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WHEN IS THE TREND THE CYCLE?

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Abstract

A literature debates the explanations for the cyclical properties of emerging markets using either trend shocks (Aguiar and Gopinath 2007) or financial frictions (Neumeyer and Perri 2004; Garcia-Cicco, Pancrazi, and Uribe 2010). We state a formal proposition that makes explicit the parametric assumptions needed for consumption to behave (exactly) as in a random-walk, permanent income model. The result is general and applies to economies with endogenous investment and production. The proposition offers a fresh perspective on the debate regarding the sources of emerging market fluctuations, and reconciles diverging findings in the literature. Moreover, we quantitatively explore the business cycle properties of the RBC model when one moves away from the parametric assumptions suggested by the proposition.

Keywords: Aggregate productivity, permanent income, trend.

JEL codes: E21, E27, E32.

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1 Introduction

A large literature seeks to explain a set of important and salient features of emerging market business cycles based on a class of baseline RBC open economy models. Different from developed countries, emerging markets tend to exhibit a larger volatility of consumption than output, and a volatile and countercyclical current account. These features of the data represent a challenge to standard, frictionless, models.

There are two strands of this literature. One strand of the literature, following the seminal contribution by Aguiar and Gopinath (2007), asserts that a standard RBC small open economy à la Schmitt-Grohe and Uribe (2003) subject to permanent productivity shocks (called also nonstationary technology shocks or trend shocks) is capable of accounting for the aforementioned features of emerging economies. This appealing economic insight has proven quite influential. The logic is that, due to the permanent income assumption, permanent shocks induce large movements in consumption and a volatile and countercyclical current account. This point has been extended in work by Boz, Daude, and Durdu (2011), Naoussi and Tripier (2013), among others.[1] A related strand of the literature, initiated by Neumeyer and Perri (2004) and Uribe and Yue (2006), asserts instead that financial frictions are the explanation to this large volatility. More recent work by Garcia-Cicco, Pancrazi, and Uribe (2010), Alvarez-Parra, Brandao-Marques, and Toledo (2013), and Chang and Fernandez (2013), among others, has also provided quantitative models of financial frictions. Most often used are purely exogenous shocks to the interest rate faced by the domestic economy, which are in fact able to generate realistic business cycle dynamics.

Our main contribution to this debate is analytical. We prove a proposition that makes explicit the conditions under which the permanent income hypothesis holds exactly in the specification by Aguiar and Gopinath (2007) (henceforth AG). This is important, because as AG emphasize, the permanent income hypothesis forms the basis of an explanation grounded on trend shocks.

Our proposition states that, when the sensitivity of the interest rate to movements in the stock of debt goes to zero and preferences are separable, consumption dynamics are only determined by the long-run of level of productivity (up to a constant):

$$c_t = constant \cdot \text{long-run level of productivity}_t$$

Somewhat surprisingly, in this parameter region consumption is completely disconnected from the rest of the model, and, it is highly sensitive to trend shocks. To see this, consider a positive trend shock. Because such a shock raises income at in-

[1] See Durdu (2013) for a recent survey.

finity, consumption reacts strongly on impact. Thus, not only consumption tends to be volatile, but the small open economy finances this increase in consumption by borrowing from the rest of the world, i.e., the net exports are countercyclical. So long the interest rate does not increase following this increase in debt, this effect is quantitatively powerful. Instead, if the interest rate increases, this effect is muted through the consumption Euler equation, and the model may not be able to fit the facts.

The proposition offers a fresh perspective on the debate regarding the sources of emerging market fluctuations.

First, it clarifies the key economic role of the interest rate sensitivity. It is important that, after an accumulation of debt, the interest rate does not increase by much, or not at all. So long this is true, the effect of trend shocks is quantitatively powerful. Notice that AG (and several other following up papers) indeed fix the sensitivity of the interest rate using a single parameter ψ, calibrated to a very small value of $\psi = 0.001$. So far, this strand of the literature has attributed to ψ just the technical role of delivering stationarity.[2] Our proposition assigns it an *economic* role, the one of delivering the random-walk permanent income behavior of consumption.

From a quantitative point of view, the analytical condition on the behavior of interest rate is an invitation to explore how insensitive the interest rate ought to be for trend shocks to have traction. We thoroughly study the implications of increasing the value of ψ, while leaving all other parameters used by AG unchanged (and keeping their exact specification.) We show that when this parameter has a higher value, but still rather small (say 0.1), consumption already features excess smoothness, the volatility of output being higher than the one of consumption. Also, the ratio of the variances of net exports to output goes down to 0.19, whereas in the benchmark results it is 0.71. So, net exports volatility is reduced considerably. In addition, the correlation of net exports and output is also reduced (although to a lesser extent). Thus, assigning a moderate value to ψ overturns most of the quantitative insights. Using the value $\psi = 1$ delivers excess smoothness, a ratio of the variances of net exports to output of 0.10, and correlation of net exports to output more than 2 times smaller, overturning the results in AG. To sum up, the permanent shocks explanation is overturned when the sensitivity of the interest rate in the AG model is increased.

An influential paper by Garcia-Cicco, Pancrazi, and Uribe (2010) (henceforth GPU), appearing after AG, estimates ψ to 2.8 using Bayesian methods. However,

[2]Our reading of this literature is that, tacitly, it is comfortable with assigning a very low value to ψ in order to mimic the behavior of the nonstationary model in which the interest rate is simply fixed. Moving away from the nonstationary model to a stationary one provides computational advantages and allows researchers to match second moments (of endogenous model variables in levels) as in the standard RBC closed-economy model. See for instance the discussion in Mendoza (1991) or Schmitt-Grohe and Uribe (2003).

the model used by GPU is *not* exactly the same as in AG, because it is augmented with interest rate shocks, preference shocks, and spending shocks. To clarify the different findings in AG and GPU, in our numerical exercises, we also consider this estimate ($\psi = 2.8$) *in exactly the same model as in AG.* It delivers similar results as using $\psi = 1$, overturning the findings in AG. Thus, our proposition underlines the pivotal role played by the behavior of the interest rate.

To the best of our knowledge, this is the first paper to take a theoretical approach to analyze the performance of the RBC model in mimicking the economies of emerging markets. In addition, the paper generalizes previous theoretical results (for instance by Campbell and Deaton 1989, or Galí 1991 for endowment economies) by showing that the random-walk permanent income hypothesis for consumption holds exactly, and quite generally, in economies with endogenous investment and production.

Our proposition is closely related to the result by Engel and West (2005) in the context of asset prices. In fact, Engel and West (2005) showcase the condition that the discount factor approaches 1 to guarantee that asset prices manifest a random walk behavior. Our theorem below also requires this condition to hold.

Our paper is related to a large body of existing literature on emerging market business cycles. In addition to those already mentioned, for more recent discussion, see Chen and Crucini (2016), Rothert (2020), Hevia (2014), Dogan (2019), Seoane (2016), Drechsel and Tenreyro (2018), Akinci (2021), among others.

Chen and Crucini (2016) stress that the main limitation of the small open economy model is the lack of attention paid to the role of TFP spillovers from the large economies and instead propose a large aggregate economic region in general equilibrium with a small open economy to capture the international correlation of business cycles. In a similar general equilibrium framework, Rothert (2020) allows the domestic and foreign tradable goods to be imperfect substitutes to account for the behavior of the real exchange rates. There, the impact of the trend shocks on aggregate consumption expenditure becomes smaller in emerging economies as expansionary productivity shocks reduce the relative price of domestic goods, dampening the impact on the country's income. Extending the 'Business Cycle Accounting' methodology by Chari et al. (2007) to an open economy setting, Hevia (2014) suggests that RBC models with just productivity shocks do not provide a successful benchmark to understand emerging market business cycles as productivity shocks in RBC models do not distort the labor-consumption margin.

In a two-country international real business cycle model with investment and consumption goods, Dogan (2019) shows that investment-specific technology shocks play an important role in our understanding of emerging market business cycles. Seoane (2016) proposes a small open economy model with tradable and non-tradable sectors, endogenously accounting for the real exchange rate, and shows that stationary productivity shocks and the country premium explain a large share of the variability observed in the data for emerging economies. Drechsel and Tenreyro (2018) consider a second sector to capture the separate role of commodities in the economy. There, commodity price and stationary productivity shocks are the most important source of fluctuations for output, consumption, and investment, while the contribution of nonstationary productivity shocks remains non-negligible. In a model with an endogenously evolving time-varying country risk premium, Akinci (2021) shows that nonstationary productivity shocks are non-negligible but not dominant in explaining the economic fluctuations in output, consumption, and investment.

The rest of the paper is organized as follows. We setup the model and the log-linear equilibrium in Section 2. We present the proposition and provide an interpretation in Section 3. We explore several quantitative implications of the proposition in Section 4. Section 5 concludes. The proof of our proposition is quite lengthy and it is therefore relegated to the appendix.

2 The Setup and Solution Method

For convenience, it seems natural to use exactly the same model as AG. Thus, we reproduce here the published setup of the model. Then, we reproduce the normalization and log-linearization reported by Aguiar and Gopinath (undated) (henceforth AGb). We use the notation adopted there.

2.1 Aguiar and Gopinath's 2007 Model

This is a single-good, single-asset, small open economy. Technology is characterized by a Cobb-Douglas production function that uses capital K_t and labor L_t as inputs:

$$Y_t = e^{z_t} K_t^{1-\alpha} (\Gamma_t L_t)^{\alpha}$$

where $\alpha \in (0,1)$ represents labor's share of output, and z_t and Γ_t are productivity processes. Specifically, level productivity z_t follows

$$z_t = \rho_z z_{t-1} + \epsilon_t^z$$

with $\rho_z < 1$, and ϵ_t^z is a stationary technology shock, labeled also cycle or transitory shock, and modeled as an i.i.d. draw from a normal distribution with zero mean and standard deviation σ_z. Trend productivity is modeled with a nonstationary process

$$\Gamma_t = e^{g_t} \Gamma_{t-1}$$

where

$$g_t = \rho_g g_{t-1} + \epsilon_t^g$$

and $\rho_g < 1$. ϵ_t^g is a nonstationary technology shock, labeled also trend or permanent shock, and modeled as an i.i.d. draw from a normal distribution with zero mean and standard deviation σ_g. AG allow for a deterministic trend in g_t (denoted μ_g, see AG p. 80.), but to simplify the algebra in the proof of our main result below, we set this deterministic trend to zero ($\mu_g = 0$). Our result extends easily to the general case with unrestricted μ_g.

Period utility is Cobb-Douglas

$$u_t = \frac{[C_t^\gamma (1 - L_t)^{1-\gamma}]^{1-\sigma}}{1 - \sigma}$$

where $0 < \gamma < 1$. The resource constraint is

$$C_t + K_{t+1} = Y_t + (1 - \delta)K_t - \frac{\phi}{2}\left(\frac{K_{t+1}}{K_t} - 1\right)^2 K_t - B_t + Q_t B_{t+1}$$

where K_{t+1} is capital, δ is the capital depreciation rate, B_t represents debt due in period t, q_t is the time t price of debt due in period $t + 1$ and adjustment costs in capital are captured by

$$\frac{\phi}{2}\left(\frac{K_{t+1}}{K_t} - 1\right)^2 K_t$$

where $\phi > 0$ is a parameter.

The price of debt is sensitive to the level of outstanding debt, taking the form used by Schmitt-Grohe and Uribe (2003):

$$\frac{1}{Q_t} = 1 + r_t = 1 + r^* + \psi\left[\exp\left(\frac{B_{t+1}}{\Gamma_t} - b\right) - 1\right] \tag{1}$$

where r^* is the world interest rate, b represent an exogenous steady-state level of normalized debt, and $\psi > 0$ governs the elasticity of the interest rate to debt.

Normalization and Recursive Formulation. For a variable X_t, we write its detrended counterpart by normalizing the variable using previous period's trend productivity:

$$\hat{X}_t = \frac{X_t}{\Gamma_{t-1}}$$

In normalized form, the representative agent's problem is written recursively as

$$V(\hat{K}, \hat{B}, z, g) = \max_{\{\hat{C}, L, \hat{K}', \hat{B}'\}} \left\{ \frac{[\hat{C}^\gamma (1 - L)^{1-\gamma}]^{1-\sigma}}{1 - \sigma} + \beta e^{g\gamma(1-\sigma)} EV(\hat{K}', \hat{B}', z', g') \right\}$$

subject to

$$\hat{C} + e^g \hat{K}' = \hat{Y} + (1-\delta)\hat{K} - \frac{\phi}{2}\left(e^g \frac{\hat{K}'}{\hat{K}} - 1\right)^2 \hat{K} - \hat{B} + e^g q \hat{B}'$$

where a prime on a variable $'$ denotes the value of the variable at $t+1$.

Log-linearization. For nonstationary variables, we define the following log-deviations from stationary steady state quantities:

$$\hat{c}_t \equiv \log(C_t/\Gamma_{t-1}) - \log(\bar{C}/\bar{\Gamma})$$

$$\hat{y}_t \equiv \log(Y_t/\Gamma_{t-1}) - \log(\bar{Y}/\bar{\Gamma})$$

$$\hat{x}_t \equiv \log(X_t/\Gamma_{t-1}) - \log(\bar{X}/\bar{\Gamma})$$

$$\hat{k}_{t+1} \equiv \log(K_{t+1}/\Gamma_t) - \log(\bar{K}/\bar{\Gamma})$$

For variables that are already stationary, we define the following log-deviations

$$\hat{n}_t \equiv \log(N_t) - \log(\bar{N})$$

$$\hat{l}_t \equiv \log(L_t) - \log(\bar{L})$$

$$\hat{q}_t \equiv \log(Q_t) - \log(\bar{Q})$$

We then define the absolute deviation of the net exports-to-output ratio

$$\hat{nx}_t = NX_t/Y_t - \bar{NX}/\bar{Y}$$

We also define the absolute deviation of

$$\hat{b}_{t+1} = \frac{B_{t+1}}{\Gamma_t} - \frac{\bar{B}}{\bar{\Gamma}}$$

These definitions for steady state deviations are identical to the ones used in AGb, with the exception of the last one. There, we use an absolute deviation the relative (log) deviation used in the original paper, in order to allow for $\bar{B}/\bar{\Gamma} = 0$. This allows us to obtain general expressions in the proposition below, but our results do not rely on this specification.

The resulting log-linearized model is fully characterized by the following set of equations:

The dynamics of productivity, including the cycle shocks and the trend shocks:

$$
\begin{aligned}
z' &= \rho_z z + \epsilon^z \\
g' &= \rho_g g + \epsilon^g
\end{aligned}
$$

The first-order condition in k', which corresponds to equation 12 in AGb:

$$
\begin{aligned}
0 = &(\gamma(1-\sigma)-1)\mathbb{E}\hat{c}' + (1-\gamma)(1-\sigma)\mathbb{E}\hat{l}' + \beta\phi\mathbb{E}g' \\
&+ \beta(1-\alpha)\frac{\bar{Y}}{\bar{K}}\mathbb{E}\hat{y}' + \beta\phi\mathbb{E}\hat{k}'' \\
&- \left(\beta\left((1-\alpha)\frac{\bar{Y}}{\bar{K}} + \phi\right) + \phi\right)\hat{k}' - (\gamma(1-\sigma)-1)\hat{c} \\
&- (1-\gamma)(1-\sigma)\hat{l} + (\gamma(1-\sigma)-1-\phi)g + \phi\hat{k}
\end{aligned}
\tag{AGb12}
$$

The first-order condition in b', which corresponds to equation 17 in AGb:

$$
\begin{aligned}
0 = &(\gamma(1-\sigma)-1)\mathbb{E}\hat{c}' + (1-\gamma)(1-\sigma)\mathbb{E}\hat{l}' + (\gamma(1-\sigma)-1)g \\
&- (\gamma(1-\sigma)-1)\hat{c} - (1-\gamma)(1-\sigma)\hat{l} - \hat{q}
\end{aligned}
\tag{AGb17}
$$

Other equations that describe the dynamics of the log-linearized model are:

$$0 = \hat{y} - \hat{n} - \hat{c} + \hat{l}$$

$$0 = \frac{\bar{Y}}{\bar{\Gamma}}\hat{y} + \bar{Q}\hat{b}' + \bar{Q}\frac{\bar{B}}{\bar{\Gamma}}(g+\hat{q}) - \hat{b} - \frac{\bar{X}}{\bar{\Gamma}}\hat{x} - \frac{\bar{C}}{\bar{\Gamma}}\hat{c} \tag{AGb20}$$

$$\frac{\bar{X}}{\bar{\Gamma}}\hat{x} = \frac{\bar{K}}{\bar{\Gamma}}\left(\hat{k}' - (1-\delta)\hat{k} + g\right) \tag{AGb21}$$

$$\hat{y} = z + (1-\alpha)\hat{k} + \alpha(g+\hat{n}) \tag{AGb22}$$

$$\bar{L}\hat{l} = -\bar{N}\hat{n}$$

$$\hat{q} = -\psi\bar{Q}\hat{b}' \tag{AGb24}$$

$$\Delta nx = (1 - \bar{N}X/\bar{Y})\hat{y} - \frac{\bar{X}}{\bar{Y}}\hat{x} - \frac{\bar{C}}{\bar{Y}}\hat{c}$$

where we have followed the notation in AGb. The model is exactly the same as in AGb except for (AGb20) and (AGb24). They are different because of the way we normalize the level of debt.

3 The Proposition

We consider the case with $\gamma = 1$, which corresponds to labor supply being exogenously given. One can trivially restate the arguments for the case $\gamma \neq 1$ but $\sigma = 1$, which corresponds to endogenous labor supply, but additively separable from consumption (log-log preferences). Our result holds when preferences belong to one of these two cases. (Numerical simulations also show that the result does not hold outside these two cases.)

If $\gamma = 1$, $n_t = 0$. Under this assumption, the linearization of the production technology (AGb22) becomes

$$\hat{y}_t = z_t + (1-\alpha)\,\hat{k}_t + \alpha g_t \tag{2}$$

We will solve for a log-linearized solution of the system using the state state space $\mathbf{X}_t = \begin{bmatrix} b_t & \hat{k}_t & \zeta_t & \zeta_{t-1} & z_t \end{bmatrix}'$, where $\zeta_t = \log(\Gamma_t)$ and

$$b_t \;=\; \hat{b}_t + \frac{\bar{B}}{\bar{\Gamma}}\zeta_{t-1}$$

For further use we let $\mathbf{X}_t^0 = \begin{bmatrix} b_t & \hat{k}_t \end{bmatrix}'$ and $\mathbf{X}_t^1 = \begin{bmatrix} \zeta_t & \zeta_{t-1} & z_t \end{bmatrix}'$. It is also important to notice that

$$g_t \;=\; \zeta_t - \zeta_{t-1}$$

Using the definition of log-consumption, we have

$$c_t = \hat{c}_t + \zeta_{t-1}$$

Following standard log-linearization techniques, for example as presented in Blanchard and Kahn (1980) and Uhlig (1999), the solution to the log-linearized model (AGb12)-(AGb24) takes the form:

$$\begin{aligned} c_t &= D_c\mathbf{X}_t \\ b_{t+1} &= D_b\mathbf{X}_t \\ \hat{k}_{t+1} &= D_k\mathbf{X}_t \end{aligned} \tag{3}$$

In particular,

$$\begin{aligned} c_t &= D_c\mathbf{X}_t = D_c^0\mathbf{X}_t^0 + D_c^1\mathbf{X}_t^1 \\ &= D_{c,b}b_t + D_{c,k}\hat{k}_t + D_{c,\zeta 1}\zeta_t + D_{c,\zeta 2}\zeta_{t-1} + D_{c,z}z_t \end{aligned} \tag{4}$$

Denote by $\zeta_{t+\infty}$ the expected long-run level of productivity, i.e.

$$\zeta_{t+\infty} = \lim_{j\to\infty}\mathbb{E}[\zeta_{t+j}] = \frac{\zeta_t - \rho_g\zeta_{t-1}}{1-\rho_g}$$

We claim that as $\bar{Q} \to 1$ and $\psi \to 0$, consumption is only a function of long-run productivity. Specifically,

$$c_t = \left(\frac{1 - \bar{X}/\bar{Y}}{\bar{C}/\bar{Y}} \right) \zeta_{t+\infty} \tag{5}$$

The result is expressed formally as follows.

Proposition 1

$$\lim_{\bar{Q}\to 1} \lim_{\psi \to 0} D_{c,k} = 0$$

$$\lim_{\bar{Q}\to 1} \lim_{\psi \to 0} D_{c,b} = 0$$

$$\lim_{\bar{Q}\to 1} \lim_{\psi \to 0} D_{c,\zeta 1} = \left(\frac{1 - \bar{X}/\bar{Y}}{\bar{C}/\bar{Y}} \right) \left(\frac{1}{1-\rho_g} \right)$$

$$\lim_{\bar{Q}\to 1} \lim_{\psi \to 0} D_{c,\zeta 2} = \left(\frac{1 - \bar{X}/\bar{Y}}{\bar{C}/\bar{Y}} \right) \left(\frac{-\rho_g}{1-\rho_g} \right)$$

$$\lim_{\bar{Q}\to 1} \lim_{\psi \to 0} D_{c,z} = 0$$

3.1 Interpretation

The proposition states that when the interest rate becomes insensitive to changes in debt holdings, consumption is only determined by the long-run level of productivity, as expressed by equation (5). This means, at this limit, the level of debt holdings or the stock of capital do not matter for the determination of consumption. The result also requires that in the steady-state the world interest rate goes to zero ($\bar{Q} \to 1$, following from $\beta \to 1$), which allows the agent to roll-over any existing stock of debt to infinity, thereby allowing to maintain consumption at the long-run level of income (determined by long-run output).[3]

A corollary is that consumption only reacts to permanent shocks, and does not react to transitory shocks. Although not immediately obvious by looking at the proposition, after a permanent shock to productivity consumption *jumps* to its long-run level $\gamma_{t+\infty}$ and stays there. This can be seen in a probably more transparent way by considering the case of zero steady-state debt holdings (or zero steady-state net exports). The following Corollary considers this case and derives the resulting behavior *of* $\hat{c}_t$, the normalized log-deviation of consumption.

[3] Campbell and Deaton (1989) and Galí (1991) considered a version of this result in the case of endowment economies. They also focus on the limit of zero interest rate to study the empirical relationship between the variance of consumption changes and of permanent income shocks (transitory income shocks will be negligible in this limit) in a simple setting with constant interest rate and without capital accumulation.

Corollary 1 *If* $\bar{C}/\bar{Y}+\bar{X}/\bar{Y}=1$,

$$\lim_{\bar{Q}\to 1}\lim_{\psi\to 0}\hat{c}_t=\frac{1}{1-\rho_g}g_t$$

Proof. Proposition 1 shows that in the limit

$$c_t=\left(\frac{1-\bar{X}/\bar{Y}}{\bar{C}/\bar{Y}}\right)\left(\frac{1}{1-\rho_g}\right)\zeta_t-\left(\frac{1-\bar{X}/\bar{Y}}{\bar{C}/\bar{Y}}\right)\left(\frac{\rho_g}{1-\rho_g}\right)\zeta_{t-1}$$

from which, given $\bar{C}/\bar{Y}+\bar{X}/\bar{Y}=1$,

$$c_t=\frac{1}{1-\rho_g}\zeta_t-\frac{\rho_g}{1-\rho_g}\zeta_{t-1}$$

Using the definition of $\hat{c}_t$:

$$\begin{aligned}\hat{c}_t &= c_t-\zeta_{t-1}\\ &= \frac{1}{1-\rho_g}\zeta_t-\frac{\rho_g}{1-\rho_g}\zeta_{t-1}-\zeta_{t-1}\\ &= \frac{1}{1-\rho_g}\left(\zeta_t-\zeta_{t-1}\right)=\frac{1}{1-\rho_g}g_t\end{aligned}$$

■

The constant $(1-\bar{X}/\bar{Y})/(\bar{C}/\bar{Y})$ in front of equation (5) is simply a factor that adjusts the size of deviations to the value of steady-state variables, which depend on the steady-state capital-to-output and consumption-to-output ratios (both exogenous). When $\bar{C}/\bar{Y}+\bar{X}/\bar{Y}=1$, this constant is equal to 1. Using this and expressing the result in terms of normalized log-deviations of consumption instead of log-consumption allows to obtain a simple intuitive expression for the behavior of consumption in the AG model, where consumption is equal to the expected cumulated sum of permanent productivity increases $(1/(1-\rho_g))\cdot g_t$.

Even though our proposition focuses on consumption, it has indirect implications for the behavior of net exports. The key point is that persistent permanent shocks embed a large wealth effect that generates large short-run volatility on consumption, while they have relatively small effects on output. Thus, the implication is countercyclical and volatile net exports.[4]

[4]Notice that transitory shocks are sort of a nuisance in the model because they generate extra output volatility (and little consumption volatility), competing with the main channel emphasized here. However, they turn out to be useful to match output volatility.

4 Quantitative Explorations

In this section, we explore the quantitative implications of the proposition. We proceed in three steps. First, we explore the robustness of the ability of the AG model to match three key moments: the relative variance of consumption to output, the relative variance of net exports with respect to output, and the correlation of net exports and output. We look at what happens when the interest rate is more sensitive than the usual calibration in the literature. Second, we look at the implication of using separable preferences. Third, we explore an endowment economy and check its ability to match the three moments just mentioned.

4.1 The Importance of the Sensitivity of the Interest Rate ψ

Proposition 1 requires $\psi \longrightarrow 0$ in order for consumption to react strongly to changes in the trend of productivity, thereby causing net exports to be highly volatile and countercyclical. We now quantitatively explore this point, and study what happens when ψ is assigned a higher value than the one used in AG (0.001).

All values of the rest of the parameters in this section are the ones used in AG, with the exception of ψ (sensitivity of the interest rate) and γ, as required by Proposition 1. The other benchmark parameter values are presented in Table 1.

Table 1: Benchmark Parameter Values

	Parameter	Value
Non-productivity Parameters		
σ	Intertemporal elasticity of substitution	2
B/Y	Steady state level of normalized debt	0.10
α	Labor's share of output	0.68
δ	Depreciation rate	0.05
ϕ	Capital adjustment costs	4.00
Productivity Processes		
ρ_g	Persistence permanent shock	0.01
ρ_z	Persistence transitory shock	0.95
σ_g	Standard dev. permanent shock	2.81
σ_z	Standard dev. transitory shock	0.48

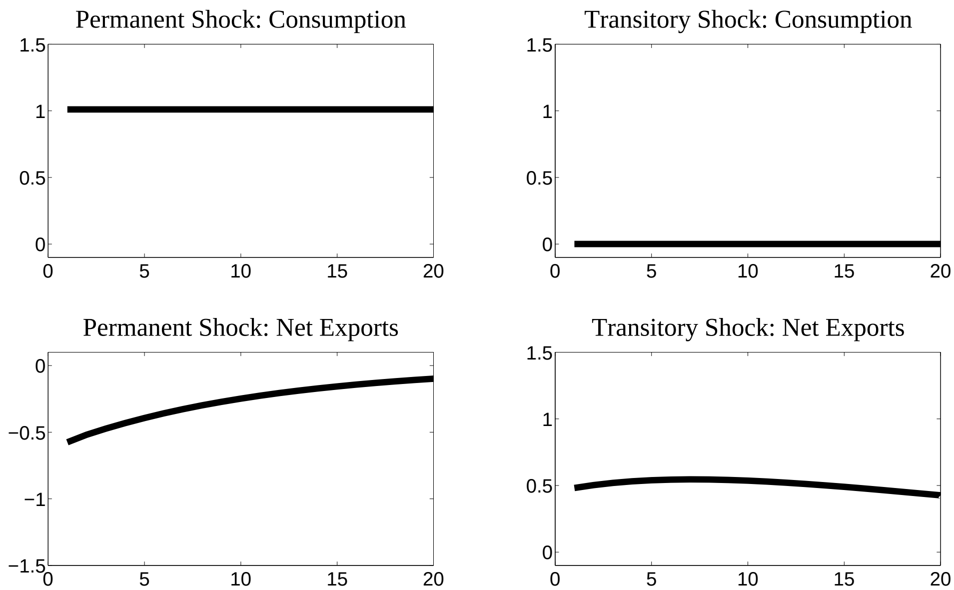
Notes: These parameters for shock processes are the same used for Table 4 of AG, Specification 1, Mexico.

We first check our theoretical result numerically. As shown in Figure 1, imposing ψ very small (and β close to 1 together with $\psi/(1-\beta)$ very small[5]) ensures the

[5] These are equivalent to $\lim_{\bar{Q}\to 1} \lim_{\psi\to 0}$ (in that order).

random walk behavior of consumption. Consumption jumps on impact following a permanent shock, but does not move at all following a transitory shock. These sharp results are quite striking given the complex structure of the rest of the model.[6]

Figure 1: Impulse Responses: Consumption and Net Exports



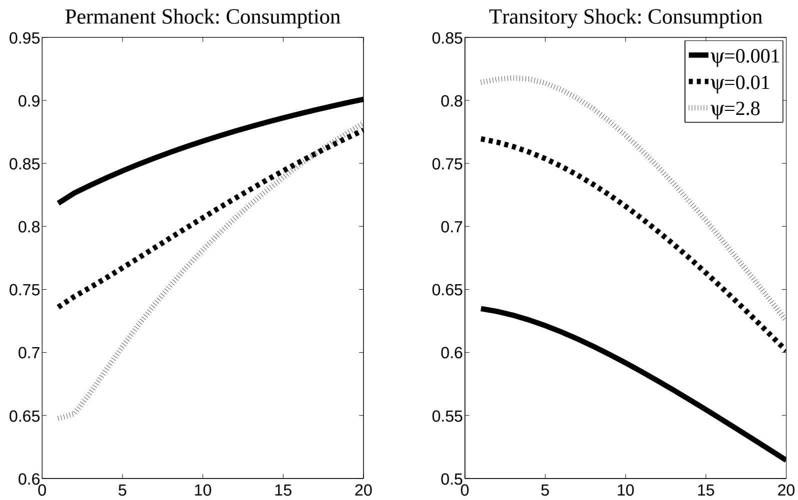
Notes: The lines depict responses of the model in AG in the limiting case of Proposition 1 where we set $\beta = 0.99999$ and $\psi = 10^{-12}$. We also set $\gamma = 1$ which implies that labor supply is exogenously given. The standard deviations of technology shocks are normalized to 1, similar to Figure 3 in AG. The other parameters are from Table 1.

Figure 2 examines the role played by the parameter ψ in determining the sensitivity of consumption to trend shocks by depicting impulse responses of consumption using different values of ψ. We consider the value used in AG (0.001) along with some larger values (0.01, 2.8). 2.8 is the value estimated by Garcia-Cicco, Pancrazi, and Uribe (2010) (henceforth GPU). Consumption does not immediately reach its long-run level with a permanent shock when the parameter ψ takes the values we choose here, these values being substantially larger than the one we used previously in Figure 1 ($\psi = 10^{-12}$). Notice that, crucially, the larger ψ, the smaller the response of consumption following a permanent shock, and the larger the response following a transitory shock. This tends to overturn the results.

Further examining the role played by the parameter ψ, we reproduce Figure 3 of AG with different values of ψ. Again, we consider $\psi = 0.001$, $\psi = 0.01$, and $\psi = 2.8$ and obtain impulse responses of (A) the ratio of net exports to GDP, (B) the ratio of consumption to GDP, and (C) the ratio of investment to GDP following a 1 percent

[6]For this Figure, we also set $\beta = 0.99999$ and $\gamma = 1$ (see p. 10.) We have also verified that setting $\sigma = 1$ instead delivers exactly the same results for consumption, and similar results for net exports.

Figure 2: Impulse Responses: Consumption (Varying ψ)



Notes: For all specifications, we set $\beta = 0.98$ and $\sigma = 2$. Also, the standard deviations of technology shocks are normalized to one. Other parameter values are given in Table 1.

shock on ϵ^g and ϵ^z. We clearly observe that these ratios vary substantially with the parameter ψ. Most importantly, the response of net exports to a permanent g shock is muted with large values of ψ. The reason is the muted response of consumption. These numerical results reveal a similar behavior of investment, which also features a muted response.

Finally, Table 2 reports a set of moments[7] using different parameter values of ψ, ranging from 10^{-12} to 2.8, the value estimated by Garcia-Cicco, Pancrazi, and Uribe (2010). We use exactly same parameters used in AG, except for ψ.[8] As shown in the table, the volatility of consumption $\sigma(c)$, the relative volatility of consumption to output $\sigma(c)/\sigma(y)$, and that of net exports to output $\sigma(NX)/\sigma(y)$ depend greatly on the parameter ψ. Also, similar to the results shown in Figure 3, the relative volatility of investment respect to output $\sigma(I)/\sigma(y)$ is decreasing with ψ.

To sum up, consistent with Proposition 1, this subsection has numerically illustrated the role played by the interest rate sensitivity parameter ψ for the results in AG. The main point is that a small value of ψ ensures the random walk behavior of consumption and leads the model to generate the key moments emphasized in AG. Larger values of ψ tend to reverse this, the model losing the ability to generate those moments. A value of ψ of around 0.1 already generates important difficulties at this task. Larger values overturn the results.

[7]The same shown in Table 5 in AG, plus $\sigma(c)$.

[8]This means that we no longer set σ_g and σ_z to one. Instead, following Table 4 in AG, we set $\sigma_g = 2.81$ and $\sigma_z = 0.48$.

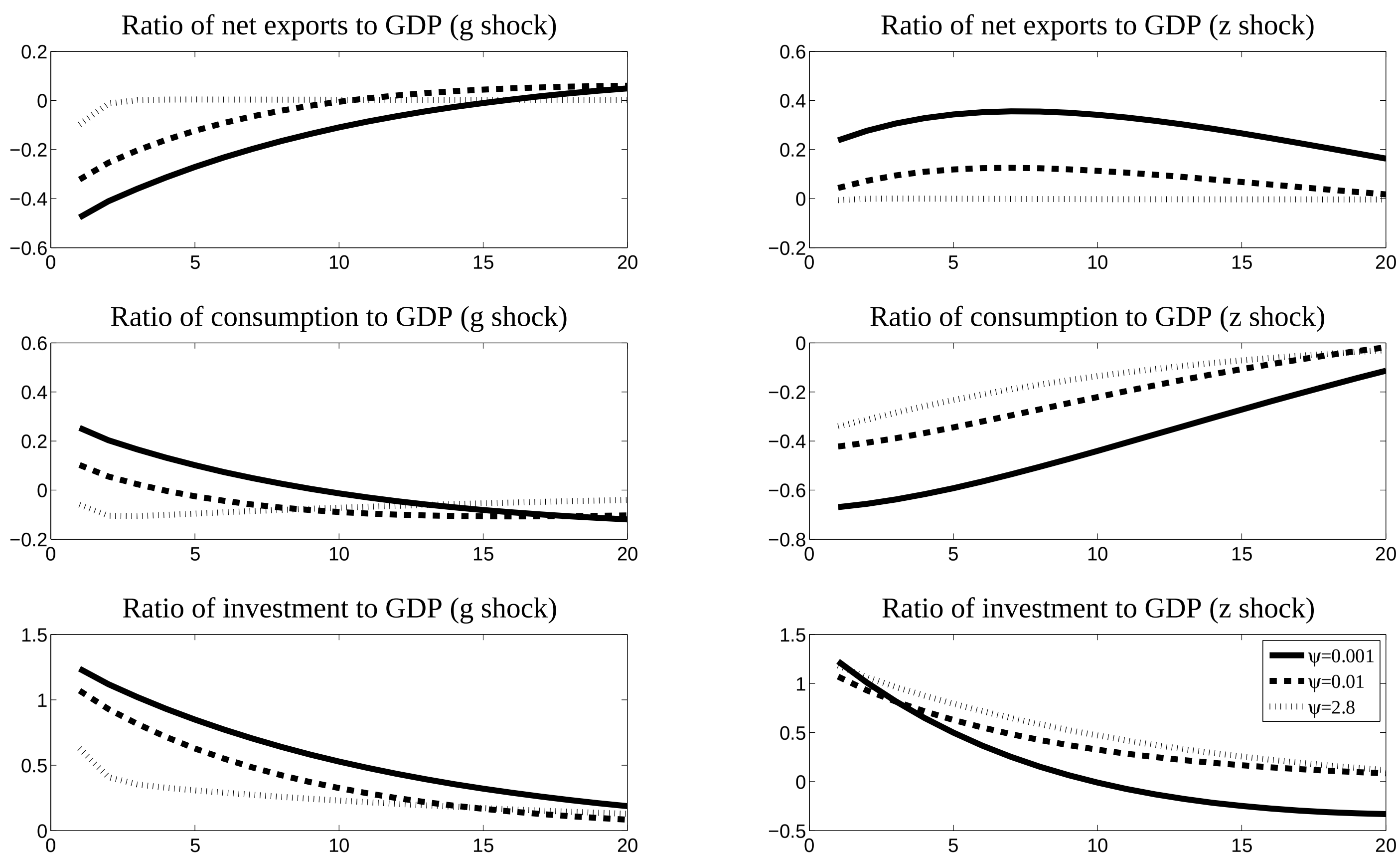


Figure 3: Impulse Responses: (A) Ratio of net exports to GDP, (B) Ratio of consumption to GDP, (C) Ratio of investment to GDP

Notes: For all specifications, we set $\sigma = 2$, $\beta = 0.98$, and the standard deviations of technology shocks are normalized to one. All other parameters as shown in Table 1. We then produce the impulse responses of (A) ratio of net exports to GDP, (B) ratio of consumption to GDP, and (C) ratio of investment to GDP with different values of ψ: $\psi = \{0.001, 0.01, 2.8\}$.

Table 2: Moments - Emerging Market: Mexico

	AG	(1)	(2)	(3)	(4)	(5)	(6)	GPU
ψ	0.001	10^{-12}	0.00001	0.0001	0.01	0.1	1	2.8
			Emerging Market: Mexico					
$\sigma(y)$	2.40	2.29	2.29	2.31	2.59	2.73	2.76	2.76
$\sigma(\Delta y)$	1.73	1.65	1.66	1.67	1.86	2.00	2.05	2.05
$\sigma(c)/\sigma(y)$	1.26	1.39	1.39	1.37	1.06	0.91	0.88	0.88
$\sigma(I)/\sigma(y)$	2.60	2.67	2.67	2.67	2.26	1.80	1.62	1.61
$\sigma(NX)/\sigma(y)$	0.71	0.84	0.84	0.82	0.45	0.19	0.10	0.10
$\rho(y)$	0.78	0.78	0.78	0.78	0.77	0.76	0.75	0.75
$\rho(\Delta y)$	0.13	0.14	0.14	0.13	0.10	0.08	0.07	0.07
$\rho(y, NX)$	-0.66	-0.65	-0.65	-0.66	-0.63	-0.49	-0.35	-0.31
$\rho(y, c)$	0.94	0.92	0.92	0.93	0.98	0.99	1.00	1.00
$\rho(y, I)$	0.92	0.92	0.92	0.92	0.93	0.96	0.97	0.98
$\sigma(c)$	3.03	3.18	3.18	3.16	2.75	2.48	2.43	2.43

Notes: AG refers to specification 1 in Table 5 of AG and GPU refers to Garcia-Cicco, Pancrazi, and Uribe (2010). In GPU, ψ is estimated to be 2.8. Thus, we keep all other parameters as same as in AG and choose ψ to be 2.8. Similarly, for our specifications (1) to (6) are obtained by using different parameter values of ψ.

4.2 Implications of Separable Preferences

Second, we investigate whether alternative preferences to those originally used by AG do a better job at generating consumption volatility. This is indeed suggested by Proposition 1 because only when preferences are separable consumption has a random-walk behavior and thus jumps on impact to the long-run level of productivity implied by the trend shock. The parametrization in AG sets $\sigma = 2$, which implies Cobb-Douglass, non-separable preferences, which can possibly dampen the reaction of consumption on impact.

Table 3: Moments - Separable Preferences Rather than Cobb-Douglas

	AG ($\sigma = 2$)	AG: Separable preferences ($\sigma = 1$)
$\sigma(c)/\sigma(y)$	1.26	1.35
$\sigma(NX)/\sigma(y)$	0.71	0.82
$\rho(NX, y)$	-0.66	-0.62

Notes: All parameters are those from AG, except for σ in the second column.

To investigate this point we simulate moments with $\sigma = 1$ (and letting all other features and parameter values in the original AG model, including the productivity parameters). Table 3 shows that using separable preferences generates an increase in the volatility of consumption to output by 9%. This finding confirm the intuition provided by Proposition 1. Also, this produces an increase in the volatility of net exports to output by 11%. There is slight fall in the correlation between net exports and output from -0.66 to -0.62.

Should one then advocate the use of separable preference in this type of exercises? Actually, one important caveat of separable preferences is the negative comovement of labor supply and trend productivity due to the wealth effect. Avoiding this issue is one reason to resort to other preferences, as in AG. Figure 4 shows that, after a permanent shock, labor supply falls. The next subsection discusses this issue further.

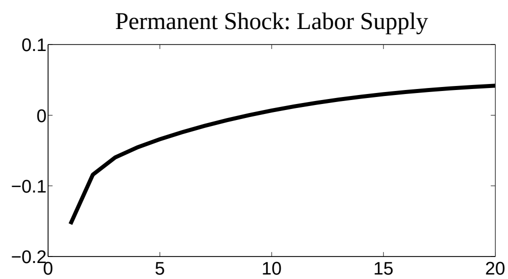


Figure 4: Impulse Responses: Labor Supply

Notes: All parameters are those from AG, except for σ, which is set to 1.

4.3 A Simple Specification

We showed previously that even though separable preferences allow to more easily obtain consumption and net exports volatility, labor supply falls. Recalling Jaimovich and Rebelo (2009), one immediate reaction is that the model requires more frictions to perform well on this dimension. However, an easier and interesting fix is to remove labor supply altogether, i.e., to consider an endowment economy. Notice however that this would also require removing investment from the model. From the point of view of Proposition 1 this is not a problem because its proof does not use any particular form of the economy's supply side.

Table 4 shows the results. It compares the moments generated by the baseline emerging market calibration in AG to a calibration of the simple model we propose. The calibration is in Table 5. The calibrated simple moments does well in replicating the moments in AG, with a larger ratio of the volatility of consumption to the volatility of output $\sigma(c)/\sigma(y)$, a slightly smaller ratio of the volatility of net exports to the volatility of output $\sigma(NX)/\sigma(y)$, and the same correlation between net exports and output $\rho(NX, y)$. So, one could recur to the simple model to match these moments in the data.

Table 4: Moments - Simple Model

	AG	Simple Model
$\sigma(c)/\sigma(y)$	1.26	1.55
$\sigma(NX)/\sigma(y)$	0.71	0.66
$\rho(NX, y)$	-0.66	-0.66

Notes: The first column (AG) is from Column of Table 4 in AG (equivalently, Specification 1, Mexico, Table 5). For the simple model, we use the following parameter values $\rho_g = 0.40$, $\rho_z = 0.95$, $\beta = 0.99$, $\psi = 0.0001$. Other parameter values are from Table 1. The simple model does not include labor supply nor capital (endowment economy).

Table 5: Calibration

	Parameter	Value
Non-productivity Parameters		
B/Y	Steady state level of normalized debt	0.10
Productivity Processes		
ρ_g	Persistence permanent shock	0.40
ρ_z	Persistence transitory shock	0.95
σ_g	Standard dev. permanent shock	2.81
σ_z	Standard dev. transitory shock	0.48

Notes: All other parameters are from AG. Specifically, we set $\beta = 0.98$ and $\psi = 0.001$.

5 Conclusions

In our view, the main conclusions can be drawn from these exercises is the importance of efforts towards a precise and well-identified estimation of the sensitivity of the interest rate ψ. In this direction, recently Miyamoto and Nguyen (2017) estimate using data for 17 developing and developed countries and find that ψ features a credible interval that is bounded away from 0, but the point estimate greatly varies across countries. So far, these estimates have been obtained via structural estimation. Therefore, identification has remained dependent on the exact specification and details of the model. Finding complementary approaches to identifying the sensitivity of the interest rate is a fruitful research avenue.

We finalize this discussion by highlighting the importance of trend shocks to fit the data in other contexts. We emphasize that trend shocks have an important conceptual advantage, the one of resting on a well-established economic mechanism (the permanent income hypothesis) in order to (at least qualitatively) generate highly volatile consumption and net exports, and countercyclical net exports. Trend shocks have been successfully used in "cousin" literatures to match interesting facts implied by consumption dynamics.[9] Thus, we remain under the impression that efforts in the direction of improving the propagation of trend shocks with the help of extra frictions constitute a fruitful research avenue. We look forward to developments in this direction.

[9] See, for instance, Blanchard and Quah (1989), Gali (1999), or Blanchard, L'Huillier, and Lorenzoni (2013).

A Main Proof

The following equations help determine the steady state values and will be important for the proof of Proposition 1:

$$
\begin{aligned}
\bar{Q} &= \frac{1}{1+r^*} = \beta \\
(1-\alpha)\frac{\bar{Y}}{\bar{K}} &= \frac{1}{\bar{Q}} - 1 + \delta \\
\frac{\bar{C}}{\bar{Y}} &= 1 - \delta\frac{\bar{K}}{\bar{Y}} + (\bar{Q}-1)\frac{\bar{B}}{\bar{Y}} \\
\bar{N} &= \left(1 + \frac{\bar{C}}{\bar{Y}}\left(\frac{1-\gamma}{\alpha\gamma}\right)\right)^{-1} \\
\frac{\bar{K}}{\bar{\Gamma}} &= \left(\frac{\bar{K}}{\bar{Y}}\right)^{\frac{1}{\alpha}}\bar{N} \\
\frac{\bar{Y}}{\bar{\Gamma}} &= \left(\frac{\bar{Y}}{\bar{K}}\right)\left(\frac{\bar{K}}{\bar{\Gamma}}\right) \\
\frac{\bar{C}}{\bar{\Gamma}} &= \left(\frac{\bar{C}}{\bar{Y}}\right)\left(\frac{\bar{Y}}{\bar{\Gamma}}\right) \\
\frac{\bar{X}}{\bar{\Gamma}} &= \delta\frac{\bar{K}}{\bar{\Gamma}} \\
\bar{NX}/\bar{Y} &= 1 - \frac{\bar{C}}{\bar{Y}} - \frac{\bar{X}}{\bar{Y}}
\end{aligned}
$$

Proof of Proposition 1. Substituting $\hat{y}_t$ from (2) and substituting $\hat{x}_t$ from (AGb21) in the linearization of the budget constraint (AGb20) implies

$$
\begin{aligned}
0 = {} & \frac{\bar{Y}}{\bar{\Gamma}}\left(z_t + (1-\alpha)\hat{k}_t + \alpha g_t\right) + \bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\left(g_t - \psi\bar{Q}\hat{b}_{t+1}\right) \\
& + \bar{Q}\hat{b}_{t+1} - \hat{b}_t - \frac{\bar{K}}{\bar{\Gamma}}\left(\hat{k}_{t+1} - (1-\delta)\hat{k}_t + g_t\right) - \frac{\bar{C}}{\bar{\Gamma}}\hat{c}_t
\end{aligned}
$$

Combining with the definition of b_{t+1} and c_t, we obtain

$$\begin{aligned}
\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)b_{t+1} &= b_t - \left(\alpha\frac{\bar{Y}}{\bar{\Gamma}} - \frac{\bar{K}}{\bar{\Gamma}} + \bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)g_t - \left(\frac{\bar{Y}}{\bar{\Gamma}}(1-\alpha) + \frac{\bar{K}}{\bar{\Gamma}}(1-\delta)\right)\hat{k}_t \\
&\quad - \frac{\bar{Y}}{\bar{\Gamma}}z_t - \frac{\bar{C}}{\bar{\Gamma}}\gamma_{t-1} + \bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)\gamma_t - \frac{\bar{B}}{\bar{\Gamma}}\gamma_{t-1} + \begin{bmatrix}\bar{C}/\bar{\Gamma} & \bar{K}/\bar{\Gamma}\end{bmatrix}\begin{bmatrix}c_t \\ \hat{k}_{t+1}\end{bmatrix} \qquad (6) \\
&= \begin{bmatrix}1 & -\left(\frac{\bar{Y}}{\bar{\Gamma}}(1-\alpha) + \frac{\bar{K}}{\bar{\Gamma}}(1-\delta)\right)\end{bmatrix}\mathbf{X}_t^0 \\
&\quad + \begin{bmatrix}-\left(\alpha\frac{\bar{Y}}{\bar{\Gamma}} - \frac{\bar{K}}{\bar{\Gamma}} + \psi\left(\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)^2\right) & \left(\alpha\frac{\bar{Y}}{\bar{\Gamma}} - \frac{\bar{K}}{\bar{\Gamma}} + \bar{Q}\frac{\bar{B}}{\bar{\Gamma}} - \frac{\bar{C}}{\bar{\Gamma}} - \frac{\bar{B}}{\bar{\Gamma}}\right) & -\frac{\bar{Y}}{\bar{\Gamma}}\end{bmatrix}\mathbf{X}_t^1 \\
&\quad + \frac{\bar{C}}{\bar{\Gamma}}D_c^0\mathbf{X}_t^0 + \frac{\bar{K}}{\bar{\Gamma}}D_k^0\mathbf{X}_t^0 + \frac{\bar{C}}{\bar{\Gamma}}D_c^1\mathbf{X}_t^1 + \frac{\bar{K}}{\bar{\Gamma}}D_k^1\mathbf{X}_t^1
\end{aligned}$$

We regroup the coefficients on $\mathbf{X}_t^0$ and $\mathbf{X}_t^1$ to obtain:

$$D_b^0 = \frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)}\left(\bar{D}_b + \frac{\bar{C}}{\bar{\Gamma}}D_c^0 + \frac{\bar{K}}{\bar{\Gamma}}D_k^0\right) \qquad (7)$$

where $\bar{D}_b = \begin{bmatrix}1 & -\left(\frac{\bar{Y}}{\bar{\Gamma}}(1-\alpha) + \frac{\bar{K}}{\bar{\Gamma}}(1-\delta)\right)\end{bmatrix}$ and

$$D_b^1 = \frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)}\left(\begin{bmatrix}-\left(\alpha\frac{\bar{Y}}{\bar{\Gamma}} - \frac{\bar{K}}{\bar{\Gamma}} + \psi(\bar{Q}\frac{\bar{B}}{\bar{\Gamma}})^2\right) & \left(\alpha\frac{\bar{Y}}{\bar{\Gamma}} - \frac{\bar{K}}{\bar{\Gamma}} + (\bar{Q}-1)\frac{\bar{B}}{\bar{\Gamma}} - \frac{\bar{C}}{\bar{\Gamma}}\right) & -\frac{\bar{Y}}{\bar{\Gamma}}\end{bmatrix} + \frac{\bar{C}}{\bar{\Gamma}}D_c^1 + \frac{\bar{K}}{\bar{\Gamma}}D_k^1\right)$$

More explicitly:

$$\begin{aligned}
D_{b,b} &= \frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)}\left(1 + \frac{\bar{K}}{\bar{\Gamma}}D_{k,b} + \frac{\bar{C}}{\bar{\Gamma}}D_{c,b}\right) \\
D_{b,k} &= \frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)}\left(\frac{\bar{K}}{\bar{\Gamma}}D_{k,k} + \frac{\bar{C}}{\bar{\Gamma}}D_{c,k} - \left(\frac{\bar{Y}}{\bar{\Gamma}}(1-\alpha) + \frac{\bar{K}}{\bar{\Gamma}}(1-\delta)\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
D_{b,z} &= \frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)}\left(\frac{\bar{K}}{\bar{\Gamma}}D_{k,z} + \frac{\bar{C}}{\bar{\Gamma}}D_{c,z} - \frac{\bar{Y}}{\bar{\Gamma}}\right) \\
D_{b,\zeta 1} &= \frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)}\left(\frac{\bar{K}}{\bar{\Gamma}}D_{k,\zeta 1} + \frac{\bar{C}}{\bar{\Gamma}}D_{c,\zeta 1} - \left(\alpha\frac{\bar{Y}}{\bar{\Gamma}} - \frac{\bar{K}}{\bar{\Gamma}} + \psi\left(\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)^2\right)\right) \\
D_{b,\zeta 2} &= \frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)}\left(\frac{\bar{K}}{\bar{\Gamma}}D_{k,\zeta 2} + \frac{\bar{C}}{\bar{\Gamma}}D_{c,\zeta 2} + \left(\alpha\frac{\bar{Y}}{\bar{\Gamma}} - \frac{\bar{K}}{\bar{\Gamma}} + \bar{Q}\frac{\bar{B}}{\bar{\Gamma}} - \frac{\bar{C}}{\bar{\Gamma}} - \frac{\bar{B}}{\bar{\Gamma}}\right)\right)
\end{aligned}$$

From the log-linearizing equation of the Euler equation with respect to k' (AGb12), using the assumption that $\gamma = 1$ and $\hat{l} \equiv 0$, we have

$$\begin{aligned} 0 &= -\sigma \mathbb{E}_t \left[c_{t+1} - c_t\right] + \beta\phi \mathbb{E}_t \left[g_{t+1}\right] \\ &\quad + \beta \left(1-\alpha\right) \frac{\bar{Y}}{\bar{K}} \mathbb{E}_t \left[z_{t+1} + \left(1-\alpha\right) \hat{k}_{t+1} + \alpha g_{t+1}\right] + \beta\phi \mathbb{E}_t \left[\hat{k}_{t+2}\right] \\ &\quad - \left(\beta \left(\left(1-\alpha\right) \frac{\bar{Y}}{\bar{K}} + \phi\right) + \phi\right) \hat{k}_{t+1} - \left(\sigma + \phi\right) g_t + \phi \hat{k}_t \end{aligned} \tag{8}$$

We use the fact that

$$\begin{aligned} \mathbb{E}_t \left[g_{t+1}\right] &= \rho_g g_t \\ \mathbb{E}_t \left[z_{t+1}\right] &= \rho_z z_t \end{aligned}$$

and $g_t = \zeta_t - \zeta_{t-1}$, together with

$$\begin{aligned} \mathbb{E}_t \left[\hat{k}_{t+2}\right] &= \mathbb{E}_t \left[D_k \mathbf{X}_{t+1}\right] \\ &= \mathbb{E}_t \left[D_{k,b} D_b \mathbf{X}_t + D_{k,k} D_k \mathbf{X}_t + D_{k,z} z_{t+1} + D_{k,\zeta 1} \zeta_{t+1}\right] + D_{k,\zeta 2} \zeta_t \\ &= \left(D_{k,b} D_b + D_{k,k} D_k\right) \mathbf{X}_t + D_{k,z} \rho_z z_t + ((1+\rho_g) D_{k,\zeta 1} + D_{k,\zeta 2}) \zeta_t - \rho_g D_{k,\zeta 1} \zeta_{t-1} \\ \mathbb{E}_t \left[\hat{k}_{t+1}\right] &= D_k \mathbf{X}_t \\ \mathbb{E}_t \left[c_{t+1}\right] &= \mathbb{E}_t \left[D_c \mathbf{X}_{t+1}\right] \\ &= \left(D_{c,b} D_b + D_{c,k} D_k\right) \mathbf{X}_t + D_{c,z} \rho_z z_t + ((1+\rho_g) D_{c,\zeta 1} + D_{c,\zeta 2}) \zeta_t - \rho_g D_{c,\zeta 1} \zeta_{t-1} \\ \mathbb{E}_t \left[c_t\right] &= D_c \mathbf{X}_t \end{aligned}$$

to simplify (8) to

$$\begin{aligned} 0 &= -\sigma \left(\left(D_{c,b} D_b + D_{c,k} D_k\right) \mathbf{X}_t + D_{c,z} \rho_z z_t + ((1+\rho_g) D_{c,\zeta 1} + D_{c,\zeta 2}) \zeta_t - \rho_g D_{c,\zeta 1} \zeta_{t-1} - D_c \mathbf{X}_t\right) \\ &\quad + \beta\phi\rho_g \left(\zeta_t - \zeta_{t-1}\right) + \beta \left(1-\alpha\right) \frac{\bar{Y}}{\bar{K}} \left(\rho_z z_t + \alpha \rho_g \left(\zeta_t - \zeta_{t-1}\right)\right) + \beta \left(1-\alpha\right) \frac{\bar{Y}}{\bar{K}} \left(1-\alpha\right) D_k \mathbf{X}_t \\ &\quad + \beta\phi \left(\left(D_{k,b} D_b + D_{k,k} D_k\right) \mathbf{X}_t + D_{k,z} \rho_z z_t + ((1+\rho_g) D_{k,\zeta 1} + D_{k,\zeta 2}) \zeta_t - \rho_g D_{k,\zeta 1} \zeta_{t-1}\right) \\ &\quad - \left(\beta \left(\left(1-\alpha\right) \frac{\bar{Y}}{\bar{K}} + \phi\right) + \phi\right) D_k \mathbf{X}_t - \left(\sigma + \phi\right) \left(\zeta_t - \zeta_{t-1}\right) + \phi \hat{k}_t \end{aligned}$$

Now, extracting the components related to $\mathbf{X}_t^0$ from this equation, we have

$$\begin{aligned} 0 &= -\sigma\left(\left(D_{c,b}D_b^0 + D_{c,k}D_k^0\right)\mathbf{X}_t^0 - D_c^0\mathbf{X}_t^0\right) \\ &\quad +\beta\left(1-\alpha\right)\frac{\bar{Y}}{\bar{K}}\left(1-\alpha\right)D_k^0\mathbf{X}_t^0 + \beta\phi\left(\left(D_{k,b}D_b^0 + D_{k,k}D_k^0\right)\mathbf{X}_t^0\right) \\ &\quad -\left(\beta\left(\left(1-\alpha\right)\frac{\bar{Y}}{\bar{K}} + \phi\right) + \phi\right)D_k^0\mathbf{X}_t^0 + \phi\hat{k}_t \end{aligned}$$

for all $\mathbf{X}_t^0$. This implies

$$\begin{aligned} 0 &= -\sigma\left(\left(D_{c,b}D_b^0 + D_{c,k}D_k^0\right) - D_c^0\right) \\ &\quad +\beta\left(1-\alpha\right)\frac{\bar{Y}}{\bar{K}}\left(1-\alpha\right)D_k^0 + \beta\phi\left(D_{k,b}D_b^0 + D_{k,k}D_k^0\right) \\ &\quad -\left(\beta\left(\left(1-\alpha\right)\frac{\bar{Y}}{\bar{K}} + \phi\right) + \phi\right)D_k^0 + \phi\overline{D}_k \end{aligned} \tag{9}$$

where $\overline{D}_k = \begin{bmatrix} 0 & 1 \end{bmatrix}$. This equation helps determines D_k^0, i.e., $D_{k,b}$ and $D_{k,k}$ as functions of $D_{c,b}$ and $D_{c,k}$. In particular when $D_{c,b}$ and $D_{c,k}$ are close to zero, we have the Taylor expansion:

$$\begin{aligned} D_{k,b} &= \alpha_1 D_{c,b} + \beta_1 D_{c,k} + o\left(D_{c,b}\right) + o\left(D_{c,k}\right) \\ D_{k,k} &= D_{k,k}^* + \alpha_2 D_{c,b} + \beta_2 D_{c,k} + o\left(D_{c,b}\right) + o\left(D_{c,k}\right) \end{aligned} \tag{10}$$

where $D_{k,k}^*$ is the solution of

$$0 = -\left(\beta\left(\left(1-\alpha\right)\alpha\frac{\bar{Y}}{\bar{K}} + \phi\right) + \phi\right)D_{k,k}^* + \beta\phi(D_{k,k}^*)^2 + \phi$$

i.e. equation (9) for $D_{k,k}^0$ when $D_c^0 = 0$.

Armed with the solution (10), we now use the first order condition for b' (AGb17), again with $\gamma = 1$ and $\hat{l} \equiv 0$:

$$0 = \sigma\mathbb{E}_t\left[c_{t+1} - c_t\right] - \psi\bar{Q}b_{t+1} + \psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\zeta_t \tag{11}$$

and extract the coefficients on $\mathbf{X}_t^0$ to obtain

$$0 = \sigma\left(\left(D_{c,b}D_b^0 + D_{c,k}D_k^0\right) - D_c^0\right) - \psi\bar{Q}D_b^0$$

Substituting D_b^0 from (7) into this equation, we arrive at

$$0 = \left(D_{c,b} - \frac{\psi}{\sigma}\bar{Q}\right) \frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)} \left(\overline{D}_b + \frac{\bar{C}}{\bar{\Gamma}}D_c^0 + \frac{\bar{K}}{\bar{\Gamma}}D_k^0\right) + D_{c,k}D_k^0 - D_c^0$$

We separate the equations for $D_{c,b}$ and $D_{c,k}$ to obtain

$$0 = \left(D_{c,b} - \frac{\psi}{\sigma}\bar{Q}\right) \frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)} \left(\overline{D}_{b,b} + \frac{\bar{C}}{\bar{\Gamma}}D_{c,b}\right) - D_{c,b} + \left\{\left(D_{c,b} - \frac{\psi}{\sigma}\bar{Q}\right) \frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)} \frac{\bar{K}}{\bar{\Gamma}} + D_{c,k}\right\} D_{k,b} \quad (12)$$

and

$$0 = \left(D_{c,b} - \frac{\psi}{\sigma}\bar{Q}\right) \frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)} \left(\overline{D}_{b,k} + \frac{\bar{C}}{\bar{\Gamma}}D_{c,k}\right) - D_{c,k} + \left\{\left(D_{c,b} - \frac{\psi}{\sigma}\bar{Q}\right) \frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)} \frac{\bar{K}}{\bar{\Gamma}} + D_{c,k}\right\} D_{k,k} \quad (13)$$

Now, we use the results above to show that as $\bar{Q} \to 1$ and $\psi \to 0$, $D_{c,b} \to 0$. Indeed, we first solve for $D_{c,k}$ from the second equation (13):

$$D_{c,k} = \frac{\left(D_{c,b} - \frac{\psi}{\sigma}\bar{Q}\right) \frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)} \left(\overline{D}_{b,k} + \frac{\bar{K}}{\bar{\Gamma}}D_{k,k}\right)}{1 - \left(D_{c,b} - \frac{\psi}{\sigma}\bar{Q}\right) \frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)} \frac{\bar{C}}{\bar{\Gamma}} - D_{k,k}}$$

As $\psi \to 0$, this equation simplifies to

$$D_{c,k} = \frac{D_{c,b}\frac{1}{\bar{Q}}\left(\overline{D}_{b,k} + \frac{\bar{K}}{\bar{\Gamma}}D_{k,k}\right)}{1 - D_{c,b}\frac{1}{\bar{Q}}\frac{\bar{C}}{\bar{\Gamma}} - D_{k,k}} \quad (14)$$

In addition, equation (12) simplifies to

$$0 = D_{c,b}\frac{1}{\bar{Q}}\left(\overline{D}_{b,b} + \frac{\bar{C}}{\bar{\Gamma}}D_{c,b}\right) - D_{c,b} + \left\{D_{c,b}\frac{1}{\bar{Q}}\frac{\bar{K}}{\bar{\Gamma}} + D_{c,k}\right\} D_{k,b} \quad (15)$$

Plugging (14) into (15) and grouping by $D_{c,b}$ (also by definition $\overline{D}_{b,b} = 1$), we have

$$0=\frac{1}{\bar{Q}}\left(1+\frac{\bar{C}}{\bar{\Gamma}}D_{c,b}\right)-1+\left\{\frac{1}{\bar{Q}}\frac{\bar{K}}{\bar{\Gamma}}+\frac{\frac{1}{\bar{Q}}\left(\overline{D}_{b,k}+\frac{\bar{K}}{\bar{\Gamma}}D_{k,k}\right)}{1-D_{c,b}\frac{1}{\bar{Q}}\frac{\bar{C}}{\bar{\Gamma}}-D_{k,k}}\right\}D_{k,b}$$

Equivalently,

$$\bar{Q}-1=\frac{\bar{C}}{\bar{\Gamma}}D_{c,b}+\left\{\frac{\bar{K}}{\bar{\Gamma}}+\frac{\overline{D}_{b,k}+\frac{\bar{K}}{\bar{\Gamma}}D_{k,k}}{1-D_{c,b}\frac{1}{\bar{Q}}\frac{\bar{C}}{\bar{\Gamma}}-D_{k,k}}\right\}D_{k,b}$$

As $\bar{Q}\to 1$ and $\psi\to 0$, Lemma 1 below shows that (10) holds with $\alpha_1=\beta_1=0$, and $0<D^*_{k,k}<1$. Therefore,

$$\bar{Q}-1 \quad = \quad \frac{\bar{C}}{\bar{\Gamma}}D_{c,b}+o(D_{c,b})$$

Therefore, as $\bar{Q}\to 1$, $D_{c,b}\to 0$.

Then, (10) and (14) imply that

$$\begin{aligned}
\lim_{\bar{Q}\to 1}\lim_{\psi\to 0}D_{c,k} &= 0\\
\lim_{\bar{Q}\to 1}\lim_{\psi\to 0}D_{k,b} &= 0\\
\lim_{\bar{Q}\to 1}\lim_{\psi\to 0}D_{k,k} &= \lim_{\bar{Q}\to 1}D^*_{k,k}\\
&= \frac{(\alpha\delta/\phi+2)-\sqrt{(\alpha\delta/\phi+2)^2-4}}{2}
\end{aligned}$$

where the last limit is given in Lemma 1.

We now move on to compute D^1_c.

Rearranging equation (11) and using the conjecture for c_{t+1}, we obtain:

$$0=\left(D_{c,b}-\frac{\psi\bar{Q}}{\sigma}\right)b_{t+1}+D_{c,k}\hat{k}_{t+1}+D^1_cA\mathbf{X}^1_t-c_t+\frac{\psi\bar{Q}\bar{B}/\bar{\Gamma}}{\sigma}\zeta_t$$

and substituting b_{t+1} from (6), we have

$$\begin{aligned}
(1-\bar{x})\,c_t \quad &= \quad \frac{\bar{x}}{\bar{C}/\bar{\Gamma}}\left[b_t-\left(\alpha\frac{\bar{Y}}{\bar{\Gamma}}-\frac{\bar{K}}{\bar{\Gamma}}+\psi\left(\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)^2\right)\zeta_t+\left(\alpha\frac{\bar{Y}}{\bar{\Gamma}}-\frac{\bar{K}}{\bar{\Gamma}}+\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}-\frac{\bar{C}}{\bar{\Gamma}}-\frac{\bar{B}}{\bar{\Gamma}}\right)\zeta_{t-1}-\frac{\bar{Y}}{\bar{\Gamma}}z_t\right]\\
&\quad -\frac{\bar{x}}{\bar{C}/\bar{\Gamma}}\left[\left(\frac{\bar{Y}}{\bar{\Gamma}}(1-\alpha)+\frac{\bar{K}}{\bar{\Gamma}}(1-\delta)\right)\hat{k}_t-\frac{\bar{K}}{\bar{\Gamma}}\hat{k}_{t+1}\right]\\
&\quad +D^1_cA\mathbf{X}^1_t+D_{c,k}\hat{k}_{t+1}+\frac{\psi\bar{Q}\bar{B}/\bar{\Gamma}}{\sigma}\zeta_t \qquad (16)
\end{aligned}$$

where

$$A = \begin{bmatrix} 1+\rho_g & -\rho_g & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \rho_z \end{bmatrix}$$

and

$$\bar{x} = \frac{(D_{c,b} - \frac{\psi \bar{Q}}{\sigma})\bar{C}/\bar{\Gamma}}{\bar{Q}\left(1 - \psi \bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)}$$

Collecting the terms for ζ_t from (16), we have

$$\begin{aligned} (1-\bar{x})\, D_{c,\zeta 1} \quad &= \quad (1+\rho_g)D_{c,\zeta 1} + D_{c,\zeta 2} + \frac{\psi \bar{Q}\bar{B}/\bar{\Gamma}}{\sigma} \\ &\quad - \frac{\bar{x}}{\bar{C}/\bar{\Gamma}}\left(\alpha \frac{\bar{Y}}{\bar{\Gamma}} - \frac{\bar{K}}{\bar{\Gamma}} + \psi \left(\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)^2\right) \\ &\quad + \left(\frac{\bar{x}}{\bar{C}/\bar{\Gamma}}\frac{\bar{K}}{\bar{\Gamma}} + D_{c,k}\right) D_{k,\zeta 1} \end{aligned}$$

which leads to

$$\begin{aligned} (\rho_g + \bar{x})D_{c,\zeta 1} + D_{c,\zeta 2} \quad &= \quad \frac{\bar{x}}{\bar{C}/\bar{\Gamma}}\left(\alpha \frac{\bar{Y}}{\bar{\Gamma}} - \frac{\bar{K}}{\bar{\Gamma}} + \psi \left(\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)^2\right) \\ &\quad - \left(\frac{\bar{x}}{\bar{C}/\bar{\Gamma}}\left(\frac{\bar{K}}{\bar{\Gamma}}\right) + D_{c,k}\right) D_{k,\zeta 1} - \frac{\psi \bar{Q}\bar{B}/\bar{\Gamma}}{\sigma} \end{aligned} \tag{17}$$

Similarly, collecting the terms for ζ_{t-1} from (16), we have

$$\begin{aligned} (1-\bar{x})\, D_{c,\zeta 2} \quad &= \quad -\rho_g D_{c,\zeta 1} + \frac{\bar{x}}{\bar{C}/\bar{\Gamma}}\left(\alpha \frac{\bar{Y}}{\bar{\Gamma}} - \frac{\bar{K}}{\bar{\Gamma}} + \bar{Q}\frac{\bar{B}}{\bar{\Gamma}} - \frac{\bar{C}}{\bar{\Gamma}} - \frac{\bar{B}}{\bar{\Gamma}}\right) \\ &\quad + \left(\frac{\bar{x}}{\bar{C}/\bar{\Gamma}}\left(\frac{\bar{K}}{\bar{\Gamma}}\right) + D_{c,k}\right) D_{k,\zeta 2} \end{aligned}$$

which leads to

$$\begin{aligned} \frac{\rho_g}{1-\bar{x}}D_{c,\zeta 1} + D_{c,\zeta 2} \quad &= \quad \frac{\bar{x}}{1-\bar{x}}\left(\frac{1}{\bar{C}/\bar{\Gamma}}\right)\left(\alpha \frac{\bar{Y}}{\bar{\Gamma}} - \frac{\bar{K}}{\bar{\Gamma}} + \bar{Q}\frac{\bar{B}}{\bar{\Gamma}} - \frac{\bar{C}}{\bar{\Gamma}} - \frac{\bar{B}}{\bar{\Gamma}}\right) \\ &\quad + \frac{1}{1-\bar{x}}\left(\frac{\bar{x}}{\bar{C}/\bar{\Gamma}}\left(\frac{\bar{K}}{\bar{\Gamma}}\right) + D_{c,k}\right) D_{k,\zeta 2} \end{aligned} \tag{18}$$

Substituting $D_{c,\zeta 2}$ from (18) into (17) and using the following steady state relations:

$$\frac{\bar{C}}{\bar{\Gamma}} + \frac{\bar{X}}{\bar{\Gamma}} + \frac{\bar{B}}{\bar{\Gamma}} = \frac{\bar{Y}}{\bar{\Gamma}} + \bar{Q}\frac{\bar{B}}{\bar{\Gamma}}$$

we obtain

$$
\begin{aligned}
D_{c,\zeta 1} \quad = \quad & \left(\frac{1-\bar{x}}{1-\bar{x}-\rho_g}\right)\left(\frac{1}{\bar{C}/\bar{\Gamma}}\right)\left(\alpha\frac{\bar{Y}}{\bar{\Gamma}}-\frac{\bar{K}}{\bar{\Gamma}}+\psi\left(\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)^2\right) \\
& -\left(\frac{1}{1-\bar{x}-\rho_g}\right)\left(\frac{1}{\bar{C}/\bar{\Gamma}}\right)\left(\alpha\frac{\bar{Y}}{\bar{\Gamma}}-\frac{\bar{K}}{\bar{\Gamma}}+\frac{\bar{X}}{\bar{\Gamma}}-\frac{\bar{Y}}{\bar{\Gamma}}\right) \\
& -\left(\frac{1-\bar{x}}{1-\bar{x}-\rho_g}\right)\left(\frac{1}{\bar{C}/\bar{\Gamma}}\right)\left(\frac{1}{\bar{C}/\bar{\Gamma}}\right)(D_{k,\zeta 1}+D_{k,\zeta 2}) \\
& -\left(\frac{1}{1-\bar{x}-\rho_g}\right)\left(\frac{1-\bar{x}}{\bar{x}}\right)\left(\frac{1}{\bar{C}/\bar{\Gamma}}\right)D_{c,k}\left(D_{k,\zeta 1}+D_{k,\zeta 2}\right) \\
& -\left(\frac{1}{1-\bar{x}-\rho_g}\right)\left(\left(\frac{\bar{x}}{\bar{C}/\bar{\Gamma}}\right)\left(\frac{\bar{K}}{\bar{\Gamma}}\right)+D_{c,k}\right)D_{k,\zeta 2} \qquad (19)
\end{aligned}
$$

Now as ψ goes to zero,

$$\lim_{\psi\to 0}\bar{x} \quad = \quad \left(D_{c,b}/\bar{Q}\right)\left(\bar{C}/\bar{\Gamma}\right)$$

Then, as $\bar{Q}$ goes to one, as shown above, we have

$$\lim_{\bar{Q}\to 1}\lim_{\psi\to 0}\bar{x} \quad = \quad \lim_{\bar{Q}\to 1}\lim_{\psi\to 0}\left(D_{c,b}/\bar{Q}\right)\left(\bar{C}/\bar{\Gamma}\right)=0$$

In addition, Lemma 2 shows that as $\bar{Q}\to 1$ and $\psi\to 0$, $D_{k,\zeta 1}+D_{k,\zeta 2}=0$. Therefore (19) implies that

$$\lim_{\bar{Q}\to 1}\lim_{\psi\to 0}D_{c,\zeta 1}=\left(\frac{1-\bar{X}/\bar{Y}}{\bar{C}/\bar{Y}}\right)\left(\frac{1}{1-\rho_g}\right)$$

Similarly, from (18),

$$
\begin{aligned}
\lim_{\bar{Q}\to 1}\lim_{\psi\to 0}D_{c,\zeta 2} \quad & = \quad -\rho_g\lim_{\bar{Q}\to 1}\lim_{\psi\to 0}D_{c,\zeta 1} \\
& = \quad \left(\frac{1-\bar{X}/\bar{Y}}{\bar{C}/\bar{Y}}\right)\left(\frac{-\rho_g}{1-\rho_g}\right)
\end{aligned}
$$

Finally, collecting the terms for z_t from (16), we have

$$(1-\bar{x})\,D_{c,z} \quad = \quad \rho_z D_{c,z}-\frac{\bar{x}}{\bar{C}/\bar{\Gamma}}\left(\frac{\bar{Y}}{\bar{\Gamma}}\right)+\left(\frac{\bar{x}}{\bar{C}/\bar{\Gamma}}\left(\frac{\bar{K}}{\bar{\Gamma}}\right)+D_{c,k}\right)D_{k,z}$$

and rearranging the equations, we have

$$D_{c,z} = \frac{1}{(1-\bar{x}-\rho_z)}\left(-\frac{\bar{x}}{\bar{C}/\bar{\Gamma}}\left(\frac{\bar{Y}}{\bar{\Gamma}}\right)+\left(\frac{\bar{x}}{\bar{C}/\bar{\Gamma}}\left(\frac{\bar{K}}{\bar{\Gamma}}\right)+D_{c,k}\right)D_{k,z}\right)$$

As as ψ goes to zero and $\bar{Q}$ goes to one, we have already shown that $\bar{x}$ goes to zero and that $D_{c,k}$ goes to zero such that in the limit $D_{c,z}$ becomes

$$\begin{aligned}\lim_{\bar{Q}\to 1}\lim_{\psi\to 0} D_{c,z} &= \lim_{\bar{Q}\to 1}\lim_{\psi\to 0}\frac{1}{(1-\bar{x}-\rho_z)}\left(-\frac{\bar{x}}{\bar{C}/\bar{\Gamma}}\left(\frac{\bar{Y}}{\bar{\Gamma}}\right)+\left(\frac{\bar{x}}{\bar{C}/\bar{\Gamma}}\left(\frac{\bar{K}}{\bar{\Gamma}}\right)+D_{c,k}\right)D_{k,z}\right)\\ &= \frac{1}{1-\rho_g}\times 0 = 0\end{aligned}$$

This completes the proof.

■

Lemma 1 *Consider the Taylor expansion in* (10). *As* $\bar{Q}\to 1$ *and* $\psi\to 0$, $\alpha_1,\beta_1\to 0$, *and* $0 < D^*_{k,k} < 1$.

Proof. First of all, from the equation that determines $D^*_{k,k}$ in (10), as $\bar{Q}\to 1$ and $\psi\to 0$, this equation becomes

$$0 = (D^*_{k,k})^2 - \left(\frac{\alpha\delta}{\phi}+2\right)D^*_{k,k}+1$$

since $(1-\alpha)\frac{\bar{Y}}{\bar{K}} = \delta$. This equation gives

$$D^*_{k,k} = \frac{(\alpha\delta/\phi+2)-\sqrt{(\alpha\delta/\phi+2)^2-4}}{2} = \frac{2}{(\alpha\delta/\phi+2)+\sqrt{(\alpha\delta/\phi+2)^2-4}} \in (0,1)$$

Now, we can use the solution for $D_{k,b}$ and $D_{k,k}$ from (10) to obtain the constant α_1 and β_1. We collect the terms for b_t from (9):

$$\begin{aligned}0 &= -\sigma\left((D_{c,b}D_{b,b})-D_{c,b}\right)+\beta\delta(1-\alpha)D_{k,b}\\ &\quad +\beta\phi(D_{k,b}D_{b,b}+D_{k,k}D_{k,b})-\left(\beta(\delta+\phi)+\phi\right)D_{k,b}\end{aligned}$$

Substituting $D_{b,b}$ from (7) and let $\tilde{D}_{k,k} = D_{k,k}-D^*_{k,k}$, we rewrite the last equation as

$$
\begin{aligned}
0 &= -\sigma\left(D_{c,b}\left(\frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)}\left(1+\frac{\bar{K}}{\bar{\Gamma}}D_{k,b}+\frac{\bar{C}}{\bar{\Gamma}}D_{c,b}\right)\right)-D_{c,b}\right)+\beta\delta(1-\alpha)D_{k,b} \\
&\quad +\beta\phi\left(D_{k,b}\left(\frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)}\left(1+\frac{\bar{K}}{\bar{\Gamma}}D_{k,b}+\frac{\bar{C}}{\bar{\Gamma}}D_{c,b}\right)\right)+\left(\tilde{D}_{k,k}+D^*_{k,k}\right)D_{k,b}\right) \\
&\quad -(\beta(\delta+\phi)+\phi)D_{k,b}
\end{aligned}
$$

Ignoring the second order terms such as $D_{c,b}D_{k,b}$, $D^2_{c,b}$, $D^2_{k,b}$, and $\tilde{D}_{k,k}D_{k,b}$ in the first order approximation, the last equation becomes

$$
\begin{aligned}
0 &= -\sigma\left(D_{c,b}\frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)}-D_{c,b}\right)+\beta\delta(1-\alpha)D_{k,b} \\
&\quad +\beta\phi\left(D_{k,b}\frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)}+D^*_{k,k}D_{k,b}\right)-(\beta(\delta+\phi)+\phi)D_{k,b}
\end{aligned}
$$

Therefore, we have

$$
\left(\beta(\delta+\phi)+\phi-\beta\delta(1-\alpha)-\beta\phi\left(\frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)}\right)-D^*_{k,k}\beta\phi\right)D_{k,b}=-\sigma\left(\frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)}-1\right)D_{c,b}
$$

or equivalently,

$$
\left(\beta\delta\alpha+\phi\left(1-\frac{\beta}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)}\right)+\beta\phi(1-D^*_{k,k})\right)D_{k,b}=-\sigma\left(\frac{1}{\bar{Q}\left(1-\psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\right)}-1\right)D_{c,b}
$$

Then, as ψ goes to zero, since $\beta=\bar{Q}$, we have

$$
\left(\beta\delta\alpha+\beta\phi(1-D^*_{k,k})\right)D_{k,b}=-\sigma\left(\frac{1}{\bar{Q}}-1\right)D_{c,b}
$$

Also, $\bar{Q}\to 1$, the coefficient on $D_{k,b}$ on the right-hand side is

$$
\delta\alpha+\phi\left(1-D^*_{k,k}\right)>0
$$

and on the left-hand side

$$
\frac{1}{\bar{Q}}-1\to 0
$$

Therefore, $\alpha_1,\ \beta_1 \to 0$.

■

Lemma 2 *As $\bar{Q} \to 1$ and $\psi \to 0$, $D_{k,\zeta 1} + D_{k,\zeta 2} = 0$.*

Proof. Combining the Euler equation with respect to k' (8) and the Euler equation with respect to b' (11), we have

$$\begin{aligned} 0 \;&=\; \sigma(\zeta_t - \zeta_{t-1}) + \psi\bar{Q}b_{t+1} - \psi\bar{Q}\frac{\bar{B}}{\bar{\Gamma}}\zeta_t \\ &\quad +\beta\phi\rho_g(\zeta_t - \zeta_{t-1}) + \beta(1-\alpha)\frac{\bar{Y}}{\bar{K}}\left(\rho_z z_t + (1-\alpha)\hat{k}_{t+1} + \alpha\rho(\zeta_t - \zeta_{t-1})\right) \\ &\quad +\beta\phi\mathbb{E}[\hat{k}_{t+2} - \hat{k}_{t+1}] \\ &\quad -\left(\beta(1-\alpha)\frac{\bar{Y}}{\bar{K}} + \phi\right)\hat{k}_{t+1} - (\sigma+\phi)(\zeta_t - \zeta_{t-1}) + \phi\hat{k}_t \end{aligned}$$

As ψ goes to zero, this equation simplifies to

$$\begin{aligned} 0 \;&=\; \left(\beta\phi\rho_g + \beta(1-\alpha)\frac{\bar{Y}}{\bar{K}}\alpha\rho_g - \phi\right)(\zeta_t - \zeta_{t-1}) + \beta(1-\alpha)\frac{\bar{Y}}{\bar{K}}\rho_z z_t \\ &\quad +\beta\phi\mathbb{E}\left[\hat{k}_{t+2} - \hat{k}_{t+1}\right] + \phi\hat{k}_t - (\alpha\delta+\phi)\hat{k}_{t+1} \end{aligned} \tag{20}$$

We use the conjecture for $\hat{k}$

$$\begin{aligned} \mathbb{E}\left[\hat{k}_{t+2}\right] &= D_{k,b}D_b\mathbf{X}_t + D_{k,k}D_k\mathbf{X}_t + \rho_z D_{k,z}z_t + ((1+\rho_g)D_{k,\zeta 1} + D_{k,\zeta 2})\zeta_t - \rho_g D_{k,\zeta 1}\zeta_{t-1} \\ \mathbb{E}\left[\hat{k}_{t+1}\right] &= D_{k,b}b_t + D_{k,k}\hat{k}_t + D_{k,z}z_t + D_{k,\zeta 1}\zeta_t + D_{k,\zeta 2}\zeta_{t-1} \end{aligned}$$

to collect the terms for ζ_t from (20):

$$\left(\beta\phi\rho_g + \beta(1-\alpha)\frac{\bar{Y}}{\bar{K}}\alpha\rho_g - \phi\right) - (\alpha\delta+\phi)D_{k,\zeta 1} + \beta\phi(D_{k,b}D_{b,\zeta 1} + D_{k,k}D_{k,\zeta 1} + \rho_g D_{k,\zeta 1} + D_{k,\zeta 2}) = 0 \tag{21}$$

Similarly, collecting the terms for ζ_{t-1}:

$$-\left(\beta\phi\rho_g + \beta(1-\alpha)\frac{\bar{Y}}{\bar{K}}\alpha\rho_g - \phi\right) - (\alpha\delta+\phi)D_{k,\zeta 2} + \beta\phi(D_{k,b}D_{b,\zeta 2} + D_{k,k}D_{k,\zeta 2} - \rho_g D_{k,\zeta 1} - D_{k,\zeta 2}) = 0 \tag{22}$$

Combining (21) and (22), as $\bar{Q} \to 1$, and consequently $\beta \to 1$, we have

$$(\alpha\delta + \phi)(D_{k,\zeta 1} + D_{k,\zeta 2}) - \phi D_{k,k}(D_{k,\zeta 1} + D_{k,\zeta 2}) = 0$$

which leads to

$$(D_{k,\zeta 1} + D_{k,\zeta 2})(\alpha\delta + \phi(1 - D_{k,k})) = 0$$

As $\bar{Q} \to 1$ and $\psi \to 0$, $D_{k,k} \to D^*_{k,k} < 1$, therefore

$$D_{k,\zeta 1} + D_{k,\zeta 2} = 0$$

■

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