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MANY-TO-MANY MATCHING ON A SKILL-SHARING PLATFORM

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Many-to-Many Matching on a Skill-Sharing Platform*

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Abstract

Each agent in a market needs to supplement his skill with a particular skill of another agent to complete his project. A platform matches the agents and allows members of the same match to share their skills. A match is valuable to an agent if he is matched with any agent who possesses a skill complementary to his own skill. When the platform uses the divide-and-conquer pricing strategy, we study the properties of incentive compatible mechanisms in relation to the reciprocal property of the complementary relationships among different skills, and when the market expands in its size.

Key words: platform, network externalities, divide and conquer, revenue maximization, ex post IC.
JEL Codes: D42, D47, D62, D82, L12

1 Introduction

Platforms as intermediaries of economic activities are gaining importance in the modern economy with the development of information technology. Two-sided platforms that match firms and workers, men and women, entrepreneurs and investors, and so on, are all popular forms of matching platforms that help subscribers realize economic gains. Among them, this paper focuses on skill-sharing platforms for agents who are entrepreneurs endowed with heterogeneous skills. Each entrepreneur has a private project that yields a positive value to them when completed. To complete their own project, however, each agent must supplement their own skill with another skill, which may be possessed by other agents. For example, an agent with

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a food delivery project needs to complement his cooking skill with a financial skill, or an agent with an advertising business project needs to complement his IT skill with a graphic design skill, and so on. The platform matches entrepreneurs and allows subscribing members of the same match to share their skills. Our analysis considers both a one-sided market in which every agent is ex ante homogeneous and draw their skills from the same set, and a two-sided market in which there exist two classes of agents who draw their skills from disjoint sets. In a two-sided market, we assume that complementary skills of an agent on one side can be possessed only by agents on the other side. We may suppose, for example, agents on one side are endowed with dexterity skills while those on the other side are equipped with hard skills.

The platform is a monopolist and designs a mechanism that operates in two stages. In the first stage, it collects information about the private skill type from each agent and then matches them based on their reported types. In the second stage, the platform presents to the agents the matching along with discriminatory subscription prices, which the agents accept or reject. We require (ex post) incentive compatibility in the first stage so that it is weakly dominant for each agent to report their skill types truthfully. Addition of the second stage addresses the possibility of coordination failures under network externalities when the platform cannot force the matching on the agents: Whether an agent should subscribe or not depends on the decisions of other agents in the same match who have a complementary skill required for his own project. Subscription for a positive price is worthwhile to the agent only if other subscribing agents have useful skills for him.[1] We require that the mechanism resolve such uncertainty by making acceptance a unique equilibrium outcome when the agents strategically respond to the subscription offers, and say that a mechanism is *uniquely enforceable* if it has this property along with incentive compatibility.

The relationship between any skill and the set of its complementary skills is captured by what we call a *complementary-skill network*. Specifically, a complementary-skill network is a directed network with a link from skill s to skill t if and only if t is complementary for skill s. The complementary-skill network can take many different forms. In particular, the network is *perfectly reciprocal* if for any pair of skills s and t, "s is complementary for t" $\Rightarrow$ "t is complementary for s," and is *perfectly non-reciprocal* if for any pair of skills s and t, "s is complementary for t" $\Rightarrow$ "t is *not* complementary for s." There are numerous intermediate patterns between these

[1]The equilibrium multiplicity in the game of adoption decisions is a central concern in the analysis of network externalities. See Dybvig and Spatt (1983). The multiplicity problem is also known as "chicken and egg" in the two-sided market literature (Caillaud and Jullien (2003)).

two extremes. Furthermore, we say that a complementary-skill network is *r-regular* if the number of complementary skills is the same and equals r for every skill.

In the complete information benchmark where the platform observes the agents' skill types, unique enforceability requires that the platform combine matching with the divide-and-conquer pricing strategy: A small subsidy is offered to some members of a match and positive price is charged to others for whom the skills of the subsidized agents are complementary. In the optimum, the number of agents in the first set is minimized whereas that in the latter set is maximized. We say that a mechanism extracts full surplus under incomplete information if it yields the same expected revenue as the optimal mechanism under complete information.

Under incomplete information, we present a class of uniquely enforceable matching mechanisms also based on the divide-and-conquer pricing strategy. These mechanisms, called *single-match mechanisms,* create at most one match. Specifically, this mechanism targets a fixed set I_1 of agents as the core agents, and identifies the set I_2 of agents for whom the skills possessed by the members of I_1 are complementary. It then identifies the set I_3 of agents for whom the skills possessed by the members of I_2 are complementary, and so on, and creates a single match involving all these agents. The core agents in I_1 are offered a small subsidy, whereas the agents in $I_2, I_3, \ldots,$ are charged a price close to the value of the completed project. The set I_1 of core agents is a choice variable in the formulation of a single-match mechanism. The standard pricing strategy in a two-sided market that offers a subsidy to all agents on one-side and charges positive prices to all agents on the other side corresponds to one single-match mechanism in which many agents belong to I_1. In the other extreme, a single-match mechanism may set $I_1 = \{i_1\}$ for some single agent i_1.

The revenue performance of any given single-match mechanism generally depends on the choice of the set I_1 of subsidized agents as well as the underlying complementary-skill network. For example, the single-match mechanism performs poorly if the number of complementary skills for each skill is small and if I_1 has a few agents since then the skills collectively possessed by I_1 will not be complementary to many other agents either directly or indirectly. On the other hand, it will perform better if each skill has many complementary skills. Suppose then that the complementary-skill network is regular so that each skill has the same number of complementary skills. In such a case, the probability distribution of the number of agents involved in the match is uniquely determined *independently* of the specification of reciprocity as mentioned above. This rather surprising conclusion, which is a consequence of the percolation theorem (McDiarmid, 1981), implies revenue equivalence for any pair of complementary-skill networks that are both r-regular.

In comparison with the first-best benchmark under complete information, a small degree of non-reciprocity of the complementary-skill network implies that the optimal mechanism under incomplete information fails to extract full surplus when the market size is fixed. This observation prompts us to study the asymptotic properties of the above mechanisms when the market expands in its size. We show that the single-match mechanism based on a single agent i_1 is asymptotically optimal provided that the underlying complementary-skill network is a supergraph of any 1-regular network. In particular, as the market expands, the probability that i_1's skill is complementary to *every* agent either directly or indirectly approaches one, implying that the platform can charge a price close to the value of the completed project to every agent but i_1 with probability close to one. This result uses the fact that when the number of agents becomes large, the complementary-agent network is connected with probability close to one provided that the complementary-skill network is perfectly reciprocal. This result can be generalized to the complementary-skill network that is not perfectly reciprocal because of the revenue equivalence result described above.

The contribution of the paper can be summarized as follows: First, it offers a first analysis of a model in which the deterministic complementary relationships among the skills translate to random complementary relationships among the agents through the random realization of their types, and identifies how the reciprocity property of the complementary skills may or may not be important from the perspective of revenue maximization by a monopolistic platform. Specifically, it shows that the application of the percolation theorem yields a revenue equivalence result as mentioned above.

Second, the single-match mechanism generalizes the standard divide-and-conquer strategy in a two-sided market that subsidizes all agents on one side and charges positive prices to agents on the other side. Our result demonstrates its effectiveness under incomplete information and also shows that a single subsidy is all it takes to attract all agents when the market expands in its size provided that the underlying complementary-skill network remains constant and satisfies some mild condition as described above.

The paper is organized as follows. We discuss the related literature in Section 2. Section 3 presents a model and Section 4 describes the requirements of unique enforceability. The benchmark case of complete information is analyzed in Section 5, and the implication of incomplete information is discussed in Section 6. We formulate the single-match mechanisms in Section 7 and present the revenue equivalence result in Section 8. The asymptotic optimality of these mechanisms is presented in Section 9. We conclude with a discussion in Section 10.

2 Related Literature

The present paper belongs to the literature on a monopolistic platform that provides privately informed agents access to each other for subscription prices. The pioneering work by Damiano and Li (2007) studies a two-sided market where each side of the market consists of agents with heterogeneous quality, and the value of a match between any pair of agents is the product of the two individual qualities. The platform maximizes revenue by creating multiple "rooms" for agents, and each agent is randomly matched with the agents from the other side who are assigned the same room. Adopting the framework of Damiano and Li (2007), Hoppe et al. (2011) analyze the performance of the coarse matching scheme in which the market is divided into only two matches. Gomes and Pavan (2016) study efficient and profit-maximizing platforms for many-to-many matching in a two-sided market where each agent has a private type that determines the match value. Board (2009) studies matching in a one-sided market, and shows that the profit maximizing planner creates too many matches from a welfare point of view under various specifications of the match value as a function of the qualities of its members. Veiga (2013) shows that the profit maximizing planner always creates a single match when a larger match always has a higher quality than a smaller match. The common assumption in these papers is that agents have vertically differentiated qualities so that agents with high qualities are unanimously preferred. In contrast, agents in our model are horizontally differentiated and different agents prefer to be matched with different skill types. Our framework is hence closer to that in the two-sided matching literature where each agent is assumed to have a heterogeneous preference ordering over agents on the other side.[2] Extensive literature on matching theory is on the design of a mechanism that matches agents with private preferences over other agents. Most closely related among them are the papers on coalition formation, which consider the problem of partitioning the set of agents into matches and the value of a match to any member is a function of the types of other members. For example, Cechlárová and Romero-Medina (2001) assume that the value of a match to any member is equal to either his most-preferred member, or his least-preferred member, Alcalde and Revilla (2004) introduce a preference which evaluates matches based on the best subset of them, and Dimitrov et al. (2006) and Rodríguez-Álvarez (2009) suppose that each agent evaluates the quality of a match based on the numbers of friends and enemies in it. While the primary focus of the matching literature is on the stability of matching, Marx and Schummer (2021) present the revenue analysis of a monopolistic two-sided platform when it matches agents one-to-one using a deferred acceptance algorithm.

[2] Another difference is that the literature studies a screening mechanism which determines an agent's assignment as a function only of his type and not of the type profile of all agents.

Unlike these papers, however, the present paper assumes that the agents' preferences are over the skills possessed by other agents rather than over the agents themselves. One key difference from the matching literature implied by this assumption is that the present model is one of interdependent values since the value of a match to any agent is a function of not only his own type, but also the types of the agents he is matched with.

A platform is a good with network externalities in the sense that its value to each agent depends positively on the adoption decisions of other agents. Monopoly sale of a network good under incomplete information is studied by Aoyagi (2013), where the value of the good to each agent is the product of the agent's private type and the size of adoption. Some models of network goods monopoly express local network externalities by networks of agents as in the present paper. Among them, Candogan et al. (2012) characterize the relationship between the location of a buyer in the network and the price he faces under imperfect and perfect price discrimination, and Bloch and Quérou (2013) examine the optimality of price discrimination when each buyer is privately informed about the stand-alone valuation of the monopolist's good. Aoyagi (2018), Chen et al. (2018), and Chen et al. (2020) formulate models of price competition between sellers of goods with local network externalities. Among them, Aoyagi (2018) examines the extent to which the divide-and-conquer pricing strategy influences the equilibrium price configuration. These models of local network externalities, however, assume that the agent network is fixed and publicly observed unlike in the current paper.

3 Model

The market consists of the set I of $n \geq 2$ agents. Each agent i owns a project and also is endowed with a private skill θ_i drawn from a finite set Θ_i. Denote by Θ the set of skill profiles $\theta = (\theta_i)_{i \in I}$.

The market is *one-sided* if the agents are ex ante homogeneous and *two-sided* if there exist two ex ante heterogeneous classes of agents. If the market is two-sided, we refer to the two class as side A and side B, and use the same symbols to denote the corresponding subsets of agents. Let $n_a = |A|$ and $n_b = |B|$ denote the numbers of agents on the two sides ($n = n_a + n_b$).

Let Σ be the finite set of skills. In a one-sided market, each agent draws his skill from Σ so that $\Theta_i = \Sigma$ for every $i \in I$. In a two-sided market, on the other hand, Σ consists of an even number of skills and is partitioned into two groups Σ_A and Σ_B of the same size. An agent draws his skill from Σ_A if he is on side A and from Σ_B if he is on side B: $\Theta_i = \Sigma_A$ if $i \in A$ and $\Theta_i = \Sigma_B$ if $i \in B$. In what follows, we denote by $S \geq 2$ the total number of skills ($S = |\Sigma|$) if the market is one-sided, and the number of skills on each side $S = |\Sigma_A| = |\Sigma_B|$ if the market is two-sided. In each case, we assume that each skill type is equally likely for every agent:

$$\Pr(\theta_i = s) = \frac{1}{S} \quad \text{for every } s \in \Sigma \text{ and } i \in I.$$

Each project yields $v > 0$ to its owner if completed and 0 otherwise. Agent i's project can be successfully completed if his skill θ_i is complemented with a particular skill of a different kind. A *complementary skill* for any skill s is a skill that leads to the completion of the project owned by the agent who has skill s. For any skill $s \in \Sigma$, denote by $\mathcal{C}(s) \subset \Sigma$ the set of complementary skills for skill s. In a two-sided market, complementary skills exist only on the other side of the market so that $\mathcal{C}(s) \subset \Sigma_B$ if $s \in \Sigma_A$ and $\mathcal{C}(s) \subset \Sigma_A$ if $s \in \Sigma_B$. Define Z_{st} to be the binary variable that equals one if and only if skill t is complementary to skill s:

$$Z_{st} = \begin{cases} 1 & \text{if } t \in \mathcal{C}(s), \\ 0 & \text{otherwise.} \end{cases}$$

It is useful to interpret Z as a directed graph which has skills as nodes: There is a link from skill s to skill t if and only if $Z_{st} = 1$. We call this graph a *complementary-skill network*. In a two-sided market, a complementary-skill network is a directed *bipartite* graph with node partition (Σ_A, Σ_B) since no link exists within Σ_A or within Σ_B. A complementary-skill network is *r-regular* if $|\mathcal{C}(s)| = r$ for every $s \in \Sigma$. Equivalently, it is r-regular if the indegree and outdegree of every node equals $r \geq 1$:

$$\sum_{t'} Z_{st'} = \sum_{s'} Z_{s't} = r.$$

A pair of skills s, $t \in \Sigma$ are *reciprocal* if $Z_{st} = 1 \Rightarrow Z_{ts} = 1$, and *non-reciprocal* if $Z_{st} = 1 \Rightarrow Z_{ts} = 0$. A complementary-skill network Z is *perfectly reciprocal* if every pair of skills are reciprocal, *perfectly non-reciprocal* if every pair of skills are non-reciprocal, and *partially reciprocal* if there exist both reciprocal and non-reciprocal pairs. Figures 1 and 2 illustrate some complementary-skill networks in a one-sided market and a two-sided market, respectively.

The realizations of random skill types are independent across agents. Given any type profile $\theta \in \Theta$ and any pair of agents i and j, define $X_{ij}(\theta)$ by

$$X_{ij}(\theta) = Z_{\theta_i \theta_j}.$$

In other words, $X_{ij}(\theta) = 1$ if and only if agent j possesses a complementary skill for agent i. Let X_{ij} be the corresponding random variable. Just as in the case of

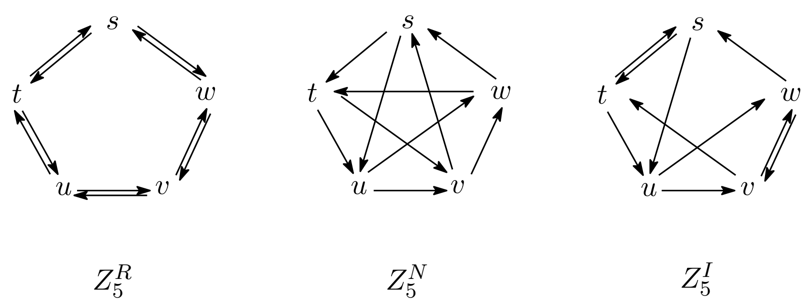

Figure 1: Complementary-skill networks in a one-sided market

$\Sigma = \{s, t, u, v, w\}$. The networks are all 2-regular, and perfectly reciprocal (Z_5^R), perfectly non-reciprocal (Z_5^N), and imperfectly reciprocal (Z_5^I).

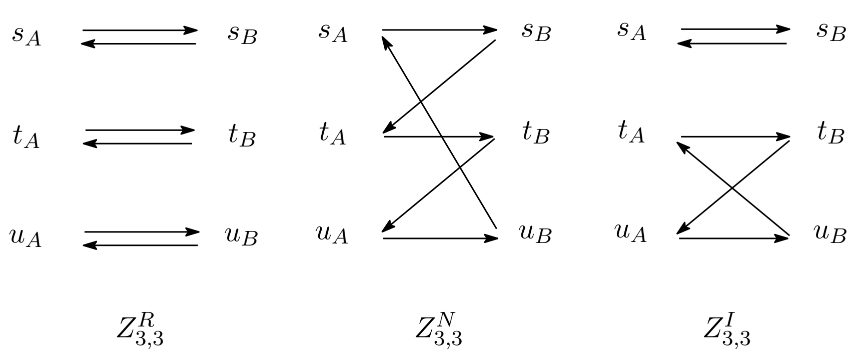

Figure 2: Complementary-skill networks in a two-sided market

$\Sigma_A = \{s_A, t_A, u_A\}$ and $\Sigma_B = \{s_B, t_B, u_B\}$. The networks are all 1-regular, and perfectly reciprocal ($Z_{3,3}^R$), perfectly non-reciprocal ($Z_{3,3}^N$), and partially reciprocal ($Z_{3,3}^I$).

the complementary-skill network Z, we identify $X = (X_{ij})_{i,j \in I}$ as a directed graph which now has the agents as nodes: $X_{ij} = 1$ if and only if this graph has a directed link $i \to j$. We refer to X as a *complementary-agent network.* When the market is two-sided, $X(\theta)$ is a directed *bipartite* graph on the partition (A, B) for any θ. Unlike Z, however, the complementary-agent network X is randomly determined by θ. In $X(\theta)$, agent i is *strongly connected* to agent j, denoted $i \rightsquigarrow_\theta j$, if there is a directed path from i to j: There exists a sequence of agents $i_0, \dots, i_K$ $(K \geq 1)$ in I such that $i_0 = i$, $i_K = j$, and for every i_k and i_{k+1} $(k = 0, \dots, K-1)$, there is a directed link $i_k \to i_{k+1}$ $(\Leftrightarrow X_{i_k i_{k+1}}(\theta) = 1)$. We write $\rightsquigarrow$ instead of $\rightsquigarrow_\theta$ when θ is evident. A subset $H \subset I$ of agents is *strongly connected* if $i \rightsquigarrow j$ for every pair (i, j) of agents in H. $H \subset I$ is a *strong component* if it is strongly connected and there exists no $H' \not\supset H$ that is strongly connected. i is *connected* to j if there is a path from i to j when the direction of each link is ignored. $H \subset I$ is *component* if any pair of agents in H are connected, and no $H' \not\supset H$ has such a property.

A *matching* $g = (g_i)_{i \in I} \in \mathcal{G} \equiv \{0, 1, \dots, n\}^n$ is the partition of agents. Agent i is assigned to match k if $g_i = k \geq 1$, and not assigned to any match if $g_i = 0$. For $k \geq 1$, let $G_k = \{j : g_j = k\}$ denote the set of agents who are assigned to match k, and $G = \cup_{k \geq 1} G_k$ the set of agents who are assigned to some match. We say that the platform *offers subscription* to agent i if $i \in G$. We will use g and G interchangeably to express a matching. Agents i and j $(i \neq j)$ can share their skills if and only if they belong to the same match: $g_i = g_j \geq 1$. Given the skill profile $\theta \in \Theta$, agent i's valuation $u_i(g, \theta)$ of the matching $g = (g_i)_{i \in I}$ is determined as follows:

$$u_i(g, \theta) = \begin{cases} v & \text{if } i \in G_k(\theta) \text{ and } \sum_{j \in G_k(\theta)} X_{ij}(\theta) \geq 1 \text{ for some } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the value of a matching to any agent equals the value v of the completed project if a member of his match has a complementary skill for him, and equals zero otherwise. Note that the value of a match to any agent is independent of the exact number of agents who possess complementary skills for him or the number of agents who find those skills useful. In other words, once in a match each skill is treated as a public good.[3]

A (many-to-many) *matching mechanism* of the platform solicits reports from the agents about their skill types, and then offers a matching along with subscription prices as a function of the reported types. Formally, it is characterized by a pair of a *matching rule* $g : \Theta \to \mathcal{G}$ and a transfer rule $t = (t_i)_{i \in I} : \Theta \to \mathbf{R}^I$: $g(\theta)$ is a matching, and $t_i(\theta)$ is monetary transfer from i, both when the reported type profile

[3]Since a skill of no agent is available to any agent outside the match, we may consider the skill to be an excludable public good.

is θ.

The platform's cost of providing subscription service to any single agent equals c, which is assumed to satisfy

$$0 < c < \frac{v}{2}. \tag{1}$$

It follows that the net social surplus is negative ($= -c$) when the platform offers subscription to a single (unmatched) agent i, but is positive ($\geq v - 2c$) when it matches two agents i and j either of whom finds the other's skill complementary ($\theta_j \in \mathcal{C}(\theta_i)$). Denote by $R(\theta \mid \Gamma)$ the platform's payoff when it employs mechanism Γ and the agents' type profile is θ:

$$R(\theta \mid \Gamma) = \sum_{i \in I} t_i(\theta) - |G(\theta)|c.$$

The platform's expected payoff $R(\Gamma)$ under the mechanism Γ is defined accordingly:

$$R(\Gamma) = E\big[R(\theta \mid \Gamma)\big].$$

4 An Enforceable Mechanism

A mechanism Γ with (g, t) is *ex post incentive compatible* (ex post IC) if for every θ_i, $\theta_i' \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$, and $i \in I$,

$$u_i(g(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}) - t_i(\theta_i, \theta_{-i}) \geq u_i(g(\theta_i', \theta_{-i}), \theta_i, \theta_{-i}) - t_i(\theta_i', \theta_{-i}),$$

and *individually rational* if for every $\theta_i \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$, and $i \in I$,

$$u_i(g(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}) - t_i(\theta_i, \theta_{-i}) \geq 0.$$

As mentioned in the Introduction, we address equilibrium multiplicity under adoption externalities by considering a version of a revelation-suggestion mechanism as in Myerson (1982). Specifically, we suppose that the platform makes take-it-or-leave-it offers to the agents by presenting the matching and subscription prices $((g(\theta), t(\theta))$ along with the reported type profile θ, and the agents play a *subscription game* in which they simultaneously decide to accept or reject the offers. If any agent rejects the offer, then he takes the outside option whose value is normalized to zero. Formally, let $A_i = \{0, 1\}$ denote the set of actions available to agent $i \in I$ in the subscription game, where $a_i = 0$ and $a_i = 1$ represent rejection and acceptance of the offer, respectively.[4] The platform makes offers expecting acceptance by every agent,

[4] If agent i is not offered subscription $g_i = 0$, then his choice a_i only determines whether or not he accepts the transfer t_i. For any such i, hence, $a_i = 1$ implies a non-positive transfer $t_i \leq 0$, and his decision is irrelevant to the adoption decisions of those who are offered subscription.

or equivalently, action profile $a^* = (1, \ldots, 1)$. When the agents instead choose the action profile $a \neq a^*$, the realized matching is different from what is proposed. We suppose that the action profile a in response to the offer g results in a matching $g^a = (g_i^a)_{i \in I}$ such that for every i,

$$g_i^a = \begin{cases} g_i & \text{if } a_i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

That is, only those agents who accept their subscription offers will be matched.[5] Accordingly, given the offer (g, t) and the reported type profile θ, agent i's payoff from the action profile a is given by

$$U_i(a, \theta \mid \Gamma) = \begin{cases} u_i(g^a, \theta) - t_i & \text{if } a_i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We say that a mechanism Γ with (g, t) is *uniquely acceptable* if for every type profile θ, the action profile $a^* = (1, \ldots, 1)$ is the unique Nash equilibrium of the subscription game $(I, A, (U_i(a, \theta \mid g(\theta), x(\theta)))_{i \in I})$ for every $\theta \in \Theta$. Since agent i's payoff $u_i(g(\theta), \theta) - t_i(\theta)$ in equilibrium must be non-negative, (g, t) is individually rational if it is uniquely acceptable. A mechanism is *uniquely enforceable* if it is ex post IC and uniquely acceptable.[6] By ex post IC, no unilateral deviation that involves misreporting in stage 1 and acceptance in stage 2 is profitable. No unilateral deviation that involves rejection in stage 2 is profitable either since any such deviation yields zero, whereas truthful reporting and acceptance yield at least zero by unique acceptability. We hence have the following observation.

Proposition 1. *If Γ is uniquely enforceable, then it implements (g, t) in PBE of the two-stage game.*

5 Complete Information

We begin our analysis with the benchmark case where the platform has complete information about the agents' skill profile θ, or equivalently, the underlying complementary-agent network $X(\theta)$. In this case, we only require the mechanism Γ to be uniquely acceptable. Define $R^*(\theta)$ to be the supremum of the platform's payoff from such mechanisms:

$$R^*(\theta) = \sup \{R(\theta \mid \Gamma) : \Gamma \text{ is uniquely acceptable}\},$$

[5] We may alternatively suppose that $g_i^a = 0$ if $a \neq a^*$ so that no agent subscribes when some agent rejects the offer.

[6] Note that unique enforceability does not imply the uniqueness of a PBE in the two-stage game, making our requirement different from that for unique (full) implementation.

and $R^* = E[R^*(\theta)]$.

For any set $F \subset I$ of agents and $i \notin F$, we write $i \rightsquigarrow F$ if $i \rightsquigarrow_\theta j$ for some $j \in F$. Define $(F^*, Y^*) \equiv (F^*(\theta), Y^*(\theta))$ to be a solution to the following maximization problem:

$$\max_{(F,Y)} |Y|(v-c) - |F|c \quad \text{subject to} \quad \begin{cases} F,\, Y \subset I, \\ Y = \{j \notin F : j \rightsquigarrow_\theta F\}. \end{cases} \tag{2}$$

(2) has a solution since I is finite, and gives the maximal payoff that the platform can achieve from any uniquely acceptable mechanism as seen in Proposition 2 below. The intuition is as follows: The objective function corresponds to the platform's payoff when it offers subscription for free to agents in F, but for the price v to agents in Y. Since Y is chosen so that its members are strongly connected to some agent in F, agents in Y are indeed willing to pay up to v for subscription if it also includes F. The platform then maximizes its payoff by taking Y as large as possible (since $v - c > 0$), while taking F as small as possible (since $c > 0$).

We next show that for any $\varepsilon > 0$, there exists a uniquely acceptable mechanism Γ such that $R(\theta \mid \Gamma) = |Y^*(\theta)|(v-c) - |F^*(\theta)|c - \varepsilon$ for the solution $(F^*(\theta), Y^*(\theta))$ to (2). Let the matching rule g be defined by

$$g_i(\theta) = \begin{cases} 1 & \text{if } i \in F^*(\theta) \cup Y^*(\theta), \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

In other words, g matches all agents in F^* with those strongly connected to them. The transfer rule t is given by

$$t_i(\theta) = \begin{cases} -\frac{\varepsilon}{n} & \text{if } i \notin Y^*(\theta), \\ v - \frac{\varepsilon}{n} & \text{if } i \in Y^*(\theta), \end{cases} \tag{4}$$

In other words, agents in $F^*(\theta)$ are offered a small subsidy whereas agents in $Y^*(\theta)$ are charged a fee close to their full valuation v.[7]

Proposition 2. *Suppose that the platform has complete information about θ. Then*

$$R^*(\theta) = |Y^*(\theta)|(v-c) - |F^*(\theta)|c.$$

Furthermore, Γ with (g, t) defined in (2) and (4) is uniquely acceptable, and satisfies $R(\theta \mid \Gamma) = R^*(\theta) - \varepsilon$ for every θ.

[7] As specified, agents who are offered no subscription are also given a small subsidy. This is to ensure that the equilibrium of the adoption game is unique, but is irrelevant for the decisions of agents who are offered subscription.

Note that the maximized payoff is strictly positive for every component of two or more agents since $2c < v$. When the complementary-agent network $X(\theta)$ is as described in Figure 3, for example, (2) has the solution $F^* = \{5, 6\}$ and $Y^* = \{1, 2, 4, 7, 8, 9, 10\}$, and its value equals $R^* = 7(v - c) - 2c = 7v - 9c$. Note that agent 3 is not offered subscription since it is not possible to make positive profits out of him, and also any agent who can be attracted by agent 3 (*i.e.,* agent 9) can also be attracted by another agent (*i.e.,* agent 5) who is already offered subscription. In the subscription stage, acceptance is a strictly dominant action for every agent in F^*, and is an iteratively strictly dominant action for agents in Y^*. The number of iteration required for each agent in Y^* to find out that acceptance is an optimal action equals the length of the shortest directed path that connects him with agents in F^*.

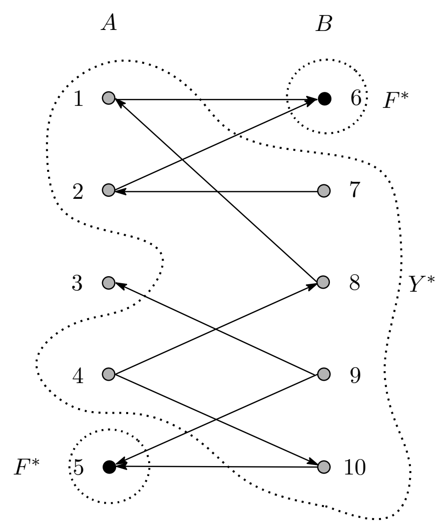

Figure 3: Optimal mechanism under complete information

The figure depicts a two sided-market but the construction is the same in a one-sided market.

6 Limits on Rent Extraction under Incomplete Information

We now return to the incomplete information environment in which the realization of s is not observable to the platform. Specifically, we show that when there exists minimal imperfection in the reciprocity of complementary skills, the optimal mechanism Γ leaves informational rents to the agents as shown in the following proposition.[8]

Proposition 3. *Suppose that the complementary-skill network Z is such that there exist skills s, s', $t \in \Sigma$ such that*

$$Z_{ss'} = 0, \quad (Z_{st}, Z_{ts}) = (1, 0), \quad \textit{and} \quad (Z_{s't}, Z_{ts'}) = (0, 1).$$

Then there exists $\kappa \equiv \kappa(n, Z) > 0$ such that if the expected revenue under a uniquely enforceable mechanism Γ is bounded away from the optimal level by κ: $R(\Gamma) \leq R^* - \kappa$.

Note that when the market is two-sided, $Z_{st} = Z_{ts'} = 1$ in the second and third conditions implies that the skills s and s' are on the same side and hence implies the first condition $Z_{ss'} = 0$. The proposition is based on the observation that for a mechanism to prevent agent j with skill type θ_j from misreporting his type as θ'_j against some skill profile θ_{-j} of others, it has to either exclude j from subscription or create a separate match for j when he reports θ'_j so that he would not gain access to any agent who has a complementary skill for the skill type θ_j. In either case, it results in a loss of revenue for the platform since exclusion implies less agents paying positive subscription prices and creation of another match requires at least one agent to be offered free subscription. We should emphasize that κ, which is interpreted as a lower bound for the agents' informational rents, depends on both Z and n. For a fixed market size n, if the complementary-skill network is such that $Z_{st} = 0$ for most skill pairs (s, t), then the complementary-agent network is empty (*i.e.,* $X_{ij} = X_{ji} = 0$ for every pair $(i, j) \in A \times B$) with probability close to one, and hence the platform's payoff approaches zero even under complete information. On the other hand, if $Z_{st} = 1$ for most skill pairs (s, t), the complementary-agent network is complete (*i.e.,* $X_{ij} = X_{ji} = 1$ for every pair $(i, j) \in A \times B$) with probability close to one, and the mechanism described in the next section that creates a single match will become optimal. Our primary interest hence is on what happens for the less extreme specifications of Z.

7 Single-Match Mechanism

In this section, we describe a class of uniquely enforceable mechanisms under incomplete information. Specifically, we consider a *single-match mechanism* which specifies some non-empty subset of agents I_1 and matches with I_1 all agents who are strongly connected to I_1, provided that the set of such agents is non-empty. The mechanism offers a subsidy to every agent in I_1 while charging a positive price

[8] When the complementary-skill network is perfectly reciprocal, it is an open question whether or not there exists a mechanism that achieves the first-best.

to other matched agents. Formally, the single-match mechanism $\Gamma^s(I_1)$ based on $I_1 \neq \emptyset$ has the matching rule $g : \Theta \to \{0,1\}$ and the transfer rule $t : \Theta \to \mathbf{R}^n$ such that

$$g_i(\theta) = \begin{cases} 1 & \text{if } \{j : j \rightsquigarrow_\theta I_1\} \neq \emptyset \text{ and } i \in I_1 \cup \{j : j \rightsquigarrow_\theta I_1\}, \\ 0 & \text{otherwise,} \end{cases}$$

and for $\varepsilon > 0$,

$$t_i(\theta) = \begin{cases} v - \frac{\varepsilon}{n} & \text{if } i \rightsquigarrow_\theta I_1, \\ -\frac{\varepsilon}{n} & \text{otherwise.} \end{cases}$$

Figure 4 illustrates a single-match mechanism based on $I_1 = \{6\}$. Note that agent i_1 is the only agent who may possibly enjoy informational rents close to v under this mechanism. It is also important that the specification of I_1 cannot be contingent on the realization of θ. If the skill types were observable, it is tempting to target the most "popular" agents as I_1 (*i.e.,* agent i whose skill is found complementary by the largest number of agents). [9] With unobservable types, however, choice of I_1 according to their popularity based on the agents' reported types potentially creates a serious incentive problem. For example, if the most popular agent i is tied with another agent j, then it can create room for profitable misreporting by j: Instead of reporting his true type θ_j for which $X_{ji}(\theta_j, \theta_{-j}) = 1$, j can report θ'_j such that $X_{ji}(\theta'_j, \theta_{-j}) = 0$ and $\sum_k X_{kj}(\theta'_j, \theta_{-j}) \geq \sum_k X_{kj}(\theta)$. Such misreporting reduces i's votes by one, while not reducing j's votes, making j the uniquely most popular agent under (θ'_j, θ_{-j}).

Under the single-match mechanism, note that agent i_1 has no incentive to misreport his type since it does not change the outcome in any way. Any agent j for whom i_1's skill is complementary also has no misreporting incentive: If j reports a type for which i_1's type is not complementary, he will be offered no subscription, whereas if he reports a type for which i_1's type is complementary, it will result in the same outcome as truth-telling. The same reasoning applies to other agents who are offered subscription, implying the following conclusion.[10]

Proposition 4. *For any $I_1 \neq \emptyset$, the single-match mechanism $\Gamma^s(I_1)$ based on I_1 is uniquely enforceable.*

[9] Bernstein and Winter (2012) present comprehensive analysis of the optimal divide-and-conquer pricing strategy under complete information when the externalities are heterogeneous and discuss its relationship with the popularity of each agent.

[10] Unique enforceability of the single match mechanism hinges on the ability of the platform to make public the profiles of the agents' reported types θ and price offers $t(\theta)$: This enables the agents to identify the uniquely optimal action choice through iterative reasoning. See Miklós-Thal and Shaffer (2017).

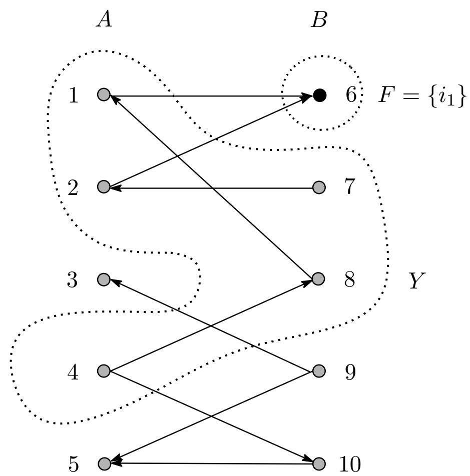

Figure 4: Single match mechanism based on $i_1 = 6$

When the set I_1 of subsidized agents is small compared with the market size n, $\Gamma^s(I_1)$ is typically inefficient unless p is very large since with non-negligible probability, only a few agents are strongly connected to I_1: The platform can charge a positive price only to a small number of agents. In Figure 4, for example, a single-match mechanism based on $i_1 = 4$, 7 or 9 ends up with matching no agents and fails to make positive profits. When the market is very large, on the other hand, it is clear that for any skill, some members of a very large match have the skills complementary to it. It follows that such a match is valuable to every agent and the agents would be willing to pay up to v for subscription if they were not concerned about coordination failures. There is, however, no guarantee that such coordination is achieved when the platform offers a subscription price close to v to every agent. As seen in Section 9, however, the problem of the single match being not too large and that of coordination failures can simultaneously be resolved when the market grows in its size.

8 Percolation Theorem and Revenue Equivalence

In this section, we invoke the percolation theorem of McDiarmid (1981) to establish that when the complementary skill network Z is r-regular for some r, the single-match mechanism Γ^s introduced in Section 7 yields the same expected revenue regardless of the reciprocity property of Z.

Recall that $S = |\Sigma|$ is the number of total skills in the case of the one-sided market and the number of skills on each side $S = |\Sigma_A| = |\Sigma_B|$ in the case of

$X_{ij} \setminus X_{ji}$	1	0
1	$\frac{2q}{S^2}$	$\frac{r}{S} - \frac{2q}{S^2}$
0	$\frac{r}{S} - \frac{2q}{S^2}$	$1 - \frac{2r}{S} + \frac{2q}{S^2}$

Table 1: Joint distributions of X_{ij} and X_{ji} when Z is r-regular

$(i, j) \in A \times B$ in the case of a two-sided market.

the two-sided market. When the complementary-skill network Z is r-regular, the probability that there is a link between any pair (i, j) of agents on the different sides is given by

$$\Pr(X_{ij} = 1) = \frac{r}{S}.$$

The key implication of the regularity of Z is the independence of the occurrence of links between any different pairs of agents.

Lemma 5. *Suppose that the complementary-skill network Z is regular. Then X_{ij} and $X_{i'j'}$ are independent unless $(i, j) = (i', j')$ or $(i, j) = (j', i')$.*

This result is an important step toward the application of the percolation theorem below since it requires the independence of the link occurrence between different pairs. As can be seen from the proof of Lemma 5, a sufficient condition for such independence is that $\Pr(s \in \mathcal{C}(\theta_i))$ and $\Pr(\theta_i \in \mathcal{C}(t))$ are all the same for every $i \in I$ and s, $t \in \Sigma$. Regularity of Z ensures that this holds under our assumption that every skill type is equally likely.

Turning now to the two links X_{ij} and X_{ji} of opposite directions between the same pair of agents (i, j), we see that they are *not* independent. In fact, different specifications of the reciprocity of the complementary-skill network Z imply different degrees of correlation between them. Clearly, $X_{ij} = X_{ji}$ when Z is perfectly reciprocal, and $X_{ij}X_{ji} = 0$ when Z is perfectly non-reciprocal. More generally, define q to be the number of reciprocal pairs of skills in Z as follows.[11]

$$q = \frac{1}{2}\left|\{(s, t) : Z_{st} = Z_{ts} = 1\}\right|.$$

When Z is r-regular, the joint distribution of (X_{ij}, X_{ji}) is a function of q as described in Table 1.

The percolation theorem of McDiarmid (1981, Theorem 4.2) shows that for any set $J \subset I \setminus I_1$ of agents, the probability that all agents in J are strongly connected with I_1 in X is the same regardless of the joint distribution of X_{ij} and X_{ji} as long

[11] For example, $q = 5$, 0, and 2 in Z_5^R, Z_5^N and Z_5^I, respectively, in Figure 1.

as the link occurrence between different pairs of agents is independent.[12] It then follows from Lemma 5 that when Z is r-regular, the expected revenue from the single-match mechanism is also independent of q, or equivalently the reciprocity property of Z.

Proposition 6. *(Revenue equivalence of the single-match mechanism) Let $I_1 \subsetneq I$ be given, and suppose that Γ^s is a single-match mechanism based on I_1. If the complementary-skill networks Z and Z' are both r-regular for some $r \geq 1$, then the expected revenue $R(\Gamma^s)$ from Γ^s is the same under Z and Z'.*

One implication of Proposition 6 is that the expected revenue from the standard pricing strategy in the two-sided market that subsidizes all agents on one side and charges the positive price v on the other side is independent of the reciprocity property of the underlying complementary-skill network as long as it is regular.

9 Asymptotic Optimality of Single-Match Mechanisms

We now investigate the performance of the single-match mechanism when the market expands in its size. In the case of a two-sided market, we assume that market expansion takes place while keeping the relative size of each side in balance. Specifically, we say that a sequence of two-sided markets is *balanced* if there exists $\rho > 1$ such that for every n,

$$\frac{1}{\rho} < \frac{n_b}{n_a} < \rho.$$

Define $\Theta^{\rightsquigarrow I_1}$ to be the set of type profiles such that every agent is strongly connected to I_1:

$$\Theta^{\rightsquigarrow I_1} = \{\theta : i \neq i_1 \Rightarrow i \rightsquigarrow_\theta I_1\}.$$

Lemma 7. *Suppose that the complementary-skill network Z is perfectly reciprocal and regular. Consider any sequence of one-sided markets or any balanced sequence of two-sided markets for $n = 2, \ldots$, and let I_1 be a non-empty subset of agents which may also be a function of the size n of the market. Then for every $\varepsilon > 0$, there exists $N > 0$ such that if $n > N$, then*

$$\Pr\big(\Theta^{\rightsquigarrow I_1}\big) > 1 - \varepsilon.$$

The intuition behind the result is as follows. When the complementary-skill network is regular and perfectly reciprocal, the corresponding complementary-agent

[12] To gain some intuition, assume $I = \{1, 2, 3\}$ and verify that the probability that both 2 and 3 are strongly connected to 1 equals $3(\frac{r}{S})^2 - 2(\frac{r}{S})^3$ regardless of the value of q.

network X is such that $X_{ij} = 1$ if and only if $X_{ji} = 1$, so that we can identify $X(\theta)$ with an undirected graph $G(\theta)$ which has a link between i and j if and only if $X_{ij} = X_{ji} = 1$ in $X(\theta)$. Note that every agent $j \notin I_1$ is strongly connected to I_1 in X if G is connected. Furthermore, since Lemma 5 implies that X_{ij} and $X_{i'j'}$ between two different pairs (i, j) and (i', j') are independent when Z is regular, G is the standard Erdős-Rényi type random graph. Lemma 7 is then established if the probability that G is connected approaches one as $n \to \infty$. This holds in both one-sided and two-sided markets when $\Pr(X_{ij} = 1)$ is held constant as n increases.

Now denote by $\Theta^s_\varepsilon(n)$ the set of type profiles θ at which the expected revenue from the single-match mechanism Γ^s described above is within ε of the optimal level:

$$\Theta^s_\varepsilon = \{\theta : \, R(\theta \mid \Gamma^s) > R^*(\theta) - \varepsilon\}.$$

Let $\mathcal{Z}$ be the set of all complementary-skill networks such that every $\hat{Z} \in \mathcal{Z}$ is a supergraph of some 1-regular complementary-skill network Z. In other words, $\hat{Z} \in \mathcal{Z}$ is a complementary-skill network obtained by adding directed links to some 1-regular complementary-skill network Z.

Proposition 8. *Let the complementary-skill network $Z \in \mathcal{Z}$ be given. For any sequence of one-sided markets, or any balanced sequence of two-sided markets, the single-match mechanism Γ^s based on a single agent $I_1 = \{i_1\}$ is asymptotically optimal: For every $\varepsilon > 0$, there exists $N > 0$ such that if $n > N$, then*

$$\Pr(\theta \in \Theta^s_\varepsilon) > 1 - \varepsilon. \tag{5}$$

When every agent is strongly connected to i_1 in X, the single-match mechanism based on i_1 charges a subscription price close to v to all agents but i_1, and hence its revenue is close to the bound R^* achieved under complete information. Since the probability that X is strongly connected to i_1 approaches one by Lemma 7 when Z is 1-regular and perfectly reciprocal, the single-match mechanism Γ^s is asymptotically optimal for such a Z. It then follows from Proposition 6 that the same holds true if $\hat{Z}$ is any complementary-skill network that is 1-regular but not perfectly reciprocal. Finally, take any $\bar{Z} \in \mathcal{Z}$. By assumption, $\bar{Z}$ can be obtained by adding directed links to some 1-regular complementary-skill network Z. Let $\bar{X}$ and X be the complementary-agent networks corresponding to $\bar{Z}$ and Z, respectively. For any $\theta \in \Theta$, $X(\theta)$ is a subgraph of $\bar{X}(\theta)$ since whenever there is a directed link between i and j in $X(\theta)$, there is a link of the same direction between them in $\bar{X}(\theta)$. It follows that if an agent is strongly connected to i_1 in $X(\theta)$, then he is also strongly corrected to i_1 in $\bar{X}(\theta)$. We can hence conclude that the single-match mechanism is asymptotically optimal for any $Z \in \mathcal{Z}$.

As mentioned earlier, the standard pricing strategy in a two-sided market takes the form of the divide-and-conquer strategy that corresponds to the single-match mechanism Γ^s based on either $I_1 = A$ or B. What is the performance of such a strategy in a large two-sided market? For concreteness, suppose the two sides are of the same size $n_a = n_b = \frac{n}{2}$ and consider $\Gamma^s(A)$ based on $I_1 = A$. If the complementary-agent network X is strongly connected, then it is clear that all agents on side B subscribe for a full price under $\Gamma^s(A)$ while all but one agent subscribe for a full price under the single-match mechanism $\Gamma^s(i_1)$ based on a single agent. Since when $Z \in \mathcal{Z}$, X is strongly connected with probability close to one for n large, it follows that $\Gamma^s(i_1)$ raises approximately twice as much revenue as $\Gamma^s(A)$. When the market is not so large, on the other hand, the number of agents in B who are strongly connected to A is on average much larger than the number of agents who are strongly connected to i_1, and hence $\Gamma^s(A)$ would yield a higher expected revenue than $\Gamma^s(i_1)$.

As an application of Theorem 8, consider the following model which takes a different interpretation of a skill type of each agent. Suppose that there exists a set Λ of two or more skills that are required to complete the project of any agent regardless of his type. The skill type of each agent corresponds to a subset of Λ:

$$\Sigma = \{s : \, s \subset \Lambda, \, s \neq \Lambda, \, \emptyset\}.$$

The skill type t is complementary to the skill type s if all the required skills are covered by them. Hence, the complementary-skill network Z is given by

$$Z_{st} = \begin{cases} 1 & \text{if } s \cup t = \Lambda, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that Z is perfectly reciprocal but not regular since different skill types with different numbers of elements have different numbers of complementary skills. Figure 5 illustrates a complementary-skill network when $|\Lambda| = 3$. Note that the skill type $\{\lambda\}$ has one complementary skill type $\{\mu, \nu\}$, whereas the skill type $\{\lambda, \mu\}$ has three complementary skill types $\{\nu\}$, $\{\lambda, \nu\}$, and $\{\mu, \nu\}$.

The complementary-skill network Z is a supergraph of a 1-regular network: $Z \in \mathcal{Z}$. In fact, if we define $\hat{Z}$ by

$$\hat{Z}_{st} = \begin{cases} 1 & \text{if } s \cup t = \Sigma \text{ and } s \cap t = \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

then $\hat{Z}$ is 1-regular and a subgraph of Z. It follows that the single-match mechanism $\Gamma^s(i_1)$ based on a single agent is asymptotically optimal by Proposition 8.

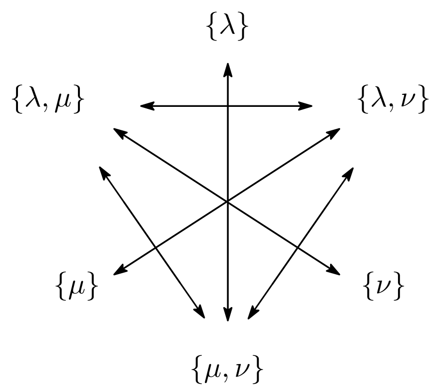

Figure 5: Complementary-skill network Z when $\Lambda = \{\lambda, \mu, \nu\}$.

10 Conclusion

The complementary-skill network describes the relationship between any skill and the set of their complementary skills that are required to complete the project owned by each agent. Through random and private realization of the skill type of each agent, the complementary-skill network translates into the random complementary network over agents endowed with heterogeneous skills. Agents share their skills when matched by a platform and hence a match has positive value to any agent if it involves another agent who has a complementary skill to his own skill.

While divide-and-conquer in a two-sided market typically subsidizes all agents on one side and charges a positive price to the other side, the uniquely enforceable mechanisms we study generalizes such a pricing scheme by allowing the platform to subsidize any subset I_1 of agents. We show that the platform's revenue under this class of mechanisms satisfies the following properties. First, when the set I_1 of subsidized agents is fixed, the platform's expected revenue from the single-match mechanism is independent of the reciprocity properties of the complementary-skill network as long as each skill type has the same number of complementary skills. Second, provided that each skill type has at least one complementary skill and each skill is complementary to at least one skill type, the expected revenue associated with the single-match mechanism that subsidizes just a single agent $I_1 = \{i_1\}$ approaches the first-best level that is achievable under the complete information benchmark as the market expands in its size.

Possible extensions are as follows. First, we have assumed that an agent's skill becomes a public good within a match so that its value to each agent is independent of the level of congestion. More generally, we may suppose that a match value for an agent is higher in when more members of his match has complementary skills, or lower when more members of his match compete for the same complementary skills. In a large market, however, the simpler specification that we adopt is not too restrictive in the environment of Proposition 8 since the single-match mechanism matches every agent with high probability in a large market and since the fraction of each skill type averages out because of the law of large numbers.

Second, we have assumed that there is a single complementary skill for each agent.[13] We may suppose instead that each agent needs to be matched with two or more agents with different skills to complete their projects. For concreteness, suppose that each agent has a single skill but needs to access two other skills of other agents. The single match mechanism based on a single agent i_1 is clearly not uniquely enforceable since subscription is not a dominant action for any agent even conditional on i_1's subscription decision. It is not clear if providing subsidies to two agents is useful. On the other hand, the single-match mechanism that subsidizes all agents on one side in a two-sided market is uniquely enforceable even with this modification. The smallest number of subsidies that ensure unique enforceability is an open question.

Appendix

Proof of Proposition 2. We first show that

$$R^*(\theta) \leq |Y^*(\theta)|(v-c) - |F^*(\theta)|c.$$

Suppose to the contrary that a uniquely acceptable mechanism $\hat{\Gamma}$ with $(\hat{g}, \hat{t})$ yields the payoff $R(\theta \mid \hat{\Gamma}) > |Y(\theta)|(v-c) - |F(\theta)|c$. Let $\hat{Y} = \{i \in I : \hat{t}_i(\theta) > 0\}$ be the set of agents who are offered subscription for strictly positive prices, and $\hat{F} = \{i \in I : \hat{t}_i(\theta) \leq 0\}$ be the set of agents who are offered subscription for non-positive prices. Note first that there exists $i_1 \in \hat{Y}$ who is not strongly connected to $\hat{F}$ since otherwise, $(\hat{F}, \hat{Y})$ would be feasible in (2). Let $J_1 = \{i_1\}$, and let $J_2 \subset \hat{Y}$ be the set of agents in $\hat{Y}$ to whom i_1 is strongly connected: $J_2 = \{j \in \hat{Y} : i_1 \rightsquigarrow j\}$. Since i_1 is charged a positive price, IR implies that $J_2 \neq \emptyset$, and furthermore, since i_1 is not strongly connected to $\hat{F}$, no $j \in J_2$ is strongly connected to $\hat{F}$. Let then $J_3 \subset \hat{Y}$ be the set of agents j to whom agents in J_2 are strongly connected. In the same way, we can iteratively construct a sequence $J_4, J_5, \ldots$ of subsets of $\hat{Y}$ so that no agent in those subsets is strongly connected to $\hat{F}$. Since $\hat{Y}$ is finite, however, we will have $J_{k+1} \subset \cup_{\ell=1}^{k} J_\ell$ for some k. We then have a contradiction to unique acceptability since then for agents i in the set $\cup_{\ell=1}^{k} J_\ell$, rejection $a_i = 0$ is a Nash equilibrium action since they are all charged positive subscription prices.

[13]However, each skill type may correspond to multiple skills as in the last example of Section 9.

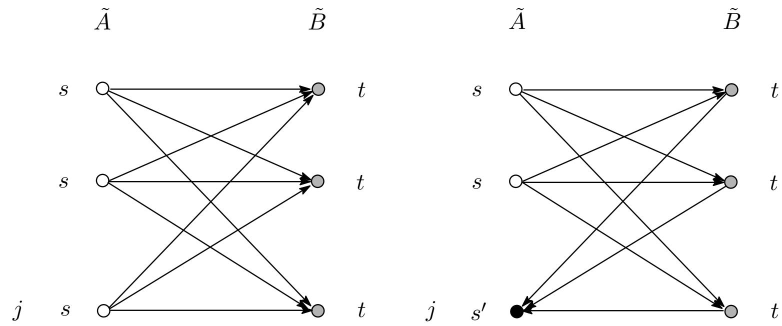

Figure 6: Type profiles θ (left) and θ' (right)

$(\tilde{A}, \tilde{B}) = (A, B)$ if the market is two-sided. Only agent j changes his type from s to s'.

The mechanism Γ with (g, t) defined in (3) and (4) is clearly uniquely acceptable: For $i \in F^*(\theta)$, $t_i(\theta) < 0$ so that subscription $a_i = 1$ is a dominant action. For $j \in Y^*(\theta)$, $a_j = 1$ is an iteratively dominant action since $Y^*(\theta)$ consists of all agents who are strongly connected to some $i \in F^*(\theta)$: For any $j \in Y^*(\theta)$, either $i \in \theta_j$ for some $i \in F^*(\theta)$ or there exists $k \in Y(\theta)$ with $k \in C_{\rightsquigarrow F^*(\theta)}(\theta)$. Rejection is dominated in the second round of the iterative elimination procedure in the first case, whereas in the second case, it is iteratively dominated for j in one round after it is dominated for k. Finally, the platform's payoff under Γ equals

$$R(\theta \mid \Gamma) = |Y^*(\theta)|(v - c) - |F^*(\theta)|c - \varepsilon.$$

Since ε is arbitrary, $R^*(\theta) \geq |Y^*(\theta)|(v - c) - |F^*(\theta)|c$. We hence obtain the stated conclusion. $\square$

Proof of Proposition 3. Let $\delta = \left(\frac{1}{S}\right)^n > 0$ so that $P(\theta) \geq \delta$ for every $\theta \in \Theta$. By assumption, we can take s, $s' \in \Sigma_A$ and $t \in \Sigma_B$ such that

$$Z_{ss'} = 0, \quad (Z_{st}, Z_{ts}) = (1, 0), \quad \text{and} \quad (Z_{s't}, Z_{ts'}) = (0, 1).$$

Let $(\tilde{A}, \tilde{B})$ be a binary partition of the set I of agents such that both $\tilde{A}$ and $\tilde{B}$ are non-empty. Specifically, we take $\tilde{A} = A$ and $\tilde{B} = B$ if the market is two-sided. Let $\tilde{n}_a = |\tilde{A}|$ and $\tilde{n}_b = |\tilde{B}|$. Take any agent $j \in \tilde{A}$ and consider the type profiles θ and θ' as follows (Figure 6):

$$\theta_i = \begin{cases} s & \text{if } i \in \tilde{A}, \\ t & \text{if } i \in \tilde{B}. \end{cases} \quad \text{and} \quad \theta_i = \begin{cases} s & \text{if } i \in \tilde{A} \setminus \{j\}, \\ s' & \text{if } i = j, \\ t & \text{if } i \in \tilde{B}. \end{cases}$$

Since every agent in $\tilde{A}$ is strongly connected to every agent in $\tilde{B}$ but no agent in $\tilde{B}$ is strongly corrected to any agent in $\tilde{A}$ under θ, the optimal mechanism Γ^* under complete information creates a single match consisting of all agents in $\tilde{A}$ and one agent in $\tilde{B}$, and the supremum of the revenue from such a mechanism equals $R^*(\theta) = \tilde{n}_a v - (\tilde{n}_a + 1)c$. On the other hand, since every agent $i \neq j$ is strongly connected to j under θ', the optimal mechanism Γ^* under complete information again creates a single match and its revenue equals $R^*(\theta') = (n-1)v - nc$.

Let $R(\cdot)$ denote the supremum of the revenue from a uniquely enforceable mechanism under incomplete information. We claim that $\max\{R^*(\theta) - R(\theta),\, R^*(\theta') - R(\theta')\} \geq c$. We derive a contradiction by supposing that there exists a uniquely enforceable mechanism Γ such that $R^*(\theta) - R(\theta \mid \Gamma) < c$ and $R^*(\theta') - R(\theta' \mid \Gamma) < c$.

Consider first θ. Since the subscription price can be positive only for agents in $\tilde{A}$, if Γ offers subscription to a total of m_a agents in $\tilde{A}$ and m_b agents in $\tilde{B}$, and offers non-positive prices to K_a agents in $\tilde{A}$, then

$$R(\theta \mid \Gamma) \leq (m_a - K_a)\, v - (m_a + m_b)\, c = m_a v - (m_a + 1)\, c - K_a v - (m_b - 1)\, c.$$

If $m_a < \tilde{n}_a$, $K_a \geq 1$, or $m_b > 1$, then $R(\theta) \leq \tilde{n}_a v - (\tilde{n}_a + 2)c$.[14] Since this implies that $R^*(\theta) - R(\theta \mid \Gamma) \geq c$, it follows that $m_a = \tilde{n}_a$, $K_a = 0$, and $m_b = 1$ must hold. In other words, Γ creates a single match consisting of all agents in $\tilde{A}$ and one agent in $\tilde{B}$, and charges positive prices to the agents in $\tilde{A}$.

Consider next θ'. If Γ offers subscription to a total of m agents (in $\tilde{A} \cup \tilde{B}$), and offers non-positive prices to K of them, then

$$R(\theta) \leq (m - K)\, v - m\, c = (m-1)\, v - mc - (K-1)\, v.$$

If $m < n$, or $K \geq 2$, then $R(\theta) \leq (n-2)v - (n-1)c$. Since this implies that $R^*(\theta) - R(\theta \mid \Gamma) \geq v - c > c$, it follows that $m = n$ and $K = 1$ must hold. In other words, Γ creates a single match consisting of all agents and offers a non-positive price to one of them. Note that the agent who is offered free subscription must be agent j since $Z_{s's} = Z_{s't} = 0$.

To summarize, under Γ, the subscription price for $j \in \tilde{A}$ is positive when the reported profile is θ, but is non-positive when it is θ'. In both cases, however, j is a member of the single match that also includes an agent on side A. Since $Z_{st} = 1$, against θ_{-j}, j has an incentive to misreport his type as s' when his true type is s. This is a contradiction.

Since $\max\{R^*(\theta) - R(\theta),\, R^*(\theta') - R(\theta')\} \geq c$, the expected revenue under Γ

[14] Note that $kv - (k+1)c$ is strictly increasing in k.

satisfies

$$\begin{aligned}R^* - R(\Gamma) &= \sum_{\tilde{\theta}} \Pr(\tilde{\theta}) \left\{R^*(\tilde{\theta}) - R(\tilde{\theta})\right\} \\ &\geq \Pr(\theta) \left\{R^*(\theta) - R(\theta)\right\} + \Pr(\theta') \left\{R^*(\theta') - R(\theta')\right\} \\ &\geq \delta c.\end{aligned}$$

The proof is complete once we let $\kappa = \delta c$. $\square$

Proof of Proposition 4. To see that Γ^s is ex post IC, note first that agent i_1 has no incentive to misreport since the allocation $(g(\theta), x(\theta))$ is independent of his report. Take any $i \neq i_1$.

1) If $i \rightsquigarrow_\theta i_1$, his transfer equals $t_i(\theta) = v - \frac{\varepsilon}{n}$ and hence his payoff is given by

$$u_i(g(\theta), \theta) - t_i(\theta) = v - (v - \frac{\varepsilon}{n}) = \frac{\varepsilon}{n} > 0.$$

If i reports θ_i' such that $i \rightsquigarrow_{(\theta_i', \theta_{-i})} i_1$, then his payoff is unchanged. If i reports θ_i' such that $i \not\rightsquigarrow_{(\theta_i', \theta_{-i})} i_1$, then his payoff equals $\frac{\varepsilon}{n}$. It follows that i has no incentive to misreport.

2) If $i \not\rightsquigarrow_\theta i_1$, then his transfer equals $-\frac{\varepsilon}{n}$ and hence his payoff is given by $u_i(g(\theta), \theta) - t_i(\theta) = \frac{\varepsilon}{n}$. If he reports θ_i' such that $i \not\rightsquigarrow_{(\theta_i', \theta_{-i})} i_1$, then his payoff is unchanged. If he reports θ_i' such that $i \rightsquigarrow_{(\theta_i', \theta_{-i})} i_1$, then his payoff equals $-(v - \frac{\varepsilon}{n}) < 0$. In either case, i has no incentive to misreport.

We have hence shown that Γ^s is ex post IC. To see that Γ^s is uniquely acceptable, note that $a_i = 1$ is a strictly dominant action for $i = i_1$, and $a_i = 1$ is iteratively strictly dominant for any i such that $i \rightsquigarrow_\theta i_1$. $\square$

Proof of Lemma 5. Independence of types implies that X_{ij} and $X_{i'j'}$ are independent if $i \neq i'$ and $j \neq j'$. Suppose that Z is r-regular for some $r \geq 1$. For any $i \in I$ and $s \in \Sigma$,

$$\Pr(\theta_i \in \mathcal{C}(s)) = \sum_{t \in \Sigma} \mathbf{1}_{\{t \in \mathcal{C}(s)\}} \Pr(\theta_i = t) = \frac{1}{S} \sum_{t \in \Sigma} Z_{st} = \frac{r}{S},$$

and

$$\Pr(s \in C(\theta_i)) = \sum_{t \in \Sigma} \mathbf{1}_{\{s \in \mathcal{C}(t)\}} \Pr(\theta_i = t) = \frac{1}{S} \sum_{t \in \Sigma} Z_{st} = \frac{r}{S}.$$

To see the independence of X_{ij} and X_{ik} ($j \neq k$), since the types θ_i, θ_j and θ_k are independent, note that

$$\begin{aligned}\Pr(X_{ij} = 1, X_{ik} = 1) &= \sum_{s\in\Sigma} \Pr(\theta_i = s,\, \theta_j, \theta_k \in \mathcal{C}(s)) \\ &= \sum_{s\in\Sigma} \Pr(\theta_i = s)\, \Pr(\theta_j \in \mathcal{C}(s))\, \Pr(\theta_k \in \mathcal{C}(s)) \\ &= \sum_{s\in\Sigma} \Pr(\theta_i = s) \left(\frac{r}{S}\right)^2 \\ &= \left(\frac{r}{S}\right)^2.\end{aligned}$$

It hence follows that $\Pr(X_{ij} = 1, X_{ik} = 1) = \Pr(X_{ij} = 1)\, \Pr(X_{ik} = 1)$. We can similarly show that $\Pr(X_{ij} = 1, X_{ik} = 0) = \Pr(X_{ij} = 1)\, \Pr(X_{ik} = 0)$ and so on. The proofs of the independence of X_{ji} and X_{ki}, and that of X_{ji} and X_{ik} ($j \neq k$) are similar and omitted. □

Proof of Proposition 6. Since Z is r-regular, X_{ij} and $X_{i'j'}$ are independent unless $(i, j) = (i', j')$ or (j', i') by Lemma 5 and $\Pr(X_{ij} = 1) = \frac{r}{S}$ independently of the reciprocity property of Z. For any $I_1 \subsetneq I$, consider the single-match mechanism Γ^s based on I_1, and let $J \subset I \setminus I_1$ be any subset of agents outside I_1. Define Θ_J to be the set of type profiles such that agent $i \notin I_1$ is strongly connected to I_1 if and only if $i \in J$:

$$\Theta_J = \{\theta \in \Theta :\, i \rightsquigarrow_\theta I_1 \Leftrightarrow i \in J\}.$$

By Theorem 4.2 of McDiarmid (1981), then, for any $J \subset I \setminus I_1$, the probability $\Pr(\theta \in \Theta_J)$ is independent of the specification of the joint distribution of (X_{ij}, X_{ji}) ($i \neq j$). Since the expected revenue $R(\Gamma^s)$ from Γ^s is given by

$$R(\Gamma^s) = \sum_{J\subset I\setminus\{i_1\}} \Pr(\theta \in \Theta_J)\, |J| v - \varepsilon,$$

it is also independent of the specification of Z as long as it is r-regular. □

Proof of Lemma 7. As noted in the text, X is identified with an undirected graph G and $\theta \in \Theta^{\rightsquigarrow I_1}$ is implied by the connectedness of G. When the market is one-sided, G is the standard Erdős-Rényi type random graph, and the probability that G is connected approaches one as $n \to \infty$ since $\Pr(X_{ij} = 1) = \frac{r}{S}$ remains constant as n increases.[15] When the market is two-sided, on the other hand, Wright (1982, Theorem 1) shows that there exists $K > 0$ (independent of n_a and n_b) such that the

[15]See, for example, Diestel (2000, p. 239).

probability that a random bipartite graph is connected approached one provided $\min\{n_a, n_b\} > K \log \max\{n_a, n_b\}$. This implies that

$$\lim_{\substack{n\to\infty \\ \frac{1}{\rho}<\frac{n_a}{n_b}<\rho}} \Pr(G(\theta) \text{ is connected}) = 1.$$

$\square$

Proof of Proposition 8. Suppose first that Z is 1-regular and perfectly reciprocal. Since $\Theta_\varepsilon^s \supset \Theta^{\rightsquigarrow i_1}$, we have $\Pr(\theta \in \Theta_\varepsilon^s) \geq \Pr(\theta \in \Theta^{\rightsquigarrow i_1}) \to 1$ as $n \to \infty$ by Lemma 7. It then follows from Proposition 6 that for any Z that is 1-regular but not necessarily perfectly reciprocal, we have $\Pr(\theta \in \Theta_\varepsilon^s) \to 1$. Finally, if $\hat{Z}$ is a supergraph of any 1-regular Z, then $i \to i_1$ in $X(\theta)$ implies $i \to i_1$ in $\hat{X}(\theta)$ so that we again have $\Pr(\theta \in \Theta_\varepsilon^s) \to 1$ as $n \to \infty$. $\square$

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[15] Although the theorem assumes that $\Pr(X_{ij} = X_{ji} = 1) = \frac{1}{2}$, the same holds when $\Pr(X_{ij} = X_{ji} = 1) = \frac{r}{S}$ since the measure induced by the latter is absolutely continuous with respect to the measure induced by the former.

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