminus \{i\}} v_j(x_j(v))$.

Remark 2: Let f and f' be two Vickrey rules. Then we have (i) $\sum_{i \in N} v_i(x_i(v)) = \sum_{i \in N} v_i(x_i'(v))$, and (ii) for each $i \in N$, $t_i(v) = t_i'(v)$. (i) follows from Definition 2(i) that the Vickrey assignment is efficient. However, it may not be unique since two assignments can be welfare-equivalent. On the other hand, the Vickrey payment $t(v)$ is always unique.

In the following, we define the "smooth connectedness" property of a domain, which is introduced by Holmström (1979).

Definition 3: A domain $V' \equiv V_1' \times \cdots \times V_n' \subseteq V$ is **smoothly connected** if for each $i \in N$, each $v_{-i} \in V_{-i}'$, each $v_i \in V_i'$, and each $v_i' \in V_i'$, there is one-dimensional parameterized family $V_i(v_i, v_i') \equiv \{v_i(\cdot; y) \in V_i' : y \in [0,1]\}$ of valuation functions such that:
(i) for each $x \in L$, $v_i(x;0) = v_i(x)$, and $v_i(x;1) = v_i'(x)$.
(ii) for each $x \in L$ and each $y \in [0,1]$, $\partial v_i(x;y) \backslash \partial y$ exists.
(iii) for each $y \in [0,1]$ and each $\widehat{x} = (\widehat{x}_i, \widehat{x}_{-i}) \in X^*(v_i, v_i'; v_{-i})$, there is $K \in (0, +\infty)$ such that $|\partial v_i(\widehat{x}_i; y) \backslash \partial y| \leq K$, where $X^*(v_i, v_i'; v_{-i}) \equiv \{\widehat{x} \in X : \exists y \in [0,1]$ s.t. $v_i(\widehat{x}_i; y) + \sum_{j \in N \setminus \{i\}} v_j(\widehat{x}_j) = \max_{\widetilde{x} \in X}\{v_i(\widetilde{x}_i; y) + \sum_{j \in N \setminus \{i\}} v_j(\widetilde{x}_j)\}\}$.

$\sum_{i \in N} v_i(x_i(v))$. Since $\sum_{i \in N} v_i(x_i^*) > \sum_{i \in N} v_i(x_i(v))$, then $\sum_{i \in N} t_i' > \sum_{i \in N} t_i(v)$. Thus, by (2) in footnote 3, $(x(v), t(v))$ is not Pareto-efficient, which implies that f is not Pareto-efficient.

It is not hard to verify that the quasi-linear domain V is smoothly connected. Although Holmström (1979) studies the public good model, his result implies that in our model, when each agent has quasi-linear preferences and receives at most one slot, Theorem H holds.

Theorem H (Holmström, 1979): Let $V' \subseteq V$ be smoothly connected. Then, a rule f on V' satisfies *efficiency*, *strategy-proofness*, and *individual rationality* if and only if it is a Vickrey rule.

However, the quasi-linear single-peaked domain $V^{SP}(\pi)$ is not smoothly connected. The following example details this point.

Example 1. Let $N = \{1\}$, $M = \{m_1, m_2\}$, and $\pi = (\pi(1), \pi(2)) = (m_1, m_2)$. Under our assumption on the quasi-linear domain, it holds that $V^{SP}(\pi) = V_1^{SP}(\pi) = \{v_1(\cdot) \in V_1 : v_1(0) = 0 \text{ and } v_1(m_1), v_i(m_2) > 0 \text{ such that } v_1(m_1) \neq v_1(m_2)\}$.

A domain $V' \subseteq V$ is **path connected** if for each $i \in N$, each $v_{-i} \in V'_{-i}$, each $v_i \in V'_i$, and each $v'_i \in V'_i$, there is one-dimensional parameterized family $V_i(v_i, v'_i) \equiv \{v_i(\cdot; y) \in V'_i : y \in [0,1]\}$ of valuation functions such that (i′) Condition (i) of Definition 3 holds and (ii′) for each $x \in L$ and each $y \in [0,1]$, $v_i(x; y)$ is continuous with respect to y. It is easy to see that smooth connectedness implies path connectedness. Thus, the violuation of path connectedness implies the violation of smooth connectedness.

In the following, we show that $V^{SP}(\pi)$ is not path connected. By contradiction, suppose that $V^{SP}(\pi)$ is path connected. Now consider two valuation functions $v_1(\cdot)$ and $v'_1(\cdot)$ such that $v_1(0) = 0$, $v_1(m_1) = 2$, and $v_1(m_2) = 1$; $v'_1(0) = 0$, $v'_1(m_1) = 1$, and $v'_1(m_2) = 2$. Then, there is one-dimensional parameterized family $V_1(v_1, v'_1) \equiv \{v_1(\cdot; y) \in V^{SP}(\pi) : y \in [0,1]\}$ of valuation functions satisfying (i′) and (ii′).

For each $y \in [0,1]$, let $h(y) \equiv v_1(m_1; y) - v_1(m_2; y)$. Then, by (ii′), $h(\cdot)$ is continuous. Moreover, by (i′), $h(0) = 1$ and $h(1) = -1$. Thus, by the intermediate-value theorem, there is $y \in [0,1]$ such that $h(y) = v_1(m_1; y) - v_1(m_2; y) = 0$, i.e., $v_1(m_1; y) = v_1(m_2; y)$, contradicting $v_1(\cdot; y) \in V^{SP}(\pi)$. Thus, $V^{SP}(\pi)$ is not path connected, and so it is not smoothly connected.

Example 1 implies that on the quasi-linear single-peaked domain, Theorem H cannot be applied. The next section provides a characterization of the Vickrey rule on the quasi-linear single-peaked domain. The following remark uses Example 1 to show that Theorem H cannot be applied to the "quasi-linear single-dipped domain" and "the linear preference domain with two extreme peaks," either.

Remark 3: (i) A valuation function $v_i \in V_i$ is *single-dippedness according to* π if there is $k \in M$ such that (i) for each pair $m, m' \in M$ such that $k >_\pi m >_\pi m'$, $v_i(k) < v_i(m) < v_i(m')$, and (ii) for each pair $m, m' \in M$ such that $m' >_\pi m >_\pi k$,

$v_i(k) < v_i(m) < v_i(m')$. Let $V_i^{SD}(\pi)$ be the class of agent i's valuation functions satisfying single-dippedness according to π. Let $V^{SD}(\pi) \equiv V_1^{SD}(\pi) \times \cdots \times V_n^{SD}(\pi)$ be the *quasi-linear single-dipped domain according to* π. In Example 1, $V^{SD}(\pi) = V^{SP}(\pi)$ and so Theorem H cannot be applied to $V^{SD}(\pi)$.
(ii) A single-peaked domain $V^{SP}(\pi)$ satisfies the *two-common-extreme-peakedness of slots according to* π if for each $i \in N$ and each $v_i \in V_i^{SP}(\pi)$, either $v_i(\pi(1)) >_\pi \cdots >_\pi v_i(\pi(m^*))$ or $v_i(\pi(m^*)) >_\pi \cdots >_\pi v_i(\pi(1))$. That is, the slots $\pi(1)$ and $\pi(m)$ are two peaks across all the agents. Let $V^L(\pi) \subseteq V^{SP}(\pi)$ be *the linear preference domain with two extreme peaks according to* π. In Example 1, $V^L(\pi) = V^{SP}(\pi)$ and so Theorem H cannot be applied to $V^L(\pi)$.

We end this section by providing two remarks. The first remark illustrates two proper subdomains of the quasi-linear single-peaked domains that satisfy the smooth connectedness condition.

Remark 4: (i) A single-peaked domain $V^{SP}(\pi)$ satisfies the *common-peakedness of slots according to* π if there is $k \in M$ such that for each $i \in N$, each $v_i \in V_i^{SP}(\pi)$, and each $m \in M \backslash \{k\}$, $v_i(k) > v_i(m)$. It means that the "peak" of valuations over slots is the same across all the agents. The single-peaked domain with the common-peakedness satisfies the smooth connectedness condition. Thus Theorem H can be applied.
(ii) A single-peaked domain $V^{SP}(\pi)$ satisfies *the common-ranking of slots according to* π if for each $i \in N$, each $v_i \in V_i^{SP}(\pi)$, and each pair $m, m' \in M$ with $m >_\pi m'$, $v_i(m) > v_i(m')$. It means that there is a common ranking of slots across all the agents. The single-peaked domain with the common-ranking property satisfies the smooth connectedness condition. Thus Theorem H can be applied.

The second remark illustrates a proper superdomain of the quasi-linear single-peaked domain that satisfy the smooth connectedness condition.

Remark 5: For each pair $m, m' \in M$, $m \geq_\pi m'$ means that slot m is ranked as high as slot m' according to π. A valuation function $v_i \in V_i$ is *single-plateaued according to* π if there are $k, k' \in M$ with $k \geq_\pi k'$ such that:
(i) for each $m \in M$ such that $k \geq_\pi m \geq_\pi k'$, $v_i(k) = v_i(m) = v_i(k')$.
(ii) for each $m, m' \in M$ such that $k' >_\pi m >_\pi m'$, $v_i(k) > v_i(m) > v_i(m')$.
(ii) for each $m, m' \in M$ such that $m' >_\pi m >_\pi k$, $v_i(k) > v_i(m) > v_i(m')$.

Let $V_i^{SPL}(\pi)$ be the class of agent i's valuation functions satisfying single-plateauedness according to π. Let $V^{SPL}(\pi) \equiv V_1^{SPL}(\pi) \times \cdots \times V_n^{SPL}(\pi)$ be *the quasi-linear single-plateaued domain according to* π. It is easily seen that $V^{SP}(\pi) \subsetneq V^{SPL}(\pi)$. The quasi-linear single-plateaued domain satisfies the smooth connectedness condition. Thus Theorem H can be applied. We conjecture that this is the minimal domain that includes the single-peaked domain where Theorem H can be applied.

4 Main result

We are ready establish our main result.

Theorem 1: A rule f on $V^{SP}(\pi)$ satisfies *efficiency*, *strategy-proofness*, and *individual rationality* if and only if it is a Vickrey rule.

On the quasi-linear domain, a Vickrey rule satisfies the three properties (Holmström, 1979). Thus the "if" part holds. We only need to show the "only if" part: A rule f on $V^{SP}(\pi)$ satisfying efficiency, strategy-proofness, and individual rationality is a Vickrey rule. We borrow the proof technique from Chew and Serizawa (2007) and apply it to our setting. We establish four lemmas to complete the proof.

Lemma 1 follows from *individual rationality.* It says that an agent who gets the dummy pays nothing. Lemma 2 follows from *efficiency.* It says that any efficient rule selects an assignment satisfying Definition 2(i). Lemma 3 follows from *strategy-proofness.* It says that if an agent truncates her valuations by following the "z_i*-favoring transformation,*" the new bundle that the agent gets remains the same as her original bundle. Lemma 4 says that if a rule satisfies *efficiency*, *strategy-proofness*, and *individual rationality*, then the payment of each agent is equal to the expression given in Definition 2(ii). In consequence, Lemmas 2 and 4 together establish the "only if" part. We relegate the complete proof to Section 4.1.

A rule f on V satisfies *budget balance* if for each $v \in V$, $\sum_{i \in N} t_i(v) = 0$. In general a Vickrey rule on the quasi-linear domain does not satisfy budget balance. It is also easy to see that a Vickrey rule on $V^{SP}(\pi)$ does not satisfy budget balance either. Therefore, from Theorem 1, we have the following result.

Corollary 1: There is no rule f on $V^{SP}(\pi)$ satisfying *efficiency*, *strategy-proofness*, *individual rationality*, and *budget balance.*

In the following, we discuss the indispensability of the three axioms. We show that the "only if" part of Theorem 1 fails if one of the axioms is dropped.

Example 2: (i) (Dropping *efficiency*) Let f be the "no-trade rule" such that for each $v \in V^{SP}(\pi)$, it assigns $(0, 0)$ to each agent. This rule satisfies strategy-proofness and individual rationality, but not efficiency.
(ii) (Dropping *strategy-proofness*) A rule f on $V^{SP}(\pi)$ is the zero-payment rule if for each $v \in V^{SP}(\pi)$, $x(v) \in \arg\max_{x' \in X} \sum_{i \in N} v_i(x'_i)$, and for each $i \in N$, $t_i(v) = 0$. This rule satisfies efficiency and individual rationality, but not strategy-proofness.
(iii) (Dropping *individual rationality*) Let $n = m + 1$ and $e = (e_i)_{i \in N} \in R^N_{++}$. A rule f on $V^{SP}(\pi)$ is a Vickrey rule with a positive entry fee e if for each $v \in V$, $x(v) \in \arg\max_{x' \in X} \sum_{i \in N} v_i(x'_i)$, and for each $i \in N$, $t_i(v) \equiv \sigma_{-i}(v) - \sigma^*_{-i}(v) + e_i$. A

Vickrey rule with positive entry fee e satisfies efficiency and strategy-proofness. Since $n = m + 1$, there is $i \in N$ such that $x_i = 0$. Since i pays a positive entry fee $e_i > 0$, then $0 > v_i(0) - e_i = -e_i$, violating individual rationality.

4.1 Proof of Theorem 1

In the following, we state four lemmas and show their proofs to formally establish the "only if" part of Theorem 1.

Since it is easy to verify that Lemma 1 and Lemma 2 hold, we omit their proofs.

Lemma 1: Let f satisfy *individual rationality.* Let $v \in V^{SP}(\pi)$. Then, for each $i \in N$, if $x_i(v) = 0$, then $t_i(v) = 0$.

Lemma 2: Let f satisfy *efficiency.* Let $v \in V^{SP}(\pi)$. Then $x(v) \in \arg\max_{x' \in X} \sum_{i \in N} v_i(x'_i)$.

Lemma 3 shows the implication of *strategy-proofness* via "*z_i-favoring transformation.*"

Definition 4: Given $z_i = (x_i, t_i) \in M \times \mathbb{R}_+$ and $v_i \in V^{SP}(\pi)$, $v'_i \in V^{SP}(\pi)$ is a *z_i-favoring transformation* of v_i at z_i if for each $y \in M \backslash \{x_i\}$, $v'_i(x_i) - v'_i(y) > t_i$.

Let $I(v_i, z_i)$ be the set of *z_i-favoring transformations* of v_i at z_i.

Lemma 3: Let f satisfy *strategy-proofness.* Let $v \in V^{SP}(\pi)$. Let $i \in N$, $x_i(v) \in M$, and $v'_i \in I(v_i, f_i(v))$. Then $f_i(v'_i, v_{-i}) = f_i(v)$.

Proof: Let $v' = (v'_i, v_{-i})$ and $y = x_i(v')$. *Strategy-proofness* implies

$$v_i(x_i(v)) - t_i(v) \geq v_i(y) - t_i(v'). \tag{1}$$

$$v'_i(y) - t_i(v') \geq v'_i(x_i(v)) - t_i(v). \tag{2}$$

In the following, we show that $x_i(v) = y$. By contradiction, suppose that $x_i(v) \neq y$. Since

$$\begin{aligned} t_i(v) - t_i(v') &\geq v'_i(x_i(v)) - v'_i(y) && \text{(by (2))} \\ &> t_i(v) && \text{(by } v'_i \in I(v_i, f_i(v))\text{)} \end{aligned}$$

then $t_i(v') < 0$, contradicting $f(v') \in Z$.

Thus $x_i(v) = y$. Then by (1), $t_i(v') \geq t_i(v)$, and by (2), $t_i(v) \geq t_i(v')$. Thus $t_i(v) = t_i(v')$ and so $f_i(v'_i, v_{-i}) = f_i(v)$. **Q.E.D.**

Lemma 4: Let f satisfy *efficiency*, *strategy-proofness*, and *individual rationality.* Let $v \in V^{SP}(\pi)$. Then, for each $i \in N$, $t_i(v) = \sigma_{-i}(v) - \sigma^*_{-i}(v)$.

Proof: Let $i \in N$. If $x_i(v) = 0$, by *efficiency*, it is easy to see that $\sigma_{-i}(v) = \sigma^*_{-i}(v)$. By *individual rationality* and Lemma 1, $t_i(v) = 0 = \sigma_{-i}(v) - \sigma^*_{-i}(v)$.

Let $x_i(v) \in M$. To show $t_i(v) = \sigma_{-i}(v) - \sigma^*_{-i}(v)$, we show the impossibility of $\sigma_{-i}(v) - \sigma^*_{-i}(v) < t_i(v)$ (Case 1) or $0 \leq t_i(v) < \sigma_{-i}(v) - \sigma^*_{-i}(v)$ (Case 2).

Case 1: $\sigma_{-i}(v) - \sigma^*_{-i}(v) < t_i(v)$

By Definition 2, $\sigma_{-i}(v) - \sigma^*_{-i}(v) \geq 0$. Let $v'_i \in V_i^{SP}$ be such that:

(1-i) $0 \leq \sigma_{-i}(v) - \sigma^*_{-i}(v) < v'_i(x_i(v)) < t_i(v)$,

(1-ii) for each $y \in M \backslash \{x_i(v)\}$, $0 < v'_i(y) < v'_i(x_i(v)) - \sigma_{-i}(v) + \sigma^*_{-i}(v)$.[5]

Let $v' = (v'_i, v_{-i})$ and $y = x_i(v')$. In the following, we show that $x_i(v) = y$. By contradiction, suppose that $x_i(v) \neq y$. We consider two cases.

Case A: $y = 0$.

In this case, $v'_i(y) = 0$, and so $v'_i(y) + \sum_{j \in N \backslash \{i\}} v_j(x_j(v')) = \sum_{j \in N \backslash \{i\}} v_j(x_j(v'))$. Since

$$\begin{aligned}
\sum_{j \in N \backslash \{i\}} v_j(x_j(v')) &\leq \sigma_{-i}(v) && \text{(by } v_i' = v_i) \\
&< \sigma^*_{-i}(v) + v'_i(x_i(v)) && \text{(by (1-i))} \\
&\leq \max_{x \in X}(\sum_{j \in N \backslash \{i\}} v_j(x_j) + v'_i(x_i)),
\end{aligned}$$

then $v'_i(y) + \sum_{j \in N \backslash \{i\}} v_j(x_j(v')) < \max_{x \in X}(\sum_{j \in N \backslash \{i\}} v_j(x_j) + v'_i(x_i))$, contradicting that $x(v')$ is *efficient*.

Case B: $y \in M \backslash \{x_i(v)\}$.

In this case, we have

$$\begin{aligned}
v'_i(y) + \sum_{j \in N \backslash \{i\}} v_j(x_j(v')) &\leq v'_i(y) + \sigma_{-i}(v) \\
&< v'_i(x_i(v)) + \sigma^*_{-i}(v) && \text{(by (1-ii))} \\
&\leq \max_{x \in X}(\sum_{j \in N \backslash \{i\}} v_j(x_j) + v'_i(x_i)).
\end{aligned}$$

Thus $v'_i(y) + \sum_{j \in N \backslash \{i\}} v_j(x_j(v')) < \max_{x \in X}(\sum_{j \in N \backslash \{i\}} v_j(x_j) + v'_i(x_i))$, contradicting that $x(v')$ is *efficient*.

Altogether, it holds that $x_i(v) = y = x_i(v')$. By *individual rationality*, $t_i(v') \leq v'_i(x_i(v))$. By (i), $t_i(v') \leq v'_i(x_i(v)) < t_i(v)$. Thus $v_i(y) - t_i(v') > v_i(x_i(v)) - t_i(v)$, contradicting *strategy-proofness*.

Case 2: $0 \leq t_i(v) < \sigma_{-i}(v) - \sigma^*_{-i}(v)$

In such a case, there is $\delta > 0$ such that $t_i(v) + \delta < \sigma_{-i}(v) - \sigma^*_{-i}(v)$. It is easy to see that there is $v'_i \in V_i^{SP}$ such that

(2-i) $v'_i(x_i(v)) = t_i(v) + \delta$,

(2-ii) for each $y \in M \backslash \{x_i(v)\}$, $0 < v'_i(y) < \delta$.

[5] Note that by (i), $v'_i(x_i(v)) - \sigma_{-i}(v) + \sigma^*_{-i}(v) > 0$.

Thus, for each $y \in M \backslash \{x_i(v)\}$, we have $v'_i(x_i(v)) - v'_i(y) > t_i(v)$. Since $t_i(v) \geq 0$, $v'_i \in I(v_i, f_i(v))$. By Lemma 3, $f_i(v) = f_i(v')$ where $v' = (v'_i, v_{-i})$.

First, we show that $\sum_{j \in N \backslash \{i\}} v_j(x_j(v')) \leq \sigma^*_{-i}(v)$. By contradiction, suppose that $\sum_{j \in N \backslash \{i\}} v_j(x_j(v')) > \sigma^*_{-i}(v)$. Recall that $x_i(v) = x_i(v')$. Let $\overline{x} = (x_i(v), x_{-i}(v')) \in X$. Then we have

$$\sum_{i \in N} v_i(\overline{x}_i) = \sum_{j \in N \backslash \{i\}} v_j(x_j(v')) + v_i(x_i(v)) > \sigma^*_{-i}(v) + v_i(x_i(v)),$$

which implies that $x(v) \notin \underset{x' \in X}{\arg\max} \sum_{i \in N} v_i(x'_i)$, contradicting *efficiency*.

Thus we have

$$\begin{aligned} \sum_{j \in N \backslash \{i\}} v_j(x_j(v')) + v'_i(x_i(v)) &\leq \sigma^*_{-i}(v) + v'_i(x_i(v)) \\ &< \sigma_{-i}(v) && \text{(by (2-i))} \\ &\leq \max_{x \in X}\Big(\sum_{j \in N \backslash \{i\}} v_j(x_j) + v'_i(x_i)\Big) \end{aligned}$$

Thus $x(v') \notin \underset{x \in X}{\arg\max}(\sum_{j \in N \backslash \{i\}} v_j(x_j) + v'_i(x_i))$, contradicting *efficiency*. **Q.E.D.**

5 Concluding remarks

We consider the problem of deciding how to assign slots to agents and how much each agent should pay, with particular attention to the situation where agents have single-peaked valuations of slots. We aim at identifying the rules satisfying efficiency, strategy-proofness, and individual rationality. Holmström (1979) shows that in the domain satisfying the smooth connectedness condition, the Vickrey rule is the unique rule that satisfies above-mentioned three properties. Since quasi-linear domain is a smoothly connected domain, Holmström's characterization can be applied. Nevertheless, the quasi-linear single-peaked domain is not connected. Therefore it is not clear whether the above characterization still holds. We establish that on the quasi-linear single-peaked domain, the Vickrey rule is still the only rule satisfying efficiency, strategy-proofness, and individual rationality.

Demange et al. (1986) provide two forms of auctions which implement an allocation selected by the Vickrey rule in the quasi-linear environment. These auctions work as well for the quasi-linear single-peaked domain. It is interesting to see whether the single-peakedness helps design auctions with a simpler form. Morimoto and Serizawa (2015) show that the minimum price rule is a natural extension of the Vickrey rule to the non-quasi-linear environment, which can be implemented by the Serial Vickrey mechanism of Zhou and Serizawa (2020). It is open to see whether the minimum price rule is still the only rule that satisfies our three axioms on the non-quasi-linear singled-peaked domain.

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