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A Generalized Uzawa Growth Theorem and Capital-Augmenting Technological Change

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Abstract

We prove a generalized, multi-factor version of the Uzawa steady-state growth theorem. The theorem implies that neoclassical growth models need at least three factors of production to be consistent with empirical evidence on both the capital-labor elasticity of substitution and the existence of capital-augmenting technical change. We also build and calibrate a three-factor endogenous growth model with directed technical change and show that it converges to a balanced growth path that is consistent with the empirical evidence. Our results indicate that natural resources including land and directed technical change play a central role in explaining balanced growth.

Keywords: Balanced Growth, Uzawa Steady-State Growth Theorem, Directed Technical Change, Land, Natural Resources

JEL Classification Codes E13, E22, O33, O41

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1 Introduction

The neoclassical growth model was developed to explain a set of stylized macroeconomic facts that can be classified under the umbrella of balanced growth (Solow, 1956, 1994). As conventionally understood, the Uzawa (1961) steady state growth theorem says that on the balanced growth path (BGP) of a neoclassical growth model, all technological change must be labor-augmenting, unless the aggregate production function is Cobb-Douglas (Jones, 2005; Jones and Scrimgeour, 2008). This creates a significant problem for the neoclassical growth model, because data from the United States strongly suggest that (i) there is capital-augmenting technical change on the BGP and (ii) the aggregate production function is not Cobb-Douglas (see, e.g., Antras et al., 2004; Oberfield and Raval, 2014; Grossman et al., 2017).

The standard neoclassical growth model assumes that there are only two factors of production, labor and reproducible capital. In reality, there are many other factors of production, including various types of land, energy, and other natural resources. These factors do not fit well in the notion of capital in the neoclassical growth model in that they cannot be readily accumulated (or reproduced) through savings. In this paper, we examine whether incorporating more factors of production makes it possible for neoclassical models to be consistent with the empirical regularities mentioned above.

We start by proving a multi-factor version of the Uzawa (1961) steady state growth theorem. When building macroeconomic models, researchers have incomplete knowledge of how the aggregate production function evolves over time due to technological change. We show that, if an economy has a BGP, the Uzawa theorem provides guidance on how to choose a simple *representation* of the ever-changing production function. We call this the *Uzawa Representation*. The Uzawa Representation gives the correct relationship between aggregate inputs and aggregate output on the BGP, while capturing steady state technological change through factor-augmenting terms on inputs other than reproducible capital. The Uzawa Representation has the same derivatives and elasticity of substitution (EoS) as the true production function, suggesting that it can be useful for some economic analyses. But, this particular representation cannot match evidence on capital-augmenting technical change, which is observed in the data. If we interpret this representation as the true production function, the puzzle arises as discussed above. By distinguishing the true production function and the Uzawa representation, however, our approach does not rule out the existence of other representations that might have more accurate descriptions of technological change.

To identify other possible representations that are a better match with data, we prove a further generalized version of the Uzawa theorem. The generalized theorem demonstrates that there are a *continuum* of representations with capital-augmenting technical change, as long as reproducible capital has a unitary EoS with at least one other factor. From this broader class of *Factor-Augmenting Representations*, it is possible to choose a representation that matches the empirically observed speed of capital-augmenting technological progress. In other words, the factor-augmenting representations can be simultaneously consistent with balanced growth, a non-unitary EoS between capital and labor, and capital-augmenting technical change. We also provide conditions under which these Factor-Augmenting Representations have the same derivatives and EoS as the true production function. Thus, these representations are more suitable for economic analyses than standard specifications that are based on the neoclassical growth model.

To demonstrate the practical importance of theorems, we formulate and analyze a model economy with three factors of production and endogenous directed technical change. We provide a micro foundation for the model such that the resultant aggregate production function has the form of a Factor-Augmenting Representation that fits the Generalized Uzawa theorem. The model exhibits a non-unitary EoS between capital and labor, but nonetheless has a BGP with a positive rate of capital-augmenting technical change, a result that is not possible with existing models.[1] We also calibrate the model to U.S. data and numerically show that the economy converges to the BGP from a wide range of initial conditions. The global stability result has a particular theoretical importance, because it demonstrates that the direction of technological change endogenously conforms to a required condition embedded in the generalized Uzawa theorem. If technological change were exogenous, a knife-edge condition would be required. In this sense, the model suggests that endogenous directed technical change is essential in explaining balanced growth in neoclassical models.

Existing work on the Uzawa theorem tends to emphasize its restrictive nature, interpreting the theorem as giving the only possible form of technological progress in the neoclassical growth model. In contrast, our results should be useful in applied macroeconomic research. The Generalized Uzawa theorem provides guidance on how to choose functional forms for the aggregate production function and technical change that together are a good representation of the true production function along the BGP. In particular, we provide representations that, along the BGP, have the correct speeds of technological progress, levels of inputs and output, first order derivatives and elasticities of substitution between factors of production.

Our results are relevant for a wide range of studies. Growth-, development-, and business-

[1] Grossman et al. (2017) is the sole exception, which we discuss in more detail below.

cycle accounting analyses all highlight the need to distinguish between different types of technology when describing macroeconomic outcomes (e.g., Greenwood et al., 1997; Krusell et al., 2000; Fisher, 2006; Caselli and Feyrer, 2007; Hsieh and Klenow, 2010). Our results allow such analyses to be conducted with production functions that are simultaneously consistent with balanced growth facts, capital-augmenting technical change, and the EoS between capital and labor. Previous approaches can only be consistent with two out of these three. This is especially important for understanding the relationship between technology and inequality, because labor- and capital-augmenting technical change have different impacts on factor shares when the EoS is different than one. For example, Karabarbounis and Neiman (2014) suggest that capital augmenting technical change is responsible for the decline in the labor share in many countries over the last several decades. On a separate note, we find that the three factor model with endogenous and directed technical change converges slowly to the steady state, when compared to standard neoclassical growth models. This is because of the slow dynamics of technology and the existence of fixed factors of production. The transition dynamics in the standard neoclassical model are too fast to explain patterns observed in data (King and Rebelo, 1993; Banerjee and Moll, 2010). These slow dynamics may also have important implications for transition dynamics following exogenous shocks (Leon-Ledesma and Satchi, 2019).

Related Literature. This paper is related to a long literature on balanced growth and the Uzawa steady state growth theorem. Although the theorem is well known, Uzawa (1961) does not provide a clear statement or proof of the theorem. A simple and intuitive proof was proposed by Schlicht (2006) and updated by Jones and Scrimgeour (2008), Acemoglu (2008), Irmen (2016), and Grossman et al. (2017). With the exception of Acemoglu (2008), the literature has been concerned only with whether a particular production function can match the level of output on the BGP. We contribute to this literature in several ways. First, we extend the theorem to multiple factors of production. Second, we prove a generalized version of theorem that stresses the difference between the true production function and representations of the production function. Third, we derive conditions under which representations have the same first-order derivatives and EoS as the true production function.

As noted above, the existing literature has treated the Uzawa theorem as a restrictive condition. As a result, many studies have tried to explain why the economy might endogenously conform to the two-factor version of theorem. These studies frequently use models with directed technical change. In particular, Acemoglu (2003) and Irmen and Tabaković

(2017) provide models where capital-augmenting technical change disappears in the long run, while Jones (2005) and Leon-Ledesma and Satchi (2019) specify models that are Cobb-Douglas in the long-run. We build on these works by presenting an endogenous growth model that converges to a BGP that is consistent with data on both the existence of capital-augmenting technical change and the less-than-unitary EoS between capital and labor. Our model specification builds on the work of Irmen (2017) and Irmen and Tabaković (2017).

To the best of our knowledge, Grossman et al. (2017) provide the only other attempt to square the Uzawa steady state growth theorem with data on the EoS and the capital-augmenting technical change. In their model, schooling is both labor-augmenting and capital-dis-augmenting. In this setting, they show that there is a scope for additional capital-augmenting technological change. Our results indicate that there is a wider scope for ways in which the neoclassical growth model can be made to be consistent with the data on EoS and capital-augmenting technical change. Indeed, their results can be understood as a particular case of the two-factor Uzawa theorem (see subsection 5.2).

Our results also stress the importance of natural resources for understanding macroeconomic outcomes. Historically, natural resources were only included in aggregate macroeconomic analyses when the research question under study was explicitly about those natural resources. For example, energy is generally only included in growth models when studying the depletion of finite resources (e.g., Hotelling, 1931; Heal, 1976) or climate change (e.g., Golosov et al., 2014). Our results suggest a much broader importance of non-accumulable factors: they must be incorporated into models of economic growth in order to recreate the balanced growth facts that originally motivated aggregate growth modeling (Solow, 1956, 1994). As noted above, our results also indicate that directed technical change, as in Acemoglu (2002), is essential to explaining balanced growth. There is a growing literature that combines directed technical change and natural resources to ask resource-related questions (e.g., Acemoglu et al., 2012; André and Smulders, 2014; Hassler et al., forthcoming).[2] Our results suggests that these models could serve as the basis for a much wider set of analyses.

Roadmap. The remainder of the paper proceeds as follows. Section 2 discusses the evidence motivating this study. Section 3 proves a multi-factor version of the Uzawa steady state growth theorem. In Section 4, we generalize theorem, proving that neoclassical models can have a positive rate of capital-augmenting on the BGP. Section 5 presents three applications of these results, focusing on simple cases and existing literature. Section 6 introduces a

[2]See also, Smulders and De Nooij (2003), Di Maria and Valente (2008), and Lemoine (2020).

full economic model with endogenous direction of technological change. Section 7 calibrates the model to the U.S. economy and demonstrates that the model economy endogenously converges to a BGP with positive capital-augmenting technical change. Section 8 discusses the broader implications of our findings.

2 Empirical Motivation

The neoclassical growth model, first developed by Solow (1956) and Swan (1956), is the central building block of much contemporary research in economic growth. Such models are designed to explain a set of stylized facts, known as 'balanced' or 'steady' growth (Jones, 2016). The main stylized fact is that income per capita has grown at a constant rate over long periods of time. Panel (a) in figure 1 presents U.S. data from 1950-2012, which clearly demonstrate this pattern.[3] It also demonstrates that other macroeconomics aggregates have grown at similar rates to GDP, capturing the notion of balance.

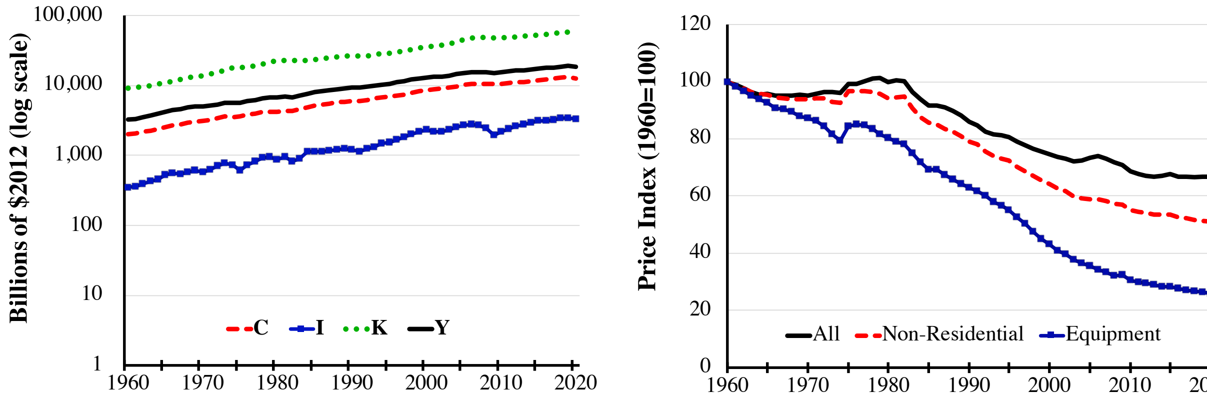
To explain these facts, the neoclassical growth model focuses on an aggregate production function that has constant returns to scale (CRS) in two factors, reproducible capital and labor.[4] The ability of the neoclassical growth model to provide a simple explanation for these facts has led to its widespread adoption (Jones and Romer, 2010). The model, however, relies on some strong assumptions, including those described by the Uzawa (1961) steady-state growth theorem. As conventionally stated, the Uzawa theorem requires that on a BGP, all technological progress must be labor-augmenting, unless the aggregate production function is Cobb-Douglas.

Given the restrictive nature of these conditions for balanced growth, it is natural to ask whether they are consistent with data. A long literature has estimated the elasticity between capital and labor in a two-factor production function and rejected the Cobb-Douglas specification. Most of papers in the literature argue that the elasticity is less than one. For example, Oberfield and Raval (2014) estimate the macro elasticity of around 0.7 using firm-level micro data, and Antras et al. (2004) estimates an elasticity of 0.6 directly from macro time-series data.

Panel (b) of Figure 1 demonstrates that the relative price of investment goods has been

[3]See Papell and Prodan (2014), Jones (2016), and others for longer time series and data from other countries.

[4]Intuitively, the key to explaining balanced growth is that capital is reproducible (i.e., it is accumulated from saved output). Thus, capital 'inherits' the constant growth rate of output, implying that the capital-output ratio will be constant in the long-run (Jones and Scrimgeour, 2008). Their joint growth rate is then determined by population growth and technological progress.



(a) BGP

(b) K-Augmenting Technology

Figure 1: Balanced Growth with capital-augmenting technical change. This figure presents some of the main features of balanced growth in the United States. Panel (a) demonstrates that real output, investment, consumption and the capital stock have grown at roughly constant rates over long periods of time. These empirical patterns summarize the notion of balanced growth. Panel (b) demonstrates that the price of investment goods, and equipment in particular, have been falling relative to the price of consumption goods in the United States. This indicates capital-augmenting technical change has been occurring along the BGP. See appendix B for details on data sources.

falling in the United States. This is a type of capital-augmenting technical change. Intuitively, in a setting with perfect competition, decreases in the relative price of capital goods reflect improvement in the efficiency of the investment goods sector. As a result, one unit of the final (consumption) good can be transformed into increasingly more units of effective capital goods. In standard neoclassical growth models, however, we typically choose to measure the quantity of capital in the same units as output. The declining relative price of capital goods, therefore, should imply that the efficiency of a given unit of capital has increased.[5] A long literature demonstrates that declining investment prices are a quantitatively important source of growth in the United States (e.g., Greenwood et al., 1997; Krusell et al., 2000).[6] As a result, there is broad consensus that capital-augmenting technical change has been pervasive in the United States over at least the last half a century, even as the economy

[5]Depending on how we measure the amount of capital, the declining relative price of capital goods can also be interpreted as investment-specific technological change (IST). Lemma 1 in Section 3 formalizes the notion that these two types of technological change are equivalent to each other. This result is also discussed by Grossman et al. (2017), who demonstrate this equivalence in a proof of the Uzawa steady-state growth theorem.

[6]See He et al. (2008) and Maliar and Maliar (2011) for discussions of the Uzawa steady state growth theorem in this context. Karabarbounis and Neiman (2014) show that declining investment prices are a widespread phenomenon in cross-country data. They use this fact to estimate the EoS between capital and labor and find a value that is greater than one.

Factor	Share	Source
Natural Resources (incl. Land)	8%	Caselli and Feyrer (2007)
Land	5%	Valentinyi and Herrendorf (2008)
Energy	4%	Golosov et al. (2014)

Table 1: This table presents some estimates of U.S. factor shares for inputs other than reproducible capital and labor. Definitions and methodologies vary. *Natural Resources* in Caselli and Feyrer (2007) encompasses land, energy, and other materials.

exhibited signs of balanced growth.

These findings create a puzzle. Given that the EoS between capital and labor is not equal to one, the Uzawa theorem implies that any two-factor neoclassical growth model that is consistent with balanced growth is necessarily at odds with evidence on capital-augmenting technical change. Put differently, the standard neoclassical growth model cannot explain the broader set of stylized growth facts that we observe in the United States.

In this paper, we examine production functions with additional factors of production, beyond reproducible capital and labor. It obvious that other factors – such as land, energy, and other natural resources – exist in the production process. Table 1 collects some evidence on the importance of these factors in the United States.[7] Broadly speaking, estimates suggest that non-reproducible factors other than labor account for about 8-9% of total factor payments. In particular, Caselli and Feyrer (2007) estimate the factor share of all natural resources and find 8%. Meanwhile, work in climate economics investigates the factor share of energy only, disregarding land. Golosov et al. (2014) summarize this evidence as pointing to a 4% share for energy. Finally, Valentinyi and Herrendorf (2008) focus only on land and find a share of 5%.

3 A Multi-factor Uzawa Theorem

In this section, we prove that the steady-state growth theorem by Uzawa (1961) (hereafter, the Uzawa theorem) extends to multi-factor environments that explicitly consider inputs beyond labor and reproducible capital. As shown by Solow (1956), sustained economic growth requires the shape of the production function to change over time, which economists usually call technological change. Given the existence of a BGP, the Uzawa theorem provides a convenient representation of the evolution of the production function. We stress the im-

[7]Estimating factor shares for inputs other than labor is notoriously difficult and often requires structural assumptions. Our intention is not to endorse any particular estimate. Instead, we simply note that there is ample evidence that factors other than labor and reproducible capital play a non-negligible role in production.

portance of making a clear distinction between this representation and the true production function, for which we often have limited information. In particular, we also prove a new set of propositions that clarify the conditions under which the representation given by the Uzawa theorem matches important properties of the true production function, implying that the representation serves as a good approximation of the true production function in economic analysis.

Neoclassical Growth Model

The Uzawa theorem depends on two assumptions: (a) the economy is described by a neoclassical growth model, and (b) the model has a balanced growth path (BGP). We start with a description of a neoclassical growth model, which is defined broadly to incorporate a wide range of dynamic macroeconomic models. For readability and consistency with the following sections, we consider a discrete time setting, where $t = 0, 1, 2, \ldots$, but it is straightforward to consider the continuous-time equivalents of the results.

Definition 1. *A* **multi-factor neoclassical growth model** *is an economic environment that satisfies:*

1. *Output, Y_t, is produced from capital, K_t, and $J \geq 1$ kinds of other inputs, $\{X_{j,t}\}_{j=1}^{J}$:*[8,9]

$$Y_t = F(K_t, X_{1,t}, ..., X_{J,t}; t). \tag{1}$$

In any $t \geq 0$, it has constant returns to scale (CRS) in all inputs, $K_t, X_{1,t}, ..., X_{J,t}$, and each input has a positive and diminishing marginal product.

2. *The amount of capital, K_t, evolves according to*

$$K_{t+1} = Y_t - C_t - R_t + (1 - \delta)K_t,\ \ K_0 > 0, \tag{2}$$

where $C_t > 0$ is consumption, $R_t \geq 0$ is expenditure other than capital investment or consumption (e.g., R&D inputs), and $\delta \in [0, 1]$ is the depreciation rate. The term $Y_t - C_t - R_t$ on the RHS represents physical capital investment.

[8] If we allow $J = 0$, the only constant-returns-to-scale production function is in the form of $Y_t = A_{K,t}K_t$. Although it cannot satisfy decreasing marginal products of its input (K_t), this AK functional form is also subject to the Uzawa theorem, in the sense that $A_{K,t}$ must be constant on the BGP (i.e., there may not be any technological change).

[9] We follow Uzawa (1961) and Jones and Scrimgeour (2008) by including t as an argument in F. Alternatively, We can write equation (1) as $Y_t = F_t(K_t, X_{1,t}, ..., X_{J,t})$ to highlight that F_t changes with t.

There are a number of points to note regarding Definition 1. First, production function $F(\cdot;t)$ in (1) depends on t, capturing technological progress. Importantly, we place no restrictions on how the shape of $F(\cdot;t)$ changes over time. As discussed below, the Uzawa theorem provides insight about how to approximate the time dependence of $F(\cdot;t)$ with standard factor-augmenting terms.

Second, if J equals 1 and $X_{1,t}$ is interpreted as labor, L_t, then equation (1) reduces to a familiar two-factor neoclassical production function, $Y_t = F(K_t, L_t; t)$. In addition, if we assume L_t grows exogenously, Definition 1 essentially coincides with the definition of a neoclassical growth model in Schlicht (2006) and Jones and Scrimgeour (2008), who provide a simple statement and proof of the two-factor Uzawa theorem.

Third, we allow for the R_t term in (2). If we set $R_t = 0$, equation (2) is in line with the previous definitions of the Uzawa theorem. This generalization is not essential for the proof of the Uzawa Theorem, but it accommodates the possibility of endogenous growth, which we examine in later sections. In production function (1), any technological change is captured by the last t term. If we think technology can be affected by the R&D expenditure, then such expenditure would be included in R_t in (2). Similarly, the evolution of factors $X_{j,t}$, including population L_t, can be either exogenous or dependent on particular types of expenditure, such as child-raising costs. Such costs are also included in R_t.

Fourth, the only reason why capital, K_t, is distinguished from other production factors $X_{1,t}, ..., X_{J,t}$ is that we explicitly specify its accumulation process as in (2), which guarantees that K_t can be accumulated linearly with the output. From theoretical viewpoint, K_t needs not to be limited to physical capital.[10] We will show that the Uzawa theorem holds regardless of the evolution process for other inputs. The $X_{j,t}$'s can be either endogenous or exogenous.

Lastly, we measure the amount of capital by its value in terms of final output. Specifically, equation (2) implicitly normalizes the unit of period $t+1$ capital so that period t final output can always be converted to the same units of period $t+1$ capital. Note that this is merely a choice of units, and therefore should not limit the applicability of our results. For example, Greenwood et al. (1997) and Grossman et al. (2017) consider investment-specific technological change, which enables more capital to be produced from a unit of final output. The following Lemma shows that, by change of variables, Definition 1 can accommodate such a case.

[10] It can be any combination of factors that can be accumulated linearly with the output. For example, in the pre-industrial Malthusian economy where population was proportional to output (e.g., Galor, 2011; Li et al., 2016), labor could be included in K_t, not in $X_{j,t}$.

Lemma 1. *Consider an economic environment where physical capital $\breve{K}_t$ accumulates according to*

$$\breve{K}_{t+1} = (Y_t - C_t - R_t)q_{t+1} + (1-\breve{\delta})\breve{K}_t, \tag{3}$$

where $q_t > 0$ is the investment-specific technology. The production function is given by $Y_t = \breve{F}(\breve{K}_t, X_{1,t}, ..., X_{J,t}; t)$, which has CRS and positive and diminishing returns to all inputs.

If the growth factor of investment-specific technological change $g_q = q_{t+1}/q_t$ is constant,[11] *this environment fits Definition 1 through a change of variables of $K_t \equiv \breve{K}_t/q_t$ and $\delta = (\breve{\delta} + g_q - 1)/g_q$.*

Proof. See Appendix A.2. □

The change of variables effectively normalizes the unit of capital so that capital $K_t \equiv \breve{K}_t/q_t$ is always measured in terms of the previous period's final goods. The depreciation rate, δ, after this normalization should be higher than $\breve{\delta}$, because positive investment specific technological change decreases the value of older capital. Thus, the new value of δ must accommodate changes in both the numerator (physical depreciation) and denominator (falling investment price) of $K_t \equiv \breve{K}_t/q_t$. In the rest of the paper, we describe the economy in terms of Definition 1, where any investment-specific technological change is included in the change of the shape of the production function $F(\cdot; t)$.

Balanced Growth Path

Now, we turn to the second requirement of the Uzawa theorem, the BGP.

Definition 2. *A* **balanced growth path (BGP)** *in a multi-factor neoclassical growth model is a path along which all quantities, $\{Y_t, K_t, X_{1,t}, ..., X_{J,t}, C_t, R_t\}$, grow at constant exponential rates for all $t \geq 0$. On the BGP, we denote the growth factor of output by $g \equiv Y_t/Y_{t-1}$, and the growth factors of any variable $Z_t \in \{K_t, X_{1,t}, ..., X_{J,t}, C_t, R_t\}$ by $g_Z \equiv Z_t/Z_{t-1}$. A* **non-degenerate** *balanced growth path is a BGP with $g_K > 1 - \delta$.*

From (2), condition $g_K > 1 - \delta$ means that physical capital investment $Y_t - C_t - R_t$ is strictly positive along the BGP. The rest of the paper focuses on this non-trivial case. We call it a non-degenerate BGP and simply mention it as a BGP when there is no risk of confusion. Note that, while a BGP requires variables to grow at constant rates, it does not

[11] We assume g_q to be constant for simplicity. This condition is not necessary if we extend Definition 1 to allow the depreciation rate to change over time. As long as we focus on the BGP, the results will not be affected.

require them to grow at the same rate. Still, the following lemma confirms that capital and consumption need to grow at the same speed as output to maintain a BGP.

Lemma 2. *On any non-degenerate BGP in a multi-factor neoclassical growth model, the capital-output ratio K_t/Y_t and the consumption-output ratio C_t/Y_t are constant and strictly positive.*

Proof. See Appendix A.3. □

The proof utilizes the assumption of $C_0 > 0$ from Definition 1. If $R_0 > 0$, we can similarly show that R_t/Y_t is constant.

Uzawa Representation and Its Properties

Having defined the neoclassical growth model and the BGP, we are ready to present a multi-factor version of the Uzawa Theorem.

Proposition 1. (A Multi-Factor Uzawa Theorem) *Consider a non-degenerate BGP in a multi-factor neoclassical growth model, and define $\widetilde{A}_{X_j,t} \equiv (g/g_{X_j})^t$ where $j = 1, ..., J$. Then, on the BGP,*

$$Y_t = \widetilde{F}(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t}) \text{ holds for all } t \geq 0, \tag{4}$$

where $\widetilde{F}(\cdot) \equiv F(\cdot; 0)$.

Proof. From the definition of $\widetilde{A}_{X_j,t} \equiv (g/g_{X_j})^t$, the growth factor of $\widetilde{A}_{X_j,t}X_{j,t}$ is g for all j. The growth factor of K_t is also g from Lemma 2. Therefore, all the arguments in function $\widetilde{F}(\cdot)$ are multiplied by g each period. This means that the RHS of (4) is multiplied by g each period since $\widetilde{F}(\cdot) \equiv F(\cdot; 0)$ has CRS. Note that in period 0, equation (4) holds true because it is identical with (1). Therefore, (4) holds for all $t \geq 0$, where both sides are multiplied by g in every period. □

It is important to understand what the theorem does and does not imply. Recall that the neoclassical production function $F(\cdot; t)$ in (1) is a time-varying function that potentially depends on t in complex ways. If the economy is on the BGP, the Uzawa theorem says that there should be a simple *representation* of this dependence of function $F(\cdot; t)$ on t, which holds at least along this particular BGP. We call this representation, which is given by (4), the *Uzawa representation*. It consists of a time-invariant function $\widetilde{F}(\cdot)$ and exponentially growing $\widetilde{A}_{X_j,t}$ terms. At $t = 0$, equation (4) coincides with the true production function (1).

The Uzawa representation illustrates how the production function evolves from there as t changes.

However, caution is needed when interpreting $\widetilde{F}(\cdot)$ as a production function for beyond $t = 0$, because Proposition 1 only guarantees that the value of $\widetilde{F}(\cdot)$ coincides with that of the true production function $F(\cdot; t)$ exactly on a particular BGP. As is clear from the proof of the proposition, function $\widetilde{F}(\cdot)$ contains no information about what will happen when inputs deviate even slightly from the BGP. As a result, there is no guarantee that the derivatives of function $\widetilde{F}(\cdot)$, even on the BGP, are equal to the derivatives of the production function $F(\cdot; t)$, apart from time $t = 0$. Without further information, therefore, the Uzawa theorem has little use in economic analysis.

In the following two propositions, we extend theorem by focusing on the conditions under which the Uzawa representation has the 'correct' marginal properties. We start by looking at first-order derivatives.

Proposition 2. (Derivatives of the Uzawa representation) *Let $F_Z(\cdot; t)$ denote the partial derivative of function $F(\cdot; t)$ with respect to its argument $Z \in \{K_t, X_{1,t}, ..., X_{J,t}\}$.*[12] *If the share of factor Z, i.e., $s_{Z,t} = F_Z(\cdot; t)Z_t/Y_t$, is constant on a non-degenerate BGP of a multi-factor neoclassical growth model, then the following holds on the BGP:*

$$\frac{\partial}{\partial Z_t}\widetilde{F}(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t}) = F_Z(K_t, X_{1,t}, ..., X_{J,t}; t) \text{ for all } t \geq 0. \tag{5}$$

Proof. See Appendix A.4. □

If the factor shares are constant on the BGP, equation (5) says that $\widetilde{F}(\cdot)$ has the same derivatives as the true production function $F(\cdot; t)$ on the BGP. We can also show that the elasticity of substitution (EoS) between capital and other production factors in the Uzawa representation $\widetilde{F}(\cdot)$ coincides with the EoS in the true production function $F(\cdot; t)$ on the BGP, if the latter does not change over time.[13] Let us first define the EoS when there are more than two inputs.[14]

[12] Appendix A.1 discusses the details regarding notation for derivatives.

[13] To the best of our knowledge, Acemoglu (2008) is the only example of previous work considering first-order properties implied by the Uzawa Theorem. He looks at first-order conditions in the two-factor case, providing a special case of Proposition 2.

[14] When there are more than two production factors, there are various ways to define the elasticity of technical substitution. See Stern (2011) for a concise taxonomy. The elasticity in (6) is calculated using the inverse of the symmetric elasticity of complementarity (SEC), defined in Stern (2010), which has a desirable property of symmetry between the two variables.

Definition 3. *The* **Elasticity of Substitution** *between capital K_t and input X_j in multi-factor neoclassical production function $F(K, X_1, ..., X_J; t)$ in (1) is defined by*

$$\sigma_{KX_j,t} = -\left.\frac{d\ln(K_t/X_{j,t})}{d\ln\left(F_K(K_t, X_{1,t}, ..., X_{J,t}; t)/F_{X_j}(K_t, X_{1,t}, ..., X_{J,t}; t)\right)}\right|_{Y_t, \mathbf{X}_{-j,t}: const}, \tag{6}$$

where $\mathbf{X}_{-j,t} \equiv \{X_{1,t}, ..., X_{J,t}\} \backslash X_{j,t}$ represents the inputs other than K_t and $X_{j,t}$.

Using this definition, we have the following result.

Proposition 3. (Elasticity of Substitution in the Uzawa Representation) *Let $\widetilde{\sigma}_{KX_j,t}$ denote the EoS in the Uzawa representation, as in Definition 3. If the EoS of the true production function, $\sigma_{KX_j,t}$ for some $j \in \{1, ..., J\}$, is constant over time on the BGP, then $\widetilde{\sigma}_{KX_j,t} = \sigma_{KX_j,t}$ holds for all $t \geq 0$ on the BGP.*

Proof. See Appendix A.5. □

Benefits and Limitations of the Uzawa Theorem

Propositions 1-3 demonstrate that the Uzawa representation $\widetilde{F}(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t})$ captures key elements of the true production function and is potentially useful for economic analysis. When developing a dynamic macroeconomic model, researchers need to take a stand on how to represent technical change. In other words, they need to decide how the shape of the production function will evolve over time. This is a challenging task that can influence the results, especially in a quantitative setting.

Given the requirement that the model should have a BGP, the multi-factor Uzawa theorem provides guidance in choosing an effective representation of the evolution of the production function. As shown in equation (1), the definition of the neoclassical growth model allows the aggregate production function to evolve in any way. The Uzawa representation captures this evolution only through factor-augmenting terms. Proposition 1 demonstrates that the Uzawa representation matches the level of all the key variables on the BGP, recreating an important set of stylized facts. Proposition 2 implies that the Uzawa representation has the correct derivatives, and therefore factor shares, as long as factor shares are constant on the BGP. Relatedly, Proposition 3 says that the Uzawa representation has the correct EoS between capital and other variables, as long as that elasticity is constant. Thus, the Uzawa representation can be useful as a local approximation of the true function around the BGP.

The Uzawa representation is based on function $\widetilde{F}(\cdot)$. In Propositions 1-3, we set $\widetilde{F}(\cdot) = F(\cdot;0)$, which may suggest that the period-0 production function must be known to the researcher. However, this assumption is made for the sake of clarity and is not necessary. The following remarks state that we can apply propositions even when only the local properties of the period-0 production function are known.

Remark 1. *The proof of Proposition 1 holds as long as $\widetilde{F}(\cdot)$ is any CRS function that matches the level of inputs and output in period 0 of the BGP; i.e., $\tilde{F}(K_0, X_{1,0}, ..., X_{J,0}) = Y_0$.*

Remark 2. *The proof of Proposition 2 only requires $\widetilde{F}(\cdot)$ to match the first derivative of $F(\cdot;0)$ in period 0 of the BGP, i.e., $\tilde{F}_Z(K_0, X_{1,0}, ..., X_{J,0}) = F_Z(K_0, X_{1,0}, ..., X_{J,0}; 0)$, in addition to the condition in Remark 1.*

Remark 3. *The proof of Proposition 3 only requires the EoS of $\widetilde{F}(\cdot)$ evaluated in the period 0 of the BGP to match that of $F(\cdot;0)$, in addition to the conditions in Remark 1.*

Thus, when building models for macroeconomic research, economists can construct the Uzawa representation that is consistent with BGP data, even if they do not know the whole shape of period-0 true production function, $F(.;0)$. They can simply pick any CRS production function (e.g., a CES production function) and calibrate it to match the level, derivatives (or factor shares), and the EoS from the data at a particular point in time, which can be regarded as time-0 BGP values. Then, Uzawa representation will continue to match these moments on the BGP, as long as the factor shares and the EoS are stationary.

Correctly modeling the level of output, factor prices, factor shares, and EoS between inputs is likely to be sufficient to answer many research questions in macroeconomics. In this sense, the Uzawa theorem is a positive result, highlighting the usefulness — rather than the limitations — of neoclassical growth models. Still, this particular representation may miss other important properties of the true production function. One of such properties is the direction of the technological change. The next section further generalizes the Uzawa growth theorem to overcome this limitation.

4 A Generalized Uzawa Growth Theorem

By viewing the $\widetilde{A}_{X_j,t}$ as the factor $X_{j,t}$-augmenting technology terms, Proposition 1 implies that it is always possible to interpret the time variation of the true production function $F(\cdot;t)$ on the BGP in terms of exponential augmentation of production factors. It is tempting to conclude that there should be no technological change that enhances the productivity of capital on the BGP, because there is no $\widetilde{A}_{K,t}$ term in (4). This reasoning is insufficient, because Proposition 1 does not establish uniqueness. As a result, it does not rule out the existence of better representations of the true production function.

In this section, we prove a further generalized version of the Uzawa theorem that allows for representations with capital-augmenting technical change. We explore the possibility that the true production function has more than one *factor-augmenting representation* and identify a condition under which there will be a representation that matches the data on the capital-augmenting technological change shown in Section 2. To satisfy this condition while remaining consistent with empirical evidence, it is essential to include factors of production beyond labor and reproducible capital in the aggregate production function.

Factor-Augmenting Technological Change and Factor Substitution

We start by defining a factor-augmenting representation. Note that we still do not know the functional form of $F^{\text{AUG}}(\cdot)$ below. Rather, the following definition sets out the goal of this section.

Definition 4. *A* **Factor-Augmenting Representation** *of the true production function (1) is a combination of a time-invariant constant-returns-to-scale function $F^{AUG}(\cdot)$ and the growth factors of factor-augmenting technologies $\gamma_K > 0$ and $\gamma_{X_j} > 0$, $j \in \{1, ..., J\}$, such that the paths of output and inputs on a BGP satisfy*

$$Y_t = F^{AUG}(A_{K,t}K_t, A_{X_1,t}X_{1,t}, ..., A_{X_J,t}X_{J,t}) \text{ holds for all } t \geq 0, \tag{7}$$

where $A_{K,t} = (\gamma_K)^t$ and $A_{X_j,t} = (\gamma_{X_j})^t$.

By comparing (4) with (7), it is clear that the Uzawa representation is a special case of factor-augmenting representation. As we explain below, (4) assumes that all effective factors grow at the same rate of g, while (7) permits different growth rates among different effective factors. In other words, the Uzawa representation hypothesizes that there is no factor substitution taking place when the economy grows along the BGP. The homothetic expansion of every effective input is the simplest interpretation of a steadily growing economy, but it does not necessarily constitute the best description of reality.

To see this, suppose that every effective input, including effective capital, grows at the same speed as the output. Recall that physical capital is already growing at the same speed as output on the BGP (Lemma 2). Then, there is no room for additional capital-augmenting technological progress to further augment its effectiveness. As discussed in

Section 2, however, there is clear evidence that the productivity of capital, measured in terms of output as in our model, has steadily been increasing on the BGP. Thus, the interpretation of the BGP as being a homothetic expansion of every input is at odds with a well-established stylized fact.

Motivated by this contradiction, we now consider a broader range of possibilities in which effective inputs grow at different constant rates. To have balanced growth with non-homothetic expansion of production factors, it is necessary to further restrict the possible functional forms of the factor-augmenting representation. Before moving to formal propositions, we provide a heuristic discussion that highlights the key intuition. Suppose that the true production function can be represented in the factor-augmenting way (7) on the BGP with the correct derivatives and EoS. Then, the growth rate of output can approximately be written as follows:[15]

$$g \equiv Y_{t+1}/Y_t \approx s_{k,t}\gamma_K g_K + \sum_{j=1}^{J} s_{X_j,t}\gamma_{X_j} g_{X_j}, \tag{8}$$

where $s_{k,t} \equiv F_K(K_t, X_{1,t}, ..., X_{J,t}; t)K_t/Y_t$ is the share of capital at time t and similarly for $s_{X_j,t}$.

Equation (8) says that the growth rate of the output is the weighted-average of the growth rates of different effective factors, where the weights are factor shares. When the effective factors grow at different speeds, $\gamma_K g_K$ and $\gamma_{X_j} g_{X_j}$'s are different. Specifically, let us assume that effective capital grows faster than output due to K-augmenting technological change ($\gamma_K g_K > g$). Then there must be at least one effective factor that is growing slower than output. Let us say that this factor is X_1 (i.e.,$\gamma_{X_1} g_{X_1} < g$) and that all the other effective factors are growing at the same rate as output. Then, dividing the factor augmenting representation (7) by Y_t gives

$$1 = F^{AUG}\left(\frac{A_{K,t}K_t}{Y_t}, \frac{A_{X_1,t}X_{1,t}}{Y_t}, constants\right). \tag{9}$$

In this form, it is evident the growing effective capital-output ratio $A_{K,t}K_t/Y_t$ permits production of unit output with the shrinking effective X_1-output ratio $A_{X_1,t}X_{1,t}/Y_t$. In other words, factor substitution is occurring.

Now, let us check if this on-going factor substitution is consistent with the definition of

[15]This decomposition is obtained by Taylor-expanding the RHS of (7) for $t+1$ with respect to every effective factor, around the period t values for the variables, and dividing the result by the RHS of (7) for t. The Taylor expansion is exact when the variables in t and $t+1$ are sufficiently close, or equivalently, in continuous time.

the BGP. On the BGP, output grows at a constant rate, g, which means that the RHS of (8) must also be constant. Given $\gamma_K g_K > \gamma_{X_1} g_{X_1}$, the RHS of (8) only remains constant when the factor shares $s_{k,t}$ and $s_{X_1,t}$ do not change over time. This happens if and only if the EoS of $F^{AUG}(\cdot)$ between K and X_1, defined in Definition 3, is equal to one. To summarize, for K-augmenting technological change to happen on the BGP in a factor-augmenting representation, the functional form of $F^{AUG}(\cdot)$ needs to have a unitary EoS between capital and some other factor.[16] In this case, it is possible to have balanced growth even when effective capital grows faster than output.

Once we obtain a factor-augmenting representation, we hope to use it as an approximation of the true production function. In particular, as in Proposition 3, the representation is especially useful if the EoS of $F^{AUG}(\cdot)$ matches that of the true production function. This is only possible when the true production function $F(\cdot;t)$ has a unitary EoS between capital and some other factor, because we already know that $F^{AUG}(\cdot)$ must have a unitary EoS. As discussed in section 2, there is a great deal of evidence suggesting that the EoS between capital and labor is different than one. However, our definition of neoclassical growth model allows for any number of inputs. Once we consider the realistic case with more than two factors of production, it becomes more likely that at least one input has a unitary EoS with capital.

A Steady-State Growth Theorem with K-augmenting Technical Change

Here, we formally construct a function that can be used as a basis for a factor-augmenting representation. Consider factors of production other than capital, $\{X_{1,t}, ..., X_{J,t}\}$, and suppose that some of them are substitutable with capital, K_t, with unitary elasticity in the period 0 production function, $F(\cdot;0)$. Without loss of generality, we reorder these factors so that the first $j^* \in \{1, ..., J\}$ of them can be substituted with capital with the unitary EoS.

If capital is substitutable with j^* other factors with unit elasticity, we can interpret them as if they are combined together in the Cobb-Douglas fashion to form an intermediate input. The intermediate input, which we call the *capital composite*, will then be one argument in the final production function. Using the share of factors in period 0, $s_{K,0} \equiv F_K(K_0, X_{1,0}, ..., X_{J,0}; 0)K_t/Y_t$ and $s_{X_j,0} \equiv F_K(K_0, X_{1,0}, ..., X_{J,0}; 0)X_{j,t}/Y_t$, we define

[16] In the Uzawa representation, $\gamma_K g_K = \gamma_{X_j} g_{X_j}$ holds for all j. Because the production function is assumed to have CRS (which guarantees $s_{K,t} + \sum_{j=1}^{J} s_{X_j,g} = 1$), the RHS is always constant. Therefore, we can use $F(\cdot;0)$ as the Uzawa representation without checking its EoS properties (see Proposition 1) at the cost that it cannot accommodate the possibility of K-augmenting technological change.

period-0 relative shares within the capital composite:

$$\alpha = s_{K,0}/(s_{K,0} + \sum_{j=1}^{j^*} s_{X_j,0}), \quad \xi_j = s_{X_j,0}/(s_{K,0} + \sum_{j=1}^{j^*} s_{X_j,0}). \tag{10}$$

Using these relative shares, we can represent the production function in a nested form:[17]

$$\overline{F}(k, x_1, ..., x_J) \equiv \widehat{F}\left(k^\alpha \prod_{j=1}^{j^*} x_j^{\xi_j}, x_{j^*+1}, ..., x_J\right). \tag{11}$$

The first argument of the $\widehat{F}(\cdot)$, $m = k^\alpha \prod_{j=1}^{j^*} x_j^{\xi_j}$, represents the capital composite, which combines capital and the other j^* factors that have a unitary EoS with capital. Capital composite m is an argument in the outside function $\widehat{F}(\cdot)$, along with other factors $x_{j^*+1}, ..., x_J$. The shape of the outside function $\widehat{F}(\cdot)$ is defined using the period-0 production function $F(\cdot;0)$:[18]

$$\widehat{F}(m, x_{j^*+1}, ..., x_J) \equiv F\left(\left(\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha}, X_{1,0}, ..., X_{j^*,0}, x_{j^*+1}, ..., x_J; 0\right). \tag{12}$$

The first argument of $\widehat{F}(\cdot)$, m, collects the j^* relevant inputs and combines them with capital in the first argument. As a result, function $\widehat{F}(\cdot)$ has j^* fewer arguments than $F(\cdot;0)$. Note that the RHS of (12) includes the BGP values $X_{j,0}$, $J = 1, \ldots, j^*$, which are treated as constants. Changes in the $x_{j,0}$ terms only matter through m.

As the following lemma shows, the nested representation, $\overline{F}(\cdot)$ with $\widehat{F}(\cdot)$, approximates the true production function around the BGP in period 0.

[17]In this section, we use lowercase letters $k, x_1, ..., x_J$ to denote variables, while uppercase letters $K_t, X_{1,t}, ..., X_{J,t}$ are the BGP values, unless otherwise noted.

[18]$F(\cdot;0)$ needs to satisfy $\sigma_{KX_j,0} = 1$ for $j = 1, ..., j^*$. Other than that, the following analysis only requires the local properties of $F(\cdot;0)$ are known to researchers. See Remarks 1-3 for a related discussions in the context of the multi-factor Uzawa theorem.

Lemma 3. (Nested representation of the production function at $t = 0$)

a. $\overline{F}(K_0, X_{1,0}, ..., X_{J,0}) = F(K_0, X_{1,0}, ..., X_{J,0}; 0)$.

b. For any $Z \in \{K, X_1, ..., X_J\}$, $\overline{F}_Z(K_0, X_{1,0}, ..., X_{J,0}) = F_Z(K_0, X_{1,0}, ..., X_{J,0}; 0)$.

c. For any $j = 1, ..., j^*$, $\overline{\sigma}_{KX_j,0} = \sigma_{KX_j,0}$, *where* $\overline{\sigma}_{KX_j,0}$ *is the EoS of function* $\overline{F}(k, x_1, ..., x_J)$ *between* k *and* x_j, *evaluated at the period-0 BGP.*

d. Functions $\widehat{F}(m, x_{j^*+1}, ..., x_J)$ *and* $\overline{F}(k, x_1, ..., x_J)$ *have constant returns to scale.*

Proof. See Appendix A.6. □

Properties a, b, and c respectively confirm that the nested representation $\overline{F}(\cdot)$ matches the period-0 true production function, $F(\cdot; 0)$, in terms of the level of inputs and output, the first derivatives for any input, and the EoS between K and any other input X_j, when the function is evaluated around the period-0 BGP.[19] Property d confirms the CRS property.

Thanks to the CRS property, the nested representation can be used not only for period 0, but also for representing how the production function evolves from there along the BGP. The following proposition establishes that, with the nested representation $\overline{F}(\cdot)$, there are multiple ways to represent the technological change in factor-augmenting fashion.

Proposition 4. (A Generalized Uzawa Growth Theorem) *Suppose that* $\sigma_{KX_j,0} = 1$ *for* $j = 1, ..., j^*$. *On a non-degenerate BGP, let* $\gamma_K > 0$ *and* $\gamma_{X_j} > 0$, $j \in \{1, ..., j^*\}$, *be* **any** *combination that satisfies the technology condition*

$$(\gamma_K g)^\alpha \prod_{j=1}^{j^*} (\gamma_{X_j} g_{X_j})^{\xi_j} = g. \tag{13}$$

For $j \in \{j^* + 1, ..., J\}$, *let* $\gamma_{X_j} = g/g_{X_j}$. *With* γ_K *and each* γ_{X_j}, *define* $A_{K,t} = (\gamma_K)^t$ *and* $A_{X_j,t} = (\gamma_{X_j})^t$. *Also, define function* $\overline{F}(\cdot)$ *by (10) and (11). Then, on the BGP,*

$$Y_t = \overline{F}\left(A_{K,t}K_t, A_{X_1,t}X_{j,t}, ..., A_{X_J,t}X_{J,t}\right) \text{ for all } t \geq 0. \tag{14}$$

Proof. See Appendix A.7. □

Note that (14) constitutes a factor augmenting representation, as defined by Definition 4.[20] Thus, Proposition 4 characterizes the set of factor-augmenting representations of the

[19] Namely, when $\{k, x_1, ..., x_J\}$ are at the period-0 BGP values $\{K_0, X_{1,0}, ..., X_{J,0}\}$.

[20] Recall that function $\overline{F}(\cdot)$ has CRS from Lemma 3.

true production function along the BGP. When there is no factor that is substitutable with capital with unit elasticity at time 0 (i.e., $j^* = 0$), then Proposition 4 becomes identical to Proposition 1.[21] However, given that there are many factors of production in reality, it seems plausible that at least one of them is substitutable with capital with unit elasticity ($j^* \geq 1$). In this case, there are several aspects of the proposition that warrant further discussion.

First, unlike Proposition 1, the generalized theorem implies that there is a continuum of representations. Namely, factor-augmenting terms, γ_K and γ_{X_j} for $j = 1, ..., j^*$, can be any combination that satisfy condition (13). This enables applied researchers to pick the representation that is most consistent with data on technical change. The Uzawa representation is a special case of the factor-augmenting representation with $\gamma_K = 0$.

Second, condition (13) implies that the amount of *effective* capital composite,

$$M_t = (A_{K,t}K_t)^{\alpha} \prod_{j=1}^{j^*} (A_{X_j,t}X_{j,t})^{\xi_j},$$

must grow at the same speed of output, g. By taking logs, it can be expressed in a log-linear form:

$$\alpha \log \gamma_K + \sum_{j=1}^{j^*} \xi_j \log \gamma_{X_j} = (1-\alpha) \log g - \sum_{j=1}^{j^*} \xi_j \log g_{X_j}. \tag{15}$$

When the growth rates of the factor-augmenting technologies are exogenous, this log-linear condition is restrictive. In a model where the direction of technical change is endogenous, however, this condition can be endogenously satisfied once the economy converges to the BGP. We will examine this possibility in sections 6 and 7.

Third, similar to the original Uzawa Theorem (Proposition 1), equation (14) is not a functional relationship. It only states that the level of inputs and outputs in this representation match those of the true production function on the BGP. The following propositions establish that, under conditions similar to Propositions 2 and 3, the factor-augmenting representation (14) gives the correct first derivatives and the correct EoS between capital and other factors around the BGP.

Proposition 5. (Derivatives of the Factor-Augmenting Representation) *Suppose that $\sigma_{KX_j,0} = 1$ for $j = 1, ..., j^*$. If the share of factor $Z_t \in \{K_t, X_{1,t}, ..., X_{J,t}\}$, i.e., $s_{Z,t} = F_Z(\cdot;t)Z_t/Y_t$, is constant on a non-degenerate BGP of a multi-factor neoclassical growth*

[21] If $j^* = 0$, condition $\alpha + \sum_{j=1}^{j^*} \xi_j = 1$ in Lemma 3 implies $\alpha = 1$. Then, condition (13) reduces to $\gamma_K = 1$, which means $A_{K,t} = 1$ for all t. Then, (14) becomes identical to (4).

model, the following holds on the BGP:

$$\frac{\partial}{\partial Z_t}\overline{F}\left(A_{K,t}K_t, A_{X_1,t}X_{1,t}, ..., A_{X_J,t}X_{J,t}\right) = F_Z(K_t, X_{1,t}, ..., X_{J,t}; t) \text{ for all } t \geq 0. \tag{16}$$

Proof. See Appendix A.8. □

Proposition 6. (The EoS of the Factor-Augmenting Representation) *Suppose that $\sigma_{KX_j,0} = 1$ for $j = 1, ..., j^*$ and let $\overline{\sigma}_{KX_j,t}$ denote the EoS in the factor-augmenting representation*
$\overline{F}\left(A_{K,t}K_t, A_{X_1,t}X_{1,t}, ..., A_{X_J,t}X_{J,t}\right)$. *If the EoS of the true production function, $\sigma_{KX_j,t}$ for some $j \in \{1, ..., J\}$, is constant over time on the BGP, then $\overline{\sigma}_{KX_j,t} = \sigma_{KX_j,t}$ holds for all $t \geq 0$ on the BGP.*

Proof. See Appendix A.9. □

Summary and Comparison to Uzawa Theorem

It is informative to contrast the results in this section with those in Section 3. The Uzawa theorem (Proposition 1) shows that if the economy exhibits balanced growth, as observed in many countries, there always exists a representation of the evolution of the production function, $\widetilde{F}(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t})$. This simple representation, called the *Uzawa Representation*, explains the balanced growth by homothetic expansion of every effective production factor. In other words, the Uzawa representation hypothesizes that no factor substitution is taking place along the BGP.

While the Uzawa representation matches the behavior of the true, often unknown, production function around the BGP (Propositions 2 and 3), it fails to explain one critical aspect of growth. In the Uzawa representation, the productivity of capital does not improve, because there is no $\widetilde{A}_K$ term. Our generalized theorem in Proposition 4 clarifies that the Uzawa theorem is only a single possibility out of a continuum of possible factor-augmenting representations, as long as the production function allows factor substitution on the BGP (which requires that at least one factor of production has a unitary EoS with capital).[22] Every candidate representation can explain the observed quantities on the BGP, but they differ in the rates of factor-augmenting technological progress among different production factors. So, it is possible to choose a candidate representations that matches the rate of capital-augmenting technological progress observed in data. Given the evidence of positive

[22]The Uzawa representation, where $\gamma_K = 1$ and $\gamma_{X_j} = g/g_{X_j}$ for all j, satisfies condition (15).

capital-augmenting technological change, the Uzawa representation will be ruled out as an appropriate representation of technological change.

Propositions 5 and 6 guarantee that, if factor shares and the EoS are stationary, the chosen factor-augmenting representation will have correct derivatives and EoS. Thus, the representation constitutes a local approximation of the actual production function along the BGP, similar to the Uzawa representation. Therefore, it should be at least as useful as the Uzawa representation in any economic analysis. Indeed, the factor-augmenting representation will be more useful in many applications, especially when the questions at hand require understanding the evolution of productivity. In the reminder of the paper, we explain the use of new propositions in several concrete settings.

5 Three Simple Examples

So far, we have presented our results in as general a setting as possible. To incorporate these results into neoclassical models suitable for economic analysis, it is necessary to specify the production factors included in the production function. This section presents three examples that explore the simplest way to make neoclassical models consistent with aggregate data on the relative price of capital and the EoS between capital and labor. In subsection 5.1, we explain why a standard neoclassical economy only with two factors cannot accomplish this goal. Then, subsection 5.2 discusses the approach taken by Grossman et al. (2017) as a special case of the 2-factor neoclassical environment. Finally, subsection 5.3 shows that the conflict between data and neoclassical models can be resolved when including factors of production beyond labor and reproducible capital. Throughout this section, we describe only the production side of the economy and do not specify the source of technological change. We develop a full macroeconomic model with endogenous technological change in Section 6.

5.1 Standard 2-Factor Neoclassical Growth Model

Suppose that the true production function uses only two kinds of inputs, capital, K_t, and labor, L_t, i.e., $Y_t = F(K_t, L_t; t)$. The production function $F(\cdot; t)$ depends on time due to the technological change. Then, Proposition 1 says that, on any BGP with positive investment, technological change can always be represented as $Y_t = \widetilde{F}(K_t, A_{L,t}L_t)$. However, if these two factors are substitutable with unit elasticity ($\sigma_{KL} = 1$), Proposition 4 shows there are other possible factor-augmenting representations of the same BGP:[23]

$$Y_t = \overline{A}(A_{K,t}K_t)^\alpha(A_{L,t}L_t)^{1-\alpha}, \text{where } \overline{A} > 0 \text{ is a constant,} \tag{17}$$

which includes an Uzawa representation $Y_t = \overline{A}K_t^\alpha(\widetilde{A}_{L,t}L_t)^{1-\alpha}$ as a special case. Given the growth factors of output and labor on the BGP, condition (13) implies that any combination of $\gamma_K = A_{K,t+1}/A_{K,t}$ and $\gamma_L = A_{L,t+1}/A_{L,t}$ is consistent with the BGP as long as they satisfy $\gamma_K^\alpha(\gamma_L g_L)^{1-\alpha} = g^{1-\alpha}$. By rewriting (17) as $Y_t = A_t K_t^\alpha L_t^{1-\alpha}$, where the TFP A_t is given by $A_t \equiv \overline{A}A_{K,t}^\alpha A_{L,t}^{1-\alpha}$, it is clear that various combinations of capital- and labor-augmenting technological changes give the same rate of growth for TFP and, therefore, output.

This result confirms the widely understood version of the Uzawa theorem: on a BGP, all technological progress must be labor-augmenting, unless the production function is Cobb-Douglas. As we have seen in Section 2, this theoretical result is in contradiction with two stylized facts: (i) the productivity of capital has been steadily increasing, and (ii) the EoS between capital and labor is less than one, ruling out the Cobb-Douglas production function. No standard two-factor production function can reconcile these two stylized facts on a BGP.

5.2 Inclusion of Schooling in a Two-Factor model

Grossman et al. (2017) propose a possible solution to this contradiction by including schooling, $s_t \geq 0$, in a standard two-factor production function. Their result can be understood intuitively in terms of our analytical framework. While they start their analysis from a factor-augmenting representation, it is worthwhile to consider an underlying time-varying true production function in the form of (1):[24]

$$Y_t = F(K_t, L_t; t) = F^s(D(s_t)^a K_t, D(s_t)^{-b} L_t; t), \tag{18}$$

where $a > 0$, $b > 0$, $D(\cdot) \in [0, 1]$, and $D'(\cdot) < 0$. With $D(s_t)$ terms, the RHS of (18) specifies the production function beyond the general form $F(K_t, L_t; t)$. When schooling s_t increases, the multiplier $D(s_t)^a$ on K_t shrinks, raising the marginal product of capital. The opposite holds for labor. In this way, Grossman et al. (2017) specified a certain type of complementarity between schooling and capital.

[23]When there are two factors ($J = 1$) and they are substitutable with unit elasticity ($j^* = 1$), equation (14) in Proposition 4 implies that $Y_t = \widehat{F}\left((A_{K,t}K_t)^\alpha(A_{L,t}L_t)^{1-\alpha}\right)$. Because function $\widehat{F}(\cdot)$ has CRS and has only one argument, we can write $\widehat{F}(x) = \overline{A}x$ for some $\overline{A} > 0$, which gives (17).

[24]They considered not only factor-augmenting technological progress, but also investment-specific technological change. Definition 1 can include both cases as we have shown in Lemma 1.

Note that s_t is not a production factor in the neoclassical sense, because the production function has CRS only in capital and labor. Still, as the $D(s_t)$ term changes over time, it affects the amount of output produced from given K_t and L_t. This is a particular form of technological change, and we can consider $D(s_t)$ as being included in the t term of $F(K_t, L_t; t)$, as in the middle part of (18). Therefore, it falls within the definition of a two-factor neoclassical growth model (i.e., Definition 1 with $J = 1$).

From Proposition 1, this production function has an Uzawa representation $Y_t = \widetilde{F}(K_t, \widetilde{A}_{L,t}L_t)$ with $\widetilde{A}_L = (g/g_L)^t$ on a BGP, where both effective factors K_t and $\widetilde{A}_{L,t}L_t$ grow at the same speed as output. The production function in Grossman et al. (2017) can be interpreted in the following way, keeping the multiplier $D(s_t)$ term in the expression:

$$Y_t = \widetilde{F}(K_t, \widetilde{A}_{L,t}L_t) = \widetilde{F}(A_{K,t}D(s_t)^a K_t, A_{L,t}D(s_t)^{-b}L_t). \tag{19}$$

Comparing the arguments in the RHS to those in the middle, we immediately obtain $A_{K,t} = D(s_t)^{-a}$ and $A_{L,t} = \widetilde{A}_{L,t}D(s_t)^b$ on the BGP. Because the multiplier $D(s_t)^a$ shrinks as s_t increases, the capital-augmenting technology $A_{K,t}$ must grow so as to exactly offset the shrinking $D(s_t)^a$ term. Conversely, the labor-augmenting term $A_{L,t}$ should grow slower than that in the Uzawa representation $\widetilde{A}_{L,t}$ because the multiplier $D(s_t)^{-b}$ is also augmenting labor.[25] In this sense, there is no overall growth in capital productivity in the Grossman et al. (2017) formulation.

Within the limits of the two-factor Uzawa theorem, Grossman et al. (2017) propose a new interpretation of the production function, which provides the first possible solution to the contradiction raised by the Uzawa theorem. In their formulation, it is important that schooling enters the production function precisely in the form of (19), where the same function $D(s_t)$ appears both before capital and labor, with the powers of opposite signs. In addition, the functional form of $D(s_t)$ and the dynamic path s_t in equilibrium must be specified such that $D(s_t)$ shrinks exponentially over time.

Future empirical work could inform our understanding of long-run economic growth by testing whether the formulation (18) is consistent with data. In this paper, we propose a wider class of functions that are consistent with balanced growth. The next subsection

[25] From these observations, the main result of (Grossman et al., 2017, proposition 2) can easily be obtained as follows. Taking the growth factor of the both sides of $A_K = D(s_t)^{-a}$ gives $\gamma_K = g_D^{-a}$. From this we obtain a discrete time equivalent of their Proposition 2(ii): $g_D = \gamma_K^{-1/a}$. Note that Grossman et al. (2017) assumed $L_t = D(s_t)N_t$, which means $g_L = g_D g_N$. Because effective labor $A_{L,t}D(s_t)^{-b}L_t$ in (19) must grow at the same rate as output, $g = \gamma_L g_D^{-b} g_L = \gamma_L g_D^{1-b} g_N = \gamma_L \gamma_K^{(b-1)/a} g_N$, which is a discrete time equivalent of their proposition 2(i).

discusses a particularly simple example.

5.3 A Simple Three-Factor Model with Natural Resources

As shown in Section 2, a significant portion of GDP is paid to production factors that do not fit well in the notion of K_t or L_t. Thus, it is natural to consider production functions with more than two factors. Adding these additional factors makes it possible to reconcile neoclassical models with the data. While labor cannot be substituted by capital with unitary elasticity ($\sigma_{KL} \neq 1$), Proposition 4 only requires that there is a *single* production factor satisfies this requirement. In this case, there exist factor-augmenting representations of the production function that have capital-augmenting technological change ($\gamma_K > 1$).

Let us consider the simplest extension of the standard neoclassical production function,

$$Y_t = F_t(K_t, L_t, X_t; t), \text{ where } X_t = X_0 g_X^t \text{ for all } t, X_0 > 0, g_X > 0. \tag{20}$$

Here, we have a third production factor X_t, which is either growing ($g_X > 1$), shrinking ($g_X \in (0,1)$), or constant ($g_X = 1$). One example of such a factor is land. In that case, g_X represents the growth factor of the available land space. If the total area of available land asymptotes to an upper bound in the long run, then g_X would be one on the BGP. Any kind of natural resources, or a collection of natural resources (including land), can be a candidate for X_t. If X_t is non-renewable, $g_X \in (0,1)$ will likely hold, while a renewable energy source (e.g., sunlight) could have $g_X = 1$.

Among many candidates for the third production factor, we focus on those that have a unitary EoS with capital: $\sigma_{KX} = 1$. For concreteness, we refer to this factor natural resources. Then, Proposition 4 implies that, along a non-degenerate BGP, the technological change can be represented in a factor-augmenting fashion:

$$Y_t = \widehat{F}\Big(\big(A_{K,t}K_t\big)^{\alpha} \big(A_{X,t}X_t\big)^{1-\alpha}, A_{L,t}L_t \Big), \ \alpha \in (0,1), \tag{21}$$

where the growth factor of technology variables must satisfy $\gamma_L = g/g_L$ and $\gamma_K^{\alpha}(\gamma_X g_X)^{1-\alpha} = g^{1-\alpha}$. As in (15), the latter condition can be written in a log-linear form:

$$\log \gamma_K = \frac{1-\alpha}{\alpha} \left(\log \gamma_L + \log g_L - \log \gamma_X - \log g_X \right). \tag{22}$$

Thus, there must be a positive capital-augmenting technological change on a BGP ($\gamma_K > 1$), as long as the economy is growing faster than the effective input of the third factor ($g =$

$\gamma_L g_L > \gamma_X g_X$).

This finding raises an important question: even if there is a factor with $\sigma_{X_j K} = 1$, will the rates of technological change γ_K, γ_X, and γ_L be determined so as to satisfy the log-linear condition (22)? If their values are exogenously given, then this is a knife-edge case. If growth rates are endogenous, however, this need not pose any additional restrictions on the model. In Section 6, we develop a growth model with endogenous and directed technical change, where γ_K, γ_X, and γ_L are endogenously chosen. We will confirm that, on the BGP, condition (22) is satisfied. In Section 7, we calibrate a version of the model to moments from the long-term U.S. data and show that the BGP with positive capital-augmenting technical change is both locally and globally stable. These two sections jointly demonstrate that regardless of the initial state of technologies, condition (22) is always satisfied in the long run.

6 A Full Model with Directed Technological Change

So far, we have discussed the implications of the generalized Uzawa theorem focusing on the production sector. In this section, we develop a complete endogenous growth model where the direction of technological progress is determined by profit-maximizing firms. We show that the log-linear technology condition (22) is endogenously satisfied on the BGP. We base this section on a streamlined version of the model of tasks developed by Irmen (2017) and Irmen and Tabaković (2017) and expand it to incorporate three production factors. There are two benefits from our specification. First, we can analyze intentional R&D within a perfectly competitive economy, which fills a gap between the standard neoclassical growth model (perfectly competitive) and standard endogenous growth theory (imperfect competition). Second, our model of tasks will be scale independent, which implies that the model has a BGP even when the amount of labor is changing.

6.1 The Model

There are non-overlapping generations of representative firms, each of which exists for only one period. A representative firm performs two types of tasks, M-tasks and N-tasks. The number of M-tasks, as well as that of N-tasks, determines the amount of final output. The M-tasks require effective capital $A_{K,t}K_t$ and effective natural resource $A_{X,t}X_t$ as inputs, where $A_{K,t}$ and $A_{X,t}$ are the representative firm's capital-augmenting and labor-augmenting technologies (explained in detail below). The number of M-tasks it can complete is given by

$$M_t = \left(A_{K,t}K_t\right)^{\alpha}\left(A_{X,t}X_t\right)^{1-\alpha}, \alpha \in (0,1). \tag{23}$$

We refer to the RHS as the amount of the *capital composite*, which combines effective capital and effective natural resources (including land) with unit elasticity. An N-task uses only effective labor, $A_{L,t}L_t$, where $A_{L,t}$ is the labor-augmenting technology of the representative firm. The number of N-tasks is simply

$$N_t = A_{L,t}L_t. \tag{24}$$

By performing M_t and N_t tasks, the representative firm produces

$$Y_t = \widehat{F}(M_t, N_t) = \widehat{F}\Big(\left(A_{K,t}K_t\right)^{\alpha}\left(A_{X,t}X_t\right)^{1-\alpha}, A_{L,t}L_t\Big) \tag{25}$$

units of output, where $\widehat{F}(\cdot)$ is a standard neoclassical production function that has CRS and satisfies the Inada conditions.[26]

Now, we explain how the factor-augmenting technologies $\{A_{K,t}, A_{X,t}, A_{L,t}\}$ are determined. Technical knowledge can be kept within the firm for only one period, after which it becomes public. Thus, the representative firm at time t can freely use the technology of the period $t-1$ firm, $\{A_{K,t-1}, A_{X,t-1}, A_{L,t-1}\}$. In addition, the period t firm can improve each of factor-augmenting technologies through R&D. We assume that tasks are differentiated and require separate R&D investments.[27] To enhance the capital-augmenting technology for an M-task by a factor of $\gamma_{K,t} \equiv A_{K,t}/A_{K,t-1} \geq 1$, the firm need to invest $i_K(\gamma_{K,t})$ units of final goods. The firm faces similar choices when improving $A_{X,t}$ for each M-task, and also when enhancing $A_{L,t}$ for each N-task. The R&D cost functions for natural resources and labor-augmenting technologies are defined accordingly as $i_X(\cdot)$ and $i_L(\cdot)$.

It is reasonable to think that the marginal cost of improving the technology is small when the size of innovation is small, but it becomes increasingly expensive when aiming for bigger innovations.[28] To capture this, we assume R&D cost functions $i_K(\cdot)$, $i_X(\cdot)$ and $i_L(\cdot)$, have

[26] There are two ways to represent the production function in intensive forms: $f(M/N) = \widehat{F}(M/N, 1)$ and $h(N/M) = \widehat{F}(1, N/M)$. We assume that both $f(\cdot)$ and $h(\cdot)$ satisfy the Inada conditions.

[27] From the symmetry of tasks within each group (M or N) and from the convexity of the R&D cost functions as assumed in (26), it is always optimal to choose the same levels of $A_{K,t}$, $A_{X,t}$, and $A_{L,t}$ across individual tasks. Therefore, we omit subscripts for technologies for individual tasks.

[28] This can be explained by congestion in R&D activities. When many researchers are devoted to improvements in the same task at the same time, some of them will end up inventing the same innovation. The

the following properties:

$$i_Z(\gamma_Z) > 0, i'_Z(\gamma_Z) > 0, i''_Z(\gamma_Z) > 0 \text{ for all } \gamma_Z \geq 1,$$
$$i_Z(1) = 0, i'_Z(1) = 0, i_Z(\infty) = \infty \text{ for } Z = K, X, L. \tag{26}$$

Note that the R&D costs must be incurred for each of the M_t and N_t tasks. This means the total R&D cost for the representative firm is the sum of

$$\begin{aligned} R_{K,t} &= M_t \cdot i_K(A_{K,t}/A_{K,t-1}), \\ R_{X,t} &= M_t \cdot i_X(A_{X,t}/A_{X,t-1}), \\ R_{L,t} &= N_t \cdot i_L(A_{L,t}/A_{L,t-1}). \end{aligned} \tag{27}$$

The objective of the representative firm is to maximize the single period profit net of R&D costs, because it lives only for one period and its knowledge will become public next period. By taking the output in each period as *numéraire*, the period profit is given by

$$\pi_t = \widehat{F}(M_t, N_t) - R_{K,t} - R_{X,t} - R_{L,t} - r_t K_t - \tau_t X_t - w_t L_t, \tag{28}$$

where r_t, τ_t, and w_t are interest rate, payment for a unit of natural resources (e.g., land rent), and wage rate, respectively.

We keep the demand side of the economy as standard as possible. There is a representative household. The size of the representative household (i.e., population) evolves according to[29]

$$L_t = L_0 g_L^t, \; L_0 > 0, \; g_L > 1 - \delta : \text{given}. \tag{29}$$

As in the Ramsey-Cass-Koopman model, the period utility of the household is given by the product of the number of household members and the per capita period felicity function:

$$u_t = L_t u(C_t/L_t), \tag{30}$$

where $C_t/L_t > 0$ is per capita consumption. We assume the felicity function $u(\cdot)$ takes the

risk of duplication become more prominent as R&D inputs increase, which makes the R&D cost function $i_K(\cdot)$ convex. See Horii and Iwaisako (2007) for a simple micro foundation.

[29] We assume $g_L > 1 - \delta$ so as to avoid the possibility of a degenerate BGP, where physical capital investment becomes zero or even negative in the long run. (See Definition 2). Note that, as long as $\delta > 0$, condition $g_L > 1 - \delta$ allows declining population. However, population should not fall faster than the speed of capital depreciation.

CRRA form. Then, the intertemporal objective function of the household can be written as

$$U = \sum_{t=0}^{\infty} L_t \beta^t \frac{(C_t/L_t)^{1-\theta} - 1}{1-\theta}, \tag{31}$$

where $\theta > 0$ is the degree of the relative risk aversion (i.e., the inverse of the intertemporal elasticity of substitution) and $\beta \in (0, 1)$ is the discount factor.[30]

The representative household owns capital, K_t, and natural resources, X_t, in addition to labor, L_t. The household also owns the representative firm and receives the profit, π_t, although in equilibrium profits will be zero due to perfect competition. For simplicity, we assume that the supply of natural resources is exogenous:[31]

$$X_t = X_0 g_X^t,\ X_0,\ g_X > 0:\ \text{given.} \tag{32}$$

As in the case of population, its available quantity can be either constant $g_X = 1$, shrinking $g_X \in (0, 1)$, or growing $g_X > 1$. Physical capital accumulates through the savings of the household:

$$K_{t+1} = (r_t + 1 - \delta)K_t + \tau_t X_t + w_t L_t + \pi_t - C_t, \quad K_0 > 0 : \text{given}, \tag{33}$$

where $(r_t + 1 - \delta)K_t + \tau_t X_t + w_t L_t + \pi_t$ represents the household's income. The household is subject to the no-Ponzi game condition. Specifically, the present value of its asset holding as $T \to \infty$ should not be negative:

$$\lim_{T\to\infty} \left(\prod_{t=1}^{T} (r_t + 1 - \delta) \right)^{-1} K_{T+1} \geq 0. \tag{34}$$

This completes the description of the model economy.

Before proceeding to the analysis of the model, we demonstrate that it conforms to our definition of the multi-factor neoclassical growth model, given by (1) and (2) in Definition 1. First, the aggregate production function (25) has exactly the same form as (21), which

[30]While we tentatively assume $\beta \in (0, 1)$, it needs to be significantly smaller than 1 since otherwise U will become infinite in a representative household model with growing population and per capita consumption. The actual upper bound for β will be derived in Proposition 8.

[31]As we have shown in Table 1, among the estimated 8% factor share of natural resources, a majority (5%) is from land. Since the supply of land is mostly constant, we assume X_t is exogenous in this baseline scenario. Our theory is also applicable to the case where X_t is depleted or expanded endogenously (See robustness scenario f in Section 7.1). Note that, although X_t is exogenous, its effective amount $A_{X,t}X_t$ as a production factor can be enhanced endogenously, through R&D for A_X.

belongs to the definition of the multi-factor neoclassical production function (1). In fact, Proposition 4 guarantees that, if the EoS between K_t and X_t is unity and the economy has a BGP in equilibrium, then the aggregate production function can always be represented in the form of (25) at least along the BGP. Our microeconomic setting gives an example of such an economy. Second, by substituting (28) into (33), we obtain the evolution of capital in the same form as (2), where the total R&D expenditure is defined as $R_t = R_{K,t} + R_{X,t} + R_{L,t}$. The difference between Definition 1 and the current model is that we now have a complete description of the economy, including how the speed and direction of technological change are determined. We are now ready to explore whether this economy can generate a BGP in equilibrium, paying special attention to whether there is a BGP with a strictly positive rate of capital-augmenting technological progress.

6.2 R&D by Firms and the Direction of Technological Progress

We start by examining the behavior of the representative firm in the economy described above, focusing on the role of R&D. The representative firm maximizes profit (28) subject to the production and R&D functions (23)–(27) with respect to $\{K_t, X_t, L_t, A_{K,t}, A_{X,t}, A_{L,t}\}$, taking as given prices, $\{r_t,\ \tau_t,\ w_t\}$, and lagged technology levels, $\{A_{K,t-1},\ A_{X,t-1},\ A_{L,t-1}\}$. For convenience, we define $\mu_t \equiv M_t/N_t$, which is the relative task intensity in final good production. It also represents the ratio of effective capital composite to effective labor $\mu_t = \left(A_{K,t}K_t\right)^{\alpha}\left(A_{X,t}X_t\right)^{1-\alpha}/A_{L,t}L_t$. Then, because $\widehat{F}(\cdot)$ in (25) is a CRS function, we can write it in intensive form, $\widehat{F}(M_t, N_t)/N_t = \widehat{F}(\mu_t, 1) \equiv f(\mu_t)$, $\widehat{F}_M(M_t, N_t) = f'(\mu_t)$, and $\widehat{F}_N(M_t, N_t) = f(\mu_t) - \mu_t f'(\mu_t)$.[32]

Using this notation, we can conveniently express the first order conditions for factor demand. The firm demands capital, natural resources, and labor so as to satisfy[33,34]

$$r_t = \left(\alpha M_t/K_t\right)\left(f'(\mu_t) - i_K(\gamma_{K,t}) - i_X(\gamma_{X,t})\right), \tag{35}$$

[32] $\widehat{F}_M(\cdot)$ and $\widehat{F}_N(\cdot)$ represent the partial derivatives of function $\widehat{F}(\cdot)$ with respect to its first and second arguments, respectively.

[33] The RHS of (35) represents the (net) marginal product of K_t in producing output Y_t. It is given by the product of two parts. The first part, $\alpha M/K$, is the marginal product of K_t in increasing the number of M-tasks performed in the firm. The second part is the net marginal product of M_t in producing the final output. Note that, in the second part, the innovation cost for an M-task, $i_K(\gamma_{K,t}) + i_X(\gamma_{X,t})$, is subtracted from the "gross" marginal product of M_t, $f'(\mu_t)$. When the firm performs more M-tasks, it chooses to pay R&D costs to increase $A_{K,t}$ and $A_{X,t}$ in these tasks so as to keep up with other M-tasks. Similarly, in (36), $(1-\alpha)M/X$ is the marginal product of X_t in performing more M-tasks.

[34] By substituting (35), (36), and (37) into (28), it can be confirmed that the firm achieves zero profit, $\pi_t = 0$. This is due to the CRS property of the firm's problem.

$$\tau_t = ((1-\alpha)M_t/X_t)\left(f'(\mu_t) - i_K(\gamma_{K,t}) - i_X(\gamma_{X,t})\right), \tag{36}$$

$$w_t = A_{L,t}(f(\mu_t) - \mu_t f'(\mu_t) - i_L(\gamma_{L,t})). \tag{37}$$

Now, we turn to R&D, starting with the condition for improving the labor-augmenting technology $A_{L,t}$. The representative firm chooses $A_{L,t}$, or equivalently the speed of technological progress $\gamma_{L,t} \equiv A_{L,t}/A_{L,t-1} \geq 1$, according to first order condition $\partial\pi_t/\partial A_{L,t} = 0$. Simplifying this condition yields:[35]

$$\text{R\&D for N-tasks:} \quad \gamma_{L,t} i'_L(\gamma_{L,t}) + i_L(\gamma_{L,t}) = f(\mu_t) - \mu_t f'(\mu_t). \tag{38}$$

As we formally prove in Proposition 7 below, condition (38) has a unique solution for $\gamma_{L,t}$ as a function of $\mu_t = M_t/N_t$, and it is strictly increasing in μ_t. Intuitively, when the firm is performing relatively few N-tasks (i.e., when $\mu_t \equiv M_t/N_t$ is higher), the benefit of increasing $A_{L,t}$ to perform another N-task is larger, and therefore it is optimal to improve the labor-augmenting technology $A_{L,t}$ at a faster pace (i.e., $\gamma_{L,t}$ should be higher).

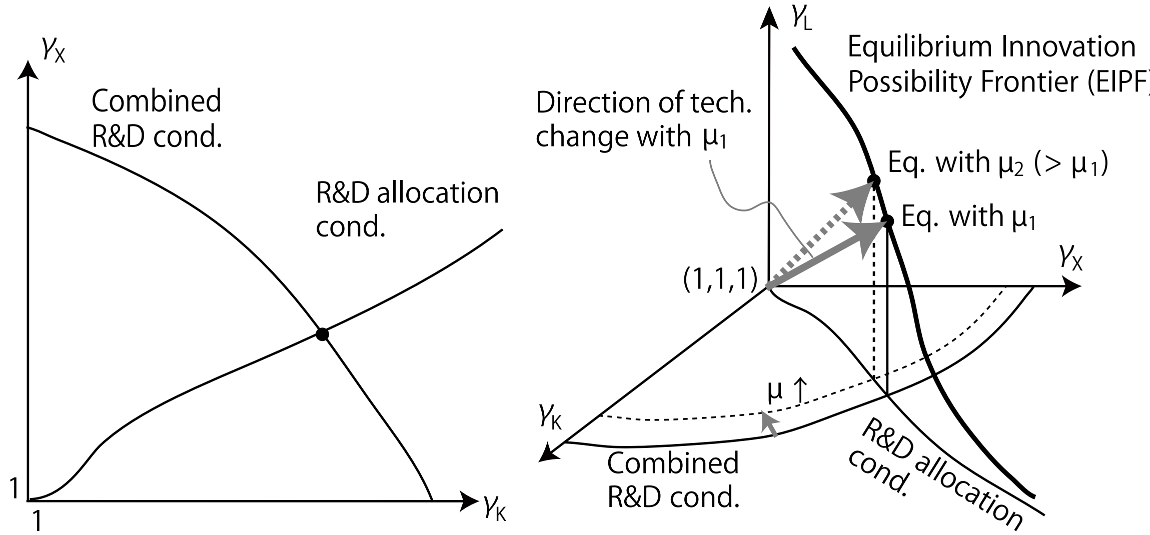
Next, we examine the R&D investments for K- and X-augmenting technologies. As in the case of labor-augmenting technology, $\gamma_{K,t} \equiv A_{K,t}/A_{K,t-1}$ and $\gamma_{X,t} = A_{X,t}/A_{X,t-1}$ need to satisfy the first order conditions, $\partial\pi_t/\partial A_{K,t} = 0$ and $\partial\pi_t/\partial A_{X,t} = 0$. Combining these two equations, we obtain two intuitive conditions that determine the allocation of relative R&D effort between K- and X-augmenting technologies, as well as the condition that specifies the optimal combined amount of R&D for M-tasks:[36]

$$\text{R\&D allocation:} \quad \frac{\gamma_{K,t} i'_K(\gamma_{K,t})}{\gamma_{X,t} i'_X(\gamma_{X,t})} = \frac{\alpha}{1-\alpha}, \tag{39}$$

$$\text{Combined R\&D:} \quad (\gamma_{K,t} i'_K(\gamma_{K,t}) + i_K(\gamma_{K,t})) + (\gamma_{X,t} i'_X(\gamma_{X,t}) + i_X(\gamma_{X,t})) = f'(\mu_t). \tag{40}$$

[35]The firm's private benefit from improving technology $A_{L,t}$ is the ability to perform a larger number of N-tasks, which increases the final output $Y_t = \widehat{F}(M_t, N_t)$. The RHS of (38) shows the marginal benefit, $\widehat{F}_N(M_t, N_t) = f(\mu_t) - \mu_t f'(\mu_t)$. The LHS corresponds to the marginal cost of performing a larger number of N-tasks through augmenting labor efficiency $A_{L,t}$ (given labor employment L_t). This can be broken into two components. First, by intensifying the R&D efforts in existing N-tasks to raise labor efficiency, the representative firm can decrease labor inputs by just enough to perform one additional N-task. The cost associated with this activity is given by the first term $\gamma_{L,t} i'_L(\gamma_{L,t})$, which we call the *intensive* marginal R&D cost. The saved labor is then used to perform a new N-task, which means the representative firm needs to invest in R&D for one more N-task, which costs $i_L(\gamma_{L,t})$. This *extensive* marginal R&D cost is represented by the second term in the LHS.

[36]The first order condition for $A_{K,t}$ yields $(\gamma_{K,t}/\alpha) i'_K(\gamma_{K,t}) + i_K(\gamma_{K,t}) + i_X(\gamma_{X,t}) = f'(\mu_t)$, whereas that for $A_{X,t}$ gives $(\gamma_{X,t}/1-\alpha) i'_X(\gamma_{X,t}) + i_K(\gamma_{K,t}) + i_X(\gamma_{X,t}) = f'(\mu_t)$. Condition (39) is obtained by subtracting the second equation from the first. Condition (40) is from adding α times the first equation and $(1-\alpha)$ times the second equation.



(a) Equilibrium innovation in K- and X-augmenting technologies. $\gamma_K \equiv A_{K,t}/A_{K,t-1}$ and $\gamma_X \equiv A_{X,t}/A_{X,t-1}$ are determined by the intersection of the R&D allocation and combined R&D conditions.

(b) When μ_t increases from μ_1 to μ_2, the equilibrium direction of technological change moves from the solid arrow to the dashed arrow, closer to the γ_L axis. The set of all these equilibrium points gives the EIPF curve. The bottom (γ_K, γ_X) plane corresponds to panel (a).

Figure 2: Determination of the direction of the technological change and the Equilibrium Innovation Possibility Frontier (EIPF) Curve

Condition (39) gives the optimal allocation of R&D investment between K- and X-augmenting technologies. Observe that $\gamma_{K,t} i'_K(\gamma_{K,t})$ and $\gamma_{X,t} i'_X(\gamma_{X,t})$ on the LHS are strictly increasing in γ_{K_t} and $\gamma_{X,t}$, respectively. Therefore, this condition can be expressed as an upward sloping curve in the $(\gamma_{K,t}, \gamma_{X,t})$ space, as depicted in Figure 2(a).[37] Condition (40) specifies the optimal combined size of R&D investments.[38] Since the LHS is increasing increasing both in γ_{K_t} and $\gamma_{X,t}$, the locus of $(\gamma_{K,t}, \gamma_{X,t})$ that satisfies this condition is depicted

[37] As the RHS of condition (39) shows, the allocation should depend on the relative contribution of capital and natural resources in performing M-tasks. When capital's relative contribution is higher (i.e., when α is higher), more resources should be allocated to R&D for the capital-augmenting technology. In addition, the slope and convexity of the R&D cost function also affects the optimal allocation. For example, if it is relatively difficult to improve the efficiency of natural resources, i.e., if the marginal R&D cost $i'_X(\gamma_{X,t})$ increases more rapidly with its argument than $i'_K(\gamma_{K,t})$, then it is optimal not to improve $A_{X,t}$ as fast as $A_{K,t}$.

[38] Capital and natural resources are used in M-tasks, and therefore improving K- and X-augmenting technologies will enable the firm to perform more M-tasks. This marginal benefit is represented by the RHS of (40), $f'(\mu_t) = \widehat{F}_M(M_t, N_t)$. The LHS is the marginal cost of R&D, which has two parts, $\gamma_{K,t} i'_K(\gamma_{K,t}) + i_K(\gamma_{K,t})$ and $\gamma_{X,t} i'_X(\gamma_{X,t}) + i_X(\gamma_{X,t})$, because both K- and X-augmenting technologies receive some R&D according to the allocation condition (39). In each of the two parts, the first term represents the intensive marginal R&D cost, whereas the second term is the extensive marginal R&D cost, as in condition (38).

by the downward-sloping curve. The intersection of the R&D allocation condition and the combined R&D condition gives the optimal rates of innovation for K- and X-augmenting technologies. Combined with $\gamma_{L,t}$ given by (38), we have the direction of technological change in equilibrium.

We can also study how the direction of technological change moves endogenously, focusing on μ_t, the ratio of effective capital composite to effective labor. The following proposition summarizes the result.

Proposition 7. (Direction of Technological Change)
In the endogenous growth model defined in Section 6.1, the growth factors of each of the factor augmenting technologies are function of $\mu_t = M_t/N_t$, *denoted by* $\widehat{\gamma}_K(\mu_t)$, $\widehat{\gamma}_X(\mu_t)$, *and* $\widehat{\gamma}_L(\mu_t)$. *These functions satisfy:*
(a) $\widehat{\gamma}'_K(\mu_t) < 0$ *for all* $\mu_t > 0$, $\widehat{\gamma}_K(0) = \infty$, *and* $\widehat{\gamma}_K(\infty) = 1$.
(b) $\widehat{\gamma}'_X(\mu_t) < 0$ *for all* $\mu_t > 0$, $\widehat{\gamma}_X(0) = \infty$, *and* $\widehat{\gamma}_X(\infty) = 1$.
(c) $\widehat{\gamma}'_L(\mu_t) > 0$ *for all* $\mu_t > 0$, $\widehat{\gamma}_L(0) = 1$, *and* $\widehat{\gamma}_L(\infty) = \infty$.

Proof. See Appendix A.10. □

Figure 2(b) illustrates the direction of the technological change in the 3-dimensional space. The γ_K-γ_X plane depicted at the bottom of the figure is the same as in panel (a). For a given value of μ_t, the intersection gives the value of $\widehat{\gamma}_K(\mu)$ and $\widehat{\gamma}_X(\mu)$. In addition, the vertical distance between this point and equilibrium point shows the size of L-augmenting innovation, $\widehat{\gamma}_L(\mu)-1$. As μ_t increases, the combined R&D locus shifts inward,[39] which lowers $\gamma_{K,t}$ and $\gamma_{X,t}$. At the same time $\gamma_{L,t}$ increases because $\widehat{\gamma}'_L(\mu_t) > 0$. This way, the direction of the technological change moves from the solid arrow to the dashed arrow in Figure 2(b).

The dependence of the direction on μ_t can be interpreted in terms of relative scarcity of effective factors. Proposition 7 says that the direction of technological progress is chosen so that it enhances effective factors which are in relatively short supply. In other words, firms are 'induced' to do more innovation that enhances the relatively scarce effective production factors.[40]

The thick downward-sloping curve in Figure 2(b) depicts the locus of all equilibrium points that correspond to various values of μ_t. This is the equilibrium innovation possibility frontier (EIPF). Depending on the equilibrium value of μ_t, the direction of technological

[39] The shift occurs because the RHS of (40) is decreasing in μ_t.
[40] This notion of induced innovation was first introduced by Hicks (1932). See Acemoglu (2002) for more discussion.

change is captured by some point on this curve.[41] To see how μ_t is determined in each period, we now consider the equilibrium dynamics.

6.3 Equilibrium Dynamics

The equilibrium path of this economy is given by the sequence of output, consumption, production factors, technologies, and R&D investments, $\{Y_t, C_t, K_t, X_t, L_t, A_{K,t}, A_{X,t}, A_{L,t}, R_{K,t}, R_{X,t}, R_{X,t}\}_{t=0}^{\infty}$, which satisfy the representative firm's optimization problem, the representative consumer's utility maximization problem, and the market clearing conditions for output and production factors. The economy is endowed with K_0, X_0 and L_0 at time 0, as well as the initial levels of publicly available technologies, $A_{K,-1}$, $A_{X,-1}$, and $A_{L,-1}$.

While the equilibrium involves many variables, we can analytically characterize its dynamic path in terms of only three: relative task intensity $\mu_t = M_t/N_t$, the amount of capital per effective labor $k_t \equiv K_t/A_{L,t}L_t$, and consumption per effective worker $c_t \equiv C_t/A_{L,t}L_t$. In Appendix A.11, we derive the dynamics these three variables in detail with intuitive explanations. The following summarizes the result.

First, the dynamics of relative task intensity μ_t can be written as

$$\mu_{t+1} = G\left(\frac{g_X^{1-\alpha}}{g_L}\left(\frac{v(\mu_t) - c_t}{k_t} + 1 - \delta\right)^{\alpha}\mu_t\right) \equiv \psi^{\mu}(\mu_t, k_t, c_t), \tag{41}$$

where $v(\mu_t) \equiv f(\mu_t) - \mu_t(i_K(\widehat{\gamma}_K(\mu_t)) + i_X(\widehat{\gamma}_X(\mu_t))) - i_L(\widehat{\gamma}_L(\mu_t))$ is the net output per effective labor, and $G(\cdot)$ is the inverse function of $\Gamma(\mu) = \widehat{\gamma}_L(\mu)\mu/\widehat{\gamma}_K(\mu)^{\alpha}\widehat{\gamma}_X(\mu)^{1-\alpha}$.[42] Intuitively, the relative task intensity M_{t+1}/N_{t+1} is determined in two steps. The relative supply of production factors determines the relative numbers of task that the representative firm can perform, given the previous period's technology levels. This is the argument of function $G(\cdot)$ in (41). The firm can also increase M_t and N_t by factor-augmenting R&D, which affects the equilibrium μ_{t+1}. The latter effect is captured by $G(\cdot)$.

[41] In this model, not only the direction within the EIPF, but also the EIPF itself is determined endogenously form the firm's profit condition. In most models of direction of technological change, it is assumed that innovation requires a certain type of exogenously given resource (e.g., scientists). In these cases, the innovation possibility frontier is derived from the resource constraint. To the contrary, in our model, the total amount of R&D input (R_t) is determined in equilibrium through profit maximization, and hence the frontier is called the 'equilibrium' innovation possibility frontier. Any innovation beyond this frontier is not profitable, although it might be materialistically feasible.

[42] In appendix A.11, we confirm $G(\cdot)$ is well defined and strictly increasing.

Second, the dynamics of capital per effective worker, k_t, are

$$k_{t+1} = \frac{1}{g_L \widehat{\gamma}_L(\psi^\mu(\mu_t, k_t, c_t))} \left(v(\mu_t) - c_t + (1-\delta)k_t\right) \equiv \psi^k(\mu_t, k_t, c_t), \tag{42}$$

where $\psi^\mu(\mu_t, k_t, c_t)$ is μ_{t+1} from (41).[43] The expression $(v(\mu_t) - c_t + (1-\delta)k_t)$ is the sum of the net saving and the un-depreciated part of existing capital, per effective labor in period t. It must be divided by $g_L \widehat{\gamma}_L(\mu_{t+1})$, because of the growth of effective labor between period t and $t+1$.

Third, the Euler equation for the consumer optimization gives the dynamics for c_t:

$$\begin{aligned} c_{t+1} = \frac{\beta^{1/\theta} c_t}{\widehat{\gamma}_L(\psi^\mu(\mu_t, k_t, c_t))} \Bigg(\frac{\alpha \psi^\mu(\mu_t, k_t, c_t)}{\psi^k(\mu_t, k_t, c_t)} \Big(f'(\psi^\mu(\mu_t, k_t, c_t)) - i_K(\widehat{\gamma}_K(\psi^\mu(\mu_t, k_t, c_t))) \\ - i_X(\widehat{\gamma}_X(\psi^\mu(\mu_t, k_t, c_t))) \Big) + 1 - \delta \Bigg)^{1/\theta} \equiv \psi^c(\mu_t, k_t, c_t). \end{aligned} \tag{43}$$

The seemingly complex expression inside $(\cdots)^{1/\theta}$ is simply $r_{t+1} + 1 - \delta$, expressed in terms of period t variables. Equations (41), (42), and (43) constitute the equilibrium mapping from $\{\mu_t, k_t, c_t\}$ to $\{\mu_{t+1}, k_{t+1}, c_{t+1}\}$ for all $t \geq 0$.

In appendix A.11, we explain starting levels of μ_0 and k_0 are given by initial factor endowments and initial technology levels. In addition, transversality condition and the non-Ponzi game condition jointly require

$$\lim_{T \to \infty} (\beta g_L)^T \left(\prod_{t=0}^{T} \widehat{\gamma}_L(\mu_t) \right)^{1-\theta} \widehat{\gamma}_L(\mu_{T+1}) c_T^{-\theta} k_{T+1} = 0. \tag{44}$$

These three boundary conditions, μ_0, k_0, and (44), pin down the equilibrium path of $\{\mu_t, k_t, c_t\}$. The next subsection will examine the property of the equilibrium path, focusing on the BGP.

6.4 The Balanced Growth Path

Now, we are ready to characterize the BGP of this economy. We will show that the direction of technological progress is endogenously chosen so that in equilibrium there is a unique BGP with a positive rate of capital-augmenting technical change.

Lemma 4. *Define a BGP as an equilibrium path where the growth factors of* $\{Y_t,\ K_t,\ X_t,$

[43] This expressions shows that the RHS is a function of only period t variables.

L_t, C_t, R_t, M_t, $N_t\}$ *are all constant.*[44] *Then, on any BGP, the values of* μ_t*,* k_t *and* c_t *must be constant.*

Proof. See Appendix A.12. □

We denote the BGP values of μ_t, k_t and c_t by μ^*, k^* and c^*, respectively. Their values are obtained by substituting $\mu_{t+1} = \mu_t = \mu^*$, $k_{t+1} = k_t = k^*$ and $c_{t+1} = c_t = c^*$ into (41), (42), and (43).

First, from (41) and (42), the BGP value of $\mu_t \equiv M_t/N_t$ will satisfy

$$1 = \frac{\left(g_X \widehat{\gamma}_X(\mu^*)\right)^{1-\alpha} \left(\widehat{\gamma}_K(\mu^*)\right)^{\alpha}}{(g_L \widehat{\gamma}_L(\mu^*))^{1-\alpha}.} \equiv \Phi(\mu^*). \tag{45}$$

Proposition 7 implies $\Phi'(\mu^*) < 0$ with $\Phi(0) = \infty$ and $\Phi(\infty) = 0$. Therefore, there exists a unique value of $\mu^* > 0$ that satisfies $\Phi(\mu^*) = 1$, and hence condition (45). An intuitive way to interpret (45) is to multiply the both of its sides by $(g_L \widehat{\gamma}_L(\mu^*))^{\alpha}$.

$$\left(g_X \widehat{\gamma}_X(\mu^*)\right)^{1-\alpha} \left(\widehat{\gamma}_K(\mu^*) g_L \widehat{\gamma}_L(\mu^*)\right)^{\alpha} = g_L \widehat{\gamma}_L(\mu^*) \quad (= g^*). \tag{46}$$

The LHS represents the growth factor of M_t on the BGP, while the RHS is that for N_t. Therefore, this condition means that the relative factor intensity $\mu^* = M_t/N_t$ is determined so that M_t and N_t grow at the same speed. This condition singles out a point on the Equilibrium Innovation Possibility Frontier (recall Figure 2), which determines the direction of technological change on the BGP. Note that, due to the CRS property of production function $Y_t = \widehat{F}(M_t, N_t)$, the value of equation (46) also represents the economic growth factor $g^* \equiv Y_{t+1}/Y_t$.

Second, from the Euler equation (43), the BGP value of $k_t = K_t/(A_t L_t)$ is

$$k^* = \frac{\beta\alpha\mu^*(f'(\mu^*) - i_K(\widehat{\gamma}_K(\mu^*)) - i_X(\widehat{\gamma}_X(\mu^*)))}{\widehat{\gamma}_L(\mu^*)^{\theta} - \beta(1-\delta)}. \tag{47}$$

Intuitively, the capital-effective labor ratio on the BGP is determined from the interest rate r^* that yields constant consumption per effective labor on the BGP.[45] Third, from (42) and

[44] Here, we slightly extend Definition 2 by requiring constancy of the growth factors of M_t and N_t, i.e., the numbers of tasks performed in the economy.

[45] Using (35), condition (47) is shown to be equivalent to $r^* + 1 - \delta = \beta^{-1}\widehat{\gamma}_L(\mu^*)^{\theta}$. Here, the RHS is the marginal rate of intertemporal substitution given that consumption per effective labor is constant (which must be true on the BGP).

$g_L\widehat{\gamma}_L(\mu^*) = g^*$ in (46), the BGP value of $c^* = C_t/A_{L,t}L_t$ must satisfy

$$c^* = v(\mu^*) - (g^* - 1 + \delta)k^*. \tag{48}$$

These three equations describe the unique BGP in this economy. The following proposition shows that the BGP uniquely exists when the discount factor is sufficiently smaller than 1.

Proposition 8. *There exists a value of $\overline{\beta} > 0$ such that whenever $\beta \in (0, \overline{\beta})$, there exists a unique BGP that satisfies $\mu^* > 0$, $k^* > 0$, $c^* > 0$, and the terminal condition (44).*[46]

Proof. See Appendix A.13. The exact expression for the upper bound $\overline{\beta}$ is given by (A.51). □

The most important implication from this model is that the technology condition (22) in Section 5.3 is now an endogenous outcome. Specifically, BGP condition (45) is equivalent to (22), except that the speed of technological progress is endogenously determined by profit-maximizing producers. This difference has important implications for the plausibility of capital-augmenting technological progress on the BGP. As discussed in Section 5.3, if the the rates of innovation for the three factor-augmenting technologies are exogenously given, and then (22) becomes a knife-edge condition. In contrast, this section has shown that, once we consider endogenous technical change, this condition is necessarily satisfied when the economy is on the BGP, which exists if discount factor β is sufficiently less than one. Thus, if we can show that the model economy converges to the BGP, then condition (22) is naturally satisfied in the long run. We do so in the next section.

7 Numerical Analysis and Stability

In this section, we investigate the local and global stability of the three-factor endogenous growth model. Our primary objective is to show that the model economy converges to a BGP with capital-augmenting technical change, where log-linear relationship (22) is endogenously

[46] There are two reasons why the existence of the BGP requires an upper bound for β (or, equivalently a lower bound for $\rho = (1-\beta)/\beta$). First, on the BGP, the amount of consumption for the household $C_t = A_{L,t}L_tc^*$ increases over time, causing the instantaneous utility to grow. Therefore, if β is too close to one, the intertemporal utility U in (31) becomes infinity, which means that the household's problem is not well defined. Second, as effective labor $A_{L,t}L_t$ grows, the household accumulates more capital K_t so as to prevent the dilution of capital per effective labor, k^*. However, when β is too large (i.e., when the discount rate ρ is too small), the BGP requires a too low real interest rate, or a too high level of k^*, to the extent that preventing the dilution is impossible even when all net output is invested in K_t. We rule out these extreme cases by assuming an upper bound for β.

satisfied. We also illustrate how having multiple technologies (including K-augmenting technology) affects the transition dynamics. To accomplish these goals, we present a series of numerical examples for which we can check stability computationally. Whenever possible, we ensure that our numerical examples are consistent with macroeconomic data characterizing the BGP of the United States. We stress, however, that this is not a complete calibration, and the results would be insufficient for a precise quantitative analysis.

7.1 Calibration

Functional Forms

We assume that the aggregate production function takes a CES form: $\widehat{F}(M_t, N_t) = (\eta M_t^{\frac{\epsilon-1}{\epsilon}} + N_t^{\frac{\epsilon-1}{\epsilon}})^{\frac{\epsilon}{1-\epsilon}}$, where $\epsilon > 0$ and $\eta \in (0,1)$. Output in the economy (25) can be written as

$$Y_t = \left\{\eta \left((A_{K,t}K_t)^{\alpha} (A_{X,t}X_t)^{1-\alpha}\right)^{\frac{\epsilon-1}{\epsilon}} + (1-\eta)(A_{L,t}L_t)^{\frac{\epsilon-1}{\epsilon}}\right\}^{\frac{\epsilon}{1-\epsilon}}. \tag{49}$$

Next, we assume power function for R&D costs[47]

$$i_Z(\gamma_Z) = \zeta_Z\big(\gamma_Z - 1\big)^{\lambda}, \quad \zeta_Z > 0,\ \lambda > 1,\ Z = K, X, L. \tag{50}$$

We allow R&D cost parameter ζ_Z to differ across types of technology. We normalize ζ_K to 1, and calibrate ζ_X and ζ_L. The degree of convexity, λ, is assumed to be the same across the three types of technology.

With these functional form assumptions, our model has 11 parameters, $\{\epsilon, \eta, \alpha, \lambda, \zeta_L, \zeta_X, \beta, \theta, \delta, g_L, g_X\}$. To calibrate the model, we also need to determine the period length, measured in years, denoted by χ. The period length in our model has important economic meaning, because it represents the duration for which a firm can monopolize the benefit from its R&D investments. Including χ, we have 12 parameters.

Exogenous Parameters

We set five parameters exogenously. Their values are given in Table 2. In the CES production function (49), we take $\epsilon = 0.7$ as the baseline value. This is a common estimate for the EoS between labor and reproducible capital (e.g., Antras et al., 2004; Oberfield and Raval, 2014). The mapping between these estimates and a structural parameter in our model is not

[47]Note that function (50) satisfies condition (26).

Parameter	Baseline	Alternate	Description	Explanation / Source
ϵ	0.7	0.9, 1.2	EoS b/w M and N	Oberfield and Raval (2014)
λ	2.0	1.5, 2.25	R&D Cost convexity	Quadratic
θ	1.0	0.5, 2	Inverse of IES	Log Preferences
$g_L^{1/\chi}$	1.01		Population growth	BEA 1960-2020 average
$g_X^{1/\chi}$	1.0	0.99	Growth of X	Fixed Supply of Land

Table 2: Exogenous Parameters

Target Moment (in annual values)	Value	Model Variable	Source
Capital output ratio	2.9	$K/(Y/\chi)$	BEA 1960-2020 average
Labor share of income	63%	$\kappa_L \equiv wL/Y$	BEA 1960-2020 average
Share of R&D payments in GDP	2.7%	$\kappa_R \equiv \frac{R_K+R_X+R_L}{Y}$	BEA 1960-2020 average
Consumption of fixed capital in GDP	14%	$\delta K/Y$	BEA 1960-2020 average
Growth rate of income per capita	1.9%	$\gamma_L^{1/\chi} - 1$	BEA 1960-2020 average
Decline in the relative price of capital	0.66%	$\gamma_K^{1/\chi} - 1$	BEA 1960-2020 average
Return on investment	4%	$(1 + r - \delta)^{1/\chi} - 1$	McGrattan *et al.* (2003)

Table 3: Target Moments for Calibration

exact and we show robustness with $\epsilon = 0.9$ and $\epsilon = 1.2$.[48] In the baseline calibration, we assume the R&D cost function is quadratic ($\lambda = 2$) and also check robustness with $\lambda = 1.5$ and $\lambda = 2.25$. Quadratic cost is a common assumption, and it is consistent with existing empirical work in endogenous growth (Acemoglu et al., 2018; Akcigit and Kerr, 2018). As for utility function (31), we take log preferences ($\theta = 1$) as the baseline and also consider cases where the intertemporal EoS is higher or lower than 1 ($\theta = 0.5$ and 2). Population growth is set to the 1960-2020 average in the U.S. (1% per year). When one period in the model corresponds to χ years, this means $g_L^{1/\chi} = 1.01$. We do not have a good data for the growth rate of factor X, which we interpret as natural resources, including land. Given that land is a major factor of production,[49] we take $g_X = 1$ as a benchmark (i.e., constant X). We also consider the case where natural resources are depleted 1% per year ($g_X^{1/\chi} = 0.99$).

Data

We calibrate the remaining parameters so that the model variables on the BGP match data from the U.S. Table 3 reports the target moments and model variables in annualized values

[48] Karabarbounis and Neiman (2014) and Piketty (2014) estimate the EoS between reproducible capital and labor and find an elasticity that is greater than one.

[49] As shown in Table 1, the factor share of land is estimated to be around 5%, while the share of all natural resources (including land) is 8%.

Parameter	Calibrated	Annualized	Description
χ	3.94		Period Length (years)
β	0.923	0.980	Discount Factor
δ	0.190	5.21%	Depreciation Rate
α	0.767		Capital Share within K-X composite
η	0.685		CES Distribution parameter
ζ_X	0.279		Cost parameter for A_X R&D
ζ_L	20.8		Cost parameter for A_L R&D

Table 4: Calibrated Parameters for Baseline Scenario

(e.g,. aggregate output per year is Y/χ, where one period in the model is χ years). For the capital-output ratio (2.9), labor share of income (63%), R&D share of income (2.7%), consumption of fixed capital as a share of GDP (14%), and real GDP per capita growth (1.9%), we use data from the Bureau of Economic Analysis (BEA) to calculate the arithmetic averages of the annual levels in the 1960-2020 period. To measure the growth rate of capital-augmenting technology, we calculate the annual decline in the relative price of all capital goods from 1960-2020 (0.66%). Finally, we set the rate of return on investment $(r^* - \delta)$ equal to the return on bonds (4%) from McGrattan and Prescott (2003).

Calibration Results

There are seven remaining parameters to calibrate, $\{\delta, \beta, \alpha, \eta, \zeta_L, \zeta_X, \chi\}$, which we identify with the seven moments in Table 3. We do so in two steps. First, we use equilibrium conditions to derive four analytical relationships among these parameters. This leaves us with three undetermined parameters, $\{\zeta_L, \eta, \chi\}$. In the second step, we numerically pin them down so that the target moments in Table 3 match the corresponding model variables on the BGP. The details of the calibration procedure are presented in Appendix A.14.

Table 4 presents the results of the two-step calibration procedure with the baseline assumptions. Period length χ is 3.94 years, which is time it takes for a firm's R&D outcome to be overtaken by a new innovation. Discount factor β is 0.923 ($\beta^{\frac{1}{\chi}} = 0.98$/year). The depreciation rate δ is 0.19 per period, which is about 5% per year. The share parameter α is 0.76. As reported in Table 5, this number implies that capital share in the GDP is 26.3%, whereas the natural resource share (including land) is 8%. Although these shares were not targeted in the calibration, they are consistent with the values reported in Table 1.

Variable	Value	Description
κ_K	26.3%	Capital Share
κ_X	8.0%	Natural Resource Share (incl. Land)
$\gamma_X^{1/\chi} - 1$	0.72%	Tech. Change in A_X per year

Table 5: Untargeted Variables in Calibrated Model

		(a)	(b)	(c)	(d)	(e)	(f)	(g)
Param-eters	Base-line	λ =1.5	λ =2.25	ϵ =0.9	ϵ =1.2	θ =0.5	θ =2	$g_X^{1/\chi}$ =0.99
χ	3.94	2.92	4.20	3.94	3.94	3.94	3.94	3.72
$\beta^{1/\chi}$	0.98	0.98	0.98	0.98	0.98	0.97	0.998	0.98
$\delta^{1/\chi}$	5.21%	5.07%	5.25%	5.21%	5.21%	5.21%	5.21%	5.18%
α	0.77	0.76	0.77	0.77	0.77	0.77	0.77	0.77
η	0.69	0.38	0.79	0.41	0.27	0.69	0.69	0.70
ζ_L	20.82	1.88	61.74	10.05	7.64	20.82	20.82	25.69
ζ_X	0.28	0.29	0.28	0.28	0.28	0.28	0.28	0.11
κ_K	26.3%	26.1%	26.4%	26.3%	26.3%	26.3%	26.3%	26.2%
κ_X	8.00%	8.25%	7.93%	8.00%	8.00%	8.00%	8.00%	8.05%
$\gamma_X^{1/\chi} - 1$	0.72%	0.80%	0.69%	0.72%	0.72%	0.72%	0.72%	1.75%

Table 6: Calibrated Parameters for the Robustness Scenarios

Robustness

Changing the free parameters, we present calibration results with $\lambda \in \{1.5, 2.25\}$, $\theta \in \{0.5, 2\}$, and $\epsilon \in \{0.9, 1.2\}$. We also calibrated the model under the assumption that natural resources X are depleted by 1% per year; i.e., $g_X^{1/\chi} = 0.99$. In each case, we change one parameter from the baseline value and then re-calibrate the model. In all cases, we find the set of parameters with which the model matches all the target moments in Table 3. The results are reported in Table 6 as scenarios (a)–(g).

7.2 Local Stability

Using parameters calibrated for the baseline setting and alternative scenarios, we can now examine the local stability of the model. Recall that the dynamic system is characterized by three variables $\{\mu_t, k_t, c_t\}$, which evolve according to equations (41), (42) and (43). Also, note that the initial values of μ_0 and k_0 are pre-determined, whereas c_0 should be chosen endogenously so that the system satisfies the transversality condition (44). In this system,

Scenario	Eigenvalues			BGP-Stability
	Stable		Unstable	
Baseline	0.602	0.970	1.672	Saddle/Determinate
(a) $\lambda = 1.5$	0.667	0.957	1.479	Saddle/Determinate
(b) $\lambda = 2.25$	0.587	0.971	1.722	Saddle/Determinate
(c) $\epsilon = .9$	0.633	0.971	1.610	Saddle/Determinate
(d) $\epsilon = 1.2$	0.664	0.974	1.550	Saddle/Determinate
(e) $\theta = 0.5$	0.496	0.969	2.034	Saddle/Determinate
(f) $\theta = 2$	0.692	0.971	1.454	Saddle/Determinate
(g) $g_X = 0.99$	0.620	0.964	1.629	Saddle/Determinate

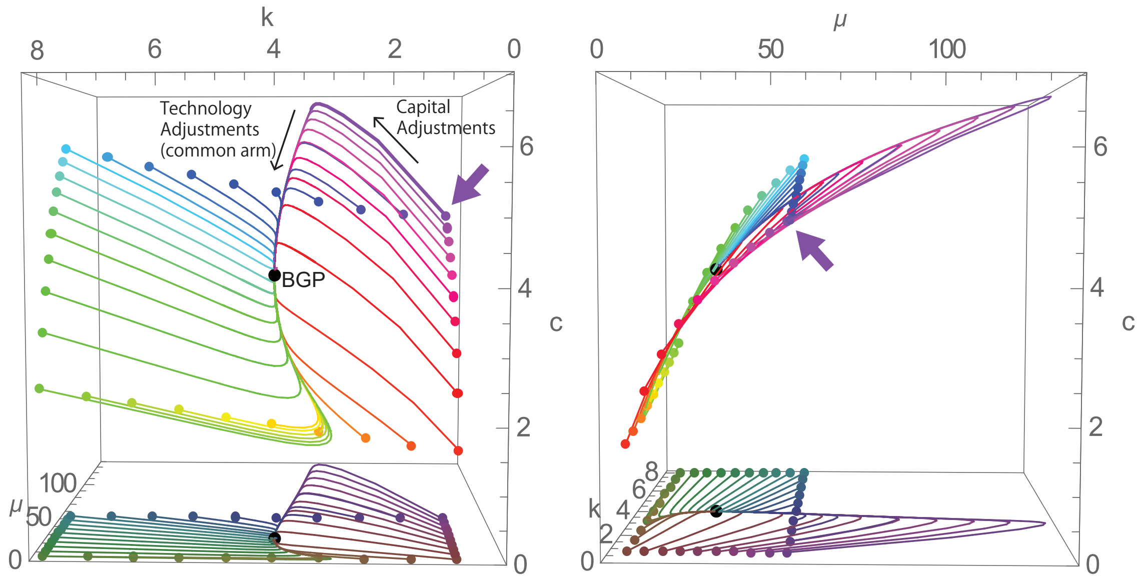
Table 7: Eigenvalues and Local Stability of the Calibrated Model

the BGP is saddle-stable and determinate if the Jacobian matrix evaluated at the BGP has two stable eigenvalues with absolute values less than one and one unstable eigenvalue with absolute value greater than one.

Table 7 summarizes the results of the local stability analysis. In all cases, we find that the BGP is saddle-stable and determinate: when state variables are near the BGP, they will converge to the BGP along the unique saddle path. In Subsection 6.4, we demonstrate that one of the conditions for balanced growth, (45), is equivalent to the technology condition (22). Therefore, the saddle stability of the BGP implies that the technology condition is endogenously satisfied as the economy converges to the BGP.

On this equilibrium path converging to the BGP, firms choose the intensities of three types of R&D, γ_K, γ_X, and γ_L, and hence the direction of the technological change, to maximize profits. The capital-augmenting technology A_K is still growing on the BGP, because firms always benefit from improving A_K. This naturally explains the observed long-term decline in the relative price of capital, which theoretically corresponds to the capital-augmenting technological change.

The saddle stability is confirmed in all seven alternative scenarios. It demonstrates that our explanation of capital-augmenting technological change is robust to changes in parameters, although we still need to maintain the assumption that there is a production factor X (e.g. natural resources or land) that can be substituted with conventional capital K with unit elasticity. It is particularly interesting to note that we find stability even when the EoS between labor and the capital-composite, ϵ, is greater than one (scenario d). Most directed technical change growth models require a low elasticity to be stable, especially when allowing for the possibility of capital-augmenting technical change (e.g., Acemoglu,



(a) Dynamic equilibrium paths from various starting points, viewed from k-c plane

(b) The same figure as (a), viewed from μ-c plane

Figure 3: Global Stability of the Calibrated Model (Baseline Setting).

2003; Grossman et al., 2017).[50]

7.3 Transition Dynamics and Global Stability

Local stability only examines convergence within the neighborhood of the BGP. In this subsection, we go one step further and demonstrate that convergence to the BGP occurs even when the initial states are far away. We call this property global stability. With three factors of production, this is not a trivial exercise, because the transitional dynamics may take various patterns depending on the initial combination of μ_0 and k_0.[51] They are determined by initial stock of production factors K_0, X_0 and L_0, as well as initial technology levels $A_{K,-1}$, $A_{X,-1}$, $A_{L,-1}$. Depending on the initial state of technology or resources, μ_0 and k_0 will take a wide range of combinations.

To cover various possibilities, we consider a large rectangular area in μ-k plane surrounding the BGP: namely, $\mu_t \in [0.2\mu^*, 2\mu^*]$ and $k_t \in [0.2k^*, 2k^*]$. We choose 36 points on the

[50] An exception is the model by Irmen and Tabaković (2017), which has an elasticity greater than one. As explained above, their model has capital-augmenting technical change on the transition path, but not on the steady state.

[51] Values of μ_0 and k_0 are respectively defined in equations (A.42) and (A.43) in Appendix A.11. In typical macro models with two factors of production, the dynamics can be written in terms of k_t and c_t, where k_t is the only state variable. In this case, the transition dynamics only have two possible patterns, depending on whether k_0 is higher or lower than the steady-state value. In either case, k_t typically converges monotonically to the steady state.

border of rectangular area and calculate the transition dynamics from each of them. We use a forward shooting method to determine the value of c_0 that eventually satisfies the transversality condition (44) as $t \to \infty$. The equilibrium path from each starting point is depicted in Figure 3, where the parameters are from baseline calibration in Section 7.1. Because the graph is three-dimensional, we depicted the same graph from two angles. We also provide the projection of the paths to the bottom μ-k plane in darker colors.

From each of 36 starting pair of μ_0 and k_0, we always find a unique level of c_0 such that the path from $\{\mu_0, k_0, c_0\}$ leads to the BGP (i.e., $\{\mu^*, k^*, c^*\}$). If c_0 is higher the resource constraint is eventually violated (k_t becomes negative), and if c_0 is lower the TVC is violated (c_t converges to zero). This means that convergence to the BGP is the only possible long-term outcome in equilibrium. These findings suggest that, as long as the initial μ_0 and k_0 is on or within the border of the rectangle, the economy necessarily converges to the BGP. Since the rectangle is reasonably large, we call it global stability.

There are couple of properties worth observing from the figure. First, the convergence is not monotonic. To illustrate this, let us focus on the path that starts form the upper right corner in Figure 3(a), as indicated by a thick arrow ($\mu_0 = 2\mu^*$ and $k_0 = 0.2k^*$).[52] Although initial level of μ_0 is double the steady-state level, μ_t initially increases further, going out of the rectangular area. This phenomenon can be interpreted as follows. At the initial state, the capital composite is abundant even though the reproducible capital is scarce. This happens when natural resources are so abundant that it more than offsets the capital scarcity. In this setting, the consumption of reproducible capital (i.e., the depreciation of K_t) is small, and savings from ample production leads to more accumulation K_t, which increases the capital composite further. This process continues until the level of k_t come close to the steady state level. This is the first stage of convergence. In the second stage, the ratio of capital composite to effective labor μ_t gradually falls to the steady state level. This is because a high μ_t means that effective labor is relatively scarce, and the firms have more incentives to improve A_L through R&D, rather than A_K or A_X. This tendency continues until μ_t reaches μ^*. Once μ_t comes to μ^*, firms have incentives to improve all types of technologies in a 'balanced' way such that the ratio of capital composite to effective labor does not change further. This illustrates how firms, in the long run, choose the direction of technological change that satisfies the BGP condition (45), or equivalently the technology condition (22).

The figure also shows that, even though the stable manifold[53] is two-dimensional, the

[52] In the color PDF version of the article, the path that we now focus is depicted in purple.

[53] The stable manifold is the set of points in the (μ, k, c) space that converges to the BGP. In Figure 3, all converging paths are on the (same) stable manifold.

equilibrium paths first converges to a common one-dimensional arm (or curve), and then converges to the BGP along the arm. This is because the system has two stable eigenvalues with significantly different magnitudes. In the baseline calibration, stable eigenvalues are 0.602 and 0.970. Given that one period χ is 3.94 years, those eigenvalues means the speed of convergence is 12% and 0.7% per year, respectively. As we discussed in the above example (the path starting from the upper-right corner, indicated by a thick arrow), the convergence to the BGP typically goes through two stages, and each stage corresponds to a different eigenvalue. The initial adjustment towards the common arm is driven mainly by capital accumulation. It is relatively fast: the distance from the common arm declines 12% every year. However, the second stage, along the common arm, is very slow. In the baseline example, the convergence speed is only 0.7% per year, which means it takes about 90 years to halve the distance. This adjustment takes much longer than capital accumulation because it is driven by the difference in the speed of technological change among A_K, A_X, and A_L. Note that these numbers are just for illustration, because the eigenvalues depend on the free parameters, as shown in Table 7. Still, this result suggests that, without considering endogenous technical change for various production factors (including capital-augmenting technological change), neoclassical growth models may overestimate the speed of convergence to the steady state by large margins.

8 Conclusions and Future Research

The relative price of investment has been falling in the U.S. for long periods of time, indicating the existence of capital-augmenting technological change on the balanced growth path. Due to the Uzawa steady state theorem, however, this fact had to be ignored in macroeconomic models that incorporate empirically relevant value for the elasticity of substitution between labor and capital.

This paper shows that this limitation can be overcome once we take the realistic step of adding more factors of production. For example, Caselli and Feyrer (2007) estimate that, among capital share of 26%, only 18% is reproducible capital (i.e., the factor that can be accumulated by savings). The other 8% cannot fall within the category of L or K in neoclassical growth models. We have shown that once we explicitly introduce other factors than L and K into the model, such as natural resources and land, we do not need to ignore capital-augmenting technological change either, as long as there is at least one production factor that has a unitary elasticity of substitution with reproducible capital.

For a model with multiple factor-augmenting technologies to have a BGP, the growth rates of these technologies need to satisfy a log-linear condition. In an endogenous growth model where technologies are improved by profit-maximizing R&D, we show that the condition is naturally satisfied as an long-term outcome of the equilibrium path, regardless of the initial conditions.

Implications for macroeconomic research

The generalized Uzawa growth theorem is not a mere theoretical curiosity. In standard macroeconomic models, the counter-factual notion that there is a single type of technology has implicitly placed severe restrictions on our ability to describe the economy. The literature on growth accounting, for example, emphasizes the role of investment specific technical change. Without utilizing our results, such analyses must focus on the Cobb-Douglas case (e.g., Greenwood et al., 1997) or be inconsistent with balanced growth (e.g., Krusell et al., 2000).[54] The same is true of business-cycle- and development accounting analyses (e.g., Fisher, 2006; DiCecio, 2009; Hsieh and Klenow, 2010; Schoellman, 2011).

In addition to improving descriptions of the economy, our results may also be useful for modeling medium-run transition dynamics. Standard models capture transition dynamics purely through capital accumulation, but directed technical change introduces another dimension of adjustment. Our numerical example (Subsection 7.3) suggests that adjustments among technologies are significantly slower than capital accumulation. This result implies that shocks have highly persistent effects when they have disproportionate effects on different effective production factors, because technological adjustment will be required for the economy to be back to the BGP (see also, Leon-Ledesma and Satchi, 2019). Existing studies suggest that the transition dynamics in standard neoclassical models are too rapid to match data (e.g., King and Rebelo, 1993; Banerjee and Moll, 2010). Our results indicate that adding natural resources and directed technical change can allow researchers to build models that match BGP data, while generating slower transition dynamics.

Models of economic growth with capital-augmenting technical change are also useful for understanding the relationship between technological progress and inequality (e.g., Karabarbounis and Neiman, 2014; Acemoglu and Restrepo, 2016; Hémous and Olsen, forthcoming). An important question in this literature is how to best model the impact of technology on labor market outcomes (Acemoglu and Autor, 2011; Acemoglu and Restrepo, 2018). There

[54]See, He et al. (2008) and Maliar and Maliar (2011) for discussions of how to make these models consistent with balanced growth.

is a consensus in the literature that capital- and labor-augmenting technical change have different impacts on the labor share of income. Thus, including both types of technical change greatly increases the usefulness of these models in studying impacts of new technologies on workers.

Estimating σ_{KX_j}

To utilize the generalized Uzawa Growth theorem, we need some factor, X_j, that has a unit-elastic EoS with reproducible capital. An obvious next step for future research would be to determine whether such a factor exists. In our illustrative growth model, we assumed that this third factor was an amalgamation of all factors not included in reproducible capital or labor. Proposition 4, however, only requires that at least one factor of production has this property. Thus, it is necessary to test many different factors to see if they satisfy this property. The obvious candidates would be land, energy, and natural materials, but it could also be any subcategory of these types of inputs. Given the vast array of choices and the limited restrictions created by theory, there is a reasonable possibility that such a factor exists.

What if there is no X_j with $\sigma_{KX_j} = 1$?

Despite the wide range of factors used in production, it is still a open question whether there exist a production factor that has unit-elastic substitution with reproducible capital. If future research determines there is no such factor, the generalized Uzawa Growth theorem again implies that the speed of capital-augmenting technological change must be exactly zero on any BGP.

This result does not fit the data on the falling relative price of investment. In this case, it is necessary to question whether the assumptions of the theorem hold in reality. A remarkable property of the Uzawa theorem is that it depends on very few assumptions: (1) the economy can be expressed by a neoclassical growth model, and (2) there is a BGP. The NIPA data strongly suggests the existence of the BGP. Therefore, we can narrow down the concern to the assumption in the neoclassical growth model, as given in Definition 1.

The definition consists of two parts, aggregate production function $Y_t = F(K_t, X_{1,t}, ..., X_{J,t}; t)$ and resource constraint $K_{t+1} = Y_t - C_t - R_t + (1 - \delta)K_t$. In the Uzawa theorem, the latter is only utilized in Lemma 2, which showed that the K/Y ratio must be constant in the BGP. The result of this lemma is clearly visible in the NIPA data depicted by Figure 1, where Y and K grows at the same rate, confirming the statement that "capital inherits the trend of output" (Jones and Scrimgeour, 2008). Therefore, as long as an economic analysis uses aggregate output Y and aggregated capital K, as defined in NIPA, this resource constraint seems to do no additional harm.[55]

The remaining suspect, then, is the aggregate production function $Y_t = F(K_t, X_{1,t}, ..., X_{J,t}; t)$. It assumes that there is a mapping from aggregated factor inputs to the aggregate output. While the vast majority of all macroeconomic models use some form of aggregate production function, it is not a weak assumption. For example, Figure 1 shows that the movement of the relative price of capital depends on the type of capital (e.g., equipment and structure). Even within the equipment category, the relative prices of different capital goods change dramatically over time. The same can be said for the left hand side of the production function, i.e., aggregated output Y_t. The aggregate production function implicitly assumes that capital and output can be aggregated and that there are stable relationships between these aggregates. If there is no X_j with $\sigma_{KX_j} = 1$, it might suggest that aggregation created the problem. Exploring disaggregated models seems important in this respect.

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[55] Another concern with the resource constraint might be that it assumes a one-to-one conversion between C_t and K_{t+1}. However, as proven in Lemma 1, if the conversion rate changes over time on the BGP, we can redefine the units of capital so Definition 1 still applies. In this sense, it does not pose a real restriction.

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Online Appendix

"A Generalized Uzawa Growth Theorem and Capital-Augmenting Technological Change"
by Gregory Casey and Ryo Horii
January 2022

A Proofs of Propositions and Lemmas

A.1 Notation for derivatives

Throughout this paper, $F_K(\cdot;t)$ denotes the partial derivative of function $F(\cdot;t)$ with respect to its first argument, whereas $F_{X_j}(\cdot;t)$ denotes the partial derivative of $F(\cdot;t)$ with respect to its $1+j$th argument. The same applies to other functions, such as $\widetilde{F}(\cdot)$.

Following the convention in economics, $\frac{\partial}{\partial K_t}$ and $\frac{\partial}{\partial X_{j,t}}$ represent the partial derivatives with respect to variables K_t and $X_{j,t}$, respectively. For example, if $\widetilde{F}(\cdot)$ is the production function, $\frac{\partial}{\partial X_{j,t}}\widetilde{F}(\cdot)$ gives the marginal product of factor $X_{j,t}$.

Note that these two definitions are different when the argument of function is not a single variable. For example, using the chain rule, we have

$$\frac{\partial}{\partial X_{j,t}}\widetilde{F}(K_t, \widetilde{A}_{X_{1},t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t}) = \widetilde{A}_{X_j,t}\widetilde{F}_{X_j}(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t}). \tag{A.1}$$

A.2 Proof of Lemma 1

Note that $\delta = (\breve{\delta} + g_q - 1)/g_q$ means $1 - \breve{\delta} = (1-\delta)q_{t+1}/q_t$. Dividing equation (3) by q_{t+1} and using the above result, we have $K_{t+1} = \breve{K}_{t+1}/q_{t+1} = (Y_t - C_t - R_t) + (1-\breve{\delta})\breve{K}_t/q_{t+1} = (Y_t - C_t - R_t) + (1-\delta)K_t$, which coincides with (2).

Production function $\breve{F}(\breve{K}_t, X_{1,t}, ..., X_{J,t}; t) = \breve{F}(q_{t-1}K_t, X_{1,t}, ..., X_{J,t}; t)$ is a CRS function of $K_t, X_{1,t}, ..., X_{J,t}$ and depends on time both through the shape of $\breve{F}(\cdot;t)$ and through the growth of q_t. Therefore, we can define function $F(K_t, X_{1,t}, ..., X_{J,t}; t) \equiv \breve{F}(q_{t-1}K_t, X_{1,t}, ..., X_{J,t}; t)$, where dependence of $F(\cdot;t)$ on t includes the effect from q_{t-1}. From the assumptions on $\breve{F}(\cdot)$, function $F(\cdot;t)$ obviously satisfies the required marginal product properties in (1).

A.3 Proof of Lemma 2

Using the notation in Definition 2, equation (2) can be written as $K_0 g_K^{t+1} = Y_0 g^t - C_0 g_C^t - R_0 g_R^t + (1-\delta)K_0 g_K^t$. Dividing all terms by g^t and rearranging them gives

$$Y_0 = C_0(g_C/g)^t + R_0(g_R/g)^t + K_0(g_K+\delta-1)(g_K/g)^t. \tag{A.2}$$

Because all three terms on the right hand side (RHS) of (A.2) are non-negative exponential functions of t, every one of them needs to be constant for the sum of all the terms to become constant (Y_0). For the first term $C_0(g_C/g)^t$ to be constant, $g_C = g$ must hold since $C_0 > 0$ from Definition 1. This means $C_t/Y_t = C_0/Y_0 > 0$. For the third term $(g_K+\delta-1)(g_K/g)^t$ to be constant, $g_K = g$ must hold since $K_0 > 0$ and $g_K > 1-\delta$. This implies $K_t/Y_t = K_0/Y_0 > 0$. If $R_0 > 0$, $g_R = R$ must hold since otherwise the second term cannot be constant.

A.4 Proof of Proposition 2

Because the production function in period 0 is $F(\cdot;0) \equiv \widetilde{F}(\cdot)$, we can write the share of factor Z in period 0 as

$$s_{Z,0} = \widetilde{F}_Z(K_0, X_{1,0}, ..., X_{J,0})\frac{Z_0}{Y_0}, \tag{A.3}$$

where $\widetilde{F}_Z(\cdot)$ represents the derivative of function $\widetilde{F}(\cdot)$ with respect to its argument (see Appendix Section A.1). Note that, since function $\widetilde{F}(\cdot)$ has constant returns to scale, its partial derivative function $\widetilde{F}_Z(\cdot)$ must be homogeneous of degree 0 (See Theorem M.B.1 in Mas-Colell et al., 1995). Therefore, the value of $\widetilde{F}_Z(\cdot)$ will be unchanged when all of its arguments are multiplied by the same factor $g^t = Y_t/Y_0 = K_t/K_0 = \widetilde{A}_{X_j,t}X_{j,t}/X_{j,0}$. (Here we used $g_K = g$ from Lemma 2.) Applying this for (A.3) gives

$$s_{Z,0} = \widetilde{F}_Z(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t})\frac{Z_0}{Y_0}.$$

In addition, because the effective amount of production factors and the output grow at the same speed, $Z_0/Y_0 = \widetilde{A}_{Z,t}Z_t/Y_t$ holds on the BGP. (In the case of $Z_t = K_t$, we define $\widetilde{A}_{K,t} \equiv 1$.) Therefore,

$$s_{Z,0} = \widetilde{F}_Z(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t})\widetilde{A}_{Z,t}\frac{Z_t}{Y_t} = \frac{\partial \widetilde{F}(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t})}{\partial Z_t}\frac{Z_t}{Y_t}, \tag{A.4}$$

where the validity of the second equality is guaranteed by the chain rule.[56] Recall that we assumed that the share is constant over time, which means

$$s_{Z,0} = s_{Z,t} = F_Z(K_t, X_{1,t}, ..., X_{J,t}; t)\frac{Z_t}{Y_t}. \tag{A.5}$$

By comparing (A.4) and (A.5), we obtain (5).

A.5 Proof of Proposition 3

As in Definition 3, the EoS between K_t and X_j, $j \in \{1, ..., J\}$, in the Uzawa Representation $\widetilde{F}(K_t, \widetilde{A}_{X_{1},t}X_{1,t}, ..., \widetilde{A}_{X_{J},t}X_{J,t})$ is defined as

$$\widetilde{\sigma}_{KX_j,t} = -\left.\frac{d\ln(K_t/X_{j,t})}{d\ln\left(\frac{\widetilde{F}_K(K_t,\widetilde{A}_{X_1,t}X_{1,t},...,\widetilde{A}_{X_J,t}X_{J,t})}{\widetilde{A}_{X_j,t}\widetilde{F}_{X_j}(K_t,\widetilde{A}_{X_1,t}X_{1,t},...,\widetilde{A}_{X_J,t}X_{J,t})}\right)}\right|_{Y_t,\mathbf{X}_{-j,t}:\text{const}}. \tag{A.6}$$

We used (A.1) for calculating the marginal product of X_j in the denominator. Note that, in addition to output Y_t and other production factors $\mathbf{X}_{-j,t}$, we keep technologies $\widetilde{A}_{X_1,t}, ..., \widetilde{A}_{X_J,t}$ fixed when calculating the EoS.

In this proof, we evaluate the value of (A.6) on the BGP. This means Y_t and $\mathbf{X}_{-j,t}$ are their BGP values, but we still need to consider (infinitesimally) small perturbations of K_t and $X_{j,t}$ from these BGP values. To make this distinction, let $Y_t, K_t, X_{1,t}, ..., X_{J,t}$ denote the specific BGP values, and k and x_j the variables to be perturbed. Then, (A.6) can be written as[57]

$$\widetilde{\sigma}_{KX_j,t} = -\left.\frac{d\ln(k/x_j)}{d\ln\left(\frac{\widetilde{F}_K(k,\widetilde{A}_{X_1,t}X_{1,t},...,\widetilde{A}_{X_j,t}x_j,...,\widetilde{A}_{X_J,t}X_{J,t})}{\widetilde{A}_{X_j,t}\widetilde{F}_{X_j}(k,\widetilde{A}_{X_1,t}X_{1,t},...,\widetilde{A}_{X_j,t}x_j,...,\widetilde{A}_{X_J,t}X_{J,t})}\right)}\right|_{\substack{\widetilde{F}(k,\widetilde{A}_{X_1,t}X_{1,t},...,\widetilde{A}_{X_j,t}x_j,...,\widetilde{A}_{X_J,t}X_{J,t})=Y_t\\ k=K_t,x_j=X_{j,t}}}. \tag{A.7}$$

Condition $\widetilde{F}(k, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_j,t}x_j, ..., \widetilde{A}_{X_J,t}X_{J,t}) = Y_t$ says that k and x_j need to move to ensure that this equality is satisfied. The other conditions $k = K_t, x_j = X_{j,t}$ say that, after the differentiation is complete, the EoS is evaluated at the BGP values.

Now, consider a change of variables: $k' = g^{-t}k$ and $x'_j = g^{-t}\widetilde{A}_{X_j,t}x_j$. Then, k in (A.7) is

[56] See Appendix A.1.

[57] We omit condition "$\mathbf{X}_{-j,t}$: const" because we already made it clear that $X_{j,t}$'s are the BGP values, not variables.

replaced by $k = g^t k'$ and x_j is by $(g^t/\widetilde{A}_{X_j,t})x'_j$. Specifically, k/x_j in the numerator becomes $\widetilde{A}_{X_j,t}k'/x'_j$. In the denominator,

$$\begin{aligned}\widetilde{F}_K(k, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_j,t}x_j, ..., \widetilde{A}_{X_J,t}X_{J,t}) &= \widetilde{F}_K(g^t k', g^t X_{1,0}, ..., g^t x'_j, ..., g^t X_{J,0}) \\ &= \widetilde{F}_K(k', X_{1,0}, ..., x'_j, ..., X_{J,0}),\end{aligned}$$

where we used the definition of $\widetilde{A}_{X_j,t} \equiv g^t X_{j,0}/X_{j,t}$ and the homogeneity of degree 0 property of the $\widetilde{F}_K(\cdot)$ function.[58] Similarly,

$$\widetilde{F}_{X_j}(k, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_j,t}x_j, ..., \widetilde{A}_{X_J,t}X_{J,t}) = \widetilde{F}_{X_j}(k', X_{1,0}, ..., x'_j, ..., X_{J,0}).$$

Note that, using the CRS property of $\widetilde{F}(\cdot)$, condition $\widetilde{F}(k, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_j,t}x_j, ..., \widetilde{A}_{X_J,t}X_{J,t}) = Y_t$ can be simplified as

$$\widetilde{F}(g^t k', g^t X_{1,0}, ..., g^t x'_j, ..., g^t X_{J,0}) = g^t \widetilde{F}(k', X_{1,0}, ..., x'_j, ..., X_{J,0}) = Y_t.$$

Since $Y_t = g^t Y_0$, the condition reduces to $\widetilde{F}(k', X_{1,0}, ..., x'_j, ..., X_{J,0}) = Y_0$. The point of evaluation, $k = K_t$, becomes $g^t k' = K_t$, or $k' = g^{-t}K_t = K_0$. Similarly, $x_j = X_{j,t}$ becomes $x'_j = g^{-t}\widetilde{A}_{X_j,t}X_{j,t} = X_{j,0}$. Therefore, (A.7) can be expressed in terms of k' and x'_j as follows:

$$\widetilde{\sigma}_{KX_j,t} = - \left. \frac{d\ln(\widetilde{A}_{X_j,t}k'/x'_j)}{d\ln\left(\frac{\widetilde{F}_K(k', X_{1,0}, ..., x'_j, ..., X_{J,0})}{\widetilde{A}_{X_j,t}\widetilde{F}_{X_j}(k', X_{1,0}, ..., x'_j, ..., X_{J,0})}\right)} \right|_{\substack{\widetilde{F}(k', X_{1,0}, ..., x'_j, ..., X_{J,0}) = Y_0 \\ k' = K_0, x'_j = X_{j,0}}}. \tag{A.8}$$

Recall that we keep technology $\widetilde{A}_{X_j,t}$ fixed when calculating the EoS. We can eliminate $\widetilde{A}_{X_j,t}$ from the numerator from $d\ln(\widetilde{A}_{X_j,t}k'/x'_j) = d(\ln(k'/x'_j) + \ln\widetilde{A}_{X_j,t}) = d\ln(k'/x'_j)$. In the same way, $\widetilde{A}_{X_j,t}$ in the denominator can also be eliminated (or replaced by $A_{X_j,0} \equiv 1$). Finally, using $\widetilde{F}(\cdot) \equiv F(\cdot; 0)$, (A.8) can be written as

$$\widetilde{\sigma}_{KX_j,t} = - \left. \frac{d\ln(k'/x'_j)}{d\ln\left(\frac{F_K(k', X_{1,0}, ..., x'_j, ..., X_{J,0}; 0)}{F_{X_j}(k', X_{1,0}, ..., x'_j, ..., X_{J,0}; 0)}\right)} \right|_{\substack{F(k', X_{1,0}, ..., x'_j, ..., X_{J,0}; 0) = Y_0 \\ k' = K_0, x'_j = X_{j,0}}}. \tag{A.9}$$

Then, comparing with Definition 3, it turns out that the RHS of (A.9) exactly matches the definition of $\sigma_{KX_j,0}$, evaluated at the period-0 BGP. Since it is assumed that $\sigma_{KX_j,t}$ does

[58] For the homogeneity of degree 0 property, see the proof of Proposition 2 in appendix A.4.

not change over time, we have $\widetilde{\sigma}_{KX_j,t} = \sigma_{KX_j,0} = \sigma_{KX_j,t}$.

A.6 Proof of Lemma 3

Proof of part a

By substituting $K_0, X_{1,0}, ..., X_{J,0}$ into (11) and then using (12),

$$\begin{aligned}
\overline{F}(K_0, X_{1,0}, ..., X_{J,0}) &= \widehat{F}\left(K_0^\alpha \prod_{j=1}^{j^*} X_{j,0}^{\xi_j}, X_{j^*+1,0}, ..., X_{J,0}\right) \\
&= F\left(\left(\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} \left(K_0^\alpha \prod_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{1/\alpha}, X_{1,0}, ..., X_{J,0}; 0\right) \\
&= F\left(K_0, X_{1,0}, ..., X_{J,0}; 0\right).
\end{aligned}$$

Proof of part b

Let $M_0 = K_0^\alpha \prod_{j=1}^{j^*} X_{j,0}^{\xi_j}$ denote the amount of capital composite m in period 0, and $\widehat{F}_M(\cdot)$ denote the derivative of function $\widehat{F}(\cdot)$ with respect to its first argument. By differentiating both sides of (12) by m with the chain rule and substituting the period-0 BGP values for $k, x_{j^*+1}, ..., x_J$,

$$\begin{aligned}
\widehat{F}_M\left(M_0, X_{j^*+1,0}, ..., X_{J,0}\right) &= F_K(K_0, X_{1,0}, ..., X_{J,0}; 0)\left(\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} \frac{1}{\alpha} M_0^{(1-\alpha)/\alpha} \\
&= F_K(K_0, X_{1,0}, ..., X_{J,0}; 0)\frac{K_0}{\alpha M_0},
\end{aligned} \tag{A.10}$$

where the last equality follows from the definition of $M_0 = K_0^\alpha \prod_{j=1}^{j^*} X_{j,0}^{\xi_j}$. Now, consider the case of $Z = K$. By differentiating both sides of (11) by k with the chain rule and substituting the period-0 BGP values,

$$\begin{aligned}
\overline{F}_K(K_0, X_{1,0}, ..., X_{J,0}) &= \widehat{F}_M\left(M_0, X_{j^*+1,0}, ..., X_{J,0}\right) \alpha M_0 / K_0 \\
&= F_K(K_0, X_{1,0}, ..., X_{J,0}; 0),
\end{aligned}$$

where the last equality is from (A.10). Similarly, for the case of $Z = X_j$, where $j \in \{1, ..., j^*\}$,

$$\begin{aligned}
\overline{F}_{X_j}(K_0, X_{1,0}, ..., X_{J,0}) &= \widehat{F}_M\left(M_0, X_{j^*+1,0}, ..., X_{J,0}\right) \xi_j M_0 / X_{j,0} \\
&= F_K(K_0, X_{1,0}, ..., X_{J,0}; 0)\frac{\xi_j}{\alpha}\frac{K_0}{X_{j,0}}.
\end{aligned} \tag{A.11}$$

Note that, from the definitions of α and ξ_j in (10), $\xi_j/\alpha = s_{X_j,0}/s_{K,0}$. Therefore, (A.11) becomes

$$F_K(K_0, X_{1,0}, ..., X_{J,0}; 0)\frac{F_{X_j}(K_0, X_{1,0}, ..., X_{J,0}; 0)X_{j,0}}{F_K(K_0, X_{1,0}, ..., X_{J,0}; 0)K_0}\frac{K_0}{X_{j,0}} = F_{X_j}(K_0, X_{1,0}, ..., X_{J,0}; 0).$$

Finally, consider the case of $Z = X_j$, where $j \in \{j^* + 1, ..., J\}$. Similarly to the proof of part a, we can confirm that $\overline{F}(K_0, X_{1,0}, ..., X_{j^*,0}, x_{j^*+1}, ...x_J) = F\,(K_0, X_{1,0}, ..., X_{j^*,0}, x_{j^*+1}, ..., x_J; 0)$ for any $x_{j^*+1}, ..., x_J$. This means that they are identical functions of $x_{j^*+1}, ..., x_J$, and have the same derivatives with respect to these variables. Therefore, for $j \in \{j^* + 1, ..., J\}$, we have $\overline{F}_{X_j}(K_0, X_{1,0}, ..., X_{J,0}) = F_{X_j}(K_0, X_{1,0}, ..., X_{J,0}; 0)$.

Proof of part c

The EoS for function $\overline{F}(\cdot)$ between capital and factor j, evaluated at the period-0 BGP, is defined as

$$\overline{\sigma}_{KX_j,0} = -\left.\frac{d\ln(k/x_j)}{d\ln\left(\frac{\overline{F}_K(k,X_{1,0},...,x_j,...,X_{J,0})}{\overline{F}_{X_j}(k,X_{1,0},...,x_j,...,X_{J,0})}\right)}\right|_{\substack{\overline{F}(k,X_{1,0},...,x_j,...,X_{J,0})=Y_0 \\ k=K_0, x_j=X_{j,0}}}, \tag{A.12}$$

where k and x_j are variables to be perturbed and $Y_0, K_0, X_{1,0}, ..., X_{J,0}$ are the period-0 BGP values.

Let us first examine $\overline{\sigma}_{KX_j,0}$ for the case of $j \in \{1, ..., j^*\}$. In this case, factors $X_{j^*+1,0}, ..., X_{J,0}$ are fixed at the BGP values. Using (11), function $\overline{F}(k, X_{1,0}, ..., x_j, ..., X_{J,0})$ can be written as $\widehat{F}(m, X_{j^*+1,0}, ..., X_{J,0})$, where m is the amount of capital composite, defined as $m = k^\alpha x_j^{\xi_j} \prod_{j' \in \{1,...,j^*\}\setminus j} X_{j',0}^{\xi_{j'}}$. Using the chain rule, its derivative with respect to k becomes

$$\begin{aligned}\overline{F}_K(k, X_{1,0}, ..., x_j, ..., X_{J,0}) &= \frac{\partial}{\partial k}\widehat{F}(m, X_{j^*+1,0}, ..., X_{J,0}) \\ &= \widehat{F}_M(m, X_{j^*+1,0}, ..., X_{J,0})\frac{\partial m}{\partial k} \\ &= \widehat{F}_M(m, X_{j^*+1,0}, ..., X_{J,0})\alpha\frac{m}{k}.\end{aligned}$$

Similarly, $F_{X_j}(k, X_{1,0}, ..., x_j, ..., X_{J,0}) = \widehat{F}_M(m, X_{j^*+1,0}, ..., X_{J,0})\xi_j\frac{m}{x_j}$. Substituting these into (A.12) gives

$$\overline{\sigma}_{KX_j,0} = -\left.\frac{d\ln(k/x_j)}{d\ln\left(\frac{\alpha}{\xi_j}\frac{x_j}{k}\right)}\right|_{\substack{\overline{F}(k,X_{1,0},...,x_j,...,X_{J,0})=Y_0 \\ k=K_0, x_j=X_{j,0}}}. \tag{A.13}$$

Since α and ξ_j are constant parameters, the denominator can be simplified as $d\ln((\alpha/\xi_j)(x_j/k)) = d(\ln(\alpha/\xi_j)+\ln(x_j/k)) = d\ln(x_j/k)$. Using this, (A.13) gives $\overline{\sigma}_{KX_j,0} = 1$. Recall that $\sigma_{KX_j,0} = 1$ because $j \in \{1,...,j^*\}$. Therefore, $\overline{\sigma}_{KX_j,0} = \sigma_{KX_j,0}$ holds.

Next, we examine $\overline{\sigma}_{KX_j,0}$ for the case of $j \in \{j^*+1,...,J\}$. In this case, equations (11) and (12) imply

$$\overline{F}(k, X_{1,0}, ..., x_j, ..., X_{J,0}) = F(k, X_{1,0}, ..., x_j, ..., X_{J,0}; 0),$$

for any $k > 0$ and $x_j > 0$. Therefore, the EoS of function $\overline{F}(k, X_{1,0}, ..., x_j, ..., X_{J,0})$ between k and x_j is identical with that of function $F(k, X_{1,0}, ..., x_j, ..., X_{J,0}; 0)$. This means $\overline{\sigma}_{KX_j,0} = \sigma_{KX_j,0}$.

Proof of part d

Let us first consider the CRS property of function $\widehat{F}(m, x_{j^*+1}, ..., x_J)$. We multiply every argument by an arbitrary factor of $\lambda > 0$. From (12),

$$\begin{aligned}
&\widehat{F}(\lambda m, \lambda x_{j^*+1}, ..., \lambda x_J) \\
&= F\left(\lambda^{1/\alpha}\left(\prod\nolimits_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha}, X_{1,0}, ..., X_{j^*,0}, \lambda x_{j^*+1}, ..., \lambda x_J; 0\right) \\
&= \lambda F\left(\lambda^{(1-\alpha)/\alpha}\left(\prod\nolimits_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha}, \frac{X_{1,0}}{\lambda}, ..., \frac{X_{j^*,0}}{\lambda}, x_{j^*+1}, ..., x_J; 0\right),
\end{aligned} \tag{A.14}$$

where the last equality comes from the CRS property of the period-0 true production function $F(\cdot;0)$. (All the arguments are divided by λ.) Our objective it to show that the last line of (A.14) coincides with $\lambda\widehat{F}(m, x_{j^*+1}, ..., x_J)$. Using (12), this desired condition can be written as

$$\begin{aligned}
&F\left(\left(\prod\nolimits_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha}, X_{1,0}, ..., X_{j^*,0}, x_{j^*+1}, ..., x_J; 0\right) \\
&= F\left(\lambda^{(1-\alpha)/\alpha}\left(\prod\nolimits_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha}, \frac{X_{1,0}}{\lambda}, ..., \frac{X_{j^*,0}}{\lambda}, x_{j^*+1}, ..., x_J; 0\right).
\end{aligned} \tag{A.15}$$

In the following, we establish this equality by focusing on the isoquants of function $F(\cdot;0)$.

Recall that we defined j^* such that the period-0 true production function $F(k, x_1, x_2, \dots, x_J; 0)$ satisfies $\sigma_{KX_j} = 1$ for $j = 1, ..., j^*$. For concreteness, let us focus on capital k and x_1. From Definition 3, $\sigma_{KX_1} = 1$ means that equation $d\ln(F_K/F_{X_1})/d\ln(k/x_1) = -1$ holds when the output and other inputs are kept constant.[59] In other words, this differential equation is satisfied on the isoquant curve in the k-x_1 space. Integrating equation $d\ln(F_K/F_{X_1})/d\ln(k/x_1) = -1$ gives $\ln(F_K/F_{X_1}) = -\ln(k/x_1) + \widetilde{\xi}_1$, where $\widetilde{\xi}_1$ is a constant of integration. Taking the exponential of the both sides gives

$$F_K/F_{X_1} = (\exp\widetilde{\xi}_1)(x_1/k). \tag{A.16}$$

From the definition of the isoquant curve, the amount of output must be constant: $dY = F_K dk + F_{X_1}dx_1 = 0$. Rearranging and using (A.16), we have the slope of the isoquant curve as $dx_1/dk = -F_K/F_{X_1} = -(\exp\widetilde{\xi}_1)(x_1/k)$. Integrating this differential equation by separation of variables gives $\ln k = -(1/\exp\widetilde{\xi}_1)\ln x_1 + \widetilde{y}_1$, where $\widetilde{y}_1$ is another constant of integration.[60] By taking the exponential,

$$k = (\exp\widetilde{y}_1)x_1^{-1/\exp\widetilde{\xi}_1}. \tag{A.17}$$

Equation (A.17) defines an isoquant curve with two parameters, $\widetilde{y}_1$ and $\widetilde{\xi}_1$. The value $\widetilde{\xi}_1$ can be pinned down by the factor share. Using (A.16), the relative share between k and x_1 is written as $kF_K/x_1F_{X_1} = \exp\widetilde{\xi}_1$. The result does not depend on k or x_1, which means that the relative share is constant on the isoquant curve. Also, notice that the value of $\widetilde{\xi}_1$ must be the same across all isoquant curves, since otherwise they intersect with each other, which is impossible by the definition of the isoquant curve. From (10), we know that the relative share in period 0 is α/ξ_1. Using these, the isoquant curve (A.17) can be written as

$$k = (\exp\widetilde{y}_1)x_1^{-\xi_1/\alpha}. \tag{A.18}$$

The remaining parameter $\widetilde{y}_1$ specifies the location of the isoquant curve. Now, consider a particular isoquant curve that goes through $k = \left(\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha}$ and $x_1 = X_{1,0}$, which means $\exp\widetilde{y}_1 = \left(\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha}X_{1,0}^{\xi_1/\alpha}$. From (A.18), we can confirm that this isoquant curve also goes through $k' = \lambda^{\xi_1/\alpha}\left(\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha}$ and $x_1' = X_{1,0}/\lambda$.[61] Since

[59] To minimize notation we omit the arguments of the functions $F_K(k, x_1, ..., x_J; 0)$ and $F_{X_1}(k, x_1, ..., x_J; 0)$.

[60] This integration can be done by separation of variables. Rearranging the equation $dx_1/dk = -(\exp\widetilde{\xi}_1)(x_1/k)$, we have $(1/k)dk = -(1/\exp\widetilde{\xi}_1)(1/x_1)dx_1$. Integrating both sides of this equation separately gives $\int(1/k)dk = -(1/\exp\widetilde{\xi}_1)\int(1/x_1)dx_1$. Since $\int(1/k)dk = \ln k + constant$ and $\int(1/x_1)dx = \ln x_1 + constant$, we obtain the result in the text.

[61] This can be confirmed by substituting k' and x_1' into (A.18). It yields the same $\exp\widetilde{y}_1$ as in the previous sentence.

the output is the same on an isoquant curve, we have

$$\begin{aligned}
&F\left(\left(\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha}, X_{1,0}, ..., X_{j^*,0}, x_{j^*+1}, ..., x_J; 0\right) \\
&= F\left(\lambda^{\xi_1/\alpha}\left(\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha}, \frac{X_{1,0}}{\lambda}, ..., X_{j^*,0}, x_{j^*+1}, ..., x_J; 0\right).
\end{aligned} \tag{A.19}$$

By repeating this operation for $j = 2, ..., j^*$ and using $\sum_{j=1}^{j^*} \xi_j = 1 - \alpha$ from (10), we obtain (A.15). This establishes the CRS property of function $F(m, x_{j^*+1}, ..., x_J)$.

Next, we prove the CRS property of function $\overline{F}(k, x_1, ..., x_J)$. From (11),

$$\begin{aligned}
\overline{F}(\lambda k, \lambda x_1, ..., \lambda x_J) &= \widehat{F}\left((\lambda k)^\alpha \prod_{j=1}^{j^*} (\lambda x_j)^{\xi_j}, \lambda x_{j^*+1,0}, ..., \lambda x_{J,0}\right) \\
&= \widehat{F}\left(\lambda k^\alpha \prod_{j=1}^{j^*} x_j^{\xi_j}, \lambda x_{j^*+1,0}, ..., \lambda x_{J,0}\right) \\
&= \lambda \widehat{F}\left(k^\alpha \prod_{j=1}^{j^*} x_j^{\xi_j}, x_{j^*+1,0}, ..., x_{J,0}\right) \\
&= \lambda \overline{F}(k, x_1, ..., x_J).
\end{aligned}$$

The second equality utilizes $\alpha + \sum_{j=1}^{j*} \xi_j = 1$ from (10), whereas the third equality is from the CRS property of function $\widehat{F}(\cdot)$.

A.7 Proof of Proposition 4

Using (11), the RHS of equation (14) can be written as

$$\widehat{F}\left((A_{K,t}K_t)^\alpha \prod_{j=1}^{j^*} (A_{X_j,t} X_{j,t})^{\xi_j}, A_{X_{j^*+1},t} X_{j^*+1,t}, ..., A_{X_J,t} X_{J,t}\right). \tag{A.20}$$

The first argument of function $\widehat{F}(\cdot)$ represent the effective amount of capital composite on the BGP. It is multiplied by g each period from condition (13). Also, all the other arguments of $\widehat{F}(\cdot)$ are multiplied by g each period because it is assumed that $\gamma_{X_j} = g/g_{X_j}$ for $j \in \{j^* + 1, ..., J\}$. Since $\widehat{F}(\cdot)$ has CRS from property d of Lemma 3, (A.20) is multiplied by g each period.

Also, the LHS of (14), Y_t, is multiplied by g every period by the definition of the BGP. In period 0, (14) holds from property a of Lemma 3. Therefore, (14) holds for all $t \geq 0$.

A.8 Proof of Proposition 5

The proof relies on Lemma 3, but otherwise it proceeds similarly to the proof for Proposition 2. Let us first consider the case of $Z_t = K_t$. Using property b of Lemma 3 and (11), the share of factor K in period 0 can be written as (11),

$$\begin{aligned} s_{K,0} &= F_K(K_0, X_{1,0}, ..., X_{J,0}; 0)\frac{K_0}{Y_0} \\ &= \overline{F}_K(K_0, X_{1,0}, ..., X_{J,0})\frac{K_0}{Y_0} \\ &= \frac{\partial}{\partial K_0}\widehat{F}\left(K_0^{\alpha}\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}, X_{j^*+1,0}, ..., X_{J,0}\right)\frac{K_0}{Y_0}. \end{aligned} \tag{A.21}$$

Let $\widehat{F}_M(\cdot)$ be the derivative of function $\widehat{F}(\cdot)$ with respect to its first argument. Note that, in (A.21), the first argument is the capital composite in period 0, $M_0 = K_0^{\alpha}\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}$. Using the chain rule, (A.21) becomes

$$s_{K,0} = \widehat{F}_M\left(M_0, X_{j^*+1,0}, ..., X_{J,0}\right)\frac{dM_0}{dK_0}\frac{K_0}{Y_0} = \widehat{F}_M\left(M_0, X_{j^*+1,0}, ..., X_{J,0}\right)\frac{\alpha M_0}{Y_0}, \tag{A.22}$$

where the second equality follows from $dM_0/dK_0 = \alpha M_0/K_0$.

Recall that $\widehat{F}(\cdot)$ has CRS from Lemma 3, and therefore its derivative $\widehat{F}_M(\cdot)$ is a homogeneous function of degree 0. Let $M_t = (A_{K,t}K_t)^{\alpha}\prod_{j=1}^{j^*}(A_{X_j,t}X_{j,t})$ denote the effective amount of capital composite in period t. From condition (13), M_t grows by a factor of g every period. The same applies to the effective amounts of factors not in the capital composite: $A_{X_J,t}X_{j,t}$ for $j = j^* + 1, ..., J$. Therefore, when we consider function $\widehat{F}_M\left(M_t, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_J,t}X_{J,t}\right)$, every argument is multiplied by g every period, which does not change the value of $F_M(\cdot)$ over time due to homogeneity of degree 0. Therefore, (A.22) can be written as

$$s_{K,0} = \widehat{F}_M\left(M_t, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_J,t}X_{J,t}\right)\frac{\alpha M_0}{Y_0}. \tag{A.23}$$

Note that, because M_t and Y_t grow at the same speed, the last term can be transformed as $\alpha M_0/Y_0 = \alpha M_t/Y_t = (\alpha M_t/K_t)(K_t/Y_t)$. In addition, $\alpha M_t/K_t$ in the latter expression represents dM_t/dK_t, which can be confirmed by differentiating $M_t = (A_{K,t}K_t)^{\alpha}\prod_{j=1}^{j^*}(A_{X_j,t}X_{j,t})$

by K_t. Therefore, (A.23) becomes

$$\begin{aligned}
s_{K,0} &= \widehat{F}_M\left(M_t, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_J,t}X_{J,t}\right)\frac{dM_t}{dK_t}\frac{K_t}{Y_t} \\
&= \frac{\partial}{\partial K_t}\widehat{F}\left((A_{K,t}K_t)^\alpha \prod_{j=1}^{j^*}(A_{X_j,t}X_{j,t}), A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_J,t}X_{J,t}\right)\frac{K_t}{Y_t} \\
&= \frac{\partial}{\partial K_t}\overline{F}_K\left(A_{K,t}K_t, A_{X_1,t}X_{1,t}, ..., A_{X_J,t}X_{J,t}\right)\frac{K_t}{Y_t},
\end{aligned} \tag{A.24}$$

where the second equality uses the chain rule, and the third is from the definition of function $\overline{F}(\cdot)$ in (11). Note that the share of capital is the same in period t and 0, which implies

$$s_{K,0} = s_{K,t} = F_K(K_t, X_{1,t}, ..., X_{J,t}; t)\frac{K_t}{Y_t}. \tag{A.25}$$

By comparing (A.24) with (A.25), we obtain (16) for the case of $Z_t = K_t$. The proof of the proposition for the case of $Z_t = X_{j,t}$, $j \in \{1, ..., j^*\}$ proceeds exactly the same way as above, with only the modification that K_t is replaced by $X_{j,t}$ and α by ξ_j.

Finally, the case of $Z_t = X_{j,t}$, $j \in \{j^*+1, ..., J\}$, can be confirmed in a similar way as in Proposition 2, because the value of $A_{X_j,t}$ is the same as $\widetilde{A}_{X_j,t}$ in the Uzawa theorem. In particular, we use $\widehat{F}(M_t, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_J,t}X_{J,t})$ instead of $\widetilde{F}(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t})$, and define $\widehat{F}_{X_j}(\cdot)$, $j \in \{j^*, ..., J\}$, as the derivative of function $\widehat{F}(\cdot)$ with respect to its $(j - j^* + 1)$th argument.[62] Except for these slight modifications, the proof proceeds exactly as in Appendix A.4.

A.9 Proof of Proposition 6

Similarly to Definition 3, the EoS $\overline{\sigma}_{KX_j,t}$ on the BGP is defined as

$$\overline{\sigma}_{KX_j,t} = -\left.\frac{d\ln(k/x_j)}{d\ln\left(\frac{A_{K,t}\overline{F}_K(A_{K,t}k, A_{X_1,t}X_{1,t},...,A_{X_j,t}x_j,...,A_{X_J,t}X_{J,t})}{A_{X_j,t}\overline{F}_{X_j}(A_{K,t}k, A_{X_1,t}X_{1,t},...,A_{X_j,t}x_j,...,A_{X_J,t}X_{J,t})}\right)}\right|_{\substack{\overline{F}(A_{K,t}k, A_{X_1,t}X_{1,t},...,A_{X_j,t}x_j,...,A_{X_J,t}X_{J,t})=Y_t \\ k=K_t, x_j=X_{j,t}}}. \tag{A.26}$$

where $Y_t, K_t, X_{1,t}, ..., X_{J,t}$ indicate the BGP values, and k and x_j are the variables to be perturbed.[63]

[62] $\widehat{F}_{X_j}(\cdot)$ needs to be defined this way because j^* arguments are eliminated from function $\widehat{F}(\cdot)$ in definition (12). Also, note that similarly to function $\widetilde{F}(\cdot)$, function $\widehat{F}(\cdot)$ has a CRS property from Lemma 3.

[63] In definition (A.26), condition $\overline{F}(A_{K,t}k, A_{X_1,t}X_{1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t}) = Y_t$ means that k and x_j are perturbed so that output Y_t is unchanged from the BGP value. Condition $k = K_t, x_j = X_{j,t}$ says that

Let us first consider the case of $j \in \{1, ..., j^*\}$. In this case, factors $X_{j^*+1,t}, ..., X_{J,t}$ are fixed at the BGP values. Now, we simplify the denominator of (A.26), particularly focusing on the fraction inside $\ln(\cdot)$. Using (11), function $\overline{F}(A_{K,t}k, A_{X_1,t}X_{1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t})$ can be written as $\widehat{F}(m, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_J,t}X_{J,t})$, where m is the effective amount of capital composite, $m = (A_{K,t}k)^\alpha (A_{X_j,t}x_j)^{\xi_j} \prod_{j' \in \{1,...,j^*\} \setminus j} (A_{X_{j'},t}X_{j',0})^{\xi_{j'}}$. Note that $dm/dk = \alpha m/k$. Using these properties and the chain rule, we have

$$
\begin{aligned}
&A_{K,t}\overline{F}_K(A_{K,t}k, A_{X_1,t}X_{1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t}) \\
&\qquad = \frac{\partial}{\partial k}\overline{F}(A_{K,t}k, A_{X_1,t}X_{1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t}) \\
&\qquad = \frac{\partial}{\partial k}\widehat{F}(m, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_J,t}X_{J,t}) \\
&\qquad = \widehat{F}_M(m, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_J,t}X_{J,t})\alpha\frac{m}{k}.
\end{aligned}
\tag{A.27}
$$

Similarly,

$$
\begin{aligned}
&A_{X_j,t}\overline{F}_{X_j}(A_{K,t}k, A_{X_1,t}X_{1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t}) \\
&\qquad = \widehat{F}_M(m, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_J,t}X_{J,t})\xi_j\frac{m}{x_j}.
\end{aligned}
\tag{A.28}
$$

Substituting (A.27) and (A.28) into (A.26) gives

$$
\overline{\sigma}_{KX_j,t} = -\left.\frac{d\ln(k/x_j)}{d\ln\left(\frac{\alpha}{\xi_j}\frac{x_j}{k}\right)}\right|_{\substack{\overline{F}(A_{K,t}k, A_{X_1,t}X_{1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t}) = Y_t \\ k=K_t, x_j = X_{j,t}}}.
\tag{A.29}
$$

Since α and ξ_j are constant parameters, the denominator can be simplified as $d\ln((\alpha/\xi_j)(x_j/k)) = d(\ln(\alpha/\xi_j) + \ln(x_j/k)) = d\ln(x_j/k)$. Using this, (A.29) gives $\overline{\sigma}_{KX_j,t} = 1$. Recall that $\sigma_{KX_j,0} = 1$ because $j \in \{1, ..., j^*\}$, and that $\sigma_{KX_j,t}$ does not change over time on the BGP. Therefore, $\sigma_{KX_j,t} = \sigma_{KX_j,0} = 1 = \overline{\sigma}_{KX_j,t}$ holds.

Next, we examine $\overline{\sigma}_{KX_j,t}$ for the case of $j \in \{j^*+1, ..., J\}$. Similarly to the proof of Proposition 3, consider a change of variables: $k' = g^{-t}k$ and $x'_j = g^{-t}A_{X_j,t}x_j$. Then, k in (A.26) is replaced by $k = g^t k'$ and x_j is replaced by $(g^t/A_{X_j,t})x'_j$. In the numerator, k/x_j becomes $A_{X_j,t}k'/x'_j$. In the denominator, by the same operations as in (A.27), $A_{K,t}\overline{F}_K(\cdot)$

the EoS is evaluated at the BGP values. It is also important to keep in mind the notation for derivatives: $A_{K,t}\overline{F}_K(A_{K,t}k, A_{X_1,t}X_{1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t})$ is the partial derivative of $\overline{F}(\cdot)$ with respect to k, which corresponds to $F_K(\cdot;t)$ in Definition 3.

can be written as

$$\widehat{F}_M(m, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t})\alpha\frac{m}{k}, \tag{A.30}$$

where $m = (A_{K,t}k)^\alpha \prod_{j'=1}^{j^*}(A_{X_{j'},t}X_{j',0})^{\xi_{j'}}$. The definition of m does not include x_j because $j \in \{j^*+1, ..., J\}$ means that x_j is not a part of capital composite. Instead, $A_{X_j,t}x_j$ appears in (A.30) as the $(j-j^*+1)$th argument of the $\widehat{F}(\cdot)$ function. Using $x_j = (g^t/A_{X_j,t})x'_j$, $A_{X_j,t}x_j$ can be written as $g^t x'_j$. Since $M_t = (A_{K,t}K_t)^\alpha \prod_{j=1}^{j^*}(A_{X_j,t}X_{j,t})$ grows by a factor of g every period, the capital composite m can also be written as

$$m = \left(g^t k'/K_t\right)^\alpha M_t = (k'/K_0)^\alpha g^t M_0 = g^t m',$$

where $m' = (k')^\alpha \prod_{j'=1}^{j^*}(A_{X'_j,0}X_{j',0})$. Other effective factors also grow by a factor of g: $A_{X_{j'},t}X_{j',t} = g^t X_{j',0}$ for $j' \in \{j^*+1, ..., J\}\backslash j$. Using these, (A.30) becomes

$$\begin{aligned}
&\widehat{F}_M(g^t m', g^t X_{j^*+1,0}, ..., g^t x'_j, ..., g^t X_{J,0})\alpha\frac{g^t m'}{g^t k'} \\
&\quad= \widehat{F}_M(m', X_{j^*+1,0}, ..., x'_j, ..., X_{J,0})\alpha\frac{m'}{k'} \\
&\quad= \frac{\partial}{\partial k'}\widehat{F}(m', X_{j^*+1,0}, ..., x'_j, ..., X_{J,0}) \\
&\quad= \overline{F}_K(k', X_{1,0}, ..., x'_j, ..., X_{J,0}),
\end{aligned}$$

where the first equality is from the homogeneity of degree 0 property of the $\widehat{F}_M(\cdot)$ function, the second equality is from the chain rule and $dm'/dk' = \alpha m'/k'$, and the last equality is from the definition of $\overline{F}(\cdot)$ in (11).

Likewise, $A_{X_j,t}\overline{F}_{X_j}(\cdot)$ in the denominator of (A.26) can be expressed in terms of k' and

x'_j as

$$\begin{aligned}
&\frac{\partial}{\partial x_j}\overline{F}(A_{K,t}k, A_{X_{1,t},t}X_{1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t}) \\
&\quad = \frac{\partial}{\partial x_j}\widehat{F}(m, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t}) \\
&\quad = \widehat{F}_{X_j}(m, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t})\frac{dA_{X_j,t}x_j}{dx_j} \\
&\quad = \widehat{F}_{X_j}(g^t m', g^t X_{j^*+1,0}, ..., g^t x'_j, ..., g^t X_{J,0})A_{X_j,t} \\
&\quad = \widehat{F}_{X_j}(m', X_{j^*+1,0}, ..., x'_j, ..., X_{J,0})A_{X_j,t} \\
&\quad = \frac{\partial}{\partial x'_j}\widehat{F}(m', X_{j^*+1,0}, ..., x'_j, ..., X_{J,0})A_{X_j,t} \\
&\quad = A_{X_j,t}\overline{F}_{X_j}(k', X_{1,0}, ..., x'_j, ..., X_{J,0}).
\end{aligned}$$

The definition in (A.26) evaluates $\bar{F}(\cdot)$ and its arguments at their BGP values.[64] We also need to re-write these conditions in terms of their period-0 values. Note that

$$\begin{aligned}
\overline{F}(A_{K,t}k, &A_{X_1,t}X_{1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t}) \\
&= \widehat{F}(m, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t}) \\
&= \widehat{F}(g^t m', g^t X_{j^*+1,0}, ..., g^t x'_j, ..., g^t X_{J,0})A_{X_j,t} \\
&= g^t\widehat{F}(m', X_{j^*+1,0}, ..., x'_j, ..., X_{J,0})A_{X_j,t} \\
&= g^t\overline{F}(k', X_{1,0}, ..., x'_j, ..., X_{J,0}).
\end{aligned}$$

Therefore, condition $\overline{F}(A_{K,t}k, A_{X_1,t}X_{1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t}) = Y_t$ can be substituted by

$$\overline{F}(k', X_{1,0}, ..., x'_j, ..., X_{J,0}) = Y_0.$$

The point of evaluation, $k = K_t$, becomes $g^t k' = K_t$, or $k' = g^{-t}K_t = K_0$. Similarly, $x_j = X_{j,t}$ becomes $x'_j = g^{-t}A_{X_j,t}X_{j,t} = X_{j,0}$. Using all these results, (A.26) can be expressed in terms of k' and x'_j as follows:

$$\overline{\sigma}_{KX_j,t} = -\left.\frac{d\ln(A_{X_j,t}k'/x'_j)}{d\ln\left(\frac{\overline{F}_K(k', X_{1,0}, ..., x'_j, ..., X_{J,0})}{A_{X_j,t}\overline{F}_{X_j}(k', X_{1,0}, ..., x'_j, ..., X_{J,0})}\right)}\right|_{\substack{\overline{F}(k', X_{1,0}, ..., x'_j, ..., X_{J,0}) = Y_0 \\ k' = K_0, x'_j = X_{j,0}}}. \tag{A.31}$$

[64] I.e., the conditions that are written to the right of "|".

We can eliminate constant $A_{X_j,t}$ from the numerator because $d\ln(A_{X_j,t}k'/x'_j) = d(\ln(k'/x'_j) + \ln A_{X_j,t}) = d\ln(k'/x'_j)$. In the same way, $A_{X_j,t}$ in the denominator can also be eliminated. Also, recall that $A_{K,0} = A_{X_j,0} = 1$. Then, comparing (A.31) with (A.26), it turns out that the RHS of (A.31) coincides with $\overline{\sigma}_{KX_j,0}$. From Lemma 3, $\overline{\sigma}_{KX_j,0} = \sigma_{KX_j,0}$ holds in period 0. In addition, it is assumed that $\sigma_{KX_j,0}$ does not change over time. Therefore, $\overline{\sigma}_{KX_j,t} = \overline{\sigma}_{KX_j,0} = \sigma_{KX_j,0} = \sigma_{KX_j,t}$.

A.10 Proof of Proposition 7

Proof of (c), as well as existence and uniqueness of $\widehat{\gamma}_L(\mu_t)$

As explained in the main text, the representative firm chooses $\gamma_{L,t}$ so as to satisfy

$$\text{R\&D for N-tasks:}\quad \gamma_{L,t}i'_L(\gamma_{L,t}) + i_L(\gamma_{L,t}) = f(\mu_t) - \mu_t f'(\mu_t). \tag{38}$$

Let us denote the LHS of (38) by $\Psi_L(\gamma_{L,t})$ because it depends only on $\gamma_{L,t}$. Then, $\Psi'_L(\gamma_{L,t}) = \gamma_{L,t}i''_L(\gamma_{L,t}) + 2i'_L(\gamma_{L,t}) > 0$ for all $\gamma_{L,t} > 1$ from $i_L(\gamma_{L,t}) > 0$ and $i'_L(\gamma_{L,t}) > 0$ in (26). When $\gamma_{L,t} = 1$, the properties of $i_L(\cdot)$ imply $\Psi_L(1) = i'_L(1) + i_L(1) = 0$. Also, $\Psi_L(\infty) \equiv \lim_{\gamma_{L,t}\to\infty}\Psi_L(\gamma_{L,t}) = \infty$ from $i_L(\infty) = \infty$ and $\gamma_{L,t}i'_L(\gamma_{L,t}) > 0$.[65] Then, since $\Psi_L(\cdot)$ is differentiable and strictly increasing, we can define its inverse function $\Psi_L^{(-1)}(\cdot)$, which is also differentiable and strictly increasing with $\Psi_L^{(-1)}(0) = 1$ and $\Psi_L^{(-1)}(\infty) = \infty$. Using this function, condition (38) can be solved for $\gamma_{L,t}$:

$$\gamma_{L,t} = \Psi_L^{(-1)}(f(\mu_t) - \mu_t f'(\mu_t)) \equiv \widehat{\gamma}_L(\mu_t). \tag{A.32}$$

Note that $f(\mu_t) - \mu_t f'(\mu_t)$ represents the marginal product of N_t in the production function, i.e., $\widehat{F}_N(\mu_t, 1)$. We can express the production function $Y_t = \widehat{F}(M_t, N_t)$ in an intensive form with respect to $N_t/M_t \equiv \nu_t$, instead of $\mu_t = M_t/N_t$. Namely, output per M_t can be expressed as $Y_t/M_t = \widehat{F}(M_t, N_t)/M_t = \widehat{F}(1, \nu_t) \equiv h(\nu_t)$. Since $\widehat{F}(\cdot)$ is CRS, its first derivative $\widehat{F}_N(\cdot)$ is homogeneous of degree 0. Using this property, $h'(\nu) = F_N(1, N_t/M_t) = F_N(M_t/N_t, 1) = F_N(\mu_t, 1) = f(\mu_t) - \mu_t f'(\mu_t)$. From the definition of the production function $\widehat{F}(\cdot)$, its alternate intensive form, $h(\nu_t)$, satisfies the Inada conditions. Therefore, $\lim_{\mu_t\to 0} f(\mu_t) - \mu_t f'(\mu_t) = \lim_{\nu_t\to\infty} h'(\nu_t) = 0$, and $\lim_{\mu_t\to\infty} f(\mu_t) - \mu_t f'(\mu_t) = \lim_{\nu_t\to 0} h'(\nu_t) = \infty$. Substituting these into (A.32) gives

[65] Similarly to the main text, we employ an abuse of notation by writing $i_L(\infty)$ to represent $\lim_{\gamma_L\to\infty} i_L(\gamma_L)$. We will employ similar abbreviations as long as they cause no confusion.

$\gamma_L(0) = \Psi_L^{(-1)}(0) = 1$ and $\gamma_L(\infty) = \Psi_L^{(-1)}(\infty) = \infty$.

Finally, we show $\gamma_L'(\mu_t) > 0$. The derivative of $f(\mu_t) - \mu_t f'(\mu_t)$ with respect to μ_t is $-\mu_t f''(\mu_t)$. It is positive for all $\mu_t > 0$ since the production function satisfies the Inada conditions, which include $f''(\mu_t) < 0$. Since $\Psi_L^{(-1)'}(\cdot) > 0$, this means $\gamma_L'(\mu_t) > 0$.

Proof of (a) and (b), as well as existence and uniqueness of $\widehat{\gamma}_K(\mu_K)$ and $\widehat{\gamma}_X(\mu_X)$

The representative firm chooses $\gamma_{K,t}$ and $\gamma_{X,t}$ according to the following two conditions:

$$\text{R\&D allocation: } \frac{\gamma_{K,t} i_K'(\gamma_{K,t})}{\gamma_{X,t} i_X'(\gamma_{X,t})} = \frac{\alpha}{1-\alpha}, \quad \alpha \in (0,1), \tag{39}$$

$$\text{Combined R\&D: } (\gamma_{K,t} i_K'(\gamma_{K,t}) + i_K(\gamma_{K,t})) + (\gamma_{X,t} i_X'(\gamma_{X,t}) + i_X(\gamma_{X,t})) = f'(\mu_t). \tag{40}$$

Let us define $\Omega_K(\gamma_{K,t}) \equiv \gamma_{K,t} i_K'(\gamma_{K,t})$ and similarly $\Omega_X(\gamma_{X,t}) \equiv \gamma_{X,t} i_K'(\gamma_{X,t})$. Then, from properties in (26), we can confirm $\Omega_K'(\gamma_{K,t}) > 0$ for $\gamma_{K,t} > 1$, $\Omega_K(1) = 0$ and $\Omega_K(\infty) = \infty$. Similar conditions hold also for $\Omega_X(\cdot)$. Then, since $\Omega_X(\cdot)$ is differentiable and strictly increasing, we can define its inverse function $\Omega_X^{(-1)}(\cdot)$, which is also differentiable and strictly increasing with $\Omega_X^{(-1)}(0) = 1$ and $\Omega_X^{(-1)}(\infty) = \infty$. Using this inverse function, condition (39) can be solved for $\gamma_{X,t}$ as

$$\gamma_{X,t} = \Omega_X^{(-1)}\left(\frac{\alpha}{1-\alpha}\Omega_K(\gamma_{K,t})\right) \equiv \Omega(\gamma_{K,t}). \tag{A.33}$$

Now let us focus on condition (40). Let us define $\Psi_K(\gamma_{K,t}) \equiv \gamma_{K,t} i_K'(\gamma_{K,t}) + i_K(\gamma_{K,t})$ and likewise $\Psi_X(\gamma_{X,t}) \equiv \gamma_{X,t} i_K'(\gamma_{X,t}) + i_K(\gamma_{X,t})$. Using these and (A.33), the LHS of condition (40) can be expressed as a function only of $\gamma_{K,t}$:

$$\Psi_K(\gamma_{K,t}) + \Psi_X(\Omega(\gamma_{K,t})) \equiv \Psi(\gamma_{K,t}).$$

Note that the properties of $\Omega_K(\cdot)$ and $\Omega_X^{(-1)}(\cdot)$ imply that $\Omega(\gamma_{K,t}) > 0$ for all $\gamma_{K,t} > 1$, $\Omega(0) = 0$ and $\Omega(\infty) = \infty$. Also, in the same way that we derived the properties of $\Psi_L(\gamma_{L,t})$ earlier in this proof, we can confirm $\Psi_K(\gamma_{K,t}) > 0$ for all $\gamma_{K,t} > 1$, $\Psi_K(1) = 0$, $\Psi_K(\infty) = \infty$, and similar properties for $\Psi_X(\gamma_{X,t})$. From these, we have $\Psi(\gamma_{K,t}) > 0$ for all $\gamma_{K,t} > 1$, $\Psi(1) = 0$, $\Psi(\infty) = \infty$. On the RHS of (40), $f'(\mu_t)$ satisfies the usual Inada conditions. The results we have obtained so far can be summarized as

$\gamma_{K,t}$	1	$\cdots$	∞
$\Psi'(\gamma_{K,t})$		$+$	
$\Psi(\gamma_{K,t})$	0	$\nearrow$	∞

μ_t	0	$\cdots$	∞
$f''(\mu_t)$		$-$	
$f'(\mu_t)$	∞	$\searrow$	0

The tables above implies that condition (40), $\Psi(\gamma_{K,t}) = f'(\mu_t)$, gives a 1 to 1 correspondence between $\mu_t \in (0, \infty)$ and $\gamma_{K,t} \in (1, \infty)$ that satisfies property (a): $\widehat{\gamma}'_K(\mu_t) < 0$ for all $\mu_t > 0$, $\widehat{\gamma}_K(0) = \infty$, and $\widehat{\gamma}_K(\infty) = 1$.

Given $\widehat{\gamma}_K(\mu_t)$, equation (A.33) uniquely determines $\gamma_{X,t} = \Omega(\widehat{\gamma}_K(\mu_t)) \equiv \widehat{\gamma}_X(\mu_t)$. From the properties of $\Omega(\cdot)$ and $\widehat{\gamma}_K(\cdot)$ above, we can confirm that property (b) is satisfied: $\widehat{\gamma}'_X(\mu_t) < 0$ for all $\mu_t > 0$, $\widehat{\gamma}_X(0) = \infty$, and $\widehat{\gamma}_X(\infty) = 1$.

A.11 Derivation of Equilibrium Dynamics

In this section, we derive the equilibrium dynamics of the endogenous growth model in Section 6. To do so, it is convenient to define the net aggregate output in the economy as $V_t = \widehat{F}(M_t, N_t) - R_{K,t} - R_{X,t} - R_{L,t}$, which is aggregate output minus total R&D costs in the economy.

The net output per effective labor can be written as a function of μ_t:

$$V_t/N_t = f(\mu_t) - \mu_t(i_K(\widehat{\gamma}_K(\mu_t)) + i_X(\widehat{\gamma}_X(\mu_t))) - i_L(\widehat{\gamma}_L(\mu_t)) \equiv v(\mu_t). \tag{A.34}$$

Then, substituting profits (28) into the budget constraint (33), we can express the growth of aggregate capital supply in terms of μ_t, k_t and c_t:

$$\frac{K_{t+1}}{K_t} = \frac{V_t + (1-\delta)K_t - C_t}{K_t} = \frac{v(\mu_t) - c_t}{k_t} + 1 - \delta. \tag{A.35}$$

Dynamics for $\mu_{\mathbf{t+1}}$. The growth factor of μ_{t+1} is defined by $\mu_{t+1}/\mu_t = (M_{t+1}/M_t)/(N_{t+1}/N_t)$. By using (23), (24), (29), (32) and (A.35), its value in equilibrium can be written as

$$\frac{\mu_{t+1}}{\mu_t} = \frac{(g_X \widehat{\gamma}_X(\mu_{t+1}))^{1-\alpha}}{g_L \widehat{\gamma}_L(\mu_{t+1})} \left(\widehat{\gamma}_K(\mu_{t+1}) \left(\frac{v(\mu_t) - c_t}{k_t} + 1 - \delta \right) \right)^{\alpha}, \tag{A.36}$$

where $\widehat{\gamma}_K(\mu_t)$, $\widehat{\gamma}_X(\mu_t)$, and $\widehat{\gamma}_L(\mu_t)$ are the rates of technological progress defined in Proposition 7. While equation (A.36) gives a relationship between the period-t variables $\{\mu_t, k_t, c_t\}$

and μ_{t+1}, it is not easy to understand how μ_{t+1} is determined since both sides of the equation depend on μ_{t+1}.

To interpret it intuitively, let us decompose the dynamic relationship in (A.36) into two steps. First, we define the pre-R&D relative factor intensity by

$$\mu_{t+1}^{\text{pre}} \equiv \frac{\left(A_{K,t}K_{t+1}\right)^{\alpha}\left(A_{X,t}X_{t+1}\right)^{1-\alpha}}{A_{L,t}L_{t+1}} = \frac{g_X^{1-\alpha}}{g_L}\left(\frac{v(\mu_t)-c_t}{k_t}+1-\delta\right)^{\alpha}\mu_t, \tag{A.37}$$

where the last equality is from (29), (32), (A.35) and the definition of μ_t. It is the value of μ_{t+1} before technologies are improved from their period-t state. Second, μ_{t+1}^{pre} and the post-R&D value of μ_{t+1} are related by the growth of technological levels $\widehat{\gamma}_K(\mu_t)$, $\widehat{\gamma}_X(\mu_t)$, and $\widehat{\gamma}_L(\mu_t)$ as follows:

$$\mu_{t+1}^{\text{pre}} = \frac{\widehat{\gamma}_L(\mu_{t+1})}{\widehat{\gamma}_K(\mu_{t+1})^{\alpha}\widehat{\gamma}_X(\mu_{t+1})^{1-\alpha}}\mu_{t+1} \equiv \Gamma(\mu_{t+1}). \tag{A.38}$$

Note that Proposition 7 implies that function $\Gamma(\mu_{t+1})$ is a strictly increasing differentiable function with $\lim_{\mu\to 0}\Gamma(\mu) = 0$ and $\lim_{\mu\to\infty}\Gamma(\mu) = \infty$. Therefore, its inverse function is well-defined for all $\mu_{t+1}^{\text{pre}} > 0$, and is a strictly increasing differentiable function. We write this inverse function by $\mu_{t+1} = G(\mu_{t+1}^{\text{pre}})$. Combined with (A.37), the dynamic relationship (A.36) can be written as (41).

Dynamics for $\mathbf{k_{t+1}}$. From (29) and (A.35), the growth factor of $k_t \equiv K_t/A_{L,t}L_t$ is given by

$$\frac{k_{t+1}}{k_t} = \frac{1}{g_L\widehat{\gamma}_L(\mu_{t+1})}\left(\frac{v(\mu_t)-c_t}{k_t}+1-\delta\right). \tag{A.39}$$

While μ_{t+1} is present in the RHS, we can replace it with (41) so that the RHS depends only on the variables in period t, which gives (42).

Dynamics for $\mathbf{c_{t+1}}$. The representative household maximizes the intertemporal utility function (31) subject to the budget constraint (33) and the no-Ponzi game condition (34). From (31), $\partial U/\partial C_t = \beta^t(C_t/L_t)^{-\theta}$. Therefore, the Euler equation for this problem is $(C_t/L_t)^{-\theta} = (r_{t+1}+1-\delta)\beta(C_{t+1}/L_{t+1})^{-\theta}$, which simplifies to

$$C_t^{-\theta} = (r_{t+1}+1-\delta)\beta g_L^{\theta} C_{t+1}^{-\theta}. \tag{A.40}$$

By substituting the market interest rate (35) into the Euler equation (A.40) and then applying it to the definition $c_t \equiv C_t/A_{L,t}L_t$, we obtain the growth factor of consumption per effective labor:

$$\frac{c_{t+1}}{c_t} = \frac{\beta^{1/\theta}}{\widehat{\gamma}_L(\mu_{t+1})} \left(\frac{\alpha \mu_{t+1}}{k_{t+1}} \left(f'(\mu_{t+1}) - i_K(\widehat{\gamma}_K(\mu_{t+1})) - i_X(\widehat{\gamma}_X(\mu_{t+1})) \right) + 1 - \delta \right)^{1/\theta}. \tag{A.41}$$

By replacing the period-$(t+1)$ variables in the RHS by (41) and (42), we can rewrite equation (A.41) as (43).

Boundary Conditions. To obtain the equilibrium path of $\{\mu_t, k_t, c_t\}_{t=0}^{\infty}$, we need three boundary conditions. First, since K_0, X_0, L_0, $A_{K,-1}$,$A_{X,-1}$, and $A_{L,-1}$ are given, we can construct μ_0^{pre}, the pre-R&D relative task intensity for period 0. Using it with the inverse function of Γ from (A.38), we have the initial value of μ_t:

$$\mu_0 = G\left(\frac{(A_{K,-1}K_0)^{\alpha}(A_{X,-1}X_0)^{1-\alpha}}{A_{L,-1}L_0} \right). \tag{A.42}$$

Second, using μ_0, the initial value of k_t is readily obtained by

$$k_0 = \frac{K_0}{\widehat{\gamma}_L(\mu_0)A_{L,-1}L_0}. \tag{A.43}$$

Finally, the initial value of c_t must be chosen so as to satisfy the no-Ponzi game condition (34) and the transversality condition

$$\lim_{T\to\infty} \beta^T \left(\frac{C_T}{L_T} \right)^{-\theta} K_{T+1} \leq 0. \tag{A.44}$$

In the remainder of this section, we establish that conditions (34) and (A.44) jointly mean (44). The Euler equation (A.40) implies that $r_t + 1 - \delta = (C_{t-1}/L_{t-1})^{-\theta}/\beta(C_t/L_t)^{-\theta}$. Through repeated multiplication,

$$\prod_{t=1}^{T}(r_t + 1 - \delta) = \frac{(C_0/L_0)^{-\theta}}{\beta^T(C_T/L_T)^{-\theta}}.$$

Using this, the no-Ponzi game condition (34) becomes

$$\left(\frac{C_0}{L_0} \right)^{\theta} \lim_{T\to\infty} \beta^T \left(\frac{C_T}{L_T} \right)^{-\theta} K_{T+1} \geq 0. \tag{A.45}$$

Since $C_0/L_0 > 0$, we can divide the both sides of (A.45) by $(C_0/L_0)^{\theta}$ to eliminate this term. Then, it turns out that (A.45) has exactly the same form as the transversality condition

(A.44), except that the direction of the inequality is opposite. By combining (A.44) and (A.45), therefore, we have a unified terminal condition

$$\lim_{T\to\infty} \beta^T \left(\frac{C_T}{L_T}\right)^{-\theta} K_{T+1} = 0. \tag{A.46}$$

Using $c_t \equiv C_t/\left(A_{L,t}L_t\right)$ and $k_t \equiv K_t/\left(A_{L,t}L_t\right)$, the expression on the LHS can be written as

$$\begin{aligned}
\beta^T \left(\frac{C_T}{L_T}\right)^{-\theta} K_{T+1} &= \beta^T \left(A_{L,T}c_T\right)^{-\theta} A_{L,T+1}L_{T+1}k_{T+1} \\
&= \beta^T \left(A_{L,T}c_T\right)^{-\theta} A_{L,T}\widehat{\gamma}_L(\mu_{T+1})L_0 g_L^{T+1} k_{T+1} \\
&= L_0 g_L (\beta g_L)^T A_{L,T}^{1-\theta} c_T^{-\theta} \widehat{\gamma}_L(\mu_{T+1}) k_{T+1} \\
&= L_0 g_L A_{L,-1} (\beta g_L)^T \left(\prod_{t=0}^{T} \widehat{\gamma}_L(\mu_t)\right)^{1-\theta} c_T^{-\theta} \widehat{\gamma}_L(\mu_{T+1}) k_{T+1}.
\end{aligned}$$

By substituting the last expression into (A.46) and then dividing both sides by $L_0 g_L A_{L,-1} > 0$, we obtain (44).[66]

A.12 Proof of Lemma 4

Consider a BGP. We will show in turn that μ_t, k_t and c_t must be constant. First, from the definition of a BGP, $N_{t+1}/N_t = (A_{L,t+1}L_{t+1})/(A_{L,t}L_t) = \widehat{\gamma}_L(\mu_{t+1})g_L$ is constant. To keep the RHS of the latter equation constant, μ_t must also be constant, since $\widehat{\gamma}_L(\cdot)$ is a strictly increasing function from Proposition 7.

Second, since growth factors of C_t and N_t are constant, the growth factor of $c_t = C_t/A_tL_t = C_t/N_t$ is also constant. This, in turn, means that the LHS of the Euler equation (A.41) is constant. Then, for the RHS of (A.41) to be constant, k_t must be constant, since we already know that μ_t is constant as shown above.

Third, the growth factor of $k_t = K_t/A_tL_t = K_t/N_t$ is constant on the BGP, which means the LHS of (A.39) is constant. For its RHS to be constant, given that μ_t and k_t are already shown to be constant, c_t also needs to be constant.

[66] Initial population L_0 and initial level of technology $A_{L,-1} > 0$ are exogenously given (see Subsection 6.3). Population growth factor $g_L > 0$ is also exogenous.

A.13 Proof of Proposition 8

Proof of $\mu^* > 0$

In the text, we have already shown that there exists a unique $\mu^* > 0$ such that $\Phi(\mu^*) = 1$ holds, since Proposition 7 implies $\Phi'(\mu^*) < 0$ with $\Phi(0) = \infty$ and $\Phi(\infty) = 0$. Therefore, there exists a unique value of $\mu^* > 0$.

Proof of $k^* > 0$

The value of k^* is explicitly given by equation (47), shown again here:

$$k^* = \frac{\beta\alpha\mu^*(f'(\mu^*) - i_K(\widehat{\gamma}_K(\mu^*)) - i_X(\widehat{\gamma}_X(\mu^*)))}{\widehat{\gamma}_L(\mu^*)^\theta - \beta(1-\delta)}. \tag{47}$$

We now show that both the numerator and the denominator of the RHS are positive. Note that the combined R&D condition (40) is satisfied on the BGP. By rearranging terms, it gives

$$f'(\mu^*) - i_K(\widehat{\gamma}_K(\mu^*)) - i_X(\widehat{\gamma}_X(\mu^*)) = \widehat{\gamma}_K(\mu^*) i'_K(\widehat{\gamma}_K(\mu^*)) + \widehat{\gamma}_X(\mu^*) i'_X(\widehat{\gamma}_X(\mu^*)) > 0, \tag{A.47}$$

where the inequality follows from Proposition 7 and (26). Given $\beta \in (0,1)$, $\alpha \in (0,1)$, and $\mu^* > 0$, this means that the numerator of (47) is strictly positive. Now, note that $\widehat{\gamma}_L(\mu^*) > 1$ from Proposition 7. Combined with $\theta > 0$, $\beta \in (0,1)$ and $\delta \in [0,1]$, it turns out that the denominator of (47) is also strictly positive.

Proof of $c^* > 0$

The value of c^* is given by

$$c^* = v(\mu^*) - (g^* - 1 + \delta)k^*. \tag{48}$$

We first show $v(\mu^*) > 0$. Combining the R&D conditions (38) and (40), we have

$$\gamma_{L,t} i'_L(\gamma_{L,t}) + i_L(\gamma_{L,t}) + \mu_t \left((\gamma_{K,t} i'_K(\gamma_{K,t}) + i_K(\gamma_{K,t})) + (\gamma_{X,t} i'_X(\gamma_{X,t}) + i_X(\gamma_{X,t})) \right) = f(\mu_t).$$

Rearranging and then evaluating this condition at $\mu_t = \mu^*$ gives

$$\begin{aligned} v(\mu^*) &= f(\mu*) - i_L(\widehat{\gamma}_L(\mu^*)) - \mu^* \left(i_K(\widehat{\gamma}_K(\mu^*)) - i_X(\widehat{\gamma}_X(\mu^*)) \right) \\ &= \widehat{\gamma}_L(\mu^*) i'_L(\widehat{\gamma}_L(\mu^*)) + \mu^* \left(\widehat{\gamma}_K(\mu^*) i'_K(\widehat{\gamma}_K(\mu^*)) + \widehat{\gamma}_X(\mu^*) i'_X(\widehat{\gamma}_X(\mu^*)) \right) > 0, \end{aligned}$$

where the inequality follows from $\mu^* > 0$ and (26).

Note that $g^* = \widehat{\gamma}_L(\mu^*)g_L$ is greater than $1-\delta$ because $\widehat{\gamma}_L(\mu^*) > 1$ and $g_L > 1-\delta$ from (29). Therefore, $(g^* - 1 + \delta)$ in (48) is positive. From this, $c^* > 0$ is equivalent to

$$k^* < \frac{v(\mu^*)}{g^* - 1 + \delta}.$$

Using (47), we can rewrite this condition in terms of β:

$$\beta < \widehat{\gamma}_L(\mu^*)^\theta \left(\frac{\alpha\mu^*}{v(\mu^*)} \left(f'(\mu^*) - i_K(\widehat{\gamma}_K(\mu^*)) - i_X(\widehat{\gamma}_X(\mu^*))\right)(g^* - 1 + \delta) + 1 - \delta \right)^{-1} \equiv \overline{\beta}_1. \tag{A.48}$$

Note that $\overline{\beta}_1 > 0$ from (A.47) and $g^* > 1 - \delta > 0$. Observe also that $\overline{\beta}_1$ does not depend on β itself since μ^* is determined entirely by the production side (see equation 45). Therefore, if $\beta > 0$ is sufficiently small, condition (A.48) holds and $c^* > 0$.

Terminal Condition

On the BGP, the terminal condition (44) becomes

$$\lim_{T\to\infty} \left(\beta g_L \widehat{\gamma}_L(\mu^*)^{1-\theta}\right)^T \widehat{\gamma}_L(\mu^*)(c^*)^{-\theta}k^* = 0. \tag{A.49}$$

Given that $\widehat{\gamma}_L(\mu^*) > 1$, $c^* > 0$ and $c^* > 0$, this condition is equivalent to

$$\beta < \frac{1}{g_L \widehat{\gamma}_L(\mu^*)^{1-\theta}} \equiv \overline{\beta}_2. \tag{A.50}$$

Note that $\overline{\beta}_2 > 0$ and that it does not depend on β since μ^* is determined entirely by the production side of the model. Therefore, if $\beta > 0$ is sufficiently small, condition (A.50) holds and the terminal condition (44) is satisfied.

Combining conditions (A.48) and (A.50), we have confirmed the unique existence of BGP with $\mu^* > 0$, $k^* > 0$, $c^* > 0$, and the terminal condition (44) whenever

$$\beta < \overline{\beta} \equiv \min\{\overline{\beta}_1, \overline{\beta}_2\}, \tag{A.51}$$

where $\overline{\beta} > 0$ is a constant that does not depend on β.

A.14 Calibration Procedure

There are seven parameters to calibrate, $\{\delta, \beta, \alpha, \eta, \zeta_L, \zeta_X, \chi\}$, which we identify with the seven moments in Table 3. We do so in two steps.

Step 1: Analytical calibration. Given period length χ, exogenous parameters, and moments, we analytically derive the values of four parameters $\{\delta, \beta, \alpha, \eta\}$. The depreciation rate is determined by data on the consumption of fixed capital and the capital-output ratio:

$$\delta = \frac{\delta K/Y}{K/Y} = \frac{0.14}{2.9}\chi \equiv \overline{\delta}(\chi). \tag{A.52}$$

Evaluating the Euler equation (A.40) on the BGP gives the discount factor β:[67]

$$\beta = \frac{\gamma_L^\theta}{1+r-\delta} = \frac{(1.019^\chi)^{1.0}}{1.04^\chi} \equiv \overline{\beta}(\chi). \tag{A.53}$$

Similarly, the first order conditions of the representative firm, (35) and (36), gives the share parameter α:

$$\alpha = \frac{\kappa_K}{\kappa_K+\kappa_X} = \frac{(r-\delta)(K/Y)+\delta K/Y}{1-\kappa_L-\kappa_R} = \frac{(1.04^\chi-1)(2.9\chi)+0.14}{1-0.63-0.027} \equiv \overline{\alpha}(\chi), \tag{A.54}$$

where we used the identity $\kappa_K+\kappa_X+\kappa_L+\kappa_R = 1$.

Next, BGP relationship (45), which is equivalent to the technology condition (22), gives the growth rate for the unobserved endogenous variable γ_X:

$$\gamma_X = \gamma_K^{-\frac{\alpha}{1-\alpha}}\frac{g_L\gamma_L}{g_X} = \left((1.066)^{-\frac{\overline{\alpha}(\chi)}{1-\overline{\alpha}(\chi)}}\frac{(1.01)(1.019)}{1.0}\right)^\chi \equiv \overline{\gamma}_X(\chi). \tag{A.55}$$

Using (A.55) and the R&D allocation condition (39), the R&D cost parameter ζ_X can be derived as follows.

$$\zeta_X = \frac{1-\alpha}{\alpha}\cdot\frac{\zeta_K\gamma_K(\gamma_K-1)^{\lambda-1}}{\gamma_X(\gamma_X-1)^{\lambda-1}} = \frac{1-\overline{\alpha}(\chi)}{\overline{\alpha}(\chi)}\cdot\frac{(1(1.0066)(0.0066)^{2.0-1})^\chi}{\overline{\gamma}_X(\chi)(\overline{\gamma}_X(\chi)-1)^{2.0-1}} \equiv \overline{\zeta}_X(\chi). \tag{A.56}$$

Step 2: Minimization. Among the 12 parameters of the model, five of them are given by Table 2, and four are given as functions of χ, in (A.52), (A.53), (A.54) and (A.56). This leaves us with three remaining parameters, $\{\zeta_L, \eta, \chi\}$. We calibrate them so as to minimize

[67] Equation (A.53) assumes that parameter θ takes the baseline value of 1.0. When doing robustness checks with $\theta = 0.8$ and 1.2, the numbers in this equation are adjusted accordingly. The same applies for (A.54)-(A.57) when using alternative parameter values or calibration targets.

the squared sum of percent difference (error) between the target moments in Table 3 and the corresponding model variables on the BGP.

Let us define the squared sum of percent error as[68]

$$SSE = \left(\frac{K/(Y/\chi) - 2.9}{2.9}\right)^2 + \left(\frac{\kappa_L - 0.63}{0.63}\right)^2 + \left(\frac{\kappa_R - 0.027}{0.027}\right)^2 + \left(\frac{\gamma_L^{1/\chi} - 1 - 0.019}{0.019}\right)^2 + \left(\frac{\gamma_K^{1/\chi} - 1 - 0.0066}{0.0066}\right)^2. \tag{A.57}$$

In (A.57), endogenous variables K/Y, κ_L, κ_R, γ_L and γ_X represent their respective BGP values, when the model is solved given all 12 parameters. Using exogenous parameters and the results of analytical calibration, we determine the remaining three parameters as the solution to the following minimization problem:

$$\begin{aligned} \{\zeta_L, \eta, \chi\} = &\underset{\zeta_L,\eta,\chi}{\operatorname{argmin}}\, SSE \quad s.t. \\ &\{\epsilon, \lambda, \theta, g_L, g_X\} : \text{given by Table 2}, \\ &\delta = \overline{\delta}(\chi),\ \beta = \overline{\beta}(\chi),\ \alpha = \overline{\alpha}(\chi),\ \zeta_X = \overline{\zeta}_X(\chi). \end{aligned} \tag{A.58}$$

We have done this minimization numerically utilizing 'FindMinimum' function of Mathematica. The minimized value of SSE is virtually zero (precisely, of order of 10^{-22}), implying that we obtained the set of parameters that fits all the moments in Table 3.

Robustness Scenarios. To check the robustness of the result, we repeat the analytical calibration (A.52)-(A.56) with modified values for the exogenous parameters. In all robustness scenarios, the modified version of minimization problem (A.58) yields almost zero. This means that the model can match all the target moments in those scenarios.

B Data Sources

All data are originally from the Bureau of Economic Analysis (BEA) and retrieved from FRED, Federal Reserve Bank of St. Louis, https://fred.stlouisfed.org. We reference series by their codes in FRED. **Panel a of Figure 1** shows annual values of Real GDP (GDPCA), real investment (GPDICA), and real personal consump-

[68] Among seven moments in Table 3, we use five moments to define the squared sum of errors. The other two moments, consumption of fixed capital in GDP and return on investments, always match the data given that other moments are correct, since we impose relationship (A.52) and (A.53). In numerical calibration, we confirmed that these two moments matches the data exactly.

tion expenditures (PCECCA). The real capital stock is calculated as the net stock of fixed assets at current cost (K1TTOTL1ES000) divided by the GDP price deflator (A191RD3A086NBEA). **Panel b of Figure 1** shows three different price deflators for gross private investment– all (A006RD3A086NBEA), non-residential (A008RD3A086NBEA), and equipment (Y033RD3A086NBEA) – relative to the price deflator for personal consumption expenditures (DPCERD3A086NBEA). 1950 values are normalized to 100. In addition to the variables listed above, the **calibration** utilizes data on nominal consumption of fixed capital (GDICONSPA), labor compensation, and R&D expenditure (Y694RC1A027NBEA) all relative to nominal GDP (GDPA) in **Table 3**, as well as population (B230RC0A052NBEA) in **Table 2**. Labor compensation is calculated as compensation of employees (A033RC1A027NBEA) plus proprietors' income with inventory valuation and capital consumption adjustments (A041RC1A027NBEA).