ISSN (Print) 0473-453X
Discussion Paper No. 1125
ISSN (Online) 2435-0982

THE DYNAMICS OF THE HOUSE PRICE-TO-INCOME RATIO: THEORY AND EVIDENCE

Charles Ka Yui Leung
Edward Chi Ho Tang

March 2021

The Institute of Social and Economic Research
Osaka University
6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan

The Dynamics of the House Price-to-Income Ratio: Theory and Evidence

Charles Ka Yui LEUNG, City University of Hong Kong

Edward Chi Ho TANG, Hong Kong Shue Yan University

This version: March 2021

Abstract

The house price-to-income ratio (PIR) is widely used as an affordability indicator. This paper complements the cross-sectionally focused literature by proposing a tractable model for the PIR dynamics. Our model predicts that the PIR is very persistent and is correlated to the lagged aggregate output. Cross-country analysis confirms this prediction and provides evidence for a long-term, positive and significant relationship between PIR and aggregate production. Our results hint at the construction of an early warning system for housing market mispricing. Our tractable formulation of a stochastic money growth rule may carry independent research interest.

JEL Classification Number: E30, O40, R30

Keywords: housing affordability, output dynamics, endogenous house price, wage rigidity, monetary policy rule.

Acknowledgment:

This project is motivated by a conversation with John Quigley, whose generosity, insights and wisdom will always be missed. The authors are grateful to Wing Leong Teo for his participation in an earlier stage, and Nan-Kuang Chen and Fred Kwan, for many discussions. The authors also want to thank (alphabetical order) Ken Chan, Tommy Chan, Kuang-Liang Chang, Mario Crucini, Kristopher Gerardi, Richard Green, Cheuk-Yin Ho, Mingming Jiang, Ronald Jones, Yuichiro Kawaguchi, Calvin Lin, Joe Ng, Shin-Kun Peng, Daniel Preve, Zhigang Tao, Byron Tsang, Ping Wang, Jing Wu, Zan Yang, Matthew Yiu, Eden Yu, Ziqi Zheng, seminars participants of AREUEA International Conference, Asian Econometric Society meeting, CEANA meeting, Conference on Real Estate and Financial Stability, Economic Linkage Conference, Midwest Macro Conference, City University of Hong Kong, Tsinghua University, especially two anonymous referees as well as the financial support from the City University of Hong Kong, the Joint Usage/Research Center at ISER, Osaka University and Grant-in-aid for Research Activity, Japan Society for the Promotion of Science (15H05728, 20H05631), and the International Joint Research Promotion Program of Osaka University. Part of the research is conducted when Leung visited ISER, Osaka University, and Hoover Institution. Their hospitality is gratefully acknowledged. The usual disclaimer applies.

Correspondence:

Leung: Department of Economics and Finance, City University of Hong Kong, Kowloon Tong, Hong Kong, email: kycleung@cityu.edu.hk; Tang: Department of Economics and Finance, Hong Kong Shue Yan University, Hong Kong, email: chtang@hksyu.edu.

1 Introduction

The importance of House Price-to-Income-Ratio (PIR), which is also called the House Price-to-Earning Ratio, can hardly be overstated. For instance, Demographia (2016) reports that "the Median Multiple (a house price-to-income ratio) is widely used for evaluating urban markets, and has been recommended by the World Bank and the United Nations and is used by the Joint Center for Housing Studies, Harvard University. Similar house price-to-income ratios... are used to compare affordability between markets by the Organization for Economic Cooperation and Development, the International Monetary Fund, international credit rating services, media outlets (such as *The Economist*) and others." Yet, despite its importance, formal modeling of PIR is relatively rare.[1] This paper constructs a simple dynamic, stochastic general equilibrium model (DSGE), where we can analytically link the PIR to the output growth. Since both variables are observable, we also bring this testable implication of the model to the data.

An essential application of the model developed here is housing affordability. PIR is one of the most commonly used measures of "affordability."[2] The reasons are clear. The data requirement for PIR is minimal, and the ratio easy to calculate and interpret. Thus, PIR is often computed and compared across countries or regions, or cities within the same state. In addition to the cross-sectional comparison, it is sometimes calculated for a fixed area or country across different periods. It indicates whether housing affordability improves (or deteriorates) over time. Since the housing affordability literature mostly takes a reduced-form approach and focuses on cross-sectional relationships, this paper

[1] See Chen and Cheng (2017), Leung (2004), Leung and Ng (2019), among others, for a review of the literature.

[2] In the literature, "affordability" may carry different meanings in different contexts and can be measured differently. Among others, see Hulchanski (1995), Quigley and Raphael (2004).

can complement the literature in the following ways. First, we provide a simple DSGE model and derive the *equilibrium level of PIR* in the model. Since prices (e.g., house prices, wages) and quantities (e.g., physical capital stock, housing stock) are endogenous in DSGE models, the equilibrium PIR is naturally tied to the movement of "economic fundamentals." As a result, the dynamics of PIR become *predictable*. This model also enables us to address concerns such as the "deterioration of housing affordability" (DHA) in different countries.[3]

Second, we study how rigid wages may affect housing affordability. Some authors argue that the existence of wage rigidity, which some empirical works confirmed, would worsen housing affordability. It is because the wage cannot respond fast enough with the flexible house price. Our model can allow for both flexible and rigid wages.

Third, some authors claim that monetary policies would affect the housing market.[4] We propose a formulation of the monetary policy in this paper. The money growth rate is no longer a constant but instead a function of the previous money growth rate and other macroeconomic variables.[5] It generalizes some previous work and may carry an independent research interest.

Fourth, we confront our theoretical model with data. More specifically, we show a robust relationship between the PIR and real GDP in a dynamic panel data setting. We also perform a panel cointegration test across countries. Thus, it provides not only an empirical validation but also a suggestion for future research directions. In the past, housing affordability studies tend to focus on the cross-sectional difference. The dynamics of housing affordability, on the other hand, maybe under-explored. This study complements the literature by considering PIR dynamics. More specifically, we establish a coherent theoretical

[3] The word "deterioration" suggests a comparison across different periods. Hence, a dynamic model may be more appropriate for the analysis of DHA.

[4] The literature is too large to be reviewed here. Among others, see Jordà et al. (2015).

[5] The previous literature, such as Friedman (1969), focuses on the case of a constant money growth rate.

framework where the PIR changes over time. The model finds support from cross-country data. It complements the previous work, such as Chen and Cheng (2017), which focuses on the United States' case. It may also suggest that a dynamic equilibrium perspective on housing policy could provide additional insights.

This paper is related to several strands of the literature. For instance, academic researchers, media, and policy share the concerns of housing affordability and its potential deterioration (DHA) (Asal, 2019; Australian Government, 2021; CBC News, 2015; McGee, 2009; Moody, 2015; National Housing Conference, 2012; National Housing Federation, 2012; RBC, 2015; Walks, 2014). There are concerns for housing affordability in Australia, Canada, Sweden, U.K., and U.S.. Edvinsson et al. (2021) construct the real estate price index for Sweden from 1818 to 2018. They show that the current house price cycles in Sweden share some similarities with her history and suggest that government intervention may be useful. Second, there is extensive literature on wage rigidity (Barattieri et al., 2014; Bils et al., 2013; Dickens et al., 2007). Third, there are studies on the empirical determinants of housing prices (Oikarinen, 2009; Stadelmann, 2010; Agnello and Schuknecht, 2011). While this paper is built on their insights, it has a very different focus. Rather than searching for the empirical determinants of housing price, which may include aggregate output, labor wage, monetary policy, etc., this paper attempts to relate the PIR and the production in a dynamic equilibrium model when all these variables are determined endogenously. Furthermore, since our paper establishes a panel cointegration relationship between PIR and macroeconomic variables, we can measure the long-run relationship's short-run deviations. If the short-run deviations are persistent or even growing over time, they should alert both academic researchers and policy-makers. In other words, it might be a preliminary step towards the construction of an early warning system for the housing market mispricing.[6]

The organization of this paper is simple. We first study a tractable dynamic, stochastic general equilibrium (DSGE) model with flexible wage and examine how the house price-to-wage ratio will evolve in the model economy. We then compare the case with short-term sticky wages. We then confront the model predictions concerning the PIR and output dynamics with data. The last section concludes with all proofs reserved in the appendix.

2 A Benchmark Model

This section will present a simple DSGE model. We will first provide an informal description, followed by the introduction of a mathematical model. Loosely speaking, the model developed here is a combination of Greenwood and Hercowitz (1991) and Benassy (1995), and hence a brief explanation will be sufficient.[7] Time is discrete, and the horizon is infinite. The population is assumed to be constant to simplify the exposition, and one can comfortably relax this assumption. There are several goods in the economy: the non-durable consumption goods C_t, the business (or physical) capital K_t, residential capital (or housing) H_t. The representative agent derives utility from the level of non-durable consumption, the amount of housing, and also the number of labor hours L_t and the amount of real cash balance $\frac{M_t}{P_t}$.[8] The government prints nominal money M_t according to a specific money supply rule, which will be

[6] Early warning system (EWS) has been extensively discussed in the context of the banking crisis and financial crisis. See Bussière and Fratzscher (2006), Cumperayot and Kouwenberg (2013), Lang and Schmidt (2016), and the reference therein.

[7] See also Leung (2007, 2014) for related studies.

[8] The money-in-utility-function formulation can be easily justified as the transaction demand for money. There is extensive literature on this, and interested readers could consult Croushore (1993), Feenstra (1986) for more details.

explained later. Firms combine labor and business capital to produce output Y_t. All agents in the model economy maximize their utility or profit.

Our formal description of the model begins with the household side. The representative household in the model that maximizes the expected value of the discounted sum of utility:

$$\max . E_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, H_t + H_t^r, L_t, \frac{M_t}{P_t}\right), \tag{1}$$

where C_t is the amount of non-durable consumption, H_t is the amount of housing stock, H_t^r is the amount of housing that are rented, L_t is the amount of labor hours (or efforts) devoted in goods production, $\frac{M_t}{P_t}$ is the real money balance, as M_t denotes the nominal money balance and P_t is the general price level. In this paper, a simple utility functional form is assumed,

$$U\left(C_t, H_t + H_t^r, L_t, \frac{M_t}{P_t}\right) = \ln C_t + \omega_1 \ln\left(H_t + H_t^r\right) + \omega_2 \ln\left(1 - L_t\right) + \omega_3 \ln \frac{M_t}{P_t}, \tag{2}$$

$\omega_i > 0$, $i = 1, 2, 3$, are parameters governing the relative importance of housing, leisure, and money holding in the utility function. The maximization problem of the representative household is subject to several constraints. First, both the business capital K_t and housing stock H_t are durable and can only adjust gradually.[9] The following equations capture this observation that the future amount of stock (whether the business capital or housing) depends positively on the amount of current stock level and investment,

$$K_{t+1} = K_t^{1-\delta_k} I_{k,t}^{\delta_k}, \tag{3}$$

$$H_{t+1} = \left(H_t + H_t^m\right)^{1-\delta_h} I_{h,t}^{\delta_h}, \tag{4}$$

[9] There is a vast literature on the gradual adjustment of capital stock and housing stock. Among others, see Cooley (1995), Hanushek and Quigley (1979).

where δ_k, δ_h are the depreciation rates of business capital and housing stock, $0 \leq \delta_k, \delta_h \leq 1$, $I_{k,t}$, $I_{h,t}$ are the investment in business capital and housing. Second, this formulation also captures the idea that holding fixed the amount of existing stock, the marginal rate of return of investment on the future capital is diminishing, which can also be interpreted as a form of adjustment cost.[10] Notice also that in (4), the amount of housing stock purchased from the secondary market, H_t^m, can also influence the amount of future housing stock. Thus, this formulation also captures the idea that one can accumulate housing stock through investment and direct purchase from the market. On top of the restrictions of (3) and (4), the household is also subject to the usual budget constraint,

$$R_t K_t + W_t L_t + \frac{\mu_t M_{t-1}}{P_t} \geq I_{k,t} + I_{h,t} + C_t + P_{ht} H_t^m + R_{ht} H_t^r + \frac{M_t}{P_t} \tag{5}$$

where R_t is the real rental rate of capital, W_t is the real wage rate, P_{ht} and R_{ht} are the real price and real rental rate of housing, respectively. Following Benassy (1995), μ_t a stochastic multiplicative monetary shock.

This dynamic optimization problem above can be solved using the Dynamic Programming method,

$$V\left(K_t, H_t, \frac{M_{t-1}}{P_t}\right) = \max U\left(C_t, H_t + H_t^r, L_t, \frac{M_t}{P_t}\right) + \beta E_t V\left(K_{t+1}, H_{t+1}, \frac{M_t}{P_{t+1}}\right),$$

subject to (3), (4), (5). The first order conditions are easy to derive and details are provided in the appendix.

To be compatible with the literature, the rest of the model is straightforward.

[10] In practice, the adjustment cost would include all the growth management policies, physical constraints such as cliffs on real estate development, and the regulations on permits. See Leung and Teo (2011), Saiz (2010) for more discussion.

There is an aggregate production technology, which exhibits constant returns to scale in capital and labor,

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}, \tag{6}$$

where α is the capital share, $0 < \alpha < 1$, and A_t is the productivity shock. The logarithm of the productivity shock follows an AR (1) process,

$$a_t = \rho a_{t-1} + \varepsilon_{at}, \tag{7}$$

where $a_t = \ln A_t$, ρ measures the persistence of the productivity shock, $0 \leq \rho \leq 1$. To further simplify the exposition, we assume that $E(\varepsilon_{at}) = 0$, $Var(\varepsilon_{at}) = \sigma_a^2$, which is a constant, and $E(\varepsilon_{at}\varepsilon_{as}) = 0$ whenever $s \neq t$. With competitive factor markets, the real rental rate and wage rate will be equalized to the corresponding marginal product,

$$R_t = \frac{\partial Y_t}{\partial K_t} = \alpha \frac{Y_t}{K_t}, \tag{8}$$

$$W_t = \frac{\partial Y_t}{\partial L_t} = (1-\alpha) \frac{Y_t}{L_t}. \tag{9}$$

And it is easy to see that the economic profit is zero in this model economy,

$$\Pi_t^d \equiv Y_t - R_t K_t - W_t L_t = 0. \tag{10}$$

Following Benassy (2002), we assume that the monetary stock's growth rate is a random variable. The household will hold all the money printed by the government. Mathematically, it means that

$$M_t = \mu_t M_{t-1}. \tag{11}$$

We will provide details on the monetary growth rate μ_t in a later section. At this point, we can take it as given.[11] Now, combining (6), (8), (9), (10) and (11), (5) can be simplified as:

$$Y_t = I_{k,t} + I_{h,t} + C_t. \tag{12}$$

To solve the model, we need to impose some market-clearing conditions. Following Lucas (1978), there is no net trade among identical households, whether in the sale market or rental market of housing. It follows that

$$H_t^m = H_t^r = 0. \tag{13}$$

For future reference, we use small letter to denote the natural log of capital letter variables. For instance, $c_t = \ln C_t$, $y_t = \ln Y_t$, $m_t = \ln M_t$, etc. The following proposition summarizes the equilibrium dynamics of the model (the proof can be found in the appendix).

Proposition 1 *With flexible prices and wages, the dynamic system depends on the joint dynamics of output and capital stock, which can be summarized by the following vector dynamics equation,*

$$\overrightarrow{y}_t = B_0 + B_1 \overrightarrow{y}_{t-1} + \overrightarrow{a}_t, \tag{14}$$

where B_0, B_1 are matrices of constant, the transpose of the vector $\overrightarrow{y}_t$ is (y_t, k_t), and $\overrightarrow{a}_t$ represents a vector of shock. In addition, we can show that $\overrightarrow{a}_t$ is serially correlated,

$$\overrightarrow{a}_t = \rho^j \overrightarrow{a}_{t-j} + \sum_{i=0}^{j-1} \rho^i \overrightarrow{\varepsilon}_{a,t-i}, \tag{15}$$

[11] Since all prices are flexible in this section, monetary policy will be neutral. Among others, see Walsh (2010). In the following section, with the sticky wage, monetary policy will not be neutral, and we will explicitly formulate the monetary policy.

where $E\left(\overrightarrow{\varepsilon}'_{a,t-i}\right) = (0,\,0),\ E\left(\overrightarrow{\varepsilon}'_{a,t}\overrightarrow{\varepsilon}_{a,s}\right) = 0$ *whenever* $s \neq t$.[12]

Besides, we can also study the house price dynamics in this model.

Proposition 2 *In this model, the house price positively correlates to the output and negatively to the housing stock. Formally, it is characterized by the following equation,*

$$p_{ht} = b^{ph} + y_t - h_t. \tag{16}$$

Notice that we have already derived the output level, the real wage, and the model's house price equation. We now focus on the real wage rate-to-house price ratio, which is often used as an "affordability index."[13]

Proposition 3 *The real wage-to-house price ratio (in log and in real terms) can be expressed as the follows,*

$$\begin{aligned} \ln\left(\frac{W_t}{P_{ht}}\right) &= w_t - p_{ht} \\ &= b^{wp} + h_t, \end{aligned} \tag{17}$$

where b^{wp} *is a constant.*

Notice that in log form, the commonly used house price-to-income ratio (PIR) is simply $\ln\left(\frac{P_{ht}}{W_t}\right) = p_{ht} - w_t = -(w_t - p_{ht})$. Thus, our real wage-to-house price ratio will inform us directly about the widely discussed PIR. For mathematical convenience, we would proceed with the real wage-to-house price ratio.[14] Several observations are immediate from (17). First, even in a

[12] Throughout this paper, we use $\overrightarrow{x}'$ to represent the transpose of the vector $\overrightarrow{x}$.

[13] The real wage rate-to-house price ratio measures how many hours (or any time units) of labor a household needs to give up to exchange for a housing unit. Sometimes people would use the reciprocal of it, i.e., the house price-to-wage rate ratio.

[14] Some researchers argue that a more appropriate measure would be the wage income-to-house price ratio, i.e., $W_t L_t / P_{ht}$, rather than the wage rate-to-house price ratio W_t / P_{ht}. As we have shown in the appendix, the labor hours are constant at the flexible wage equilibrium. Hence, the two ratios would only differ by a fixed factor. We will revisit the difference between the two ratios when the wage is rigid in the short run.

stationary environment, the real wage-house price is not constant but would vary according to the housing stock. Second, if the stock of housing suddenly decreases (say, due to unexpected natural disasters), the real wage-to-house price ratio will also decrease. The intuition is simple. If some disasters destroy some housing stock, agents need to be re-allocated to existing shelters. However, housing stock cannot adjust soon enough to meet the demand, which increases the house price. Other things being equal, the real wage-to-house price ratio will drop.

In principle, one can estimate equations such as (14), (16), or (17) directly. In practice, variables such as the business capital stock k_t and housing stock h_t are not available in some countries. Even if the data on capital stock and housing stock is available, they adjust slowly and are typically measured infrequently. Therefore, we need to derive other *testable implications* of the model. The following proposition takes a step in this direction.

Proposition 4 *The real wage-to-house price ratio is related to lagged output level of the economy,*

$$w_t - p_{ht} = b^{wp\prime} + \delta_h \sum_{i=0}^{t-1} \left(1-\delta_h\right)^i y_{t-1-i} + \left(1-\delta_h\right)^{t-1} h_0, \tag{18}$$

where $b^{wp\prime}$ is a constant, h_0 is the amount of initial housing stock in the model economy.

The intuition behind this proposition is simple. Capital accumulation and house construction are both endogenous in the model. Therefore, the consumer-workers optimally allocate the resource in the two activities. Since house price is related to the construction activities on the one hand, and the output and equilibrium wage depends on the amount of physical capital, on the other hand, the PIR and GDP are naturally related. Notice that as $t \longrightarrow \infty$,

$(1-\delta_h)^{t-1} h_0 \longrightarrow 0$ as $0 < (1-\delta_h) < 1$. Thus, the importance of the initial stock of housing diminishes over time, and the real wage-to-house price ratio will depend on the series of lagged output, $\{y_{t-1-i}\}$. In particular, an increase in the previous period's real output will increase the real wage-to-house price in some subsequent periods. The intuition is simple. Higher levels of previous periods' real output will lead to higher demand for housing and a higher level of business capital, and the latter tends to lift the wage. Given the assumptions made in this model economy, the wage effect dominates the housing demand effect and the wage-to-house price ratio increases. We will further examine this theoretical prediction's robustness in the following section and then will confront the theory with data.

3 The case with short-term rigid wages

The previous section studies a model with perfectly flexible prices and wages, and hence the monetary policy is neutral. Recent evidence, however, suggests that nominal wages are sticky. For instance, based on the micro-evidence from the 1970s to the early 2000s in twelve countries, Dickens et al. (2007) conclude that the nominal wages are indeed sticky rather than flexible.[15] The differential flexibility between the house price and the wage may influence our conclusion in the present context. Therefore, we introduce sticky wages in this section. To facilitate the comparison, the model economy we would consider is the same as in the previous section, except for the short-term nominal rigidity in wages. As shown in Walsh (2010), monetary policy would affect real economic activities if wages are sticky. Here our focus is on how the equilibrium dynamics of PIR could interact with the monetary policy. We adopt a tractable formulation

[15] The twelve countries are Austria, Belgium, Denmark, Finland, France, Germany, Italy, Norway, Portugal, Sweden, Switzerland, and the United Kingdom.

of Benassy (2002), assuming that *wages are set one period in advance.* More specifically, the contract wage is set equal to the expected value of the Walrasian wage ($E_{t-1}w_t^{n*}$), and hence *the labor market will clear ex-ante.* Mathematically, the nominal wage becomes,

$$w_t^n = E_{t-1}w_t^{n*} = b^w + E_{t-1}m_t. \tag{19}$$

We will provide more discussion on the details of the expected value of money supply $E_{t-1}m_t$ later.[16] We will proceed *as if* $E_{t-1}m_t$ is known to the agent. With short-term nominal wage rigidity, the real wage in the economy becomes,

$$w_t = w_t^n - p_t = E_{t-1}w_t^{n*} - p_t = b^w + E_{t-1}m_t - p_t. \tag{20}$$

After all the shocks realize, the firms will hire labor at the pre-committed wage. Profit maximization will imply that the factor returns (wage and capital rental rate) are still equalized to the corresponding marginal products, and hence (8), (9) are still valid. The following proposition dictates the equilibrium labor supply under this slight change in the economic environment,

Proposition 5 *If (19) holds, the equilibrium labor supply under short-term wage rigidity depends on the "forecast error" in monetary supply,*

$$l_t = l + \varepsilon_{mt}, \tag{21}$$

[16] Notice that this formulation implies that the nominal wage rigidity is symmetric. Abbritti and Fahr (2013) argue that the nominal wage is downward rigid but upward flexible. Babecký et al. (2012) find that when employers do not cut "wages," they would cut other benefits to reduce labor costs. Hence, from the employee perspective, the "net income" is reduced. Hofmann et al. (2012) find that the US wage indexation degree varies across different periods. Elsby and Solon (2019) study the microdata across countries and find that a nominal wage cut is typical annually. Thus, given the diverse opinions on downward wage rigidity, it might not be a bad idea to study flexible and rigid wages and show how they might affect the house price-to-income ratio dynamics.

where ε_{mt} is the forecast error term in money supply at time t,

$$\varepsilon_{mt} \equiv m_t - E_{t-1} m_t. \tag{22}$$

Furthermore, we can show that the joint dynamics of output and capital stock is summarized by the following vector dynamics equation,

$$\overrightarrow{y}_t = B_0 + B_1 \overrightarrow{y}_{t-1} + \overrightarrow{a_t^r}, \tag{23}$$

where B_0, B_1 are matrices of constant, the transpose of the vector $\overrightarrow{y}_t$ is (y_t, k_t), and $\overrightarrow{a_t^r}$ represents a vector of shock. And $\overrightarrow{a_t^r}$ is serially correlated.

Notice that the form of (23) is identical to (14). The only difference is that the forecast error term in money supply at time t, ε_{mt}, is a part of the vector of shocks, $\overrightarrow{a_t^r}$. It is reasonable to expect that the forecast error term ε_{mt} depends on how the monetary policy is conducted. Thus, to fully understand the dynamics of the system, we must formally introduce the monetary policy.

4 Monetary policy

In the literature, a famous formulation of the monetary policy is to adopt a version of the Taylor rule (e.g., Koenig et al., 2012; Walsh, 2010). While it has many merits, a drawback is that an analytical solution is typically unavailable, and the model would need to be solved numerically. To keep the model tractable, we formulate the monetary policy as a money growth rule.[17] On the other hand, a stochastic money growth policy rule (SMG) seems to be under-explored.

[17] In the literature, researchers discussed whether the Taylor rule and a constant money growth rule suggested by Friedman (1969) are equivalent. For instance, see Carlstrom and Fuerst (1995), Schabert (2003), Auray and Feve (2008), among others. See also Nelson (2008) for a review.

Drawing lessons from the literature, our formulation of SMG is analogous to the Taylor rule. In particular, we assume that the growth rate of the nominal money supply μ_t depends on its lag, the inflation rate P_t/P_{t-1}, and the output level Y_t, relative to the corresponding steady-state values. Formally, it means that

$$\left(\frac{\mu_t}{\mu}\right) = \left(\frac{\mu_{t-1}}{\mu}\right)^{\rho^{\mu}} \left(\frac{P_t/P_{t-1}}{\pi}\right)^{\phi_{\pi}} \left(\frac{Y_t}{Y}\right)^{\phi_y} e^{\varepsilon_{\mu t}}, \tag{24}$$

where μ, π, Y are the steady-state level of $\mu_t, P_t/P_{t-1}, Y_t$ respectively, with ρ^{μ}, ϕ_{π}, ϕ_y being the policy parameters. For instance, if the inflation rate P_t/P_{t-1} deviates from the steady-state inflation rate π, the money growth rate would respond, and the parameter ϕ_{π} governs how much the money supply should react to the "excessive" inflation rate. Similarly, if the aggregate output level Y_t falls short of its steady-state level Y, the money growth rate μ_t might increase relative to its steady-state level μ. The parameter ϕ_y captures how sensitive is the monetary growth rate to the deviation of the aggregate output from its steady-state level. Clearly, if $\rho^{\mu} = \phi_{\pi} = \phi_y = 0$, the monetary growth rate is an i.i.d. process. For simplicity, we assume that the "innovation term" of the monetary policy $\varepsilon_{\mu t}$ has a zero mean and constant volatility, $E(\varepsilon_{\mu t}) = 0$, $Var(\varepsilon_{\mu t}) = \sigma_{\mu}^2$. The innovation terms are also uncorrelated over time, $E(\varepsilon_{\mu t}\varepsilon_{\mu s}) = 0$ whenever $s \neq t$. Furthermore, we assume that $\{\varepsilon_{at}\}$ and $\{\varepsilon_{\mu t}\}$ are independent. A merit of this formulation is that the model remains very tractable. Another merit of using SMG is that the gross monetary growth rate $\mu_t = M_t/M_{t-1}$ is, by definition, positive. Hence, we do not worry about the zero lower bound (ZLB) of the nominal interest rate.[18] Like other results, the following lemma is proved in the appendix.

Lemma 6 *Given (24) holds, the forecast error of money supply in (22), ε_{mt}, is*

[18] Recently, Hirose and Inoue (2016) show that if we ignore the ZLB in an estimated DSGE model with the usual interest rate-rule-type monetary policy, other parameters may be estimated with bias.

shown to be an "weighted sum" of the forecast error of productivity, $(a_t - E_{t-1}a_t)$*, and the monetary innovation term* $\varepsilon_{\mu t}$*,*

$$\begin{aligned} \varepsilon_{mt} &\equiv m_t - E_{t-1}m_t \\ &= \omega^a (a_t - E_{t-1}a_t) + \omega^\mu \varepsilon_{\mu t} \\ &= \omega^a \varepsilon_{at} + \omega^\mu \varepsilon_{\mu t}, \end{aligned} \tag{25}$$

where ω^a*,* ω^μ *are constant.*

Equipped with these results, we can prove our main result, which related the real wage-to-house price ratio to the output dynamics and other random terms in this economy.

Proposition 7 *If (19) and (24) hold, the real wage-to-house price ratio (WPR) is related to lagged output level of the economy, as well as the forecast error of productivity shock* ε_{at}*, and that of money supply* $\varepsilon_{\mu t}$*,*

$$w_t - p_{ht} = b^{wp\prime} + \delta_h \sum_{i=0}^{t-1} (1-\delta_h)^i y_{t-1-i} + (1-\delta_h)^{t-1} h_0 - \widehat{\varepsilon}_t, \tag{26}$$

where $b^{wp\prime}$ *is a constant,* h_0 *is the amount of initial housing stock in the model economy,* $\widehat{\varepsilon}_t$ *is a stochastic residual term.*

Clearly, (18) and (26) are very similar. In other words, the wage rigidity does not significantly alter the PIR dynamics in this setup. To put it another way, the PIR does not provide "extra information" about the housing market; it is merely a *"summary statistics"* of the aggregate output in previous periods. In principle, as both PIR and aggregate output are observable in the data, we will seek empirical confirmation of (26). In practice, however, the right-hand side of (26) contains the whole series of past output $\{y_t\}_{t=0}^{t-1}$, and it would pose a challenge in applied work. We, therefore, derive the following proposition from

(26), which are much easier to implement in empirical works, as we will further discuss it in the subsequent empirical section.

Proposition 8 *The following equations characterize the real wage-to-house price ratio (WPR):*

(a) The variance of the WPR is a "weighted sum" of the variance of output and some residual term,

$$var\left(w_t - p_{ht}\right) = \widetilde{\delta}_t var\left(y_t\right) + var\left(\widehat{\varepsilon}_t\right), \tag{27}$$

where $\widetilde{\delta}_t > 0$ depends on t and is non-stochastic.

(b) The dynamics of the WPR is a "weighted average" of its lagged value and output level:

$$\left(w_{t+1} - p_{h,t+1}\right) = \widetilde{b^{wp\prime}} + \delta_h y_t + \left(1 - \delta_h\right)\left(w_t - p_{ht}\right) + \left[\widetilde{\varepsilon_{t+1}}\right], \tag{28}$$

where $\widetilde{b^{wp\prime}}$ is a constant term, $\widetilde{\varepsilon_{t+1}}$ is a stochastic term. It can be shown that $\widetilde{\varepsilon_{t+1}}$ is serially correlated, i.e. $E[\widetilde{\varepsilon_{t+1}}\widetilde{\varepsilon_t}] \neq 0$.

Notice that the lagged output coefficient is δ_h, which is positive but small. On the other hand, the coefficient of lagged WPR is $\left(1 - \delta_h\right)$, which is lower than but close to unity. The dynamics of WPR should be persistent.

Some researchers argue that a more appropriate measure would be the wage income-to-house price ratio, i.e. $\left(W_t L_t / P_{ht}\right)$, rather than the wage rate-to-house price ratio (W_t/P_{ht}). The following result shows that the dynamics of the two ratios are very similar in log form.

Corollary 9 *In log form, the wage income-to-house price ratio is very similar* *to the wage rate-to-house price ratio,*

$$w_{t+1}+l_{t+1}-p_{h,t+1} = b''+\delta_h\sum_{i=0}^{t-1}(1-\delta_h)^i y_{t-1-i}+(1-\delta_h)^{t-1}h_0$$
$$= \widetilde{b''}+\delta_h y_t+(1-\delta_h)(w_t+l_t-p_{ht}). \quad (29)$$

Therefore, our empirical work's choice variable will depend on data availability, present in the next section.

5 Empirical Evidence

The theoretical analysis has provided several testable implications, and this section verifies them with international data. Since the model presumes a well-functioned capital market, it may be more appropriate to employ data from more developed economies. Moreover, time-series data on house prices from developed countries are more accessible. Thus, our data on real GDP (in millions of US dollars) and wage index are collected from OECD.stat, while the housing price indices are obtained from the Bank of International Settlements.[19] Altogether, there are 15 countries in our study, including Australia, Canada, Denmark, Finland, France, Germany, Italy, Japan, Netherlands, New Zealand, Norway, Spain, Sweden, UK, and the US.[20] Our data is in quarterly frequency and covers the period from 1997Q1 – 2018Q4. Following Cooley (1995), the seasonal component is removed from the time series. Our choice of the sampling period balances the desire to maximize the number of countries included and the discipline to include countries that can meet some econometric test requirements. Figure 1 provides a visualization of the evolution of PIR across

[19] The data can be downloaded from https://www.bis.org/

[20] In an earlier version, we also include Ireland. However, we find that the results dramatically change once Ireland is removed. Hence, following the suggestion of a referee, we remove Ireland from our sample.

OECD countries over our sampling period. Several observations are in order. While there are outliers over the years, they are not that many. The median PIR fluctuates over the years in a relatively smooth manner. The range of PIR between the 25th and 75th percentile tends to increase over time, which seems consistent with OECD countries' diverse economic performance during and after the Great Recession.

(Figure 1, 2, Table 1 about here)

For testable implications, we begin with (27). Notice that the stochastic residual term $\widehat{\varepsilon_t}$ is not directly observable. Thus, other things being equal, (27) predicts that the variance of wage-to-house price ratio and the variance of output are positively correlated.[21] Since the original time series may not be stationary, we employ the Hodrick-Prescott (HP) filter to extract the cyclical components for regression.[22] Figure 2 visually suggests that a country with a volatile output also tends to have a less volatile PIR. However, the coefficient is not significant. We then compute the correlation between the output variance and PIR variance for the full sample and many sub-sample, each with one country removed. The idea is to detect whether any outlier drives the correlation. Table 1 reports the result. The correlation coefficients are all negative, as predicted by the theory, but none is statistically significant. The insignificance result may be due to our small sample size, or some ECB policies have distorted the real estate markets after the economic crises (Acharya et al., 2019).

Now, we turn to the relationship between PIR and output dynamics. While the two formulations, (26) and (28), are equivalent mathematically, it is better to use the latter. Testing (26) directly would introduce many lags on the right

[21] We have conducted further analysis on the residual terms, based on our estimation of the production function and monetary policy function across countries. Those results will be available upon request.

[22] In an earlier version, we used the Christiano-Fitzgerald bandpass filter. The results are similar. See Christiano and Fitzgerald (2003), Hodrick and Prescott (1997) for more details.

hand, which makes the estimation difficult given the relatively short time series. On the other hand, equation (28) expresses the wage-to-house price ratio as a weighted average of its lag and lagged output level. Its data requirement is less stringent, and therefore, we prefer to estimate (28).

We consider the dynamic panel data approach (DPDA) as an appropriate econometric methodology. It enables us to identify and measure effects that are not detectable in pure cross-section or pure time-series data. It also allows us to control for individual heterogeneity. As equation (28) contains a lagged dependent variable as an explanatory variable, strict exogeneity of the regressors no longer holds (Hsiao, 2015). Hence, it will be proper to apply the dynamic panel data model.[23] Notice also that equation (28) is directly derived from our simple DSGE model. Hence, our dynamic panel regression can be interpreted as a "structural estimation" (of one aspect of a dynamical system). However, it can also be interpreted as a "reduced-form estimation," where the PIR is related to its past value and economic fundamental, proxied by the GDP.[24]

Table 2 confirms the theoretical predictions. It shows that the coefficients of lagged output and lagged dependent variables are found to be positive and significant. Consistent with the theory, the coefficient of the lagged PIR is smaller than but close to unity. On the other hand, the lagged output coefficient, which is supposed to be δ_h, is much lower. This result holds for the full sample and most of the sub-samples, each with one being removed. Moreover, it is worthy to note that DPDA requires instruments. Consistently, J-statistics probabilities are always between 0.34 and 0.46, suggesting that the instruments are all

[23] Among others, see Arellano and Bond (1991), Baltagi (2013) for more discussion.

[24] Also, to keep our model tractable, we abstract away from many institutional details, varying across countries. They include whether public housing units and housing vouchers are provided, whether (and how much) mortgage payments are tax-deductible, etc. Among others, see Green (2014), Malpezzi (1999b).

valid.[25]

(Table 2 about here)

Finally, we would like to examine a long-run relationship between the wage-to-house price ratio and the real GDP (both in log form). The justification is clear. If we can identify a long-run relationship between the wage-to-house price ratio and GDP, we can also detect "short-run deviations from the long-run relationship." Such deviations might be used as one of the proxies to measure whether the whole housing market "deviates" from its long-run situation.[26]

To examine whether a long-run relationship exists between the wage-to-house price ratio and GDP (both in logarithm), we proceed in several steps. First, we check the stationarity of the series. As suggested by Cheng and Kwan (2000), Kwan (2007), performing a panel unit root test is more powerful than the unit root test for individual time series. All the panel unit root tests suggest that the two log series are indeed I(1).[27] Second, we proceed to the panel cointegration test. It adopts three types of panel cointegration tests, including Pedroni (1999, 2004), Kao (1999), and Johansen Fisher Panel Cointegration Test (Maddala

[25] There is an additional issue here. The theory predicts that the sum of coefficients for the lag $(w-p)$ term and y term should sum to unity. Table 2 shows that in the full sample (i.e., when all countries are included), the coefficients' sum is equal to (0.94+0.11), which is larger than unity. However, when we turn to different sub-samples, we might have a different conclusion. While the lag $(w-p)$ coefficients are all close to unity, the y term's coefficients vary across different sub-samples. For instance, when we exclude Canada, the coefficient for the lag $(w-p)$ term is 0.93, but the coefficient of the y term is insignificant, and hence the sum is 0.93, which is less than unity. A similar pattern is found when Finland, New Zealand, or the U.S. is excluded from the sample.

We then turn to Table 3a, when the size of government is controlled for. Again, the coefficients' sum is equal to (0.94+0.16), which is larger than unity in the full sample. The coefficients of the lag $(w-p)$ are close to unity in all sub-samples. However, when Canada, or New Zealand, or the U.S. is excluded from the sample, the y term's coefficient is insignificant, and hence the sum of the coefficients is less than unity. Thus, we conclude that whether the sum of the two coefficients is larger than or less than unity might depend on the inclusion of a few countries in the sample.

[26] Clearly, this is the idea behind the "error correction model." Among others, see Engle and Granger (1987), Malpezzi (1999a).

[27] See the appendix for details.

and Wu, 1999). The majority of the result suggests a long-run relationship between $(w_{it} - p_{it})$ and y_{it}. Third, it estimates a long-run relationship using the group-mean fully-modified OLS method (FMOLS), group-mean dynamic OLS (DOLS), and Static OLS method (SOLS). FMOLS employs a *semi-parametric correction* to eliminate the problems caused by the long-run correlation between the cointegrating equation and stochastic regressors' innovation. DOLS involves augmenting the cointegrating regression with leads and lags for the change of variables. The resulting cointegrating equation error term is orthogonal to the entire history of the stochastic regressor innovations. When the leads and lags are set to none, it becomes SOLS. Since SOLS produces biased, *super-consistent estimates* (Tsionas, 2019), it is used as a supplement.[28] The coefficients of FMOLS and SOLS are significant and positive, verifying a long-run relationship between PIR and the real GDP (Table *3a*). To ensure the robustness of our results, we exclude a country one-at-a-time. The results are presented in Table 3a. Our results of the panel cointegration are robust.

(Figure 3, Table 3a about here)

We should interpret the panel cointegration test results with cautions. It shows that the countries in our sample, as a group, display a cointegration relationship. However, some sub-samples may not have such a cointegration relationship (Pesaran, 2015, chapter 31). To understand the evolution of the wage-to-house price ratio in different countries, we study the residual terms from our Group-mean Fully-modified OLS. We take the first-difference of residuals and then divide each series by the corresponding standard deviation to become comparable across countries. Figure 3 shows that these detrended and normalized residual terms are stationary. For most countries, these terms do

[28] FMOLS can be interpreted as an extension of Phillips and Hensen (1990). Among others, see Pedroni (2000, 2001) for more discussion.

not significantly and persistently deviate from zero, suggesting that the PIR's growth rate is roughly constant in the long run.[29] In other words, we do not find evidence of persistent bubbles.

6 Robustness Checks

The previous section has empirically confirmed our theoretical results. As our evidence comes from a panel dataset, the concern is the existence of "outlier(s)," which might impact the results. Constrained by data availability, we attempt to address this concern by performing the following tests.[30] First, we re-run the dynamic panel regression, controlling for government size (captured by the government expenditure ratio to GDP ratio in log). Second, we exclude a country one-at-a-time. The results are presented in Table 3b. It is safe to conclude that the results from the dynamic panel regression are robust.

(Table 3b about here)

7 Concluding Remarks

The house price-to-income ratio (PIR) is widely used in the media and policy institutions to indicate the property market's condition. Yet formal modeling is disproportionately rare. Existing studies also incline to concentrate on cross-sectional, reduced-form regression. This paper attempts to bridge the gap. First, it constructs a simple DSGE model and studies the *endogenous* dynamics of the house price-to-wage ratio. We confirm the prediction that the PIR is very persistent (close to the unit root) and is positively related to the previous period GDP with the data of OECD countries. We show that the empirical result is

[29] Australia, Canada, Sweden seem to be the exception.

[30] As it is well known, many cross-country macroeconomic variables are in annual frequency, while our dataset is quarterly.

robust. We further identify a long-run relationship between PIR and GDP. Our robustness checks also indicate the importance of semi-parametric correction, which means that some nonlinearity may exist in the data that our current model has yet to capture. Perhaps more importantly, our panel cointegration results on the long-run relationship between PIR and GDP imply that we can track the short-run deviations (SRD) in each period. If the SRD is persistent and even growing over time, it might suggest that more careful investigation is necessary for policy considerations. In other words, our cointegration results might provide another indicator for housing market mispricing and could be included in the "early warning system" for a possible housing-related crisis.[31]

Further research can extend the model in several ways. We can develop an environment where some agents may be subject to collateral constraints. Another possibility is to consider inventory accumulation in a sticky-price environment. Furthermore, we can build models in which agents live in different cities or have different income paths. One would model information frictions in both housing and labor markets as well. Finally, it can consider a richer set of government policies and compare their costs and benefits. The pursuit of these possibilities would further enrich our understanding of housing affordability.[32]

[31] There is emerging literature on the "early warning system" for the housing-related crisis. For instance, Yiu et al. (2013) propose a time series test for real-time bubble detection. Based on a search-theoretic model, Leung and Tse (2017) suggest that the increase in the cost of funds for speculators could lead to a significant house price adjustment. Huang et al. (2018) build another search-theoretic model and suggest that the price-rent ratio and turnover rate are essential indicators for a housing-related crisis.

[32] Among others, see Chen and Cheng (2017), Teo (2009), Leung and Teo (2011), Lubik and Teo (2012).

References

[1] Abbritti, M. and S. Fahr, 2013, Downward wage rigidity and business cycle asymmetries, *Journal of Monetary Economics*, 60(7), 871-886.

[2] Acharya, V., Eisert, T., Eufinger, C. and C. W. Hirsch, 2019, Whatever it takes: The real effects of unconventional monetary policy, *Review of Financial Studies*, 32 (9), 3366–3411.

[3] Agnello, L. and L. Schuknecht, 2011, Booms and busts in housing markets: Determinants and implications, *Journal of Housing Economics*, 20, 171-190.

[4] Andrews, D., Caldera Sanchez, A. and A. Johansson, 2011, Housing markets and structural policies in OECD countries, OECD Economics Department Working Papers No. 836.

[5] Arellano, M. and S. Bond, 1991, Some tests of specification for panel data: Monte Carlo evidence and an application to employment equations, *Review of Economic Studies*, 58(2), 277-297.

[6] Asal, M., 2019, Is there a bubble in the Swedish housing market? *Journal of European Real Estate Research*, 12(1), 32-61.

[7] Auray, S. and P. Feve, 2008, On the observational (non)equivalence of money growth and interest rate rules, *Journal of Macroeconomics*, 30, 801-816.

[8] Australian Government, 2021, Report on Government Services Productivity Commission, Part G: Housing and homelessness, https://www.pc.gov.au/research/ongoing/report-on-government-services/2021/housing-and-homelessness.

[9] Babecký, J., Du Caju, P., Kosma, T., Lawless, M., Messina, J. and T. Rõõm, 2012, How do European firms adjust their labour costs when nominal wages are rigid?, *Labour Economics*, 19(5), 792-801.

[10] Baltagi, B. H., 2013, *Econometric Analysis of Panel Data*, New York: Wiley.

[11] Barattieri, A., S. Basu and P. Gottschalk, 2014, Some evidence on the importance of sticky wages, *American Economic Journal: Macroeconomics*, 6(1), 70-101.

[12] Benassy, J. P., 1995, Money and wage contracts in an optimizing model of the business cycle, *Journal of Monetary Economics*, 35, 303-15.

[13] Benassy, J. P., 2002, *The Macroeconomics of Imperfect Competition and Nonclearing Markets: A Dynamic General Equilibrium Approach*, Cambridge: MIT Press.

[14] Bils, M., P. J. Klenow and B. A. Malin, 2013, Testing for Keynesian labor demand, *NBER Macroeconomics Annual,* 27(1), 311 - 349.

[15] Bussière, M., and M. Fratzscher, 2006, Towards a new early warning system of financial crises, *Journal of International Money and Finance,* 25, 953–973.

[16] Carlstrom, C. and T. Fuerst, 1995, Interest rate rules vs. money growth rules: A welfare comparison in a cash-in-advance economy, *Journal of Monetary Economics*, 36, 247-267.

[17] CBC News, 2015, Toronto, Vancouver housing affordability deteriorates to 'risky' levels, https://www.cbc.ca/news/business/toronto-vancouver-housing-affordability-deteriorates-to-risky-levels-rbc-1.3209786.

[18] Chao, C. C. and E. S. H. Yu, 2015, Housing markets with foreign buyers, *Journal of Real Estate Finance and Economics*, 50, 207-218.

[19] Chen, N. K. and H. L. Cheng, 2017, House price to income ratio and fundamentals: Evidence on long-horizon forecastability, *Pacific Economic Review*, 22(3), 293-311.

[20] Cheng, L. K. and Y. K. Kwan, 2000, What are the determinants of the location of foreign direct investment? The Chinese experience, *Journal of International Economics*, 51, 379-400.

[21] Christiano, L. and T. Fitzgerald, 2003, The band pass filter, *International Economic Review*, 44(2), 435–465.

[22] Cooley, T. ed., 1995, *Frontiers of Business Cycle Research*, Princeton: Princeton University Press.

[23] Croushore, D., 1993, Money in the utility function: Functional equivalence to a shopping-time model, *Journal of Macroeconomics*, 15(1), 175-182.

[24] Cumperayot, P. and R. Kouwenberg, 2013, Early warning systems for currency crises: A multivariate extreme value approach, *Journal of International Money and Finance*, 36, 151–171.

[25] Demographia, 2016, 12th Annual Demographia International Housing Affordability Survey 2016, http://demographia.com/dhi2016.pdf.

[26] Dickens, W., L. Goette, E. Groshen, S. Holden, J. Messina, M. Schweitzer, J. Turunen, and M. E. Ward, 2007, How wages change: Micro evidence from the international wage flexibility project, *Journal of Economic Perspectives,* 21(2), 195-214.

[27] Edvinsson, R., Eriksson, K. and G. Ingman, 2021, A real estate price index for Stockholm, Sweden 1818–2018: Putting the last decades housing price boom in a historical perspective, *Scandinavian Economic History Review*, 69(1), 83-101.

[28] Elsby, M. and G. Solon, 2019, How prevalent is downward rigidity in nominal wages? International evidence from payroll records and pay slips, *Journal of Economic Perspectives*, 33(3), 185-201.

[29] Engle, R. and C. Granger, 1987, Co-integration and error correction: Representation, estimation and testing, *Econometrica* 55, 251–276.

[30] Feenstra, R. C., 1986, Functional equivalence between liquidity costs and the utility of money, *Journal of Monetary Economics*, 17, 271-291.

[31] Friedman, M., 1969, *The Optimum Quantity of Money: And Other Essays*, Chicago: Aldine.

[32] Green, R., 2014, *Introduction to Mortgages and Mortgage Backed Securities*, Waltham: Academic Press.

[33] Green, R. and S. Wachter, 2005, The American mortgage in historical and international context, *Journal of Economic Perspectives*, 19(4), 93-114.

[34] Greenwood, J. and Z. Hercowitz, 1991, The allocation of capital and time over the business cycle, *Journal of Political Economy*, 99, 1188-1214.

[35] Hanushek, E. and J. Quigley, 1979, The dynamics of housing market: A stock adjustment model of housing consumption, *Journal of Urban Economics*, 6(1), 90-111.

[36] Hirose, Y. and A. Inoue, 2016, The zero lower bound and parameter bias in an estimated DSGE model, *Journal of Applied Econometrics*, 31(4), 630-651.

[37] Hodrick, R. and E. C. Prescott, 1997, Postwar U.S. business cycles: An empirical investigation, *Journal of Money, Credit, and Banking*, 29(1), 1–16.

[38] Hofmann, B., Peersman, G., and R. Straub, 2012, Time variation in U.S. wage dynamics, *Journal of Monetary Economics*, 59(8), 769-783.

[39] Hsiao, C., 2015, *Analysis of Panel Data*, 3rd. ed., Cambridge: Cambridge University Press.

[40] Huang, D. J., Leung, C. K. Y. and C. Y. Tse, 2018, What accounts for the differences in rent-price ratio and turnover rate? A search-and-matching approach, *Journal of Real Estate Finance and Economics*, 57(3), 431-475.

[41] Hulchanski, J. D., 1995, The concept of housing affordability: Six contemporary uses of the housing expenditure-to-income ratio, *Housing Studies*, 10, 471-492.

[42] IMF, 2015, Global Housing Watch, http://www.imf.org/external/research/housing/

[43] Jordà, Ò., Schularick, M., and A. Taylor, 2015, Betting the house: Monetary policy, mortgage booms and housing prices, https://voxeu.org/article/monetary-policy-and-housing-prices-lessons-140-years-data.

[44] Kao, C. D., 1999. Spurious regression and residual-based tests for cointegration in panel data, *Journal of Econometrics*, 90, 1-44.

[45] Koenig, E. F., Leeson, R., and G. A. Kahn, ed., 2012, *The Taylor Rule and the Transformation of Monetary Policy*, Stanford: Hoover Institution Press.

[46] Kwan, Y. K., 2007, The Direct substitution between government and private consumption in East Asia, in *Fiscal Policy and Management in East Asia*, in T. Ito and A. K. Rose (ed.), Chapter 2, 45-58, Chicago: University of Chicago Press.

[47] Lang, M. and P. G. Schmidt, 2016, The early warnings of banking crises: Interaction of broad liquidity and demand deposits, *Journal of International Money and Finance*, 61, 1–29.

[48] Leung, C., 2004, Macroeconomics and housing: A review of the literature, *Journal of Housing Economics*, 13, 249-267.

[49] Leung, C. K. Y., 2007, Equilibrium correlations of asset price and return, *Journal of Real Estate Finance and Economics*, 34, 233-256.

[50] Leung, C. K. Y., 2014, Error correction dynamics of house price: An equilibrium benchmark, *Journal of Housing Economics*, 25, 75-95.

[51] Leung, C. K. Y., and J. C. Y. Ng, 2019, Macroeconomic Aspects of Housing, in J. Hamilton, A. Dixit, S. Edwards, and K. Judd ed., *Oxford Research Encyclopedia of Economics and Finance*, Oxford University Press. doi: http://dx.doi.org/10.1093/acrefore/9780190625979.013.294

[52] Leung, C. K. Y., Ng, J. C. Y., and E. C. H. Tang, 2020, Why is the Hong Kong housing market unaffordable? Some Stylized facts and estimations, *Quarterly Bulletin*, Central Bank of the Republic of China (Taiwan), 42(1), 5-58.

[53] Leung, C. K. Y. and W. L. Teo, 2011, Should the optimal portfolio be region-specific? A multi-region model with monetary policy and asset price co-movements, *Regional Science and Urban Economics*, 41, 293-304.

[54] Leung, C. K. Y. and C. Y. Tse, 2017, Flipping in the housing market, *Journal of Economic Dynamics and Control*, 76(C), 232-263.

[55] Lubik, T.A. and W. L. Teo, 2012, Inventories, inflation dynamics and the New Keynesian Phillips curve, *European Economic Review*, 56(3), 327-346.

[56] Maddala, G. S. and S. Wu, 1999, A comparative study of unit root tests with panel data and a new simple test, *Oxford Bulletin of Economics and Statistics*, 61, 631-652.

[57] Malpezzi, S., 1999a, A simple error correction model of house prices, *Journal of Housing Economics*, 8, 27-62.

[58] Malpezzi, S., 1999b, Economic analysis of housing markets in developing and transition economies, in P. C. Cheshire & E. S. Mills ed., *Handbook of Regional and Urban Economics*, edition 1, volume 3, chapter 44, pages 1791-1864, New York: Elsevier.

[59] McGee, J., 2009, Why didn't Canada's housing market go bust? *Economic Commentary, Federal Reserve Bank of Cleveland*.

[60] Moody, 2015, Deteriorating Australian housing affordability is credit negative for Australian RMBS, https://www.emis.cow/php/search/doc?dcid=50888473&ebsco=l.

[61] National Housing Conference, 2012, Paycheck to paycheck: Wages and the Cost of Housing in America, http://www.nhc.org/chp/p2p/

[62] National Housing Federation, 2012, House prices in London rise three times as much as incomes over ten years, http://www.housing.org.uk/our_regions/london_region/london_news/house_price_affordability.aspx

[63] Nelson, E., 2008, Friedman and Taylor on monetary policy rules: A comparison, *Federal Reserve Bank of St. Louis Review* (March/April), 95-116.

[64] Oikarinen, E., 2009, Household borrowing and metropolitan housing price dynamics–Empirical evidence from Helsinki, *Journal of Housing Economics*, 18, 126-139.

[65] Pedroni, P., 1999, Critical values for cointegration tests in heterogeneous panels with multiple regressors, *Oxford Bulletin of Economics and Statistics*, 61, 653-670.

[66] Pedroni, P., 2000, Fully Modified OLS for Heterogeneous Cointegrated Panels, in B. Baltagi ed., *Nonstationary Panels, Panel Cointegration and Dynamic Panels*, 15, Amsterdam: Elsevier, 93-130.

[67] Pedroni, P., 2001, Purchasing power parity tests in cointegrated panels, *Review of Economics and Statistics*, 83, 727-731.

[68] Pedroni, P., 2004, Panel cointegration; asymptotic and finite sample properties of pooled time series tests with an application to the PPP hypothesis, *Econometric Theory*, 20, 597-625.

[69] Pesaran, M. H., 2015, *Time Series and Panel Data Econometrics*, Oxford: Oxford University Press.

[70] Phillips, P. C. B. and B. E. Hansen, 1990, Statistical inference in instrumental variables regression with I(1) processes. *Review of Economics Studies*, 57, 99-125.

[71] Quigley, J. M. and S. Raphael, 2004, Is housing unaffordable? Why isn't it more affordable? *Journal of Economic Perspectives*, 18, 191-214.

[72] RBC (Royal Bank of Canada), 2015, Housing trends and affordability, http://www.rbc.com/newsroom/_assets-custom/pdf/20150831-ha.pdf.

[73] Saiz, A., 2010, The geographic determinants of housing supply. *Quarterly Journal of Economics*, 125(3), 1253-1296.

[74] Schabert, A., 2003, On the equivalence of money growth and interest rate policy, University of Glasgow Working Paper.

[75] Stadelmann, D., 2010, Which factors capitalize into house prices? A Bayesian averaging approach, *Journal of Housing Economics*, 19, 180-204.

[76] Teo, W. L., 2009, A DSGE model with durable goods, research notes, National Taiwan University.

[77] Tsionas, M., 2019, *Panel Data Econometrics: Theory*, New York: Academic Press.

[78] Walks, A., 2014, Canada's housing bubble story: Mortgage securitization, the state, and the global financial crisis, *International Journal of Urban and Regional Research*, 38(1), 256–284.

[79] Walsh, C., 2010, *Monetary Theory and Policy*, 3rd. ed., Cambridge: MIT Press.

[80] Yiu, M. S., J. Yu, and L. Jin, 2013, Detecting bubbles in Hong Kong residential property market, *Journal of Asian Economics*, 28, 115-124.

PIR Figures and Tables

Table 1. Correlation between variance of PIR and variance of output (Cross-sectional)

		Correlation between var($w_t - p_{ht}$) and var (y_t)
Full sample		-0.2917
All countries except	Australia	-0.2984
	Canada	-0.3197
	Denmark	-0.4300
	Finland	-0.2584
	France	-0.3293
	Germany	-0.2175
	Italy	-0.3015
	Japan	-0.2736
	Netherlands	-0.2903
	New Zealand	-0.2569
	Norway	-0.2941
	Spain	-0.2938
	Sweden	-0.2772
	UK	-0.3018
	US	-0.2624

Note: All coefficients are insignificant. Cyclical components (from HP filter) are used.

Table 2. Dynamic Panel Data Regression Result

Dependent variable: $w_{t+1} - p_{h,t+1}$

		Instruments – Lags 2 to 4 of dependent variable included			
		$w_t - p_{ht}$	y_t	J-statistics	Prob. (J-stat)
Full sample		0.9440 ***	0.1127 **	13.89	0.38
All countries except	Australia	0.9562 ***	0.1357 ***	13.11	0.36
	Canada	0.9363 ***	0.0694	11.83	0.46
	Denmark	0.9365 ***	0.1013 **	12.80	0.38
	Finland	0.9460 ***	0.0559	12.48	0.41
	France	0.9505 ***	0.1391 ***	13.32	0.35
	Germany	0.9548 ***	0.1235 *	13.38	0.34
	Italy	0.9643 ***	0.1737 ***	13.47	0.34
	Japan	0.9527 ***	0.1062 *	12.47	0.41
	Netherlands	0.9481 ***	0.1433 ***	12.90	0.38
	New Zealand	0.9556 ***	0.0908	12.83	0.38
	Norway	0.9529 ***	0.1262 ***	12.99	0.37
	Spain	0.9502 ***	0.1029 ***	12.72	0.39
	Sweden	0.9544 ***	0.1274 ***	13.41	0.34
	UK	0.9523 ***	0.1064 *	12.67	0.39
	US	0.9381 ***	0.0663	12.62	0.40

Note: ***, ** and * denotes 1%, 5% and 10% statistical significance respectively.
Cyclical components (from HP filter) are used.

Table 3a. Panel Co-integration Test Between PIR and GDP (Full Sample and Robustness Check)

		Group-mean Fully-modified OLS	Group-mean Dynamic OLS	Static OLS
Full sample		0.0024 ***	0.0003	0.0040 ***
All countries except	Australia	0.0017 **	0.0004	0.0032 ***
	Canada	0.0016 **	-0.0002	0.0030 ***
	Denmark	0.0024 ***	0.0002	0.0041 ***
	Finland	0.0025 ***	0.0002	0.0041 ***
	France	0.0026 ***	0.0005	0.0040 ***
	Germany	0.0029 ***	0.0006	0.0046 ***
	Italy	0.0035 ***	0.0012	0.0051 ***
	Japan	0.0031 ***	0.0008	0.0048 ***
	Netherlands	0.0033 ***	0.0011	0.0048 ***
	New Zealand	0.0015 **	-0.0007	0.0030 ***
	Norway	0.0019 **	-0.0004	0.0034 ***
	Spain	0.0036 ***	0.0015	0.0051 ***
	Sweden	0.0011	-0.0010	0.0026 ***
	UK	0.0022 ***	0.0001	0.0037 ***
	US	0.0025 ***	0.0003	0.0041 ***

Note: *** and ** denote 1% and 5% statistical significance respectively.

Table 3b. Dynamic Panel Data Regression Result (Robustness check)

Dependent variable: $w_{t+1} - p_{h,t+1}$

		Instruments – Lags 2 to 4 of dependent variable included				
		$w_t - p_{ht}$	y_t	gov_t	J-statistics	Prob. (J-stat)
Full sample		0.9446 ***	0.1569 **	0.0451	13.92	0.31
All countries except	Australia	0.9568 ***	0.1891 ***	0.0532	13.16	0.28
	Canada	0.9312 ***	0.0898	0.0492	11.55	0.40
	Denmark	0.9368 ***	0.1369 **	0.0332	12.87	0.30
	Finland	0.9382 ***	0.1380 *	0.0966 *	12.26	0.34
	France	0.9461 ***	0.2135 ***	0.1011	12.92	0.30
	Germany	0.9469 ***	0.1970 **	0.1084	12.35	0.34
	Italy	0.9393 ***	0.2270 **	0.1629 **	11.67	0.39
	Japan	0.9485 ***	0.1582 **	0.0635	12.42	0.33
	Netherlands	0.9421 ***	0.1954 ***	0.1046 *	12.16	0.35
	New Zealand	0.9499 ***	0.0779	0.1482	10.66	0.47
	Norway	0.9481 ***	0.1852 ***	0.1068	12.37	0.34
	Spain	0.9459 ***	0.1497 **	0.0694 **	12.64	0.32
	Sweden	0.9553 ***	0.1856 ***	0.0617	13.51	0.26
	UK	0.9521 ***	0.1963 *	0.0935	12.64	0.32
	US	0.9380 ***	0.1187	0.0695	12.47	0.33

Note: ***, ** and * denotes 1%, 5% and 10% statistical significance respectively.
Cyclical components (from HP filter) are used.

Figure 1. Boxplot of PIR

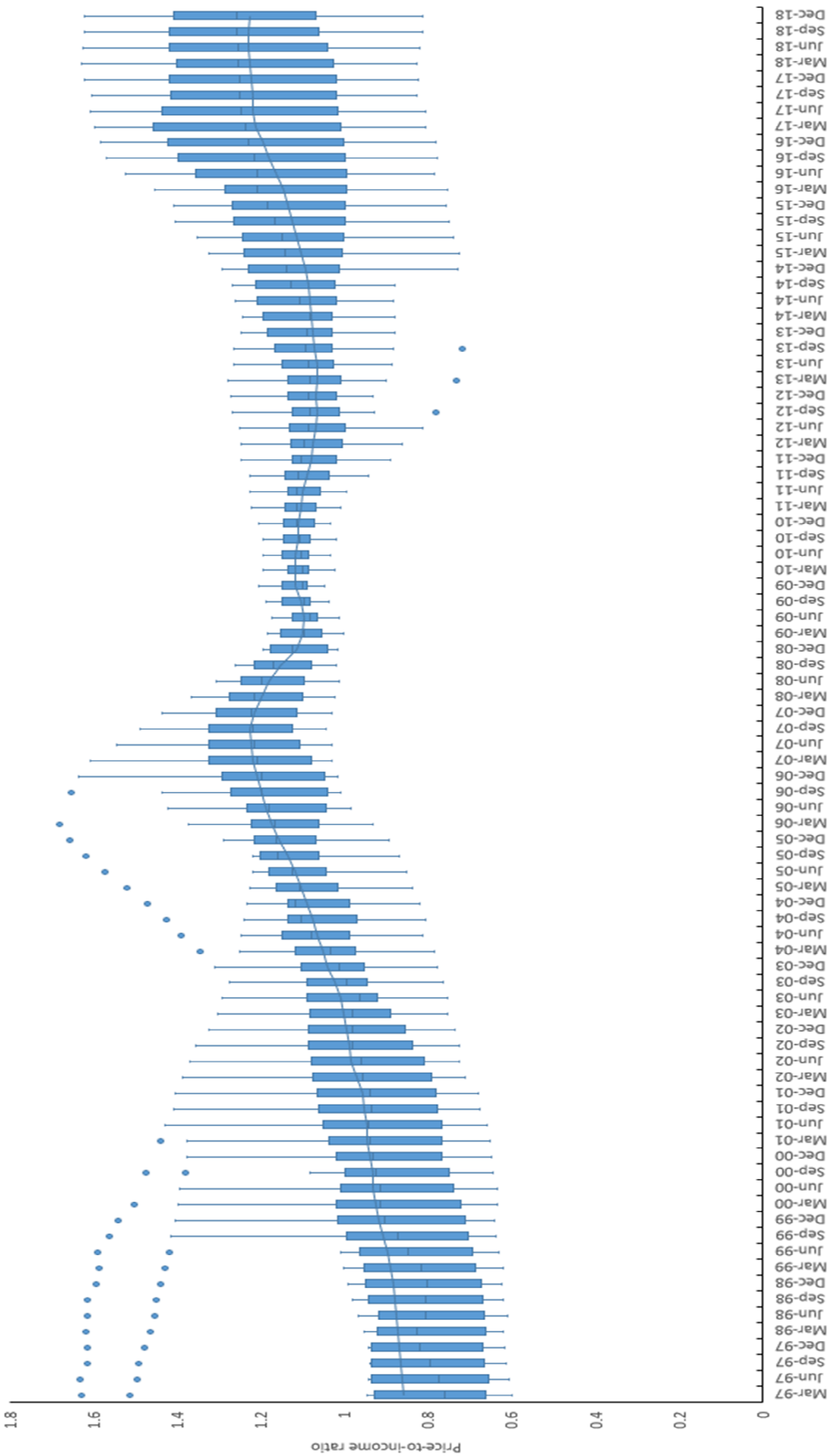


Figure 2. Variance of Wage-to-House Price ratio and Variance of real GDP

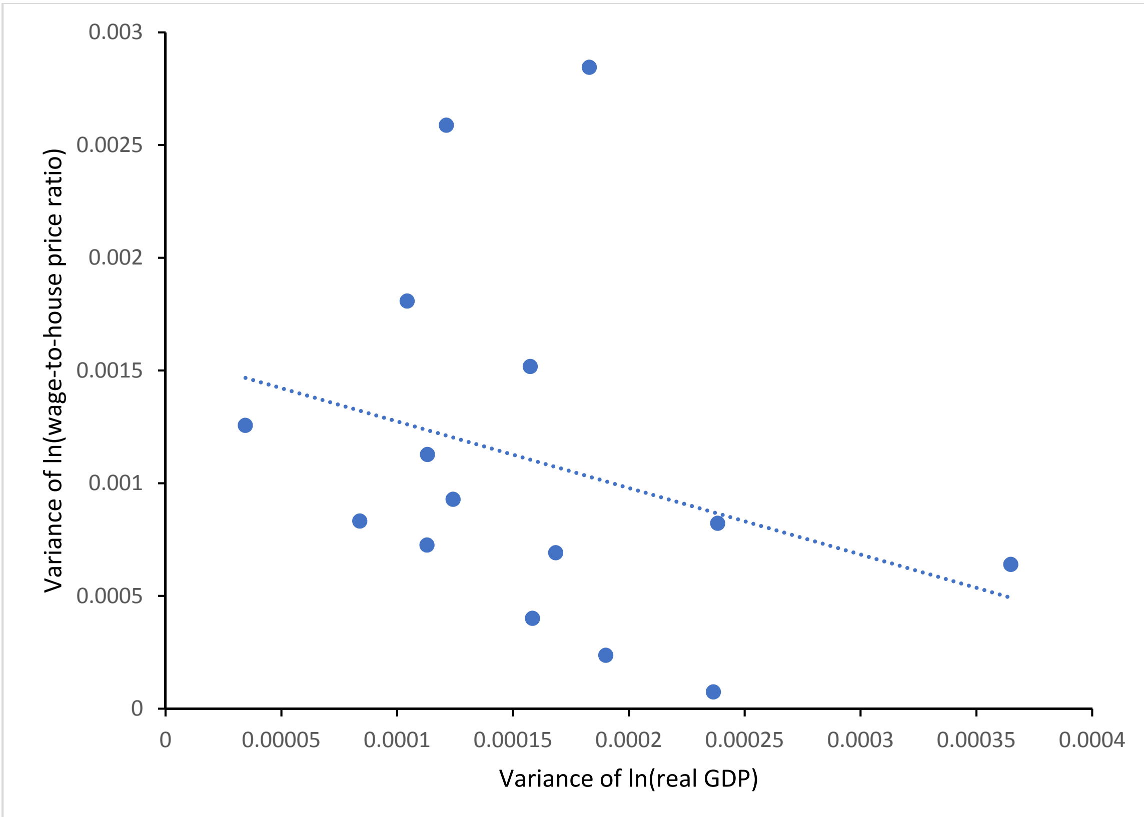
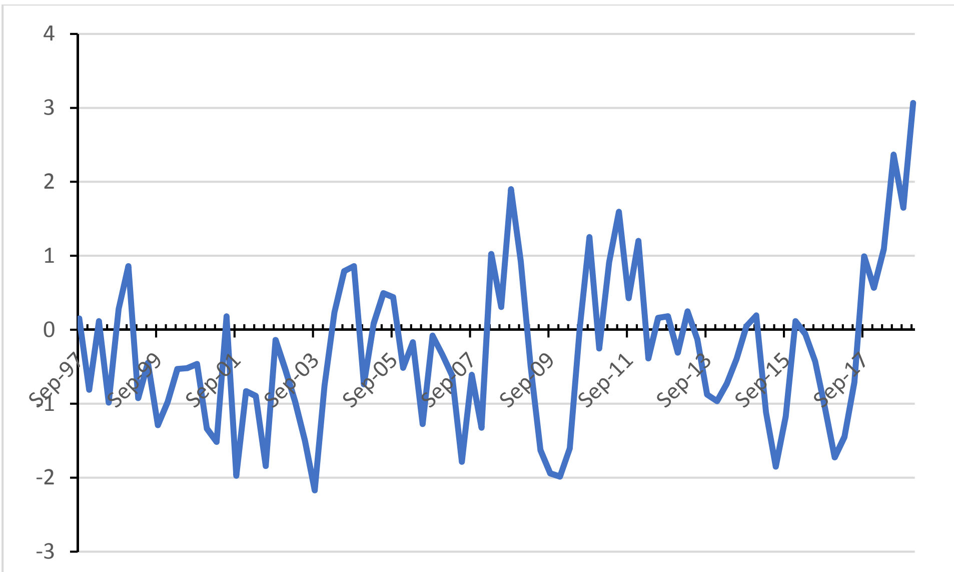


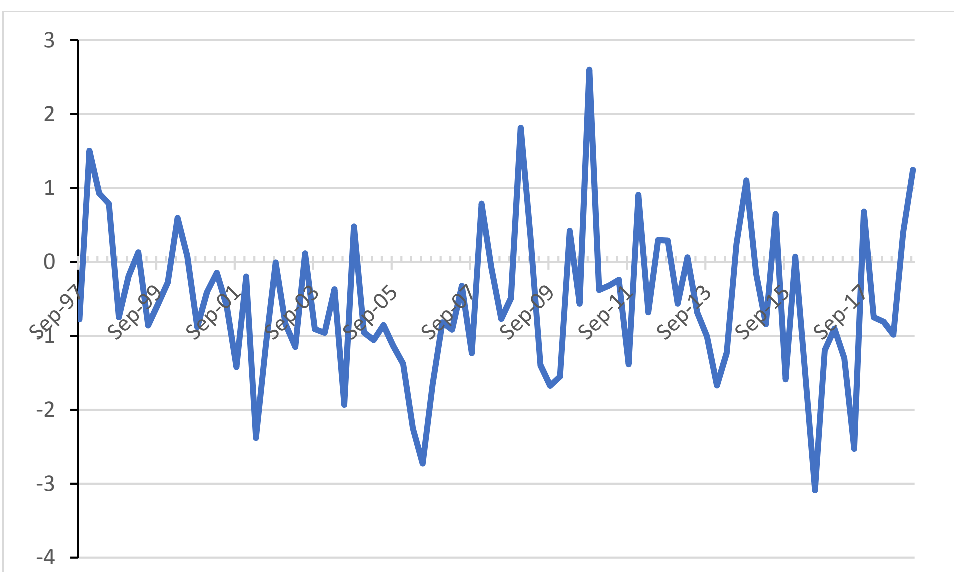
Figure 3. Detrended residual (based on our Group-mean Fully-modified OLS)

The residuals terms are obtained after performing group-mean fully-modified OLS. Then, we take the first-difference and divide each series by the corresponding standard deviation so that they become comparable.

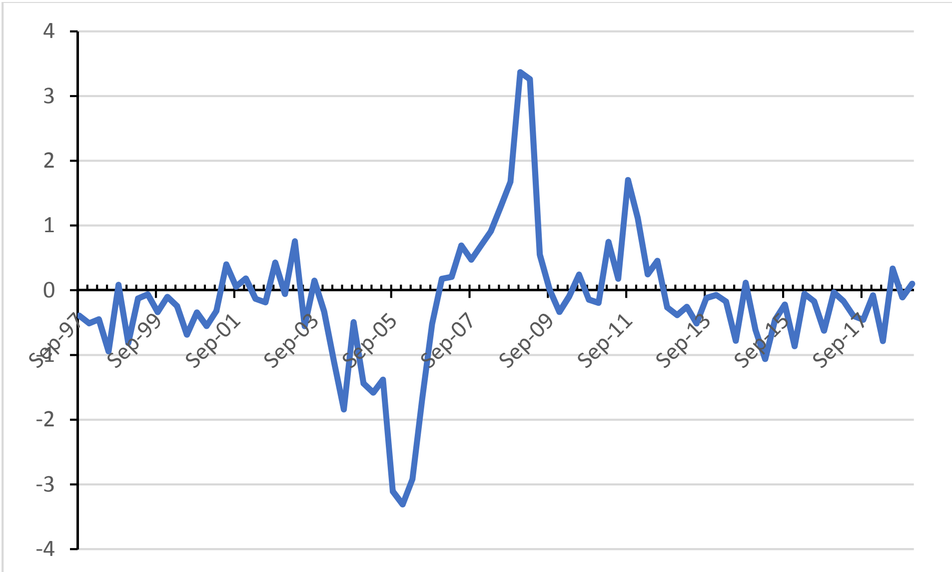
Australia



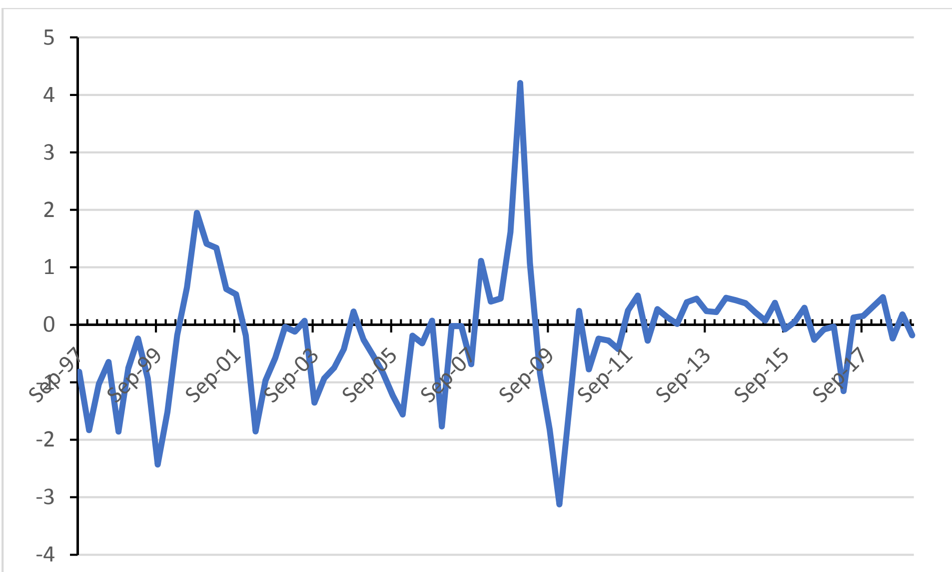
Canada



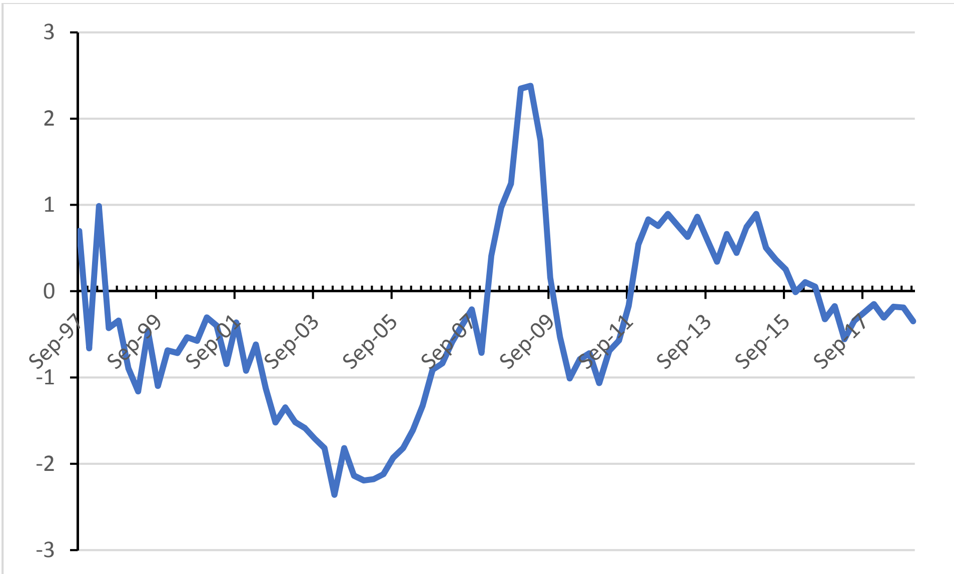
Denmark



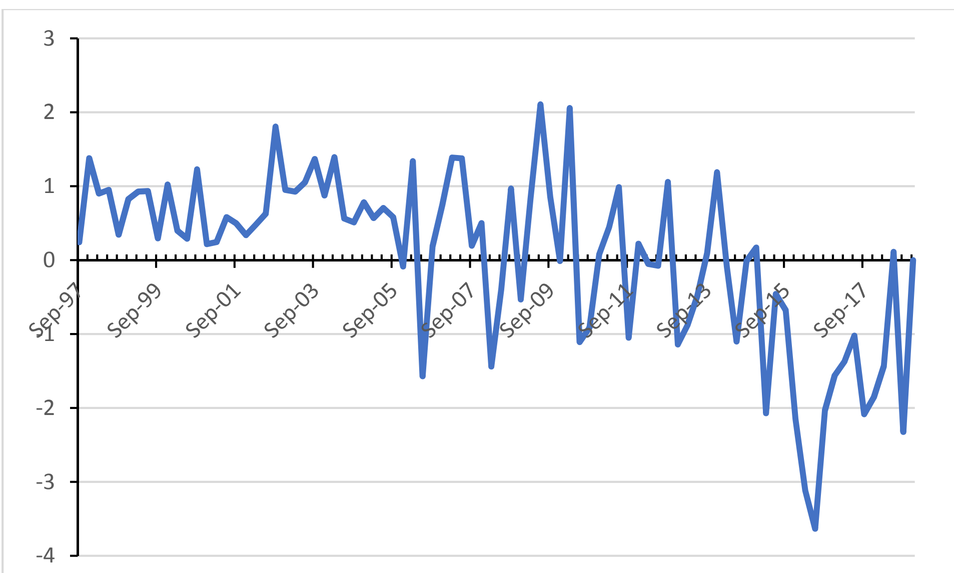
Finland



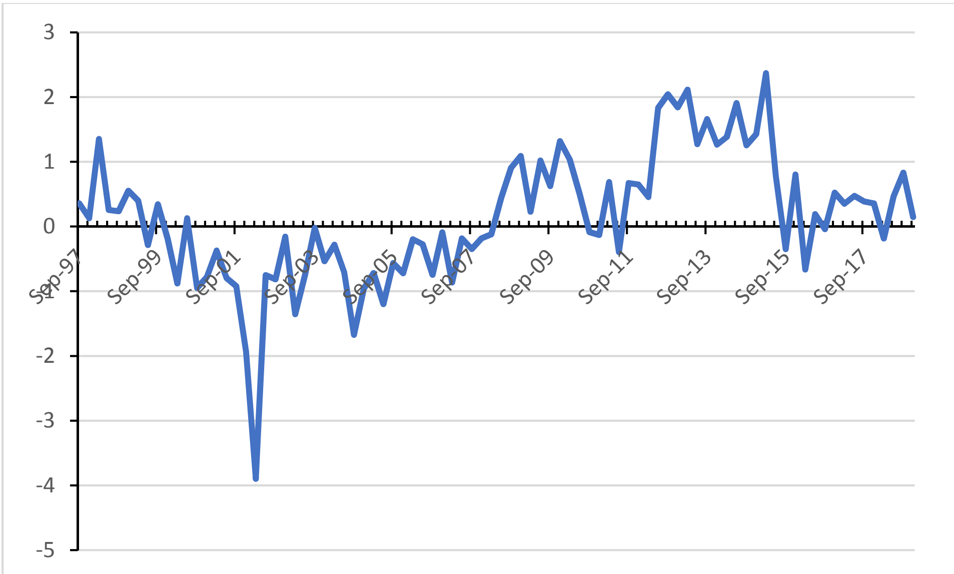
France



Germany

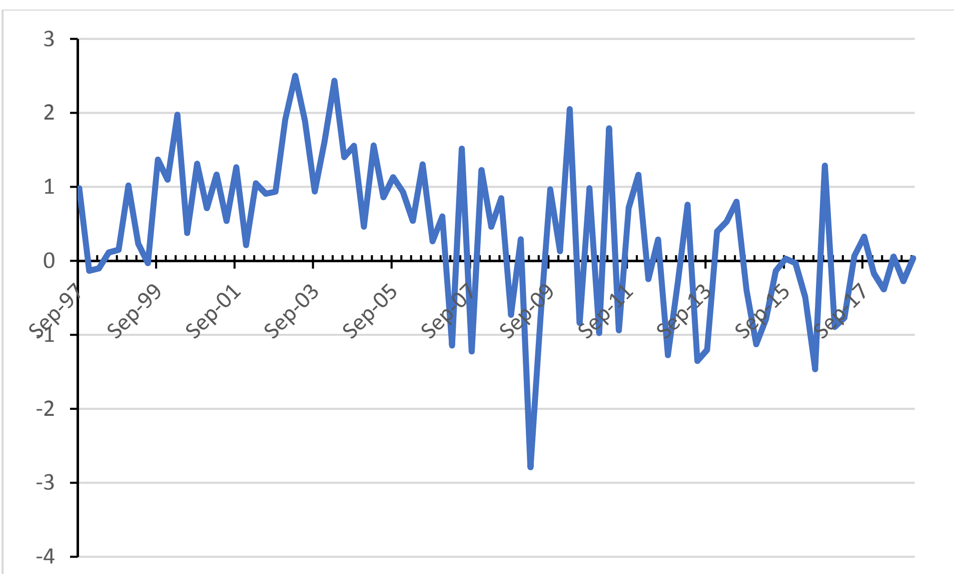


Italy



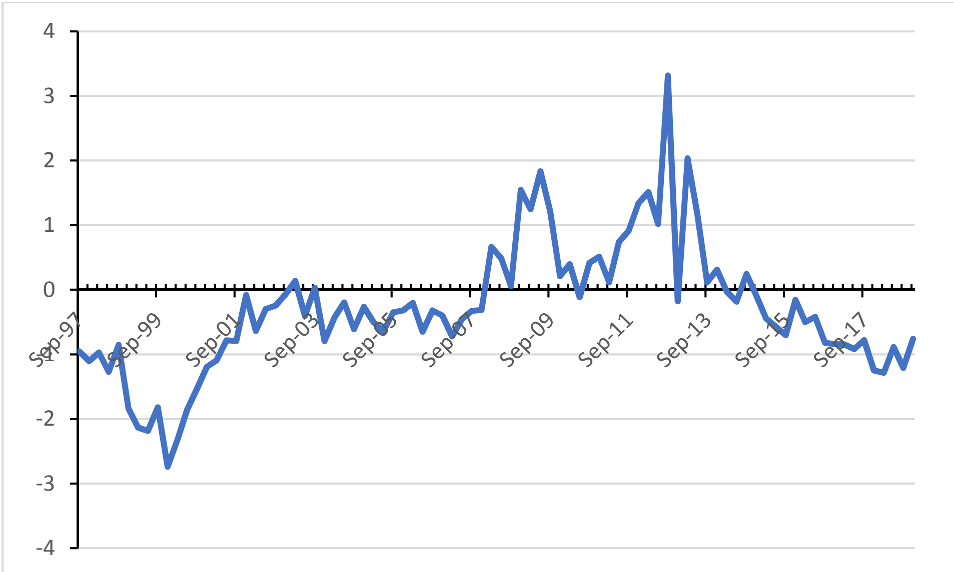
3
2
1
0
-1
-2
-3
-4
-5
Sep-97
Sep-99
Sep-01
Sep-03
Sep-05
Sep-07
Sep-09
Sep-11
Sep-13
Sep-15
Sep-17

Japan

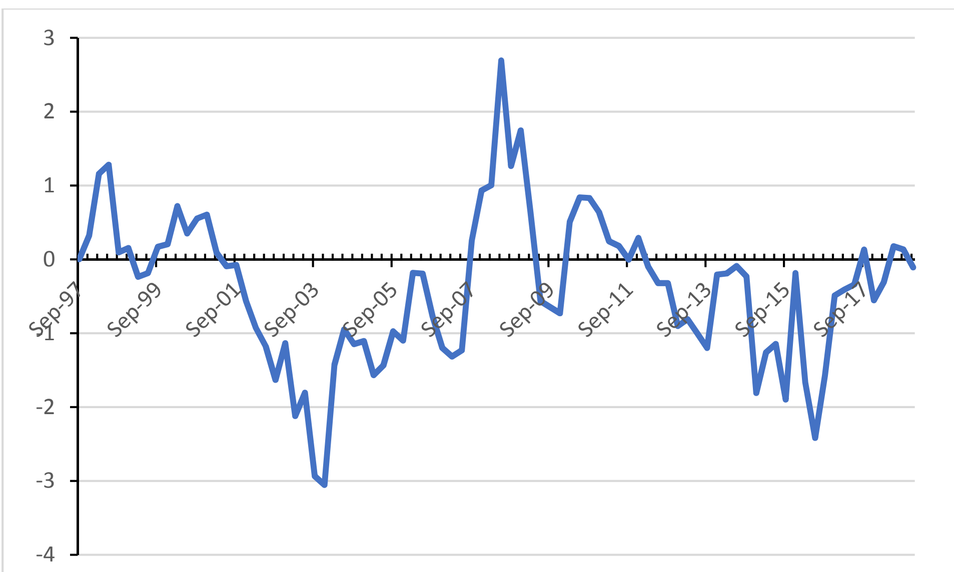


3
2
1
0
-1
-2
-3
-4
Sep-97
Sep-99
Sep-01
Sep-03
Sep-05
Sep-07
Sep-09
Sep-11
Sep-13
Sep-15
Sep-17

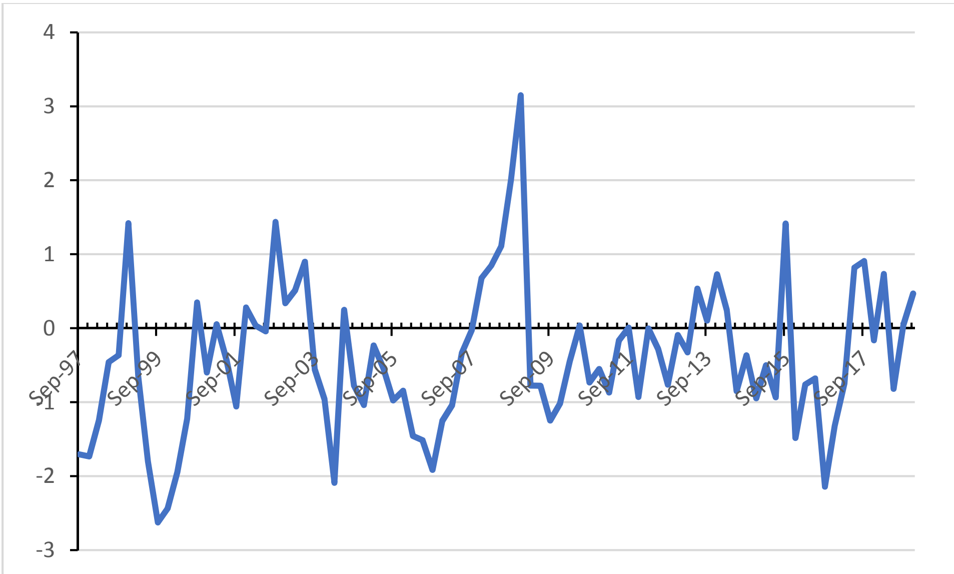
Netherlands



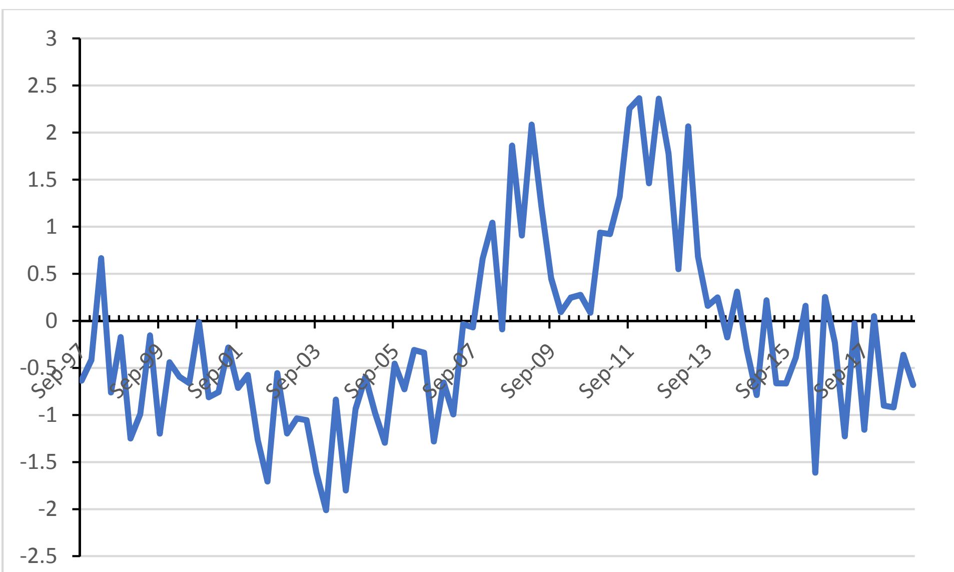
New Zealand



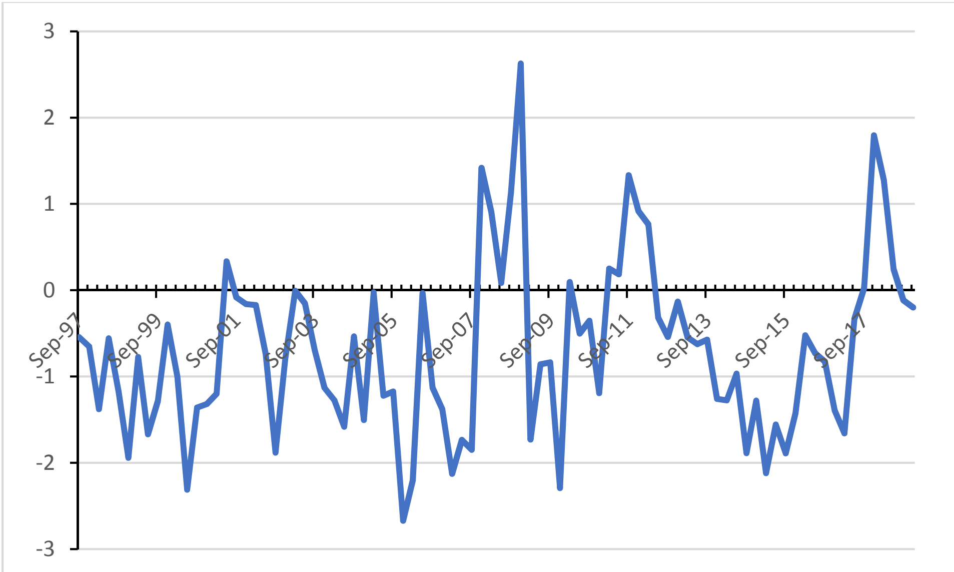
Norway



Spain

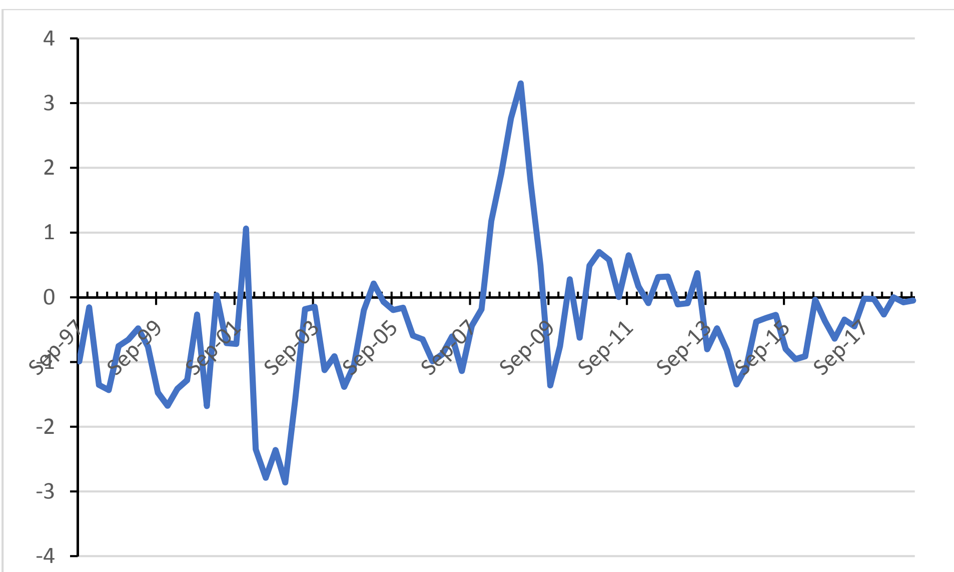


Sweden



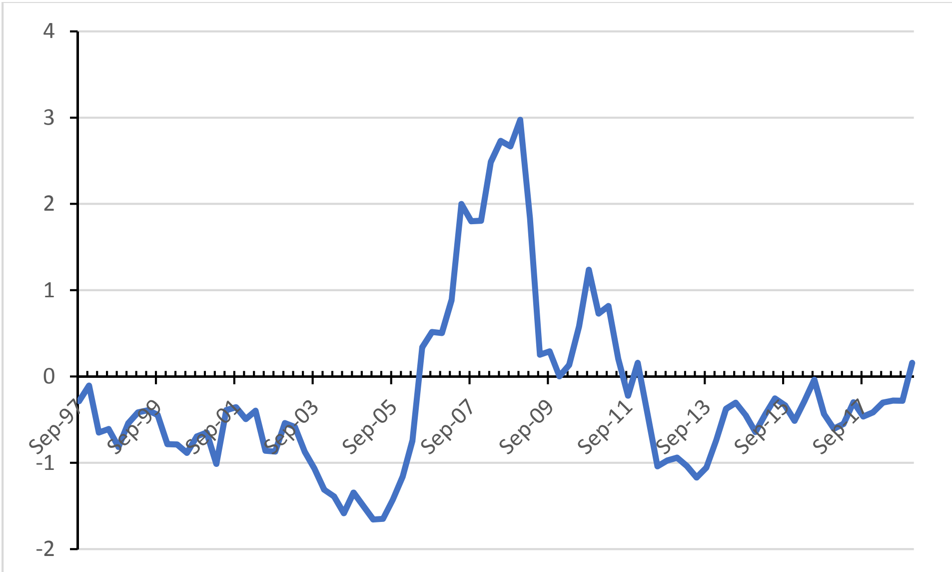
3
2
1
0
-1
-2
-3
Sep-97
Sep-99
Sep-01
Sep-03
Sep-05
Sep-07
Sep-09
Sep-11
Sep-13
Sep-15
Sep-17

UK



4
3
2
1
0
-1
-2
-3
-4
Sep-97
Sep-99
Sep-01
Sep-03
Sep-05
Sep-07
Sep-09
Sep-11
Sep-13
Sep-15
Sep-17

US


4
3
2
1
0
-1
-2
Sep-97
Sep-99
Sep-01
Sep-03
Sep-05
Sep-07
Sep-09
Sep-11
Sep-13

This appendix has several parts.

Appendix A provides the proofs of the analytical results.

Appendix B provides additional empirical results.

Appendix C provides the results when Ireland is included in the sample.

A Proof

A.1 Solution of the representative household problem and the proof of (14)

First, we will show that solution of the dynamic programming problem of the representative household can be summarized by the following proposition.

Proposition 10 *When the wages and prices are all flexible, the equilibrium dynamics of the model is characterized by the following equations:*

$$\begin{aligned} c_t &= b^c + y_t, \\ i_{kt} &= b^k + y_t, \\ i_{ht} &= b^h + y_t, \\ m_t - p_t &= b^m + y_t, \end{aligned} \tag{30}$$

where b^x, $x = c, h, k, m$, are all constant. Effort is constant over time,

$$l_t = l. \tag{31}$$

The log real wage w_t and log nominal wage w_t^n depend on the money supply,

$$w_t = b^w + m_t - p_t, \tag{32}$$

$$w_t^n = b^w + m_t, \tag{33}$$

where b^w is a constant.

Once we prove these statements, we can easily prove (14). We now provide the proof of (30), (31), (32), (33), (14).

The solution method adopted here is similar to that in Leung (2007). To prove all these results, we need to first obtain the first order conditions. Then combine them with the market clearing conditions. And then we will manipulate the algebra and obtain the equilibrium dynamics of the model. Recall that the dynamic maximization problem of the representative agent household can be formulated as,

$$V\left(K_t, H_t, \frac{M_{t-1}}{P_t}\right) = \max U\left(C_t, H_t + H_t^r, L_t, \frac{M_t}{P_t}\right) + \beta E_t V\left(K_{t+1}, H_{t+1}, \frac{M_t}{P_{t+1}}\right), \tag{34}$$

subject to (3), (4), (5).

The first order conditions are easy to derive,

FOC C_t
$$\lambda_{1t} = C_t^{-1} \tag{35}$$

FOC L_t
$$\omega_2 \left(1 - L_t\right)^{-1} = \lambda_{1t} W_t \tag{36}$$

FOC K_{t+1}

$$\lambda_{3t} = \beta E_t \left[\lambda_{1,t+1} R_{t+1} + (1 - \delta_k) \left(\frac{\lambda_{3,t+1} K_{t+2}}{K_{t+1}} \right) \right] \tag{37}$$

FOC I_{kt}

$$\lambda_{1t} = \lambda_{3t} \delta_k \frac{K_{t+1}}{I_{kt}} \tag{38}$$

FOC H_t^m

$$P_{ht} \lambda_{1t} = \lambda_{2t} \left(1 - \delta_h\right) \frac{H_{t+1}}{H_t + H_t^m} \tag{39}$$

FOC H_t^r
$$R_{ht} \lambda_{1t} = \frac{\omega_1}{H_t + H_t^r} \tag{40}$$

FOC H_{t+1}

$$\lambda_{2t} = \beta E_t \left[\frac{\omega_1}{H_{t+1} + H_{t+1}^r} + (1 - \delta_h) \frac{\lambda_{2,t+1} H_{t+2}}{H_{t+1} + H_{t+1}^m} \right] \tag{41}$$

FOC $I_{h,t}$

$$\lambda_{1t} = \lambda_{2t} \delta_h \frac{H_{t+1}}{I_{ht}} \tag{42}$$

FOC wrt M_t

$$\frac{\lambda_{1t}}{P_t} = \frac{\omega_3}{M_t} + \beta E_t \left(\frac{\lambda_{1,t+1} \mu_{t+1}}{P_{t+1}} \right). \tag{43}$$

At the equilibrium, (5) holds with equality,

$$R_t K_t + W_t L_t + \frac{\mu_t M_{t-1}}{P_t} = I_{k,t} + I_{h,t} + C_t + P_{ht} H_t^m + R_{ht} H_t^r + \frac{M_t}{P_t}. \tag{44}$$

Recall also that on the firm side, the production technology is constant returns to scale,

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha},$$

where α is the capital share, $0 < \alpha < 1$. The logarithm of the productivity shock follows an AR (1) process,

$$a_t = \rho a_{t-1} + \varepsilon_{at},$$

where $a_t = \ln A_t$, ρ measures the persistence of the productivity shock, $0 \leq \rho \leq 1$, and ε_{at} is a white noise (zero mean and constant variance). With competitive factor markets, we have (8), (9),

$$R_t = \frac{\partial Y_t}{\partial K_t} = \alpha \frac{Y_t}{K_t},$$

$$W_t = \frac{\partial Y_t}{\partial L_t} = (1-\alpha) \frac{Y_t}{L_t}.$$

In addition, we have market clearing conditions (for rental and secondary sale markets), (13):

$$H_t^m = H_t^r = 0.$$

On the government side, the money printed will be held by the household eventually, (11):

$$M_t = \mu_t M_{t-1}.$$

Equipped with all these equations, we are now ready to solve the model.

Using (13), (41) can be written as

$$\lambda_{2t} H_{t+1} = \beta E_t \left[\omega_1 + (1-\delta_h) \lambda_{2,t+1} H_{t+2}\right] \tag{45}$$

Iterating the equation above forward, we have

$$\lambda_{2t} H_{t+1} = \frac{\omega_1 \beta}{1-\beta(1-\delta_h)}, \tag{46}$$

where we impose a transversality condition

$$\lim_{j \to \infty} \left[\beta(1-\delta_h)\right]^j E_t \lambda_{2,t+j+1} H_{t+j+2} = 0. \tag{47}$$

Substituting (35),(46) into (42), we have

$$\frac{I_{ht}}{C_t} = \frac{\delta_h \omega_1 \beta}{1-\beta(1-\delta_h)}. \tag{48}$$

Substituting (35) into (38), we have

$$\lambda_{3t}K_{t+1} = \frac{I_{kt}}{C_t}\frac{1}{\delta_k}. \tag{49}$$

Substituting (35), (8) (12) (48) and (49) into (37), we have

$$\begin{aligned}
\lambda_{3t}K_{t+1} &= \beta E_t\left[\alpha\frac{Y_{t+1}}{C_{t+1}} + (1-\delta_k)\lambda_{3,t+1}K_{t+2}\right], \\
\frac{I_{kt}}{C_t}\frac{1}{\delta_k} &= \beta E_t\left[\alpha\left(1+\frac{I_{k,t+1}}{C_{t+1}}+\frac{I_{h,t+1}}{C_{t+1}}\right) + (1-\delta_k)\frac{I_{k,t+1}}{C_{t+1}}\frac{1}{\delta_k}\right] \\
\frac{I_{kt}}{C_t}\frac{1}{\delta_k} &= \beta E_t\left[\alpha\left(1+\frac{I_{k,t+1}}{C_{t+1}}+\frac{\delta_h\omega_1\beta}{1-\beta(1-\delta_h)}\right) + (1-\delta_k)\frac{I_{k,t+1}}{C_{t+1}}\frac{1}{\delta_k}\right] \\
\frac{I_{kt}}{C_t} &= \beta\alpha\delta_k\left(1+\frac{\delta_h\omega_1\beta}{1-\beta(1-\delta_h)}\right) + \left[\beta\alpha\delta_k + \beta(1-\delta_k)\right]E_t\left(\frac{I_{k,t+1}}{C_{t+1}}\right) \\
&= \beta\alpha\delta_k\frac{1-\beta(1-\delta_h)+\delta_h\omega_1\beta}{1-\beta(1-\delta_h)}\frac{1}{1-\beta\alpha\delta_k-\beta(1-\delta_k)}, \qquad (50)
\end{aligned}$$

where we impose transversality condition

$$\lim_{j\to\infty}\left[\beta\alpha\delta_k + \beta(1-\delta_k)\right]^j E_t\left(\frac{I_{k,t+j+1}}{C_{t+j+1}}\right) = 0. \tag{51}$$

Notice that as $0 < \alpha < 1$, $[\beta\alpha\delta_k + \beta(1-\delta_k)] < 1$, and $[\beta\alpha\delta_k + \beta(1-\delta_k)]^j \to 0$ as $j \to \infty$.

Using (48) and (50), (12) can be re-written as

$$\begin{aligned}
\frac{Y_t}{C_t} &= \frac{I_{kt}}{C_t} + \frac{I_{ht}}{C_t} + 1 \\
&= \beta\alpha_1\delta_k\frac{1-\beta(1-\delta_h)+\delta_h\omega_1\beta}{1-\beta(1-\delta_h)}\frac{1}{1-\beta\alpha_1\delta_k-\beta(1-\delta_k)} + \frac{\delta_h\omega_1\beta}{1-\beta(1-\delta_h)} + 1 \\
&= \frac{\beta\alpha_1\delta_k\left(1-\beta(1-\delta_h)+\delta_h\omega_1\beta\right) + \left[1-\beta\alpha_1\delta_k+\beta(1-\delta_k)\right]\delta_h\omega_1\beta}{\left[1-\beta(1-\delta_h)\right]\left[1-\beta\alpha_1\delta_k-\beta(1-\delta_k)\right]} \\
&\quad + \frac{\left[1-\beta(1-\delta_h)\right]\left[1-\beta\alpha_1\delta_k+\beta(1-\delta_k)\right]}{\left[1-\beta(1-\delta_h)\right]\left[1-\beta\alpha_1\delta_k-\beta(1-\delta_k)\right]} \\
&= \frac{\left[1-\beta(1-\delta_h)+\delta_h\omega_1\beta\right]\left[1-\beta(1-\delta_k)\right]}{\left[1-\beta(1-\delta_h)\right]\left[1-\beta\alpha_1\delta_k-\beta(1-\delta_k)\right]} \\
C_t &= S^c Y_t \qquad (52)
\end{aligned}$$

where

$$S^c = \left(1 - \frac{\alpha\beta\delta_k}{1-\beta(1-\delta_k)}\right)\left(\frac{1-\beta(1-\delta_h)}{1-\beta(1-\delta_h)+\omega_1\beta\delta_h}\right). \tag{53}$$

Clearly, we need $0 < S^c < 1$ so that $0 < C_t < Y_t$. The following lemma formulates the idea.

Lemma 11

$$C_t = S^c Y_t, \text{ where } 0 < S^c < 1. \tag{54}$$

Proof: Notice that as $0 < \beta, \delta_h < 1$, $\omega_1 > 0$, *we have* $0 < \frac{1-\beta(1-\delta_h)}{1-\beta(1-\delta_h)+\omega_1\beta\delta_h} < 1$. *It suffices to show that* $0 < \frac{\alpha\beta\delta_k}{1-\beta(1-\delta_k)} < 1$. *Notice that* $0 < \alpha,\ \beta,\ \delta_k < 1$. *Thus,* $0 < \frac{\alpha\beta\delta_k}{1-\beta(1-\delta_k)}$. *Notice that as* $0 < \beta,\ \delta_k$,

$$\begin{aligned} & \alpha\beta\delta_k + \beta(1-\delta_k) \\ < \;& \beta\delta_k + \beta(1-\delta_k) \;\; \text{as } \alpha < 1 \\ = \;& \beta < 1. \end{aligned}$$

In other words, $\alpha\beta\delta_k + \beta(1-\delta_k) < 1$, *which means that* $\frac{\alpha\beta\delta_k}{1-\beta(1-\delta_k)} < 1$. *Since* $S^c = \left(1 - \frac{\alpha\beta\delta_k}{1-\beta(1-\delta_k)}\right)\left(\frac{1-\beta(1-\delta_h)}{1-\beta(1-\delta_h)+\omega_1\beta\delta_h}\right)$, *and* $0 < \left(1 - \frac{\alpha\beta\delta_k}{1-\beta(1-\delta_k)}\right)$, $\left(\frac{1-\beta(1-\delta_h)}{1-\beta(1-\delta_h)+\omega_1\beta\delta_h}\right) < 1$. *It follows that* $0 < S^c < 1$.

Equipped with these results, we can prove the following lemma:

Lemma 12

$$I_{kt} = S^k Y_t, I_{ht} = S^h Y_t, \text{ where } 0 < S^k, S^h < 1. \tag{55}$$

Proof: We combine (50) and (52), and we have

$$\begin{aligned} I_{kt} &= \beta\alpha_1\delta_k \frac{1-\beta(1-\delta_h)+\delta_h\omega_1\beta}{1-\beta(1-\delta_h)} \frac{1}{1-\beta\alpha_1\delta_k-\beta(1-\delta_k)} C_t \\ &= S^k Y_t \end{aligned} \tag{56}$$

where

$$\begin{aligned}
S^k &\equiv \beta\alpha\delta_k\left(\frac{1-\beta\left(1-\delta_h\right)+\delta_h\omega_1\beta}{1-\beta\left(1-\delta_h\right)}\right)\frac{1}{1-\beta\alpha\delta_k-\beta\left(1-\delta_k\right)}S^c \\
&= \beta\alpha\delta_k\left(\frac{1-\beta\left(1-\delta_h\right)+\delta_h\omega_1\beta}{1-\beta\left(1-\delta_h\right)}\right)\frac{1}{1-\beta\alpha\delta_k-\beta\left(1-\delta_k\right)} \\
&\quad *\left(1-\frac{\alpha\beta\delta_k}{1-\beta\left(1-\delta_k\right)}\right)\left(\frac{1-\beta\left(1-\delta_h\right)}{1-\beta\left(1-\delta_h\right)+\omega_1\beta\delta_h}\right) \\
&= \frac{\alpha\beta\delta_k}{1-\beta\left(1-\delta_k\right)}, \qquad (57)
\end{aligned}$$

and we have just shown that $0<\frac{\alpha\beta\delta_k}{1-\beta(1-\delta_k)}<1$. *Thus, we have* $0<S^k<1$.

Similarly, (48) can be written as:

$$\begin{aligned}
I_{ht} &= \frac{\delta_h\omega_1\beta}{1-\beta\left(1-\delta_h\right)}C_t \\
&= S^h Y_t, \qquad (58)
\end{aligned}$$

where

$$\begin{aligned}
S^h &\equiv \frac{\delta_h\omega_1\beta}{1-\beta\left(1-\delta_h\right)}S^c \\
&= \frac{\delta_h\omega_1\beta}{1-\beta\left(1-\delta_h\right)}\left(1-\frac{\alpha\beta\delta_k}{1-\beta\left(1-\delta_k\right)}\right)\left(\frac{1-\beta\left(1-\delta_h\right)}{1-\beta\left(1-\delta_h\right)+\omega_1\beta\delta_h}\right) \\
&= \left(1-\frac{\alpha\beta\delta_k}{1-\beta\left(1-\delta_k\right)}\right)\left(\frac{\omega_1\beta\delta_h}{1-\beta\left(1-\delta_h\right)+\omega_1\beta\delta_h}\right). \qquad (59)
\end{aligned}$$

We have just proved that $0<\frac{\alpha\beta\delta_k}{1-\beta(1-\delta_k)}<1$. *In addition,* $0<\beta,\delta_h<1$, $\omega_1>0$, *we have* $0<\frac{\omega_1\beta\delta_h}{1-\beta(1-\delta_h)+\omega_1\beta\delta_h}<1$. *Thus, we have* $0<S^h<1$.

By the same token, we can prove the following results,

Lemma 13

$$\left(\frac{M_t}{P_t}\right)=S^m Y_t,\ \text{where } 0<S^m. \qquad (60)$$

Proof: Substituting (35), (11) into (43), we have

$$\begin{aligned}
\frac{1}{C_t} &= \frac{\omega_3 P_t}{M_t} + \beta E_t \left(\frac{P_t M_{t+1}}{C_{t+1} P_{t+1} M_t} \right) \\
\frac{M_t}{P_t C_t} &= \omega_3 + \beta E_t \left(\frac{M_{t+1}}{P_{t+1} C_{t+1}} \right) \\
\frac{M_t}{P_t C_t} &= \frac{\omega_3}{1-\beta} \\
\frac{M_t}{P_t} &= \frac{\omega_3}{1-\beta} C_t = S^m Y_t \qquad (61)
\end{aligned}$$

where

$$S^m \equiv \frac{\omega_3}{1-\beta} S^c, \qquad (62)$$

with S^c is defined by (53), and we have implicitly assumed the transversality condition,

$$\lim_{j \to \infty} \beta^j E_t \left(\frac{M_{t+j}}{P_{t+j} C_{t+j}} \right) = 0. \qquad (63)$$

Since $0 < S^c$, ω_3, $(1-\beta) > 0$, it follows that $0 < S^m$.

We now take natural log of (54), (55) and (60), and use small letter variables to denote log of capital variables, and we have (30).

Now we move to derive results regarding the labor market, including both equilibrium labor supply and wages. Notice that the values of consumption, investment, real money balance all depend on the output level y_t, and by taking log of (6), we have

$$y_t = a_t + \alpha k_t + (1-\alpha) l_t, \qquad (64)$$

and thus to determine the dynamics of the output, we need to know the dynamics of both labor hours as well as business capital stock. In the case of flexible wages, we can dictate the equilibrium labor hours by substituting (35), (9) into (36) and make use of (52)

$$\begin{aligned}
\omega_2 (1-L_t)^{-1} &= \frac{1}{C_t} \alpha_2 \frac{Y_t}{L_t} \\
\omega_2 (1-L_t)^{-1} &= \frac{\alpha_2}{S^c} \frac{1}{L_t} \\
\frac{S^c \omega_2}{\alpha_2} &= \frac{1-L_t}{L_t} \\
\frac{\alpha_2 + S^c \omega_2}{\alpha_2} &= \frac{1}{L_t} \\
L_t &= \frac{\alpha_2}{\alpha_2 + S^c \omega_2} = L. \qquad (65)
\end{aligned}$$

Taking log of (65) gives us (31),

$$l_t = l,$$

where

$$l = \ln\left(\frac{\alpha_2}{\alpha_2 + S^c\omega_2}\right). \tag{66}$$

We then combine (61), (65) with (9) and obtain the following expression,

$$W_t = \alpha_2 \frac{1}{L}\frac{1}{S^m}\frac{M_t}{P_t}. \tag{67}$$

Taking log on both sides and we verify (32). In addition, it is straightforward to show that

$$\begin{aligned} b^w &= \ln\left(\alpha_2 \frac{1}{L}\frac{1}{S^m}\right) \\ &= \ln \alpha_2 - l - \ln S^m, \end{aligned} \tag{68}$$

where l is defined by (66) and S^m is defined by (62).

Recall that W_t is the real wage. Let W_t^n be the nominal wage. By definition, the real wage is defined as the nominal wage divided by the price level, (67) implies that

$$W_t^n = \alpha_2 \frac{1}{L}\frac{1}{S^m} M_t. \tag{69}$$

So the equilibrium nominal wage is proportional to the money supply. Or, in log form,

$$w_t^n = b^w + m_t,$$

which is (33).

Now we need to study the dynamics of the capital stock. Take log of (3) and make use of (56), we have

$$\begin{aligned} k_{t+1} &= (1-\delta_k)\,k_t + \delta_k i_{k,t} \\ &= (1-\delta_k)\,k_t + \delta_k\left(\ln S^k + y_t\right) \\ &= b^{k\prime} + (1-\delta_k)\,k_t + \delta_k y_t, \end{aligned} \tag{70}$$

where $b^{k\prime}$ is a constant. Notice that when we combine (31) with (64), we have

$$y_t = b^y + \alpha k_t + a_t, \tag{71}$$

where $b^y \equiv (1-\alpha)l$, which is a constant. Putting (70) and (71) together, with appropriate adjustment in the time period, we have

$$\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_t \\ k_t \end{pmatrix} = \begin{pmatrix} b^y \\ b^{k\prime} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \delta_k & (1-\delta_k) \end{pmatrix}\begin{pmatrix} y_{t-1} \\ k_{t-1} \end{pmatrix} + \begin{pmatrix} a_t \\ 0 \end{pmatrix},$$

or,

$$\begin{aligned}
\begin{pmatrix} y_t \\ k_t \end{pmatrix} &= \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}^{-1}\left\{\begin{pmatrix} b^y \\ b^{k\prime} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \delta_k & (1-\delta_k) \end{pmatrix}\begin{pmatrix} y_{t-1} \\ k_{t-1} \end{pmatrix} + \begin{pmatrix} a_t \\ 0 \end{pmatrix}\right\} \\
&= \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\left\{\begin{pmatrix} b^y \\ b^{k\prime} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \delta_k & (1-\delta_k) \end{pmatrix}\begin{pmatrix} y_{t-1} \\ k_{t-1} \end{pmatrix} + \begin{pmatrix} a_t \\ 0 \end{pmatrix}\right\} \\
&= \begin{pmatrix} b^y + \alpha b^{k\prime} \\ b^{k\prime} \end{pmatrix} + \begin{pmatrix} \alpha\delta_k & \alpha(1-\delta_k) \\ \delta_k & (1-\delta_k) \end{pmatrix}\begin{pmatrix} y_{t-1} \\ k_{t-1} \end{pmatrix} + \begin{pmatrix} a_t \\ 0 \end{pmatrix},
\end{aligned}$$

which is (14). Thus,

$$\overrightarrow{a}_t = \begin{pmatrix} a_t \\ 0 \end{pmatrix}. \tag{72}$$

By (7),

$$a_t = \rho a_{t-1} + \varepsilon_{at}.$$

Thus, we have

$$\overrightarrow{a}_t = \rho^j \overrightarrow{a}_{t-j} + \sum_{i=0}^{j-1} \rho^i \overrightarrow{\varepsilon}_{a,t-i},$$

where

$$\overrightarrow{\varepsilon}_{a,t} = \begin{pmatrix} \varepsilon_{at} \\ 0 \end{pmatrix},$$

and by assumption, $E(\varepsilon_{at}) = 0$, $E(\varepsilon_{at}, \varepsilon_{as}) = 0$ whenever $s \neq t$. Thus, we have proved (15).

A.2 Proof of (16), (17) and (18)

The proof will proceed in a few steps. First, we will describe the relationship between house price and house rent. Then we will relate the house price to the output and housing stock.

Using (39) (40), (13) and (46), we have the relationship between housing rent and housing price,

$$\begin{aligned} \frac{R_{ht}}{P_{ht}} &= \frac{\omega_1}{\lambda_{2t}(1-\delta_h) H_{t+1}} = \frac{\omega_1}{(1-\delta_h)\frac{\omega_1 \beta}{1-\beta(1-\delta_h)}} = \frac{1-\beta(1-\delta_h)}{\beta(1-\delta_h)} \\ R_{ht} &= \frac{1-\beta(1-\delta_h)}{\beta(1-\delta_h)} P_{ht}. \end{aligned}$$

Since house price and rent are proportional to each other, it suffices to study the house price.

Combining (39) and (42), and using (13), we have

$$P_{ht} = \frac{1-\delta_h}{\delta_h} \frac{I_{ht}}{H_t},$$

and when we combine it with (55) and take log, we have (16),

$$p_{ht} = \alpha_{ph} + y_t - h_t.$$

The proof of (17) is easy. It can be obtained by simply combining (30), (32), (67) and (16).

To derive (18), we first recall (17) that

$$\ln\left(\frac{W_t}{P_{ht}}\right) = w_t - p_{ht} = b^{wp} + h_t.$$

And we can combine (4), (13) and (30),

$$\begin{aligned}
h_{t+1} &= (1-\delta_h) h_t + \delta_h i_{ht} \\
&= (1-\delta_h) h_t + \delta_h \left(b^h + y_t\right) \\
&= \delta_h b^h + \delta_h y_t + (1-\delta_h) h_t \\
&= \delta_h b^h + \delta_h y_t + (1-\delta_h) \left[\delta_h b^h + \delta_h y_{t-1} + (1-\delta_h) h_{t-1}\right] \\
&= \delta_h b^h (1 + (1-\delta_h)) + \delta_h y_t + (1-\delta_h) \delta_h y_{t-1} \\
&\quad + (1-\delta_h)^2 h_{t-1} \\
&= ... \\
&= b^{h\prime} + \delta_h \sum_{i=0}^{t} (1-\delta_h)^i y_{t-i} + (1-\delta_h)^t h_0, \qquad (73)
\end{aligned}$$

where $b^{h\prime}$ is a constant, h_0 is the amount of initial housing stock in the model economy. We then substitute the last expression into (17) and we get

$$w_t - p_{ht} = b^{wp\prime} - \delta_h \sum_{i=0}^{t-1} (1-\delta_h)^i y_{t-1-i} - (1-\delta_h)^{t-1} h_0,$$

where $b^{wp\prime}$ is a constant, and we have completed the proof of (18).

A.3 Proof of (21), (23)

Recall that under the modified economic environment, we assume that wages are set one period in advance. The contract wage is fixed one period in advance in such a manner that it will clear the market ex ante. This means that the contract wage is set equal to the expected value of the Walrasian wage ($E_{t-1}w_t^{n*}$), i.e. (19),

$$w_t^n = E_{t-1}w_t^{n*} = b^w + E_{t-1}m_t,$$

and the real wage in this case is described by (20),

$$w_t = w_t^n - p_t = E_{t-1}w_t^{n*} - p_t = b^w + E_{t-1}m_t - p_t,$$

where the formula of b^w is given by (68).

Notice that the factor return is still equalized to the corresponding marginal product as the firms maximize profit and the factor market is competitive. In particular, (9) still holds. In log form, it can be expressed as

$$w_t = y_t - l_t + \ln \alpha_2. \tag{74}$$

At the same time, money market clearing ensures that (60) holds. In log form, it is expressed as

$$m_t - p_t = \ln S^m + y_t. \tag{75}$$

We then combine (20), (74), (75), with (68), and obtain

$$l_t = l + \varepsilon_{mt},$$

which is (21), when we define ε_{mt} according to (22),

$$\varepsilon_{mt} \equiv m_t - E_{t-1} m_t.$$

To derive (23), we first recall that the aggregate production function can be written in log form, which is (64),

$$y_t = a_t + \alpha k_t + (1 - \alpha) l_t.$$

Combine it with (21), we have

$$\begin{aligned} y_t &= (1-\alpha) l + a_t + \alpha k_t + (1-\alpha) \varepsilon_{mt} \\ &= b^y + a_t + \alpha k_t + (1-\alpha) \varepsilon_{mt}. \end{aligned} \tag{76}$$

Thus, to understand the dynamics of y_t, we also need to understand the dynamics of k_t. We begin our investigation with the law of motion of the business capital k_t. Taking log of (3) and combine it with (30), we have

$$\begin{aligned} k_{t+1} &= (1-\delta_k) k_t + \delta_k \left(\ln S^k + y_t \right) \\ &= \delta_k \ln S^k + (1-\delta_k) k_t + \delta_k y_t, \end{aligned}$$

which is (70), with S^k is deinfed in (57).

Now we can combine (76) with (70), and we have

$$\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_t \\ k_t \end{pmatrix} = \begin{pmatrix} b^y \\ b^{k\prime} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \delta_k & (1-\delta_k) \end{pmatrix}\begin{pmatrix} y_{t-1} \\ k_{t-1} \end{pmatrix} + \begin{pmatrix} a_t + (1-\alpha)\,\varepsilon_{mt} \\ 0 \end{pmatrix}, \tag{77}$$

which is (23). Notice also that the "residual term" is $\begin{pmatrix} a_t + (1-\alpha)\,\varepsilon_{mt} \\ 0 \end{pmatrix}$, which is serially correlated because a_t is serially correlated, by (7).

A.4 Proof of (25)

Recall from (24) that the money growth rate is now a function of other economic variables,

$$\left(\frac{\mu_t}{\mu}\right) = \left(\frac{\mu_{t-1}}{\mu}\right)^{\rho^\mu} \left(\frac{P_t/P_{t-1}}{\pi}\right)^{\phi_\pi} \left(\frac{Y_t}{Y}\right)^{\phi_y} e^{\varepsilon_{\mu t}},$$

which, combined with (11), can be re-written in log form,

$$(m_t - m_{t-1}) - \ln\mu = \rho^\mu\,(m_{t-1} - m_{t-2} - \ln\mu) + \phi_\pi\,(p_t - p_{t-1} - \ln\pi) + \phi_y\,(y_t - y) + \varepsilon_{\mu t}. \tag{78}$$

When we take expectation on both sides based on the period $(t-1)$ information, we will get

$$\begin{aligned} E_{t-1}m_t &= m_{t-1} + \rho^\mu\,(m_{t-1} - m_{t-2}) + (1-\rho^\mu)\ln\mu + \phi_\pi\,(E_{t-1}p_t - p_{t-1} - \ln\pi) \\ &\quad + \phi_y\,(E_{t-1}y_t - y)\,. \end{aligned} \tag{79}$$

Therefore, by (78) and (79), we have

$$\begin{aligned} \varepsilon_{mt} &\equiv m_t - E_{t-1}m_t \\ &= \phi_\pi\,(p_t - E_{t-1}p_t) + \phi_y\,(y_t - E_{t-1}y_t) + \varepsilon_{\mu t}. \end{aligned} \tag{80}$$

In words, it means that the "forecast error" on the money growth rate is a "weighted sum" of the forecast error of the general price level $(p_t - E_{t-1}p_t)$, the forecast error of the aggregate output level $(y_t - E_{t-1}y_t)$, and the innovation term $\varepsilon_{\mu t}$. Thus, to gain a better understanding of the ε_{mt} term, we need to compute the two forecast errors, $(p_t - E_{t-1}p_t)$ and $(y_t - E_{t-1}y_t)$.

Based on (76),

$$y_t = b^y + a_t + \alpha k_t + (1-\alpha)\,\varepsilon_{mt}.$$

Thus,

$$(y_t - E_{t-1}y_t) = (a_t - E_{t-1}a_t) + (1-\alpha)\,\varepsilon_{mt}. \tag{81}$$

Now we need to compute $(p_t - E_{t-1}p_t)$. Recall from (30) that

$$m_t - p_t = b^m + y_t,$$

which means that

$$(m_t - E_{t-1}m_t) - (p_t - E_{t-1}p_t) = (y_t - E_{t-1}y_t).$$

By definition, it means that

$$(p_t - E_{t-1}p_t) = \varepsilon_{mt} - (y_t - E_{t-1}y_t). \tag{82}$$

Now substitute (81), (82) into (80), we have

$$\varepsilon_{mt} = \phi_\pi\left(\varepsilon_{mt} - (y_t - E_{t-1}y_t)\right) + \phi_y\,(y_t - E_{t-1}y_t) + \varepsilon_{\mu t},$$

or

$$\begin{aligned}(1-\phi_\pi)\,\varepsilon_{mt} &= \left(\phi_y - \phi_\pi\right)(y_t - E_{t-1}y_t) + \varepsilon_{\mu t} \\ &= \left(\phi_y - \phi_\pi\right)\left[(a_t - E_{t-1}a_t) + (1-\alpha)\,\varepsilon_{mt}\right] + \varepsilon_{\mu t},\end{aligned}$$

which means that

$$\left[(1-\phi_\pi) - \left(\phi_y - \phi_\pi\right)(1-\alpha)\right]\varepsilon_{mt} = \left(\phi_y - \phi_\pi\right)(a_t - E_{t-1}a_t) + \varepsilon_{\mu t},$$

or

$$\begin{aligned}\varepsilon_{mt} &= \frac{\left(\phi_y - \phi_\pi\right)}{\left[(1-\phi_\pi) - \left(\phi_y - \phi_\pi\right)(1-\alpha)\right]}(a_t - E_{t-1}a_t) \\ &\quad + \frac{1}{\left[(1-\phi_\pi) - \left(\phi_y - \phi_\pi\right)(1-\alpha)\right]}\varepsilon_{\mu t}.\end{aligned}$$

In words, it means that the "forecast error" of money supply ε_{mt}, is a *weighted sum* of the forecast error of productivity, $(a_t - E_{t-1}a_t)$, and the innovation term in monetary growth $\varepsilon_{\mu t}$. In fact, we can give even more details about the forecast error of productivity, $(a_t - E_{t-1}a_t)$. Recall from (7) that

$$a_t = \rho a_{t-1} + \varepsilon_{at}.$$

It is straightforward to show that

$$(a_t - E_{t-1}a_t) = \varepsilon_{at}.$$

Thus, we complete the proof of (25), with

$$\begin{aligned}
\omega^a &\equiv \frac{\left(\phi_y - \phi_\pi\right)}{\left[(1-\phi_\pi) - \left(\phi_y - \phi_\pi\right)(1-\alpha)\right]} \\
&= \frac{1}{\left[\frac{(1-\phi_\pi)}{\left(\phi_y - \phi_\pi\right)} - (1-\alpha)\right]}, \\
\omega^\mu &\equiv \frac{1}{\left[(1-\phi_\pi) - \left(\phi_y - \phi_\pi\right)(1-\alpha)\right]}.
\end{aligned}$$

A.5 Proof of (26)

Before we prove the desired result, we need to have some technical results.

Corollary 14 *Given (24) holds, the forecast error of money supply, ε_{mt}, shares the same properties of $\varepsilon_{\mu t}$ in the following sense,*

$$E\left(\varepsilon_{mt}\right) = 0,\ Var\left(\varepsilon_{mt}\right) \text{ is a constant, } E\left(\varepsilon_{mt}\varepsilon_{ms}\right) = 0 \text{ whenever } s \neq t. \quad (83)$$

The proof is simple.

- $E\left(\varepsilon_{mt}\right) = E\left(\omega^a\varepsilon_{at} + \omega^\mu\varepsilon_{\mu t}\right) = \omega^a E\left(\varepsilon_{at}\right) + \omega^\mu E\left(\varepsilon_{\mu t}\right) = 0.$
- $Var\left(\varepsilon_{mt}\right) = Var\left(\omega^a\varepsilon_{at} + \omega^\mu\varepsilon_{\mu t}\right) = \left(\omega^a\right)^2 Var\left(\varepsilon_{at}\right) + \left(\omega^\mu\right)^2 Var\left(\varepsilon_{\mu t}\right)$ *since* $\{\varepsilon_{at}\}$ *and* $\{\varepsilon_{\mu t}\}$ *are independent. And since both* $Var\left(\varepsilon_{at}\right)$, $Var\left(\varepsilon_{\mu t}\right)$ *are constant, so is* $Var\left(\varepsilon_{mt}\right)$.
- $E\left(\varepsilon_{mt}\varepsilon_{ms}\right) = E\left[\left(\omega^a\varepsilon_{at} + \omega^\mu\varepsilon_{\mu t}\right)\left(\omega^a\varepsilon_{as} + \omega^\mu\varepsilon_{\mu s}\right)\right] = \left(\omega^a\right)^2 E\left(\varepsilon_{at}\varepsilon_{as}\right) + \left(\omega^\mu\right)^2 E\left(\varepsilon_{\mu t}\varepsilon_{\mu s}\right)$ *since* $\{\varepsilon_{at}\}$ *and* $\{\varepsilon_{\mu t}\}$ *are independent. By assumption,* $E\left(\varepsilon_{at}\varepsilon_{as}\right) = E\left(\varepsilon_{\mu t}\varepsilon_{\mu s}\right) = 0$, *whenever* $s \neq t$. *Thus,* $E\left(\varepsilon_{mt}\varepsilon_{ms}\right) = 0$.

Equipped with all these results, we know that the forecast error of money supply, ε_{mt}, behaves like an i.i.d. process. In other words, we do not need to worry about the serial correlations. This piece of knowledge will simplify the computations significantly.

Now recall (77) that

$$\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_t \\ k_t \end{pmatrix} = \begin{pmatrix} b^y \\ b^{k\prime} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \delta_k & (1-\delta_k) \end{pmatrix}\begin{pmatrix} y_{t-1} \\ k_{t-1} \end{pmatrix} + \begin{pmatrix} a_t + (1-\alpha)\,\varepsilon_{mt} \\ 0 \end{pmatrix}.$$

And by (25), we have

$$\varepsilon_{mt} = \omega^a \varepsilon_{at} + \omega^\mu \varepsilon_{\mu t}.$$

Combine the two expressions, we can describe the dynamics of the output. Our focus, however, is the real wage-to-house price ratio. Thus, we should try to relate the real wage and the house price to the output and innovation terms.

We begin with the real wage. Recall that if (19) holds, then the real wage process is described by (20),

$$w_t = w_t^n - p_t = E_{t-1} w_t^{n*} - p_t = b^w + E_{t-1} m_t - p_t.$$

Since (75) holds, we also have

$$p_t = m_t - \ln S^m - y_t.$$

Combine the two expressions, we have

$$w_t = w_t^n - p_t = b^w + E_{t-1} m_t - m_t + \ln S^m + y_t.$$

By definition and (25), we have

$$m_t - E_{t-1} m_t = \varepsilon_{mt} = \omega^a \varepsilon_{at} + \omega^\mu \varepsilon_{\mu t}.$$

Thus, the real wage (in log) can be expressed as

$$w_t = w_t^n - p_t = (b^w + \ln S^m) + y_t - \widehat{\varepsilon}_t, \tag{84}$$

where

$$\widehat{\varepsilon}_t \equiv \omega^a \varepsilon_{at} + \omega^\mu \varepsilon_{\mu t} = \varepsilon_{mt}. \tag{85}$$

Now we need to derive the equilibrium house price p_{ht} in this model economy. It is not difficult to see that the house price equation changes very little with rigid wage. One can start from the individual dynamic optimization problem, i.e. to maximize (34) subject to (3), (4), (5), as before. The only change is that the wage is pre-set by the wage contract. From the perspective of the representative household, however, the real wage and other prices are taken as given anyway. Thus, the short run wage rigidity may not have much impact on the housing market.

In the first glance, this result may be surprising. Notice that the nominal rigid wage will impact the ex post labor supply, as shown in (21). However, since consumption, housing and leisure all separable to each other in the utility function, this change would not have a direct effect on the marginal utility of consumpton or that of housing. In fact, following the proof of proposition 1 and 2, we find that the equation (16) still holds,

$$p_{ht} = \alpha_{ph} + y_t - h_t.$$

Now since (35) to (43) still apply, following the proof of proposition 1, it is straightforward to show that (73) still applies, and hence $h_t = b^{h\prime} + \delta_h \sum_{i=0}^{t-1} (1-\delta_h)^i y_{t-1-i} + (1-\delta_h)^{t-1} h_0$,

$$p_{ht} = \alpha'_{ph} + y_t - \delta_h \sum_{i=0}^{t-1} (1-\delta_h)^i y_{t-1-i} - (1-\delta_h)^{t-1} h_0, \tag{86}$$

where $\alpha'_{ph} \equiv \left(\alpha_{ph} - b^{h\prime}\right)$, which is a constant.

Combining (84) and (86) delivers (26).

A.6 Proof of (28)

The proof is simple. Recall (26) that

$$w_t - p_{ht} = b^{wp\prime} + \delta_h \sum_{i=0}^{t-1} (1-\delta_h)^i y_{t-1-i} + (1-\delta_h)^{t-1} h_0 - \widehat{\varepsilon_t}.$$

Update one more period and we get

$$\begin{aligned}
& w_{t+1} - p_{h,t+1} \\
= \quad & b^{wp\prime} + \delta_h \sum_{i=0}^{t} (1-\delta_h)^i y_{t-i} + (1-\delta_h)^t h_0 - \widehat{\varepsilon_{t+1}} \\
= \quad & b^{wp\prime} + \delta_h y_t + \delta_h \sum_{i=1}^{t} (1-\delta_h)^i y_{t-i} + (1-\delta_h)^t h_0 - \widehat{\varepsilon_{t+1}} \\
= \quad & b^{wp\prime} + \delta_h y_t + (1-\delta_h)\delta_h \sum_{i=0}^{t-1} (1-\delta_h)^i y_{t-1-i} + (1-\delta_h)^t h_0 - \widehat{\varepsilon_{t+1}} \\
= \quad & b^{wp\prime} + \delta_h y_t + (1-\delta_h)\left[(w_t - p_{ht}) - b^{wp\prime} - (1-\delta_h)^{t-1} h_0 + \widehat{\varepsilon_t}\right] \\
& + (1-\delta_h)^t h_0 - \widehat{\varepsilon_{t+1}} \\
= \quad & \delta_h b^{wp\prime} + \delta_h y_t + (1-\delta_h)(w_t - p_{ht}) + \left[(1-\delta_h)\widehat{\varepsilon_t} - \widehat{\varepsilon_{t+1}}\right],
\end{aligned}$$

which is (28), if we define

$$\begin{aligned}\widetilde{b^{wp\prime}} &\equiv \delta_h b^{wp\prime},\\ \widetilde{\varepsilon_{t+1}} &\equiv (1-\delta_h)\,\widehat{\varepsilon_t} - \widehat{\varepsilon_{t+1}}.\end{aligned}$$

Notice that by (85), we have $\widehat{\varepsilon_t} = \varepsilon_{mt}$, and by (83), we have $E(\varepsilon_{mt}\varepsilon_{ms}) = 0$ whenever $s \neq t$. Therefore, $E\left[\widetilde{\varepsilon_{t+1}}\widetilde{\varepsilon_t}\right] = E\left[\left[(1-\delta_h)\,\widehat{\varepsilon_t} - \widehat{\varepsilon_{t+1}}\right]\left[(1-\delta_h)\,\widehat{\varepsilon_{t-1}} - \widehat{\varepsilon_t}\right]\right]$ $= (1-\delta_h)\,E\left[(\widehat{\varepsilon_t})^2\right] > 0.$

A.7 Proof of (27)

The proof is simple.[33] Recall (26) that

$$w_t - p_{ht} = b^{wp\prime} + \delta_h \sum_{i=0}^{t-1} (1-\delta_h)^i\, y_{t-1-i} + (1-\delta_h)^{t-1} h_0 - \widehat{\varepsilon_t}.$$

For un-conditional moment, we know that $var(y_t) = var(y_{t-i})$, $i = 1, 2, ...$ Thus, taking variance on both sides, we have

$$\begin{aligned} var(w_t - p_{ht}) &= var\left(b^{wp\prime} + \delta_h \sum_{i=0}^{t-1} (1-\delta_h)^i\, y_{t-1-i} + (1-\delta_h)^{t-1} h_0 - \widehat{\varepsilon_t}\right)\\ &= var\left(\delta_h \sum_{i=0}^{t-1} (1-\delta_h)^i\, y_{t-1-i} - \widehat{\varepsilon_t}\right)\\ &= (\delta_h)^2\, var\left(\sum_{i=0}^{t-1} (1-\delta_h)^i\, y_{t-1-i}\right) - 2\delta_h covar\left(\sum_{i=0}^{t-1} (1-\delta_h)^i\, y_{t-1-i}, \widehat{\varepsilon_t}\right)\\ &\quad + var(\widehat{\varepsilon_t}) \end{aligned} \tag{87}$$

Notice that if we can show $covar\left(\sum_{i=0}^{t-1} (1-\delta_h)^i\, y_{t-1-i}, \widehat{\varepsilon_t}\right) = 0$, then by (87), we will have

[33] The authors are deeply indebted to Fred Kwan, whose suggestions lead to this result.

$$\begin{aligned}
& var\left(w_t - p_{ht}\right) \\
= \; & \left(\delta_h\right)^2 var\left(\sum_{i=0}^{t-1}\left(1-\delta_h\right)^i y_{t-1-i}\right) - 2\delta_h covar\left(\sum_{i=0}^{t-1}\left(1-\delta_h\right)^i y_{t-1-i}, \widehat{\varepsilon}_t\right) \\
& + var\left(\widehat{\varepsilon}_t\right) \\
= \; & \left(\delta_h\right)^2 var\left(y_t\right)\left(\sum_{i=0}^{t-1}\left(1-\delta_h\right)^{2i}\right) + var\left(\widehat{\varepsilon}_t\right) \\
= \; & \widetilde{\delta}_t var\left(y_t\right) + var\left(\widehat{\varepsilon}_t\right),
\end{aligned}$$

which is (27), where

$$\begin{aligned}
\widetilde{\delta}_t &\equiv \left(\delta_h\right)^2\left(\sum_{i=0}^{t-1}\left(1-\delta_h\right)^{2i}\right) \\
&= \left(\delta_h\right)^2 \frac{1-\left(1-\delta_h\right)^{2t}}{1-\left(1-\delta_h\right)^2}. \qquad (88)
\end{aligned}$$

Notice that $\widetilde{\delta}_t$ depends on t in a deterministic manner. Clearly, since $0 \leq \delta_h < 1$, $0 < \left(1-\delta_h\right)^2 < 1$. $\widetilde{\delta}_t > 0$.

Thus, we focus on the co-variance term, $covar\left(\sum_{i=0}^{t-1}\left(1-\delta_h\right)^i y_{t-1-i}, \widehat{\varepsilon}_t\right)$. By definition,

$$\begin{aligned}
& covar\left(\sum_{i=0}^{t-1}\left(1-\delta_h\right)^i y_{t-1-i}, \widehat{\varepsilon}_t\right) \\
= \; & \sum_{i=0}^{t-1} covar\left(\left(1-\delta_h\right)^i y_{t-1-i}, \widehat{\varepsilon}_t\right) \\
= \; & \sum_{i=0}^{t-1}\left(1-\delta_h\right)^i covar\left(y_{t-1-i}, \widehat{\varepsilon}_t\right). \qquad (89)
\end{aligned}$$

Notice that by definition, the "shocks" in period t, $\widehat{\varepsilon}_t$, should be independent of previous period output. In other words, we have

$$E\left(\widehat{\varepsilon}_t | y_{t-1-i}\right) = 0, \; i = 0, 1, ... \qquad (90)$$

From (90), by the law of iterated expectation, we have the following

$$E(\widehat{\varepsilon_t}) = E(E(\widehat{\varepsilon_t}|y_{t-1-i})) = 0,\ i = 0, 1, ... \tag{91}$$

Given that, we can now compute the co-variance terms,

$$\begin{aligned} & covar(y_{t-1-i}, \widehat{\varepsilon_t}) \\ = & E(y_{t-1-i}\widehat{\varepsilon_t}) - E(y_{t-1-i}) E(\widehat{\varepsilon_t}) \\ = & E(y_{t-1-i}\widehat{\varepsilon_t}) \text{ by (91)} \\ = & E(E(y_{t-1-i}\widehat{\varepsilon_t}|y_{t-1-i})) \\ = & E(y_{t-1-i}E(\widehat{\varepsilon_t}|y_{t-1-i})) \\ = & E(y_{t-1-i} * 0) \text{ by (90)} \\ = & 0. \end{aligned} \tag{92}$$

Substitute (92) into (89), we have $covar\left(\sum_{i=0}^{t-1} (1-\delta_h)^i y_{t-1-i}, \widehat{\varepsilon_t}\right) = 0$, and hence by (87), we have (27).

A.8 Proof of Variance Bound of $(w_t - p_{ht})$

The proof is simple.[34] Recall the formula (27), and (88),

$$var(w_t - p_{ht}) = \widetilde{\delta_t} var(y_t) + var(\widehat{\varepsilon_t}),$$

where

$$\widetilde{\delta_t} = (\delta_h)^2 \frac{1-(1-\delta_h)^{2t}}{1-(1-\delta_h)^2},$$

and by (85), $\widehat{\varepsilon_t} \equiv \omega^a \varepsilon_{at} + \omega^\mu \varepsilon_{\mu t} = \varepsilon_{mt}$, which means that

$$var(\widehat{\varepsilon_t}) = (\omega^a)^2 var(\varepsilon_{at}) + (\omega^\mu)^2 var(\varepsilon_{\mu t}),$$

as the technological shock ε_{at} and monetary policy innovation $\varepsilon_{\mu t}$ are assumed to be independent.

Clearly, since $0 \leq \delta_h < 1$, $0 < (1-\delta_h)^2 < 1$. $\widetilde{\delta_t} > 0$. Moreover, $\widetilde{\delta_t}$ has a

[34] The authors are deeply indebted to Richard Green, whose suggestions lead to this result.

limit,

$$\begin{aligned}
\widetilde{\delta} &\equiv \lim_{t\to\infty} \widetilde{\delta}_t = (\delta_h)^2 \lim_{t\to\infty} \left(\frac{1-(1-\delta_h)^{2t}}{1-(1-\delta_h)^2} \right) \\
&= (\delta_h)^2 \frac{1}{1-(1-\delta_h)^2} \\
&= \frac{\delta_h}{2-\delta_h}.
\end{aligned}$$

Again, since $0 \leq \delta_h < 1 < 2$, $\widetilde{\delta} \geq 0$. In fact, it is easy to that $\widetilde{\delta}$ is also bounded. If we see $\widetilde{\delta}$ as a function of δ_h, $\widetilde{\delta}(\delta_h = 0) = 0$, $\widetilde{\delta}(\delta_h = 1) = 1$. $\partial\widetilde{\delta}/\partial\delta_h = 2(2-\delta_h)^{-2} > 0$, as $0 \leq \delta_h < 1 < 2$. Thus, $0 < \widetilde{\delta} < 1$.

A.9 Proof of (29)

Notice that by definition,

$$\begin{aligned}
& \ln\left(\frac{W_t L_t}{P_{ht}}\right) \\
&= w_t + l_t - p_{ht} \\
&= (w_t - p_{ht}) + l_t \\
&= (w_t - p_{ht}) + l + \varepsilon_{mt}, \text{ by (21)} \\
&= \left(b^{wp\prime} + \delta_h \sum_{i=0}^{t-1} (1-\delta_h)^i y_{t-1-i} + (1-\delta_h)^{t-1} h_0 - \widehat{\varepsilon_t} \right) \\
& \quad +l + \varepsilon_{mt}, \text{ by (26)} \\
\text{where } \widehat{\varepsilon_t} &= \varepsilon_{mt}, \text{ by (85)} \\
&= b'' + \delta_h \sum_{i=0}^{t-1} (1-\delta_h)^i y_{t-1-i} + (1-\delta_h)^{t-1} h_0,
\end{aligned}$$

which is very similar to (26), except for a different constant term. Furthermore,

$$
\begin{aligned}
& w_{t+1} + l_{t+1} - p_{h,t+1} \\
= \quad & b'' + \delta_h \sum_{i=0}^{t} (1 - \delta_h)^i \, y_{t-i} + (1 - \delta_h)^t \, h_0 \\
= \quad & b'' + \delta_h y_t + \delta_h \sum_{i=1}^{t} (1 - \delta_h)^i \, y_{t-i} + (1 - \delta_h)^t \, h_0 \\
= \quad & b'' + \delta_h y_t + (1 - \delta_h) \, \delta_h \sum_{i=0}^{t-1} (1 - \delta_h)^i \, y_{t-1-i} + (1 - \delta_h)^t \, h_0 \\
= \quad & b'' + \delta_h y_t + (1 - \delta_h) \left[(w_t + l_t - p_{ht}) - b'' - (1 - \delta_h)^{t-1} \, h_0 \right] \\
& + (1 - \delta_h)^t \, h_0 \\
= \quad & \delta_h b'' + \delta_h y_t + (1 - \delta_h) \, (w_t + l_t - p_{ht}) \, ,
\end{aligned}
$$

which means that the wage income-to-house price ratio can also be written as a moving average form, as the case of wage rate-to-house price ratio.

Appendix B

Table B1: Panel Unit Root Tests

	$w_t - p_{ht}$	$\Delta(w_t - p_{ht})$	y_t	$\Delta(y_t)$
IPS W-statistic	1.03	-10.47 ***	-1.48	-21.40 ***
ADF-Fisher Chi-Square	33.49	184.75 ***	34.13	399.68 ***
PP-Fisher Chi-Square	13.84	248.04 ***	37.86	451.27 ***

Notes: H_0: Each country follows an individual unit root process. H_1: At least one country's process is trend stationary. Exogenous variables: individual effects, individual linear trends. The lag length is selected by Hannan-Quinn Criterion. *** denotes 1% statistical significance.

Table B2: Panel Integration Test Results

Kao and Johansen tests confirm that PIR and output are cointegrated.

Pedroni tests have several versions. Some confirm that PIR and output are cointegrated.

(i) Kao Residual Cointegration Test

	Statistic
ADF	-3.3909 ***

(ii) Johansen Fisher Panel Cointegration Test

Hypothesized Number of Cointegrating Equations	Trace test	Max. Eigen Test
None	69.22 ***	67.87 ***
At most 1	39.23	39.23

(iii) Pedroni Residual Cointegration Test

(Individual intercept only)

	Statistic
Panel v-Statistic	-0.6423
Panel rho-Statistic	1.4477
Panel PP-Statistic	1.3332
Panel ADF-Statistic	1.7225
Group rho-Statistic	1.2828
Group PP-Statistic	1.4188
Group ADF-Statistic	1.0762

(No intercept or trend)

	Statistic
Panel v-Statistic	-1.5217
Panel rho-Statistic	0.1961
Panel PP-Statistic	-1.3246 *
Panel ADF-Statistic	1.7248 **
Group rho-Statistic	2.5939
Group PP-Statistic	-1.4091 *
Group ADF-Statistic	-2.1855 **

Note: *** and ** denote 1% and 5% statistical significance respectively.

(Not for publications)

Appendix C: Results when Ireland is included

Table C1. Correlation between variance of PIR and variance of output (Cross-sectional)

		Correlation between var($w_t - p_{ht}$) and var (y_t)
Full sample		0.2308
All countries except	Australia	0.3103
	Canada	0.2226
	Denmark	0.2910
	Finland	0.2604
	France	0.2267
	Germany	0.2977
	Ireland	-0.2057
	Italy	0.2287
	Japan	0.2357
	Netherlands	0.2200
	New Zealand	0.2117
	Norway	0.2707
	Spain	0.2189
	Sweden	0.2447
	UK	0.2229
	US	0.2715

Note: All coefficients are insignificant. Cyclical components are used.

Table C2. Dynamic Panel Data Regression Result

Dependent variable: $w_{t+1} - p_{h,t+1}$

		Instruments – Lags 2 to 4 of dependent variable included			
		$w_t - p_{ht}$	y_t	J-statistics	Prob. (J-stat)
Full sample		0.9687 ***	0.1895 ***	14.67	0.40
All countries except	Australia	0.9800 ***	0.2063 ***	16.32	0.29
	Canada	0.9716 ***	0.1853 ***	12.33	0.58
	Denmark	0.9622 ***	0.1752 ***	14.71	0.40
	Finland	0.9659 ***	0.1743 ***	14.26	0.36
	France	0.9704 ***	0.1981 ***	13.89	0.53
	Germany	0.9823 ***	0.2479 ***	13.55	0.41
	Ireland	0.9623 ***	0.1812 ***	13.48	0.41
	Italy	0.9798 ***	0.2313 ***	14.04	0.52
	Japan	0.9781 ***	0.2196 ***	14.22	0.36
	Netherlands	0.9686 ***	0.1985 ***	13.56	0.41
	New Zealand	0.9696 ***	0.1875 ***	13.54	0.41
	Norway	0.9722 ***	0.1969 ***	13.53	0.41
	Spain	0.9736 ***	0.2017 ***	14.44	0.34
	Sweden	0.9674 ***	0.1886 ***	14.05	0.37
	UK	0.9712 ***	0.1893 ***	13.72	0.39
	US	0.9706 ***	0.2032 ***	15.57	0.34

Note: *** denotes 1% statistical significance. Cyclical components are used.

Table C3a. Dynamic Panel Data Regression Result (Robustness check)

Dependent variable: $w_{t+1} - p_{h,t+1}$

		Instruments – Lags 2 to 4 of dependent variable included				
		$w_t - p_{ht}$	y_t	gov_t	J-statistics	Prob. (J-stat)
Full sample		0.9663 ***	0.2920 ***	0.1100 ***	14.22	0.43
All countries except	Australia	0.9790 ***	0.2553 ***	0.0407	13.86	0.38
	Canada	0.9734 ***	0.3226 ***	0.1385 ***	11.48	0.57
	Denmark	0.9619 ***	0.2668 ***	0.0807	13.66	0.32
	Finland	0.9664 ***	0.2612 ***	0.0683 **	12.84	0.46
	France	0.9754 ***	0.2508 ***	-0.0056	12.76	0.39
	Germany	0.9768 ***	0.3291 ***	0.1013	13.98	0.38
	Ireland	0.9617 ***	0.2075 ***	0.0212	13.46	0.33
	Italy	0.9859 ***	0.3167 ***	0.0189	12.82	0.38
	Japan	0.9926 ***	0.2926 ***	-0.0072	8.52	0.81
	Netherlands	0.9690 ***	0.2815 ***	0.0671	13.33	0.35
	New Zealand	0.9656 ***	0.2674 ***	0.0990 ***	14.59	0.33
	Norway	0.9732 ***	0.3152 ***	0.1197 ***	13.90	0.38
	Spain	0.9790 ***	0.2965 ***	0.0377	12.92	0.37
	Sweden	0.9674 ***	0.2924 ***	0.0937 ***	14.01	0.37
	UK	0.9830 ***	0.2868 ***	0.0288	12.82	0.38
	US	0.9749 ***	0.2999 ***	0.0697	15.44	0.28

Note: *** and ** denote 1% and 5% statistical significance respectively. Cyclical components are used.

Table C3b. Panel Co-integration Test Between PIR and GDP (Robustness Check)

		Group-mean Fully-modified OLS	Group-mean Dynamic OLS	Static OLS
Full sample		0.0016 ***	-0.0008	0.0028 ***
All countries except	Australia	0.0009 **	-0.0007	0.0020 ***
	Canada	0.0008 **	-0.0014	0.0018 **
	Denmark	0.0016 ***	-0.0010	0.0028 ***
	Finland	0.0017 ***	-0.0009	0.0029 ***
	France	0.0017 ***	-0.0007	0.0028 ***
	Germany	0.0021 ***	-0.0006	0.0033 ***
	Ireland	0.0028 ***	0.0003	0.0040 ***
	Italy	0.0026 ***	0.0000	0.0038 ***
	Japan	0.0022 ***	-0.0004	0.0035 ***
	Netherlands	0.0024 ***	-0.0001	0.0036 ***
	New Zealand	0.0007	-0.0018	0.0019 **
	Norway	0.0010 ***	-0.0015	0.0022 ***
	Spain	0.0027 ***	0.0002	0.0038 ***
	Sweden	0.0003	-0.0021	0.0015 **
	UK	0.0014 ***	-0.0011	0.0025 ***
	US	0.0017 ***	-0.0009	00029 ***

Note: *** and ** denote 1% and 5% statistical significance respectively.