

**SIGNALING UNDER
DOUBLE-CROSSING PREFERENCES:
THE CASE OF
DISCRETE TYPES**

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March 2022

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Signaling under Double-Crossing Preferences: The Case of Discrete Types

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March 3, 2022

Abstract. The class of double-crossing preferences, where signaling is cheaper for higher types than for lower types at low signaling levels and the opposite is true at high signaling levels, underlines the phenomenon of countersignaling. We show that under the D1 refinement, the equilibrium signaling action must be quasi-concave in type and generally exhibits pooling, with intermediate types choosing higher actions than higher and lower types. We provide an algorithm to systematically construct an equilibrium and use this algorithm to establish its existence for this general class of preferences with an arbitrary discrete-type distribution.

Keywords. countersignaling; weak pairwise-matching condition; minimum allocation; low types separate and high types pairwise-pool

JEL Classification. D82; I21

The first author acknowledges financial support from JSPS KAKENHI (20K01560). The second author acknowledges financial support from JSPS KAKENHI (18H03640, 19H01468, 20H05631, 21H04398), and the Joint Usage/Research Center for Behavioral Economics at ISER and Osaka University for its International Joint Research Promotion Program.

1. Introduction

Despite its widespread use in signaling models, the single-crossing property imposes strong restrictions on the structure of preferences, and its validity is not necessarily evident in economic applications.¹ To bridge this gap in the literature, in Chen et al. (2021b), we introduce a double-crossing property and provide a general analysis of signaling under this property. That paper identifies some fundamental features of double-crossing preferences, which enables us to obtain a full characterization of equilibria and establish equilibrium existence under such environments, when agent types are distributed continuously. From the practical point of view, however, discrete-type models are often more useful, or even indispensable, as many applied settings are naturally cast in discrete types. For this framework to be of use for applied work, it is important to discern which insights of the continuous-type model hold generally and which need amendment when agent types are discrete.

In this paper, we extend the continuous-type model of Chen et al. (2021b) to an arbitrary number of types. The extension is not as straightforward as it may appear. Specifically, there are two technical issues that confront us. One issue is that in Chen et al. (2021b), we first establish the continuity of the equilibrium action above some threshold and then exploit this feature to derive other equilibrium properties. This approach is not applicable to a discrete-type model that has no corresponding notion of continuity. The other issue is that with discrete types, agents may randomize over different actions, so we need to explicitly consider mixed strategies. This substantially enlarges the strategy space, as it is now a set of probability distributions over actions, whereas the equilibrium strategy can be represented simply by a (piecewise continuous) function of agent type in the continuous-type model. These issues prevent us from applying the results of Chen et al. (2021b) directly to a discrete-type model, both for obtaining a characterization and for establishing equilibrium existence, and force us to explore different avenues.

Central to our analysis are the general characterization and existence results that hold for any number of types. In Section 3, we provide a characterization of equilibria that satisfy the D1 refinement (hereafter, D1 equilibria): types above some threshold are pairwise

¹Hörner (2008) notes, “Little is known about equilibria when single-crossing fails, as may occur in applications.” There are now several applications which deal with the situation where the single-crossing property fails to hold (Feltovich et al., 2002; Kolev and Prusa, 2002; Daley and Green, 2014; Bobtcheff and Levy, 2017; Frankel and Kartik, 2019; Chen et al., 2021a; Degan and Li, 2021).

matched, and the equilibrium action is quasi-concave in type. Our characterization result thus confirms that the key insights of Chen et al. (2021b) hold in a qualitative sense, but also reveals that there are some properties that are specific to the discrete-type model. The most symbolic difference is captured by what we call the *weak pairwise-matching condition*, which plays an essential role in ensuring global incentive compatibility. In the continuous-type model, pairwise matching is characterized by a bijective mapping, where all types above the threshold pool with some other type. In the discrete-type model, the mapping is often an injection, and the lower end of types may be left unmatched when types are not dense enough. This result illuminates a fundamental property of signaling under double-crossing preferences: it is the higher end of types above the threshold who need to be pooled with lower types, but not vice versa. Understanding this property is crucial when we construct an equilibrium.

In Section 5, we provide an algorithm to construct a D1 equilibrium that works for any double-crossing preferences and any number of types, and show by construction that a D1 equilibrium always exists. In the continuous-type model, equilibrium is essentially characterized by a system of differential equations, and the problem to find an equilibrium reduces to a standard initial value problem, to which we can apply standard arguments to establish the existence of solutions. No such techniques can be used when types are discrete. In the discrete-type model, we must consider the possibility of mixed strategies, and equilibrium is characterized by a set of probability distributions over actions. The strategy space thus expands substantially as the number of types increases. Moreover, since the equilibrium conditions (i.e., incentive compatibility and D1) in the discrete-type model generally appear as inequality constraints, there is a continuum of possible actions for each type to be considered, making it complicated to pin down an equilibrium.

A conceptual advancement on this front is what we call the *minimum allocation*. Our equilibrium construction centers around this notion. Given our characterization result on pairwise matching, we choose types in the lower end of the type distribution to match with the highest remaining type in each round of our algorithm, and move down until we run out of types to match. The problem is that in each round, we need to determine which fractions of types to match with the highest remaining type and which fractions to remain fully separated, so as to satisfy all the equilibrium conditions. The adjustment process could be highly complicated and easily become intractable, as there are a multitude of possible choices in each round due to inequality constraints. We show that this process can be substantially simplified and guaranteed to always end up with a well-behaved

equilibrium by searching for the minimum allocation repeatedly in each round.

One important economic insight we gain from the class of models with double-crossing preferences is countersignaling—a phenomenon where higher and lower types are pooled at a low signaling action while some intermediate types separate by choosing a higher action (Feltovich et al., 2002; Araujo et al., 2007). While real-life examples of countersignaling abound,² analyzing this phenomenon in general terms has proved to be difficult and elusive, and our understanding of countersignaling has been limited to specific contexts despite its potential social implications: for instance, Feltovich et al. (2002)—the seminal work on countersignaling—consider only three types and restrict attention to pure-strategy equilibria. Our analysis provides a flexible framework to understand countersignaling under general payoff and distribution functions and can be used to analyze this perverse yet pervasive phenomenon in a variety of applied contexts.

2. Model

We consider a model with an arbitrary number $I \geq 2$ of types. The type space is $\{1, \dots, I\}$ where we use i to denote a generic type. The type of an agent is his private information.

In the following analysis, it is often convenient to distinguish between agent n and type i . We suppose that there is a continuum of agents with unit measure, each indexed by $n \in [0, 1]$. Agents are partitioned into types $i = 1, \dots, I$ by (F_0, \dots, F_I) , where $F_0 = 0$, $F_I = 1$, and $F_i < F_{i+1}$ for $i = 0, 1, \dots, I - 1$. The partition (F_0, \dots, F_I) summarizes the type distribution with $F_{i'} = \mathbb{P}(i \leq i')$. Let $\iota(n)$ indicate the type of agent n , where

$$\iota(n) = \begin{cases} i & \text{if } n \in [F_{i-1}, F_i) \text{ for } i < I, \\ I & \text{if } n \in [F_{I-1}, F_I]. \end{cases}$$

We adopt the above convention because it is useful for representing mixed strategies. Suppose type i randomizes between actions a'' and a' with probabilities z and $1 - z$ respectively. This is equivalent to having agent $n'' \in [F_{i-1}, (1 - z)F_{i-1} + zF_i)$ choose a'' and agent $n' \in [(1 - z)F_{i-1} + zF_i, F_i)$ choose a' .³ Under this convention, we need to pay special attention to agents in the set $B := \{F_0, \dots, F_{I-1}\}$. We refer to B as the set of “threshold

²Feltovich et al. (2002) raise a number of examples drawn from common observations, such as “The nouveau riche flaunt their wealth, but the old rich scorn such gauche displays,” and provide experimental evidence in support of countersignaling. Also see Araujo et al. (2007) and the references therein. Dixit and Nalebuff’s (2008) book, *The Art of Strategy*, has a section devoted to countersignaling.

³In this example, if $i = I$, then our convention would have agent $n' = F_I$ also choose action a' .

agents.”

Let $u(a, t, i)$ denote the payoff to an agent of type i , where a is the signaling action he chooses and t is the market’s perception of his type, or his “reputation,” i.e., $t = \mathbb{E}[i | a] \in [1, I]$. Our specification of *double-crossing preferences* follows Chen et al. (2021b).

Assumption 1. $u : \mathbb{R}_+ \times [1, I] \times \{1, \dots, I\} \rightarrow \mathbb{R}$ is twice continuously differentiable, strictly increasing in t and strictly decreasing in a .

Assumption 2. For any $i' > i''$, there exists a continuous function $D(\cdot; i', i'') : [1, I] \rightarrow \mathbb{R}_+$ such that:

(a) if $a < a' \leq D(t'; i', i'')$, then

$$u(a, t, i'') \leq u(a', t', i'') \implies u(a, t, i') < u(a', t', i');$$

(b) if $a > a' \geq D(t'; i', i'')$, then

$$u(a, t, i'') \leq u(a', t', i'') \implies u(a, t, i') < u(a', t', i').$$

Assumption 3. For any t , $D(t; i', i'')$ strictly decreases in i' and in i'' .

Assumption 1 states that signaling is costly and that the agent benefits from a higher reputation. The “dividing line” $D(\cdot; i', i'')$ specified in Assumption 2 partitions the (a, t) -space into two regions for each pair of types. For actions to the left of the dividing line, the standard single-crossing property holds for types i' and i'' . To the right of the dividing line, the *reverse single-crossing property* holds: whenever the lower type i'' is indifferent between two allocations, the higher type i' strictly prefers the one with the lower action. We place no restriction on the shape of $D(\cdot; i', i'')$, other than that it is a continuous function of t .

When there are only two types, Assumptions 1 and 2 provide enough structure to analyze signaling under double-crossing preferences because $D(\cdot; 2, 1)$ is the only dividing line. When there are more than two types, there are $I(I - 1)/2$ dividing lines to be considered. The analysis would clearly become unmanageable without further restriction. Assumption 3 provides the restriction we adopt.

The restrictions imposed by Assumptions 1 to 3 can also be expressed in terms of the marginal rate of substitution between a and t . Let

$$m(a, t, i) := -\frac{u_a(a, t, i)}{u_t(a, t, i)}$$

represent the marginal rate for type i at (a, t) , and let $t = \phi(a, u_0, i)$ represent the indifference curve of type i at utility level u_0 . The marginal rate of substitution is always positive under Assumption 1. If type i'' attains utility level u_0 at (a_0, t_0) , then for $i' > i''$, the difference in their marginal rates along the indifference curve of type i'' is

$$m(a, \phi(a, u_0, i''), i') - m(a, \phi(a, u_0, i''), i'').$$

Chen et al. (2021b) show that Assumption 2 is equivalent to the requirement that the above difference is *strictly single-crossing from below in a* , with crossing point at $a = a_0 = D(t_0; i', i'')$. Furthermore, if an indifference curve $\phi(\cdot, u_0, i'')$ crosses the dividing line $D(\cdot; i', i'')$ at (a_0, t_0) , then $a < D(\phi(a, u_0, i''); i', i'')$ for all $a < a_0$, i.e., an indifference curve can cross a dividing line only once in the (a, t) -space.⁴

Assumption 3 is related to how the marginal rate behaves with respect to type. Specifically, it requires that for any (a, t) ,

$$m(a, t, i) - m(a, t, i - 1)$$

is *strictly single-crossing from below in i* . These restrictions are different from the standard setting with single-crossing preferences, in which a higher type always has a lower marginal rate of substitution than a lower type at all levels of a and t .

We consider equilibria that satisfy the D1 refinement (Cho and Kreps, 1987). In the standard setting with single-crossing preferences, this refinement implies that the unique equilibrium outcome is given by the least-cost separating solution, also known as the Riley outcome (Riley, 1979). The model described here is identical to that of Chen et al. (2021b), except that the type space is now discrete. The reader can refer to that paper for a fuller discussion of the modeling assumptions, as well as examples of economic environments in which preferences satisfy these assumptions.

3. Equilibrium Characterization

Let $(a(n), t(n))$ denote the allocation of agent n where $a(\cdot)$ and $t(\cdot)$ are generally step functions. It is without loss of generality to assume that $a(\cdot)$ is weakly monotone within each type interval. If $a(\cdot)$ is constant on the type interval $[F_{i-1}, F_i)$, type i adopts a pure

⁴Suppose an indifference curve $\phi(\cdot, u_0, i'')$ of type i'' crosses $D(\cdot; i', i'')$ at a_1 and a_2 . Indifference curves of the higher type i' that cross $\phi(\cdot, u_0, i'')$ must be steeper than the latter between a_1 and a_2 , which implies that type i' strictly prefers the lower action a_1 to the higher action a_2 . This would violate Assumption 2(a).

strategy; otherwise, type i randomizes over different actions. Let $Q(a)$ be the set of agents who choose action a in equilibrium. We say that $Q(a)$ is a *pooling set* if it contains more than one type. For the subsequent analysis, it is convenient to define for a given $n \in [0, 1]$,

$$\phi^*(\cdot; \iota(n)) := \phi(\cdot, u(a(n), t(n), \iota(n)), \iota(n)),$$

which is the indifference curve of type $\iota(n)$ that passes through his equilibrium allocation $(a(n), t(n))$; for brevity, we refer to it as the *equilibrium indifference curve* of type $\iota(n)$.

Chen et al. (2021b) show that equilibrium under double-crossing preferences with a continuum of types exhibits a particular form of pooling, labeled as *Low types Separate and High types Pairwise-Pool* (LSHPP). There are two important properties of an LSHPP equilibrium. First, the equilibrium action is quasi-concave in type. Second, there is a threshold type above which types are pooled in a pairwise manner, where two distinct types, or two disjoint intervals of types, choose the same action. While intuition suggests that some extended version of these properties would continue to hold for an arbitrary number of types, the results of Chen et al. (2021b) cannot be applied directly to the present context because the proof in that paper exploits the continuity of $a(\cdot)$ above the threshold type—a notion that has no counterpart in the discrete-type model.

In the continuous-type model, incentive compatibility and D1 impose tight restrictions on the feasible range of equilibrium actions. Specifically, suppose that the type space is given by $[\underline{\theta}, \bar{\theta}]$, where we use θ as the generic notation of the agent type. Then, there are two threshold types θ_0 and θ^* and a mapping $p : (\theta^*, \bar{\theta}] \rightarrow [\theta_0, \theta^*)$ such that for each type $\theta \in (\theta^*, \bar{\theta}]$, there is a paired type $p(\theta) \in [\theta_0, \theta^*)$ who chooses the same action. Due to the denseness of the type space, the mapping is always bijective, and hence $Q(a(\theta))$ is a pooling set for all $\theta \in [\theta_0, \theta^*)$. Moreover, the marginal rates of substitution must be exactly equalized at the end points of each pooling set (except when the pooling set includes the highest type): if there is any pooling at (a_p, t_p) , then types $\min Q(a_p)$ and $\max Q(a_p)$ must have the same marginal rate of substitution.

Equilibrium characterization is more involved in the discrete-type model because those nice properties do not hold exactly when there are no arbitrarily close “adjacent types.” We can still show, however, that a weaker version of pairwise matching still holds.

Definition 1. *The weak pairwise-matching condition holds if the following two conditions are satisfied:*

- (a) *There exists $n^* \in (0, 1)$ such that for any action $a(n')$ chosen by agent $n' > n^*$, there exists $n'' < n^*$ such that agent n'' chooses the same action $a(n')$.*

(b) If there is pooling at (a_p, t_p) , then $Q(a_p) = [n_1, n_2) \cup [n_3, n_4)$ with $n_2 \leq n^* \leq n_3$, where

$$m(a_p, t_p, \iota(n_3)) \geq m(a_p, t_p, \iota(n_2 - \varepsilon)),$$

$$m(a_p, t_p, \iota(n_4 - \varepsilon)) \leq \begin{cases} m(a_p, t_p, \iota(n_1)) & \text{if } n_1 \notin B, \\ m(a_p, t_p, \max\{\iota(n_1) - 1, 1\}) & \text{if } n_1 \in B, \end{cases}$$

for an arbitrarily small $\varepsilon > 0$.

Part (a) of the above definition requires that every action chosen by a type higher than $\iota(n^*)$ is also chosen by another type below $\iota(n^*)$. In other words every type above $\iota(n^*)$ is pooled with some type(s) below $\iota(n^*)$. The flip side of this, however, is not required by the weak pairwise-matching condition. In particular, suppose the lowest type that pools with type I is type i_0 . Then Definition 1(a) does not rule out the possibility that some type between i_0 and $\iota(n^*)$ may choose an action that separates from all other types. This is qualitatively different from the continuous-type case where no type above a gap type θ_0 chooses a fully separating action.

Since each agent type is defined by a left-closed, right-open interval, in Definition 1(b) $Q(\cdot)$ is also defined as the union of left-closed, right-open intervals (except when it includes agent $F_I = 1$, in which case the set is defined by a closed interval). If $Q(a_p) = [n_1, n_2) \cup [n_3, n_4)$ and n_2 is a “threshold agent,” then for a small $\varepsilon > 0$, type $\iota(n_2 - \varepsilon)$ chooses a_p but type $\iota(n_2)$ does not. This is why we write $\iota(n_2 - \varepsilon)$ and $\iota(n_4 - \varepsilon)$ in Definition 1(b). Instead of requiring equality of marginal rates of substitution, Definition 1(b) leaves some degree of freedom for the marginal rates of substitution among types that are pooled. The rationale for the direction of inequalities stated in the definition will become apparent when we explain below the restrictions on off-equilibrium beliefs imposed by the D1 refinement.

Proposition 1. *In any equilibrium, $a(\cdot)$ is weakly quasi-concave and the weak pairwise-matching condition holds.*

Proof. Quasi-concavity. Suppose to the contrary that $\min\{a(n_1), a(n_3)\} > a(n_2)$ for some $n_1 < n_2 < n_3$. There are three possibilities: (i) $a(n_3) > a(n_1) > a(n_2)$; (ii) $a(n_1) = a(n_3) > a(n_2)$; and (iii) $a(n_1) > a(n_3) > a(n_2)$.

Case (i). Since agent n_1 chooses a higher action than agent n_2 , it must be that

$$a(n_1) > D(t(n_1); \iota(n_2), \iota(n_1)).$$

Moreover, since agent n_1 chooses a lower action than agent n_3 , we have

$$a(n_1) < D(t(n_1); \iota(n_3), \iota(n_1)).$$

These two equations imply that $D(t(n_1); \cdot, \iota(n_1))$ is increasing, which is a contradiction.

Case (ii). Let (a_p, t_p) represent the pooling action and the associated reputation chosen by agents n_1 and n_3 in this case. Let \underline{i} and \bar{i} be respectively the lowest and the highest type that choose a_p in equilibrium. Note that $m(a_p, t_p, i) - m(a_p, t_p, i - 1)$ is single-crossing from below in i if and only if $m(a_p, t_p, \iota(n))$ is quasi-convex in n . Hence either type \underline{i} or type \bar{i} has the highest marginal rate of substitution among all types pooling at action a_p . Because types are discrete, actions slightly below a_p are off-path. By D1, a slight downward deviation from a_p is attributed to either type \underline{i} or type \bar{i} . If $a_p > D(t_p; \bar{i}, \underline{i})$, then the off-equilibrium belief associated with downward deviation is \bar{i} . This cannot be an equilibrium, because $\bar{i} > t_p$. So we must have $a_p \leq D(t_p; \bar{i}, \underline{i}) \leq D(t_p; \iota(n_3), \underline{i})$. But $a(n_2) < a_p$ implies $a_p > D(t_p; \iota(n_2), \underline{i})$. This means that $D(t_p; \iota(n_2), \underline{i}) < a_p \leq D(t_p; \iota(n_3), \underline{i})$, which contradicts the monotonicity of $D(t_p; \cdot, \underline{i})$.

Case (iii). Because signaling is costly, $a(n_1) > a(n_3)$ implies $t(n_1) > t(n_3)$. Since $n_1 < n_3$, either $Q(a(n_1))$ or $Q(a(n_3))$ (or both) must be a pooling set. Furthermore, if we let $n' = \sup Q(a(n_1))$ and $n'' = \inf Q(a(n_3))$, we must have $n' > n''$. If $n' > n_2$, then for agents $n_1 < n_2 < n'$, we have $a(n') = a(n_1) > a(n_2)$. This reduces to case (ii) above. If $n' \leq n_2$, then for agents $n'' < n_2 < n_3$, we have $a(n'') = a(n_3) > a(n_2)$. Again this reduces to case (ii).

Weak pairwise-matching condition. Quasi-concavity of $a(\cdot)$ implies that any pooling set must consist of at most two intervals, i.e., there exists n^* such that for any pooling action a_p , $Q(a_p) = [n_1, n_2] \cup [n_3, n_4]$ with $n_2 \leq n^* \leq n_3$.

To show that every type above $\iota(n^*)$ pools with a type below $\iota(n^*)$, suppose the opposite is true. Specifically let n' be the lowest agent in $(n^*, F_1]$ who chooses an action a' which is not chosen by agents in $[F_0, n^*]$. The corresponding reputation satisfies $t(n') \geq n'$. By quasi-concavity of $a(\cdot)$, $Q(a(n^*))$ must be either a singleton or an interval. Therefore, we have $\sup Q(a(n^*)) < n'$, and so $t(n^*) \leq t(n')$. On the other hand, quasi-concavity of $a(\cdot)$ implies $a(n') < a(n^*)$. Thus, every agent would strictly prefer $(a(n'), t(n'))$ to $(a(n^*), t(n^*))$, a contradiction.

Let the reputation corresponding to pooling action a_p be t_p . Recall that $m(a_p, t_p, \iota(n))$ is quasi-convex in n , and hence the type in the pooling set $Q(a_p)$ with the highest marginal

rate of substitution is either $\iota(n_1)$ or $\iota(n_4 - \epsilon)$ (for some small $\epsilon > 0$). When agent n_1 is not a “threshold agent” (i.e., when $n_1 \notin B$), if $m(a_p, t_p, \iota(n_4 - \epsilon)) > m(a_p, t_p, \iota(n_1))$, then a downward deviation from a_p to an off-equilibrium action $a' < a_p$ would be attributed to type $\iota(n_4 - \epsilon)$ under D1, leading to a profitable deviation. When n_1 is a “threshold agent,” the same conclusion holds unless type $\iota(n_1) - 1$ has even greater incentive to deviate to a' . Only if type $\iota(n_1) - 1$ is indifferent between his own allocation and (a_p, t_p) , and if $m(a_p, t_p, \iota(n_4 - \epsilon)) \leq m(a_p, t_p, \iota(n_1) - 1)$, would the off-equilibrium belief be assigned to type $\iota(n_1) - 1$ rather than type $\iota(n_4 - \epsilon)$, preventing such a deviation.

Finally, we show $m(a_p, t_p, \iota(n_3)) \geq m(a_p, t_p, \iota(n_2 - \epsilon))$. Note that quasi-concavity of $a(\cdot)$ implies $a(n') > a_p$ for $n' \in [n_2, n_3]$. Costly signaling in turn implies that $t(n') > t_p$. On one hand, the higher type $\iota(n')$ prefers the higher action $a(n')$ to the lower action a_p while the lower type $\iota(n_2 - \epsilon)$ has the opposite preference. Thus the standard single-crossing property holds between these two types. On the other hand, the higher type $\iota(n_3)$ prefers the lower action a_p to $a(n')$ while the lower type $\iota(n')$ has the opposite preference. Thus the reverse single-crossing property holds between n' and n_3 . These two observations imply that

$$m(a_p, t_p, \iota(n')) < \min \{m(a_p, t_p, \iota(n_2 - \epsilon)), m(a_p, t_p, \iota(n_3))\}$$

Because $m(a_p, t_p, \iota(n))$ is quasi-convex in n , this in turn implies that the type with the lowest marginal rate of substitution in the pooling set $Q(a_p)$ is either $\iota(n_2 - \epsilon)$ or $\iota(n_3)$. If, contrary to the weak pairwise-matching condition, we have $m(a_p, t_p, \iota(n_3)) < m(a_p, t_p, \iota(n_2 - \epsilon))$, then an upward deviation from a_p would be attributed to type $\iota(n_3)$ under D1. To prevent off-equilibrium deviations, it must be that $t_p \geq n_3$. But this would lead to a contradiction, because $t(n') > t_p \geq n_3$ while $Q(a(n'))$ consists of types weakly lower than $\iota(n_3)$. ■

Proposition 2. *In any equilibrium, the following properties hold:*

- (a) *Types $i \geq \iota(n^*)$ do not randomize over different allocations;*
- (b) *If type $i < \iota(n^*)$ randomizes over a set of allocations, only the lowest allocation in the set can be separating.*

Proof. Part (a). Suppose there is some type $i' > \iota(n^*)$ who randomizes over two different allocations (a_1, t_1) and (a_2, t_2) , where $a_1 < a_2$ and $t_1 < t_2$. Let i'' be the largest type below $\iota(n^*)$ who pools at (a_1, t_1) . Then, by the weak pairwise-matching condition, we have $m(a_1, t_1, i'') \leq m(a_1, t_1, i')$, which in turn implies that $\phi^*(a; i'') < \phi^*(a; i')$ for all

$a > a_1$. This is a contradiction because type i'' must strictly prefer (a_2, t_2) to (a_1, t_1) . The possibility that type $\iota(n^*)$ randomizes will be ruled out by part (b); we will come back to that point later.

Part (b). Suppose some type $i' < \iota(n^*)$ randomizes over (a_1, t_1) and (a_2, t_2) , with $a_1 < a_2$ and $t_1 < t_2$, where (a_2, t_2) is separating. Let $Q(a_1) = [n_1, n_2) \cup [n_3, n_4)$. Recall that the type with the lowest marginal rate of substitution in the pooling set $Q(a_1)$ is either $\iota(n_2 - \epsilon)$ or $\iota(n_3)$. Therefore, under D1, a deviation to a' slightly above a_1 (but below a_2) would be given an off-equilibrium belief $t' \geq \iota(n_2 - \epsilon) \geq i' = t_2$. Thus type i' would strictly prefer (a', t') to (a_2, t_2) .

Finally, we can use this argument to show that type $\iota(n^*)$ cannot randomize either. Suppose to the contrary that type $\iota(n^*)$ randomizes over (a_1, t_1) and (a_2, t_2) where $a_1 < a_2$ and $t_1 < t_2$. Note that we only need to consider the case where (a_2, t_2) is separating. (If $Q(a_2)$ is a pooling set, all type $\iota(n^*) + 1$ agents must choose (a_2, t_2) , in which case we can just redefine n^* to be in $[F_{\iota(n^*)}, F_{\iota(n^*)+1})$.) This means that there must be some agent $n' \in [F_{\iota(n^*)-1}, n^*)$ who pools with a higher type at (a_1, t_1) . We can then apply the argument in part (b) to derive a contradiction. ■

The equilibrium requirements described in Propositions 1 and 2 are illustrated in Figure 1. Agent n_0 in this figure is the lowest agent pooling with type $I = 15$. The signaling function $a(\cdot)$ below n_0 is determined by the least-cost separating solution. We pick an arbitrary agent choosing the highest action and label this agent n^* . Each type higher than $\iota(n^*)$ (i.e., types 11 to 15) adopt a pure strategy, and each type pools with some type below $\iota(n^*)$. Note that some types between $\iota(n_0)$ and $\iota(n^*)$ (specifically, types 6 and 7) may choose separating actions, and when type 7 randomize between a separating action and a pooling action, the separating action is lower. The signaling function in Figure 1 is quasi-concave.

4. Equilibrium Construction with a Small Number of Types

In contrast to the extant literature on countersignaling, the general model in this paper allows an arbitrary number of discrete types. We will provide in Section 5 an algorithm to construct an equilibrium for the general case. Because this algorithm is unavoidably complex in order to accommodate all possible double-crossing preferences and all type distributions, in this section we first illustrate the main ideas and difficulties of our equilibrium construction using very simple type distributions.

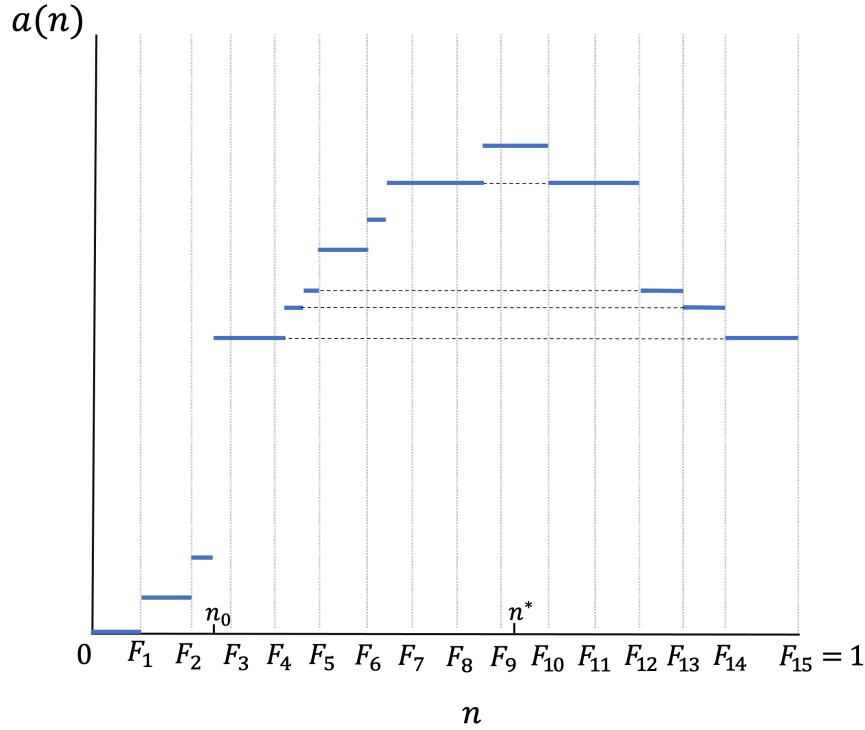


Figure 1. The signaling function is quasi-concave and satisfies the weak pairwise-matching condition.

Consider the simplest case with only two types, in which case there is only one dividing line $D(\cdot; 2, 1)$. Let $s^*(i)$ denote the least-cost separating solution, where type 1 is indifferent between $(s^*(1), 1)$ and $(s^*(2), 2)$, with $s^*(1) = 0$. When $s^*(2) \leq D(2; 2, 1)$, the equilibrium indifference curve of type 1 stays strictly to the left of the dividing line for $a \leq s^*(2)$. Because the single-crossing property prevails in the relevant region, the least-cost separating solution is an equilibrium. Also, by standard argument, no other equilibrium can satisfy D1 (Cho and Kreps, 1987).

When $s^*(2) > D(2; 2, 1)$, no separating equilibrium exists because type 2 has a higher marginal rate of substitution at $(s^*(2), 2)$, giving him an incentive to deviate to a lower action. In an equilibrium that involves some pooling, the pooling allocation must be on $D(\cdot; 2, 1)$, for otherwise the higher type would have an incentive to deviate either to a slightly higher action or to a slightly lower action.

Let (\hat{a}, \hat{t}) be the intersection of the dividing line $D(\cdot; 2, 1)$ and the indifference curve of type 1 passing through $(0, 1)$. If $\hat{t} > E[i]$, the only possible equilibrium is a semi-pooling

equilibrium in which type 1 randomizes between $(0, 1)$ and (\hat{a}, \hat{t}) and type 2 chooses (\hat{a}, \hat{t}) . Observe that the equilibrium indifference curve of type 2 is flatter than that of type 1 to the left of the dividing line $D(\cdot; 2, 1)$, and is steeper than that of type 1 to the right of the dividing line. Hence any deviation from that allocation which weakly benefits type 2 would strictly benefit type 1. Under D1, such a deviation would be attributed to type 1, which makes the deviation unprofitable. If $\hat{t} \leq E[i]$, both types strictly prefer to choose (\hat{a}, \hat{t}) , and the only equilibrium is a fully pooling equilibrium at $a_p = D(E[i]; 2, 1)$ and $t_p = E[i]$.

Proposition 3. *When there are two types, there exists a unique D1 equilibrium. Moreover, if such an equilibrium is not fully separating, the pooling allocation must lie on the dividing line $D(\cdot; 2, 1)$.*

This proposition is established in the preceding discussion. The uniqueness of D1 equilibrium in specific instances of double-crossing preferences has been pointed out in Daley and Green (2014) and Chen et al. (2021a).

Next consider a model with three types. Although it is possible to enumerate all the possible equilibrium configurations, such an exercise would be rather tedious. Instead we will explain why equilibrium construction may be difficult even in this very simple case and how our characterization results help narrowing down the set of possible equilibrium configurations. Suppose we are interested in finding an equilibrium that exhibits countersignaling with two actions (i.e., the signaling function $a(\cdot)$ is nonmonotone and takes two values). Even in this particular class of equilibria, there are still many possible configurations: some type 1 or type 3 agents (or both) may pool with type 2 agents at the high action; or some type 2 agents may pool with type 1 and type 3 agents at the low action. Our characterization result suggests, however, that many of these configurations are inconsistent with the equilibrium conditions and can hence be ruled out at the outset. More precisely, Proposition 2(a) states that neither type 2 nor type 3 can randomize, meaning that all type 2 agents must choose the high action while all type 3 agents must choose the low action in any countersignaling equilibrium with three types and two actions. This leaves us with only two possible forms of countersignaling equilibrium, depending only on whether type 1 randomizes between the two actions or not. Moreover, Proposition 2(a) also implies that the pooling action chosen by type 3 must be the second highest in any countersignaling equilibrium with three types,⁵ which further narrows down the set

⁵If the action chosen by type 3 is the highest, we cannot have countersignaling because the equilibrium

of possible equilibrium configurations.

When there are four or more types, the situation becomes much more complicated and increasingly intractable, even with the help of our characterization. For instance, with four types, even if we just consider pure-strategy countersignaling equilibria, there are five possible patterns of countersignaling.⁶ Added to the complication is that Assumption 3 does not impose any restriction on the relative location of the dividing lines $D(\cdot; 4, 1)$ and $D(\cdot; 3, 2)$. Clearly a more systematic approach to finding an equilibrium is called for whenever the model has many types.

5. Equilibrium Existence

Under single-crossing preferences, it is well known that the least-cost separating equilibrium satisfies the D1 refinement. Because of this, finding a D1 equilibrium in this environment is conceptually straightforward and equilibrium existence is never an issue, even when the number of types is very large: each separating action can be pinned down by the corresponding local incentive compatibility constraint alone, and an equilibrium can be constructed from bottom up. This is not the case under double-crossing preferences, where equilibrium may entail some pooling and the equilibrium conditions (i.e., incentive compatibility and D1) impose joint restrictions on payoff and type distribution functions. As discussed in the previous section, finding an equilibrium under double-crossing preferences by brute force can be extremely tedious, especially when there are many types. Moreover, there is no easy way to tell whether a D1 equilibrium exists for any given preferences and type distribution.

In this section, to overcome these issues, we provide a systematic way to find a D1 equilibrium under double-crossing preferences. The algorithm developed here works for any double-crossing preferences and type distribution, and can be used to establish the existence of a D1 equilibrium by construction.

action must be quasi-concave. Suppose there are two actions above the action chosen by type 3. Then, for each of these actions, there must be a type adopting a pure strategy of choosing that action by Proposition 2(a). This implies that type 1 must adopt a pure strategy and choose a higher action than type 3, which necessarily violates the weak pairwise-matching condition.

⁶We can have types 1 and 4 pool while types 2 and 3 pool at a higher action; types 1 and 4 pool while types 2 and 3 separate at two higher actions; types 1 and 2 pool with type 4 while type 3 separates; types 1 pools with types 3 and 4 while type 2 separates; or types 2 and 4 pool while type 1 separates at a lower action and type 3 separates at a higher action.

5.1. An overview of the algorithm for equilibrium construction

Because the signaling function is quasi-concave, our construction finds the actions chosen by agents near the two ends of the agent distribution first. Then, in successive rounds, we find the actions chosen by agents nearer the middle part of the agent distribution until the highest equilibrium action is pinned down. One of the agents choosing that highest action corresponds to the equilibrium n^* . Each round of this construction is designed to respect local incentive compatibility (each agent has no incentive to deviate to adjacent allocations) and the weak pairwise-matching condition. We show later that these two restrictions together are sufficient to guarantee global incentive compatibility.

We also make sure that Bayes' rule for belief consistency is satisfied at any equilibrium pooling action. Suppose $Q(a_p) = [n_1, n_2) \cup [n_3, n_4)$ for some a_p . Bayes' rule requires that for all $n \in Q(a_p)$,

$$t(n) = \frac{\int_{n_1}^{n_2} \iota(n) dn + \int_{n_3}^{n_4} \iota(n) dn}{(n_2 - n_1) + (n_4 - n_3)}.$$

It is particularly convenient to define

$$\mu(n'', n'; i) := \frac{\int_{n''}^{n'} \iota(n) dn + \int_{F_{i-1}}^{F_i} \iota(n) dn}{(n' - n'') + (F_i - F_{i-1})},$$

for $n'' \leq n' \leq F_{i-1}$. This formula gives the expected type when agents from n'' to n' pool with all type i agents. When agents in $[p, q)$ pool with all agents of type j , Bayes' rule can be written as $t(n) = \mu(p, q; j)$ for $n \in [p, q)$. The following properties of $\mu(\cdot)$ will be useful: (a) $\mu(n'', n'; i)$ is continuously increasing in n'' and equals i at $n'' = n'$; and (b) $\mu(n'', n'; i)$ decreases in n' when $\mu(n'', n'; i) > \iota(n')$ and increases in n' when $\mu(n'', n'; i) < \iota(n')$.

Our algorithm for equilibrium construction exploits a key property established in Propositions 1 and 2—namely, each type $i \geq \iota(n^*)$ adopts a pure strategy and pools with some agents below n^* . We therefore propose an algorithm with multiple rounds. In the first round, we find a set of agents, say $[p_1, q_1)$, who pool with type I at some candidate allocation (a_1, t_1) . (If $p_1 > 0$, we also determine the tentative allocations $(a(n), t(n))$ for $n \in [0, p_1)$ based on that candidate allocation). In round 2, the allocations to agents in $[q_1, F_{I-1})$ remains to be determined. Among these remaining agents, we find a set of agents, $[p_2, q_2)$, who pool with type $I - 1$ at a weakly higher tentative allocation (a_2, t_2) (along with the allocations chosen by $n \in [q_1, p_2)$ if $p_2 > q_1$). In round 3, the allocations

to agents in $[q_2, F_{I-2})$ remains to be determined, and so on. The algorithm finds an equilibrium when by the end of some round, the allocations to all agents in $[0, 1]$ have been determined. It is possible that we cannot find a tentative allocation at some round k before the algorithm stops with a full solution. In that case, the algorithm specifies how the tentative allocation at some round $k' < k$ should be adjusted and the algorithm proceeds to round $k' + 1$ again with the adjusted tentative allocation for round k' as the starting point. The proof of equilibrium existence essentially boils down to showing that this algorithm will always end with a full solution for the allocations of all agents in $[0, 1]$.

To introduce the notation for this algorithm, for $k \geq 1$, define $j := I + 1 - k$ to be the highest type among all agents whose allocation remains to be determined in round k . The tentative solution in round k is represented by a 4-tuple, (a_k, t_k, p_k, q_k) , with the interpretation that agents in $[p_k, q_k)$ pool with all type j agents at allocation (a_k, t_k) . Let $\mathbf{s}_k := (a_{k-1}, t_{k-1}, n_k)$ represent the *state* in round k , which is inherited from the tentative solution in round $k-1$: (a_{k-1}, t_{k-1}) is the allocation chosen in the previous round, and $n_k = q_{k-1}$ is the lowest agent whose allocation is yet to be determined. At the beginning of the algorithm, we initialize the state to $\mathbf{s}_1 = (0, 1, 0)$ where the allocation $(0, 1)$ corresponds to the least-cost separating solution for type 1.

As we will detail below, the pair (t_k, p_k) provides enough information for finding an equilibrium, while a_k and q_k can be uniquely pinned down from (t_k, p_k) . Instead of working with the 4-tuple (a_k, t_k, p_k, q_k) , we thus denote the tentative allocation in round k simply by $\mathbf{c}_k := (t_k, p_k) \in [t_{k-1}, I] \times [q_{k-1}, F_{j-1})$. Let $(\mathbf{c}_1, \dots, \mathbf{c}_{k'})$ be the allocation path up to round k' .

Among all allocations \mathbf{c}_k in round k that will satisfy incentive compatibility and the weak pairwise-matching condition, the algorithm only considers a special subset of these allocations, which we call the *candidate set* and denote by C_k . The candidate set depends on the state $\mathbf{s}_k = (a_{k-1}, t_{k-1}, n_k)$, but since a_{k-1} and $n_k = q_{k-1}$ can be derived from \mathbf{c}_{k-1} we sometimes write $C_k(\mathbf{c}_{k-1})$ to emphasize this dependence. When the candidate set is well defined, it can be a singleton or it can contain more than one element. If C_k contains more than one element, we argue below that there is a linear order \succeq_k defined on C_k such that any two elements in this set can be compared, despite the fact that a typical element of C_k is two dimensional. We use $\bar{\mathbf{c}}_k$ and $\underline{\mathbf{c}}_k$ to denote the highest and lowest element of C_k .

We are now in a position to provide a high-level description of our main algorithm for equilibrium construction:

Main algorithm. Each round $k \geq 1$ inherits the allocation path $(\mathbf{c}_1, \dots, \mathbf{c}_{k-1})$ from earlier iterations of the algorithm. Given $\mathbf{c}_{k-1} = (t_{k-1}, p_{k-1})$, apply Bayes' rule to find the q_{k-1} which is the smallest solution to $\mu(p_{k-1}, q_{k-1}; j+1) = t_{k-1}$.

1. If $q_{k-1} = F_j$:
 - stop with a well-defined equilibrium characterized by the existing allocation path $(\mathbf{c}_1, \dots, \mathbf{c}_{k-1})$
2. If the equation $\mu(p_{k-1}, q_{k-1}; j+1) = t_{k-1}$ does not admit a solution for q_{k-1} on $[p_{k-1}, F_j]$:
 - adopt the *adjustment procedure* to modify the allocation path to $(\mathbf{c}'_1, \dots, \mathbf{c}'_{k-1})$
 - return to the beginning of round k
3. Otherwise:
 - update the state to $\mathbf{s}_k = (a_{k-1}, t_{k-1}, n_k)$ with $n_k = q_{k-1}$
 - compute the candidate set $C_k(\mathbf{c}_{k-1})$;
 - if the *minimum allocation* \mathbf{c}_k^{\min} exists in $C_k(\mathbf{c}_{k-1})$, pick $\mathbf{c}_k = \mathbf{c}_k^{\min}$; otherwise pick $\mathbf{c}_k = \bar{\mathbf{c}}_k \in C_k(\mathbf{c}_{k-1})$
 - append \mathbf{c}_k to the allocation path $(\mathbf{c}_1, \dots, \mathbf{c}_{k-1})$ and go to the beginning of round $k+1$

The remainder of this section will specify precisely how the candidate set C_k is determined (Section 5.2), define the minimum allocation \mathbf{c}_k^{\min} (as well as the highest allocation $\bar{\mathbf{c}}_k$) and explain its role in equilibrium construction (Section 5.3), and elaborate on the adjustment procedure that produces a modified allocation path from the original path (Section 5.4). Finally, we will show that the main algorithm will always stop at some finite round k and produce a well-defined allocation for all agents. Since the solution so produced satisfies Bayes' rule, local incentive compatibility, weak pairwise-matching, and is quasi-concave by construction, equilibrium existence can be guaranteed by showing that local incentive compatibility also implies global incentive compatibility (Section 5.5).

5.2. The candidate set

Since agents in $[p_k, q_k)$ pool with type j to produce reputation t_k , we define the following mapping from $\mathbf{c}_k = (t_k, p_k)$ to q_k to ensure that Bayes' rule is satisfied:

$$\pi_k(\mathbf{c}_k) := \min \{q \in [p_k, F_{j-1}] : \mu(p_k, q; j) = t_k\}.$$

Because $\mu(p_k, \cdot; j)$ is decreasing then increasing, there can be multiple values of q that satisfy the specified condition, and in that case we pick the smallest solution. It is possible that the specified condition admits no solution, in which case $\pi_k(\mathbf{c}_k)$ is undefined.

The candidate set C_k depends on the state $\mathbf{s}_k = (a_{k-1}, t_{k-1}, n_k)$, where $n_k = \pi_{k-1}(\mathbf{c}_{k-1})$. This set is undefined whenever $\pi_{k-1}(\mathbf{c}_{k-1})$ is undefined. In this subsection, we assume that C_k is well defined. The determination of the candidate set depends on whether n_k is a “threshold agent” (i.e., $n_k \in B$) or not (i.e., $n_k \notin B$). We deal with these two cases in turn.

Suppose first that $n_k \notin B$. This means that type $\iota(n_k)$ must be randomizing between a_{k-1} (determined in previous iterations of the algorithm) and a higher action a_k (to be determined). But part (b) of Proposition 2 requires that type $\iota(n_k)$ agents with $n > n_k$ cannot take a fully separating action. This implies that we must choose $p_k = n_k$ for \mathbf{c}_k . Since type $\iota(n_k)$ agents randomize, any tentative allocation (a_k, t_k) must be on the indifference curve of type $\iota(n_k)$ that passes through (a_{k-1}, t_{k-1}) . Also, since they pool with type j agents, we require the tentative allocation to be on $D(\cdot; j, \iota(n_k))$ so that it satisfies the weak pairwise-matching condition. Define $(\hat{a}(j, i; \tilde{a}_k, \tilde{t}_k), \hat{t}(j, i; \tilde{a}_k, \tilde{t}_k))$, or $(\hat{a}_k(j, i), \hat{t}_k(j, i))$ for short, to be the intersection of the relevant indifference curve and dividing line when the “reservation allocation” is $(\tilde{a}_k, \tilde{t}_k)$;⁷ i.e., it is the (a_k, t_k) that satisfies:

$$\begin{aligned} u(a_k, t_k, i) &= u(\tilde{a}_k, \tilde{t}_k, i), \\ a_k &= D(t_k; j, i). \end{aligned}$$

If $p_k = n_k$, the reservation allocation is simply the allocation chosen in the previous round, i.e., $(\tilde{a}_k, \tilde{t}_k) = (a_{k-1}, t_{k-1})$. Then, we let

$$\mathbf{c}_k = \begin{cases} (\hat{t}_k(j, \iota(n_k)), n_k) & \text{if } \hat{t}_k(j, \iota(n_k)) > t_{k-1}, \\ (t_{k-1}, n_k) & \text{if } \hat{t}_k(j, \iota(n_k)) \leq t_{k-1}. \end{cases}$$

The candidate set C_k is a singleton containing \mathbf{c}_k when $n_k \notin B$.

⁷As we will see below, the reservation allocation $(\tilde{a}_k, \tilde{t}_k)$ is determined by \mathbf{c}_{k-1} and p_k .

Note that the action a_k can be determined once \mathbf{c}_k is pinned down, and we denote this relationship by $a_k = \alpha_k(\mathbf{c}_k)$, where

$$\alpha_k(\mathbf{c}_k) = \begin{cases} \hat{a}_k(j, \iota(n_k)) & \text{if } \hat{t}_k(j, \iota(n_k)) > t_{k-1}, \\ a_{k-1} & \text{if } \hat{t}_k(j, \iota(n_k)) \leq t_{k-1}. \end{cases}$$

The case where $\alpha_k(\mathbf{c}_k) = a_{k-1}$ corresponds to “mass pooling” of the continuous-type case of Chen et al. (2021b), where more than two types pool at the same allocation.

Next suppose that $n_k \in B$. This means that type $\iota(n_k) - 1$ chooses action a_{k-1} , while types $\iota(n_k)$ and above may potentially choose separating actions. In this case, we first construct a least-cost separating solution given the initial condition.⁸ We denote this solution by $s_k^*(i)$. There are two possibilities.

Case (i). If $s_k^*(j) \leq D(j; j, j-1)$, then there is a separating equilibrium for types between $\iota(n_k)$ and j because the standard single-crossing property holds in the relevant region. In this case, the separating solution for these types, together with the allocation path derived from previous iterations of the algorithm, will constitute an equilibrium. Formally, we specify $\mathbf{c}_k = (j, F_{j-1})$ to be the only element of our candidate set C_k , together with $\alpha_k(\mathbf{c}_k) = s_k^*(j)$. With $p_k = F_{j-1}$, the solution to $\mu(p_k, q_k; j) = j$ is $q_k = F_{j-1}$, and so the main algorithm will stop with a well-defined equilibrium.

Case (ii). Otherwise, we find $i_k^* := \max\{i : s_k^*(i) < D(i; j, i)\}$; this is the largest type that can choose a separating action.⁹ If $s_k^*(\iota(n_k)) \leq D(\iota(n_k); j, \iota(n_k))$ and i_k^* is not well defined, we adopt the convention to denote $i_k^* = \iota(n_k) - 1$. Observe that when $i_k^* = \iota(n_k) - 1$, type $\iota(n_k)$ agents cannot take a separating action, and the only feasible choice of p_k is n_k . If $i_k^* \geq \iota(n_k)$, we may pick any $p_k \in [n_k, F_{i_k^*}]$. In either case, once we pick p_k , we can identify the feasible range of t_k . If $p_k \notin B$, the relevant indifference curve of type $\iota(p_k)$ is the one that passes through his reservation allocation $(s_k^*(\iota(p_k)), \iota(p_k))$, and the relevant dividing line is $D(\cdot; j, \iota(p_k))$. We can uniquely pin down the intersection $(\hat{a}_k(j, \iota(p_k)), \hat{t}_k(j, \iota(p_k)))$ and set $t_k = \hat{t}_k(j, \iota(p_k))$. If $p_k \in B$ and $\iota(p_k) > 1$, any candidate allocation must be bounded between the indifference curves of types $\iota(p_k) - 1$ and $\iota(p_k)$ that pass through

⁸In round 1, the initial condition requires $s_1^*(1) = 0$; in all subsequent rounds, type $\iota(n_k) - 1$ must be indifferent between $(s_k^*(\iota(n_k)), \iota(n_k))$ and (a_{k-1}, t_{k-1}) .

⁹To see this, note that if type $\iota_k^* + 1$ chooses a separating action, it must be to the right of $D(\iota_k^* + 1; j, \iota_k^* + 1)$ by definition. As type $\iota_k^* + 1$ has a lower marginal rate of substitution than type j , it is not possible to find a pooling allocation that satisfies the weak pairwise-matching condition.

the reservation allocation given by

$$(\tilde{a}_k, \tilde{t}_k) = \begin{cases} (s_k^*(\iota(p_k) - 1), \iota(p_k) - 1) & \text{if } p_k > n_k, \\ (a_{k-1}, t_{k-1}) & \text{if } p_k = n_k. \end{cases} \quad (1)$$

Moreover, the weak pairwise-matching condition requires that, for $p_k \in B$ such that $\iota(p_k) > 1$, any candidate allocation (a_k, t_k) must satisfy:

$$D(t_k; j, \iota(p_k) - 1) \geq a_k \geq D(t_k; j, \iota(p_k)).$$

Therefore, the feasible range of t_k is $[\max\{\hat{t}_k(j, \iota(p_k)), t_{k-1}\}, \hat{t}_k(j, \iota(p_k) - 1)]$. Finally, if $p_k \in B$ and $\iota(p_k) = 1$ (i.e., $p_k = 0$), we have a special case because there are no lower types to separate from and hence no payoff upper bound. The feasible range of t_k in this case is unbounded and is given by $[\max\{\hat{t}_k(j, 1), 1\}, \infty)$. This fact will play a crucial role in our algorithm.

To sum up, in case (ii), the candidate set C_k is the set of (t_k, p_k) such that

$$p_k \in [n_k, F_{i_k}^*] \quad \text{and} \quad t_k \begin{cases} = \hat{t}_k(j, \iota(p_k)) & \text{if } p_k \notin B, \\ \in [\max\{\hat{t}_k(j, \iota(p_k)), \tilde{t}_k\}, \hat{t}_k(j, \iota(p_k) - 1)] & \text{if } p_k \in B, \end{cases}$$

where $\hat{t}(j, 0) = \infty$.¹⁰ Even though the candidate set is more complicated in this case because C_k is not a singleton, three properties of this set are important. First, when there is a range of feasible choices for p_k , t_k is uniquely pinned down and fixed (type $\iota(p_k)$ is choosing the same (a_k, t_k) with different probabilities); and when there is a range of feasible choices for t_k , p_k is uniquely pinned down and fixed (type $\iota(p_k)$ is choosing different allocations (a_k, t_k) that are all consistent with incentive compatibility and the weak pairwise-matching condition). Second, C_k is connected in that if $(t'_k, p_k), (t''_k, p_k) \in C_k$ with $t' > t''$, then $(t_k, p_k) \in C_k$ for any p_k and $t_k \in (t''_k, t'_k)$; and if $(t_k, p'_k), (t_k, p''_k) \in C_k$ with $p' > p''$, then $(t_k, p_k) \in C_k$ for any t_k and $p_k \in (p''_k, p'_k)$. Third, if both (t'_k, p'_k) and (t''_k, p''_k) belong to C_k , then $t''_k \geq t'_k$ if and only if $p''_k \leq p'_k$. These properties of C_k are illustrated in Figure 2. They are important because they allow us to order allocations in the candidate set continuously, as we will discuss below.

¹⁰For any $\mathbf{c}_k \in C_k$, we can also determine the corresponding value of a_k , denoted $\alpha_k(\mathbf{c}_k)$. However, since the precise definition of $\alpha_k(\mathbf{c}_k)$ is cumbersome in case (ii), and is not germane to the discussion of the main algorithm, we leave the details to the Appendix.

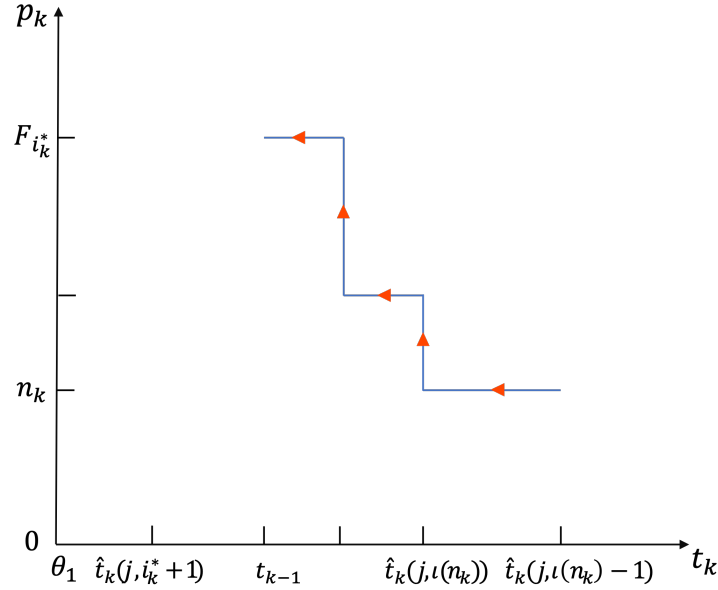


Figure 2. The candidate set C_k contains pairs (t_k, p_k) that lie on the blue line. The arrows indicate the linear order on this set: the point at the southeast end of C_k is $\bar{\mathbf{c}}_k$ and the point at the northwest end of $\underline{\mathbf{c}}_k$.

5.3. The minimum allocation

If the candidate set C_k is not a singleton, there are more than one way to pick a tentative allocation in round k that would satisfy incentive compatibility and the weak pairwise-matching condition. When this allocation is carried to the next round, and when C_{k+1} is also not a singleton, the multiplicity of tentative allocation paths expands, making the search for an equilibrium difficult. Our main algorithm takes advantage of the fact that elements of the candidate set can be linearly ordered, even though they are two-dimensional. In this subsection we formally define this order relation and introduce a way to systematically pick a particular tentative allocation from a candidate set.

Recall that $\mu(p_k, q_k; j)$ is decreasing in q_k when it is larger than $\iota(q_k)$ and is increasing in q_k when it is smaller than $\iota(q_k)$. Let $\mu_k^*(p_k) := \min_{q \in [p_k, F_{j-1}]} \mu(p_k, q; j)$ represent the lowest possible reputation when type j pools with an interval of agents starting from p_k . Because $\mu(p_k, \cdot; j)$ is continuous and strictly increasing, so is $\mu_k^*(\cdot)$. Define

$$\tau_k(\mathbf{c}_k) := t_k - \mu_k^*(p_k).$$

This function is continuous and strictly increasing in t_k and strictly decreasing in p_k . We

say that \mathbf{c}_k is “higher than” \mathbf{c}'_k , and write $\mathbf{c}_k \succeq \mathbf{c}'_k$, if and only if $\tau_k(\mathbf{c}_k) \geq \tau_k(\mathbf{c}'_k)$ (and we use \succ to denote the corresponding strict ordering). Given the properties of C_k , allocations in this set can be linearly ordered according to the relation \succeq . Figure 2 illustrates how allocations in C_k are ordered. The allocation with the highest value of t_k and lowest value of p_k is the highest element of C_k and is denoted $\bar{\mathbf{c}}_k$. The allocation with the lowest value of t_k and highest value of p_k is the lowest element of C_k and is denoted $\underline{\mathbf{c}}_k$. Of course, when C_k is a singleton the two coincide.

We say that an allocation \mathbf{c}_k is a *minimum allocation* if $\tau_k(\mathbf{c}_k) = 0$, and denote it by $\mathbf{c}_k^{\min} = (t_k^{\min}, p_k^{\min})$. This is the allocation which admits a solution in q_k to the equation $\mu(p_k, q_k; j) = t_k$ for the first time as we gradually increase \mathbf{c}_k (according to the order \succeq) in C_k . In other words, $\pi_k(\mathbf{c}_k)$ is undefined whenever $\mathbf{c}_k^{\min} \succ \mathbf{c}_k$. It is clear that there exists a unique \mathbf{c}_k^{\min} in C_k if and only if $\tau_k(\bar{\mathbf{c}}_k) \geq 0 \geq \tau_k(\underline{\mathbf{c}}_k)$. This follows from the fact that C_k is connected, and $\tau_k(\cdot)$ is continuous and strictly increasing in \mathbf{c}_k .

A *minimum allocation path* is an allocation path $(\mathbf{c}_1, \dots, \mathbf{c}_{k'})$ in which $\mathbf{c}_{k'} = \mathbf{c}_{k'}^{\min}$, i.e., an allocation path in which the minimum allocation is chosen in the latest round. The following statement establishes some convenient properties of a minimum allocation path.

Lemma 1. (a) $\pi_k(\mathbf{c}_k^{\min}) \in B$. (b) If $q_k = \pi_k(\mathbf{c}_k^{\min})$, then $t_k^{\min} \leq \iota(q_k)$. (c) If $\mathbf{c}_k = \mathbf{c}_k^{\min}$, then $0 > \tau_{k+1}(\underline{\mathbf{c}}_{k+1})$. (d) $\mathbf{c}_1^{\min} \in C_1$.

Proof. (a) Because $\mu(p_k^{\min}, q; j)$ is decreasing in q when it is larger than $\iota(q)$ and it reaches a minimum at $q = \pi_k(\mathbf{c}_k^{\min})$, we must have $\iota(q) < t_k^{\min}$ for $q \in [p_k^{\min}, \pi_k(\mathbf{c}_k^{\min})]$. Similarly, $\mu(p_k^{\min}, q; j)$ is increasing in q when it is smaller than $\iota(q)$; therefore, $\iota(q) \geq t_k^{\min}$ for $q \in (\pi_k(\mathbf{c}_k^{\min}), F_{j-1}]$. This shows that $\iota(\cdot)$ must jump up at $q = \pi_k(\mathbf{c}_k^{\min})$, and so $\pi_k(\mathbf{c}_k^{\min})$ must be a “threshold agent.” Also, if $\mu(p_k^{\min}, \cdot; j)$ is monotonically decreasing, then $\pi_k(\mathbf{c}_k^{\min}) = F_{j-1} \in B$.

(b) If $q_k = \pi_k(\mathbf{c}_k^{\min}) \in B$, then $\mu(p_k^{\min}, \cdot; j)$ must be weakly increasing on $(\pi_k(\mathbf{c}_k^{\min}), F_{\iota(q_k)}]$. This means $t_k^{\min} \leq \iota(q_k)$. This property also holds when $\mu(p_k^{\min}, \cdot; j)$ is monotonically decreasing, because $q_k = F_{j-1}$ and so $t_k^{\min} < j = \iota(q_k)$.

(c) Since $n_{k+1} = q_k$ and \mathbf{c}_k^{\min} is a minimum allocation, by parts (a) and (b), we have $n_{k+1} \in B$ and $t_k \leq \iota(n_{k+1})$. Moreover, because $n_{k+1} \in B$, the lowest allocation in the candidate set $C_{k+1}(\mathbf{c}_k^{\min})$ satisfies:

$$\tau_{k+1}(\underline{\mathbf{c}}_{k+1}) = \begin{cases} \hat{t}_k(j-1, i_{k+1}^* + 1) - \mu_{k+1}^*(F_{i_{k+1}^*}) & \text{if } i_{k+1}^* \geq \iota(n_{k+1}), \\ t_k - \mu_{k+1}^*(F_{\iota(n_{k+1})-1}) & \text{if } i_k^* = \iota(n_{k+1}) - 1. \end{cases}$$

Since $\hat{t}_k(j-1, i_{k+1}^* + 1) \leq i_{k+1}^* + 1 < \mu^*(F_{i_{k+1}^*})$, we have $\tau_{k+1}(\underline{\mathbf{c}}_{k+1}) < 0$ if $i_{k+1}^* \geq \iota(n_{k+1})$. If $i_{k+1}^* = \iota(n_{k+1}) - 1$, $t_k \leq \iota(n_{k+1}) < \mu_{k+1}^*(F_{\iota(n_{k+1})-1})$ implies $\tau_{k+1}(\underline{\mathbf{c}}_{k+1}) < 0$.

(d) The candidate set C_1 is special because once p_1 reaches 0, we do not require agent $n = 0$ to be indifferent between (a_1, t_1) and the reference allocation given by the state $(a_0, t_0) = (0, 1)$ (type 1 may strictly prefer the former allocation to the latter). In other words, C_1 is unbounded because the initial state only specifies a payoff lower bound for type $\iota(0) = 1$. Once p_1 reaches 0, therefore, the reputation t_1 can be raised without bounds along $D(\cdot; I, 1)$, and we will eventually have $\tau_1(t_1, 0) > 0$ for a sufficiently large t_1 . ■

According to our main algorithm, an allocation path $(\mathbf{c}_1, \dots, \mathbf{c}_{k'})$ constitutes an equilibrium if $\pi_{k'}(\mathbf{c}_{k'}) = F_{j'-1}$ for some k' and $j' = I + 1 - k'$. Parts (a) and (b) of Lemma 1 suggest that if we can find a minimum allocation in a round with two or three remaining types (i.e., $n_k \in [F_{j-3}, F_{j-1})$), the resulting allocation path necessarily constitutes an equilibrium. To see this, consider some round k' such that $n_{k'} \in [F_{j'-3}, F_{j'-1})$ and we find a minimum allocation $\mathbf{c}_{k'}^{\min}$. We then have either (i) $\pi_{k'}(\mathbf{c}_{k'}^{\min}) = F_{j'-1}$; or (ii) $\pi_{k'}(\mathbf{c}_{k'}^{\min}) = F_{j'-2}$ and $t_{k'}^{\min} \leq \iota(F_{j'-2}) = j' - 1$. In case (i), the allocation path $(\mathbf{c}_1, \dots, \mathbf{c}_{k'}^{\min})$ is an equilibrium. In case (ii), we let all type $j' - 1$ agents separate at the top if $t_{k'}^{\min} < j' - 1$, and pool at the same allocation $\mathbf{c}_{k'}^{\min}$ if $t_{k'}^{\min} = j' - 1$.¹¹ Since there are only a finite number of types, this means that we can find an equilibrium, as long as we can consistently find a minimum allocation in the current round by adjusting the allocation path.

Parts (c) and (d) of Lemma 1 imply that we can always start with the minimum allocation in round 1 of the main algorithm. Moreover, in the next round, we have $n_2 \in B$, which allows us to obtain $\underline{\mathbf{c}}_2$ and $\bar{\mathbf{c}}_2$. Since $\tau_2(\underline{\mathbf{c}}_2) < 0$ by Lemma 1(c), we just need to check $\tau_2(\bar{\mathbf{c}}_2)$. If $\tau_2(\bar{\mathbf{c}}_2) \geq 0$, the minimum allocation \mathbf{c}_2^{\min} exists in $C_2(\mathbf{c}_1^{\min})$. We can then continue with the minimum allocation path and proceed to round 3 with $n_3 \in B$. If we can keep on finding the minimum allocation in every round, the algorithm will eventually find an equilibrium at some point. If we fail to find the minimum allocation in some round, the candidate set for the next round is undefined, and we will have to go back to previous rounds and adopt an adjustment procedure to modify the allocation path.

¹¹If $t_{k'}^{\min} = j' - 1$, then $\mu(p_{k'}^{\min}, q; j') = t_{k'}^{\min}$ for $q \in [F_{j'-2}, F_{j'-1}]$. Therefore, we can extend pooling all the way to agent F_j .

5.4. The adjustment procedure

Starting with the minimum allocation \mathbf{c}_1^{\min} , the main algorithm keeps appending the minimum allocation in the next round to the allocation path. If at some round k' the algorithm fails to find a minimum allocation, Lemma 1(c) implies that this occurs when $\tau_{k'}(\bar{\mathbf{c}}_{k'}) < 0$. Because even the highest allocation is “too low” to satisfy the belief consistency requirement, our adjustment procedure will try to pick a “higher” allocation (according to the order \succeq) than the existing one. However, since $\bar{\mathbf{c}}_{k'}$ is already the highest allocation in $C_{k'}$, we need to go back to previous rounds to make the adjustment. The basic rule of thumb is that we always increase the allocation in the latest round before k' that has still room to be adjusted upward. Define

$$\hat{k}(k') := \max\{k \leq k' : \bar{\mathbf{c}}_k \succ \mathbf{c}_k\},$$

as the “round of adjustment.” We often simply write \hat{k} for short, and let $\hat{j} := I + 1 - \hat{k}$. Observe that \hat{k} is always well defined because C_1 is not a singleton and is not bounded.

For any round $k < k'$ with $\mathbf{c}_k = \bar{\mathbf{c}}_k$, there is no room to be adjusted upward; note that this includes all rounds with $n_k \notin B$, because C_k is a singleton and $\mathbf{c}_k = \underline{\mathbf{c}}_k = \bar{\mathbf{c}}_k$. Therefore, if $\mathbf{c}_{\hat{k}+1} = \bar{\mathbf{c}}_{\hat{k}+1}$, we can uniquely pin down $\mathbf{c}_{\hat{k}+1}$ from $\mathbf{c}_{\hat{k}}$. By repeatedly applying this process, we can uniquely pin down $\mathbf{c}_{\hat{k}+m}$ from $\mathbf{c}_{\hat{k}}$ if $\mathbf{c}_k = \bar{\mathbf{c}}_k$ for all $k = \hat{k} + 1, \dots, \hat{k} + m$. Let $\mathbf{c}_{\hat{k}+m} = \zeta_{\hat{k}}^m(\mathbf{c}_{\hat{k}})$ denote this mapping.

Lemma 2. *Suppose $\mathbf{c}_k = \bar{\mathbf{c}}_k$ for $k = \hat{k} + 1, \dots, \hat{k} + m$, such that $\zeta_{\hat{k}}^m(\cdot)$ is well defined. Then, $\zeta_{\hat{k}}^m(\cdot)$ is strictly increasing.*

Proof. When $\mathbf{c}_k = \bar{\mathbf{c}}_k$, $t_k = \hat{t}_k(j, \iota(n_k))$ and $p_k = n_k$. Consider a slight increase in $\mathbf{c}_{\hat{k}}$ according to the order \succeq . There are two ways to do this, either by increasing $t_{\hat{k}}$ or by decreasing $p_{\hat{k}}$. If $t_{\hat{k}}$ increases while $p_{\hat{k}}$ remains fixed, the indifference curve of type $\iota(n_{\hat{k}+1})$ passing through the point $(a_{\hat{k}}, t_{\hat{k}})$ shifts up, and so $t_{\hat{k}+1} = \hat{t}_{\hat{k}+1}(\hat{j} - 1, \iota(n_{\hat{k}+1}))$ (given by the intersection of this indifference curve and the dividing line $D(\cdot; \hat{j} - 1, \iota(n_{\hat{k}+1}))$) increases. Moreover, a higher $t_{\hat{k}}$ also reduces $q_{\hat{k}} = n_{\hat{k}+1}$ (because $\mu(p_{\hat{k}}, q_{\hat{k}}; \hat{j}) = t_{\hat{k}}$ and $\mu(\cdot)$ is locally decreasing in $q_{\hat{k}}$). As a consequence, $p_{\hat{k}+1} = n_{\hat{k}+1}$ also decreases. Therefore $\mathbf{c}_{\hat{k}+1}$ becomes higher according to \succeq . If $p_{\hat{k}}$ decreases while $t_{\hat{k}}$ remains fixed, this reduces $q_{\hat{k}}$ (because $\mu(p_{\hat{k}}, q_{\hat{k}}; \hat{j}) = t_{\hat{k}}$ and $\mu(\cdot)$ is increasing in $p_{\hat{k}}$). Thus $p_{\hat{k}+1} = n_{\hat{k}+1} = q_{\hat{k}}$ falls, while there is no change in $t_{\hat{k}+1}$. Again, $\mathbf{c}_{\hat{k}+1}$ becomes higher according to \succeq . By repeatedly applying this, we show that \mathbf{c}_k increases for all $k = \hat{k} + 1, \dots, \hat{k} + m$ if $\mathbf{c}_{\hat{k}}$ increases slightly. ■

We are now in a position to describe our adjustment procedure.

Adjustment procedure. The current allocation path is $(\mathbf{c}_1, \dots, \mathbf{c}_{k'})$ and the candidate set $C_{k'+1}$ is undefined. Go to round \hat{k} and raise $\mathbf{c}_{\hat{k}}$. Compute the corresponding $\mathbf{c}_{\hat{k}+m} = \zeta_{\hat{k}}^m(\mathbf{c}_{\hat{k}})$ for $m = 1, \dots, k' - \hat{k}$. Keep raising $\mathbf{c}_{\hat{k}}$ and updating the allocations for rounds $\hat{k} + 1$ through k' to the latest values of $\zeta_{\hat{k}}^m(\mathbf{c}_{\hat{k}})$ until one of the following occurs.

1. If for some $m' = 1, \dots, k' - \hat{k}$, the $n_{\hat{k}+m'}$ corresponding to $\zeta_{\hat{k}}^{m'}(\mathbf{c}_{\hat{k}})$ decreases to a point $n_{\hat{k}+m'} \in B$:
 - reset \hat{k} to $\hat{k} + m'$ and go back to the beginning of the adjustment procedure with the updated allocation path.
2. If $\mathbf{c}_{\hat{k}}$ is raised to $\bar{\mathbf{c}}_{\hat{k}}$:
 - reset \hat{k} to $\max\{k \leq \hat{k} : \bar{\mathbf{c}}_k \succ \mathbf{c}_k\}$ and go back to the beginning of the adjustment procedure with the updated allocation path.
3. If $\zeta_{\hat{k}}^{k' - \hat{k}}(\mathbf{c}_{\hat{k}})$ becomes equal to the minimum allocation $\mathbf{c}_{k'}^{\min} \in C_{k'}$:
 - exit the adjustment procedure and return to the main algorithm at round k' with the updated allocation path.

We argue that in part 2 of the adjustment procedure, $\mathbf{c}_{\hat{k}}$ will reach $\bar{\mathbf{c}}_{\hat{k}}$ when $t_{\hat{k}}$ is still below \hat{j} . Suppose we set $t_{\hat{k}} = \hat{j}$, but we still have $\tau_{k'}(\bar{\mathbf{c}}_{k'}) < 0$. If $t_{\hat{k}} = \hat{j}$, then $t_{\hat{k}+1}$ must be greater than $\hat{j} - 1$. Therefore $\tau_{\hat{k}+1}(\mathbf{c}_{\hat{k}+1}) = t_{\hat{k}+1} - \mu_{\hat{k}+1}^*(p_{\hat{k}+1}) > 0$ for any $p_{\hat{k}+1}$. Since we change $\tau_{\hat{k}+1}(\cdot)$ continuously during the adjustment procedure, and since $\tau_{\hat{k}+1}(\cdot) < 0$ at the beginning of the adjustment, there must be a smaller allocation $\mathbf{c}'_{\hat{k}}$ and the induced allocation $\mathbf{c}'_{\hat{k}+1}$ such that $\tau_{\hat{k}+1}(\mathbf{c}'_{\hat{k}+1}) = 0$. By repeating this argument, there is an even smaller allocation $\mathbf{c}''_{\hat{k}}$ and the induced allocation $\mathbf{c}''_{k'}$ such that $\tau_{k'}(\mathbf{c}''_{k'}) = 0$ where $j' := N + 1 - k'$. However, this is a contradiction because we would have reached part 3 of the adjustment procedure before we reach that point.

This argument also implies that our adjustment procedure can always return a minimum allocation path up to round k' to the main algorithm. Since we always try to find the minimum allocation, whenever we fail to find it for the first time, we know $\tau_{k'}(\bar{\mathbf{c}}_{k'}) < 0$. We then adjust the allocation path continuously until we find the minimum allocation. It is thus without loss of generality to assume $\tau_{k'}(\bar{\mathbf{c}}_{k'}) < 0$ in any round where $C_{k'+1}$ is undefined. Now suppose that we set $\mathbf{c}_k = \bar{\mathbf{c}}_k$ for all $k = 2, \dots, k' - 1$ but still have $\tau_{k'}(\bar{\mathbf{c}}_{k'}) < 0$.

In this case, we increase \mathbf{c}_1 , which is unbounded, and in particular raise t_1 to I while fixing p_1 at 0. By the same argument, there must be an allocation \mathbf{c}_1 smaller than $(I, 0)$ and the induced allocation $\mathbf{c}'_{k'}$ such that $\tau_{k'}(\bar{\mathbf{c}}_{k'}) = 0$. Given that we start from an allocation that is too low, by continuity, we must find the minimum allocation before we reach that point. Since $C_{k'+1}$ is well defined whenever the allocation in the previous round is a minimum allocation, this means that the main algorithm can proceed to round $k' + 1$ after the adjustment procedure.

5.5. Global incentive compatibility

Because the adjustment procedure can always return an allocation path to the main algorithm that allows the algorithm to proceed to the next round, the algorithm will eventually stop (the number of rounds cannot exceed I). In each round of this construction, the allocations are chosen in such a way that they satisfy Bayes' rule, local incentive compatibility, and the weak pairwise-matching condition (which guarantees the D1 refinement is met). Equilibrium existence is therefore proven if we can show that the allocation path obtained at the end of the main algorithm satisfies global incentive compatibility.

In any equilibrium, there can only be a finite number of allocations that are chosen on the equilibrium path. We thus denote by $\{(a^\ell, t^\ell)\}_{\ell=1}^L$ the set of equilibrium allocations where $a^{\ell+1} > a^\ell$ and $t^{\ell+1} > t^\ell$ for all $\ell = 1, \dots, L-1$; throughout the proof, we assume $L \geq 3$ because there is no distinction between local and global incentive compatibility when $L < 3$. We say that local incentive compatibility holds for agent $n \in Q(a_\ell)$ if he weakly prefers (a^ℓ, t^ℓ) to $(a^{\min\{\ell+1, L\}}, t^{\min\{\ell+1, L\}})$ and $(a^{\max\{\ell-1, 1\}}, t^{\max\{\ell-1, 1\}})$. Our algorithm ensures that local incentive compatibility holds for all agents.

For each ℓ , use \underline{i}^ℓ and \bar{i}^ℓ to represent respectively the lowest and highest type choosing (a^ℓ, t^ℓ) . The following fact is useful toward establishing the main existence result.

Lemma 3. *Consider some agent n' such that $\iota(n') \leq \underline{i}^L$ and his equilibrium allocation $(a^{\ell'}, t^{\ell'})$. At any point on $\phi^*(a; \iota(n'))$ for $a < a^{\ell'}$, type $\iota(n')$ has a lower marginal rate of substitution than any lower type $i < \iota(n')$.*

Proof. We claim that for any n' such that $\iota(n') \leq \underline{i}^L$, $a^{\ell'} < D(t^{\ell'}; \iota(n'), \iota(n') - 1)$, i.e., the corresponding allocation $(a^{\ell'}, t^{\ell'})$ must locate to the left of $D(t^{\ell'}; \iota(n'), \iota(n') - 1)$. There are three cases. First, if $Q(a^{\ell'})$ is a pooling set and $\ell' < L$, $Q(a^{\ell'})$ is disconnected. Then, by the weak pairwise-matching condition, $(a^{\ell'}, t^{\ell'})$ must locate to the left of $D(t^{\ell'}; \iota(n'), \iota(n') - 1)$. Second, if $Q(a^{\ell'})$ is a pooling set and $\ell' = L$, $Q(a^L)$ must be an interval. Since the lowest

type who chooses (a^L, t^L) must have the highest marginal rate of substitution, (a^L, t^L) must locate to the left of $D(t^L; \underline{i}^L, \underline{i}^L - 1)$. Finally, if $(a^{\ell'}, t^{\ell'})$ is fully separating and $Q(a^{\ell'})$ contains only one type of agents, we must again have $(a^{\ell'}, t^{\ell'})$ to the left of $D(t^{\ell'}; \iota(n'), \iota(n') - 1)$, for otherwise incentive compatibility cannot be satisfied.

Since any indifference curve can cross a dividing line once, the equilibrium indifference curve of type $\iota(n')$ cannot cross $D(\cdot; \iota(n'), \iota(n') - 1)$ at any point to the left of $a^{\ell'}$. This means $a < D(\phi^*(a; \iota(n')); \iota(n'), \iota(n') - 1)$ for all $a < a^{\ell'}$. Then, by Assumption 2, we have $a < D(\phi^*(a; \iota(n')); \iota(n'), i)$ for all $a < a^{\ell'}$ and $i < \iota(n')$, which proves the lemma. ■

Proposition 4. *There always exists a D1 equilibrium.*

Proof. We first show that local incentive compatibility implies global incentive compatibility for all types $i \leq \underline{i}^L$. Consider some agent n' such that $\iota(n') \leq \underline{i}^L$ and his equilibrium allocation $(a^{\ell'}, t^{\ell'})$. Suppose there exists some $\ell'' > \ell' + 1$ such that agent n' strictly prefers $(a^{\ell''}, t^{\ell''})$ to $(a^{\ell'}, t^{\ell'})$. Local incentive compatibility suggests that there is some agent $n'' > n'$ with $\iota(n'') > \iota(n')$ who weakly prefers $(a^{\ell''-1}, t^{\ell''-1})$ to $(a^{\ell''}, t^{\ell''})$.¹² This means that $\phi^*(\cdot; \iota(n'))$ must cross $\phi^*(\cdot; \iota(n''))$ from above at some $a < a^{\ell''}$. This contradicts Lemma 3 because type $\iota(n')$ must have a higher marginal rate of substitution than type $\iota(n'')$ at any point on $\phi^*(a; \iota(n'))$ for $a < a^{\ell''}$. Similarly, suppose there exists some $\ell'' < \ell' - 1$ such that agent n' strictly prefers $(a^{\ell''}, t^{\ell''})$ to $(a^{\ell'}, t^{\ell'})$. Then, by local incentive compatibility, there is some agent $n'' < n'$ with $\iota(n'') < \iota(n')$ who weakly prefers $(a^{\ell''+1}, t^{\ell''+1})$ to $(a^{\ell''}, t^{\ell''})$. This means that $\phi^*(\cdot; \iota(n'))$ must cross $\phi^*(\cdot; \iota(n''))$ from below at some $a < a_{\ell'+1}$. This is again a contradiction because type $\iota(n')$ must have a lower marginal rate of substitution than type $\iota(n'')$ at any point on $\phi^*(a; \iota(n'))$ for $a < a_{\ell'+1}$.

If $Q(a_L)$ is a pooling set, there exists agent n' such that $\bar{i}^L \geq \iota(n') > \underline{i}^L$. By the weak pairwise-matching condition, for any such n' , we have $m(a^L, t^L, \underline{i}^L) \geq m(a^L, t^L, \iota(n'))$. This means $\phi^*(a; \underline{i}^L) < \phi^*(a; \iota(n'))$ for all $a < a^L$. Since type \underline{i}^L prefers (a^L, t^L) to any other allocation, so does type $\iota(n')$.

Finally, consider agent n' such that $\iota(n') > \bar{i}^L$. If there is some agent $n'' \in Q(a^{\ell'})$ such that $\iota(n'') < \iota(n')$ and $m(a^{\ell'}, t^{\ell'}, \iota(n')) = m(a^{\ell'}, t^{\ell'}, \iota(n''))$, then $\phi^*(a; \iota(n')) \geq \phi^*(a; \iota(n''))$ for all a by the double-crossing property. Since agent n'' prefers $(a^{\ell'}, t^{\ell'})$ to any other allocation, so does agent n' . If there are no such agents who have the same marginal rate

¹²To see this, observe that $Q(a^{\ell''-1})$ cannot contain any type $\iota(n')$ agent because otherwise agent n' must be indifferent between $(a^{\ell'}, t^{\ell'})$ and $(a^{\ell''-1}, t^{\ell''-1})$. If this is the case, local incentive compatibility implies that agent n' weakly prefers $(a^{\ell'}, t^{\ell'})$ to $(a^{\ell''}, t^{\ell''})$.

of substitution, by the weak pairwise-matching condition, there must be agent $n'' \in Q(a^{\ell'})$ such that $\iota(n') > \iota(n'') > \underline{i}^{\ell'}$ and $m(a^{\ell'}, t^{\ell'}, \iota(n'')) < m(a^{\ell'}, t^{\ell'}, \iota(n')) < m(a^{\ell'}, t^{\ell'}, \underline{i}^{\ell'})$. This means that $\phi^*(a; \iota(n')) > \phi^*(a; \iota(n''))$ for all $a > a^{\ell'}$ and $\phi^*(a; \iota(n')) > \phi^*(a; \underline{i}^{\ell'})$ for all $a < a^{\ell'}$, i.e., $\phi^*(\cdot; \iota(n'))$ stays above the lower envelope of $\phi^*(\cdot; \iota(n''))$ and $\phi^*(\cdot; \underline{i}^{\ell'})$. Since both types $\iota(n'')$ and $\underline{i}^{\ell'}$ prefer $(a^{\ell'}, t^{\ell'})$ to any other allocation, so does type $\iota(n')$. This proves that local incentive compatibility implies global incentive compatibility for all agents. ■

We remark that except in the case of $I = 2$ described by Proposition 3, equilibrium is typically not unique. There are generally many equilibria because the candidate set specified for our main algorithm is only a subset of all the possible allocations that satisfy local incentive compatibility and the weak pairwise-matching condition. To illustrate this possibility, define

$$\begin{aligned}\iota_{\max}(a, t) &:= \min \left\{ \arg \max_{i=1, \dots, I} m(a, t, i) \right\}, \\ \iota_{\min}(a, t) &:= \min \left\{ \arg \min_{i=1, \dots, I} m(a, t, i) \right\},\end{aligned}$$

for a given allocation (a, t) ,¹³ and consider a full pooling equilibrium in which all types choose $(a_p, \mathbb{E}[i])$. This constitutes a perfect Bayesian equilibrium under some off-equilibrium beliefs if a_p satisfies

$$u(a_p, \mathbb{E}[i], 1) \geq u(0, 1, 1). \quad (2)$$

Moreover, this pooling equilibrium satisfies D1 if

$$\mathbb{E}[i] \geq \max \left\{ \iota_{\max}(a_p, \mathbb{E}[i]), \iota_{\min}(a_p, \mathbb{E}[i]) \right\}. \quad (3)$$

It is easy to see that for any given $\mathbb{E}[i]$, there is a range of a_p that satisfies (2). This means that we may have a continuum of pooling equilibria if there is a range of a_p that also satisfies (3). This is generally the case if there are three or more types. Suppose, for instance, that there are three types with $\mathbb{E}[i] \geq 2$. In this case, (3) is satisfied if $\iota_{\max}(a_p, \mathbb{E}[i]) = 1$ and $\iota_{\min}(a_p, \mathbb{E}[i]) = 2$, which hold for any allocation that is bounded between $D(\cdot; 3, 2)$ and $D(\cdot; 3, 1)$; i.e., any $(a_p, \mathbb{E}[i])$ such that $D(\mathbb{E}[i]; 3, 2) \leq a_p \leq D(\mathbb{E}[i]; 3, 1)$.

¹³When types are discrete, $\arg \max_{i=1, \dots, I} m(a, t, i)$ and $\arg \min_{i=1, \dots, I} m(a, t, i)$ may not be singleton. In this case, we always pick the smaller type because D1 has no bite and we can assign an arbitrary off-equilibrium belief.

As such, if there is some action a_p such that (2) holds with strict inequality and also $D(\mathbb{E}[i]; 3, 2) < a_p < D(\mathbb{E}[i]; 3, 1)$, both (2) and (3) continue to hold for any a in a small neighborhood of a_p . If there are only two types, on the other hand, $\mathbb{E}[i] \in (1, 2)$ and (3) can be satisfied if and only if $\iota_{\max}(a_p, \mathbb{E}[i]) = \iota_{\min}(a_p, \mathbb{E}[i]) = 1$, i.e., the allocation $(a_p, \mathbb{E}[i])$ must be exactly on the dividing line as stated in Proposition 3. Since the dividing line is a function of t , there is a unique value of a_p that can satisfy this, thereby eliminating any possibility of multiple equilibria.

6. Conclusion

In this paper we obtain a characterization of signaling equilibrium under double-crossing preferences that holds for an arbitrary number of discrete types, and based on this characterization, provide a recipe that allows an explicit construction of equilibrium, complementing our earlier work for the case of continuous types. This recipe is useful because equilibrium construction for even a small number of types (say, three or four) is generally difficult. We also illustrate why equilibrium uniqueness obtains in the special case of two types but a continuum of equilibria emerges when there are three or more types.

References

- Araujo, Aloisio, Daniel Gottlieb, and Humberto Moreira**, “A Model of Mixed Signals with Applications to Countersignalling,” *RAND Journal of Economics*, 2007, 38 (4), 1020–1043.
- Bobtcheff, Catherine and Raphael Levy**, “More Haste, Less Speed? Signaling through Investment Timing,” *American Economic Journal: Microeconomics*, 2017, 9 (3), 148–186.
- Chen, Chia-Hui, Junichiro Ishida, and Wing Suen**, “Reputation Concerns in Risky Experimentation,” *Journal of the European Economic Association*, 2021, 19 (4), 1981–2021.
- , —, and —, “Signaling under Double-Crossing Preferences,” *Econometrica*, 2021, (forthcoming).
- Cho, In-Koo and David M. Kreps**, “Signaling Games and Stable Equilibria,” *Quarterly Journal of Economics*, 1987, 102 (2), 179–221.
- Daley, Brendan and Brett Green**, “Market Signaling with Grades,” *Journal of Economic Theory*, 2014, 151, 114–145.
- Degan, Ariana and Ming Li**, “Persuasion with Costly Precision,” *Economic Theory*, 2021, 72 (3), 869–908.
- Dixit, Avinash K. and Barry Nalebuff**, *The Art of Strategy*, New York, NY: W. W. Norton, 2008.
- Feltovich, Nick, Richmond Harbaugh, and Ted To**, “Too Cool for School? Signalling and Countersignalling,” *RAND Journal of Economics*, 2002, 33 (4), 630–649.
- Frankel, Alex and Navin Kartik**, “Muddled Information,” *Journal of Political Economy*, 2019, 127 (4), 1739–1776.
- Hörner, Johannes**, “Signalling and Screening,” *The New Palgrave Dictionary of Economics*, 2d. ed., 2008, 15. Basingstoke, UK: Palgrave Macmillan.
- Kolev, Dobrin R. and Thomas J. Prusa**, “Dumping and Double Crossing: The (In)efficiency of Cost-Based Trade Policy under Incomplete Information,” *International Economic Review*, 2002, 43 (3), 895–918.

Riley, John G., "Informational Equilibrium," *Econometrica*, 1979, 47 (2), 331–359.

Appendix: The Set of Candidate Allocations

In this appendix, we provide a more detailed account of how we construct the candidate set C_k when $n_k \in B$. Since case (i) (where there is a separating equilibrium for types between $\iota(n_k)$ and j) is fully described in the main text, we here focus on case (ii) where it is not feasible to construct a separating equilibrium for types between types $\iota(n_k)$ and j . In this case, the set of candidate allocations often traces the indifference curve of some relevant type that passes through his reservation allocation. To obtain the corresponding action $a_k = \alpha(t_k)$ from the choice of t_k , it is thus convenient to define an indifference curve as a function of reputation t : define $\psi(\cdot, i; \tilde{a}_k, \tilde{t}_k)$, or $\psi_k(\cdot, i)$ for short, to be the indifference curve of type i that passes through some reservation allocation $(\tilde{a}_k, \tilde{t}_k)$. Also, as in the main text, define $(\hat{a}_k(j, i), \hat{t}_k(j, i))$ such that

$$\begin{aligned} u(\hat{a}_k(j, i), \hat{t}_k(j, i), i) &= u(\tilde{a}_k, \tilde{t}_k, i), \\ \hat{a}_k(j, i) &= D(\hat{t}_k(j, i); j, i), \end{aligned}$$

for a given reservation allocation $(\tilde{a}_k, \tilde{t}_k)$. It is important to note that the reservation allocation $(\tilde{a}_k, \tilde{t}_k)$ depends on \mathbf{c}_{k-1} and p_k ; as such, both $\psi_k(\cdot, i)$ and $(\tilde{a}_k, \tilde{t}_k)$ depend on them as well, although we do not explicitly indicate this dependence for brevity.

We have already shown that when $n_k \in B$, the candidate set C_k is given by

$$p_k \in [n_k, F_{i_k}^*] \quad \text{and} \quad t_k \begin{cases} = \hat{t}_k(j, \iota(p_k)) & \text{if } p_k \notin B, \\ \in [\max\{\hat{t}_k(j, \iota(p_k)), t_{k-1}\}, \hat{t}_k(j, \iota(p_k) - 1)] & \text{if } p_k \in B, \end{cases}$$

where we let $\hat{t}(j, 0) = \infty$. In each round of our algorithm, we need to search for a tentative allocation over this two-dimensional space C_k . Below, we will describe how we pin down the corresponding action $a_k = \alpha(t_k)$ from $\mathbf{c}_k = (t_k, p_k) \in C_k$.

First, if $p_k \notin B$, some type $\iota(p_k)$ agents choose a separating action. We can then uniquely pin down $t_k = \hat{t}_k(j, \iota(p_k))$ with $(\tilde{a}_k, \tilde{t}_k) = (s_k^*(\iota(p_k)), \iota(p_k))$. This is the easy case to deal with, because we can also uniquely pin down

$$a_k = \alpha(\hat{t}_k(j, \iota(p_k))) = \hat{a}_k(j, \iota(p_k)).$$

Second, if $p_k \in B$ and $\iota(p_k) > 1$, type $\iota(p_k)$ separates from type $\iota(p_k) - 1$ and pool with type j . Both types $\iota(p_k) - 1$ and $\iota(p_k)$ are relevant, and any candidate allocation must be bounded between $\psi_k(\cdot, \iota(p_k) - 1)$ and $\psi_k(\cdot, \iota(p_k))$ with the reservation allocation given by

(1). Moreover, to pool with type j , the weak pairwise-matching condition requires that any candidate allocation must be bounded between $D(\cdot; j, \iota(p_k))$ and $D(\cdot; j, \iota(p_k) - 1)$. The feasible range of t_k is therefore

$$t_k \in [\max\{\hat{t}_k(j, \iota(p_k)), \tilde{t}_k\}, \hat{t}_k(j, \iota(p_k) - 1)],$$

where $\tilde{t}_k \leq \hat{t}_k(j, \iota(p_k))$ only if $p_k = F_{i_k^*}$. For t_k in this range, if the reservation allocation is to the left of $D(\cdot; j, \iota(p_k))$, we first move up along the dividing line $D(\cdot; j, \iota(p_k))$ until it hits $\psi_k(\cdot, \iota(p_k) - 1)$, and then trace this indifference curve; if the reservation allocation is to the right of $D(\cdot; j, \iota(p_k))$, we simply trace $\psi_k(\cdot, \iota(p_k) - 1)$. To sum up,

$$\alpha_k(\cdot) = \max\{D(\cdot; j, \iota(p_k)), \psi_k(\cdot, \iota(p_k) - 1)\}.$$

Figure 3 illustrates how to construct C_k and also how to “increase” the allocation in this set when $i_k^* > \iota(n_k)$.

Finally, if $p_k = 0$, the set of candidate allocations is unbounded; observe that this can happen only in round 1. Any allocation on $D(\cdot; I, 1)$ can be a candidate allocation as long as it gives type 1 a payoff at least as large as the reservation allocation $(\tilde{a}_1, \tilde{t}_1) = (0, 1)$. The feasible range of t_1 is therefore $t_1 \in [\max\{\hat{t}_1(I, 1), 1\}, \infty)$ with

$$\alpha_1(\cdot) = \max\{D(\cdot; I, 1), 0\}.$$

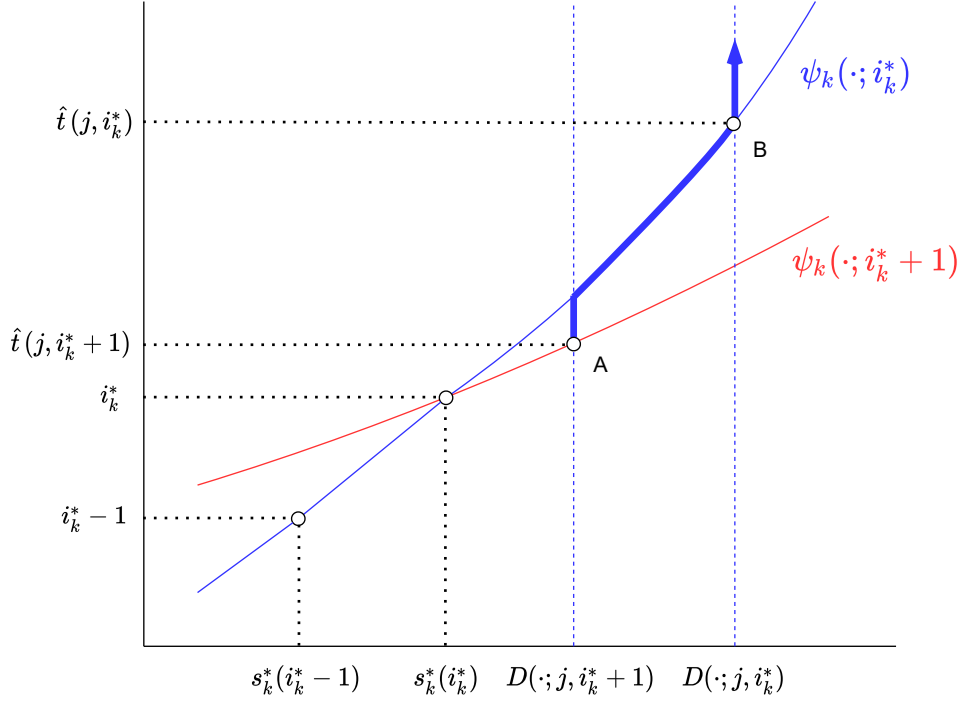


Figure 3. When p_k is set to its highest level $F_{i_k^*}$, a candidate allocation must be bounded between $\psi_k(\cdot, i_k^*)$ and $\psi_k(\cdot, i_k^* + 1)$, with the reservation allocation given by $(s_k^*(i_k^*), i_k^*)$, and also between $D(\cdot; j, i_k^* + 1)$ and $D(\cdot; j, i_k^*)$. Point A represents \underline{c}_k , which is the lowest possible allocation in round k . Starting from this point, we first move up along $D(\cdot; j, i_k^* + 1)$ until it hits $\psi_k(\cdot; i_k^*)$ and then trace $\psi_k(\cdot; i_k^*)$ until it reaches point B, as indicated by the thick line. At point B, we lower p_k from $F_{i_k^*}$ to $F_{i_k^* - 1}$. Once p_k reaches $F_{i_k^* - 1}$, we switch the reservation allocation to $(s_k^*(i_k^* - 1), i_k^* - 1)$, draw the indifference curve of type $i_k^* - 1$ that passes through $(s_k^*(i_k^* - 1), i_k^* - 1)$ and repeat the same process.