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NO PRICE ENVY IN THE MULTI-UNIT OBJECT ALLOCATION PROBLEM WITH NON-QUASI-LINEAR PREFERENCES

Hiroki Shinozaki

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The Institute of Social and Economic Research
Osaka University
6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan

No price envy in the multi-unit object allocation problem with non-quasi-linear preferences*

Hiroki Shinozaki†

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Abstract

We consider the problem of allocating multiple units of an indivisible object among a set of agents and collecting payments. Each agent can receive multiple units of the object, and has a (possibly) non-quasi-linear preference on the set of (consumption) bundles. We assume that preferences exhibit both nonincreasing marginal valuations and nonnegative income effects.

We propose a new property of fairness: *no price envy*. It extends the standard no envy test (Foley, 1967) over bundles to prices (per-unit payments), and requires no agent envy other agents' prices to his own in the sense that if he has a chance to receive some units at other agents' prices, then he gets better off than his own bundle.

First, we show that a rule satisfies *no price envy* and *no subsidy for losers* if and only if it is an *inverse uniform-price rule*. Then, we identify the unique maximal domain for *no price envy*, *strategy-proofness*, and *no subsidy for losers*: the *domain with partly constant marginal valuations*. We further establish that on the domain with partly constant marginal valuations, a rule satisfies *no price envy*, *strategy-proofness*, and *no subsidy for losers* if and only if it is a *minimum inverse uniform-price rule*.

Our maximal domain result implies that no rule satisfies *no price envy*, *strategy-proofness*, and *no subsidy for losers* when agents have preferences with nonincreasing marginal valuations. Given this negative observation, we look for a *minimally manipulable* rule among the class of rules satisfying both *no price envy* and *no subsidy for losers* in the case of preferences with nonincreasing marginal valuations. We show that a rule is minimally manipulable among the class of rules satisfying *no*

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†Graduate School of Economics, Osaka University. Email: vge017sh@student.econ.osaka-u.ac.jp

price envy and *no subsidy for losers* if and only if it is a minimum inverse uniform-price rule. Our results provide a rationale for the use of the popular minimum uniform-price rule in terms of fairness and non-manipulability.

JEL Classification Numbers. D44, D47, D63, D71, D82

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1 Introduction

1.1 Purposes

Auctions have been understood as the price discovery process by the interactions among the biddes and the seller(s) (Milgrom, 2017; Teytelboym et al., 2021). One of the virtues of the auctions held by the public sectors, such as spectrum auctions, car liscence auctions, etc., is to find "fair" prices of objects that are not traded in markets. Indeed, one of the announced goals in spectrum auctions in several countries is to find fair prices through the auctions.[1] However, the precise meaning of "fair prices" has been yet opaque, or depended on the authors.[2] In this paper, we formulate a notion of fair prices as a property of *fairness*, and investigate its implications.

Another important issue in real-life auctions is the existence of bidders whose preferences are not quasi-linear. The assumption of quasi-linear preferences make the analysis simple and tractable, but it is applicable only to unrealistic situations where agents have sufficiently large willingness to pay for the objects (no income effect), face linear borrowing costs in financial markets (no budget constraint), etc. In many real-life situations, the assumption of quasi-linear preferences does not seem plausible, and agents rather have non-quasi-linear preferences. Thus, we take agents with non-quasi-linear preferences into account.

The goals of this paper are two-fold: we attempt to (i) formulate fair prices as a property of fairness, and to (ii) identify the class of rules satisfying our property of fair prices together with the other desirable properties for agents with non-quasi-linear preferences.

[1] For example, the regulator in India (Department of Telecommunications) announced that one of the goals of auctions to allocate the lights to use scarce spectrum bands is to "obtain a market determined price of" spectrum bands "through a transparent process" (Government of India, 2021).

[2] For example, on the one hand Ausubel et al. (2014) write that "it (uniform priceing) is fair in the sense that the same price is paid by everyone", but on the other hand Burkett and Woodward (2020) write that "the uniform price auction is fair, in the sense that bidders never pay less than other bidders for the same number of units won". These two papers use the term "fair" in slightly different ways.

1.2 Main results

We consider the problem of allocating multiple units of an object to the agents with payments. Each agent can receive multiple units of the object, and has a (possibly) non-quasi-linear prefereces over (consumption) bundles, where a bundle specifies the consumption level of the object and the payments.

A preference exhibits *nonincreasing* (resp. *constant*) *marginal valuations* if the marginal willingness to pay at each bundle is no greater than (resp. equal to) the marginal willingness to sell at the bundle. A preference exhibits *nonnegative income effects* if the demand of the object does not decrease when the payments decrease. In this paper, we assume that preferences exhibit both nonincreasimg narginal valuations and nonnegative income effects, both of which are standard assumptions in the literature.

An *allocation* is a profile of each agent's bundle, and an (*allocation*) *rule* is a function from the set of preference profiles to the set of allocations.

In this paper, we regard a price as a per-unit payment. Formally, a *price* faced by an agent at a preference profile under a rule is defined as the agent's per-unit payment for the preference profile under the rule. Note that our definition of a price does not take agents who receive no object (the *losers*) into account, and we choose to leave prices of the losers undefined in this paper.[3]

Our property of fair prices incorporates prices into the *no envy* test (Foley, 1967). Formally, we say that a rule satisfies *no price envy* if no agent prefers other agents' *prices* in the sense that if he has a chance to buy some units of the object at other agent's prices, then he can get better off than his own bundle.

First, we try to identify the class of rules satisfying *no price envy* together with the other mild property. A rule satisfies *no subsidy for losers* if each loser does not receive money. In many real-life auctions, bidders are often forbidden to receive money, and in such situations *no subsidy for losers* is plausible. We can also interpret *no subsidy for losers* as a desirable property since it excludes "dummy" agents who are interested only in the participation subsidy.

A *uniform-price rule* is a rule defined for quasi-linear preferences such that for each preference profile, (i) the object is allocated so as to maximize the sum of valuations, and (ii) each agent pays the same price that is no less than the highest losing marginal valuation, and is no greater than the lowest winning marginal valuation. In this paper, we extend the uniform-price rule for quasi-linear preferences to non-quasi-linear preferences, but there are several ways to extend it. An *inverse uniform-price rule* is a new variant of such an extension, and it adopts the *inverse-demand function* of Shinozaki et al. (2020) to the uniform-price rule instead of the quasi-linear valuations.[4] A *minimum inverse*

[3] In Section 5.1.1, we will discuss the difficulty of defining the prices of the losers in detail.

[4] Baisa (2016) introduces the indirect uniform-price auction mechanism where agents submit not their preferences but bids in the same model as ours. Since our inverse uniform price rule is an allocation rule

uniform-price rule is an inverse uniform-price rule that chooses the highest losing marginal valuation as its price.

Our first result is a characterization of the inverse uniform-price rule by means of *no price envy*. We show that a rule satisfies *no price envy* together with *no subsidy for losers* if and only if it is an inverse uniform-price rule (Theorem 1).

As already noted, *no subsidy for losers* is a fairly mild property, and trivially holds in a natural model with nonnegative payments.[5] In such a model, our first result gives a redefenition of the inverse uniform-price rule by the single tight property of fairness: *no price envy*.

Next, we turn to the non-manipulability of rules. A rule is said to be *manipulable* by an agent at a preference profile if he gets better off by misreporting his preference. A rule is said to be *strategy-proof* if it is manipulable by no agent at each preference profile.

First, we search for domains that admit the existence of a rule satisfying *no price envy*, *strategy-proofness*, and *no subsidy for losers*. On the quasi-linear domain with constant marginal valuations, the minimum (inverse) uniform-price rule satisfies the three desirable properties. An interesting quastion is: How much can we extend a domain from the quasi-linear domain with constant marginal valuations while guaranteeing the existence of a rule satisfying *no price envy*, *strategy-proofness*, and *no subsidy for losers*? Thus, we investigate a *maximal domain* for *no price envy*, *strategy-proofness*, and *no subsidy for losers* that contains the quasi-linear domain with constant marginal valuations.[6]

A preference exhibits *partly constant marginal valuations* if the valuation at the status quo bundle exhibits constant marginal valuations, where the status quo bundle includes no object and no monetary transfer. Note that the domain with partly constant marginal valuations contains the quasi-linear domain with constant marginal valuations. Then, we show that the domain with partly constant marginal valuations is the unique maximal domain for *no price envy*, *strategy-proofness*, and *no subsidy for losers* containing the quasi-linear domain with constant marginal valuations (Theorem 2). Moreover, we extablish that the minimum inverse uniform-price rule is the only rule satisfying *no price envy*, *strategy-proofness*, and *no subsidy for losers* on any domain that contains the quasi-linear domain with constant marginal valuations and is contained by the domain with partly constant marginal valuations (Theorem 3).

Although our maximal domain result highlights the importance of the assumption of constant marginal valuations for a positive result, in many real-life situations it seems more natural to assume that agents have preferences with nonincreasing marginal valuations

(or a direct mechanism), there seems no direct relationship between our rule and his mechanism.

[5]For example, Chew and Serizawa (2007) consider the model with nonnegative payments.

[6]Several authors have investigated maximal domains that guarantee lists of desirable properties in many models. See, for example, Ching and Serizawa (1998), Berga and Serizawa (2000), Ehlers (2002), etc.

than assuming that they have preferences with constant marginal valuations.[7] Moreover, nonincreasing marginal valuations are a standard assumption in the multi-unit object allocation problem (Vickrey, 1961; Ausubel et al., 2014; Baisa, 2016, 2020, etc). Thus, it is worthwhile to investigate the existence of a rule satisfying *no price envy* together with the other desirable properties without relaxing the assumption of nonincreasing marginal valuations.

Our maximal domain result (Theorem 2) implies that no rule satisfies *no price envy*, *strategy-proofness*, and *no subsidy for losers* on the domain with nonincreasing marginal valuations (Corollary 1), and so we have to give up one of the three properties if we keep the assumption of the nonincreasing marginal valuations. Since *no price envy* is at the heart of the paper, and *no subsidy for losers* is a mild condition that almost all natural rules satisfy, we give up *strategy-proofness* instead of the other two properties. Then, we search for rules satisfying both *no price envy* and *no subsidy for losers* that prevent agents from misreporting their preferences as much as possible.

We adopt the manipulability measure of Pathak and Sönmez (2013) that they call the "as intensely and strongly as manipulable" relation to the class of rules satisfying *no price envy* and *no subsidy for losers*. We say that a rule is *at least as manipulable as* another rule if for each preference profile and each agent, whenever he can manipulate the latter rule, (i) the former rule is as well by him, and (ii) the gain from manipulation of the former rule is at least as large as that of the latter rule. Further, a rule is *minimally manipulable* among a given class of rules if (i) the rule is in the class, and (ii) each rule in the class is at least as manipulable as the rule. Clearly, a minimally manipulable rule is the best among a given class of rules in terms of non-manipulability, but it does not necesarily exist in general.

We first show that for each pair of rules satisfying *no price envy* and *no subsidy for losers*, a rule is at least as manipulable as another rule if and only if for each preference profile, each agent weakly prefers the outcome of the latter rule to that of the former rule (Proposition 3). Using this result, we establish that a rule is minimally manipulable among the class of rules satisfying both *no price envy* and *no subisidy for losers* if and only if it is a minimum inverse uniform-price rule (Theorem 4). Our results (Theorems 3 and 4) provide a rationale for the use of the minimum inverse unform-price rule in terms of fairness and non-manipulability.

[7] For example, in auctions for collectibles such as arts and wine, it seems natural to assume that bidders have preferences with nonincreasing marginal valuations. Also, if a firm in a procurement auction has a technology that exhibits nonincreasing returns to scale, then it has a preference with nonincreasing marginal valuations because nonincreasing returns in a procurement auction model to scale corresponds to nonincreasing marginal valuations in our model.

1.3 Related literature

1.3.1 Object allocation problems

The literature on object allocation problems mainly focuses on *efficiency* (Holmström, 1979; Chew and Serizawa, 2007; Saitoh and Serizawa, 2008; Sakai, 2008; Morimoto and Serizawa, 2015;. Baisa, 2020; Shinozaki et al., 2020). In contrast, other authors consider the properties of fairness in the object allocation problems. Ohseto (2004, 2006) consider the unit-demand identical objects model with quasi-linear preferences, and identify the classes of Groves rules satisfying *egalitarian-equivalence* and *envy-freeness*, respectively. Papài (2003) and Yengin (2012) identify the classes of Groves rules satisfying *no envy* and *egalitarian-equivalence*, respectively, in the multi-demand heterogeneous objects model with quasi-linear preferences. Sakai (2013) and Adachi (2014) characterize the generalized Vickrey rule (Saitoh and Serizawa, 2008; Sakai, 2008) by means of properties of fairness in the unit-demand identical object(s) model with non-quasi-linear preferences.

Our paper is different from the above papers in that we consider a property of fairess in the *multi-demand identical objects model* (with non-quasi-linear preferences). To the best of our knowledge, ours is the first paper that provides a characterization result by a property of fairness in such a model (with or without quasi-linear preferences).

1.3.2 Uniform-price auctions

The uniform-price auction occupies a central position both in auction theory and in prc-tical auction design. The literature on the uniform-price aucion mainly focuses on its equilibrium properties in models with quasi-linear preferences (Vickrey, 1961, Noussair, 1995; Engelbrecht-Wiggans and Kahn, 1998; Ausubel et al., 2014; Burkett and Woodward, 2020). One of the most important results in this strand of research is the inefficiency theorem: any equilibrium in the minimum uniform-price auction does not achieve an efficient allocation in general (Ausubel et al., 2014; Baisa, 2016). In a uniform-price auction, the truth-telling does not constitute an equilibrium, but it does an approximate equilibrium if there are many agents (Swinkers, 2001; Azevedo and Budish, 2019) or many objects (Tajika and Kazumura, 2019).

This paper is different from the papers in the strand of research in that we do not focus on the uniform-price auction *a priori*, but rather obtain it as a *consequence of the properties* that we consider in this paper. As far as we know, ours is the first paper that gives a characterization of the minimum uniform-price rule in terms of fairness and non-manipulability.

1.3.3 Minimal manipulability

The method of comparing rules in terms of their manipulability has been adopted to *non-strategy-proof* rules in many models such as the voting model (Kelly, 1988; Maus et al., 2007), the matching with contracts model (Chen et al., 2016), the school choice model (Pahak and Sönmez, 2013), the heterogeneous objects model with quasi-linear preferences (Day and Milgrom, 2008; Andersson et al., 2014). Manipulability measures depend on the authors, and ours extends one in Pathak and Sönmez (2013) for quasi-linear preferences to non-quasi-linear preferences.

Day and Milgrom (2008) is a closely related paper to ours. They show that in the multi-demand heterogeneous objects model with quasi-linear preferences, an agent-optimal core-selecting rule is minimally manipulable according to their manipulability measure among the class of core-selecting rules.[8] Note that an inverse uniform-price rule is core-selecting. However, our results can not be obtained by their results and proof technique since their argument crucially relies on a truncation of a preference which is not feasible in our model.[9]

Chen et al. (2016) is another related paper. They show that in the (many-to-many) matching with contracts model, a rule is at least as manipulable as another rule according to their manipulability measure if and only if for each preference profile, each agent weakly prefers the outcome of the latter rule to that of the former rule. Note that Proposition 3 in this paper is parallel to their result. However, our results does not follow from their results and technique since our manipulability measure is different from theirs (Example 3 in Section 5.2), and their argument crucially relies on the finiteness of the model.

1.3.4 Fair allocations

Finally, this paper also contributes to the literature on the theory of fair allocation by proposing a new property of fairness.[10] It is worthwhile to note that *no price envy* is closely related to *opportunity fairness* of Varian (1976) and *no envy of opportunities* of Thomson (1994) since the prices naturally defines the opportunity set, but our property is different from the other properties in that we do not take the losers' opportunity sets (the prices) into account. In Section 5.1.2, we will discuss the relationships between *no price envy* and the other two properties of fairness of opportunities in detail.

Our *no price envy* is also closely related to *envy-free pricing* of Guruswami et al.

[8] A rule is *core-selecting* if its outcome is in the core for each preference profile. A rule is *agent-optimal core-selecting rule* if it is core-selecting, and there is no other core-selecting rule such that each agent weakly prefers the outcome of the rule to that of the original rule, and some agent strictly prefers.

[9] To be precise, they consider a truncation of a preference relative to the payoff under the Vickrey rule. In our model, an agent can not report such a truncated preference since it may violate our property of preferences (object monotonicity), and hence their proof technique does not work.

[10] For the excellent survey of the theory of fair allocation, see Thomson (2011).

(2005): given a common and linear price of the object, each agent receives a bundle that is optimal at the given price. Note that in our model, *envy-free pricing* is equivalent to *opportunity fairness* of Varian (1976), and the discussion in Section 5.1.2 about the relationship between *no price envy* and *opportunity fairness* also applies to that between *no price envy* and *envy-free pricing*. In particular, we emphasize that *no price envy* is different from *envy-free pricing* in that we do not consider a common and liner price of the object *a priori*, but rather regard a per-unit payment at an agent's bundle as his price. This will enable us to apply our model to a wider range of situations than a model with common and linear price.

1.4 Organization

The remaining part of this paper is organized as follows. Section 2 introduces the model. Section 3 introduces the inverse uniform-price rule. Section 4 provides the results. Section 5 discusses the relationships between *no price envy* and other related properties of fairness, and further discuss the relationships between our manipulability measure and other measures. Section 6 concludes. Almost all proofs are relegated to Appendix, but the others can be found in the supplementary material.

2 The model

There are n agents and m units of an identical object, where $n \geq 2$ and $m \geq 2$. Let $N \equiv \{1, \ldots, n\}$ denote the set of agents. Let $M \equiv \{0, \ldots, m\}$. Agent $i \in N$ receives $x_i \in M$ units of the object. Let $t_i \in \mathbb{R}$ denote the amount of money paid by agent i. The common **consumption set** is $M \times \mathbb{R}$, and a (**consumption**) **bundle** of agent $i \in N$ is a pair $z_i \equiv (x_i, t_i) \in M \times \mathbb{R}$. Let $\mathbf{0} \equiv (0, 0)$.

2.1 Preferences

Each agent $i \in N$ has a preference relation R_i over the consumption set $M \times \mathbb{R}$. In what follows, we assume that a preference R_i is complete and transitive, and satisfies the next properties.

Object monotonicity. For each pair $x_i, x'_i \in M$ with $x_i > x'_i$ and each $t_i \in \mathbb{R}$, it holds that $(x_i, t_i)\ P_i\ (x'_i, t_i)$.

Money monotonicity. For each $x_i \in M$ and each pair $t_i, t'_i \in \mathbb{R}$ with $t_i < t'_i$, it holds that $(x_i, t_i)\ P_i\ (x_i, t'_i)$.

Possibility of compensation. For each $z_i \in M \times \mathbb{R}$ and each $x_i \in M$, there is a pair $t_i, t'_i \in \mathbb{R}$ such that $(x_i, t_i)\ R_i\ z_i$ and $z_i\ R_i\ (x_i, t'_i)$.

Continuity. For each $z_i \in M \times \mathbb{R}$, the upper contour set at z_i, $\{z'_i \in M \times \mathbb{R} : z'_i\ R_i\ z_i\}$, and the lower contour set at z_i, $\{z'_i \in M \times \mathbb{R} : z_i\ R_i\ z'_i\}$, are both closed.

All of the above properties are standard in the literature, and do not need detailed explanations. A typical class of preferences satisfying the above four properties is denoted by $\mathcal{R}$.

Given a preference $R_i \in \mathcal{R}$, a bundle $z_i \in M \times \mathbb{R}$, and $x_i \in M$, by the possibility of compensation and the continuity, we can choose a payment t_i such that $(x_i, t_i)\ I_i\ z_i$. Moreover, by money monotonicity, such a payment must be unique. Let $V_i(x_i, z_i)$ denote the payment such that $(x_i, V_i(x_i, z_i))\ I_i\ z_i$, and we call it the **valuation of x_i at z_i for R_i**. Further, given $z_i \in M \times \mathbb{R}$ and $x_i \in M$, let $v_i(x_i, z_i) \equiv V_i(x_i, z_i) - V_i(0, z_i)$. We call $v_i(x_i, z_i)$ the **net valuation of x_i at z_i for R_i**. Note that given $z_i \in M \times \mathbb{R}$, $v_i(x_i, z_i) \geq 0$ for each $x_i \in M$.

The class of preferences that has been extensively studied in the literature is that of quasi-linear preferences.

Definition 1. A preference R_i is **quasi-linear** if for each pair $(x_i, t_i), (x'_i, t'_i) \in M \times \mathbb{R}$ and each $\delta \in \mathbb{R}$, $(x_i, t_i)\ I_i\ (x'_i, t'_i)$ implies $(x_i, t_i - \delta)\ I_i\ (x'_i, t'_i - \delta)$.

Let $\mathcal{R}^Q$ denote the class of quasi-linear preferences.

The next remark (i) states that under a quasi-linear preference, the net valuation does not depend on a bundle, and (ii) provides a utility representation of a quasi-linear preference.

Remark 1. Let $R_i \in \mathcal{R}^Q$.
(i). Let $z_i, z'_i \in M \times \mathbb{R}$. Then, for each $x_i \in M$, $v_i(x_i, z_i) = v_i(x_i, z'_i)$. Thus, we simply write $v_i(x_i)$ instead of $v_i(x_i, z_i)$.
(ii). For each pair $(x_i, t_i), (x'_i, t'_i) \in M \times \mathbb{R}$, $(x_i, t_i)\ R_i\ (x'_i, t'_i)$ if and only if $v_i(x_i) - t_i \geq v_i(x'_i) - t'_i$.

Given a preference $R_i \in \mathcal{R}$, a bundle $z_i \in M \times \mathbb{R}$, and a consumption level $x_i \in M \backslash \{m\}$, the *marginal (net) valuation of x_i at z_i for R_i* is $v_i(x_i + 1, z_i) - v_i(x_i, z_i)$. The next properties of marginal valuations play an important role in this paper. The first property is a standard one in the literature on the multi-unit auctions, and means that the object is substitutable for the agent as the margianal valuation of the object does not increase in the number of units of the object. The second property means that the object is an independent good for the agent as the marginal valuation of the object is independent of the consumption level of the object.

Definition 2. (i). A preference R_i **exhibits the nonincreasing marginal valuations** if for each $z_i \in M \times \mathbb{R}$ and each $x_i \in M \backslash \{0, m\}$,

$$v_i(x_i, z_i) - v_i(x_i - 1, z_i) \geq v_i(x_i + 1, z_i) - v_i(x_i, z_i).$$

(ii). A preference R_i **exhibits the constant marginal valuations** if for each $z_i \in M \times \mathbb{R}$ and each $x_i \in M \backslash \{0, m\}$,

$$v_i(x_i, z_i) - v_i(x_i - 1, z_i) = v_i(x_i + 1, z_i) - v_i(x_i, z_i).$$

Let $\mathcal{R}^{NI}$ and $\mathcal{R}^{C}$ denote the classes of preferences that exhibit nonincreasing and constant marginal valuations, respectively. Note that $\mathcal{R}^{C} \subsetneq \mathcal{R}^{NI}$.

In this paper, we will study a typical class of non-quasi-linear preferences that exhibit income effects. Here, we introduce the notion of nonnegative income effects. Although we do not introduce the income of an agent in our model explicitly, the zero payment ($t_i = 0$) can be regarded as the initial income of the agent. Then, each payment $t_i \in \mathbb{R}$ can be regarded as the *negative* of the (relative) income. Thus, the increase of the income is equivalent to the decrease of the payment. Then, the notion of the nonnegative income effect states that if an agent's income increases (or equivalently, if an agent's payment decreases), then he demands at least as many units of the object at the new income level as at the original one.

Definition 3. A preference R_i **exhibits the nonnegative income effect** if for each pair $x_i, x'_i \in M$ with $x_i > x'_i$, each pair $t_i, t'_i \in \mathbb{R}$ with $t_i > t'_i$, and each $\delta \in \mathbb{R}_{++}$, $(x_i, t_i)\ I_i\ (x'_i, t'_i)$ implies $(x_i, t_i - \delta)\ R_i\ (x'_i, t'_i - \delta)$.

Let $\mathcal{R}^{+}$ denote the class of preferences that exhibit nonnegative income effects. Note that $\mathcal{R}^{Q} \subsetneq \mathcal{R}^{+}$, i.e., any quasi-linear preference exhibits nonnegative income effects.

Remark 2. Let $R_i \in \mathcal{R}$. For each $x_i \in M \backslash \{m\}$, let $h_i(\cdot; x_i) : \mathbb{R} \to \mathbb{R}_{++}$ be such that for each $t_i \in \mathbb{R}$, $h_i(t_i; x_i) = V_i(x_i + 1, (x_i, t_i)) - t_i$. Then, $R_i \in \mathcal{R}^{+}$ if and only if for each $x_i \in M \backslash \{m\}$, $h_i(\cdot; x_i)$ is nondecreasing in t_i.

Throughout the paper, we consider preferences that exhibit both nonincreasing marginal valuations and nonnegative income effects. Thus, hereafter we assume that $\mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^{+}$. In order to emphasize our assumption on a class of preferences, we explicity record it.

Assumption. *For each class of preferences $\mathcal{R}$, we assume that $\mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^{+}$.*

2.2 Allocations and rules

Let $X \equiv \{(x_1, \ldots, x_n) \in M^n : \sum_{i \in N} x_i = m\}$ denote the set of (feasible) object allocations. Note that we assume each object is assigned to some agent. Given $x \in X$, let

$N^+(x) \equiv \{i \in N : x_i \neq 0\}$ denote the set of agents who receive the object (**winners**) at x.

A (**feasible**) **allocation** is an n-tuple $z \equiv (z_1, \ldots, z_n) \equiv ((x_1, t_1), \ldots, (x_n, t_n)) \in (M \times \mathbb{R})^n$ such that $(x_1, \ldots, x_n) \in X$. Let Z denote the set of allocations. We denote the object allocation and the payments at $z \in Z$ by $x \equiv (x_1, \ldots, x_n)$ and $t \equiv (t_1, \ldots, t_n)$, respectively. We may write $z \equiv (x, t) \in Z$.

A **preference profile** is an n-tuple $R \equiv (R_1, \ldots, R_n) \in \mathcal{R}^n$. A set $\mathcal{R}^n$ of preference profiles is a **domain**. Given $R \in \mathcal{R}^n$ and $i \in N$, let $R_{-i} \equiv (R_j)_{j \in N \setminus \{i\}}$.

Given $z \equiv (z_i)_{i \in N} \in (M \times \mathbb{R})^n$ and $R \in \mathcal{R}^n$, let $mv^k(R, z)$ denote the k-th highest marginal valuation among the set of marginal valuations at z for R: $\{v_i(x_i + 1, z_i) - v_i(x_i, z_i) : i \in N,\ x_i \in M \setminus \{m\}\}$. When $z_i = \mathbf{0}$ for each $i \in N$, we may simply write $mv^k(R) \equiv mv^k(R, z)$. Note that by Remark 1 (i), if $R \in (\mathcal{R}^Q)^n$, then $mv^k(R, z) = mv^k(R)$.

An **allocation rule**, or a **rule** for short, on $\mathcal{R}^n$ is a function $f : \mathcal{R}^n \to Z$. We may write $f \equiv (x^f, t^f)$, where $x^f : \mathcal{R}^n \to X$ and $t^f : \mathcal{R}^n \to \mathbb{R}^n$ are the object allocation and the payment rules associated with f, respectively. The consumption bundle of agent i under a rule f at a preference profile R is denoted by $f_i(R) = (x_i^f(R), t_i^f(R))$, where $x_i^f(R)$ and $t_i^f(R)$ are the consumption level of the object and the payment of agent i under the rule f, respectively.

2.3 Properties of rules

Next, we introduce the properties of rules.

First, we introduce a standard property of fairness introduced by Foley (1967) as a benchmark. It requires that no agent should envy other agents' bundles to his own.

No envy. For each $R \in \mathcal{R}^n$ and each pair $i, j \in N$, $f_i(R)\ R_i\ f_j(R)$.

Then, we introduce a new fairness property that plays a central role in this paper. Ftrst, we define a price faced by agent $i \in N$ at a preference profile $R \in \mathcal{R}^n$ under a rule f. In this paper, we regard the per-unit payment as a price. Given a rule f, a preference profile $R \in \mathcal{R}^n$, and $i \in N^+(x^f(R))$, let $p_i^f(R) \in \mathbb{R}$ be a **price of agent** i **for** R **under** f such that

$$p_i^f(R) = \frac{t_i^f(R)}{x_i^f(R)}.$$

That it, if agent i receives the object, then his price is determined as the per unit payment. Note that we define a price only for the winners since it is difficult to define the per-unit payments (prices) of the losers in a natural way. We discuss such a difficulty in detail in Section 5.1.1. Note also that in our definition of price, the prices may be different by agent.

Now, we are ready to define our property of fairness of prices. The next property is an extension of the *no envy* test (Foley, 1967) over bundles to prices, which requires that no agent should prefer other agents' *prices* in the sense that if he has a chance to buy some units of the object at other agents' prices, then he can get better off than his bundle.

No price envy. For each $R \in \mathcal{R}^n$ and each $i \in N$, there is no $j \in N^+(x^f(R))$ such that $(x_i, p^f_j(R)x_i)\ P_i\ f_i(R)$ for some $x_i \in M$.

The next remark states that *no price envy* is independent of (i.e., neither implies nor is implied by) *no envy*.

Remark 3. In general, *no price envy* does not imply *no envy*, and vice versa.

In Section 5.1.2, we will compare *no price envy* to related properties introduced by other authors in detail.

The third property requires that no agent should have an incentive to misreport his preference.

Strategy-proofness. For each $R \in \mathcal{R}^n$, each $i \in N$, and each $R'_i \in \mathcal{R}$, it holds that $f_i(R)\ R_i\ f_i(R'_i, R_{-i})$.

Given a preference profile $R \in \mathcal{R}^n$ and $i \in N$, a rule f on $\mathcal{R}^n$ is **manipulable at** R **by** i if there is $R'_i \in \mathcal{R}$ such that $f_i(R'_i, R_{-i})\ P_i\ f_i(R)$. Note that f is *strategy-proof* if and only if f is not manipulable at each preference profile by each agent.

The fourth property is concerned with the nonnegative payments, which requires that an agent who receives no object (a **loser**) should not receive money. We regard this condition as desirable since it excludes "dummy" agents interested only in participation subsidy.

No subsidy for losers. For each $R \in \mathcal{R}^n$ and each $i \in N \backslash N^+(x^f(R))$, $t^f_i(R) \geq 0$.

The last property states that a rule should select an allocation at which no agent gets worth off than the status quo bundle $\mathbf{0}$.

Individual rationality. For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R)\ R_i\ \mathbf{0}$.

3 The inverse uniform-price rule

In this section, we define the new class of rules that we call the inverse uniform-price rules.

First, we define the uniform-price rule for quasi-linear preferences as a benchmark.

Definition 4. A rule f on $\mathcal{R}^n \subseteq (\mathcal{R}^Q)^n$ is a **uniform-price rule** if it satisfies the following two conditions.
(i). For each $R \in \mathcal{R}^n$, $x^f(R) \in \underset{x \in X}{\arg\max} \sum_{i \in N} v_i(x_i)$.
(ii). There is a function $\pi^f : \mathcal{R}^n \to \mathbb{R}_+$ such that (ii-i) for each $R \in \mathcal{R}^n$, it holds that $\pi^f(R) \in [mv^{m+1}(R), mv^m(R)]$, and (ii-ii) for each $R \in \mathcal{R}^n$ and each $i \in N$, $t_i^f(R) = \pi^f(R)x_i^f(R)$.

In words, the first condition (i) states that the object is allocated so as to maximize the sum of net valuations, and the second condition (ii) states that there is a (common) price function such that (ii-i) the price is set between the highest losing marginal valuation and the lowest winning one, and (ii-ii) each agent pays the price for the object.

We introduce the subclass of the uniform-price rule for quasi-linear preferences. The minimum uniform-price rule chooses the highest losing marginal valuation as the price.

Definition 5. A rule f on $\mathcal{R}^n \subseteq (\mathcal{R}^Q)^n$ is a **minimum uniform-price rule** if it is a uniform-price rule associated with a price function $\pi^f : \mathcal{R}^n \to \mathbb{R}_+$ such that for each $R \in \mathcal{R}^n$, $\pi^f(R) = mv^{m+1}(R)$.

Next, we extend the uniform-price rule for quasi-linear preferences to non-quasi-linear preferences. There are several possible ways to extend the uniform-price rule to non-quasi-linear preferences, and one natural way to extend the uniform-price rule is to adopt the net valuations at $\mathbf{0}$ to the uniform-price rule.[11] In this paper, we introduce an alternative extension of the uniform-price rule to non-quasi-linear preferences. To this end, we will introduce the *inverse-demand set* of Shinozaki et al. (2020).

The next remark is a slight extension of Lemmas 9 and 11 of Shinozaki et al. (2020) for a preference with decreasing marginal valuations and positive income effects to a preference with nonincreasing marginal valuations and nonnegative income effects. The proof of it can be found in the supplementary material.

Remark 4. Let $R_i \in \mathcal{R}^{NI} \cap \mathcal{R}^+$.
(i) For each $x_i \in M \backslash \{0, m\}$, there is a unique payment $t^*(x_i) \in (0, V_i(x_i, \mathbf{0})]$ such that $V_i(x_i + 1, (x_i, t^*(x_i))) - t^*(x_i) = \frac{t^*(x_i)}{x_i}$.
(ii) For each $x_i \in M \backslash \{0, m-1, m\}$, $\frac{t^*(x_i)}{x_i} \geq \frac{t^*(x_i+1)}{x_i+1}$.

[11]Note that this extension is parallel to the way that the Vickrey rule (Vickrey, 1961) for quasi-linear preferences is extended to the generalized Vickrey rule (Saitoh and Serizawa, 2008; Sakai, 2008) for non-quasi-linear preferences.

Given $R_i \in \mathcal{R}$ and $x_i \in M$, the **inverse-demand set at** x_i **for** R_i is defied as the set $P(x_i; R_i) \equiv \{p \in \mathbb{R}_+ : (x_i, px_i)\ R_i\ (x'_i, px'_i) \text{ for each } x'_i \in M\}$. Note that $P(x_i; R_i)$ may be an empty set for some $R_i \in \mathcal{R}$ and $x_i \in M$. Further, given $R_i \in \mathcal{R}$, the **inverse-demand function** is a function $p(\cdot; R_i) : M \to \mathbb{R}_+ \cup \{\infty\}$ such that for each $x_i \in M$, $p(x_i; R_i) \equiv \inf P(x_i; R_i)$, where $\inf \varnothing \equiv \infty$.

The next proposition generalizes Corollary 1 of Shinozaki et al. (2020) for a preference with decreasing marginal valuations and positive income effects to a preference with nonincreasing marginal valuations and nonnegative income effects. It identifies the inverse-demand function of a preference with nonincreasing marginal valuations and nonnegative income effects. Since the proof of the next proposition is same as that of Shinozaki et al. (2020), we omit it.

Proposition 1. *Let $R_i \in \mathcal{R}^{NI} \cap \mathcal{R}^+$. We have $p(0; R_i) = V_i(1, \mathbf{0})$, $p(m; R_i) = 0$, and for each $x_i \in M \backslash \{0, m\}$, $p(x_i; R_i) = \frac{t^*(x_i)}{x_i}$.*

By Remark 4 and Proposition 1, given $R_i \in \mathcal{R}^{NI} \cap \mathcal{R}^+$, we can define a preference $R_i^{inv} \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ such that for each $x_i \in M \backslash \{m\}$, $v_i^{inv}(x_i + 1) - v_i^{inv}(x_i) = p(x_i, R_i)$. Given $R \in (\mathcal{R}^{NI} \cap \mathcal{R}^+)^n$ and $i \in N$, let $R^{inv} \equiv (R_j^{inv})_{j \in N}$ and $R_{-i}^{inv} \equiv (R_j^{inv})_{j \in N \backslash \{i\}}$.

The next remark states that (i) if a preference $R_i \in \mathcal{R}^{NI} \cap \mathcal{R}^+$ is quasi-linear, then the transformed preference R_i^{inv} from R_i coincides with the original preference R_i, and that (ii) if a preference exhibits constant marginal valuations, then the net valuations of the original preference at $\mathbf{0}$ coincide with those of the trnasformed preference.

Remark 5. Let $R_i \in \mathcal{R}^{NI} \cap \mathcal{R}^+$.
(i). If $R_i \in \mathcal{R}^Q$ then, $R_i^{inv} = R_i$.
(ii). If $R_i \in \mathcal{R}^C$, then $v_i^{inv}(x_i) = v_i(x_i, \mathbf{0})$ for each $x_i \in M$.

Now, we are ready to define the inverse uniform-price rule. It adopts the transformed preference profile R^{inv} from the original preference profile R to the uniform-price rule.

Definition 6. A rule f on $\mathcal{R}^n$ is an **inverse(-demand-based generalized) uniform-price rule** if it satisfies the following two conditions.
(i). For each $R \in \mathcal{R}^n$, $x^f(R) \in \arg\max_{x \in X} \sum_{i \in N} v_i^{inv}(x_i)$.
(ii). There is a function $\pi^f : \mathcal{R}^n \to \mathbb{R}_+$ such that (ii-i) for each $R \in \mathcal{R}^n$, it holds that $\pi^f(R) \in [mv^{m+1}(R^{inv}), mv^m(R^{inv})]$, and (ii-ii) for each $R \in \mathcal{R}^n$ and each $i \in N$, $t_i^f(R) = \pi^f(R) x_i^f(R)$.

Note that by Remark 5 (i), the inverse uniform-price rule coincides with the uniform-price rule on $(\mathcal{R}^{NI} \cap \mathcal{R}^Q)^n$.

Finally, we introduce the two subclass of the inverse uniform-price rule.

Definition 7. A rule f on $\mathcal{R}^n$ is a **minimum inverse uniform-price rule** if it is an inverse uniform-price rule associated with a price function $\pi^f : \mathcal{R}^n \to \mathbb{R}_+$ such that for each $R \in \mathcal{R}^n$, $\pi^f(R) = mv^{m+1}(R^{inv})$.

4 Main results

In this section, we provide the main results of this paper.

Our first result states that the inverse uniform-price rule is the only rule satisfying *no price envy* and *no subsidy for losers* on a domain contained in the domain with nonincreasing marginal valuations and nonnegative income effects.

Theorem 1. *Let $\mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^{+}$. A rule f on $\mathcal{R}^n$ satisfies no price envy and no subsidy for losers if and only if it is an inverse uniform-price rule.*

Note that Theorem 1 is free from richness of a domain, and it holds for each subdomain of the domain $(\mathcal{R}^{NI} \cap \mathcal{R}^{+})^n$.

Both the properties in Theorem 1 are indispensable in the sense that if we drop one of these properties, then Theorem 1 does not hold. The following examples demonstrate this fact on any domain $\mathcal{R}^n \subseteq (\mathcal{R}^{NI} \cap \mathcal{R}^{+})^n$.

Example 1 (Dropping no price envy). Let f be a rule on $\mathcal{R}^n$ such that for each $R \in \mathcal{R}^n$, $f_1(R) = (m, 0)$ and $f_i(R) = \mathbf{0}$ for each $i \in N\backslash\{1\}$. Then, if satisfies *no subsidy for losers* and *strategy-proofness*, but violates *no price envy*. □

Example 2 (Dropping no subsidy for losers). Let f be a rule on $\mathcal{R}^n$ such that for each $R \in \mathcal{R}^n$ and each $i \in N$, (i-i) $x_i^f(R) \in \{0, m\}$, (i-ii) if $x_i(R) = m$, then it holds that $v_i(m,(m,0)) \geq \max_{j \in N\backslash\{i\}} v_j(m,(m,0))$, (ii-i) $t_i^f(R) = -\max_{j \in N\backslash\{i\}} v_j(m,(m,0))$ if $x_i^f(R) = 0$, and (ii-ii) $t_i^f(R) = 0$ if $x_i^f(R) = m$. Then, it satisfies *no price envy* and *strategy-proofness*, but violates *no subsidy for losers*. □

Since *no subsidy for losers* is a minimal condition on nonnegatice payments, Theorem 1 states that *no price envy* almost fully characterizes the inverse uniform-price rule. Indeed, in many cases, it is reasonable to incorporate nonnegatieve payments in the model, and *no subsidy for losers* trivially holds in such a model. Thus, in a model that incorporates nonnegative payments, *no price envy* is a redefinition of the inverse uniform-price rule.

4.1 Maximal domain

Next, we turn to the non-manipulability of rules. In this subsection, we identify the maximal domain for the existence of a rule satisfying *no price envy*, *strategy-proofness*, and *no subsidy for losers*.

Definition 8. A domain $\mathcal{R}^n$ is a **maximal domain** for a list of properties if (i) there is a rule on $\mathcal{R}^n$ satisfying the list of properties, and (ii) for each $\mathcal{R}' \supsetneq \mathcal{R}$, there is no rule on $(\mathcal{R}')^n$ satisfies the list of properties.[12]

[12]Note that by our assumption, $\mathcal{R}, \mathcal{R}' \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^{+}$.

The next definition inntroduces a preference whose valuation at $\mathbf{0}$ exhibits constant marginal valuations.

Definition 9. A preference $R_i \in \mathcal{R}^{NI} \cap \mathcal{R}^+$ **exhibits the partly constant marginal valuations** if $v_i(x_i+1, \mathbf{0}) - v_i(x_i, \mathbf{0}) = v_i(x_i, \mathbf{0}) - v_i(x_i-1, \mathbf{0})$ for each $x_i \in M\backslash\{0, m\}$.

Let $\hat{\mathcal{R}}^C$ denote the class of preferences that exhibits partly constant marginal valuations. Clearly, $\hat{\mathcal{R}}^C \supseteq \mathcal{R}^C \cap \mathcal{R}^+$. Thus, $\hat{\mathcal{R}}^C \supseteq \mathcal{R}^C \cap \mathcal{R}^Q$. Moreover, since we restrict our attention to preferences with both nonincreasing marginal valuations and nonnegative income effects, we must have $\hat{\mathcal{R}} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^+$. Note that by Remark 5 (ii), $R_i \in \hat{\mathcal{R}}^C$ if and only if $R_i^{inv} \in \mathcal{R}^C$.

The next proposition states that if we add one arbitrary preference that does not exhibit partly constant marginal valuations to the quasi-linear domain with constant marginal valuations, then no rule on the expanded domain satisfies *no price envy*, *strategy-proofness*, and *no subsidy for losers*.

Proposition 2. *Let $R_0 \in (\mathcal{R}^{NI} \cap \mathcal{R}^+)\backslash\hat{\mathcal{R}}$. Let $\mathcal{R}$ be such that $\mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^+$ and $\mathcal{R} \supseteq (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$. Then. no rule on $\mathcal{R}^n$ satisfies no price envy, strategy-proofness, and no subsidy for losers.*

As a result of Proposition 2, we obtain the following.

Corollary 1. *Let $\mathcal{R} \in \{\mathcal{R}^{NI} \cap \mathcal{R}^Q, \mathcal{R}^{NI} \cap \mathcal{R}^+\}$. Then, no rule on $\mathcal{R}^n$ satisfies no price envy, strategy-proofness, and no subsidy for losers.*

Proposition 2 furhter serves to obtain a maximal domain result for *no price envy*, *strategy-proofness*, and *no subsidy for losers*. The next theorem stats that if a domain includes all quasi-linear preferences with constant marginal valuations, then the domain with partly constant marginal valuations is the unique maximal domain for *no price envy*, *strategy-proofness*, and *no subsidy for losers*.

Theorem 2. *Let $\mathcal{R}$ be such that $\mathcal{R}^C \cap \mathcal{R}^Q \subseteq \mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^+$. Then, $\mathcal{R}^n$ is a maximal domain for no price envy, strategy-proofness, and no subsidy for losers if and only if $\mathcal{R} = \hat{\mathcal{R}}^C$*

Note that by Theorem 1, the minimum inverse uniform-price rule on $(\hat{\mathcal{R}}^C)^n$ satisfies both *no price envy* and *no subsidy for losers*. Moreover, we show that it satisfies *strategy-proofness* on $(\hat{\mathcal{R}}^C)^n$. Thus, there is a rule satisfying the three properties in Theorem 2 on $(\hat{\mathcal{R}}^C)^n$. Then, we exploit Proposition 2 to show the maximal domain property of $(\hat{\mathcal{R}}^C)^n$ and the uniqueness of the maximal domain.

Note also that the assumption that for any class of preferences $\mathcal{R}$, $\mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^+$ is necessary for a maximal domain result (Theorem 2). Indeed, if we add one arbitrary preference with increasing marginal valuations to the domain $(\mathcal{R}^C \cap \mathcal{R}^Q)^n$, then the *generalized Vickrey rule* satisfies *no price envy*, *strategy-proofness*, and *no subsidy for losers*.[13]

The next result further states that on a domain that includes the quasi-linear domain with constant marginal valuations and is contained by the domain with partly constant marginal valuations, the minimum inverse uniform-price rule is the only rule satisfying *no price envy*, *strategy-proofness*, and *no subsidy for losers*.

Theorem 3. *Let $\mathcal{R}$ be such that $\mathcal{R}^C \cap \mathcal{R}^Q \subseteq \mathcal{R} \subseteq \hat{\mathcal{R}}^C$. A rule on $\mathcal{R}^n$ satisfies no price envy, strategy-proofness, and no subsidy for losers if and only if it is a minimum inverse uniform-price rule.*

By using the same examples as in Theorem 1, we can show that both *no price envy* and *no subsidy for losers* are indispensable for Theorem 3. Moreover, *strategy-proofness* is also indispensable for Theorem 3 because inverse uniform-price rules which are different from the minimum one satisfies both *no price envy* and *no subsidy for losers*, but violates *strategy-proofness*.

4.2 Minimal manipulability

Although Theorem 2 states that the assumption of (partly) constaint marginal valuations is not only suffienct but also necessary in a maxiaml domain sense for the existence of a rule satisfying *no price envy*, *strategy-proofness*, and *no subsidy for losers*, in many situations it is rather natural and plausible to assume that agents have preferences with nonincreasing marginal valuations. However, Corollary 1 states that there is no such a desirable rule when agents have preferences with nonincreasing marginal valuations. Given an impossibility theorem for *no price envy*, *strategy-proofness*, and *no subsidy for losers* (Corollary 1), we must give up one of the three properties. As already stated in Section 1.2, we give up *strategy-proofness* instead of the other two properties in this paper. In terms of non-manipulability, a (*non-strategy-proof*) rule is more desirable than another rule if it is less manipulable than the other rule. Thus, we investigate a rule that satisfies *no price envy* and *no subsidy for losers*, and is *minimally manipulable* among the class of rules satisfying both the properties.

Our manipulability measure extends Pathak and Sönmez (2013)'s one. They introduce a manipulability measure that takes the gains from manipulations into account for quasi-linear preferences, and a rule is said to be *as intensively and strongly manipuble as* another rule g if (i) whenever the latter rule is manipulable at a preference profile by an agent, the former rule is as well at the preference profile by him, and (ii) the gain from manipulation of the former rule is greater than or equal to that of the latter rule.

[13] For the formal definition of the generalized Vickrey rule, see, for example, Shinozaki et al. (2020).

Now, we formalize the above manipulability measure in our setting with non-quasi-linear preferences. Given a rule f on $\mathcal{R}^n$, $R \in \mathcal{R}^n$, $i \in N$, and $R'_i \in \mathcal{R}$, the **gain from manipualtion R'_i at R by i under f** is defined as

$$G_i^f(R'_i; R) \equiv V_i(x_i^f(R'_i, R_{-i}), f_i(R)) - t_i^f(R'_i, R_{-i}).$$

The next remark states that (i) the gain from manipulation is positive if and only if the manipulation is successful, and (ii) if a preference is quasi-linear, our notion of gain from manipulation coincides with that of Pathak and Sönmez (2013).

Remark 6. Let f be a rule on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$, $i \in N$, and $R'_i \in \mathcal{R}$.
(i). We have $G_i^f(R'_i; R) > 0$ if and only if $f_i(R'_i, R_{-i})\ P_i\ f_i(R)$.
(ii). If $R_i \in \mathcal{R}^Q$, then

$$G_i^f(R'_i; R) = v_i(x_i^f(R'_i, R_{-i})) - t_i^f(R'_i, R_{-i}) - (v_i(x_i^f(R)) - t_i^f(R)).$$

The next definition generalizes the "as intensely and strongly manipulability" relation of Pathak and Sönmez (2013) for quasi-linear preferences to non-quasi-linear preferences.[14]

Definition 10. A rule f on $\mathcal{R}^n$ is **at least as manipulable as** another rule g on $\mathcal{R}^n$ if for each $R \in \mathcal{R}^n$, each $i \in N$, each $R'_i \in \mathcal{R}$, and each $\varepsilon \in \mathbb{R}_{++}$, whenever $g_i(R'_i, R_{-i})\ P_i\ g_i(R)$, there is $R''_i \in \mathcal{R}$ such that $f_i(R''_i, R_{-i})\ P_i\ f_i(R)$ and $G_i^f(R''_i; R) > G_i^g(R'_i; R) - \varepsilon$.

We will discuss the relation between our manipulabilty measure to ones proposed by other authors in detail in Section 5.1.2.

The next proposition is a key building block of the main result of this subsection, which states that under a class of rules satisfying both *no price envy* and *no subsidy for losers*, a rule is at least as manipulable as another rule if and only if each agent weakly prefers the bundle of the latter rule to that of the former rule for each preference profile.

Proposition 3. *Let $\mathcal{R}$ be such that $\mathcal{R}^{NI} \cap \mathcal{R}^Q \subseteq \mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^+$. Let f, g be a pair of rules on $\mathcal{R}^n$ satisfying both no price envy and no subsidy for losers. Then, g is at least as manipulable as f if and only if $f_i(R)\ R_i\ g_i(R)$ for each $R \in \mathcal{R}^n$ and each $i \in N$.*

Next, we investigate the minimally manipulable rule among the class of rules satisfying *no price envy* and *no subsidy for losers* according to our manipulability measure.

Definition 11. A rule f on $\mathcal{R}^n$ is **minimally manipulable among the class of rules** if (i) f is in the class, and (ii) for each rule g on $\mathcal{R}^n$ in the class, g is at lesat as manipulable as f.

[14]For simplicity of notation, we simply say that a rule f is *at least as manipulable as* another rule g instead of that f is *as intensely and strongly manipulable as* g as in Pathak and Sönmez (2013). Clearly, this will create no confusion in this paper.

If a rule f on $\mathcal{R}^n$ is *strategy-proof*, then it is minimally manipulable among any class of rules on $\mathcal{R}^n$ to which the rule belongs since no agent can manipulate the rule f at any $R \in \mathcal{R}^n$. Thus, *strategy-proofness* implies the minimal manipulability among a given class of rules.

The next result is a main result of this subsection, which states that among the class of rules satisfying *no price envy* and *no subsidy for losers*, the minimum inverse uniform-price rule is the only minimally manipulable rule.

Theorem 4. *Let $\mathcal{R}$ be such that $\mathcal{R}^{NI} \cap \mathcal{R}^{Q} \subseteq \mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^{+}$. A rule f on $\mathcal{R}^n$ is minimally manipulable among the class of rules on $\mathcal{R}^n$ satisfying no price envy and no subsidy for losers if and only if it is a minimum inverse uniform-price rule.*

Recall that Theorem 1 states that the class of rules satisfying *no price envy* and *no subsidy for losers* coincides with that of the inverse uniform-price rules. Thus, as a corollary of Theorems 1 and 4, we obtain the following.

Corollary 2. *Let $\mathcal{R}$ be such that $\mathcal{R}^{NI} \cap \mathcal{R}^{Q} \subseteq \mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^{+}$. A rule f on $\mathcal{R}^n$ is minimally manipulable among the class of inverse uniform-price rules on $\mathcal{R}^n$ if and only if it is a minimum inverse uniform-price rule.*

5 Discussion

In this section, we discuss some topics about *no price envy* and our manipulability measure.

5.1 No price envy

5.1.1 Prices for losers

In our formulation of *no price envy*, we do not take the prices of the losers into account. Here, we show the difficulty of defining the prices of the losers by pursuing some possible ways to formulate the prices.

One possible way to define the prices of the losers is to set them at zero. This formulation seems plausible since such agents pay no money for the object. However, if we set the prices of the losers at zero, then *no price envy* gets so strong that is incompatible with a minimal requirement of *no subsidy for losers* since the winners must envy the losers' prices, and is no longer an attractive property.

Instead, if we follow the interpretation of *no price envy* that agents choose an optimal bundle given his price, it seems reasonable to think that a loser faces so high price that his optimal choice at the price is to receive nothing. Now, we again confront the difficulty of defining the "high" prices that the losers face in a natural way. In our view, there is no natural way to formulate the "high" prices of the losers which keeps *no price envy* attractive.[15]

Thus, instead of defining the prices of the losers, we choose not to define them.

5.1.2 Comparison to related properties

Here, we compare our *no price envy* to related properties introduced by other authors.

Varian (1976) introduces *opportunity fairness* in the exchange economy model, which requires that each agent prefers his bundle to any bundle in other agents' budget sets whose price is determined as the *exogenously given* equilibrium price.[16] Our *no price envy* is different from *opportunity fairness* of Varian (1976) since the prices in our property is determined *endogenously by the rule*, but the price in *opportunity fairness exogenously* determined at the equilibrium price. In particular, *opportunity fairness* implies *no price envy*, but the converse is not true.

Thomson (1994) generalizes *opportunity fairness* of Varian (1976), and introduces the property of fairness that he calls *no envy of opportunities*. A rule f satisfies *no envy of opportunities* if there is a a family of opportunity sets $\mathcal{B}$ such that for each $R \in \mathcal{R}$, the following two conditions hold: (i) for each $i \in N$, there is a choice set $B_i \in \mathcal{B}$ of agent i, and (ii) for each pair $i, j \in N$, there is no $z_j \in B_j$ such that $z_j \ P_i \ f_i(R)$. Note that *no envy of opportunities* is a fairly general property, and it subsumes *opportunity fairness* of Varian (1976). However, *no price envy* neither implies nor is implied by *no envy of opportunities* since *no price envy* does not take the losers' prices into account.

5.2 Other manipulability measures

Here, we compare our manipulability measure to ones introduced by other authors.

Day and Milgrom (2008) consider the heterogeneous objects model with quasi-linear preferences, and introduce the gain from manipulations that they call the incentive profile. In our model, the *incentive profile* of a rule f on $\mathcal{R}^n$ at $R \in \mathcal{R}^n$ is defined as the profile $(\varepsilon_i^f(R))_{i \in N}$ such that for each $i \in N$,

$$\varepsilon_i^f(R) \equiv \sup_{R_i' \in \mathcal{R}} G_i^f(R_i'; R).$$

We say that a rule f on $\mathcal{R}^n$ is *DM-at least as manipulable as*[17] another rule g on $\mathcal{R}^n$ if for each $R \in \mathcal{R}^n$ and each $i \in N$, $\varepsilon_i^f(R) \geq \varepsilon_i^g(R)$.

[15] An example of a "high" price of a loser $i \notin N^+(x^f(R))$ for R under f is $p_i^f(R) = p(R_i; 0)$, i.e., the inverse-demand of 0 unit for R_i. However, again *no price envy* becomes incompatible with *no subsidy for loses* if we incorporate such losers' prices to *no price envy*.

[16] In our model, a price $p \in \mathbb{R}$ is an *equilibrium price* for $R \in \mathcal{R}^n$ if there is $z \equiv (x, t) \in Z$ such that (i) for each $i \in N$ and each $x_i' \in X$, $(x_i, px_i) \ R_i \ (x_i', px_i')$, and (ii) for each $i \in N$, $t_i = px_i$.

[17] Note that "DM" refers to Day and Milgrom (2008).

Chen et al. (2016) consider the (many-to-many) matching with contracts model, and compare stable rules according to their manipulability measure. Since the manipulability measure in Chen et al. (2016) is for the finite model, we need to slightly modify their measure so as to be comparable to our measure. To this end, we introduce a distance function to the consumption set $M \times \mathbb{R}$. Let $d : (M \times \mathbb{R})^2 \to \mathbb{R}$ be a distance function such that for each pair $(x_i, t_i), (x'_i, t'_i) \in M \times \mathbb{R}$[18]

$$d((x_i, t_i), (x'_i, t'_i)) = |x_i - x'_i| + |t_i - t'_i|.$$

According to the manipulability measure of Chen et al. (2016), a rule is at least as manipulable as another rule if whenever an agent can manipulate the latter rule at a preference profile and achieves a certain bundle, he can also manipulate the former rule and achieve the same bundle as a result of the manipulation. Now, we slightly modify the manipulability measure of Chen et al. (2016) to be able to handle the issue of tie-breaking, and say that a rule is at least as manipulable as another rule if whenever an agent can achieve a bundle as a result of the manipulation of the latter rule, he can achieve *almost the same bundle* as a consequence of the manipulation of the former rule.

Formally, we say that a rule f on $\mathcal{R}^n$ is *C-at least as manipulable as*[19] another rule g on $\mathcal{R}^n$ if for each $R \in \mathcal{R}^n$, each $i \in N$, each $R'_i \in \mathcal{R}$, and each $\varepsilon \in \mathbb{R}_{++}$, whenever $g_i(R'_i, R_{-i})\ P_i\ g_i(R)$, there is $R''_i \in \mathcal{R}$ such that $f_i(R''_i, R_{-i})\ P_i\ f_i(R)$ and

$$d(f_i(R''_i, R_{-i}), g_i(R'_i, R_{-i})) < \varepsilon.$$

We define the minimal manipulabilities according to the manipulability measures introduced by other authors.

Definition 12. (i). A rule f on $\mathcal{R}^n$ is DM-**minimally manipulable among the class of rules** if (i) f is in the class, and (ii) for each rule g on $\mathcal{R}^n$ in the class, g is DM-at lesat as manipulable as f.
(ii). A rule f on $\mathcal{R}^n$ is C-**minimally manipulable among the class of rules** if (i) f is in the class, and (ii) for each rule g on $\mathcal{R}^n$ in the class, g is C-at lesat as manipulable as f.

The next remark states that our manipulability measure is equivalent to the one introduced by Day and Milgrom (2008).

Remark 7. Let f, g be a pair of rules on $\mathcal{R}^n$. Then, the following statements are equivalent.

[18] The following discussion is valid if we replace the distance function d with the Euclidean distance function or any other equivalent distance function. Our choice of a distance function is only for simplicity of the discussion.

[19] Note that "C" refers to Chen et al. (2016).

(i). f is at least as manipulable as g.
(ii). f is DM-at least as manipulable as g.

Thus, we obtain the next result as a corollary of Theorem 4.

Corollary 3. *Let $\mathcal{R}$ be such that $\mathcal{R}^{NI} \cap \mathcal{R}^{Q} \subseteq \mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^{+}$. A rule f on $\mathcal{R}^n$ is DM-minimally manipulable among the class of rules on $\mathcal{R}^n$ satisfying no price envy and no subsidy for losers if and only if it is a minimum inverse uniform-price rule.*

The next example shows that our manipulability measure is not equivalent to C-manipulability measure over the class of rules satisfying both *no price envy* and *no subsidy for losers*. This contrasts with Theorem 3 of Chen et al. (2016), which states that in their matching with contracts model, C-manipulability measure is equivalent to DM-manipulability measure.

Example 3. Let $\mathcal{R}$ be such that $\mathcal{R}^{NI} \cap \mathcal{R}^{Q} \subseteq \mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^{+}$. Let $R^* \in (\mathcal{R}^{NI} \cap \mathcal{R}^{Q})^n$ be such that (i) $v_1^*(x_1+1) - v_1^*(x_1) = 100m$ for each $x_1 \in M \backslash \{m\}$, and (ii) for each $i \in N \backslash \{1\}$, $v_i^*(x_i+1) - v_i^*(x_i) = 1$ for each $x_i \in M \backslash \{m\}$. Let $R_1' \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that $v_1'(x_1+1) - v_1'(x_1) = 5$ for each $x_1 \in M \backslash \{m\}$. Then, $mv^{m+1}(R^*) = mv^{m+1}(R_1', R_{-1}^*) = 1$, $mv^m(R^*) = 100m$, and $mv^m(R_1', R_{-1}^*) = 5$. Then, for each inverse uniform-price rule f on $\mathcal{R}^n$, it holds that $x_1^f(R^*) = x_1^g(R^*) = x_1^f(R_1', R_{-1}^*) = m$, $\pi^f(R^*), \pi^g(R^*) \in [1, 100m]$ and $\pi^f(R_1', R_{-1}^*) \in [1, 5]$.

Let f, g be a pair of inverse uniform-price rules on $\mathcal{R}^n$ such that (i) $\pi^f(R^*) = \pi^g(R^*) = 5$, (ii) $\pi^f(R_1', R_{-1}^*) = 3$ and $\pi^g(R_1', R_{-1}^*) = 4$, and (iii) for each $R \in \mathcal{R}^n \backslash \{R^*, (R_1', R_{-1}^*)\}$, $\pi^f(R) = \pi^g(R) = mv^{m+1}(R^{inv})$. Then, $f_i(R)\ R_i\ g_i(R)$ for each $R \in \mathcal{R}^n$ and each $i \in N$. Thus, by Proposition 3, g is at least as manipulable as f.

Note that $f_1(R_1', R_{-1}^*) = (m, 3m)\ P_1^*\ (m, 5m) = f_1(R^*)$. Let $\varepsilon \in \mathbb{R}_{++}$ be such that $\varepsilon < m$. Let $R_1'' \in \mathcal{R}$ be such that $g_1(R_1'', R_{-1})\ P_1\ g_1(R) = (m, 5m)$. Then, by the definition of R_1^* and $t_1^g(R_1'', R_{-1}) \geq 0$, $x_1^g(R_1'', R_{-1}) = m$. Then, by the definition of π^g, either $\pi^g(R_1'', R_{-1}^*) = 4$ or $\pi^g(R_1'', R_{-1}^*) = 1$. Thus, we have

$$d(g_1(R_1'', R_{-1}^*), f_1(R_1', R_{-1}^*)) \geq |5m - 4m| = m > \varepsilon.$$

Thus, g is not C-at least as manipulable as f, and the the equivalnece between our manipulability measure and C-manipulability measure does not hold. □

The next theorem, however, states that even though C-manipulability measure is not equivalent to ours as the above example demonstrates, the minimum inverse uniform-price rule is still the only minimally manipulable rule among the class of rules satisfying *no price envy* and *no subsidy for losers* according to C-manipulability measure. The proof of the next theorem can be found in the supplymentary material.

Theorem 5. *Let $\mathcal{R}$ be such that $\mathcal{R}^{NI} \cap \mathcal{R}^{Q} \subseteq \mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^{+}$. A rule f on $\mathcal{R}^n$ is C-minimally manipulable among the class of rules on $\mathcal{R}^n$ satisfying no price envy and no subsidy for losers if and only if it is a minimum inverse uniform-price rule.*

In the supplementary material, we further discuss the relationship between our manipulability measure and the one introduced by Pathak and Sönmez (2013) which does not take gains from manipulations into account.

6 Conclusion

In this paper, we propose a new property of fairness: *no-price-envy*, and investigate its implications in conjunction with the other desirable properties. We identify the unique maximal domain for *no price envy*, *strategy-proofness*, and *no subsidy for losers*, and show that on the domain the minimum inverse uniform-price rule is the unique rule satisfying the three properties. Our maximal domain result implies that in the case of nonincreasing marginal valuations, no rule satisfies *no price envy*, *strategy-proofness*, and *no subsidy for losers*, but we show in such a case that the minimum inverse uniform-price rules is the only minimally manipulable rule among the class of rules satisfying *no-price-envy* and *no subsidy for losers*. These results provides a rationale for the use of the minimum uniform-price rule that is one of the most popular auction rules in real-life auctions in terms of fairness and non-manipulability.

Appendix

A Basic lemmas

In this section, we prove the basic lemmas that will be used to prove the results.

The next lemma states that if a rule satisfies *no price envy*, then it satisfies *individual rationality*.

Lemma 1 (Individual rationality). *Let f be a rule on $\mathcal{R}^n$ satisfying no price envy. Then, it satisfies individual rationality.*

Proof. Let $R \in \mathcal{R}^n$ and $i \in N$. First, if $x_i^f(R) > 0$, then *no price envy* implies that $f_i(R)\ R_i (0, p_i^f(R)0) = \mathbf{0}$. Second, if $x_i^f(R) = 0$, then there is $j \in N \backslash \{i\}$ such that $x_j^f(R) > 0$ by the feasibility. Then, *no price envy* implies that $f_i(R)\ R_i\ (0, p_j^f(R)0) = \mathbf{0}$, as desired. □

The next lemma states that if a rule satisfies *no price envy* and *no subsidy for losers*, then a loser makes no monetary transfer.

Lemma 2 (Zero payment for losers). *Let f be a rule on $\mathcal{R}^n$ satisfying no price envy and no subsidy for losers. Let $R \in \mathcal{R}^n$ and $i \in N$. If $x_i^f(R) = 0$, then $t_i^f(R) = 0$.*

Proof. Suppose $x_i^f(R) = 0$. By *no subsidy for losers*, $t_i^f(R) \geq 0$. By Lemma 1, f satisfies *individual rationality*. Then, it holds that $(0, t_i^f(R)) = f_i(R)\ R_i\ \mathbf{0}$, which implies $t_i^f(R) \leq 0$. Thus, $t_i^f(R) = 0$. □

The following lemma states that under a rule satisfying *no price envy*, winners face the equal prices.

Lemma 3 (Equal prices for winners). *Let f be a rule on $\mathcal{R}^n$ satisfying no price envy. Let $R \in \mathcal{R}^n$ and $i, j \in N^+(x^f(R))$. Then, $p_i^f(R) = p_j^f(R)$.*

Proof. Suppose by contradiction that $p_i^f(R) \neq p_j^f(R)$. If $p_i^f(R) > p_j^f(R)$, then it holds that $t_i^f(R) > p_j^f(R) x_i^f(R)$. Then, $(x_i^f(R), p_j^f(R) x_i^f(R))\ P_i\ f_i(R)$. However, this contradicts *no price envy*. Instead, if $p_i^f(R) < p_j^f(R)$, then $t_j^f(R) > p_i^f(R) x_j^f(R)$. Then,we have $(x_j^f(R), p_i^f(R) x_j^f(R))\ P_j\ f_j(R)$, which contradicts *no price envy*. □

The next lemma provides a characterization of an efficient allocation for preferences with nonincreasing marginal valuations.

Lemma 4. *Let $R \in (\mathcal{R}^{NI})^n$. Let $z \equiv (x, t) \in Z$. Then, $x \in \arg\max_{x' \in X} \sum_{i \in N} v_i(x'_i, z_i)$ if and only if for each pair $i, j \in N$ with $x_i \neq 0$ and $x_j \neq m$,*

$$v_i(x_i, z_i) - v_i(x_i - 1, z_i) \geq v_j(x_j + 1, z_j) - v_j(x_j, z_j).$$

Proof. First, we show the "if" part. Suppose $x \in \arg\max_{x' \in X} \sum_{i \in N} v_i(x'_i, z_i)$. Let $i, j \in N$ be a pair such that $x_i \neq 0$ and $x_j \neq m$. By contradiction, suppose that

$$v_i(x_i, z_i) - v_i(x_i - 1, z_i) < v_j(x_j + 1, z_j) - v_j(x_j, z_j). \tag{1}$$

Note that by $R_i, R_j \in \mathcal{R}^{NI}$, $i \neq j$. Let $x' \in X$ be such that $x'_i = x_i - 1$, $x'_j = x_j + 1$, and $x'_k = x_k$ for each $k \in N \backslash \{i, j\}$. Then, by (1),

$$\sum_{k \in N} v_k(x'_k, z_k) - \sum_{k \in N} v_k(x_k, z_k) = v_j(x_j + 1, z_j) - v_j(x_j, z_j) - (v_i(x_i, z_i) - v_i(x_i - 1, z_i)) > 0,$$

or $\sum_{k \in N} v_k(x'_k, z_k) > \sum_{k \in N} v_k(x_k, z_k)$. However, this contradicts $x \in \arg\max_{x' \in X} \sum_{k \in N} v_k(x'_k, z_k)$.

Next, we show the "only if" part. Suppose that for each pair $i, j \in N$ with $x_i \neq 0$ and $x_j \neq m$,

$$v_i(x_i, z_i) - v_i(x_i - 1, z_i) \geq v_j(x_j + 1, z_j) - v_j(x_j, z_j). \tag{2}$$

Let $x' \in X$. Let $N^> \equiv \{i \in N : x_i > x'_i\}$, $N^= \equiv \{i \in N : x_i = x'_i\}$, and $N^< \equiv \{i \in N : x_i < x'_i\}$. Note that $\{N^>, N^=, N^<\}$ is a partition of N. Note also that for each $i \in N^>$, $x_i \neq 0$, and for each $i \in N^<$, $x_i \neq m$.

By the feasibility, $\sum_{i\in N} x_i = m = \sum_{i \in N} x'_i$. Thus, we have

$$\begin{aligned} 0 &= \sum_{i\in N}(x_i - x'_i) \\ &= \sum_{i\in N^>}(x_i - x'_i) + \sum_{i \in N^<}(x_i - x'_i), \end{aligned}$$

or

$$\sum_{i\in N^>}(x_i - x'_i) = \sum_{i\in N^<}(x'_i - x_i). \tag{3}$$

Then,

$$\begin{aligned} &\sum_{i\in N} v_i(x_i, z_i) - \sum_{i \in N} v_i(x'_i, z_i) \\ &= \sum_{i\in N^>}(v_i(x_i, z_i) - v_i(x'_i, z_i)) - \sum_{i\in N^<}(v_i(x'_i, z_i) - v_i(x_i, z_i)) \\ &\geq \sum_{i\in N^>}(x_i - x'_i)(v_i(x_i, z_i) - v_i(x_i - 1, z_i)) - \sum_{i\in N^<}(x'_i - x_i)(v_i(x_i+1, z_i) - v_i(x_i, z_i)) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{by } R \in (\mathcal{R}^{NI})^n) \\ &\geq \sum_{i\in N^>}(x_i - x'_i)\Big(\min_{i\in N^>}\{v_i(x_i, z_i) - v_i(x_i - 1, z_i)\}\Big) - \sum_{i\in N^<}(x'_i - x_i)\Big(\max_{i\in N^<}\{v_i(x_i+1, z_i) - v_i(x_i, z_i)\}\Big) \\ &\geq 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{by (2) and (3)}) \end{aligned}$$

or $\sum_{i\in N} v_i(x_i, z_i) \geq \sum_{i\in N} v_i(x'_i, z_i)$. □

The next lemma gives both the lower and the upper bounds of the marginal valuations at an efficient allocation.

Lemma 5. *Let $R \in (\mathcal{R}^{NI})^n$ and $z \equiv (x, t) \in Z$ be such that $x \in \arg\max_{x' \in X} \sum_{i \in N} v_i(x'_i, z_i)$. Let $i \in N$.*
(i). If $x_i \neq 0$, then $v_i(x_i, z_i) - v_i(x_i - 1, z_i) \geq mv^m(R, z)$.
(ii). If $x_i \neq m$, then $v_i(x_i + 1, z_i) - v_i(x_i, z_i) \leq mv^{m+1}(R, z)$.

Proof. First, we show (i). Suppose $x_i \neq 0$. For each $j \in N$, let $M_j \equiv \{x'_j \in M \backslash \{m\} : x'_j \geq x_j\}$. Then, for each $j \in N$, $|M_j| = m - x_j$. Thus,

$$\sum_{j\in N}|M_j| = nm - \sum_{j\in N} x_j = nm - m = (n-1)m,$$

where the second equality follows from the feasibility. Further, for each $j \in N$, let

$$\overline{M}_j \equiv \Big\{x'_j \in M\backslash\{m\} : v_j(x'_j+1, z_j) - v_j(x'_j, z_j) \le v_i(x_i, z_i) - v_i(x_i - 1, z_i)\Big\}.$$

By $x_i \neq 0$ and $R \in (\mathcal{R}^{NI})^n$, Lemma 4 implies that for each $j \in N$, $M_j \subseteq \overline{M}_j$. Thus, $|M_j| \le |\overline{M}_j|$ for each $j \in N$. Moreover, by $x_i - 1 \in \overline{M}_i$, $|M_i| < |\overline{M}_i|$. Then,

$$(n-1)m = \sum_{j\in N} |M_j| < \sum_{j \in N} |\overline{M}_j|.$$

This means that there are more than $(n-1)m$ marginal valuations at z that is no greater than $v_i(x_i, z_i) - v_i(x_i - 1, z_i)$ among all the nm marginal valuations at z. Thus, by $nm - (n-1)m = m$, we obtain $v_i(x_i) - v_i(x_i - 1) \ge mv^m(R, z)$.

Then, we show (ii). Suppose $x_i \neq m$. For each $j \in N$, let $M_j \equiv \{x'_j \in M\backslash\{0\} : x'_j \le x_j\}$. Then, $|M_j| = x_j$ for each $j \in N$. Thus, by the feasibility,

$$\sum_{j\in N} |M_j| = \sum_{j\in N} x_j = m.$$

For each $j \in N$, let

$$\overline{M}_j \equiv \Big\{x'_j \in M\backslash\{0\} : v_j(x'_j, z_j) - v_j(x'_j - 1, z_j) \ge v_i(x_i + 1, z_i) - v_i(x_i, z_i)\Big\}.$$

By $x_i \neq m$ and $R \in (\mathcal{R}^{NI})^n$, Lemma 4 implies $M_j \subseteq \overline{M}_j$ for each $j \in N$. Moreover, by $\{x_i + 1\} \cup M_j \subseteq \overline{M}_i$, $M_i \subsetneq \overline{M}_i$. Thus,

$$m = \sum_{j\in N} |M_j| < \sum_{j\in N} |\overline{M}_j|.$$

This implies that $v_i(x_i + 1, z_i) - v_i(x_i, z_i) \le mv^{m+1}(R, z)$. □

The next lemma is a slight generalization of Lemma 10 of Shinozaki et al. (2020). Since the proof is same as that of Shinozaki et al. (2020), we omit it.

Lemma 6. *Let $R_i \in \mathcal{R}^{NI} \cap \mathcal{R}^+$ and $x_i \in M\backslash\{0, m\}$. For each $t_i \in \mathbb{R}_+$, $t_i < t^*(x_i)$ if and only if $V_i(x_i + 1, (x_i, t_i)) - t_i > \frac{t_i}{x_i}$.*

B Proof of Theorem 1

In this section, we provide the proof of Theorem 1. Throughout the section, let $\mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^+$.

B.1 "If" part

Let f be an inverse uniform-price rule on $\mathcal{R}^n$.

It is obvious that f satisfies *no subsidy for losers*. Thus, we here show that f satisfies *no price envy*. The proof is in four steps.

STEP 1. Let $R \in \mathcal{R}^n$ and $i \in N$. Let $j \in N^+(x(R))$ and $x_i \in M$. Then,

$$p_j^f(R) = \frac{t_j(R)}{x_j(R)} = \frac{x_j(R)\pi^f(R)}{x_j(R)} = \pi^f(R).$$

STEP 2. Let $x_i \in M$ be such that $x_i < x_i^f(R)$. We show $(x_i{+}1, \pi^f(R)(x_i{+}1))\ R_i\ (x_i, \pi^f(R)x_i)$.

We have

$$\pi^f(R) \leq mv^m(R^{inv}) \leq v_i^{inv}(x_i^f(R)) - v_i^{inv}(x_i^f(R)-1) \leq v_i^{inv}(x_i+1) - v_i^{inv}(x_i), \quad (1)$$

where the second inequality follows from Lemma 5 (ii), and the last one from $R_i^{inv} \in \mathcal{R}^{NI}$.

Suppose $x_i = 0$. Then,

$$\pi^f(R) \leq v_i^{inv}(1) = p(0; R_i) = V_i(1, \mathbf{0}),$$

where the inequality follows from (1), and the second equality from Proposition 1. This implies

$$(1, \pi^f(R))\ R_i\ \mathbf{0} = (0, \pi^f(R)0).$$

Suppose next $x_i > 0$. Then,

$$\pi^f(R) \leq v_i^{inv}(x_i+1) - v_i^{inv}(x_i) = p(x_i; R_i) = \frac{t^*(x_i)}{x_i},$$

where the inequality follows from (1), and the second equality from Proposition 1. This implies $\pi^f(R)x_i \leq t^*(x_i)$. This, together with Lemma 6, implies

$$V_i(x_i+1, (x_i, \pi^f(R)x_i)) - \pi^f(R)x_i \geq \frac{\pi^f(R)x_i}{x_i} = \pi^f(R),$$

or $V_i(x_i+1, (x_i, \pi^f(R)x_i)) \geq \pi^f(R)(x_i+1)$. This implies

$$(x_i+1, \pi^f(R)(x_i+1))\ R_i\ (x_i, \pi^f(R)x_i),$$

as desired.

STEP 3. Let $x_i \in M$ be such that $x_i > x_i^f(R)$. We show $(x_i{-}1, \pi^f(R)(x_i{-}1))\ R_i\ (x_i, \pi^f(R)x_i)$.

Note that

$$\pi^f(R) \geq mv^{m+1}(R^{inv}) \geq v_i^{inv}(x_i^f(R)+1) - v_i^{inv}(x_i^f(R)) \geq v_i^{inv}(x_i) - v_i^{inv}(x_i-1), \quad (2)$$

where the second inequality follows from Lemma 5 (ii), and the last one from $R_i^{inv} \in \mathcal{R}^{NI}$. Note that by $x_i > x_i^f(R)$, $x_i \geq 1$.

Suppose $x_i = 1$. Then,

$$\pi^f(R) \geq v_i^{inv}(1) = p(0; R_i) = V_i(1, \mathbf{0}),$$

where the inequaliy follows from (2), and the last equality from Proposition 1. Thus, we obtain

$$(0, \pi^f(R)0) = \mathbf{0} \ R_i \ (1, \pi^f(R)).$$

Suppose $x_i > 1$. We have

$$\pi^f(R) \geq v_i^{inv}(x_i) - v_i^{inv}(x_i-1) = p(x_i-1; R_i) = \frac{t^*(x_i-1)}{x_i-1},$$

where the inequality follows from (2), and the last inequality from Proposition 1. This gives $\pi^f(R)(x_i-1) \geq t^*(x_i-1)$. Thus, by Lemma 6,

$$V_i(x_i, (x_i-1, \pi^f(R)(x_i-1))) - \pi^f(R)(x_i-1) \leq \frac{\pi^f(R)(x_i-1)}{x_i-1} = \pi^f(R),$$

or $V_i(x_i, (x_i-1, \pi^f(R)(x_i-1))) \leq \pi^f(R)x_i$. This implies

$$(x_i-1, \pi^f(R)(x_i-1)) \ R_i \ (x_i, \pi^f(R)x_i).$$

Step 4. First, if $x_i(R) = x_i$, then $f_i(R) = (x_i, \pi^f(R)x_i)$. Thus, $f_i(R) \ R_i \ (x_i, \pi^f(R)x_i)$.

Next, if $x_i < x_i(R)$, then Step 2 gives

$$f_i(R) = (x_i(R), \pi^f x_i(R)) \ R_i \cdots R_i \ (x_i, \pi^f(R)x_i).$$

Finally, if $x_i > x_i(R)$, then Step 3 gives

$$f_i(R) = (x_i(R), \pi^f x_i(R)) \ R_i \cdots R_i \ (x_i, \pi^f(R)x_i),$$

as desired. ■

B.2 "Only if" part

We do the proof of the "only if" part in six steps.

Step 1. Let f be a rule on $\mathcal{R}^n$ satisfying both *no price envy* and *no subsidy for losers*. We define a price function $\pi^f : \mathcal{R}^n \to \mathbb{R}$ of the rule f. Let $R \in \mathcal{R}^n$. By the feasibility, $N^+(x^f(R)) \neq \varnothing$. By Lemma 3, for each pair $i, j \in N^+(x^f(R))$, $p_i^f(R) = p_j^f(R)$. Thus, we can define a function $\pi^f : \mathcal{R}^n \to \mathbb{R}$ such that for each $R \in \mathcal{R}^n$, $\pi^f(R) = p_i^f(R)$ for each $i \in N^+(x(R))$.

Step 2. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $x_i^f(R) \neq 0$. In this step, we show that $\pi^f(R) \leq v_i^{inv}(x_i(R)) - v_i^{inv}(x_i(R)-1)$. By contradiction, suppose

$$\pi^f(R) > v_i^{inv}(x_i(R)) - v_i^{inv}(x_i(R)-1). \tag{1}$$

Note that by $x_i^f(R) \neq 0$, $x_i^f(R) \geq 1$.

Suppose first $x_i^f(R) = 1$. Then, by Proposition 1,

$$v_i^{inv}(x_i^f(R)) - v_i^{inv}(x_i^f(R)-1) = p(0; R_i) = V_i(1, \mathbf{0}). \tag{2}$$

Thus, by (1),

$$\pi^f(R) > V_i(1, \mathbf{0}).$$

This implies

$$\mathbf{0} \; P_i \; (1, \pi^f(R)) = f_i(R),$$

where the equality follows from $x_i^f(R) = 1$. However, this contradicts Lemma 1.

Next, suppose $x_i^f(R) > 1$. By Proposition 1,

$$v_i^{inv}(x_i^f(R)) - v_i^{inv}(x_i^f(R)-1) = p(x_i^f(R)-1; R_i) = \frac{t^*(x_i^f(R)-1)}{x_i^f(R)-1}.$$

This, together with (1), implies

$$\pi^f(R) > \frac{t^*(x_i^f(R)-1)}{x_i^f(R)-1},$$

or $\pi^f(R)(x_i^f(R)-1) > t^*(x_i^f(R)-1)$. Thus, by Lemma 6,

$$V_i\Big(x_i^f(R), (x_i^f(R)-1, \pi^f(R)(x_i^f(R)-1))\Big) - \pi^f(R)(x_i^f(R)-1) < \frac{\pi^f(R)(x_i^f(R)-1)}{x_i^f(R)-1} = \pi^f(R),$$

or

$$V_i\Big(x_i^f(R), (x_i^f(R)-1, \pi^f(R)(x_i^f(R)-1))\Big) < \pi^f(R)x_i^f(R).$$

This implies

$$(x_i^f(R)-1, \pi^f(R)(x_i^f(R)-1)) \ P_i \ (x_i^f(R), \pi^f(R)x_i^f(R)) = f_i(R),$$

which contradicts *no price envy*.

STEP 3. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $x_i^f(R) \neq m$, In this step, we show that $\pi^f(R) \geq v_i^{inv}(x_i^f(R)+1) - v_i^{inv}(x_i^f(R))$. Suppose to the contrary that

$$\pi^f(R) < v_i^{inv}(x_i^f(R)+1) - v_i^{inv}(x_i^f(R)). \tag{3}$$

Suppose $x_i^f(R) = 0$. By Proposition 1,

$$v_i^{inv}(x_i^f(R)+1) - v_i^{inv}(x_i^f(R)) = p(0; R_i) = V_i(1, \mathbf{0}). \tag{4}$$

Then, by (3),

$$\pi^f(R) < V_i(1, \mathbf{0}). \tag{5}$$

By Lemma 2, $f_i(R) = \mathbf{0}$. Thus, by (5),

$$(1, \pi^f(R)) \ P_i \ \mathbf{0} = f_i(R),$$

which contradicts *no price envy*.

Suppose instead $x_i^f(R) > 0$. Then, by Proposition 1,

$$v_i^{inv}(x_i^f(R)+1) - v_i^{inv}(x_i^f(R)) = p(x_i^f(R); R_i) = \frac{t^*(x_i^f(R))}{x_i^f(R)}.$$

Thus, by (3),

$$\pi^f(R) < \frac{t^*(x_i^f(R))}{x_i^f(R)}.$$

This implies $\pi^f(R)x_i^f(R) < t^*(x_i^f(R))$. By Lemma 6,

$$V_i\Big(x_i^f(R)+1, (x_i^f(R), \pi^f(R)x_i^f(R))\Big) - \pi^f(R)x_i^f(R) > \frac{\pi^f(R)x_i^f(R)}{x_i^f(R)} = \pi^f(R),$$

which implies

$$V_i\Big(x_i^f(R)+1, (x_i^f(R), \pi^f(R)x_i^f(R))\Big) > \pi^f(R)(x_i^f(R)+1).$$

Thus, we have

$$(x_i^f(R)+1, \pi^f(R)(x_i^f(R)+1))\ P_i\ (x_i^f(R), \pi^f(R)x_i^f(R)) = f_i(R).$$

However, this contradicts *no price envy*.

STEP 4. Let $R \in \mathcal{R}^n$. Let $i \in N$ be such that $x_i^f(R) \neq 0$. We show that $v_i^{inv}(x_i^f(R)) - v_i^{inv}(x_i^f(R)-1) \geq mv^m(R^{inv})$.

For each $j \in N$, let

$$M_j \equiv \Big\{x_j \in M\backslash\{m\} : v_j^{inv}(x_j+1) - v_j^{inv}(x_j) \leq v_i^{inv}(x_i^f(R)) - v_i^{inv}(x_i^f(R)-1)\Big\}.$$

Now, we claim that $\sum_{j\in N}|M_j| > (n-1)m$.

First, suppose there is $j \in N$ such that $x_j^f(R) = m$. By $x_i^f(R) \neq 0$, it must hold that $x_i^f(R) = m$. Then, for each $j \in N\backslash\{i\}$, $x_j(R) = 0$. Thus, by Steps 2 and 3, for each $j \in N\backslash\{i\}$,

$$v_j^{inv}(1) \leq \pi^f(R) \leq v_i^{inv}(m) - v_i^{inv}(m-1).$$

Thus, for each $j \in N\backslash\{i\}$, by $R_j^{inv} \in \mathcal{R}^{NI}$,

$$M_j = M\backslash\{m\}.$$

By $m-1 \in M_i$, $M_i \neq \varnothing$. Thus,

$$\sum_{j\in N}|M_j| > \sum_{j\in N\backslash\{i\}}|M_j| = (n-1)m.$$

Second, suppose $x_j^f(R) \neq m$ for each $j \in N$. By Step 2 and 3,

$$v_j^{inv}(x_j^f(R)+1) - v_j^{inv}(x_j^f(R)) \leq \pi^f(R) \leq v_i^{inv}(x_i^f(R)) - v_i^{inv}(x_i^f(R)-1).$$

Thus, by $R_{-i}^{inv} \in (\mathcal{R}^{NI})^{n-1}$, for each $j \in N\backslash\{i\}$, it holds that

$$\{x_j^f(R), \ldots, m-1\} \subseteq M_j.$$

Then, $|M_j| \geq m - x_j^f(R)$ for each $j \in N\backslash\{i\}$. Moreover, by $R_i^{inv} \in \mathcal{R}^{NI}$,

$$\{x_i^f(R)-1, \ldots, m-1\} \subseteq M_i.$$

Thus, $|M_i| \geq m - x_i^f(R) + 1$. Then,

$$\sum_{j \in N} |M_j| \geq nm - \sum_{j \in N} x_j^f(R) + 1 = (n-1)m + 1 > (n-1)m,$$

where the equality follows from the feasibility.

Thus, we have established $\sum_{j \in N} |M_j| > (n-1)m$. This means that there are more than $(n-1)m$ marginal valuations for R^{inv} which is no greater than $v_i^{inv}(x_i^f(R)) - v_i^{inv}(x_i^f(R)-1)$. By $nm - (n-1)m = m$, we get

$$v_i^{inv}(x_i^f(R)) - v_i^{inv}(x_i^f(R) - 1) \geq mv^m(R^{inv}).$$

Step 5. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $x_i^f(R) \neq m$. We show that $v_i^{inv}(x_i^f(R) + 1) - v_i^{inv}(x_i^f(R)) \leq mv^{m+1}(R^{inv})$.

For each $j \in N$, let

$$M_j \equiv \Big\{x_j \in M \backslash \{0\} : v_j^{inv}(x_j^f(R)) - v_j^{inv}(x_j^f(R) - 1) \geq v_i^{inv}(x_i^f(R) + 1) - v_i^{inv}(x_i^f(R))\Big\}.$$

Note that $N^+(x^f(R)) \neq \varnothing$ by the feasibility. For each $j \in N^+(x^f(R))$, by Steps 2 and 3,

$$v_j^{inv}(x_j^f(R)) - v_j^{inv}(x_j^f(R) - 1) \geq \pi^f(R) \geq v_i^{inv}(x_i^f(R) + 1) - v_i^{inv}(x_i^f(R)).$$

Thus, for each $j \in N^+(x^f(R))$, by $R_j^{inv} \in \mathcal{R}^{NI}$,

$$\{0, \ldots, x_j^f(R) - 1\} \subseteq M_j.$$

This implies that $|M_j| \geq x_j^f(R)$ for each $j \in N^+(x^f(R))$. Note that $x_i^f(R) \in M_i$. If $i \in N^+(x^f(R))$, then $|M_i| \geq x_i^f(R) + 1$. Thus,

$$\sum_{j \in N} |M_j| \geq \sum_{j \in N^+(x^f(R))} |M_j| \geq \sum_{j \in N^+(x^f(R))} x_j^f(R) + 1 = m + 1 > m,$$

where the equality follows from the feasibility. Instead, if $i \notin N^+(x^f(R))$, then by $M_i \neq \varnothing$,

$$\sum_{j \in N} |M_j| > \sum_{j \in N^+(x^f(R))} |M_j| \geq \sum_{j \in N^+(x^f(R))} x_j^f(R) = m,$$

where the equality follows from the feasibility.

Thus, we have established $\sum_{j \in N} |M_j| > m$. This means that there are more than m marginal valuations for R^{inv} which are no less than $v_i^{inv}(x_i^f(R) + 1) - v_i^{inv}(x_i^f(R))$. Thus, we obtain

$$v_i^{inv}(x_i^f(R)+1) - v_i^{inv}(x_i^f(R)) \leq mv^{m+1}(R^{inv}).$$

STEP 6. In this step, we show that f is an inverse uniform-price rule on $\mathcal{R}^n$ whose price function is π^f, and complete the proof.

First, we show that f satisfies the first condition (i) of the inverse uniform-price rule. Let $R \in \mathcal{R}^n$. Let $i, j \in N$ be such that $x_i^f(R) \neq 0$ and $x_j^f(R) \neq m$. Then, by Steps 4 and 5,

$$v_i^{inv}(x_i^f(R)) - v_i^{inv}(x_i^f(R)-1) \geq mv^m(R^{inv}) \geq mv^{m+1}(R^{inv}) \geq v_j(x_j^f(R)+1) - v_j^{inv}(x_j^f(R)).$$

Thus, by Lemma 4, it holds that

$$x^f(R) \in \arg\max_{x \in X} \sum_{i \in N} v_i^{inv}(x_i).$$

Second, we show that f satifies the second condition (ii) of the inverse uniform-price rule. By the feasibility, there must exist a pair $i, j \in N$ such that $x_i^f(R) \neq 0$ and $x_j^f(R) \neq m$. By Steps 2 and 4,

$$\pi^f(R) \leq v_i^{inv}(x_i^f(R)) - v_i^{inv}(x_i^f(R)-1) \leq mv^m(R^{inv}).$$

Moreover, by Steps 3 and 5,

$$\pi^f(R) \geq v_j^{inv}(x_j^f(R)+1) - v_j^{inv}(x_j^f(R)) \geq mv^{m+1}(R^{inv}).$$

Thus, we obtain $\pi^f(R) \in [mv^{m+1}(R^{inv}), mv^m(R^{inv})]$.

Let $R \in \mathcal{R}^n$ and $i \in N$. If $i \in N^+(x^f(R))$, then by $\pi^f(R) = p_i^f(R)$, we get

$$t_i^f(R) = \pi^f(R) x_i^f(R).$$

Moreover, if $i \notin N^+(x^f(R))$, then by Lemma 2,

$$t_i^f(R) = 0 = \pi^f(R) x_i^f(R),$$

as desired. ■

C Proof of Proposition 2

In this section, we prove Proposition 2. Let $R_0 \in (\mathcal{R}^{NI} \cap \mathcal{R}^+) \backslash \hat{\mathcal{R}}^C$ Then. $R_0^{inv} \notin \mathcal{R}^C$. Let $\mathcal{R}$ be such that $\mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^+$ and $\mathcal{R} \supseteq (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$.

Suppose that threre is a rule f on $\mathcal{R}^n$ satisfying *no price envy*, *strategy-proofness*, and *no subsidy for losers*. Then, by $\mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^{+}$, Theorem 1 implies that f is an inverse uniform-price rule on $\mathcal{R}^n$.

STEP 1. In this step, we construct a preference profile. Let $R_1 = R_0$. By $R_1^{inv} \in \mathcal{R}^{NI} \backslash \mathcal{R}^{C}$, there is $x_1 \in M \backslash \{0, m\}$ such that

$$v_1^{inv}(x_1) - v_1^{inv}(x_1 - 1) > v_1^{inv}(x_1 + 1) - v_1^{inv}(x_1).$$

Let $A_2 \in \mathbb{R}_{++}$ be a positive number such that

$$v_1^{inv}(x_1 + 1) - v_1^{inv}(x_1) < A_2 < v_1^{inv}(x_1) - v_1^{inv}(x_1 - 1). \tag{1}$$

Let $R_2 \in \mathcal{R}^{C} \cap \mathcal{R}^{Q}$ be such that $v_2(x_2) = A_2 x_2$ for each $x_2 \in M$.

Let $\varepsilon \in \mathbb{R}_{++}$ be such that for each $x_1' \in M \backslash \{m\}$,

$$\varepsilon < v_1^{inv}(x_1' + 1) - v_1^{inv}(x_1'). \tag{2}$$

Then, $\varepsilon < A_2$. For each $i \in N \backslash \{1, 2\}$, let $R_i \in \mathcal{R}^{C} \cap \mathcal{R}^{Q}$ be such that $v_i(x_i) = \varepsilon x_i$ for each $x_i \in M$.

STEP 2. In this step, we show that $x_2^f(R) = m - x_1$.

First, we show that $x_i^f(R) = 0$ for each $i \in N \backslash \{1, 2\}$. Let $i \in N \backslash \{1, 2\}$. Suppose by contradiction that $x_i^f(R) \neq 0$. Then, $x_1^f(R) \neq m$. Then, by (2),

$$v_i(x_i^f(R)) - v_i(x_i^f(R) - 1) = \varepsilon < v_1^{inv}(x_1^f(R) + 1) - v_1^{inv}(x_1^f(R)).$$

By Lemma 4, this contradicts the first condition (i) of the inverse uniform-price rule.

Next, we show that $x_1^f(R) = x_1$ Suppose not.

First, suppose $x_1^f(R) > x_1$. Then, $x_2^f(R) \neq m$. We have

$$v_1^{inv}(x_1^f(R)) - v_1^{inv}(x_1^f(R) - 1) \leq v_1^{inv}(x_1 + 1) - v_1^{inv}(x_1) < A_2 = v_2(x_2^f(R) + 1) - v_2(x_2^f(R)),$$

where the first inequality follows from $R_1^{inv} \in \mathcal{R}^{NI}$, and the second one from (1). However, by Lemma 4, this contradicts the first condition (i) of the inverse uniform-price rule.

Second, suppose $x_1^f(R) < x_1$. Then, by $x_i^f(R) = 0$ for each $i \in N \backslash \{1, 2\}$, the feasibility implies $x_2^f(R) \neq 0$. We have

$$v_2(x_2^f(R)) - v_2(x_2^f(R) - 1) = A_2 < v_1^{inv}(x_1) - v_1^{inv}(x_1 - 1) \leq v_1^{inv}(x_1^f(R) + 1) - v_1^{inv}(x_1^f(R)),$$

where the first inequality follows from (1), and the second one from $R_1^{inv} \in \mathcal{R}^{NI}$. This contradicts the first condition (i) of the inverse uniform-price rule by Lemma 4.

Thus, $x_1^f(R) = x_1$, and for each $i \in N\backslash\{1,2\}$, $x_i^f(R) = 0$. By the feasibility, we have

$$x_2^f(R) = m - x_1^f(R) = m - x_1.$$

Step 3. By (1), (2), and $R_1^{inv} \in \mathcal{R}^{NI}$, $mv^{x_1}(R^{inv}) = v_1^{inv}(x_1) - v_1^{inv}(x_1-1)$, $mv^{m+x_1}(R^{inv}) = v_1^{inv}(x_1+1) - v_1^{inv}(x_1)$, and for each $x \in M\backslash\{0\}$, $mv^{x_1+x}(R^{inv}) = A_2$. Thus, by $x_1 \in M\backslash\{0,m\}$, $mv^m(R^{inv}) = mv^{m+1}(R^{inv}) = A_2$. Then, $f_2(R) = (m - x_1, A_2(m - x_1))$ by Step 2.

Let $A_2' \in \mathbb{R}_{++}$ be such that

$$v_1^{inv}(x_1 + 1) - v_1^{inv}(x_1) < A_2' < A_2. \tag{3}$$

Let $R_2' \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that $v_2'(x_2) = A_2' x_2$ for each $x_2 \in M$. Then, by (1) and (3),

$$v_1^{inv}(x_1 + 1) - v_1^{inv}(x_1) < A_2' < v_1^{inv}(x_1) - v_1^{inv}(x_1 - 1). \tag{4}$$

Then, in the same way as in Step 2, we can show that $x_2^f(R_2', R_{-2}) = m - x_1$. By (2), (4), and $R_1^{inv} \in \mathcal{R}^{NI}$, $mv^m(R_2', R_{-2}^{inv}) = mv^{m+1}(R_2', R_{-2}^{inv}) = A_2'$. Thus, $\pi^f(R_2', R_{-2}) = A_2'$. Then, $f_2(R_2', R_{-2}) = (m - x_1, A_2'(m - x_1))$. However, by (3),

$$f_2(R_2', R_{-2}) = (m - x_1, A_2'(m - x_1)) \; P_2 \; (m - x_1, A_2(m - x_1)) = f_2(R),$$

which contradicts *strategy-proofness*. ■

D Proof of Theorem 2

In this section, we prove Theorem 2. Let $\mathcal{R}$ be such that $\mathcal{R}^C \cap \mathcal{R}^Q \subseteq \mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^+$.

D.1 "If" part

First, we show the "if" part. Suppose $\mathcal{R} = \hat{\mathcal{R}}^C$.

We first show that there is a rule on $\mathcal{R}^n$ satisfying *no price envy*, *strategy-proofness*, and *no subsidy for losers*. Let f be a minimum inverse uniform-price rule on $\mathcal{R}^n$. Then, by $\mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^+$, Theorem 1 implies that f satisfies both *no price envy* and *no subsidy for losers*. Thus, it suffices to show that it satisfies *strategy-proofness*.

Let $R \in \mathcal{R}^n$, $i \in N$, and $R_i' \in \mathcal{R}$. We consider the following two cases.

Case 1. $x_i^f(R) \neq 0$.

If $x_i^f(R_i', R_{-i}) \neq 0$, then by Lemma 5 (i) and $R^{inv}, (R_i'^{inv}, R_{-i}^{inv}) \in (\mathcal{R}^C)^n$,

$$p_i^f(R) = mv^{m+1}(R^{inv}) = \max_{j \in N \setminus \{i\}} v_j^{inv}(1) = mv^{m+1}(R_i'^{inv}, R_{-i}^{inv}).$$

Thus, $f_i(R_i', R_{-i}) = (x_i^f(R_i', R_{-i}), p_i^f(R) x_i^f(R_i', R_{-i}))$. By *no price envy*, we get $f_i(R)\ R_i\ f_i(R_i', R_{-i})$.

Instead, if $x_i^f(R_i', R_{-i}) = 0$, then by Lemma 1, we get $f_i(R)\ R_i\ \mathbf{0} = f_i(R_i', R_{-i})$.

Case 2. $x_i^f(R) = 0$.

If $x_i^f(R_i', R_{-i}) \neq 0$, then

$$v_i^{inv}(1) \leq mv^{m+1}(R^{inv}) \leq \max_{j \in N \setminus \{i\}} v_j^{inv}(1) \leq mv^{m+1}(R_i'^{inv}, R_{-i}^{inv}), \tag{1}$$

where the first inequality follows from Lemma 5 (ii), the second one from the feasibility, Lemma 5 (i), and $R^{inv} \in (\mathcal{R}^C)^n$, and the last one from Lemma 5 (i) and $(R_i'^{inv}, R_{-i}^{inv}) \in (\mathcal{R}^C)^n$. By $R_i \in \hat{\mathcal{R}}^C$, $v_i^{inv}(1) = v_i(1, \mathbf{0}) = v_i(x_i+1, \mathbf{0}) - v_i(x_i, \mathbf{0})$ for each $x_i \in M$ with $x_i < x_i^f(R_i', R_{-i})$. Thus, by (1),

$$v_i(x_i^f(R_i', R_{-i}), \mathbf{0}) \leq x_i^f(R_i', R_{-i}) mv^{m+1}(R_i'^{inv}, R_{-i}^{inv}) = t_i^f(R_i', R_{-i}).$$

This implies $\mathbf{0}\ R_i\ f_i(R_i', R_{-i})$. Thus, by Lemma 1, $f_i(R)\ R_i\ f_i(R_i', R_{-i})$.

Instead, if $x_i^f(R_i', R_{-i}) = 0$, then $f_i(R) = \mathbf{0} = f_i(R_i', R_{-i})$.

Next, let $\mathcal{R}'$ be such that $\mathcal{R}' \supsetneq \mathcal{R}$ and $\mathcal{R}' \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^+$. By $\mathcal{R}' \not\subseteq \mathcal{R}$, there is $R_0 \in \mathcal{R}'$ such that $R_0 \notin \mathcal{R}$. Note that by $\mathcal{R}' \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^+$, $R_0 \in \mathcal{R}^{NI} \cap \mathcal{R}^+$. By $\mathcal{R} \supseteq \mathcal{R}^C \cap \mathcal{R}^Q$, $\mathcal{R}' \supseteq \mathcal{R}^C \cap \mathcal{R}^Q$. Thus, we have $\mathcal{R}' \supseteq (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$ for $R_0 \in (\mathcal{R}^{NI} \cap \mathcal{R}^+) \setminus \hat{\mathcal{R}}^C$, and Proposition 2 implies that there is no rule on $(\mathcal{R}')^n$ satifying *no price envy*, *strategy-proofness*, and *individual rationality*.

D.2 "Only if" part

Next, we prove the "only if" part. Suppose by contradiction that $\mathcal{R}$ is a maximal domain for *no price envy*, *strategy-proofness*, and *no subsidy for losers*, but $\mathcal{R} \neq \hat{\mathcal{R}}^C$.

If $\mathcal{R} \subseteq \hat{\mathcal{R}}^C$, then by $\mathcal{R} \neq \hat{\mathcal{R}}^C$, $\mathcal{R} \subsetneq \hat{\mathcal{R}}^C$. However, this contradicts that $\mathcal{R}^n$ is a maximal domain for *no price envy*, *strategy-proofness*, and *no subsidy for losers* since the minimum inverse uniform-price rule on $(\hat{\mathcal{R}}^C)^n$ satisfies the three properties.

Thus, $\mathcal{R} \not\subseteq \hat{\mathcal{R}}^C$. Then, there is $R_0 \in \mathcal{R}$ such that $R_0 \notin \hat{\mathcal{R}}^C$. By $\mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^+$, $R_0 \in \mathcal{R}^{NI} \cap \mathcal{R}^+$. Thus, by $\mathcal{R} \supseteq (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$, Proposition 2 implies no rue on

$\mathcal{R}^n$ satisfies *no price envy*, *strategy-proofness*, and *no subsidy for losers*. However, this contradicts that $\mathcal{R}$ is a maximal domain for *no price envy*, *strategy-proofness*, and *no subsidy for losers*. ■

E Proof of Theorem 3

In this section, we prove Theorem 3. Let $\mathcal{R}$ be such that $\mathcal{R}^C \cap \mathcal{R}^Q \subseteq \mathcal{R} \subseteq \hat{\mathcal{R}}^C$. By the proof of the "if" part of Theorem 2, the minimum inverse uniform-price rule satisfies *no price envy*, *strategy-proofness*, and *no subsidy for losers* on $\mathcal{R}^n$. Thus, we here show the "only if" part.

We begin with the following two lemmas.

Lemma 7. *Let f, g be a pair of inverse uniform-price rules on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $mv^{m+1}(R^{inv}) < mv^m(R^{inv})$. Let $R'_i \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that for each $x_i \in M \backslash \{m\}$,*

$$mv^{m+1}(R^{inv}) < v'_i(x_i + 1) - v'_i(x_i) < mv^m(R^{inv}).$$

Then, (i) $x^g_i(R'_i, R_{-i}) = x^f_i(R)$, and (ii) if $0 < x^f_i(R) < m$, then $\pi^g(R'_i, R_{-i}) = v'_i(x_i + 1) - v'_i(x_i)$ for each $x_i \in M \backslash \{m\}$.

Proof. First, we show (i). By contradiction, suppose $x^g_i(R'_i, R_{-i}) \neq x^f_i(R)$. We consider the following two cases.

Case 1. $x^g_i(R'_i, R_{-i}) < x^f_i(R)$

By the feasibility, there is $j \in N \backslash \{i\}$ such that $x^g_j(R'_i, R_{-i}) > x^f_j(R)$. Then,

$$\begin{aligned}
& v^{inv}_j(x^g_j(R'_i, R_{-i})) - v^{inv}_j(x^g_j(R'_i, R_{-i}) - 1) \\
& \leq v^{inv}_j(x^f_j(R) + 1) - v^{inv}_j(x^f_j(R)) && \text{(by } R^{inv}_j \in \mathcal{R}^{NI}\text{)} \\
& \leq mv^{m+1}(R^{inv}) && \text{(by Lemma 5 (ii))} \\
& < v'_i(x^g_j(R'_i, R_{-i}) + 1) - v'_i(x^g_j(R'_i, R_{-i})), && \text{(by the def. of } R'_i\text{)}
\end{aligned}$$

which contradicts Lemma 4.

Case 2. $x^g_i(R'_i, R_{-i}) > x^f_i(R)$

Then, there is $j \in N\backslash\{i\}$ such that $x_j^g(R_i', R_{-i}) < x_j^f(R)$. Then,

$$\begin{aligned}
& v_j^{inv}(x_j^g(R_i', R_{-i}) + 1) - v_j^{inv}(x_j^g(R_i', R_{-i})) \\
& \geq v_j^{inv}(x_j^f(R)) - v_j^{inv}(x_j^f(R) - 1) && (\text{by } R_j^{inv} \in \mathcal{R}^{NI}) \\
& \geq mv^m(R^{inv}) && (\text{by Lemma 5 (i)}) \\
& > v_i'(x_i^g(R_i', R_{-i})) - v_i'(x_i^g(R_i', R_{-i}) - 1), && (\text{by the def. of } R_i')
\end{aligned}$$

which contradicts Lemma 4.

Next, we show (ii). Suppose $0 < x_i^f(R) < m$. By (i) and $x_i^f(R) \neq 0$, $x_i^g(R_i', R_{-i}) \neq 0$. Thus, by Lemma 5 (i),

$$v_i'(x_i^g(R_i', R_{-i})) - v_i'(x_i^g(R_i', R_{-i}) - 1) \geq mv^m(R_i', R_{-i}^{inv}). \tag{1}$$

Similarly, by (i) and $x_i^f(R) \neq m$, $x_i^g(R_i', R_{-i}) \neq m$. Thus, by Lemma 5 (ii),

$$v_i'(x_i^g(R_i', R_{-i}) + 1) - v_i'(x_i^g(R_i', R_{-i})) \leq mv^{m+1}(R_i', R_{-i}^{inv}). \tag{2}$$

By $mv^m(R_i', R_{-i}^{inv}) \geq mv^{m+1}(R_i', R_{-i}^{inv})$, (1), (2), and $R_i' \in \mathcal{R}^C$, we obtain that for each $x_i \in M\backslash\{m\}$,

$$mv^m(R_i', R_{-i}^{inv}) = mv^{m+1}(R_i', R_{-i}^{inv}) = v_i'(x_i + 1) - v_i'(x_i).$$

Thus, $\pi^g(R_i', R_{-i}) = v_i'(x_i + 1) - v_i'(x_i)$ for each $x_i \in M\backslash\{m\}$. □

Lemma 8. *Let f, g be a pair of inverse uniform-price rules on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $x_i^f(R) = m$. Let $R_i' \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ be such that $v_i'(x_i+1) - v_i'(x_i) > mv^{m+1}(R^{inv})$ for each $x_i \in M\backslash\{m\}$. Then, (i) $x_i^g(R_i', R_{-i}) = m$ and (ii) $\pi^g(R_i', R_{-i}) \leq v_i'(m) - v_i'(m-1)$.*

Proof. First, we show (i). Suppose by contradiction that $x_i^g(R_i', R_{-i}) \neq m$. Then, by the feasibility, there is $j \in N\backslash\{i\}$ such that $x_j^g(R_i', R_{-i}) \neq 0$. Note that by $x_i^f(R) = m$, $x_j^f(R) = 0$. Then, we have

$$\begin{aligned}
mv^{m+1}(R^{inv}) & \geq v_j^{inv}(1) && (\text{by Lemma 5 (ii)}) \\
& \geq v_j^{inv}(x_j^g(R_i', R_{-i})) - v_j^{inv}(x_j^g(R_i', R_{-i}) - 1) && (\text{by } R_j^{inv} \in \mathcal{R}^{NI}) \\
& \geq mv^m(R_i', R_{-i}^{inv}) && (\text{by Lemma 5 (i)}) \\
& \geq mv^{m+1}(R_i', R_{-i}^{inv}) \\
& \geq v_i'(x_i^g(R_i', R_{-i}) + 1) - v_i'(x_i^g(R_i', R_{-i})) && (\text{by Lemma 5 (ii)}) \\
& > mv^{m+1}(R^{inv}),
\end{aligned}$$

a contradiction.

Then, we show (ii). By (i), $x_i^g(R_i', R_{-i}) = m$. Thus, by Lemma 5 (i),

$$\pi^g(R_i', R_{-i}) \leq v_i'(m) - v_i'(m-1),$$

as desired. □

Now, we turn to the proof of the "only if" part of Theorem 3. Let f be a rule on $\mathcal{R}^n$ satisfying *no price envy*, *strategy-proofness*, *no subsidy for losers*. Note that by Theorem 1, f is an inverse uniform-price rule. By contradiction, suppose f is not a minimum inverse uniform-price rule. Then, there is $R \in \mathcal{R}^n$ such that $\pi^f(R) \neq mv^{m+1}(R^{inv})$. Let $\varepsilon \in \mathbb{R}_{++}$ be such that $\varepsilon < \pi^f(R^{inv}) - mv^{m+1}(R^{inv})$. By the feasibility, there is $i \in N$ such that $x_i^f(R) \neq 0$. Let $R_i' \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that $v_i'(x_i) = (\pi^f(R) - \varepsilon)x_i$ for each $x_i \in M$.

We consider the following two cases.

Case 1. $x_i^f(R) \neq m$.

Since f is an inverse uniform-price rule, $\pi^f(R) \leq mv^m(R^{inv})$. Thus, for each $x_i \in M \backslash \{m\}$,

$$mv^{m+1}(R^{inv}) < v_i'(x_i + 1) - v_i'(x_i) < mv^m(R^{inv}),$$

where the first inequality follows from $\varepsilon < \pi^f(R) - mv^{m+1}(R^{inv})$. By Lemma 7, $x_i^f(R_i', R_{-i}) = x_i^f(R)$ and $\pi^f(R_i', R_{-i}) = \pi^f(R) - \varepsilon$. Then, by $x_i^f(R_i', R_{-i}) = x_i^f(R) \neq 0$,

$$t_i^f(R_i', R_{-i}) = \pi^f(R_i', R_{-i})x_i^f(R_i', R_{-i}) < \pi^f(R)x_i^f(R) = t_i^f(R).$$

Thus, by $x_i^f(R_i', R_{-i}) = x_i^f(R)$, $f_i(R_i', R_{-i}) \; P_i \; f_i(R)$. However, this contradicts *strategy-proofness*.

Case 2. $x_i^f(R) = m$.

By $\varepsilon < \pi^f(R) - mv^{m+1}(R^{inv})$, for each $x_i \in M \backslash \{m\}$,

$$v_i'(x_i + 1) - v_i'(x_i) > mv^{m+1}(R^{inv}).$$

Thus, by Lemma 8, $x_i^f(R_i', R_{-i}) = x_i^f(R) = m$ and

$$\pi^f(R_i', R_{-i}) \leq \pi^f(R) - \varepsilon < \pi^f(R).$$

These imply $t_i(R_i', R_{-i}) < t_i(R)$. Thus, by $x_i^f(R_i', R_{-i}) = x_i^f(R)$, $f_i(R_i', R_{-i}) \; P_i \; f_i(R)$, which contradicts *strategy-proofness*. ■

F Proofs of Theorem 4 and Proposition 3

In this section, we prove Theorem 4 and Proposition 3. Throughout the section, let $\mathcal{R}$ be such that $\mathcal{R}^{NI} \cap \mathcal{R}^{Q} \subseteq \mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^{+}$.

F.1 Preliminaries

First, we provide lemmas that will be used to prove Theorem 4 and Proposition 3.

The next lemma states that if an agent can manipulate an inverse uniform-price rule, then he must receive some units of the object as a result of the manipulation.

Lemma 9. *Let f be an inverse uniform-price rule on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$, $i \in N$, and $R'_i \in \mathcal{R}$ be such that $f_i(R'_i, R_{-i})\ P_i\ f_i(R)$. Then, $x_i^f(R'_i, R_{-i}) \neq 0$.*

Proof. By contradiction, suppose $x_i^f(R'_i, R_{-i}) = 0$. By the second condition (ii) of the inverse uniform-price rule, $f_i(R'_i, R_{-i}) = \mathbf{0}$. By Theorem 1, f satisfies *no price envy*. Thus, by Lemma 1,

$$f_i(R)\ R_i\ \mathbf{0} = f_i(R'_i, R_{-i}),$$

which contradicts $f_i(R'_i, R_{-i})\ P_i\ f_i(R)$. □

The next two lemmas state that for a given preference profile such that an agent receives some but not all units of the object under an inverse uniform-price, there is a quasi-linear preference that produces the same consumption level under another inverse uniform-price rule as the original rule.

Lemma 10. *Let f, g be a pair of inverse uniform-price rules on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $0 < x_i^f(R) < m$ and $mv^{m+1}(R^{inv}) = mv^m(R^{inv})$. Let $R'_i \in \mathcal{R}^{NI} \cap \mathcal{R}^{Q}$ be such that*

$$v'_i(x_i^f(R)+1) - v'_i(x_i^f(R)) < mv^{m+1}(R^{inv}) < v'_i(x_i^f(R)) - v'_i(x_i^f(R)-1).$$

Then, (i) $x_i^g(R'_i, R_{-i}) = x_i^f(R)$ and (ii) $v'_i(x_i^f(R)+1) - v'_i(x_i^f(R)) \leq \pi^g(R'_i, R_{-i}) \leq v'_i(x_i^f(R)) - v'_i(x_i^f(R)-1)$.

Proof. We first show (i). Suppose by contradiction that $x_i^g(R'_i, R_{-i}) \neq x_i^f(R)$. We need to consider the following two cases.

CASE 1. $x_i^g(R'_i, R_{-i}) < x_i^f(R)$

By the feasibility, there is $j \in N\backslash\{i\}$ such that $x_j^g(R_i', R_{-i}) > x_j^f(R)$. Then,

$$\begin{aligned}
& v_j^{inv}(x_j^g(R_i', R_{-i})) - v_j^{inv}(x_j^g(R_i', R_{-i}) - 1) \\
& \leq v_j^{inv}(x_j^f(R) + 1) - v_j^{inv}(x_j^f(R)) && \text{(by } R_j^{inv} \in \mathcal{R}^{NI}\text{)} \\
& \leq mv^{m+1}(R^{inv}) && \text{(by Lemma 5 (ii))} \\
& = mv^m(R^{inv}) \\
& < v_i'(x_i^f(R)) - v_i'(x_i^f(R) - 1) && \text{(by the def. of } R_i'\text{)} \\
& \leq v_i'(x_i^g(R_i', R_{-i}) + 1) - v_i'(x_i^g(R_i', R_{-i})), && \text{(by } R_i' \in \mathcal{R}^{NI}\text{)}
\end{aligned}$$

which contradicts Lemma 4.

CASE 2. $x_i^g(R_i', R_{-i}) > x_i^f(R)$

Note that there is $j \in N\backslash\{i\}$ such that $x_j^g(R_i', R_{-i}) < x_j^f(R)$ by the feasibility. Then,

$$\begin{aligned}
& v_j^{inv}(x_j^g(R_i', R_{-i}) + 1) - v_j^{inv}(x_j^g(R_i', R_{-i})) \\
& \geq v_j^{inv}(x_j^f(R)) - v_j^{inv}(x_j^f(R) - 1) && \text{(by } R_j^{inv} \in \mathcal{R}^{NI}\text{)} \\
& \geq mv^m(R^{inv}) && \text{(by Lemma 5 (i))} \\
& = mv^{m+1}(R^{inv}) \\
& > v_i'(x_i^f(R) + 1) - v_i'(x_i^f(R)) && \text{(by the def. of } R_i'\text{)} \\
& \geq v_i'(x_i^g(R_i', R_{-i})) - v_i'(x_i^g(R_i', R_{-i}) - 1), && \text{(by } R_i' \in \mathcal{R}^{NI}\text{)}
\end{aligned}$$

which contradicts Lemma 4.

Next, we show (ii). By (i) and $x_i^f(R) \neq 0$, $x_i^g(R_i', R_{-i}) \neq 0$. Thus, by Lemma 5 (i),

$$v_i'(x_i^g(R_i', R_{-i})) - v_i'(x_i^g(R_i', R_{-i}) - 1) \geq mv^m(R_i', R_{-i}^{inv}) \geq \pi^g(R_i', R_{-i}). \tag{1}$$

By (i) and $x_i^f(R) \neq m$, $x_i^g(R_i', R_{-i}) \neq m$. Thus, by Lemma 5 (ii),

$$v_i'(x_i^g(R_i', R_{-i}) + 1) - v_i'(x_i^g(R_i', R_{-i})) \leq mv^{m+1}(R_i', R_{-i}^{inv}) \leq \pi^g(R_i', R_{-i}). \tag{2}$$

Combining (1) and (2), we obtain the desired inequality. □

Lemma 11. *Let f, g be a pair of inverse uniform-price rules on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $0 < x_i^f(R) < m$. Let $\varepsilon \in \mathbb{R}_{++}$. Then, there is $R_i' \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ such that (i) $x^g(R_i', R_{-i}) = x_i^f(R)$ and (ii) $|t_i^g(R_i', R_{-i}) - t_i^f(R)| < \varepsilon$.*

Proof. We consider the following two cases.

Case 1. $mv^{m+1}(R^{inv}) < mv^{m}(R^{inv})$.

We further divide the argument into two cases.

Case 1-1. $mv^{m+1}(R^{inv}) < \pi^f(R)$.

Let $\varepsilon' \in \mathbb{R}_+$ be such that

$$x_i^f(R)\varepsilon' < \min\{\varepsilon, \pi^f(R) - mv^{m+1}(R^{inv})\}.$$

Let $R_i' \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that $v_i'(x_i) = (\pi^f(R) - \varepsilon')x_i$ for each $x_i \in M$. Then, by $\varepsilon' < \pi^f(R) - mv^{m+1}(R^{inv})$, for each $x_i \in M\backslash\{m\}$,

$$v_i'(x_i+1) - v_i'(x_i) > mv^{m+1}(R^{inv}).$$

Moreover, by $\pi^f(R) \leq mv^{m}(R^{inv})$, for each $x_i \in M\backslash\{m\}$,

$$v_i'(x_i+1) - v_i'(x_i) < mv^{m}(R^{inv}).$$

Thus, by $0 < x_i^f(R) < m$, Lemma 7 implies that $x_i^g(R_i', R_{-i}) = x_i^f(R)$ and $\pi^g(R_i', R_{-i}) = \pi^f(R) - \varepsilon'$. Then,

$$|t_i^g(R_i', R_{-i}) - t_i^f(R)| = x_i^f(R)|\pi^f(R) - \varepsilon' - \pi^f(R)| = x_i^f(R)\varepsilon' < \varepsilon,$$

where the first equality follows from $x_i^g(R_i', R_{-i}) = x_i^f(R)$, and the inequality from $x_i^f(R)\varepsilon' < \varepsilon$.

Case 1-2. $mv^{m+1}(R^{inv}) = \pi^f(R)$.

Let $\varepsilon' \in \mathbb{R}_{++}$ be such that

$$x_i^f(R)\varepsilon' < \min\{mv^{m}(R^{inv}) - mv^{m+1}(R^{inv}), \varepsilon\}.$$

Let $R_i' \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that $v_i'(x_i) = (\pi^f(R) + \varepsilon')x_i$ for each $x_i \in M$. Then, for each $x_i \in M\backslash\{m\}$, by $\varepsilon' > 0$,

$$v_i'(x_i+1) - v_i'(x_i) > mv^{m+1}(R^{inv}),$$

and by $\varepsilon' < mv^{m}(R^{inv}) - mv^{m+1}(R^{inv})$ and $\pi^f(R) = mv^{m+1}(R^{inv})$,

$$v_i'(x_i+1) - v_i'(x_i) < mv^{m}(R^{inv}).$$

Thus, as in Case 1-1, by $0 < x_i^f(R) < m$, Lemma 7 implies $x_i^g(R_i', R_{-i}) = x_i^f(R)$ and $\pi^g(R_i', R_{-i}) = p_i^f(R) + \varepsilon'$. Then, by $x_i^g(R_i', R_{-i}) = x_i^f(R)$ and $x_i^f(R)\varepsilon' < \varepsilon$, we obtain $|t_i^g(R_i', R_{-i}) - t_i^f(R)| < \varepsilon$ in the same way as in Case 1-1.

Case 2. $mv^{m+1}(R^{inv}) = mv^m(R^{inv})$.

Let $\varepsilon' \in \mathbb{R}_{++}$ be such that

$$x_i^f(R)\varepsilon' < \min\{mv^m(R^{inv}), \varepsilon\}.$$

Let $R_i' \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ be such that for each $x_i \in M\backslash\{m\}$, $v_i'(x_i+1) - v_i'(x_i) = mv^m(R^{inv}) + \varepsilon'$ if $x_i < x_i^f(R)$, and $v_i'(x_i+1) - v_i'(x_i) = mv^m(R^{inv}) - \varepsilon'$ if $x_i \geq x_i^f(R)$. Then, by $mv^m(R^{inv}) = mv^{m+1}(R^{inv})$,

$$v_i'(x_i^f(R)+1) - v_i'(x_i^f(R)) < mv^{m+1}(R^{inv}) < v_i'(x_i^f(R)) - v_i'(x_i^f(R)-1).$$

Thus, by Lemma 10, $x_i^g(R_i', R_{-i}) = x_i^f(R)$ and

$$-\varepsilon' < \pi_i^g(R_i', R_{-i}) - \pi^f(R) < \varepsilon'.$$

Thus, by $x_i^g(R_i', R_{-i}) = x_i^f(R)$ and $x_i^f(R)\varepsilon' < \varepsilon$,

$$|t_i^g(R_i', R_{-i}) - t_i^f(R)| = x_i^f(R)|\pi^g(R_i', R_{-i}) - \pi^f(R)| = x_i^f(R)\varepsilon' < \varepsilon,$$

as desired. □

The next lemma states that for a preference profile such that an agent receives all the units under an inverse uniform-price rule, there is a quasi-linear preference that produces the same consumption level under another inverse uniform-price rule as the original rule.

Lemma 12. *Let f, g be a pair of inverse uniform-price rules on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $x_i^f(R) = m$. Let $\varepsilon \in \mathbb{R}_{++}$. Then, there is $R_i' \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ such that (i) $x_i^g(R_i', R_{-i}) = m$, and (ii) $t_i^g(R_i', R_{-i}) - t_i^f(R) < \varepsilon$.*

Proof. Note that $\pi^f(R) \geq mv^{m+1}(R^{inv})$. We consider the following two cases.

Case 1. $mv^{m+1}(R^{inv}) = \pi^f(R)$.

Let $\varepsilon' \in \mathbb{R}_{++}$ be such that $m\varepsilon' < m$. Let $R_i' \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that for each $x_i \in M\backslash\{m\}$,

$$mv^{m+1}(R^{inv}) < v_i'(x_i+1) - v_i'(x_i) < mv^{m+1}(R^{inv}) + \varepsilon'.$$

Then, by Lemma 8, $x_i^g(R_i', R_{-i}) = m$ and $\pi^g(R_i', R_{-i}) \leq v_i'(m) - v_i'(m-1)$. We have

$$\pi^f(R) + \varepsilon' > v_i'(m) - v_i'(m-1) \geq mv^m(R_i', R_{-i}^{inv}) \geq \pi^g(R_i', R_{-i}), \tag{1}$$

where the first inequality follows from $mv^{m+1}(R^{inv}) = \pi^f(R)$. Then, we have

$$t_i^g(R_i', R_{-i}) - t_i^f(R) = m(\pi^g(R_i', R_{-i}) - \pi^f(R)) < m\varepsilon' < \varepsilon,$$

where the equality follows from $x_i^g(R_i', R_{-i}) = x_i^f(R) = m$, and the first inequality from (1).

Case 2. $mv^{m+1}(R^{inv}) < \pi^f(R)$.

Let $R_i' \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that for each $x_i \in M\backslash\{m\}$,

$$mv^{m+1}(R^{inv}) < v_i'(x_i + 1) - v_i'(x_i) < \pi^f(R).$$

Then, by Lemma 8, $x_i^g(R_i', R_{-i}) = m$ and $\pi^g(R_i', R_{-i}) \leq v_i'(m) - v_i'(m-1)$. Thus,

$$\pi^g(R_i', R_{-i}) \leq v_i'(m) - v_i'(m-1) < \pi^f(R).$$

By $x_i^g(R_i', R_{-i}) = x_i^f(R) = m$, this implies

$$t_i^g(R_i', R_{-i}) < t_i^f(R) < t_i^f(R) + \varepsilon,$$

or $t_i^g(R_i', R_{-i}) - t_i^f(R) < \varepsilon$. □

The next lemma states that each agent weakly prefers an outcome of an inverse uniform-price rule than that of another rule if and only if the price of former rule is no greater than that of the latter rule.

Lemma 13. *Let f, g be a pair of inverse uniform-price rules on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$ and $i \in N$. Then, $f_i(R)\ R_i\ g_i(R)$ if and only if $\pi^f(R) \leq \pi^g(R)$.*

Proof. First, we show the "if" part. Suppose $\pi^f(R) \leq \pi^g(R)$. By Lemma 3, $\pi^f(R) = p_j^f(R)$ for each $j \in N^+(x^f(R))$. By Theorem 1, f satisfies *no price envy*. Thus,

$$f_i(R)\ R_i\ (x_i^g(R), \pi^f(R)x_i^g(R))\ R_i\ g_i(R),$$

where the last relation follows from $\pi^f(R) \leq \pi^g(R)$.

Next, we show the "only if" part. Suppose by contradiction that $f_i(R)\ R_i\ g_i(R)$ but

$\pi^f(R) > \pi^g(R)$. Then, by *no price envy* of g,

$$g_i(R)\ R_i\ (x_i^f(R), \pi^g(R)x_i^f(R))\ P_i\ f_i(R),$$

where the second relation follows from $\pi^g(R) < \pi^f(R)$. However, this contradicts that $f_i(R)\ R_i\ g_i(R)$. □

Finally, the next lemma states the the set of gains of manipulation is bounded above at each preference profile.

Lemma 14. *Let f be an inverse uniform-price rule on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$ and $i \in N$. Then,* $\sup_{R_i' \in \mathcal{R}} G_i^f(R_i'; R) < \infty$.

Proof. We show that the set $G \equiv \{G_i^f(R_i'; R) : R_i' \in \mathcal{R}\}$ is bounded above. Then, the continuity of the real numbers implies that there is $\sup_{R_i' \in \mathcal{R}} G_i^f(R_i'; R)$. Let $R_i' \in \mathcal{R}$ be such that $f_i(R)\ R_i\ f_i(R_i', R_{-i})$. Then, $G_i^f(R_i'; R) \leq 0$. Instead, let $R_i' \in \mathcal{R}$ be such that $f_i(R_i', R_{-i})\ P_i\ f_i(R)$. Then, $G_i^f(R_i', R_{-i}) > 0$. By $\pi^f(R_i', R_{-i}) \geq mv^{m+1}(R_i'^{inv}, R_{-i}^{inv}) > 0$,

$$t_i^f(R_i', R_{-i}) = \pi^f(R_i', R_{-i})x_i^f(R_i', R_{-i}) \geq 0.$$

Then,

$$\max_{x_i \in M} V_i(x_i, f_i(R)) \geq V_i(x_i^f(R_i', R_{-i}), f_i(R)) - t_i^f(R_i', R_{-i}) = G_i^f(R_i'; R),$$

where the inequality follows from $t_i^f(R_i', R_{-i}) \geq 0$. Then, by $G_i^f(R_i'; R) > 0$, we have $\max_{x_i \in M} V_i(x_i, f_i(R)) > 0$. Thus, $\max\{0, \max_{x_i \in M} V_i(x_i, f_i(R))\}$ is an upper bound of the set G. □

F.2 Proof of Proposition 3

Now, we turn to the proof of Proposition 3.

Let f, g be a pair of rules on $\mathcal{R}^n$ satisfying *no price envy* and *no subsidy for losers*. Then, by Theorem 1, f and g are both inverse uniform-price rules on $\mathcal{R}^n$.

F.2.1 The "if" part

First, we show the "if" part. Suppose that $f_i(R)\ R_i\ g_i(R)$. Then, we show that g is at least as manipulable as f. Let $R \in \mathcal{R}^n$, $i \in N$, and $R_i' \in \mathcal{R}$ be such that $f_i(R_i', R_{-i})\ P_i\ f_i(R)$. Let $\varepsilon \in \mathbb{R}_{++}$.

Note that by $f_i(R)\ R_i\ g_i(R)$, for each $x_i \in M$,

$$V_i(x_i, f_i(R)) \leq V_i(x_i, g_i(R)). \tag{1}$$

By $f_i(R_i', R_{-i})\ P_i\ f_i(R)$, Lemma 9 implies $x_i^f(R_i', R_{-i}) \neq 0$. Then, we divide the argument into two cases.

Case 1. $x_i^f(R_i', R_{-i}) \neq m$.

By $0 < x_i^f(R_i', R_{-i}) < m$, Lemma 11 implies that there is $R_i'' \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ such that $x_i^g(R_i'', R_{-i}) = x_i^f(R_i', R_{-i})$ and

$$|t_i^g(R_i'', R_{-i}) - t_i^f(R_i', R_{-i})| < \varepsilon. \tag{2}$$

Then, we have

$$\begin{aligned}
&G_i^g(R_i''; R) - G_i^f(R_i'; R) \\
&= V_i(x_i^f(R_i', R_{-i}), g_i(R)) - t_i^g(R_i'', R_{-i}) - \Big(V_i(x_i^f(R_i', R_{-i}), f_i(R)) - t_i^f(R_i', R_{-i})\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{by } x_i^g(R_i'', R_{-i}) = x_i^f(R_i', R_{-i})) \\
&\geq t_i^f(R_i', R_{-i}) - t_i^g(R_i'', R_{-i}) \qquad\qquad (\text{by } (1)) \\
&> -\varepsilon, \qquad\qquad (\text{by } (2))
\end{aligned}$$

or $G_i^g(R_i''; R) > G_i^f(R_i'; R) - \varepsilon$.

Case 2. $x_i^f(R_i', R_{-i}) = m$.

By $x_i^f(R_i', R_{-i}) = m$, Lemma 12 implies that there is $R_i'' \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ such that $x_i^g(R_i'', R_{-i}) = x_i^f(R_i', R_{-i}) = m$ and

$$t_i^g(R_i'', R_{-i}) - t_i^f(R_i', R_{-i}) < \varepsilon. \tag{3}$$

Then, we have

$$G_i^g(R_i''; R) = V_i(m, g_i(R)) - t_i^g(R_i'', R_{-i}) > V_i(m, f_i(R)) - t_i^f(R_i', R_{-i}) - \varepsilon = G_i^f(R_i'; R) - \varepsilon,$$

where the inequality follows from (1) and (3).

F.2.2 The "only if" part

We show the "only if" part. Suppose by contradiction that g is at least as manipulable as f, but there are $R \in \mathcal{R}^n$ and $i \in N$ such that $g_i(R)\ P_i\ f_i(R)$. Then, by Lemma 13, $\pi^g(R) < \pi^f(R)$. The proof has three steps.

Step 1. We here show that $x_i^g(R) \neq 0$. Suppose $x_i^g(R) = 0$. Then, $t_i^g(R) = 0$ by the definition of the inverse uniform-price rule. Thus, $g_i(R) = \mathbf{0}$, and by $g_i(R)\ R_i\ f_i(R)$, $\mathbf{0}\ P_i\ f_i(R)$. However, this contradicts *individual rationality* of f, which follows from *no price envy* of f by Lemma 1.

STEP 2. Now, we claim that there is $R'_i \in \mathcal{R}$ such that $f_i(R'_i, R_{-i})\ P_i\ f_i(R)$. By $g_i(R)\ P_i\ f_i(R)$, $t_i^g(R) < V_i(x_i^g(R), f_i(R))$. Thus, we can choose $\varepsilon \in \mathbb{R}_{++}$ such that

$$\varepsilon < V_i(x_i^g(R), f_i(R)) - t_i^g(R). \tag{4}$$

By Step 1, $x_i^g(R) \neq 0$. Thus, by Lemmas 11 and 12, there is $R'_i \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ such that $x_i^f(R'_i, R_{-i}) = x_i^g(R)$ and $t_i^f(R'_i, R_{-i}) - t_i^g(R) < \varepsilon$. Then,

$$t_i^f(R'_i, R_{-i}) < t_i^g(R) + \varepsilon < V_i(x_i^f(R'_i, R_{-i}), f_i(R)),$$

where the second inequality follows from (4) and $x_i^f(R'_i, R_{-i}) = x_i^g(R)$. This implies $f_i(R'_i, R_{-i})\ P_i\ f_i(R)$.

STEP 3. Now, we will derive a contradiction, and complete the proof. We consider the following two cases.

CASE 1. For each $R''_i \in \mathcal{R}$, $G_i^g(R''_i; R) \leq 0$.

By Step 2, $G_i^f(R'_i; R) > 0$. Let $\varepsilon \in \mathbb{R}_{++}$ be such that $\varepsilon < G_i^f(R'_i; R)$. Then, for each $R''_i \in \mathcal{R}$,

$$G_i^g(R''_i; R) \leq 0 < G_i^f(R'_i; R) - \varepsilon.$$

This, together with Step 2, yileds a contadiction that g is at least as manipulable as f.

CASE 2. There is $R''_i \in \mathcal{R}$ such that $G_i^g(R''_i; R) > 0$.

Then, $\sup_{R''_i \in \mathcal{R}} G_i^g(R''_i; R) > 0$. Moreover, by $g_i(R)\ P_i\ f_i(R)$, $V_i(x_i, g_i(R)) < V_i(x_i, f_i(R))$ for each $x_i \in M$. Then, we can choose $\varepsilon \in \mathbb{R}_{++}$ such that

$$2\varepsilon < \min\Big\{\min_{x_i \in M} V_i(x_i, g_i(R)) - V_i(x_i, f_i(R)),\ \sup_{R''_i \in \mathcal{R}} G_i^g(R''_i; R)\Big\}. \tag{5}$$

By Lemma 14, for $\frac{\varepsilon}{2} > 0$, there is $R''_i \in \mathcal{R}$ such that

$$G_i^g(R''_i; R) > \sup_{\tilde{R}_i \in \mathcal{R}} G_i^g(\tilde{R}_i; R) - \frac{\varepsilon}{2}. \tag{6}$$

By (5) and (6), $G_i^g(R''_i; R) > 0$. Thus, $g_i(R''_i, R_{-i})\ P_i\ g_i(R)$, and by Lemma 9, $x_i^g(R''_i, R_{-i}) \neq 0$.

By Lemmas 11 and 12, for $\frac{\varepsilon}{2} > 0$, there is $\overline{R}_i \in \mathcal{R}^{NI} \cap \mathcal{R}^{Q}$ such that $x_i^f(\overline{R}_i, R_{-i}) = x_i^g(R_i'', R_{-i})$ and

$$t_i^f(\overline{R}_i, R_{-i}) - t_i^g(R_i'', R_{-i}) < \frac{\varepsilon}{2}. \tag{7}$$

Then, for each $\tilde{R}_i \in \mathcal{R}$,

$$\begin{aligned}
G_i^g(\tilde{R}_i; R) &\leq \sup_{\tilde{R}_i' \in \mathcal{R}} G_i^g(\tilde{R}_i'; R) \\
&< G_i^g(R_i''; R) + \frac{\varepsilon}{2} && \text{(by (6))} \\
&= V_i(x_i^f(\overline{R}_i, R_{-i}), g_i(R)) - t_i^g(R_i'', R_{-i}) + \frac{\varepsilon}{2} && \text{(by } x_i^g(R_i'', R_{-i}) = x_i^f(\overline{R}_i, R_{-i})\text{)} \\
&< V_i(x_i^f(\overline{R}_i, R_{-i}), g_i(R)) - t_i^f(\overline{R}_i, R_{-i}) + \varepsilon && \text{(by (7))} \\
&< V_i(x_i^f(\overline{R}_i, R_{-i}), f_i(R)) - t_i^f(\overline{R}_i, R_{-i}) - \varepsilon && \text{(by (5))} \\
&= G_i^f(\overline{R}_i; R) - \varepsilon.
\end{aligned}$$

However, by Step 2, this contradicts that g is at least as manipulable as f. ■

F.3 Proof of Theorem 4

Theorem 4 directly follows from Proposition 3 and Lemma 13. ■

References

[1] Adachi, T. (2014), "Equity and the Vickrey allocation rule on general preference domains." *Social Choice and Welfare*, 42(4): 813–830.

[2] Andersson, T., L. Ehlers, and L. G. Svensson (2014), "Budget balance, fairness, and minimal manipulability." *Theoretical Economics*, 9(3): 753–777.

[3] Ausubel, L. M., P. Cramton, M. Pycia, M. Rostek, and M. Weretka (2014), "Demand reduction and inefficiency in multi-unit auctions." *Review of Economic Studies*, 81(4): 1366–1400.

[4] Azevedo, E. M. and E. Budish (2019), "Strategy-proofness in the large." *Review of Economic Studies*, 86(1): 81–116.

[5] Baisa, B. (2016), "Overbidding and inefficiencies in multi-unit Vickrey auctions for normal goods." *Games and Economic Behavior*, 99: 23–35.

[6] Baisa, B. (2020), "Efficient multi-unit auctions for normal goods." *Theoretical Economics*, 15(1): 361–413.

[7] Berga, D and S. Serizawa (2000), "Maximal domain for strategy-proof rules with one pure public goods." *Journal of Economic Theory*, 90(1): 39–61.

[8] Bogomolnaia, A. and H. Moulin (2001), "A new solution to the random assignment problem." *Journal of Economic Theory*, 100(2), 295-328.

[9] Burkett, J. and K. Woodward (2020), "Uniform-price auctions with a last accepted bid pricing rule." *Journal of Economic Theory*, 185: 104954.

[10] Chen, P., M. Egesdal, M. Pycia, and M. B. Yenmez (2016), "Manipulability of stable mechanisms." *American Economic Journal: Microeconomics*, 8(2): 202–214.

[11] Chew, S. H. and S. Serizawa (2007), "Characterizing the Vickrey combinatorial auction by induction." *Economic Theory*, 33(2): 393–406.

[12] Ching, S. and S. Serizawa (1998), "A maximal domain for the existence of strategy-proof rules." *Journal of Economic Theory*, 78(1): 157–166.

[13] Day, R. and P. Milgrom (2008), "Core-selecting package auctions." *International Journal of Game Theory*, 36(3): 393–407.

[14] Department of Telecommunications. (2021), "Amended notice inviting applications – incorporating amendment to NIA, Dated 28.01.2021." Government of India, Ministry of Communications. URL: https://dot.gov.in/sites/default/files/AmendedNoticeInvitingApplications28-01-2021.pdf?download=1 (accessed 2022-02-21).

[15] L/ Ehlers (2002), "Coalitional strategy-proof house allocation." *Journal of Economic Theory*, 105(2): 298–317.

[16] Engelbrecht-Wiggans, R., and C. M. Kahn (1998), "Multiunit auctions with uniform-prices." *Economic Theory*, 12(2): 227–258.

[17] Foley, D. (1967), "Resource allocation and the public sector." *Yale Economic Essays*, 7: 45-98.

[18] Guruswami, V., J. D. Hartline, A. R. Karlin, D. Kempe, C. Kenyon, and F. McSherry (2005), "On profit-maximizing envy-free pricing." *In: Proceedings of the 16th Annual ACM-SIAM Symposium on Discrete Algorithms*, 1164-1173.

[19] Holmström, B. (1979), "Groves' sheme on restricted domains." *Econometrica*, 47(5): 1137–1144.

[20] Kelly, J. S. (1988), "Minimal manipulability and local strategy-proofness." *Social Choice and Welfare*, 5(1): 81–85.

[21] Maus, S., H. Peters, and T. Storcken (2007), "Anonymous voting and minimal manipulability." *Journal of Economic Theory*, 135(1): 533-544.

[22] Milgrom, P. (2017), "*Discovering prices: Auction design in markets with complex constraints*." Columbia University Press.

[23] Morimoto, S. and S. Serizawa (2015), "Strategy-proofness and efficiency with non-quasi-linear preferences: A characterization of minimum price Warlasian rule." *Theoretical Economics*, 10(2): 437–455.

[24] Noussair, C. (1995), "Equilibria in a multi-object uniform-price sealed bid auction with multi-unit demands." *Economic Theory*, 5(2): 337–351.

[25] Ohseto, S. (2004), "Implementing egalitarian-equivalent allocation of indivisible goods on restricted domains." *Economic Theory*, 23(3): 659–670.

[26] Ohseto, S. (2006), "Characterizations of strategy-proof and fair mechanisms for allocating indivisible goods." *Economic Theory*, 29(1): 1–121.

[27] Paṗai, S. (2003), "Groves sealed bid auctions of heterogeneous objects with fair prices." *Social Choice and Welfare*, 20(3): 371–385.

[28] Pathak, P. A. and T. Sönmez (2013), "School admissions reform in Chicago and England: Comparing mechanisms by their vulnerability to manipulation." *American Economic Review*, 103(1): 80–106.

[29] Saitoh, H. and S. Serizawa (2008), "Vickrey allocation rule with income effect." *Economic Theory*, 35(2): 391–401.

[30] Sakai, T. (2008), "Second price auctions on general preference domains." *Economic Theory*, 37(2): 347–356.

[31] Sakai, T. (2013), "An equity characterization of second price auctions when preferences may not be quasilinear." *Review of Economic Design*, 17(1): 17–26.

[32] Shinozaki, H., T. Kazumura, and S. Serizawa (2020), "Efficient and strategy-proof multi-unit object allocation with money: (Non)decreasing marginal valuations without quasi-linearity." *ISER Discussion paper No. 1097*.

[33] Swinkers, J. M. (2001), "Efficiency of large private value auctions." *Econometrica*, 69(1): 37–68.

[34] Tajika, T. and T. Kazumura (2019), "Non-manipulability of uniform price auctions with a large number of objects." *International Journal of Game Theory*, 48(2): 543–569.

[35] Teytelboym, A., S. Li, S. D. Kominers, M. Akbarpour, and P. Dworczak (2021), "Discovering auctions: Contributiuons of Paul Milgrom and Robert Wilson." *The Scandanavian Journal of Economics*, 123(3): 709–750.

[36] Thomson, W. (1994), "Notions os equal, or equivalent opportunities." *Social Choice and Welfare*, 11(2): 137–156.

[37] Thomson, W. (2011), "*Fair allocation rules*." In: Arrow, K., A. Sen, K. Suzumura, (eds) Handbook of social choice and welfare. North-Holland, Amsterdam, 393-506.

[38] Varian, H. R. (1976), "Two problems in the theory of fairness." *Journal of Public Economics*, 5: 249–260.

[39] Vickrey, W. (1961), "Counterspeculation, auctions, and competitive seald tenders." *Journal of Finance*, 16(1): 8–37.

[40] Yengin, D. (2012), "Egalitarian-equvalent Groves mechanisms in the allocation of heterogeneous objects." *Social and Choice Welfare*, 38(1): 137–160.

Supplementary material for "No price envy in the multi-unit object allocation problem with non-quasi-linear preferences"

Hiroki Shinozaki*

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In this supplementary metrial, we provide the proofs and discussions omitted in the main text (Shinozaki, 2022).

1 Proof of Theorem 5

In this section, we prove Theorem 5.

Theorem 5. *Let $\mathcal{R}$ be such that $\mathcal{R}^{NI} \cap \mathcal{R}^{Q} \subseteq \mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^{+}$. A rule f on $\mathcal{R}^n$ is C-minimally manipulable among the class of rules on $\mathcal{R}^n$ satisfying no price envy and no subsidy for losers if and only if it is a minimum inverse uniform-price rule.*

Let $\mathcal{R}$ be such that $\mathcal{R}^{NI} \cap \mathcal{R}^{Q} \subseteq \mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^{+}$.

First, we give the lemmas that will be used to prove Theorem 5.

The next lemma states that if an agent can manipulate a minimum inverse uniform-price rule, then his consumption level must be no greater than the original level as a consequence of the manipulation.

Lemma 15. *Let f be a minimum inverse uniform-price rule on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$, $i \in N$, and $R'_i \in \mathcal{R}$ be such that $f_i(R'_i, R_{-i}) \; P_i \; f_i(R)$. Then, $x_i^f(R'_i, R_{-i}) \leq x_i^f(R)$.*

Proof. Suppose by contradiction that $x_i^f(R'_i, R_{-i}) > x_i^f(R)$. By $x_i(R'_i, R_{-i}) > x_i(R)$ and the feasibility, there is $j \in N \backslash \{i\}$ such that $x_j(R'_i, R_{-i}) < x_j(R)$. We have

$$\begin{aligned} mv^{m+1}(R_i'^{inv}, R_{-i}^{inv}) &\geq v_j^{inv}(x_j(R'_i, R_{-i}) + 1) - v_j^{inv}(x_j(R'_i, R_{-i})) && \text{(by Lemma 5 (ii))} \\ &\geq v_j^{inv}(x_j(R)) - v_j^{inv}(x_j(R) - 1) && \text{(by } R_j^{inv} \in \mathcal{R}^{NI}\text{)} \\ &\geq mv^m(R^{inv}) && \text{(by Lemma 5 (i))} \\ &\geq mv^{m+1}(R^{inv}). \end{aligned}$$

*Graduate School of Economics, Osaka University. Email: vge017sh@student.econ.osaka-u.ac.jp

Thus, we obtain

$$\pi^f(R_i', R_{-i}) = mv^{m+1}(R_i'^{inv}, R_{-i}^{inv}) \geq mv^{m+1}(R^{inv}) = \pi^f(R). \tag{1}$$

Note that $p_j^f(R) = \pi^f(R)$. Note also that by Theorem 1, f satisfies *no price envy*. Then,

$$f_i(R)\ R_i\ (x_i(R_i', R_{-i}), \pi^f(R)x_i(R_i', R_{-i}))\ R_i\ (x_i(R_i', R_{-i}), \pi^f(R_i', R_{-i})x_i(R_i', R_{-i})) = f_i(R_i', R_{-i}),$$

where the first relation follows from *no price envy*, and the second one from (1). □

The next lemma further states that if an agent is successful in manipulating a minimum inverse uniform-price rule, then he can not receive all the units as a result of the manipulation.

Lemma 16. *Let f be a minimum inverse uniform-price rule on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$, $i \in N$, and $R_i' \in \mathcal{R}$ be such that $f_i(R_i', R_{-i})\ P_i\ f_i(R)$. Then, $x_i^f(R_i', R_{-i}) \neq m$.*

Proof. By contradiction, suppose $x_i^f(R_i', R_{-i}) = m$. By Lemma 15 and $f_i(R_i', R_{-i})\ P_i\ f_i(R)$, $x_i^f(R) = m$. Thus, by $f_i(R_i', R_{-i})\ P_i\ f_i(R)$,

$$\pi^f(R_i', R_{-i})m = t_i^f(R_i', R_{-i}) < t_i^f(R) = \pi^f(R)m,$$

or

$$mv^{m+1}(R_i'^{inv}, R_{-i}^{inv}) = \pi^f(R_i', R_{-i}) < \pi^f(R) = mv^{m+1}(R^{inv}). \tag{1}$$

However, by $x_i^f(R) = x_i^f(R_i', R_{-i}) = m$, Lemma 5 (i) implies

$$v_i^{inv}(m) - v_i^{inv}(m-1) \geq mv^m(R^{inv}) \text{ and } v_i'^{inv}(m) - v_i'^{inv}(m-1) \geq mv^m(R_i'^{inv}, R_{-i}^{inv}).$$

Thus, by $R_{-i}^{inv} \in (\mathcal{R}^{NI} \cap \mathcal{R}^Q)^{n-1}$,

$$mv^{m+1}(R^{inv}) = \max_{j \in N \setminus \{i\}} v_j^{inv}(1) = mv^{m+1}(R_i'^{inv}, R_{-i}^{inv}).$$

This contradicts (1). □

Finally, the next lemma states that if the prices of a pair of inverse uniform-price rules are different at a preference profile, then each agent must receive the same number of units under the two rules.

Lemma 17. *Let f, g be a pair of inverse uniform-price rules on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$. If $\pi^f(R) \neq \pi^g(R)$, then $x_i^f(R) = x_i^g(R)$ for each $i \in N$.*

Proof. We will prove a contrapositive. Suppose there is $i \in N$ such that $x_i^f(R) \neq x_i^g(R)$. Without loss of generality, assume $x_i^f(R) > x_i^g(R)$. Then,

$$\begin{aligned} mv^{m+1}(R^{inv}) &\geq v_i^{inv}(x_i^g(R)+1) - v_i^{inv}(x_i^g(R)) && \text{(by Lemma 5 (ii))} \\ &\geq v_i^{inv}(x_i^f(R)) - v_i^{inv}(x_i^f(R)-1) && \text{(by } R_i^{inv} \in \mathcal{R}^{NI}\text{)} \\ &\geq mv^m(R^{inv}), && \text{(by Lemma 5 (i))} \end{aligned}$$

Thus, by $mv^m(R^{inv}) \geq mv^{m+1}(R^{inv})$, we obtain $mv^m(R^{inv}) = mv^{m+1}(R^{inv})$. Thus, we get

$$\pi^f(R) = mv^{m+1}(R^{inv}) = mv^m(R^{inv}) = \pi^g(R),$$

as desired. □

1.1 The "if" part

First, we prove the "if" part of Theorem 5. Let f be a minimum inverse uniform-price rule on $\mathcal{R}^n$. Note that by Theorem 1, f satisfies both *no price envy* and *no subsidy for losers*. Let g be a rule on $\mathcal{R}^n$ satisfying *no price envy* and *no subsidy for losers*. By Theorem 1, g is an inverse uniform-price rule on $\mathcal{R}^n$. We show that g is C-at least as manipulable as f. Let $R \in \mathcal{R}^n$, $i \in N$, and $R'_i \in \mathcal{R}$ be such that $f_i(R'_i, R_{-i})\ P_i\ f_i(R)$. Let $\varepsilon \in \mathbb{R}_{++}$.

By $f_i(R'_i, R_{-i})\ P_i\ f_i(R)$, $t_i^f(R'_i, R_{-i}) < V_i(x_i^f(R'_i, R_{-i}), f_i(R))$. Let $\varepsilon' \in \mathbb{R}_{++}$ be such that

$$\varepsilon' < \min\Big\{V_i(x_i^f(R'_i, R_{-i}), f_i(R)) - t_i^f(R'_i, R_{-i}), \varepsilon\Big\}. \tag{1}$$

By $f_i(R'_i, R_{-i})\ P_i\ f_i(R)$, Lemmas 7 and 16 together imply $0 < x_i^f(R'_i, R_{-i}) < m$. Thus, by Lemma 11, there is $R''_i \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ such that $x_i^g(R''_i, R_{-i}) = x_i^f(R'_i, R_{-i})$ and

$$|t_i^g(R''_i, R_{-i}) - t_i^f(R'_i, R_{-i})| < \varepsilon'. \tag{2}$$

Now, we show that $g_i(R''_i, R_{-i})\ P_i\ g_i(R)$. Note that $\pi^f(R) = mv^{m+1}(R^{inv}) \leq \pi^g(R)$. Thus, by Lemma 13, $f_i(R)\ R_i\ g_i(R)$. This implies

$$V_i(x_i^g(R''_i, R_{-i}), f_i(R)) \leq V_i(x_i^g(R''_i, R_{-i}), g_i(R)). \tag{3}$$

Then,

$$\begin{aligned}
&V_i(x_i^g(R_i'', R_{-i}), g_i(R)) - t_i^g(R_i'', R_{-i}) \\
&\geq V_i(x_i^g(R_i'', R_{-i}), f_i(R)) - t_i^g(R_i'', R_{-i}) && \text{(by (3))} \\
&> V_i(x_i^g(R_i'', R_{-i}), f_i(R)) - t_i^f(R_i', R_{-i}) - \varepsilon' && \text{(by (2))} \\
&= V_i(x_i^f(R_i', R_{-i}), f_i(R)) - t_i^f(R_i', R_{-i}) - \varepsilon' \\
&> 0, && \text{(by (1))}
\end{aligned}$$

or $V_i(x_i^g(R_i'', R_{-i}), g_i(R)) > t_i^g(R_i'', R_{-i})$. This implies $g_i(R_i'', R_{-i})\ P_i\ g_i(R)$.

By $x_i^f(R_i', R_{-i}) = x_i^g(R_i'', R_{-i})$, (1), and (2), we also have

$$d(g_i(R_i'', R_{-i}), f_i(R_i', R_{-i})) = |t_i^g(R_i'', R_{-i}) - t_i^f(R_i', R_{-i})| < \varepsilon' < \varepsilon.$$

Thus, g is C-at least as manipulable as f. ■

1.2 The "only" part

Then, we prove the "only if" part. Suppose by contradiction that f is not a minimum inverse uniform-price rule on $\mathcal{R}^n$, but is a C-minimally manipulable rule on the class of rules on $\mathcal{R}^n$ satisfying *no price envy* and *no subsidy for losers*. Then, f satisfies both *no price envy* and *no subsidy for losers*. Thus, by Theorem 1, f is an inverse uniform-price rule. Since f is not a minimum inverse uniform-price rule, there is $R \in \mathcal{R}^n$ such that $\pi^f(R) > mv^{m+1}(R^{inv})$.

Let g be a minimum inverse uniform-price on $\mathcal{R}^n$. By the feasibility, there is $i \in N^+(x^f(R))$. Then, by $\pi^g(R) = mv^{m+1}(R^{inv}) < \pi^f(R)$, Lemma 17 implies $x_i^g(R) = x_i^f(R)$. We consider the following two cases.

Case 1. $x_i^f(R) \neq m$.

Let $\varepsilon \in \mathbb{R}_{++}$ be such that

$$2\varepsilon < \min\{\pi^f(R) - \pi^g(R), 1\}.$$

Let $R_i' \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that $v_i'(x_i) = (\pi^f(R) - \varepsilon)x_i$ for each $x_i \in M$. By $\pi^f(R) \leq mv^m(R^{inv})$ and $\varepsilon < \pi^f(R) - mv^{m+1}(R^{inv})$, for each $x_i \in M \backslash \{m\}$,

$$mv^{m+1}(R^{inv}) < v_i'(x_i + 1) - v_i'(x_i) < mv^m(R^{inv}).$$

Then, by $0 < x_i^f(R) < m$, Lemma 10 implies that $x_i^f(R_i', R_{-i}) = x_i^f(R)$ and $\pi^f(R_i', R_{-i}) = \pi^f(R) - \varepsilon$. Then, $\pi^f(R_i', R_{-i}) < \pi^f(R)$. Thus, by $x_i^f(R_i', R_{-i}) = x_i^f(R)$, we obtain

$f_i(R'_i, R_{-i})\ P_i\ f_i(R)$.

Let $R''_i \in \mathcal{R}$ be such that $g_i(R''_i, R_{-i})\ P_i\ g_i(R)$. Suppose $x^g_i(R''_i, R_{-i}) \neq x^f_i(R'_i, R_{-i})$. Then,

$$d(g_i(R''_i, R_{-i}), f_i(R'_i, R_{-i})) \geq |x^g_i(R''_i, R_{-i}) - x^f_i(R'_i, R_{-i})| \geq 1 > \varepsilon.$$

Instead, suppose $x^g_i(R''_i, R_{-i}) = x^f_i(R'_i, R_{-i})$. Then, by $x^f_i(R'_i, R_{-i}) = x^f_i(R) = x^g_i(R)$ and $g_i(R''_i, R_{-i})\ P_i\ g_i(R)$, $t^g_i(R''_i, R_{-i}) < t^g_i(R)$. By $x^g_i(R) \neq 0$, $\pi^g(R''_i, R_{-i}) < \pi^g(R)$. Then, by $\varepsilon < \pi^f(R) - \pi^g(R)$,

$$\pi^g(R''_i, R_{-i}) < \pi^f(R) - \varepsilon = \pi^f(R'_i, R_{-i}). \tag{4}$$

Then,

$$\begin{aligned}
&d(g_i(R''_i, R_{-i}), f_i(R'_i, R_{-i})) \\
&= |t^g_i(R''_i, R_{-i}) - t^f_i(R'_i, R_{-i})| && \text{(by } x^g_i(R''_i, R_{-i}) = x^f_i(R'_i, R_{-i})\text{)} \\
&= x^f_i(R'_i, R_{-i})(\pi^f(R'_i, R_{-i}) - \pi^g(R''_i, R_{-i})) && \text{(by (4))} \\
&> x^f_i(R'_i, R_{-i})(\pi^f(R'_i, R_{-i}) - \pi^g(R)) && \text{(by } \pi^g(R''_i, R_{-i}) < \pi^g(R)\text{)} \\
&= x^f_i(R'_i, R_{-i})(\pi^f(R) - \pi^g(R) - \varepsilon) \\
&> x^f_i(R'_i, R_{-i})\varepsilon && \text{(by } 2\varepsilon < \pi^f(R) - \pi^g(R)\text{)} \\
&\geq \varepsilon. && \text{(by } x^f_i(R'_i, R_{-i}) = x^f_i(R) \neq 0\text{)}
\end{aligned}$$

Thus, in either case, g is not C-at least as manipulable as f, which contradicts that f is C-minimally manipulable among the class of rules satisfying *no price envy* and *no subsidy for losers*.

CASE 2. $x^f_i(R) = m$.

By $x^g_i(R) = x^f_i(R) = m$ and $\pi^g(R) < \pi^f(R)$,

$$t^g_i(R) = m\pi^g(R) < m\pi^f(R) = t^f_i(R).$$

Let $\varepsilon \in \mathbb{R}_{++}$ be such that $\varepsilon < t^f_i(R) - t^g_i(R)$. By $x^g_i(R) = m$, Lemma 12 implies that there is $R'_i \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ such that $x^f_i(R'_i, R_{-i}) = m$ and $t^f_i(R'_i, R_{-i}) - t^g_i(R) < \varepsilon$. Then,

$$t^f_i(R'_i, R_{-i}) < t^g_i(R) + \varepsilon < t^f_i(R),$$

where the second inequality follows from $\varepsilon < t^f_i(R) - t^g_i(R)$. Thus, by $x^f_i(R) = x^f_i(R'_i, R_{-i}) = m$, we obtain $f_i(R'_i, R_{-i})\ P_i\ f_i(R)$.

Let $R''_i \in \mathcal{R}$ be such that $g_i(R''_i, R_{-i})\ P_i\ g_i(R)$. By Lemma 16, $x^g_i(R''_i, R_{-i}) \neq m$. Then, for $\varepsilon = 1$,

$$d(g_i(R_i'', R_{-i}), f_i(R_i', R_{-i})) \geq |x_i^g(R_i'', R_{-i}) - x_i^f(R_i', R_{-i})| \geq 1 = \varepsilon.$$

Thus, g is not C- at least as manipulable as f, a contradiction. ■

2 Proof of Remark 4

In this section, we prove Remark 4

Remark 4. Let $R_i \in \mathcal{R}^{NI} \cap \mathcal{R}^+$.
(i) For each $x_i \in M\backslash\{0, m\}$, there is a unique payment $t^*(x_i) \in (0, V_i(x_i, \mathbf{0})]$ such that $V_i(x_i + 1, (x_i, t^*(x_i))) - t^*(x_i) = \frac{t^*(x_i)}{x_i}$.
(ii) For each $x_i \in M\backslash\{0, m-1, m\}$, $\frac{t^*(x_i)}{x_i} \geq \frac{t^*(x_i+1)}{x_i+1}$.

. The proof basically follows the proof of Lemmas 8 and 10 of Shinozaki et al. (2020), but the proof here slightly generalize that of Shinozaki et al. (2020) in that we consider a preference that exhibits both nonincreasing marginal valuations and nonnegative income effects, while Shinozaki et al. (2020) consider a preference that exhibits both decreasing marginal valuations and positive income effects.

Let $R_i \in \mathcal{R}^{NI} \cap \mathcal{R}^+$.

First, we prove (i). Let $x_i \in M\backslash\{0, m\}$. Let $h : \mathbb{R} \to \mathbb{R}$ be such that $h(t_i) = V_i(x_i + 1, (x_i, t_i)) - t_i - \frac{t_i}{x_i}$ for each $t_i \in \mathbb{R}$. By object monotonicity, $(x_i + 1, 0)\ P_i\ (x_i, 0)$. This implies $V_i(x_i + 1, (x_i, 0)) > 0$. Thus, we obtain

$$h(0) = V_i(x_i + 1, (x_i, 0)) > 0. \tag{1}$$

Further, we have

$$\begin{aligned} h(V_i(x_i, \mathbf{0})) &= V_i\Big(x_i + 1, \Big(x_i, V_i(x_i, \mathbf{0})\Big)\Big) - V_i(x_i, \mathbf{0}) - \frac{V_i(x_i, \mathbf{0})}{x_i} \\ &= V_i(x_i + 1, \mathbf{0}) - V_i(x_i, \mathbf{0}) - \frac{V_i(x_i, \mathbf{0})}{x_i} \\ &= v_i(x_i + 1, \mathbf{0}) - v_i(x_i, \mathbf{0}) - \frac{v_i(x_i, \mathbf{0})}{x_i} \\ &\leq 0, \end{aligned} \tag{2}$$

where the second equality follows from $(x_i, V_i(x_i, \mathbf{0}))\ I_i\ \mathbf{0}$, the third one from the definition of the net valuation, and the inequality from $R_i \in \mathcal{R}^{NI}$.

Note that by Remark 2, $h^+(t_i; x_i) \equiv V_i(x_i + 1, (x_i, t_i)) - t_i$ is a nonincreasing function on $\mathbb{R}$. Thus, $h(\cdot)$ is strictly decreasing. Let $\bar{t}_i \in \mathbb{R}_{++}$ be such that $\bar{t}_i > V_i(x_i, \mathbf{0})$. Then, by (2),

$$h(\bar{t}_i) < h(V_i(x_i, \mathbf{0})) \leq 0. \tag{3}$$

By continuity of R_i, $h^+(\cdot; x_i)$ is continuous on $[0, \bar{t}_i]$. Thus, $h(\cdot)$ is continuous on $[0, \bar{t}_i]$ as well. Then, by (1) and (3), the intermediate value theorem implies that there is a payment $t_i^*(x_i) \in (0, \bar{t}_i)$ such that $h(t_i^*(x_i)) = 0$. This implies

$$V_i(x_i + 1, (x_i, t_i^*(x_i))) - t_i^*(x_i) = \frac{t_i^*(x_i)}{x_i}.$$

Since $h(\cdot)$ is strictly decreasing, such a payment $t_i^*(x_i)$ must be unique. Moreover, by $h(t_i^*(x_i)) = 0$ and (2), we have $t_i^*(x_i) \leq V_i(x_i, \mathbf{0})$ since $h(\cdot)$ is strictly decreasing.

Next, we prove (ii). Let $x_i \in M \backslash \{0, m-1, m\}$. Suppose by way of contradiction that $\frac{t^*(x_i)}{x} < \frac{t^*(x_i+1)}{x_i+1}$. Let $\bar{t} \equiv V_i(x_i + 1, (x_i, t^*(x_i)))$.

First, we claim that $\bar{t} < t^*(x_i+1)$. By Remark 4 (i), $V_i(x_i+1, (x_i, t^*(x_i))) = \frac{x_i+1}{x_i} t^*(x_i)$. Thus, by $\frac{t^*(x_i)}{x} < \frac{t^*(x_i+1)}{x_i+1}$, we obtain

$$\bar{t} = V_i(x_i + 1, (x_i, t^*(x_i))) = \frac{x_i + 1}{x_i} t^*(x_i) < t^*(x_i + 1). \tag{4}$$

Second, we claim that $V_i(x_i+2, (x_i+1, \bar{t})) - \bar{t} < V_i(x_i+2, (x_i+1, t^*(x_i+1))) - t^*(x_i+1)$. Note that by the definition of $\bar{t}$, $(x_i + 1, \bar{t})\ I_i\ (x_i, t^*(x_i))$. This implies

$$V_i(x_i, (x_i + 1, \bar{t})) = V_i(x_i, (x_i, t^*(x_i))) = t^*(x_i). \tag{5}$$

Then,

$$\begin{aligned}
V_i(x_i + 2, (x_i + 1, \bar{t})) - \bar{t} &\leq \bar{t} - V_i(x_i, (x_i + 1, \bar{t})) && (\text{by } R_i \in \mathcal{R}^{NI}) \\
&= \bar{t} - t^*(x_i) && (\text{by } (5)) \\
&= \frac{t^*(x_i)}{x_i} && (\text{by Remark 4 (i)}) \\
&< \frac{t^*(x_i + 1)}{x_i + 1} \\
&= V_i(x_i + 2, (x_i + 1, t^*(x_i + 1))) - t^*(x_i + 1), && (6)
\end{aligned}$$

where the last equality follow from Remark 4 (i).

We derive a contradiction to Remark 2 by (4) and (6). ■

3 Manipulability measure without gains from manipulations

In Section 6.2 of Shinozaki (2022), we compare our manipulability measure to other measures that take gains from manipulations into account. In this section, we compare it to a measure that does not take gains from manipulations into account. There are several such manipulability measures, and we comapre ours to the one introduced by Pathak and Sonmez (2013).

According to the "strongly as manipulable as" relation of Pathak and Sonmez (2013), a rule is at least as manipulable as another rule if for each preference profile and each agent, whenever he can manipulate the latter rule, he can also manipulate the former rule. Formally, we say that a rule on $\mathcal{R}^n$ is said to be *weakly at least as manipulable as* another rule g on $\mathcal{R}^n$ if for each $R \in \mathcal{R}^n$ each $i \in N$, and each $R'_i \in \mathcal{R}$, whenever $g_i(R'_i, R_{-i})\ P_i\ g_i(R)$, there is $R''_i \in \mathcal{R}$ such that $f_i(R''_i, R_{-i})\ P_i\ f_i(R)$. Clearly, if a rule f on $\mathcal{R}^n$ is at least as manipulable as another rule g on $\mathcal{R}^n$, then f is weakly at least as manipulable as g.

Definition 12. A rule f on $\mathcal{R}^n$ is **weakly minimally manipulable among the class of rules** if (i) f is in the class, and (ii) for each rule g on $\mathcal{R}^n$ in the class, g is weakly at lesat as manipulable as f.

Given a rule f on $\mathcal{R}^n$ and $i \in N$, let

$$\mathcal{M}_i^f \equiv \{R \in \mathcal{R}^n : \exists R'_i \in \mathcal{R} \text{ s.t. } f_i(R'_i, R_{-i})\ P_i\ f_i(R)\}$$

denote the set of preference profiles at which agent i can manipulate the rule f.

Remark 8. (i). A rule f on $\mathcal{R}^n$ is weakly at least as manipulable as another rule g on $\mathcal{R}^n$ if and only if for each $i \in N$, $\mathcal{M}_i^g \subseteq \mathcal{M}_i^f$.
(ii). A rule f on $\mathcal{R}^n$ is weakly minimally manipulable among a given class of rules on $\mathcal{R}^n$ if and only if (ii-i) f is in the class, and (ii-ii) for each rule g on $\mathcal{R}^n$ in the class and each $i \in N$, $\mathcal{M}_i^f \subseteq \mathcal{M}_i^g$.

Note that if a rule is minimally manipulable among a given class of rules, then it is also weakly minimally manipubale among the class. Thus, by Theorem 4, we obtain the following.

Corollary 4. *Let $\mathcal{R}$ be such that $\mathcal{R}^{NI} \cap \mathcal{R}^{Q} \subseteq \mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^{+}$. Then, a minimum inverse uniform-price rule on $\mathcal{R}^n$ is weakly minimally manipulable among the class of rules on $\mathcal{R}^n$ satisfying both no price envy and no subsidy for losers.*

The next example demonstrates that the converse of Corollary 4 does not hold, i.e., the minimum inverse uniform-price rule is not the only weakly minimally manipulable rule among the class of rules satisfying both *no price envy* and *no subsidy for losers*.

Example 4. Let $\mathcal{R}$ be such that $\mathcal{R}^{NI} \cap \mathcal{R}^{Q} \subseteq \mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^{+}$. Let f and $\hat{f}$ be minimum and maximum inverse uniform-price rules on $\mathcal{R}^n$, respectively.[1] Let $R^* \in (\mathcal{R}^{NI} \cap \mathcal{R}^{Q})^n$ be such that (i) $v_1^*(x_1) = 4x_1$ for each $x_1 \in M$, (ii) $v_2^*(1) = 3.5$ and $v_2^*(x_2) = 3.5 + 0.01(x_2 - 1)$ for each $x_2 \in M\backslash\{0,1\}$, and (iii) $v_i^*(x_i) = 0.01x_i$ for each $i \in N\backslash\{1,2\}$ and each $x_i \in M$. Let g be a rule on $\mathcal{R}^n$ such that for each $R \in \mathcal{R}$, if $R \neq R^*$, then $g(R) = f(R)$, and if $R = R^*$, then $g(R) = \hat{f}(R)$. Then, by $\pi^g(R^*) = 4 > 3.5 = \pi^f(R^*)$, g is not a minimum inverse uniform-price rule.

Note that $R^* \in \mathcal{M}_1^g \cap \mathcal{M}_1^f$, and for each $i \in N\backslash\{1\}$, $R^* \notin \mathcal{M}_i^g \cup \mathcal{M}_i^f$. Note also that for each $i \in N$ and each $R \in \mathcal{R}^n\backslash\{R^*\}$, by $g_i(R) = f_i(R)$, $R \in \mathcal{M}_i^g$ if and only if $R \in \mathcal{M}_i^f$. Thus, $\mathcal{M}_i^g = \mathcal{M}_i^f$ for each $i \in N$. Since f is weakly minimally manipulable among the class of rules on $\mathcal{R}^n$ satisfying both *no price envy* and *no subsidy for losers*, g is as well. □

References

[1] Pathak, P. A. and T. Sonmez (2013), "School admissions reform in Chicago and England: Comparing mechanisms by their vulnerability to manipulation." *American Economic Review*, 103(1): 80–106.

[2] Shinozaki, H. (2022), "No price envy and minimal manipulability in the multi-unit object allocation problem with non-quasi-linear preferences." Working paper.

[3] Shinozaki, H., T. Kazumura, and S. Serizawa (2020), "Efficient and strategy-proof multi-unit object allocation with money: (Non)decreasing marginal valuations with non-quasi-linear preferences." ISER Discussion Paper, No. 1097.

[1]The maximum inverse uniform-price rule is an inverse uniform-price rule such that for each $R \in \mathcal{R}$, $\pi^f(R) = mv^m(R^{inv})$.