ISSN (Print) 0473-453X
Discussion Paper No. 1134
ISSN (Online) 2435-0982

A CHARACTERIZATION OF MINIMUM PRICE WALRASIAN RULE IN OBJECT ALLOCATION PROBLEM FOR AN ARBITRARY NUMBER OF OBJECTS

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June 2021

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A Characterization of Minimum Price Walrasian Rule in Object Allocation Problem for an Arbitrary Number of Objects *

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June 2, 2021

Abstract

We consider the multi-object allocation problem with monetary transfers where each agent obtains at most one object (unit-demand). We focus on allocation rules satisfying *individual rationality*, *no subsidy*, *efficiency*, and *strategy-proofness*. Extending the result of Morimoto and Serizawa (2015), we show that for an arbitrary number of agents and objects, the minimum price Walrasian is characterized by the four properties on the classical domain.

JEL classification: D82, D47, D63.

Keywords: Multi-object allocation problem, Strategy-proofness, Efficiency, Minimum price Walrasian rule, Non-quasi-linear preference, Heterogeneous objects.

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*We gratefully acknowledge financial support from the Joint Usage/Research Center at ISER, Osaka University, Osaka University's International Joint Research Promotion Program (Type A), and the Japan Society for the Promotion of Science (15H05728, 19J10221, 20H05631)

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1 Introduction

This is to extend Morimoto and Serizawa (2015) to a general case of an arbitrary number of agents and objects. They consider the multi-object allocation problem with monetary transfers where each agent obtains at most one object (unit-demand). A (*consumption*) *bundle* is a pair of object and payment. Each agent has a continuous preference relation over bundles satisfying the possibility of compensation, money monotonicity, and object desirability. Such preferences are called *classical*. The *classical domain* is the class of all classical preferences.

In multi-object allocation problem, for each preference profile, *Walrasian equilibrium* exists (Alkan and Gale, 1990), and Demange and Gale (1985) show that the set of Walrasian prices has a lattice structure; that is, there is the minimum price Walrasian equilibrium for each preference profile.

An (*allocation*) *rule*, or simply *rule* chooses, for each preference profile, the object each agent receives and how much each agent pays. Demange and Gale (1985) show that a *minimum price Walrasian rule* satisfies following properties: (i) *Individual rationality* requires that each agent's bundle is at least as good as receiving nothing with no payment. Without this condition, agents does not participate the rule voluntarily. (ii) *No subsidy* means that all agents' payments are nonnegative. (iii) *Efficiency* requires that no allocation can increase the sum of payments without changing agents' welfare. (iv) *Strategy-proofness* is the incentive compatible condition, which means that no agent has incentive to misreport his preference.

Morimoto and Serizawa (2015) show that in the case where the number of agents is greater than the number of objects, the minimum price Walrasian rule is characterized by individual rationality, no subsidy, efficiency and strategy-proofness. We extend their result to a general case of an arbitrary number of agents and objects.

This article is organized as follows. Section 2 introduces the model and basic concepts and checks the properties of minimum price Walrasian rules. Our results are in Section 3. Section 4 provides proofs. Section 5 refers to related literatures, and Section 6 concludes.

2 The model

Let $N = \{1, 2, \cdots, n\}$ be the set of agents and $M = \{1, 2, \cdots, m\}$ be the set of different objects. Not consuming an object in M is called consuming the "null object". Let $L \equiv M \cup \{0\}$, where 0 denotes the null object. Each agent consumes at most one object. A typical (consumption) bundle for agent i is a pair $z_i = (x_i, t_i) \in L \times \mathbb{R}$: agent i receives object x_i and pays t_i.

Each agent has a complete and transitive preference relation R_i over $L \times \mathbb{R}$. Let I_i and P_i be the indifference relation and strict preference relation associated with R_i. A typical class of preferences is denoted by $\mathcal{R}$. We call $\mathcal{R}^n$ a **domain**. R_i is **classical** if it satisfies the following assumptions:

1. *Continuity:* For each $z_i \in L \times \mathbb{R}$, the sets $\{z'_i \in L \times \mathbb{R} : z'_i \, R_i \, z_i\}$ and $\{z'_i \in L \times \mathbb{R} : z_i \, R_i \, z'_i\}$ are closed.

2. *Possibility of compensation:* For each pair $a, b \in L$ and each $t \in \mathbb{R}$, there exist $t', t'' \in \mathbb{R}$ such that $(a, t) \; R_i \; (b, t')$ and $(b, t') \; R_i \; (a, t'')$.

3. *Money monotonicity:* For each $a \in L$ and each pair $t, t' \in \mathbb{R}$, if $t < t'$, then $(a, t) \; P_i \; (a, t')$.

4. *Object desirability:* For each $a \in M$ and each $t \in \mathbb{R}$, $(a, t) \; P_i \; (0, t)$.

Let $\mathcal{R}^C$ be the set of classical preferences. We assume that $\mathcal{R} \subseteq \mathcal{R}^C$.

Definition 1. A preference $R_i \in \mathcal{R}^C$ is **quasi-linear** if for each $(a, t), (b, t') \in L \times \mathbb{R}$ and each $\delta \in \mathbb{R}$, $(a, t) \, I_i \, (b, t')$ implies $(a, t - \delta) \, I_i \, (b, t' - \delta)$.

Let $\mathcal{R}^Q$ be the set of quasi-linear preferences. Note that $\mathcal{R}^Q \subsetneq \mathcal{R}^C$.

A preference profile is a list of preferences $R \equiv (R_1, \cdots, R_n)$. Given $i \in N$ and $N' \subseteq N$, let $R_{-i} \equiv (R_j)_{j \neq i}$ and $R_{-N'} \equiv (R_j)_{j \in N \setminus N'}$.

A (feasible) **object allocation** is an n-tuple $x = (x_1, x_2, \ldots, x_n) \in L^n$ such that for each pair $i, j \in N$, if $x_i = x_j$, then $x_i = x_j = 0$. Let A be the set of all object allocations. An **allocation** is a pair of an object allocation and a vector of payments, $z = ((x_1, x_2, \ldots, x_n), (t_1, t_2, \ldots, t_n)) \in A \times \mathbb{R}^n$. Given $z \in A \times \mathbb{R}^n$ and $i \in N$, $z_i = (x_i, t_i) \in L \times \mathbb{Z}$ denotes the bundle of agent i. An allocation $z' \in A \times \mathbb{R}^n$ **Pareto-dominates** $z \in A \times \mathbb{R}^n$ if (i) for each $i \in N$, $z'_i \, R_i \, z_i$ and (ii) $\sum_{i \in N} t'_i > \sum_{i \in N} t_i$. [1] An allocation z is **efficient** if there is no allocation that Pareto-dominates z.

An **(allocation) rule** associates an allocation to each preference profile. Formally, a **rule** is a mapping $f = (x, t) \colon \mathcal{R}^n \to A \times \mathbb{R}^n$. Given a rule f and a preference profile $R \in \mathcal{R}^n$, agent i's assignment under f at R is denoted by $f_i(R)$. Moreover, we write $f_i(R) \equiv (x_i(R), t_i(R)) \in L \times \mathbb{R}$, where $x_i(R)$ denotes i's object assignment and $t_i(R)$ denotes his payment. We define $f(R) \equiv (f_1(R), \cdots, f_n(R))$.

We introduce the properties of allocation rule.

- *Efficiency:* For each $R \in \mathcal{R}^n$, $f(R)$ is efficient for R.

- *Individual rationality:* For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R) \; R_i \; (0, 0)$.

- *No subsidy:* For each $R \in \mathcal{R}$ and each $i \in N$, $t_i(R) \geq 0$.

- *No subsidy for losers:* For each $R \in \mathcal{R}^n$ and each $i \in N$, if $x_i(R) = 0$, then $t_i(R) \geq 0$.

[1] This condition is equivalent to the following: (i) for each $i \in N$, $z'_i \, R_i \, z_i$, (ii) there is $j \in N$ such that $z'_j \, P_j \, z_j$ and (iii) $\sum_{i \in N} t'_i \geq \sum_{i \in N} t_i$.

- *Strategy-proofness:* For each $R \in \mathcal{R}^n$, each $i \in N$ and each $R'_i \in \mathcal{R}$, $f_i(R)\ R_i\ f_i(R'_i, R_{-i})$.

Let $p = (p_1, p_2, \cdots, p_m) \in \mathbb{R}^m_+$ be a price vector. We assume that the price of null object is equal to zero; that is, $p_0 = 0$. Given $i \in N$, $R_i \in \mathcal{R}$, and $p \in \mathbb{R}^m_+$, let $D(R_i, p) \equiv \{a \in L : \forall b \in L,\ (a, p_a)\, R_i\, (b, p_b)\}$ denote the demand set of agent i with R_i at p.

Next, we define the concept of Walrasian equilibrium. It is a pair of a price vector and an allocation such that each agent an object he demands and pays its price, and the price of an unassigned object is zero.

Definition 2. Given $R \in \mathcal{R}^n$, a pair $((x,t), p) \in (A \times \mathbb{R}^n) \times \mathbb{R}^m_+$ is a **Walrasian equilibrium** for R if

WE-i: for each $i \in N$, $x_i \in D(R_i, p)$ and $t_i = p_{x_i}$; and

WE-ii: for each $a \in M \setminus \{x_j\}_{j \in N}$, $p_a = 0$.

Given $R \in \mathcal{R}^n$, let $W(R)$ be the set of Walrasian equilibria for R, and define

$$Z(R) \equiv \{z \in A \times \mathbb{R}^n : \exists p \in \mathbb{R}^m_+ \text{ s.t. } (z, p) \in W(R)\}$$

and

$$P(R) \equiv \left\{p \in \mathbb{R}^m_+ : \exists z \in A \times \mathbb{R}^n \text{ s.t. } (z, p) \in W(R)\right\}.$$

Fact 1 (Alkan and Gale, 1990)**.** For each $R \in \mathcal{R}^n$, there is a Walrasian equilibrium; that is, $W(R) \neq \emptyset$.

Fact 2 (Demange and Gale, 1985)**.** For each $R \in \mathcal{R}^n$, there is $p \in \mathbb{R}^m_+$ such that for each $p' \in P(R)$, $p \leq p'$. [2]

Given $R \in \mathcal{R}^n$, we denote the minimum Walrasian price for R by $p^{min}(R)$ and define

$$Z^{min}(R) \equiv \{z \in A \times \mathbb{R}^n : (z, p^{min}(R)) \in W(R)\}.$$

We say an allocation rule f is a **minimum price Walrasian rule** if for each $R \in \mathcal{R}^n$, $f(R) \in Z^{min}(R)$.

Fact 3 (Demange and Gale, 1985)**.** The minimum price Walrasian rule f is strategy-proof.[3]

By the definition of Walrasian equilibrium, the minimum price Walrasian rule satisfies individual rationality, no subsidy and efficiency.

Fact 4 (Demange and Gale, 1985)**.** The minimum price Walrasian rule f satisfies individual rationality, no subsidy, efficiency and strategy-proofness.

[2] $p \leq p'$ means that $p_a \leq p'_a$ for each $a \in M$.

[3] Precisely, they show that the minimum price Walrasian rule f is *group strategy-proof*: that is, for each $R \in \mathcal{R}^n$ and each $N' \subseteq N$, there is no $R'_{N'} \in \mathcal{R}^{|N'|}$ such that for each $i \in N$, $f_i(R'_{N'}, R_{-N'})\, P_i\, f_i(R)$.

3 Characterization

Morimoto and Serizawa (2015) characterizes a minimum price Walrasian rule on classical domain by efficiency, individual rationality, no subsidy for losers, and strategy-proofness when the number of agent is greater than the number of objects.

Fact 5 (Theorem 2 in Morimoto and Serizawa, 2015)**.** Let $\mathcal{R} = \mathcal{R}^C$ and $n > m$. An allocation rule satisfies efficiency, individual rationality, no subsidy for losers and strategy-proofness if and only if it is a minimum price Walrasian rule on $\mathcal{R}^n$.

Fact 6 (Lemma 7 in Morimoto and Serizawa, 2015)**.** Let $\mathcal{R} = \mathcal{R}^C$ and $n > m$. If f satisfies efficiency, individual rationality, no subsidy for losers and strategy-proofness then it also satisfies no subsidy.

Remark 1. A minimum price Walrasian rule satisfies no subsidy

From Fact 6 and Remark 1, we have the following fact:

Fact 7. Let $\mathcal{R} = \mathcal{R}^C$ and $n > m$. An allocation rule satisfies efficiency, individual rationality, no subsidy and strategy-proofness if and only if it is a minimum price Walrasian rule on $\mathcal{R}^n$.

Our theorem shows that even when the number of agents is less or equal to the number of objects, a minimum price Walrasian rule is characterized by the axioms in Corollary 1.

Theorem. Let $\mathcal{R} = \mathcal{R}^C$. An allocation rule satisfies efficiency, individual rationality, no subsidy, and strategy-proofness if and only if it is a minimum price Walrasian rule.

4 Proofs

In this section, we assume that $\mathcal{R} = \mathcal{R}^C$. To prove our Theorem, we show the following Propositions:

Proposition 1. Let f satisfy four axioms in Theorem. For each $R \in \mathcal{R}^n$, each $z \in Z^{min}(R)$ and each $i \in N$, $f_i(R)\, R_i\, z_i$.

Proposition 2. Let f satisfy four axioms in Theorem. For each $R \in \mathcal{R}^n$, each $z \in Z^{min}(R)$ and each $i \in N$, $z_i\, R_i\, f_i(R)$.

By using these Propositions, we prove Theorem.

Given $i \in N$, $R_i \in \mathcal{R}$, $a \in L$ and $(b,t) \in M \times \mathbb{R}_+$, we say $V^{R_i}(a;(b,t))$ is the **compensated valuation** of a from (b,t) for R_i if $(a, V^{R_i}(a;(b,t)))\, I_i\, (b,t)$.

Fact 8. Let f satisfies individual rationality. Let $R \in \mathcal{R}^n$, $i \in N$ and $z_i \in L \times \mathbb{R}_+$. If $z_i \, P_i \, f_i(R)$, then $x_i \neq 0$.

Definition 3. Given $(a,t) \in M \times \mathbb{R}_+$, R_i' is (a,t)**-favoring** if for each $b \in M \setminus \{a\}$, $V^{R_i'}(b;(a,t)) < 0$.

Fact 9 (Lemma 8 in Morimoto and Serizawa, 2015)**.** Let f satisfy strategy-proofness and no subsidy. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $x_i(R) \neq 0$. Let $R_i' \in \mathcal{R}$ be $f_i(R)$-favoring. Then, $f_i(R_i', R_{-i}) = f_i(R)$.

Given $(a,t) \in M \times \mathbb{R}_+$ and $\varepsilon > 0$, we say $R_i \in \mathcal{R}$ is $(a,t)^{\varepsilon}$**-favoring** if (i) R_i is (a,t)-favoring, (ii) $V^{R_i}(a;(0,0)) = t + 2\varepsilon$ and (iii) for each $b \in M \setminus \{a\}$, $V^{R_i}(b;(0,0)) = \varepsilon$.[4] Especially, if $R_i \in \mathcal{R}^Q$ is $(a,t)^{\varepsilon}$-favoring, we write this preference by $R^Q((a,t),\varepsilon)$.

Remark 2. For each $(a,t) \in M \times \mathbb{R}_+$ and each $\varepsilon \in \mathbb{R}_{++}$, there is $R_i \in \mathcal{R}$ such that $R_i = R^Q((a,t),\varepsilon)$.

Remark 3. Let $(a,t) \in M \times \mathbb{R}_+$, $\varepsilon_i \in \mathbb{R}_{++}$ and $R_i = R^Q((a,t),\varepsilon_i)$. Then R_i is (a,t)-favoring.

Remark 4. Let $i \in N$, $z_i = (x_i, t_i) \in M \times \mathbb{R}_+$, $\varepsilon_i \in \mathbb{R}_{++}$ and $R_i = R^Q(z_i, \varepsilon_i)$. Then for each $(a,t) \in (M \setminus \{x_i\}) \times \mathbb{R}$, (i) $(x_i, t_i + \varepsilon_i + t) \, I_i \, (a,t)$ and (ii) $V^{R_i}(x_i;(a,t)) = t_i + \varepsilon_i + t$.

Proof. Let $(a,t) \in (M \setminus \{x_i\}) \times \mathbb{R}$.

(i) By $R_i = R^Q(z_i, \varepsilon_i)$ and $a \neq x_i$,

$$
\begin{aligned}
& (x_i, t_i + 2\varepsilon_i) \, I_i \, (a, \varepsilon_i) \\
\Leftrightarrow \quad & (x_i, t_i + \varepsilon_i) \, I_i \, (a, 0) \\
\Leftrightarrow \quad & (x_i, t_i + \varepsilon_i + t) \, I_i \, (a, t).
\end{aligned}
$$

(ii) By the definition of compensated valuation, $(x_i, V^{R_i}(x_i;(a,t))) \, I_i \, (a,t)$, and so by (i), $V^{R_i}(x_i;(a,t)) = t_i + \varepsilon_i + t$. □

Definition 4. Let $R \in \mathcal{R}^n$, $z \in A \times \mathbb{R}^n$ and $N' \equiv \{i_1, \ldots, i_K\} \subseteq N$ with $K \geq 2$. $(N', (t_i')_{i \in N'})$ is a **Pareto-dominating trading cycle** of z if (i) $z_{i_1} \, I_{i_1} \, (x_{i_K}, t_{i_1}')$, (ii) for each $k \in \{2, \ldots, K\}$, $z_{i_k} \, I_{i_k} \, (x_{i_{k-1}}, t_{i_k}')$ and (iii) $\sum_{i \in N'} t_i' > \sum_{i \in N'} t_i$.

Fact 10. Let $R \in \mathcal{R}^n$ and $z \in A \times \mathbb{R}^n$. If z has a Pareto-dominating trading cycle, then it is not efficient for R.

[4]Note that this definition is different from Kazumura et al. (2020B) and Sakai and Serizawa (2020)

4.1 Preliminary results for Proposition 1

By using the similar method of Proposition 1 in Sakai and Serizawa (2020), we prove Proposition 1. [5]

Remark 5. Let f satisfy no subsidy. Let $R \in \mathcal{R}^n$, $i \in N$, and $z_i = (x_i, t_i) \in M \times \mathbb{R}_+$ be such that R_i is z_i-favoring. If $x_i(R) \neq x_i$, then $z_i \, P_i \, f_i(R)$.

Proof. Let $x_i(R) \neq x_i$. Since R_i is z_i-favoring, $V^{R_i}(x_i(R); z_i) < 0$. By no subsidy, $t_i(R) \geq 0$. Hence $V^{R_i}(x_i(R); z_i) < t_i(R)$. By money monotonicity, $(x_i(R), V^{R_i}(x_i(R); z_i)) \, P_i \, (x_i(R), t_i(R))$, and so by the definition of compensated valuation, $z_i \, P_i \, (x_i(R), t_i(R)) = f_i(R)$. □

Fact 11. Let $R \in \mathcal{R}^n$, $i \in N$, $z \in A \times \mathbb{R}^n$ and $(a, t) \in M \times \mathbb{R}_{++}$ be such that (a) for each $j \in N$, $x_j \neq a$, (b) $(a, t) \, I_i \, (x_i, t_i)$ and (c) $t > t_i$. Then z is not efficient for R.

Proof. Let $z' \in Z$ be such that for each $j \in N \setminus \{i\}$, $z'_j = z_j$ and $z'_i = (a, t)$. Then, (i) for each $j \in N$, $z'_i \, I_i \, z_i$ and (ii) $\sum_{j \in N} t'_j = \sum_{j \neq i} t_j + t > \sum_{j \neq i} t_j + t_i = \sum_{j \in N} t_j$. Thus, z' Pareto-dominates z for R. □

Fact 12 (Lemma 5 in Morimoto and Serizawa, 2015)**.** Let $z \in A \times \mathbb{R}^n$. Let $R \in \mathcal{R}^n$ and $i, j \in N$ with $i \neq j$. If $t_i + t_j < V^{R_i}(x_j; z_i) + V^{R_j}(x_i; z_j)$, then z is not efficient for R.

Lemma 1. Let f satisfy individual rationality and strategy-proofness. Let $R \in \mathcal{R}^n$, $i \in N$ and $z_i \in M \times \mathbb{R}_+$ be such that $z_i \, P_i \, f_i(R)$. Then, (i) there are $\varepsilon_i \in (0, \frac{1}{2}(V^{R_i}(x_i; f_i(R)) - t_i))$ and $R'_i = R^Q(z_i, \varepsilon_i)$, and (ii) $x_i(R'_i, R_{-i}) \neq x_i$.

Proof. (i) By $z_i \, P_i \, f_i(R)$, $t_i < V^{R_i}(x_i; f_i(R))$. Thus, there is $\varepsilon_i \in (0, \frac{1}{2}(V^{R_i}(x_i; f_i(R)) - t_i))$ and $R'_i = R^Q(z_i, \varepsilon_i)$.

(ii) Suppose $x_i(R'_i, R_{-i}) = x_i$. Then, by individual rationality, $t_i(R'_i, R_{-i}) \leq t_i + 2\varepsilon_i$, and by (i), $t_i + 2\varepsilon_i < V^{R_i}(x_i; f_i(R))$. Thus, $t_i(R'_i, R{-}i) < V^{R_i}(x_i; f_i(R))$, and hence $f_i(R'_i, R_{-i}) \, P_i \, f_i(R)$. This contradicts strategy-proofness. Thus, $x_i(R'_i, R_{-i}) \neq x_i$. □

Lemma 2. Let f satisfy no subsidy, efficiency and strategy-proofness. Let $R \in \mathcal{R}^n$, $N' \subseteq N$, $z \in A \times \mathbb{R}^n_+$ and $(\varepsilon_i)_{i \in N'} \in \mathbb{R}^{|N'|}_{++}$ be such that for each $i \in N'$, $x_i \neq 0$ and $R_i = R^Q(z_i, \varepsilon_i)$. Then, for each $i \in N'$, there is $j \in N$ such that

(i) $x_j(R) = x_i$ and

(ii) if $i \neq j$, then $t_j(R) \geq t_i + \varepsilon_i$.

Proof. (i) Let $i \in N'$. Then $x_i \neq 0$. Suppose that for each $j \in N$, $x_j(R) \neq x_i$. By $x_i(R) \neq x_i$, $R_i = R^Q(z_i, \varepsilon_i)$ and Remark 4 (i), $(x_i, t_i + \varepsilon_i + t_i(R)) \, I_i \, (x_i(R), t_i(R))$. By $t_i + \varepsilon_i > 0$, $t_i + \varepsilon_i + t_i(R) > t_i(R)$, and hence by Fact 11, $f(R)$ is not efficient, which is a contradiction.

[5] Kazumura et al. (2020B) and Sakai and Serizawa (2020) show the dominance in agents' welfare by replacing efficiency with weak fairness condition (equal treatment of equals)

(ii) Let $i \in N'$ and $j \in N$ be such that $i \neq j$ and $x_j(R) = x_i$. Suppose that $t_j(R) < t_i + \varepsilon_i$. Let $R'_j \in \mathcal{R}$ be such that it is $f_j(R)$-favoring and for each $a \in L \setminus \{x_i\}$, $-V^{R'_j}(a; f_j(R)) < t_i + \varepsilon_i - t_j(R)$. By Fact 9, $f_j(R'_j, R_{-j}) = f_j(R)$, and so by $x_i(R'_j, R_{-j}) \neq x_i$, $-V^{R'_j}(x_i(R'_j, R_{-j}); f_j(R'_j, R_{-j})) < t_i + \varepsilon_i - t_j(R'_j, R_{-j})$. By $x_i(R'_j, R_{-j}) \neq x_i$, $R_i = R^Q(z_i, \varepsilon_i)$ and Remark 4 (ii), $V^{R_i}(x_i; f_i(R'_j, R_{-j})) = t_i + \varepsilon_i + t_i(R'_j, R_{-j})$. Hence we have

$$\begin{aligned}
& V^{R_i}(x_j(R'_j, R_{-j}); f_i(R'_j, R_{-j})) + V^{R'_j}(x_i(R'_j, R_{-j}); f_j(R'_j, R_{-j})) \\
= \; & V^{R_i}(x_i; f_i(R'_j, R_{-j})) + V^{R'_j}(x_i(R'_j, R_{-j}); f_j(R'_j, R_{-j})) \\
> \; & (t_i + \varepsilon_i + t_i(R'_j, R_{-j})) + (t_j(R'_j, R_{-j}) - t_i - \varepsilon_i) \\
= \; & t_i(R'_j, R_{-j}) + t_j(R'_j, R_{-j}).
\end{aligned}$$

By Fact 12, $f(R'_j, R_{-j})$ is not efficient for (R'_j, R_{-j}), which is a contradiction. $\square$

4.2 Proof of Proposition 1

Proof. Let $R \in \mathcal{R}^n$, $p = p^{min}(R)$ and $z \in Z^{min}(R)$. Let

$$\underline{p} \equiv \begin{cases} \min\{p_a \in \mathbb{R} : a \in M \text{ and } p_a > 0\} & \text{if } \exists a \in M \text{ such that } p_a > 0 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that there is $i \in N$ such that $z_i \, P_i \, f_i(R)$. Without loss of generality, let $i \equiv 1$.

Claim: For each $k \geq 0$, there are sets $N(k)$ and $N(k+1)$ of distinct agents such that $N(k+1) \supseteq N(k)$, $|N(k)| = k$, $|N(k+1)| = k+1$, say $N(k) = \{1, 2, \ldots, k\}$, $N(k+1) = \{1, 2, \ldots, k+1\}$, and $(\varepsilon_j)_{j \in N(k+1)} \in \mathbb{R}^{k+1}_{++}$, $R^{(k)} \equiv (R'_{N(k)}, R_{-N(k)}) \in \mathcal{R}^n$ and $R^{(k+1)} \equiv (R'_{N(k+1)}, R_{-N(k+1)}) \in \mathcal{R}^n$ such that
(i-a) $z_{k+1} \, P_{k+1} \, f_{k+1}(R^{(k)})$ and
(i-b) $x_{k+1} \neq 0$,
(ii-a) $\varepsilon_1 < \min(\{\underline{p}, \frac{1}{2}(V^{R_1}(x_1; f_1(R)) - t_1)\} \setminus \{0\})$ and $R'_1 = R^Q(z_1, \varepsilon_1)$,
(ii-b) for each $j \in N(k+1) \setminus \{1\}$, $\varepsilon_j < \min\{\varepsilon_{j-1}, \frac{1}{2}(V^{R_j}(x_j; f_j(R^{(j-1)})) - t_j)\}$ and $R'_j = R^Q(z_j, \varepsilon_j)$, and
(ii-c) for each $j \in N(k)$, $\varepsilon_{k+1} < t_j + \varepsilon_j$,
(iii) $x_{k+1}(R^{(k+1)}) \neq x_{k+1}$ and $z_{k+1} \, P'_{k+1} \, f_{k+1}(R^{(k+1)})$,
(iv) $x_{k+1}(R^{(k+1)}) \notin \{x_l\}_{l \in N(k+1)}$, and
(v) there is $j \in N \setminus N(k+1)$ such that $x_j(R^{(k+1)}) \in \{x_l\}_{l \in N(k+1)}$ and $z_j \, P_j \, f_j(R^{(k+1)})$.

We prove Claim by induction on k.

Base Case: Let $k = 0$. (i) By assumption, $z_1 \, P_1 \, f_1(R)$. Thus, (i-a) holds. By individual rationality, $z_1 \in L \times \mathbb{R}_+$, (i-a) and Fact 8, $x_1 \neq 0$. Hence, (i-b) holds.

(ii) By $z_1 \, P_1 \, f_1(R)$, $t_1 < V^{R_1}(x_1; f_1(R)))$. Thus, there is $\varepsilon_1 > 0$ such that $\varepsilon_1 < \min(\{\underline{p}, \frac{1}{2}(V^{R_1}(x_1; f_1(R)) - t_1)\} \setminus \{0\})$. By (i-b), $x_1 \neq 0$. Thus, by Remark 2, there is a preference $R'_1 = R^Q(z_1, \varepsilon_1)$. Hence, (ii-a) holds. By $k = 0$, (ii-b) and (ii-c) hold vacantly.

(iii) By (i-a), (ii-a) and Lemma 1 (ii), $x_1(R^{(1)}) \neq x_1$. By (ii-a) and Remark 3, R'_1 is z_1-favoring, and so by Remark 5, $z_1 \, P'_1 \, f_1(R^{(1)})$.

(iv) By $k = 0$, (iv) directly follows from (iii).

(v) By $x_1(R^{(1)}) \neq x_1$ and Lemma 2 (i), there is $j \in N \setminus \{1\}$ such that $x_j(R^{(1)}) = x_1$. Without loss of generality, let $j \equiv 2$. We show that $z_2 \, P_2 \, f_2(R^{(1)})$. By Lemma 2 (ii) and $(z, p) \in W(R)$, $t_2(R^{(1)}) \geq t_1 + \varepsilon_1 > t_1 = p_{x_1}$. Thus,

$$z_2 \underset{\text{by } (z,p)\in W(R)}{R_2} (x_1, p_{x_1}) \underset{\text{by } t_2(R^{(1)})>p_{x_1}}{P_2} (x_1, t_2(R^{(1)})) \underset{\text{by } x_2(R^{(1)})=x_1}{=} f_2(R^{(1)}).$$

Hence, $z_2 \, P_2 \, f_2(R^{(1)})$.

Inductive Hypothesis: Let $k \geq 1$. There are sets $N(k-1)$ and $N(k)$ of distinct agents such that $N(k) \supseteq N(k-1)$, $|N(k-1)| = k-1$, $|N(k)| = k$, say $N(k-1) = \{1, 2, \ldots, k-1\}$, $N(k) = \{1, 2, \ldots, k\}$, and $(\varepsilon_j)_{j \in N(k)} \in \mathbb{R}^k_{++}$, $R^{(k-1)} \equiv (R'_{N(k-1)}, R_{-N(k-1)}) \in \mathcal{R}^n$ and $R^{(k)} \equiv (R'_{N(k)}, R_{-N(k)}) \in \mathcal{R}^n$ such that
(i-a-k) $z_k \, P_k \, f_k(R^{(k-1)})$ and
(i-b-k) $x_k \neq 0$,
(ii-a-k) $\varepsilon_1 < \min(\{\underline{p}, \frac{1}{2}(V^{R_1}(x_1; f_1(R)) - t_1)\} \setminus \{0\})$ and $R'_1 = R^Q(z_1, \varepsilon_1)$,
(ii-b-k) for each $j \in N(k) \setminus \{1\}$, $\varepsilon_j < \min\{\varepsilon_{j-1}, \frac{1}{2}(V^{R_j}(x_j; f_j(R^{(j-1)})) - t_j)\}$ and $R'_j = R^Q(z_j, \varepsilon_j)$, and
(ii-c) for each $j \in N(k-1)$, $\varepsilon_{k+1} < t_j + \varepsilon_j$,
(iii-k) $x_k(R^{(k)}) \neq x_k$ and $z_k \, P'_k \, f_k(R^{(k)})$,
(iv-k) $x_k(R^{(k)}) \notin \{x_l\}_{l \in N(k)}$, and
(v-k) there is $j \in N \setminus N(k)$ such that $x_j(R^{(k)}) \in \{x_l\}_{l \in N(k)}$ and $z_j \, P_j \, f_j(R^{(k)})$.

Inductive Step: (i) By (iv-k), there is $j \in N \setminus N(k)$ such that $z_j \, P_j \, f_j(R^{(k)})$. Without loss of generality, let $j = k+1$. Then, (i-a) holds. By individual rationality, $z_{k+1} \in L \times \mathbb{R}_+$, (i-a) and Fact 8, $x_{k+1} \neq 0$. Thus, (i-b) holds.

(ii) The hypothesis (ii-a-k) is equivalent to (ii-a).

Next we show (ii-b). By (i-a), $t_{k+1} < V^{R_{k+1}}(x_{k+1}; f_{k+1}(R^{(k)}))$. By (ii-b-k), $\varepsilon_k > 0$. Thus, there is $\varepsilon_{k+1} > 0$ such that $\varepsilon_{k+1} < \min\{\varepsilon_k, \frac{1}{2}(V^{R_{k+1}}(x_{k+1}; f_{k+1}(R^{(k)})) - t_{k+1})\}$. By (i-b), $x_{k+1} \neq 0$. Thus, by Remark 2, there is a preference $R'_{k+1} = R^Q(z_{k+1}, \varepsilon_{k+1})$. Hence by (ii-b-k), (ii-b) holds.

Note that by (ii-b), for each $j \in N(k)$, $\varepsilon_{k+1} < \varepsilon_j$. Since for each $j \in N(k)$, $t_j \geq 0$, $\varepsilon_{k+1} < \varepsilon_j + t_j$. Hence (ii-c) holds.

(iii) By (i-a), (ii-a) and Lemma 1 (ii), $x_{k+1}(R^{(k+1)}) \neq x_{k+1}$. By Remark 3, R'_{k+1} is z_{k+1}-favoring and so by Remark 5, $z_{k+1} \, P'_{k+1} \, f_{k+1}(R^{(k+1)})$.

(iv) Suppose that $x_{k+1}(R^{(k+1)}) \in \{x_l\}_{l\in N(k+1)}$. By (iii), since $x_{k+1}(R^{(k+1)}) \neq x_{k+1}$, there is $j \in N(k+1) \setminus \{k+1\}$ such that $x_{k+1}(R^{(k+1)}) = x_j$. By $x_{k+1}(R^{(k+1)}) \neq x_{k+1}$ and (ii-b), $V^{R'_{k+1}}(x_{k+1}(R^{(k+1)}); (0,0)) = \varepsilon_{k+1}$. By individual rationality, $t_{k+1}(R^{(k+1)}) \leq V^{R'_{k+1}}(x_{k+1}(R^{(k+1)}); (0,0)) = \varepsilon_{k+1}$, and so by (ii-c), $t_{k+1}(R^{(k+1)}) \leq \varepsilon_{k+1} < t_j+\varepsilon_j$. However, by Lemma 2 (ii), $t_{k+1}(R^{(k+1)}) \geq t_j + \varepsilon_j$. This is a contradiction. Thus, $x_{k+1}(R^{(k+1)}) \notin \{x_l\}_{l\in N(k+1)}$.

(v) By (ii-a), (ii-b) and Lemma 2 (i), for each $i \in N(k+1)$, there is $j \in N$ such that $x_j(R^{(k+1)}) = x_i$. By (iv), since $x_{k+1}(R^{(k+1)}) \notin \{x_l\}_{l\in N(k+1)}$, there is $j \in N \setminus N(k+1)$ such that $x_j(R^{(k+1)}) \in \{x_l\}_{l\in N(k+1)}$.

By Lemma 2 (ii) and $(z,p) \in W(R)$, $t_j(R^{(k+1)}) \geq t_j + \varepsilon_j > t_j = p_{x_j(R^{(k+1)})}$. Thus,

$$\begin{aligned} z_j \quad & R_j \quad (x_j(R^{(k+1)}), p_{x_j(R^{(k+1)})}) && \text{(by } (z,p) \in W(R) \text{)} \\ & P_j \quad (x_j(R^{(k+1)}), t_j(R^{(k+1)})) && \text{(by } t_j(R^{(k+1)}) > p_{x_j(R^{(k+1)})} \text{)} \\ & = \quad f_j(R^{(k+1)}). \end{aligned}$$

Hence $z_j \, P_j \, f_j(R^{(k+1)})$. The proof of Claim is completed.

By the above Claim, we derive a contradiction. For $k = n$, there are $n+1$ distinct agents $N(n+1) \subseteq N$. However, since $|N(n+1)| = n+1$ and $|N| = n$, $N(n+1) \nsubseteq N$. This is a contradiction. □

4.3 Preliminary results for Proposition 2

Throughout this subsection, we assume that f satisfies four axioms in Theorem.

Lemma 3. Let $R \in \mathcal{R}^n$, $(z,p) \in W^{min}(R)$ and $M^+ \equiv \{a \in M : p_a > 0\}$. Then, for each $i \in N$,

(i) $t_i(R) \leq p_{x_i(R)}$,

(ii) if $f_i(R) \, P_i \, z_i$, then $t_i(R) < p_{x_i(R)}$ and $x_i(R) \in M^+$, and

(iii) if $x_i(R) \notin M^+$, then $t_i(R) = 0$ and $f_i(R) \, I_i \, z_i$.

Proof. Let $z' \equiv f(R)$. Note that by Proposition 1, for each $i \in N$, $z'_i \, R_i \, z_i$.

(i) Let $i \in N$. By WE-i of (z,p), $z_i \, R_i \, (x'_i, p_{x'_i})$. Thus, by $z'_i \, R_i \, z_i$, $(x'_i, t'_i) = z'_i \, R_i \, (x'_i, p_{x'_i})$ and so $t'_i \leq p_{x'_i}$.

(ii) Let $i \in N$ be such that $z'_i \, P_i \, z_i$. By WE-i of (z,p), $z_i \, R_i \, (x'_i, p_{x'_i})$. Thus, by $z'_i \, P_i \, z_i$, $(x'_i, t'_i) = z'_i \, P_i \, (x'_i, p_{x'_i})$ and so $t'_i < p_{x'_i}$. Since $t'_i \geq 0$, $0 \leq t'_i < p_{x'_i}$, and so $x'_i \in M^+$.

(iii) Let $i \in N$ be such that $x'_i \notin M^+$. First we show that $t'_i = 0$. By (i) and $x'_i \notin M^+$, $t'_i \leq p_{x'_i} = 0$. By $t'_i \geq 0$, $t'_i = 0$.

Next we show that $z'_i \, I_i \, z_i$. Suppose that $z'_i \, P_i \, z_i$. Then by (ii), $x'_i \in M^+$, which contradicts the assumption that $x'_i \notin M^+$. Thus $x'_i \notin M^+$. □

Fact 13 (Corollary 1 in Morimoto and Serizawa, 2015)**.** Let $R \in \mathcal{R}^n$ and $(z, p) \in W^{min}(R)$. Let $N^+ \equiv \{i \in N : p_{x_i} > 0\}$ and $M^+ \equiv \{a \in M : p_a > 0\}$. Then, $|N^+| = |M^+| < n$.

Lemma 4. Let $R \in \mathcal{R}^n$ and $(z, p) \in W^{min}(R)$ and $M^+ \equiv \{a \in M : p_a > 0\}$. Let $x' \in A$. Assume that there is $j_1 \in N$ such that $x_{j_1} \notin M^+$ and $x'_{j_1} \in M^+$. Then, there is K distinct agents $J \equiv \{j_1, \ldots, j_K\} \subseteq N$ with $K \geq 2$ such that (4-a) for each $k \in \{2, \ldots, K\}$, $x_{j_k} \in M^+$, (4-b) for each $k \in \{1, \ldots, K-1\}$, $x'_{j_k} = x_{j_{k+1}}$, and (4-c) $x'_{j_K} \notin M^+$.

Proof. Since $(z, p) \in W(R)$, by WE-ii of (z, p), for each $a \in M^+$, there is $i \in N$ such that $a = x_i$. By $x_{j_1} \notin M^+$ and $x'_{j_1} \in M^+$, there is $j_2 \in N \setminus \{j_1\}$ such that $x'_{j_1} = x_{j_2} \in M^+$. We consider the following procedure.

Step 1: If $x_{j_2} \notin M^+$, this procedure stops. If $x_{j_2} \in M^+$, then there is $j_3 \in N \setminus \{j_1, j_2\}$ such that $x'_{j_2} = x_{j_3} \in M^+$, and this procedure proceeds to Step 2.
Step $t \geq 2$: If $x_{j_{t+1}} \notin M^+$, this procedure stops. If $x_{j_{t+1}} \in M^+$, then there is $j_{t+2} \in N \setminus \{j_1, \ldots, j_{t+1}\}$ such that $x'_{j_{t+1}} = x_{j_{t+2}} \in M^+$, and this procedure proceeds to Step $t+1$.
Step $n-1$: This procedure stops whether $x_{j_n} \in M^+$ or not.

If this procedure stops at Step t with $t < n-1$, then the sequence $\{j_1, \ldots, j_t\}$ satisfies (4-a) to (4-c). We consider the case where the procedure stops at Step $n-1$. We show that $x'_{j_n} \notin M^+$. Suppose that $x'_{j_n} \in M^+$. Then, by Steps 1 to $n-2$, for each $i \in N$, $x'_i \in M^+$. Thus, $|M^+| \geq n$. However, by $(z, p) \in W^{min}(R)$ and Fact 13, $|M^+| < n$. This is a contradiction. Hence, $x'_{j_n} \notin M^+$, and so $\{j_1, \ldots, j_n\}$ satisfies (4-a) to (4-c). □

Given $(z, p) \in W^{min}(R)$, $R_i \in \mathcal{R}$ is z**-indifferent** (i) if $n > m$, for each $a, b \in L$, $(a, p_a)\, I_i\, (b, p_b)$, and (ii) if $n \leq m$, for each $a, b \in M$, $(a, p_a)\, I_i\, (b, p_b)$.

Given $(z, p) \in W^{min}(R)$ and $M^+ = \{a \in M : p_a > 0\}$, let $\mathcal{R}_I^-(z, p)$ be the set of z-indifferent preferences such that for each $R_i \in \mathcal{R}_I^-(z, p)$, each $a \in L \setminus M^+$ and each $(b, t) \in M^+ \times \mathbb{R}_+$ with $p_b - t > 0$, $-V^{R_i}(a; (b, t)) < p_b - t$.

Lemma 5. Let $R \in \mathcal{R}^n$, $(z, p) \in W^{min}(R)$ and $M^+ \equiv \{a \in M : p_a > 0\}$. Let $z' \equiv f(R)$. Then there is no $J \equiv \{j_1, \ldots, j_K\} \subseteq N$ with $K \geq 2$ such that (5-a) $R_{j_1} \in \mathcal{R}_I^-(z, p)$ and $z'_{i_1}\, P_{i_1}\, z_{i_1}$, (5-b) for each $k \in \{2, \ldots, K\}$, $x'_{j_{k-1}} \in D(R_{j_k}, p)$ and $z'_{j_k}\, I_{j_k}\, z_{j_k}$, and (5-c) $x'_{j_K} \notin M^+$.

Proof. Suppose that there is $J \equiv \{j_1, \ldots, j_K\} \subseteq N$ with $K \geq 2$ satisfying (5-a) to (5-c). Let $(t''_j)_{j \in J} \in \mathbb{R}^K$ be such that $t''_{i_1} \equiv V^{R_{i_1}}(x'_{i_K}; z'_{i_1})$ and for each $k \in \{2, \ldots, K\}$, $t''_{j_k} \equiv p_{x'_{j_{k-1}}}$. Then, $z'_{i_1}\, I_{i_1}\, (x'_{i_K}, t''_{i_1})$. Thus J and $(t''_j)_{j \in J}$ satisfies condition (i) of Pareto-dominating trading cycle. Note that by (5-b), for each $k \in \{2, \ldots, K\}$, $z'_{j_k}\, I_{j_k}\, z_{j_k}\, I_{j_k}\, (x'_{j_{k-1}}, p_{x'_{j_{k-1}}}) = (x'_{j_{k-1}}, t''_{j_k})$, and so $z'_{j_k}\, I_{j_k}\, (x'_{j_{k-1}}, t''_{j_k})$. Thus, J and $(t''_j)_{j \in J}$ satisfies condition (ii) of Pareto-dominating trading cycle.

By (5-a) and Lemma 3 (ii), $t'_{i_1} < p_{x'_{i_1}}$. Thus by $R_{j_1} \in \mathcal{R}_I^-(z, p)$ and (5-c), $-V^{R_{j_1}}(x'_{j_K}; z'_{i_1}) < p_{x'_{j_1}} - t'_{j_1}$, and so $t''_{j_1} > t'_{i_1} - p_{x'_{i_1}}$. Moreover, by (5-b), (5-c) and Lemma 3 (i), for each

$k \in \{2, \ldots, K\}$, $t'_{j_{k-1}} \leq p_{x'_{j_{k-1}}} = t''_{j_k}$. Thus,

$$\begin{aligned}
\sum_{j \in J} t''_j &= \sum_{k \in \{2,\ldots,K\}} t''_{x_{j_k}} + V^{R_{j_1}}(x'_{j_K}; z'_{j_1}) \\
&> \sum_{k \in \{2,\ldots,K\}} t''_{j_k} + t'_{j_1} - p_{x'_{j_1}} \\
&\qquad \text{(by } -V^{R_{j_1}}(x'_{j_K}; z'_{i_1}) < p_{x'_{j_1}} - t'_{j_1}\text{)} \\
&= \sum_{k \in \{2,\ldots,K\}} t''_{j_k} + t'_{j_1} - t''_{j_2} \\
&= \sum_{k \in \{3,\ldots,K\}} t''_{j_k} + t'_{j_1} \\
&\geq \sum_{k \in \{3,\ldots,K\}} t'_{j_{k-1}} + t'_{j_1} \\
&= \sum_{k \in \{1,\ldots,K-1\}} t'_{j_k} \\
&= \sum_{k \in \{1,\ldots,K\}} t'_{j_k} \qquad \text{(by } t'_{j_K} = 0\text{)}
\end{aligned}$$

Thus J and $(t''_j)_{j \in J}$ satisfy condition (iii) of Pareto-dominating trading cycle. By Fact 10, z' is not efficient R; that is, f is not efficient, which is a contradiction. □

Remark 6. Let $R \in \mathcal{R}^n$, $(z,p) \in W^{min}(R)$ and $M^+ \equiv \{a \in M : p_a > 0\}$. Let $z' \equiv f(R)$. Then there is no pair $i, j \in N$ such that (a) $R_i \in \mathcal{R}_I^-(z,p)$ and $z'_i \, P_i \, z_i$, and (b) $x_i \in D(R_j, p)$, $z'_j \, I_j \, z_j$ and $x'_j \notin M^+$.

Proof. This is a special case of Lemma 5 with $K = 2$. □

Fact 14 (Lemma 11 in Morimoto and Serizawa, 2015)**.** Let $R \in \mathcal{R}^n$, $(z,p) \in W^{min}(R)$. Let $N' \subseteq N$, $R'_{N'} \in \mathcal{R}_I^-(z)^{|N'|}$, and $R' \equiv (R'_{N'}, R_{-N'})$. Then, (i) $(z,p) \in W^{min}(R')$ and (ii) for each $i \in N$, $f_i(R') \, R'_i \, z_i$.

Lemma 6. Let $R \in \mathcal{R}^n$, $(z,p) \in W^{min}(R)$. Assume that there is $i \in N$ such that $f_i(R) \, P_i \, z_i$. Let $R'_i \in \mathcal{R}_I^-(z,p)$. Then, (i) $f_i(R'_i, R_{-i}) \, P'_i \, z_i$ and (ii) $x_i(R'_i, R_{-i}) \in M^+$.

Proof. (i) Suppose that $z_i \, R'_i \, f_i(R'_i, R_{-i})$. Then by Fact 14 (ii), $z_i \, I'_i \, f_i(R'_i, R_{-i})$. By $R'_i \in \mathcal{R}_I^-(z,p)$, $(x_i(R), p_{x_i(R)}) \, I'_i \, z_i$. By $f_i(R) \, P_i \, z_i$ and Lemma 3 (ii), $t_i(R) < p_{x_i(R)}$, and so $f_i(R) = (x_i(R), t_i(R)) \, P'_i \, (x_i(R), p_{x_i(R)})$. Hence,

$$f_i(R) = (x_i(R), t_i(R)) \, P'_i \, (x_i(R), p_{x_i(R)}) \, I'_i \, z_i \, I'_i \, f_i(R'_i, R_{-i});$$

that is, $f_i(R) \, P'_i \, f_i(R'_i, R_{-i})$. This contradicts strategy-proofness. Thus, $f_i(R'_i, R_{-i}) \, P'_i \, z_i$.

(ii) By $R'_i \in \mathcal{R}_I^-(z,p)$ and (i),

$$f_i(R'_i, R_{-i})\, P'_i\, z_i\, I'_i\, (x_i(R'_i, R_{-i}), p_{x_i(R'_i,R_{-i})}),$$

and so $t_i(R'_i, R_{-i}) < p_{x_i(R'_i,R_{-i})}$. By no subsidy, $0 \leq t_i(R'_i, R_{-i})$ and hence $p_{x_i(R'_i,R_{-i})} > 0$. Thus $x_i(R'_i, R_{-i}) \in M^+$. □

Lemma 7. Let $R \in \mathcal{R}^n$, $(z,p) \in W^{min}(R)$. Let $N^+ \equiv \{i \in N : p_{x_i} > 0\}$ and $\hat{N} \equiv \{i \in N : f_i(R)\, P_i\, z_i\}$. Then, $\hat{N} \subseteq N^+$,

Proof. If $\hat{N} = \emptyset$, this lemma is trivial. Assume that $\hat{N} \neq \emptyset$. Let $i \in \hat{N}$. Suppose that $i \notin N^+$; i.e., $f_i(R)\, P_i\, z_i$ and $x_i \notin M^+$. Without loss of generality, assume that $i = 1$.

Claim: For each $k \geq 0$, there are sets $N(k)$ and $N(k+1)$ of distinct agents such that $N(k+1) \supseteq N(k)$, $|N(k)| = k$, $|N(k+1)| = k+1$, say $N(k) = \{1, 2, \ldots, k\}$, $N(k+1) = \{1, 2, \ldots, k+1\}$, and $R^{(k)} \equiv (R'_{N(k)}, R_{-N(k)}) \in \mathcal{R}^n$ and $R^{(k+1)} \equiv (R'_{N(k+1)}, R_{-N(k+1)}) \in \mathcal{R}^n$ such that
(i) $f_{k+1}(R^{(k)})\, P_{k+1}\, z_{k+1}$, $x_{k+1}(R^{(k)}) \in M^+$, and $t_{k+1}(R^{(k)}) < p_{x_{k+1}(R^{(k)})}$,
(ii) for each $i \in N(k+1)$, $R'_i \in \mathcal{R}_I^-(z,p)$ and $f_{k+1}(R^{(k+1)})\, P_{k+1}^{(k+1)}\, z_{k+1}$,
(iii) for each $i \in N(k+1)$, $x_i(R^{(k+1)}) \in M^+$,
(iv) there is $J^{(k+1)} \equiv \{j_1^{(k+1)} = 1, \ldots, j_{T_{k+1}}^{(k+1)}\} \subseteq N$ with $T_{k+1} \geq 2$ such that
(a) for each $t \in \{2, \ldots, T_{k+1}\}$, $x_{j_t} \in M^+$,
(b) for each $t \in \{1, \cdots, T_{k+1} - 1\}$, $x_{j_t^{(k+1)}}(R^{(k+1)}) = x_{j_{t+1}^{(k+1)}}$ and
(c) $x_{j_{T_{k+1}}^{(k+1)}}(R^{(k+1)}) \notin M^+$,
(v) there is $j \in J^{(k+1)} \setminus N(k+1)$ such that $f_i(R^{(k+1)})\, P_i\, z_i$.

We prove Claim by induction on k.

Base Case: $k = 0$. (i) By assumption, $f_1(R)\, P_1\, z_1$. By Lemma 3 (i), $x_1(R) \in M^+$ and $t_1(R) < p_{x_1(R)}$. Thus, (i) holds.

(ii) Let $R'_1 \in \mathcal{R}_I^-(z)$. By (i) and Lemma 6 (i), $f_1(R^{(1)})\, P'_1\, z_1$. Thus (ii) holds.

(iii) By (i), (ii) and Lemma 6 (ii), $x_1(R^{(1)}) \in M^+$.

(iv) Since $x_1 \notin M^+$ and $x_1(R^{(1)}) \in M^+$, by Lemma 4, there is $J^{(1)} \equiv \{j_1 = 1, j_2, \ldots, j_T\} \subseteq N$ satisfying (a)-(c) in Claim (iv). Thus, (iv) holds.

(v) By Fact 14 (ii), for each $i \in N$, $f_i(R^{(1)})\, R_i^{(1)}\, z_i$. Suppose for each $j \in J^{(1)} \setminus \{j_1\}$, $f_j(R^{(1)})\, I_j^{(1)}\, z_j$. Then $J^{(1)}$ satisfies (5-a) to (5-c), which contradicts Lemma 5. Thus, there is $j \in J^{(1)} \setminus \{j_1\}$ such that $f_j(R^{(1)})\, P_j\, z_j$. Hence (v) holds.

Inductive Hypothesis: Let $k \geq 1$. There are sets $N(k-1)$ and $N(k)$ of distinct agents such that $N(k-1) \supseteq N(k)$, $|N(k-1)| = k-1$, $|N(k1)| = k$, say $N(k-1) = \{1, 2, \ldots, k-1\}$, $N(k) = \{1, 2, \ldots, k\}$, and $R^{(k)} \equiv (R'_{N(k-1)}, R_{-N(k-1)}) \in \mathcal{R}^n$ and $R^{(k)} \equiv (R'_{N(k)}, R_{-N(k)}) \in \mathcal{R}^n$ such that
(i-k) $f_k(R^{(k-1)})\, P_k\, z_k$, $x_k(R^{(k-1)}) \in M^+$, and $t_k(R^{(k-1)}) < p_{x_k(R^{k-1})}$,

(ii-k) for each $i \in N(k)$, $R'_i \in \mathcal{R}_I^-(z)$ and $f_k(R^{(k)})\, P_k^{(k)}\, z_k$,
(iii-k) for each $i \in N(k)$, $x_i(R^{(k)}) \in M^+$,
(iv-k) there is $J^{(k)} \equiv \{j_1^{(k)} = 1, \ldots, j_{T_k}^{(k)}\} \subseteq N \setminus \{1\}$ with $T_k \geq 2$ such that
(a) for each $t \in \{2, \ldots, T_k\}$, $x_{j_t} \in M^+$,
(b) for each $t \in \{1, \cdots, T_k - 1\}$, $x_{j_t^{(k)}}(R^{(k)}) = x_{j_{t+1}^{(k)}}$ and
(c) $x_{j_{T_k}^{(k)}}(R^{(k)}) \notin M^+$,
(v-k) there is $j \in J^{(k)} \setminus N(k)$, $f_i(R^{(k)})\, P_i^{(k)}\, z_i$, and

Induction Step: (i) By (v-k), there is $j \in J^{(k)} \setminus N(k)$ such that $f_j(R^{(k)})\, P_j\, z_j$ and $x_j \in M^+$. Let $j \equiv k+1$. Thus, (i) holds.

(ii) Let $R'_{k+1} \in \mathcal{R}_I^-(z,p)$. By (ii-k), for each $i \in N(k+1)$, $R'_i \in \mathcal{R}_I^-(z,p)$. By (i) and Lemma 6 (i), $f_{k+1}\, P_{k+1}^{(k+1)}\, z_{k+1}$. Thus (ii) holds.

(iii) By (i), (ii) and Lemma 6 (ii), $x_{k+1}(R^{(k+1)}) \in M^+$.

Next we show that for each $i \in N(k+1) \setminus \{k+1\}$, $x_i(R^{(k+1)}) \in M^+$ Suppose that there is $i \in N(k+1) \setminus \{k+1\}$ such that $x_i(R^{(k+1)}) \notin M^+$. By Lemma 3 (iii), $f_i(R^{(k+1)})\, I'_i\, z_i$, and so by $i \in N(k+1)$, $x_{k+1}(R^{(k+1)}) \in D(R'_i, p)$. Thus, by $R'_{k+1} \in \mathcal{R}_I^-(z)$, $f_{k+1}(R^{(k+1)})\, P'_{k+1}\, z_{k+1}$, $\{k+1, i\}$ satisfies (a) and (b) in Remark 6.[6] This is a contradiction.

(iv) Since $x_1 \notin M^+$ and $x_1(R^{(k+1)}) \in M^+$, by Lemma 4, there is $J^{(k+1)} \equiv \{j_1^{(k+1)} = 1, \ldots, j_{T_{k+1}}^{(k+1)}\} \subseteq N$ with $T_{k+1} \geq 2$ satisfying (a)-(c) in Claim (iv). Thus, (iv) holds.

(v) Without loss of generality, let $T \equiv T_{k+1}$ and $J \equiv \{j_1 = 1, \ldots, j_T\} = J^{(k+1)}$. First, we show that there is $j \in J$ such that $f_j(R^{(k+1)})\, P_j^{(k+1)}\, z_j$. Suppose that for each $j \in J \setminus \{j_1\}$, $f_j(R^{(k+1)})\, I_j^{(k+1)}\, z_j$. We consider two cases: $f_1(R^{(k+1)})\, P_1^{(k+1)}\, z_1$ and $f_1(R^{(k+1)})\, I_1^{(k+1)}\, z_1$.

Case 1: $f_1(R^{(k+1)})\, P_1^{(k+1)}\, z_1$. Then J satisfies (5-a) to (5-c) in Lemma 5, which is a contradiction.

Case 2: $f_1(R^{(k+1)})\, I_1^{(k+1)}\, z_1$. Let $H \equiv \{h_1, \ldots, h_{T+1}\} = \{k+1\} \cup J \subseteq N$ be such that $h_1 = k+1$ and for each $t \in \{1, \ldots, T\}$, $h_{t+1} = j_t$. Then H satisfies (5-a) to (5-c) in Lemma 5, which is a contradiction.

From Cases 1 and 2, there is $j \in J^{(k+1)}$ such that $f_i(R^{(k+1)})\, P_i^{(k+1)}\, z_i$.

Next, we show that there is $j \in J \setminus N(k+1)$ such that $f_j(R^{(k+1)})\, P_j^{(k+1)}\, z_j$. Suppose that for each $j \in J \setminus N(k+1)$, $f_j(R^{(k+1)})\, I_j^{(k+1)}\, z_j$. Let $j_K \equiv \arg\max_{j_s \in J}\{s \in \{1, \ldots, T\} : j_s \in N(k+1) \cap J$ and $f_{j_s}(R^{(k+1)})\, P_{j_s}^{(k+1)}\, z_{j_s}\}$. By (iii), $x_{i_K}(R^{(k+1)}) \in M^+$, and so by (iv), $j_T \neq j_K$. Hence, $K < T$. Let $H' \equiv \{h'_1, \ldots, h'_{T-K+1}\}$ be such that for each $t \in \{1, \ldots, T-K+1\}$, $h'_t = j_{K+t-1}$. Then H' satisfies (5-a) to (5-c) in Lemma 5, which is a contradiction. Thus, there is $j \in J^{(k+1)} \setminus N(k+1)$. We complete the proof of Claim.

[6] Precisely, since (a) $R'_{k+1} \in \mathcal{R}_I^-(z)$ and $f_{k+1}(R^{(k+1)})\, P'_{k+1}\, z_{k+1}$, and (b) $x_{k+1}(R^{(k+1)}) \in D(R'_i, p)$, $f_i(R^{(k+1)})\, I'_i\, z_i$ and $x_i(R^{(k+1)}) \notin M^+$, $\{k+1, i\}$ satisfies the conditions in Remark 6.

By using the above Claim, we prove Lemma 7. For $k = n$, there are $n+1$ distinct agents $N(n+1) \subseteq N$. However, since $|N(n+1)| = n+1$ and $|N| = n$, $N(n+1) \nsubseteq N$. This is a contradiction. □

We have the following corollary of Lemma 7.

Corollary 1. Let $R \in \mathcal{R}^n$, $(z,p) \in W^{min}(R)$. Let $N^+ \equiv \{i \in N : p_{x_i} > 0\}$. Let $N' \subseteq N$ and $R' \equiv (R'_{N'}, R_{-N'}) \in \mathcal{R}^n$ be such that for each $i \in N'$, $R'_i \in \mathcal{R}_I^-(z,p)$, and let $\hat{N}' \equiv \{i \in N : f_i(R')\, P'_i\, z_i\}$. Then, $\hat{N}' \subseteq N^+$.

Proof. By Fact 14 (i), $(z,p) \in W^{min}(R')$. Thus by Lemma 7, $\hat{N}' \subseteq N^+$. □

Given $R \in \mathcal{R}^n$, $p \in \mathbb{R}^m_+$ and $L' \subseteq L$, let $N(R,p,L') \equiv \{i \in N : D(R_i,p) \cap L' \neq \emptyset\}$. Note that $N(R,p,\emptyset) = \{i \in N : D(R_i,p) \cap \emptyset \neq \emptyset\} = \{i \in N : \emptyset \neq \emptyset\} = \emptyset$.

Remark 7. Let $R \in \mathcal{R}^n$, $p \in \mathbb{R}^m_+$ and $L_1, L_2, \ldots, L_K \subseteq L$. Then $N(R,p,\bigcup_{k=1}^K L_k) = \bigcup_{k=1}^K N(R,p,L_k)$.

Fact 15 (Theorem 1 in Morimoto and Serizawa, 2015)**.** Let $R \in \mathcal{R}^n$ and $(z,p) \in W^{min}(R)$. Let $M^+ \equiv \{a \in M : p_a > 0\}$. Then for each $M' \subseteq M^+$ with $M' \neq \emptyset$, $|N(R,p,M')| > |M'|$.

Remark 8. Let $R \in \mathcal{R}^n$ and $(z,p) \in W^{min}(R)$ and $M^+ \equiv \{a \in M : p_a > 0\}$ and $M' \subseteq M^+$ with $M' \neq \emptyset$. Let $N' \subseteq N$ and $R' \equiv (R'_{N'}, R_{-N'}) \in \mathcal{R}^n$ be such that for each $i \in N'$, $R'_i \in \mathcal{R}_I^-(z,p)$. Then $|N(R',p,M')| > |M'|$.

Proof. By Fact 14 (i), $(z,p) \in W^{min}(R')$. Thus by Fact 15, $|N(R',p,M')| > |M'|$. □

Lemma 8. Let $x \in A$ be a feasible object allocation. Let $R \in \mathcal{R}^n$ and $p \in \mathbb{R}_+$. Let $J(0) \subseteq N$ and $\{J(t)\}_{t=0}^\infty \subseteq N$ be the sequence of the set of agents such that $J(1) \equiv N(R,p,\{x_i\}_{i\in J(0)}) \setminus J(0)$ and for each $T \geq 1$, $J(T+1) \equiv N(R,p,\{x_i\}_{i \in \bigcup_{t=0}^T J(t)}) \setminus \bigcup_{t=0}^T J(t)$. Then for each $T \geq 0$, $J(T+1) = N(R,p,\{x_i\}_{i\in J(T)}) \setminus \bigcup_{t=0}^T J(t)$.

Proof. We prove this lemma by induction. This lemma holds for $T = 0$ obviously. As a base case, we consider the case $T = 1$

Base Case: $T = 1$. Then, by Remark 7

$$\begin{aligned}
J(2) &= N(R,p,\{x_i\}_{i\in(J(0)\cup J(1))}) \setminus (J(0) \cup J(1)) \\
&= \left[N(R,p,\{x_i\}_{i\in J(0)}) \cup N(R,p,\{x_i\}_{i\in J(1)})\right] \setminus (J(0) \cup J(1)) \\
&= \left[N(R,p,\{x_i\}_{i\in J(0)}) \setminus (J(0) \cup J(1))\right] \cup \left[N(R,p,\{x_i\}_{i\in J(1)}) \setminus (J(0) \cup J(1))\right] \\
&= \left[\left(N(R,p,\{x_i\}_{i\in J(0)}) \setminus J(0)\right] \setminus J(1)\right] \cup \left[N(R,p,\{x_i\}_{i\in J(1)}) \setminus (J(0) \cup J(1))\right] \\
&= (J(1) \setminus J(1)) \cup \left[N(R,p,\{x_i\}_{i\in J(1)}) \setminus (J(0) \cup J(1))\right] \\
&= \emptyset \cup \left[N(R,p,\{x_i\}_{i\in J(1)}) \setminus (J(0) \cup J(1))\right] \\
&= N(R,p,\{x_i\}_{i\in J(1)}) \setminus (J(0) \cup J(1)).
\end{aligned}$$

Inductive hypothesis: Let $T \geq 1$. Assume that for each $t \in \{1, \ldots, T\}$, $J(t) = N(R, p, \{x_i\}_{i \in J(t-1)}) \setminus \bigcup_{s=0}^{t} J(s)$.

Inductive Step: Let $T \geq 1$. Then by Remark 7 and inductive hypothesis,

$$\begin{aligned}
J(T+1) &= N(R, p, \{x_i\}_{i \in \bigcup_{t=0}^{T} J(t)}) \setminus \bigcup_{t=0}^{T} J(t) \\
&= N(R, p, \bigcup_{t=0}^{T} \{x_i\}_{i \in J(t)}) \setminus \bigcup_{t=0}^{T} J(t) \\
&= \bigcup_{t=0}^{T} N(R, p, \{x_i\}_{i \in J(t)}) \setminus \bigcup_{t=0}^{T} J(t) \\
&= \left[\bigcup_{t=0}^{T-1} [N(R, p, \{x_i\}_{i \in J(t)}) \setminus \bigcup_{s=0}^{t} J(s)] \setminus (J(t+1) \cup \cdots \cup J(T)) \right] \\
&\quad \cup \left[N(R, p, \{x_i\}_{i \in J(T)}) \setminus \bigcup_{t=0}^{T} J(t) \right] \\
&= \left[\bigcup_{t=0}^{T-1} [J(t+1) \setminus (J(t+1) \cup \cdots \cup J(T))] \right] \cup \left[N(R, p, \{x_i\}_{i \in J(T)}) \setminus \bigcup_{t=0}^{T} J(t) \right] \\
&= \underbrace{\emptyset \cup \cdots \cup \emptyset}_{T \text{ times}} \cup N(R, p, \{x_i\}_{i \in J(T)}) \setminus \bigcup_{t=0}^{T} J(t) \\
&= N(R, p, \{x_i\}_{i \in J(T)}) \setminus \bigcup_{t=0}^{T} J(t).
\end{aligned}$$

$\square$

Lemma 9. Let $R \in \mathcal{R}^n$ and $(z, p) \in W^{min}(R)$. Let $M^+ \equiv \{a \in M : p_a > 0\}$. Assume that there are $N(k) = \{1, \ldots, k\} \subseteq N$ and $R^{(k)} \equiv (R'_{N(k)}, R_{-N(k)}) \in \mathcal{R}^n$ such that for each $i \in N(k)$, $R'_i \in \mathcal{R}_I^-(z)$. Let $\hat{N}(k) \equiv \{i \in N : f_i(R^{(k)}) \, P_i \, z_i\}$. Assume that $\hat{N}(k) \neq \emptyset$. Then, $\hat{N}(k) \nsubseteq N(k)$.

Proof. For convenience, let $J(0) \equiv \hat{N}(k)$. Suppose that $J(0) = \hat{N}(k) \subseteq N(k)$. We prove this lemma by induction.

Claim: For each $T \geq 0$, there is $\{J(t)\}_{t=0}^{T+1} \subseteq N$ such that
(i) $J(T+1) \equiv N(R^{(k)}, p, \{x_j(R^{(k)})\}_{j \in J(T)}) \setminus \bigcup_{t=0}^{T} J(t) \neq \emptyset$,
(ii) for each $j \in J(T+1)$, $x_j(R^{(k)}) \in M^+$, and
(iii) $N(R^{(k)}, p, \{x_j(R^{(k)})\}_{j \in \bigcup_{t=0}^{T+1} J(t)}) \setminus \bigcup_{t=0}^{T+1} J(t) \neq \emptyset$.

Base Case: $T = 0$. (i) Let $J(1) \equiv N(R^{(k)}, p, \{x_i\}_{i \in J(0)}) \setminus J(0)$. By Lemma 3 (ii), for each $i \in J(0)$, $x_i(R^{(k)}) \in M^+$. Since $p = p^{min}(R)$, by Remark 8, $|N(R^{(k)}, p, \{x_i\}_{i \in J(0)})| > |\{x_i\}_{i \in J(0)}| = |J(0)|$. Thus, $J(1) = N(R^{(k)}, p, \{x_i\}_{i \in J(0)}) \setminus J(0) \neq \emptyset$. Hence (i) holds.

(ii) Suppose that there is $j \in J(1)$ such that $x_j(R^{(k)}) \notin M^+$. Let $h \in \hat{N}(k)$ be such that $x_h(R^{(k)}) \in D(R_j^{(k)}, p)$. By $j \notin J(0) = \hat{N}(k)$, $f_j(R^{(k)})\, I^{(k)}\, z_j$. Thus, $\{h, j\}$ satisfies (a) and (b) in Remark 6, which is a contradiction. Thus, for each $j \in J(1)$, $x_j(R^{(k)}) \in M^+$.

(iii) By (ii), $\{x_j(R^{(k)})\}_{j \in J(1)} \subseteq M^+$. Thus, by $\{x_j(R^{(k)})\}_{j \in J(0)} \subseteq M^+$, $p = p^{min}(R^{(k)})$ and Remark 8, $|N(R^{(k)}, p, \{x_j(R^{(k)})\}_{j \in J(0) \cup J(1)})| > |\{x_j(R^{(k)})\}_{j \in J(0) \cup J(1)}| = |J(0) \cup J(1)|$. Thus, $N(R^{(k)}, p, \{x_j(R^{(k)})\}_{j \in J(0) \cup J(1)}) \setminus (J(0) \cup J(1)) \neq \emptyset$.

Inductive Hypothesis: Let $T \geq 1$. Assume that for each $t \in \{1, \ldots, T\}$, there is $\{J(s)\}_{s=0}^T \subseteq N$ such that
(i-t) $J(t) = N(R^{(k)}, p, \{x_j(R^{(k)})\}_{j \in J(t-1)}) \setminus \bigcup_{s=0}^{t-1} J(s) \neq \emptyset$,
(ii-t) for each $j \in J(t)$, $x_j(R^{(k)}) \in M^+$, and
(iii-t) $N(R^{(k)}, p, \{x_j(R^{(k)})\}_{j \in \bigcup_{s=0}^t J(s)}) \setminus \bigcup_{s=0}^t J(s) \neq \emptyset$.

Inductive Step: (i) Let $J(T+1) \equiv N(R^{(k)}, p, \{x_j(R^{(k)})\}_{j \in \bigcup_{s=0}^T J(s)}) \setminus \bigcup_{s=0}^T J(s)$. By (iii-T), $J(T+1) \neq \emptyset$. By Lemma 8, $J(T+1) = N(R^{(k)}, p, \{x_j(R^{(k)})\}_{j \in J(T)}) \setminus \bigcup_{s=0}^T J(s)$. Thus, (i) holds.

(ii) Suppose that there is $j_{T+1} \in J(T+1)$ such that $x_{j_{T+1}}(R^{(k)}) \notin M^+$. Note that by (i) and (i-1) to (i-T), there is $\{h, j_1, \ldots, j_{T+1}\} \subseteq N$ such that, $h \in \hat{N}(k) = J(0)$, for each $t \in \{1, \ldots, T\}$, $j_t \in J(t)$, $x_h(R^{(k)}) \in D(R_{j_1}^{(k)}, p)$, and for each $t \in \{1, \ldots, T\}$, $x_{j_t}(R^{(k)}) \in D(R_{j_{t+1}}^{(k)}, p)$. Since for each $t \in \{1, \ldots, T+1\}$, $J(t) \cap J(0) = J(t) \cap \hat{N}(k) = \emptyset$, for each $t \in \{1, \ldots, T+1\}$, $f_{j_t}(R^{(k)})\, I_{j_t}^{(k)}\, z_{j_t}$. Let $H \equiv \{h_1, \ldots, h_{T+2}\} \subseteq N$ such that $h_1 = h$, and for each $t \in \{1, \ldots, T+1\}$, $h_{t+1} = j_t$. Then H satisfies (5-a) to (5-c) in Lemma 5, which is a contradiction.

(iii) By (ii-k) and (ii), $\{x_j(R^{(k)})\}_{j \in \bigcup_{s=0}^{T+1} J(s)} \subseteq M^+$. Thus, by Remark 8, $|N(R^{(k)}, p, \{x_j(R^{(k)})\}_{j \in \bigcup_{s=0}^{T+1} J(s)}| > |\{x_j(R^{(k)})\}_{j \in \bigcup_{s=0}^{T+1} J(s)}| = |\bigcup_{s=0}^{T+1} J(s)|$. Thus, $N(R^{(k)}, p, \hat{M} \cup \{x_j(R^{(k)})\}_{j \in N'(l+1)}) \setminus \bigcup_{s=0}^{T+1} J(s) \neq \emptyset$. We complete the proof of Claim.

By the above claim, we derive a contradiction. For $T = n$, by the definition of $\{J(s)\}_{s=0}^n$, $|\bigcup_{s=0}^n J(s)| \geq n+1$ and $\bigcup_{s=0}^n J(s) \subseteq N$. However, since $|N| = n$, $\bigcup_{s=0}^n J(s) \not\subseteq N$. This is a contradiction. $\square$

4.4 Proof of Proposition 2

Proof. Suppose that there is $i \in N$ such that $f_i(R)\, P_i\, z_i$. Without loss of generality, $i = 1$.

Claim: For each $k \geq 0$, there are sets $N(k)$ and $N(k+1)$ of distinct agents such that $N(k+1) \supseteq N(k)$, $|N(k)| = k$, $|N(k+1)| = k+1$, say $N(k) = \{1, 2, \ldots, k\}$, $N(k+1) = \{1, 2, \ldots, k+1\}$, and $R^{(k)} \equiv (R'_{N(k)}, R_{-N(k)}) \in \mathcal{R}^n$ and $R^{(k+1)} \equiv (R'_{N(k+1)}, R_{-N(k+1)}) \in \mathcal{R}^n$ such that

(i) $f_{k+1}(R^{(k)})\, P_{k+1}\, z_{k+1}$
(ii) for each $i \in N(k+1)$, $R'_i \in \mathcal{R}_I^-(z,p)$ and $f_{k+1}(R^{(k+1)})\, P'_{k+1}\, z_{k+1}$,
(iii) $\hat{N}(k+1) \neq \emptyset$ and $\hat{N}(k+1) \subseteq N^+$, and
(iv) $N(k+1) \subsetneq N^+$.

Base Case: Let $k=0$. (i) By assumption, (i) holds.

(ii) Let $R'_1 \in \mathcal{R}_I^-(z)$. By (i) and Lemma 6, $f_1(R^{(1)})\, P'_1\, z_1$.

(iii) By (ii), $1 \in \hat{N}(1)$ and so $\hat{N}(1) \neq \emptyset$. By Corollary 2, $\hat{N}(1) \subseteq N^+$.

(iv) By (iii) and Lemma 9, $\hat{N}(1) \nsubseteq N(1)$. Thus, there is $j \in \hat{N}(1) \setminus N(1)$. Hence, by (iii), since $j \in N^+$, $N(1) \subsetneq N^+$.

Inductive Hypothesis: Let $k \geq 1$. There are sets $N(k-1)$ and $N(k)$ of distinct agents such that $N(k-1) \supseteq N(k)$, $|N(k-1)| = k-1$, $|N(k)| = k$, say $N(k-1) = \{1,2,\ldots,k-1\}$, $N(k) = \{1,2,\ldots,k\}$, and $R^{(k)} \equiv (R'_{N(k-1)}, R_{-N(k-1)}) \in \mathcal{R}^n$ and $R^{(k)} \equiv (R'_{N(k)}, R_{-N(k)}) \in \mathcal{R}^n$ such that
(i-k) $f_k(R^{(k-1)})\, P_k^{(k)}\, z_k$,
(ii-k) for each $i \in N(k)$, $R'_i \in \mathcal{R}_I^-(z,p)$ and $f_k(R^{(k)})\, P'_k\, z_k$,
(iii-k) $\hat{N}(k) \neq \emptyset$ and $\hat{N}(k) \subseteq N^+$, and
(iv-k) $N(k) \subsetneq N^+$

Inductive Step: (i) By (iii-k), $\hat{N}(k) \neq \emptyset$. Thus, by Lemma 9, $\hat{N}(k) \nsubseteq N(k)$. Hence there is $j \in \hat{N}(k) \setminus N(k)$; that is, $j \notin N(k)$ and $f_j(R^{(k)})\, P_j\, z_j$. Let $j \equiv k+1$. Thus (i) holds.

(ii) Let $R'_{k+1} \in \mathcal{R}_I^-(z,p)$. Thus by (ii-k), for each $i \in N(k+1)$, $R'_i \in \mathcal{R}_I^-(z,p)$. Moreover, by (i) and Lemma 6 (i), $f_{k+1}(R^{(k+1)})\, P'_{k+1}\, z_{k+1}$. Thus (ii) holds.

(iii) By (ii), $k+1 \in \hat{N}(k+1)$ and so $\hat{N}(k+1) \neq \emptyset$. By Corollary 2, $\hat{N}(k+1) \subseteq N^+$.

(iv) By (iii) and Lemma 9, $\hat{N}(k+1) \nsubseteq N(k+1)$. Thus, there is $j \in \hat{N}(k+1) \setminus N(k+1)$. Hence, by $j \in \hat{N}(k+1)$ and (iii), $j \in N^+$. Thus, $N(k+1) \subsetneq N^+$. Therefore, we complete the proof of Claim.

By the above claim, we prove Proposition 2. For $k = n$, there are $n+1$ distinct agents $N(n+1) \subseteq N$. However, since $|N(n+1)| = n+1$ and $|N| = n$, $N(n+1) \nsubseteq N$. This is a contradiction. $\square$

4.5 Proof of Theorem 1

Proof. Let $R \in \mathcal{R}^n$ and $(z,p) \in W^{min}(R)$. By Propositions 1 and 2, for each $i \in N$, $f_i(R)\, I_i\, z_i$. We show that $(f(R),p)$ satisfies WE-i and WE-ii.

Suppose that $(f(R),p)$ does not satisfy WE-i; that is, there is $i \in N$ such that $x_i(R) \notin D(R_i,p)$ or $t_i(R) \neq p_{x_i(R)}$. Note that if $x_i(R) \notin D(R_i,p)$, then by $f_i(R)\, I_i\, z_i$ $(x_i(R), t_i(R)) = f_i(R)\, I_i\, z_i\, P_i\, (x_i(R), p_{x_i(R)})$, and so $t_i(R) < p_{x_i(R)}$. Hence we need only to consider the case $t_i(R) < p_{x_i(R)}$. Also note that by Lemma 3 (i), for each $i \in N$,

$t_i(R) \leq p_{x_i(R)}$. Thus,

$$\sum_{k\in N} t_k(R) < \sum_{k\in N} p_{x_k(R)} \leq \sum_{a\in M} p_a = \sum_{k\in N} t_k.$$

Hence, z Pareto-dominates $f(R)$, contradicting efficiency. Therefore, for each $i \in N$, $x_i(R) \in D(R_i, p)$; that is, $z_i \, I_i \, (x_i(R), p_{x_i(R)})$. Since for each $i \in N$, $f_i(R) \, I_i \, z_i$,

$$(x_i(R), t_i(R)) = f_i(R) \, I_i \, z_i \, (x_i(R), p_{x_i(R)}).$$

Hence, for each $i \in N$, $t_i(R) = p_{x_i(R)}$, which means that $(f(R), p)$ satisfies WE-i.

Next, we show that for each $a \in M$ with $p_a > 0$, there is $i \in N$ such that $x_i(R) = a$. Suppose that there is $a \in M$ such that $p_a > 0$ but for each $i \in N$, $x_i(R) \neq a$. Since $\sum_{k\in N} t_k = \sum_{b\in M^+} p_b$, and $\sum_{k\in N} t_k(R) \leq \sum_{b\in M^+} p_b - p_a$,

$$\sum_{k\in N} t_k(R) \leq \sum_{b\in M^+} p_b - p_a < \sum_{b\in M^+} p_b = \sum_{k\in N} t_k,$$

Since for each $i \in N$, $f_i(R) \, I_i \, z_i$, z Pareto-dominates $f(R)$, contradicting efficiency. Hence, for each $a \in M^+$, there is $i \in N$ such that $x_i(R) = a$, which means that $(f(R), p)$ satisfies WE-ii.

Therefore, $(f(R), p) \in W(R)$, and by $p = p^{min}(R)$, $f(R) \in W^{min}(R)$; that is, $f(R)$ is a minimum price Walrasian allocation for R. $\square$

5 Related literatures

In the cases of homogeneous objects, Saitoh and Serizawa (2008) and Sakai (2008) characterize the generalized Vickrey rule by individual rationality, no subsidy, efficiency and strategy-proofness on the classical domain; moreover, Saitoh and Serizawa (2008) also show the same characterization on the positive income domain and the negative income domain.

Zhou and Serizawa (2018) also maintain unit-demand preferences, but study the special class of preferences, *the common-tiered domains*. It says that objects are partitioned into several tiers, and if objects are equally priced, agents prefer an object in the higher tier to one in the lower. They show that when we sort objects and the tier including nth highest objects is singleton, for an arbitrary numbers of agents and objects, the minimum price Walrasian rule is the only rule satisfying same four properties on the common-tiered domains; moreover, when the number of agents is less than or equal to the number of objects including null object, on the common-tiered positive income effect domains, the minimum price Walrasian rule is also the only rule satisfying same four properties.

There is also the literature on auction with non-quasi-linear preferences admitting multi-demand in various settings. Kazumura and Serizawa (2016) study classes of preferences that include unit-demand preferences and additionally includes at least one multi-demand preference, and show that no rule satisfies the four properties on such a domain.

Malik and Mishra (2021) study the special classes of preferences, "dichotomous" domains. A preference is *dichotomous* if there is a set of objects such that the valuations of its supersets are constant and the valuations of other sets are zero. A *dichotomous domain* includes all such dichotomous preferences for a given set of objects. They show that no rule satisfies the four properties on a dichotomous domain, but that the generalized Vickrey rule is the only rule satisfying the four properties on a class of dichotomous preferences exhibiting positive income effects.

Baisa (2020) assumes that objects are homogeneous and shows that on the class of preferences exhibiting decreasing marginal valuations, positive income effect, and single-crossing property, if the preferences are parametrized by one dimensional types, there is a rule satisfying the above four properties, but that if types are multi-dimensional, no rule satisfies these properties. Shinozaki et al. (2020) also assume the homogeneity of objects, and show that on the class of preferences includes sufficiently various preferences exhibiting non-decreasing marginal valuations (*minimal richness*), the generalized Vickrey rule is the only rule satisfying the four properties, but that no rule satisfies these properties on the class of preferences that additionally includes at least one preference exhibiting decreasing marginal valuations.

There is another topic on auction with non-quasi-linear preference which focus on ex-post revenue maximization. On the unit-demand setting, Kazumura et al. (2020B) and Sakai and Serizawa (2020) show that in the class of auction rules satisfying individual rationality, no subsidy, non-wastefulness, equal treatment of equals and strategy-proofness, a minimum price Walrasian rule is the unique rule ex-post revenue maximizing rule. Recently, Kazumura et al. (2020A) develop methods to analyze strategy-proof rules in general settings including multi-demand cases.

6 Conclusion

By extending the results of Morimoto and Serizawa (2015), we showed that for an arbitrary numbers of agents and objects, the minimum price Walrasian rule is the unique rule satisfying individual rationality, no subsidy, efficiency and strategy-proofness on the classical domain. We believe that our technique will be useful for the analysis of auction in the environment of non-quesi-linear preferences.

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