Discussion Paper No. 1187

ISSN (Print) 0473-453X
ISSN (Online) 2435-0982

CHARACTERIZING PAIRWISE STRATEGY-PROOF RULES IN OBJECT ALLOCATION PROBLEMS WITH MONEY

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August 2022

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Characterizing pairwise strategy-proof rules in object allocation problems with money

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August 31, 2022

Abstract

We consider the problem of allocating a single object to the agents with payments. Agents have preferences that are not necessarily quasi-linear. We characterize the class of rules satisfying *pairwise strategy-proofness* and *non-imposition* by the *priority rule*. Our characterization result remains valid even if we replace *pairwise strategy-proofness* by either weaker *effectively pairwise strategy-proofness* or stronger *group strategy-proofness*. By exploiting our characterization, we identify the class of rules satisfying both the properties that are in addition (i) *onto*, (ii) *welfare continuous*, (iii) *minimally fair*, (iv) *constrained efficient* within the class of rules satisfying both the properties, or (v) *revenue undominated* within the class of rules satisfying the properties, and find the tension between minimal properties of efficiency, fairness, and revenue maximization under *pairwise strategy-proofness*.

JEL Classification Numbers. D44, D47, D71, D82

Keywords. Pairwise strategy-proofness, Effectively pairwise strategy-proofness, Group strategy-proofness, Non-imposition, Efficiency, Fairness, Revenue maximization, Priority rules

1 Introduction

We consider the problem of allocating a single object to the agents with payments. A *bundle* specifies whether an agent receives the object and how much he pays. Agents have (possibly) non-quasi-linear preferences over bundles, which exhibits income effects or reflects soft budget constraints. An *allocation* specifies a bundle for each agent. An (*allocation*) *rule* is a function from the set of preference profiles to the set of allocations.

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Examples include single-object auctions. One of the biggest concerns in practical auction design is to prevent bidders from collusion (Klemperer, 2002). If an auction is vulnerable to collusion, then it may produce a poor outcome in terms of the planner's goals such as efficiency, fairness, and revenue maximization. Much of collusive bidding in auctions is difficult to challenge legally (Klemperer, 2002), and thus, a "desirable" auction should not give bidders incentives to collude. In this paper, we investigate rules that prevent agents from collusive manipulations of preferences.

1.1 Main results

A rule is *pairwise strategy-proof* if no pair of agents ever benefits by misrepresenting their preferences. It is *effectively pairwise strategy-proof* if no pair of agents ever benefits by "self-imposing" misrepresentation of preferences under which no single agent in the pair has further incentives to deviate from the manipulation unilaterally (Serizawa, 2006). It is *group strategy-proof* if no coalition of agents ever benefits by misrepresenting their preferences.[1]

Clearly, *group strategy-proofness* implies *pairwise strategy-proofness*, which in turn implies *effectively pairwise strategy-proofness*. In many situations, it is difficult for agents to form a coalition of large size to manipulate the rule, and thus *group strategy-proofness* may be an unnecessarily demanding property. Thus, we investigate the class of rules satisfying (*effectively*) *pairwise strategy-proofness* in this paper.

A rule satisfies *non-imposition* if whenever an agent is uninterested in the object, his welfare is not affected by the rule (Sakai, 2008). It is a mild requirement of a rule, and almost all standard rules satisfy it.

A rule is a (*fixed-prices*) *priority rule* if there are a priority over the set of participants of the rule and a (personal) price for each participant such that an agent with the highest priority among the agents who would like to win the object with the given prices receives the object and pays his price.

We establish that a rule satisfies *pairwise strategy-proofness* and *non-imposition* if and only if it is a priority rule (Theorem 1). We further find that our characterization result remains valid even if we replace *pairwise strategy-proofness* by either weaker *effectively pairwise strategy-proofness* or stronger *group strategy-proofness* (Theorem 1). Thus, under *non-imposition*, the three group incentive properties happen to coincide with one another even though *group strategy-proofness* is seemingly much stronger than *pairwise strategy-proofness*, which in turn is stronger than *effectively pairwise strategy-proofness*

[1] In this paper, we consider the property called *strong pairwise* (or *group*) *strategy-proofness* in the literature. The weaker version of the property called *weak pairwise* (or *group*) *strategy-proofness* has been studied in the literature (e.g., Mukherjee, 2014), but is so weak in our model that it is difficult to produce a clear result without additional strong properties such as *efficiency*, *anonymity in welfare*, and *envy-freeness*.

in principle.[2]

Next, we exploit our characterization result (Theorems 1) to identify the class of rules satisfying any of our group incentive properties, *non-imposition*, and an additional property of interest.

A rule is *onto* if each agent has a chance to win the object at some preference profile. It is *welfare continuous* if the valuation of each agent at the outcome bundle of the rule does not change drastically by a small change of preference profiles. In many situation, it is difficult both for agents and the planner to estimate the complex parameters such as preferences precisely, and a "good" rule should be robust to small changes of such parameters.[3] A rule is *minimally fair* if whenever all the agents have the same preferences, each loser who receives no object does not prefer an outcome bundle of each winner. It is a minimal requirement of fairness, and arguably all the properties of fairness in the literature such as *equal treatment of equals*, *anonymity in welfare*, *egalitarian equivalence*, and *envy-freeness* imply it.[4] A rule is *constrained efficient* within a class of rules if (i) it belongs to the class, and (ii) no other rule in the class Pareto dominates it. Note that a priority rule violates (*Pareto*) *efficiency*, and so no rule satisfies any of our group incentive properties, *non-imposition*, and *efficiency*. Then, *constrained efficiency* is a natural second best property of efficiency. A rule is *revenue undominated* within a class of rules if (i) it belongs to the class, and (ii) no other rule in the class revenue dominates it.[5] A *revenue undominated* rule achieves the second best in terms of revenue maximization when there is no revenue maximizing rule or it is difficult to identify a revenue maximizing rule in a complex environment such as a non-quasi-linear one.[6][7]

Applying our characterization result (Theorem 1), we identify the class of rules satisfying any of our group incentive properties and *non-imposition* that are in addition (i) *onto*, (ii) *welfare continuous*, (iii) *minimally fair*, (iv) *constrained efficient* within the class of rules satisfying both the properties, or (v) *revenue undominated* within the class of rules satisfying the properties (Propositions 1, 2, 3, 4, and 5). In particular, our results (Proposition 3, 4, 5) highlight the tension between minimal properties of efficiency, fairness, and revenue maximization: under any of our group incentive properties and *non-imposition*, any one of the three properties (i.e., *constrained efficiency*, *minimal fairness*, and *revenue*

[2]Serizawa (2006) and Alva (2017) come to the parallel conclusions in different models from ours.

[3]Juarez (2019) discusses the importance of properties of continuity in mechanism design in detail.

[4]Note that *minimal fairness* is a weaker variant of *weak envy-freeness for equals* introduced by Sakai (2013).

[5]A rule *revenue dominates* another rule if for each preference profile, the revenue from the former rule is greater than or equal to that from the latter rule, and for some preference profile, the inequality is strict.

[6]Kazumura et al. (2020) show that in a non-quasi-linear environment, the standard revenue equivalence approach is no longer valid, and the revenue maximization problem is no longer tractable even in a canonical one-object setting.

[7]Note that if a rule *maximizes revenue* in terms of standard expected revenues among a class of rules, then it is *revenue undominated* within the class of rules.

undominance) is compatible with neither one of the other two properties (Corollary 1). This contrasts with the well known tension under (individual) *strategy-proof* rules: *expected revenue maximization* is compatible neither with (*Pareto*) *efficiency* nor with a weak property of fairness such as *minimal fairness* (Myerson, 1981), while *efficiency* and a strong property of fairness such as *envy-freeness* and *anonymity in welfare* are compatible with each other even in a non-quasi-linear environment (Vickrey, 1961; Saitoh and Serizawa, 2008; Sakai, 2008).

1.2 Related literature

Juarez (2013) considers the identical objects model with unit-demand and quasi-linear preferences, and obtains several characterizations of rules satisfying *group strategy-proofness*, *individual rationality*, *nonnegative payments* and the additional tie-breaking properties. In particular, Proposition 3 of Juarez (2013) shows that in the single-object model with quasi-linear preferences, if a rule satisfies *group strategy-proofness*, *ontoness*, *individual rationality*, and *nonnegative payments*, then it is a priority rule. Note that the combination of *individual rationality* and *nonnegative payments* implies *non-imposition*. Our main result (Theorem 1) substantially extends his result by removing the assumptions of (i) *ontoness* and (ii) quasi-linearity, and weakening (iii) *group strategy-proofness* to (*effectively*) *pairwise strategy-proofness* and (iv) the combination of *individual rationality* and *nonnegative payments* to *non-imposition*.

Some authors consider properties of fairness together with group incentives properties. Mitra (2014) considers the single-object model with quasi-linear preferences, and obtain characterizations of *weak pairwise strategy-proof* rules satisfying strong properties of fairness and no wastage of the object. Hagen (2019) considers the same model as Mitra (2014), and obtain a necessary and sufficient condition on a domain to guarantee the existence of a non-trivial rule satisfying *group strategy-proofness* and *equal treatment of equals*. Tierney (2022) considers the several objects model with non-quasi-linear preferences, and shows that if a rule satisfies *weak pairwise strategy-proofness*, *anonymity in welfare*, and a stronger variant of *welfare continuity*, then it also satisfies *envy-freeness*. Our main result (Theorem 1) is independent of theirs as we do not impose any property of fairness in our result. Although we consider a property of fairness (*minimal fairness*) in some of our result (Proposition 3), our result does not follow from theirs because we consider a weaker property of fairness than theirs.

Several authors consider coalitional manipulations of a rule that allow the possibility that agents arrange monetary transfers among agents themselves (Shummer, 2000; Bu, 2016; Hagen, 2022, etc.). Such properties are much stronger than the properties without the possibility of arranging monetary transfers such as (*effectively*) *pairwise strategy-proofness* and *group strategy-proofness*.

Some authors obtain characterizations of a variant of a priority rule by using the properties different from ours. Klaus and Nichifor (2020, 2021) consider the several objects model with non-quasi-linear preferences, and establish that a rule satisfies *strategy-proofness*, *consistency*, *non-imposition*, and the additional properties if and only if it is a serial dictatorship rule with reservation prices. Note that when there is only one object, their rule is a priority rule. Our results are independent of theirs because they consider (individual) *strategy-proofness* and *consistency* together with the additional properties, while we consider only group incentive properties.

1.3 Organization

The remainder of this paper is organized as follows. Section 2 introduces the model. Section 3 defines a priority rule. Section 4 provides the main result. In Section 5, we apply our main result to identify several classes of rules of interest. Section 6 concludes. All proofs are in the appendix.

2 Model

There are n agents and the single object. Let $N = \{1, \ldots, n\}$ and $M = \{0, 1\}$. The **consumption set** of each agent is $M \times \mathbb{R}$. A (**consumption**) **bundle** of agent $i \in N$ is $z_i = (x_i, t_i) \in M \times \mathbb{R}$.

Each agent has a preference R_i over $M \times \mathbb{R}$. The indifference and strict relations associated with R_i are denoted by I_i and P_i, respectively. We assume that preferences satisfy the following properties.

Weak desirability of the object. For each $t_i \in \mathbb{R}$, $(1, t_i)\ R_i\ (0, t_i)$.

Money monotonicity. For each $x_i \in M$ and each pair $t_i, t'_i \in \mathbb{R}$ with $t_i < t'_i$, $(x_i, t_i)\ P_i\ (x_i, t'_i)$.

Finiteness. For each $z_i \in M \times \mathbb{R}$ and each $x_i \in M$, there is $t_i \in \mathbb{R}$ such that $(x_i, t_i)\ I_i\ z_i$.

A typical class of preferences is $\mathcal{R}$. Given $z_i \in M \times \mathbb{R}$ and $x_i \in M$, let $V_i(x_i, z_i)$ denote a payment such that $(x_i, V_i(x_i, z_i))\ I_i\ z_i$. By finiteness, there is such a payment, and by money monotonicity, it is unique. We call $v_i(t_i) = V_i(1, (0, t_i)) - t_i$ the **valuation at** t_i. By weak desirability of the object, for each $t_i \in \mathbb{R}$, $v_i(t_i) \geq 0$.

A preference R_i is quasi-linear if it is represented by a utility function $u((x_i, t_i); v_i) = v_i \cdot x_i - t_i$, where $v_i \in \mathbb{R}_+$. The class of quasi-linear preferences is $\mathcal{R}^Q$. If $R_i \in \mathcal{R}^Q$, then $V_i(x'_i, (x_i, t_i)) - t_i = v_i \cdot (x'_i - x_i)$ for each $x_i, x'_i \in M$ and each $t_i \in \mathbb{R}$. In particular, if $R_i \in \mathcal{R}^Q$, then $v_i(t_i) = v_i$ for each $t_i \in \mathbb{R}$.

A class of preferences $\mathcal{R}$ is **rich** if $\mathcal{R} \supseteq \mathcal{R}^Q$. Throughout the paper, we assume that a class of preferences $\mathcal{R}$ is rich. The next example provides a list of rich classes of preferences.

Example 1. The following are examples of rich classes of preferences.

- $\mathcal{R}^Q$ itself is rich.
- A preference R_i **exhibits nonnegative income effects** if $v_i(t_i)$ is nonincreasing in t_i. Let $\mathcal{R}^+$ denote the class of preferences exhibiting nonnegative income effects. Then, $\mathcal{R}^+$ is rich.
- A preference R_i **exhibits nonpositive income effects** if $v_i(t_i)$ is nondecreasing in t_i. Let $\mathcal{R}^-$ denote the class of preferences exhibiting nonpositive income effects. Then, $\mathcal{R}^-$ is rich.
- Given an interest rate $r \in \mathbb{R}_{++}$ and an income $I_i \in \mathbb{R}_{++} \cup \{\infty\}$, a preference R_i **reflects a soft budget constraint under** (r, I_i) if it is represented by a utility function

$$u(x_i, t_i; v_i, I_i, r) = \begin{cases} v_i \cdot x_i - t_i & \text{if } t_i \leq I_i, \\ v_i \cdot x_i - I_i - (1+r)(t_i - I_i) & \text{if } t_i > I_i, \end{cases}$$

  where $v_i \in \mathbb{R}_+$. Let $\mathcal{R}^{SB}$ denote the class of preferences reflecting a soft budget constraint under some (r, I_i). Then, $\mathcal{R}^{SB}$ is rich.

An n-tuple $z = (z_i)_{i \in N} = (x_i, t_i)_{i \in N} \in (M \times \mathbb{R})^n$ is a (**feasible**) **allocation** if it satisfies $\sum_{i \in N} x_i \leq 1$. Let Z denote the set of allocations.

A **preference profile** is $R = (R_1, \ldots, R_n) \in \mathcal{R}^n$. Given $R \in \mathcal{R}^n$ and $N' \subseteq N$, let $R_{N'} = (R_i)_{i \in N'}$ and $R_{-N'} = (R_i)_{i \in N \setminus N'}$. In particular, for a given pair of distinct agents $i, j \in N$, if $N' = \{i\}$, then let $R_{-i} = R_{-N'}$, and if $N' = \{i, j\}$, then let $R_{i,j} = R_{N'}$ and $R_{-i,j} = R_{-N'}$. A **rule** is $f : \mathcal{R}^n \to Z$. Let $f_i(R) = (x_i^f(R), t_i^f(R))$ denote agent i's outcome bundle under f at R. Let $N_+^f = \{i \in N : \exists R \in \mathcal{R}^n \text{ s.t. } x_i^f(R) = 1\}$ denote the set of agents who have a chance to receive the object at some preference profile under the rule f.

We introduce the properties of rules. A **coalition** is a subset of N. A coalition N' **benefits by misrepresenting preferences** $R'_{N'} \in \mathcal{R}^{|N'|}$ **at** $R \in \mathcal{R}^n$ if for each $i \in N'$, $f_i(R'_{N'}, R_{-N'})\ R_i\ f_i(R)$, and for some $j \in N'$, $f_j(R'_{N'}, R_{-N'})\ P_j\ f_j(R)$.

Group strategy-proofness. There are no $R \in \mathcal{R}^n$, $N' \subseteq N$, and $R'_{N'} \in \mathcal{R}^{|N'|}$ such that N' benefits by misrepresenting $R'_{N'}$ at R.

Pairwise strategy-proofness. There are no $\mathcal{R} \in \mathcal{R}^n$, $N' \subseteq N$, and $R'_{N'} \in \mathcal{R}^{|N'|}$ such that $|N'| \leq 2$ and N' benefits by misrepresenting $R'_{N'}$ at R.

Effectively pairwise strategy-proofness. There are no $\mathcal{R} \in \mathcal{R}^n$, $N' \subseteq N$, and $R'_{N'} \in \mathcal{R}^{|N'|}$ such that $|N'| \leq 2$, N' benefits by misrepresenting $R'_{N'}$ at R, and if $|N'| = 2$, then for each $i \in N'$ and each $R''_i \in \mathcal{R}$, $f_i(R'_{N'}, R_{-N'})\ R_i\ f_i(R''_i, R'_{N' \setminus \{i\}}, R_{-N'})$.

Strategy-proofness. There are no $R \in \mathcal{R}^n$, $i \in N$, and $R'_i \in \mathcal{R}$ such that $\{i\}$ benefits by misrepresenting R_i at R.

Note that *group strategy-proofness* implies *pairwise strategy-proofness*, which in turn implies *effectively pairwise strategy-proofness*. Note also that *strategy-proofness* is implied by all the other three properties. Let $R^0_i \in \mathcal{R}^Q$ be such that $v^0_i = 0$.

Non-imposition. For each $R \in \mathcal{R}^n$ and each $i \in N$, if $R_i = R^0_i$, then $f_i(R)\ I_i\ (0, 0)$.

Ontoness. For each $i \in N$, there is $R \in \mathcal{R}^n$ such that $x^f_i(R) = 1$.

A sequence $(R^n)_{n \in \mathbb{N}}$ of preference profiles **converges to** a preference profile R if for each $i \in N$ and each $x_i \in M$, the sequence of functions $(V^n_i(x_i, \cdot))_{n \in \mathbb{N}}$ uniformly converges to $V_i(x_i, \cdot)$.

Remark 1. For each sequence $(R^n)_{n \in \mathbb{N}}$ in $(\mathcal{R}^Q)^n$ and each $R \in (\mathcal{R}^Q)^n$, R^n converges to R if and only if $v^n \to v$ as $n \to \infty$.

Welfare continuity.[8] For each sequence $(R^n)_{n \in \mathbb{N}}$ in $\mathcal{R}^n$ and each $R \in \mathcal{R}^n$, if $(R^n)_{n \in \mathbb{N}}$ converges to R, then for each $i \in N$ and each $x_i \in M$, $V^n_i(x_i, f_i(R^n)) \to V_i(x_i, f_i(R))$ as $n \to \infty$.

The next remark states that if a rule is defined on $(\mathcal{R}^Q)^n$, then *welfare continuity* is equivalent to the continuity of quasi-linear utility of each agent from the outcome bundle of the rule in valuation profiles.

Remark 2. A rufe f on $(\mathcal{R}^Q)^n$ is *welfare continuous* if and only if for each sequence $(R^n)_{n \in \mathbb{N}}$ in $(\mathcal{R}^Q)^n$ and each $R \in (\mathcal{R}^Q)^n$, if $v^n \to v$ as $n \to \infty$, then for each $i \in N$, $v_i \cdot x^f_i(R^n) - t^f_i(R^n) \to v_i \cdot x^f_i(R) - t^f_i(R)$ as $n \to \infty$.

Minimal fairness. For each $R \in \mathcal{R}^n$, if $R_i = R_j$ for each pair $i, j \in N$, then $f_i(R)\ R_i\ f_j(R)$ for each pair $i, j \in N$ such that $x^f_i(R) = 0$ and $x^f_j(R) = 1$.

[8] Tierney (2022) defines a stronger variant of the property with the same name.

A rule f **Pareto dominates** another rule g if for each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R)\ R_i\ g_i(R)$, and for some $R' \in \mathcal{R}^n$ and $j \in N$, $f_j(R')\ P'_j\ g_j(R')$. A rule f is **constrained efficient** within a class of rules if (i) it belongs to the class, and (ii) no rule in the class Pareto dominates it. A rule f **revenue dominates** another rule g if for each $R \in \mathcal{R}^n$, $\sum_{i\in N} t_i^f(R) \geq \sum_i t_i^g(R)$ and for some $R' \in \mathcal{R}^n$, $\sum_{i\in N} t_i^f(R') > \sum_{i\in N} t_i^g(R')$. A rule f is **revenue undominated** within a class of rules if (i) it belongs to the class, and (ii) no rule in the class revenue dominates it.

3 Priority rule

In this section, we define the priority rule. It specifies a priority over the set of participants N_+^f of the rule and the (personal) price for each participant. Then, it allocates the object to the agent with the highest priority among the agents who would like to get the object with their prices.

Definition 1. A rule f is a (**fixed-prices**) **priority rule** if there are a strict order (*priority*) $\succ^f$ over N_+^f and prices $(p_i^f)_{i\in N_+^f} \in \mathbb{R}_+^{|N_+^f|}$ such that for each $R \in \mathcal{R}^n$, the following hold.

- For each $i \in N_+^f$, if $v_j(0) \leq p_j^f$ for each $j \succ^f i$ and $v_i(0) > p_i^f$, then $x_i^f(R) = 1$.
- For each $i \in N_+^f$, if $x_i^f(R) = 1$, then $v_j(0) \leq p_j^f$ for each $j \succ^f i$ and $v_i(0) \geq p_i^f$.
- For each $i \in N_+^f$, if $x_i^f(R) = 1$, then $t_i^f(R) = p_i^f$.
- For each $i \in N$, if $x_i^f(R) = 0$, then $t_i^f(R) = 0$.

If a priority rule f satisfies $|N_+^f| = 1$, then it is a (**fixed-price**) **dictatorial rule**. If a priority rule f satisfies $N_+^f = \varnothing$, then it is the **no-trade rule**.

The above definition of a priority rule is different from Juarez (2013) in that we (i) allow the set of participants to be determined endogenously by the rule, (ii) allow preferences to be non-quasi-linear, and more importantly, (iii) add the new condition to the rule (the second condition above).[9]

4 Main result

The main result of this paper is a characterization of the class of rules satisfying any one of our group incentive properties and *non-imposition*.

[9] Juarez (2013) restricts attention to a rule that satisfies both *individual rationality* and *nonnegative payments*, and under both the conditions, our priority rule coincides with him (except for the differences (i) and (ii) above). Because we do not restrict our attention to the class of rules satisfying both the properties in this paper, we choose to add a new condition to the definition of a priority rule.

Theorem 1. *The following statements are equivalent.*
(i) A rule satisfies effectively pairwise strategy-proofness and non-imposition.
(ii) A rule satisfies pairwise strategy-proofness and non-imposition.
(iii) A rule satisfies group strategy-proofness and non-imposition.
(iv) A rule is a priority rule.

Theorem 1 implies *effectively pairwise strategy-proofness*, *pairwise strategy-proofness*, and *group strategy-proofness* are all equivalent under *non-imposition*. This observation does not follow from previous results, but is consistent with the previous findings in several models that all the properties are equivalent under some assumptions such as a richness of a domain and additional properties (Serizawa, 2006; Alva, 2017).

5 Applications

In this section, we identify the classes of rules satisfying any of our group incentive properties and *non-imposition* together with the additional properties of interest by exploiting Theorem 1. All of the following results are new.

First, we extend Proposition 3 of Juarez (2013), and characterize the class of rules satisfying any one of our group incentive properties, *ontoness*, and *non-imposition*. Since its proof is trivial, we omit it.

Proposition 1. *A rule satisfies ontoness, non-imposition, and any of effectively pairwise, pairwise, or group strategy-proofness if and only if it is a priority rule with $N_+^f = N$.*

The next result states that *welfare continuity* leads to the dictatorship or the no-trade rule.

Proposition 2. *A rule satisfies welfare continuity, non-imposition, and any of effectively pairwise, pairwise, or group strategy-proofness if and only if it is either a dictatorial rule or the no-trade rule.*

The next result states that *minimal fairness* leads to the no-trade rule. Since its proof is straightforward, we omit it.

Proposition 3. *A rule satisfies minimal fairness, non-imposition, and any of effectively pairwise, pairwise, or group strategy-proofness if and only if it is the no-trade rule.*

The next result identifies the *constrained efficient* rules within the class of rules satisfying any of our group incentive properties and *non-imposition*.

Proposition 4. *A rule is constrained efficient within the class of rules satisfying non-imposition and any of effectively pairwise, pairwise, or group strategy-proofness if and only if it s a priority rule such that $N_+^f = N$ and $p_i^f = 0$ for each $i \in N$.*

Finally, the next result identifies the *revenue undominated* rules within the class of rules satisfying any of our group incentive properties and *non-imposition*.

Proposition 5. *A rule is revenue undominated within the class of rule satisfying non-imposition and any of effectively pairwise, pairwise, or group strategy-proofness if and only if it is a priority rule such that $N_+^f = N$, $p_i^f > 0$ for each $i \in N$, and the following hold.*

- *For each $R \in \mathcal{R}^n$ if $v_i(0) \leq p_i^f$ for each $i \in N$, and $v_j(0) = p_j^f$ for some $j \in N$, then $x_i^f(R) = 1$ for $i \in N$ such that $v_i(0) = p_i^f \geq p_j^f$ for each $j \in N$ with $v_j(0) = p_j^f$.*
- *For each pair $i, j \in N$, if $i \succ^f j$, then $p_i^f \geq p_j^f$.*

The first condition above is concerned with a tie-breaking rule of a priority rule. It requires that if no agent prefers winning the object with his price to losing, and some agents are indifferent between them, then the rule should allocate the object to an agent with the highest price among the agents who are indifferent between winning and losing. The second condition states that the higher priority an agent has, the higher price he faces.

As a corollary of Propositions 3, 4 and 5, we observe the tension between minimal properties of efficiency, fairness, and revenue maximization under one of our group incentive properties and *non-imposition*. The next corollary states that if a rule satisfies any of our group incentive properties and *non-imposition*, then each one of the three properties (i.e., *constrained efficiency*, *minimal fairness*, and *revenue undominance*) is compatible with neither one of the other two properties.

Corollary 1. *Let f be a rule satisfying non-imposition and any of effectively pairwise, pairwise, and group strategy-proofness. If it satisfies any of constrained efficiency within the class of rules satisfying the properties, minimal fairness, and revenue undominance within the class of rules, then it cannot satisfy any one of the other two properties.*

6 Conclusion

In this paper, we characterize the class of rule satisfying *pairwise strategy-proofness* and *non-imposition*. Our characterization result remains valid even if we replace *pairwise strategy-proofness* by *effectively pairwise strategy-proofness* or *group strategy-proofness*. We apply our characterization to identify the classes of rules satisfying our group incentive properties, *non-imposition*, and the additional properties of interest. Then, we observe the tension between minimal properties of efficiency, fairness, and revenue maximization under any of our group incentive properties and *non-imposition*. An interesting direction of future research is to investigate the class of rules satisfying (*effectively*) *pairwise strategy-proofness* and additional properties in several objects model.[10] We believe that our results serve as a benchmark for such a direction of future research, and provide the further understanding of the classes of rules that prevent joint manipulations.

Appendix

A Proof of Theorem 1

In this section, we prove Theorem 1. Clearly, (iii) implies (ii), which in turn implies (i). Thus, it suffices to show that (iv) implies (iii) and (i) implies (iv).

A.1 (iv) implies (iii)

Let f be a priority rule. Clearly, it satisfies *non-imposition*. Thus, it suffices to show that it satisfies *group strategy-proofness*.

Let $R \in \mathcal{R}^n$, $N' \subseteq N$, and $R'_{N'} \in \mathcal{R}^{|N'|}$. Let $R' = (R'_{N'}, R_{-N'})$. If $x_i^f(R) = x_i^f(R')$ for each $i \in N'$, then $f_i(R) = f_i(R')$ for each $i \in N'$. Thus, suppose $x_i^f(R) \neq x_i^f(R')$ for some $i \in N'$.

Suppose $x_i^f(R) = 0$ for each $i \in N'$. By $x_i^f(R) \neq x_i^f(R')$ for some $i \in N'$, there is $i \in N'$ such that $x_i^f(R') = 1$. Then, $f_i(R') = (1, p_i^f)$. By $x_i^f(R) = 0$, either $v_j(0) > p_j^f$ for some $j \succ^f i$ or $v_i(0) \leq p_i^f$ by the definition of a priority rule. Suppose $v_j(0) > p_j^f$ for some $j \succ^f i$. Let $k \in N_+^f$ be the first agent in N_+^f such that $j \succ^f i$ and $v_j(0) > p_j^f$. Then, by the definition of a priority rule, $f_k(R) = (1, p_k^f)$, which contradicts that $x_j^f(R) = 0$ for each $j \in N$. Thus, $v_i(0) \leq p_i^f$. Then, $f_i(R) = (0, 0)\ R_i\ (1, p_i) = f_i(R')$. Further, for each $j \in N \backslash \{i\}$, $f_j(R) = f_j(R') = (0, 0)$. Thus, N' cannot benefit by misrepresenting $R'_{N'}$ at R.

Suppose there is $i \in N'$ such that $x_i^f(R) = 1$. By $x_j^f(R) \neq x_j^f(R')$ for some $j \in N'$, $x_i^f(R') = 0$. By $x_i^f(R) = 1$, $v_i(0) \geq p_i^f$. If $v_i(0) > p_i^f$, then $f_i(R)\ P_i\ f_i(R')$, and so by $i \in N'$, N' cannot benefit by misrepresenting $R'_{N'}$ at R. Thus, suppose $v_i(0) = p_i^f$. Then, $f_i(R) = (1, p_i^f)\ I_i\ (0, 0) = f_i(R')$.

Let $j \in N_+^f \backslash \{i\}$. We claim $v_j(0) \leq p_j^f$. If $j \succ^f i$, then by $x_i^f(R) = 1$, the definition of a priority rule implies $v_j(0) \leq p_j^f$. Suppose $i \succ^f j$. By $x_j^f(R) = 0$, either $v_k(0) > p_k^f$ for some $k \succ^f j$ or $v_j(0) \leq p_j^f$. Suppose by contradiction that there is $k \succ^f j$ such that $v_k(0) > p_k^f$. By $x_i^f(R) = 1$, for each $l \succ^f i$, $v_l(0) \leq p_l^f$. Thus, by $v_i(0) = p_i^f$, $i \succ^f k \succ^f j$. W.l.o.g.,

[10] Juarez (2013) shows that in the identical objects model, the class of rules satisfying *group strategy-proofness*, *individual rationality*, and *nonnegative payments* is so broad that it is difficult to provide a tractable characterization without an additional property of rules.

let k be the first agent in N^f_+ such that $i \succ^f k \succ^f j$ and $v_k(0) > p^f_k$. Then, for each $l \succ^f k$, $v_l(0) \leq p^f_l$, and by the definition of a priority rule, $x^f_k(R) = 1$, which contradicts $x^f_i(R) = 1$. Thus, in either case $v_j(0) \leq p^f_j$, and so $f_j(R) = (0,0)\ R_j\ f_j(R')$. Note that for each $k \in N \backslash N^f_+$, $f_k(R) = f_k(R') = (0,0)$. Thus, for each $k \in N$, $f_k(R)\ R_k\ f_k(R')$, and so N' cannot benefit by misrepresenting $R'_{N'}$ at R. ■

A.2 (i) implies (iv)

Next, we show that (i) implies (iv). The proof is in a series of lemmas.

The proofs of the following two lemmas are straightforward, and we omit them. Note that the first lemma shows the last condition of a priority rule.

Lemma 1. *Let f satisfy strategy-proofness and non-imposition. Let $R \in \mathcal{R}^n$ and $i \in N$. If $x^f_i(R) = 0$, then $t^f_i(R) = 0$.*

Lemma 2. *Let f satisfy strategy-proofness and non-imposition. Let $R \in \mathcal{R}^n$ and $i \in N$. If $x^f_i(R) = 1$, then $t^f_i(R) \geq 0$.*

In what follows, let f be a rule satisfying *effectively pairwise strategy-proofness* and *non-imposition*.

Lemma 3. *Let $i, j \in N$ be a distinct pair, $R \in \mathcal{R}^n$, and $R'_{i,j} \in \mathcal{R}^2$ be such that $x^f_i(R) = 1$, $v'_i(0) > t^f_i(R)$, and $v'_j(0) = 0$. Then, $x^f_i(R'_{i,j}, R_{-i,j}) = 1$.*

Proof. Suppose by contradiction that $x^f_i(R'_{i,j}, R_{-i,j}) = 0$. By Lemma 1, $f_i(R'_{i,j}, R_{-i,j}) = (0,0)$. By $v'_i(0) > t_i(R)$, $f_i(R)\ P'_i\ f_i(R'_{i,j}, R_{-i,j})$. By Lemmas 1 and 2 and $v'_j(0) = 0$, $f_j(R) = (0,0)\ R'_j\ f_j(R'_{i,j}, R_{-i,j})$. Thus, $\{i,j\}$ benefits by misrepresenting $R_{i,j}$ at $(R'_{i,j}, R_{-i,j})$. By Lemma 1, *strategy-proofness*, and $v'_i(0) > t^f_i(R)$, for each $R''_i \in \mathcal{R}$, $f_i(R)\ R'_i\ f_i(R''_i, R_{-i})$. By Lemmas 1 and 2 and $v'_j(0) = 0$, for each $R''_j \in \mathcal{R}$, $f_j(R) = (0,0)\ R'_j\ f_j(R''_j, R_{-j})$, contradicting *effectively pairwise strategy-proofness*. □

Lemma 4. *Let $i, j \in N$ be a distinct pair, $R \in \mathcal{R}^n$, and $R'_j \in \mathcal{R}$ be such that $x^f_i(R) = x^f_i(R'_j, R_{-j}) = 1$. Then, $t^f_i(R) = t^f_i(R'_j, R_{-j})$.*

Proof. Suppose by contradiction that $t^f_i(R) \neq t^f_i(R'_j, R_{-j})$. W.l.o.g., let $t^f_i(R) < t^f_i(R'_j, R_{-j})$. Then, $f_i(R)\ P_i\ f_i(R'_j, R_{-j})$. By Lemma 1, $f_j(R) = f_j(R'_j, R_{-j}) = (0,0)$. Thus, $\{i,j\}$ benefits by misrepresenting $R_{i,j}$ at (R'_j, R_{-j}). By *strategy-proofness*, for each $R'_i \in \mathcal{R}$, $f_i(R)\ R_i\ f_i(R'_i, R_{-i})$, and for each $R''_j \in \mathcal{R}$, $f_j(R) = f_j(R'_j, R_{-j})\ R'_j\ f_j(R''_j, R_{-j})$, which contradicts *effectively pairwise strategy-proofness*. □

The next lemmas shows the existence of prices of a priority rule.

Lemma 5. *Let $i \in N^f_+$. There is $p^f_i \in \mathbb{R}_+$ such that for each $R \in \mathcal{R}^n$, if $x^f_i(R) = 1$, then $t^f_i(R) = p^f_i$.*

Proof. W.l.o.g., let $i = 1$. By Lemma 2, for each $R \in \mathcal{R}^n$, if $x_1^f(R) = 1$, then $t_1^f(R) \geq 0$. Thus, it suffices to show that for each pair $R, R' \in \mathcal{R}^n$ with $x_i^f(R) = x_i^f(R') = 1$, $t_i^f(R) = t_i^f(R')$. Let $R, R' \in \mathcal{R}^n$ be such that $x_i^f(R) = x_i^f(R') = 1$. By richness, we can choose $R_1'' \in \mathcal{R}$ such that $v_1''(0) > \max\{t_1^f(R), t_1^f(R')\}$. By Lemma 1 and *strategy-proofness*, $f_1(R_1'', R_{-1}) = f_1(R)$ and $f_1(R_1'', R_{-1}') = f_1(R')$.

By richness, we can choose $R_2'' \in \mathcal{R}$ such that $v_2''(0) = 0$. Then, by $x_1^f(R_1'', R_{-1}) = 1$ and $v_1''(0) > t_1^f(R) = t_1^f(R_1'', R_{-1})$, Lemma 3 implies $x_1^f(R_{1,2}'', R_{-1,2}) = x_1^f(R_1'', R_{-1}) = 1$. By Lemma 4, $t_1^f(R_{1,2}'', R_{-1,2}) = t_1^f(R_1'', R_{-1,2})$. By the same argument, $t_1^f(R_{1,2}'', R_{-1,2}') = t_1^f(R_1'', R_{-1}')$.

Repeating the same arguments for agents $3, \ldots, n$,

$$t_1^f(R_1'', R_{-1}) = t_1^f(R_{1,2}'', R_{-1,2}) = \ \ldots = t_1^f(R'') = \ldots = t_1^f(R_{1,2}'', R_{-1,2}') = t_1^f(R_1'', R_{-1}').$$

By $t_1^f(R) = t_1^f(R_1'', R_{-1})$ and $t_1^f(R') = t_1^f(R_1'', R_{-1}')$, we get $t_1^f(R) = t_1^f(R')$. □

Lemma 6. *Let $R \in \mathcal{R}^n$, $N' \subseteq N_+^f$, and $R_{N'}' \in \mathcal{R}^{|N'|}$ be such that $x_i^f(R) = 1$ for some $i \in N'$ and $v_j'(0) > p_j^f$ for each $j \in N'$. Then,. $x_j^f(R_{N'}', R_{-N'}) = 1$ for some $j \in N'$.*

Proof. W.l.o.g., let $N' = \{1, \ldots, k\}$ and $i = 1$. By $x_1^f(R) = 1$ and $v_1'(0) > p_1^f$, Lemmas 1 and 5 and *strategy-proofness* imply $x_1^f(R_1', R_{-1}) = 1$. We show $x_i^f(R_{1,2}', R_{-1,2}) = 1$ for some $i \in \{1, 2\}$. Suppose by contradiction that $x_i^f(R_{1,2}', R_{-1,2}) = 0$ for each $i \in \{1, 2\}$. By Lemma 1, $f_1(R_{1,2}', R_{-1,2}) = f_2(R_{1,2}', R_{-1,2}) = (0, 0)$. By $v_1'(0) > p_1^f$ and $x_1^f(R_1', R_{-1}) = 1$, Lemma 5 implies $f_1(R_1', R_{-1}) = (1, p_1^f)\ P_1'\ (0, 0) = f_1(R_{1,2}', R_{-1,2})$. By Lemma 1 and $x_2^f(R) = x_2^f(R_1', R_{-1}) = 0$, $f_2(R_1', R_{-1}) = f_2(R_{1,2}', R_{-1,2}) = (0, 0)$. Thus, $\{1, 2\}$ benefits by misrepresenting (R_1', R_2) at $(R_{1,2}', R_{-1,2})$. By *strategy-proofnss*, for each $R_1'' \in \mathcal{R}$, $f_1(R_1', R_{-1})\ R_1'\ f_1(R_1'', R_{-1})$. By *strategy-proofness*, $v_2'(0) > p_2^f$, and $x_2^f(R_{1,2}', R_{-1,2}) = 0$, Lemmas 1 and 5 imply there is no $R_2'' \in \mathcal{R}$ such that $x_2^f(R_1', R_2'', R_{-1,2}) = 1$. Thus, by Lemma 1, for each $R_2'' \in \mathcal{R}$, $f_2(R_1', R_{-1,2}) = f_2(R_1', R_2'', R_{-1,2}) = (0, 0)$, which contradicts *effectively pairwise strategy-proofness*.

Thus, we obtain $x_i^f(R_{1,2}', R_{-1,2}) = 1$ for some $i \in \{1, 2\}$. Repeating the same arguments for agents $3, \ldots, k$ inductively, we have $x_j^f(R_{N'}', R_{-N'}) = 1$ for some $j \in N'$. □

Lemma 7. *Let $R \in \mathcal{R}^n$ and $i \in N_+^f$ be such that $v_i(0) > p_i^f$. Let $R_{-N_+^f}' \in \mathcal{R}^{|N \setminus N_+^f|}$. If $x_i^f(R) = 1$, then $x_i^f(R_{N_+^f}, R_{-N_+^f}') = 1$.*

Proof. Suppose $x_i^f(R) = 1$. W.l.o.g., let $i = 1$ and $N \setminus N_+^f = \{2, \ldots, k\}$. We claim $x_1^f(R_2', R_{-2}) = 1$. Suppose by contradiction that $x_1^f(R_2', R_{-2}) = 0$. By Lemma 1, $f_1(R_2', R_{-2}) = (0, 0)$. By $v_1(0) > p_1^f$ and Lemma 5, $f_1(R) = (1, p_1^f)\ P_1\ (0, 0) = f_1(R_2', R_{-2})$. By $2 \notin N_+^f$, $x_2^f(R) = x_2^f(R_2', R_{-2}) = 0$. By Lemma 1, $f_2(R) = f_2(R_2', R_{-2}) = (0, 0)$. Thus, $\{1, 2\}$ benefits by misrepresenting $R_{1,2}$ at (R_2', R_{-2}). By *strategy-proofness*, for each $R_1' \in \mathcal{R}$, $f_1(R)\ R_1\ f_1(R_1', R_{-1})$. By Lemma 1 and $2 \notin N_+^f$, for each $R_2'' \in \mathcal{R}$, $f_2(R) =$

$f_2(R_2'', R_{-2}) = (0,0)$, contradicting *effectively pairwise strategy-proofness*. Thus, we get $x_1^f(R_2', R_{-2}) = 1$. By $x_1^f(R) = 1$ and Lemma 5, $f_1(R) = f_1(R_2', R_{-2})$. Repeating the same arguments for agents $3, \ldots, k$, $x_1(R_{N_+^f}, R'_{-N_+^f}) = 1$. $\square$

Lemma 8. *Let $R \in \mathcal{R}^n$, $N' \subseteq N_+^f$, and $i \in N'$ be such that $x_i^f(R) = 1$ and $v_j(0) > p_j^f$ for each $j \in N'$. Then, for each $R'_{N' \setminus \{i\}} \in \mathcal{R}^{|N'|-1}$, $x_i^f(R'_{N' \setminus \{i\}}, R_{-N' \setminus \{i\}}) = 1$.*

Proof. W.l.o.g., let $N' = \{1, \ldots, k\}$ and $i = 1$. Let $R'_{N' \setminus \{1\}} \in \mathcal{R}^{|N'|-1}$. We show that $x_1^f(R_2', R_{-2}) = 1$. Suppose not. By Lemma 1, $f_1(R_2', R_{-2}) = (0,0)$. By $x_1^f(R) = 1$, Lemma 5 implies $f_1(R) = (1, p_1^f)$. Thus, by $v_1(0) > p_1^f$, $f_1(R)\ P_1\ f_1(R_2', R_{-2})$. By $x_2^f(R) = 0$, Lemma 1 implies $f_2(R) = (0,0)$. By Lemma 5 and $v_2(0) > p_2^f$, *strategy-proofness* implies that there is no $R_2'' \in \mathcal{R}$ such that $x_2^f(R_2'', R_{-2}) = 1$. Thus, $x_2^f(R_2', R_{-2}) = 0$, and by Lemma 1, $f_2(R) = f_2(R_2', R_{-2}) = (0,0)$. Thus, $\{1,2\}$ benefits by misrepresenting $R_{1,2}$ at (R_2', R_{-2}). By *strategy-proofness*, for each $R_1'' \in \mathcal{R}$, $f_1(R)\ R_1\ f_1(R_1'', R_{-1})$. Since $f_2(R) = f_2(R_2'', R_{-2}) = (0,0)$ for each $R_2'' \in \mathcal{R}$, this contradicts *effectively pairwise strategy-proofness*.

Thus, $x_1^f(R_2', R_{-2}) = 1$. Repeating the same arguments for agents $3, \ldots, k$, we obtain $x_i^f(R'_{N' \setminus \{i\}}, R_{-N' \setminus \{i\}}) = 1$. $\square$

Lemma 9. *Let $R \in \mathcal{R}^n$ and $i \in N_+^f$ be such that $x_i^f(R) = 1$ and $v_i(0) > p_i^f$. Let $N' \subseteq N_+^f \setminus \{i\}$ and $R'_{N'} \in \mathcal{R}^{|N'|}$ be such that $v_j'(0) \leq p_j^f$ for each $j \in N'$. Then, $x_i^f(R'_{N'}, R_{-N'}) = 1$.*

Proof. W.l.o.g., let $i = 1$ and $N' = \{2, \ldots, k\}$. We show $x_1^f(R_2', R_{-2}) = 1$. Suppose by contradiction that $x_1^f(R_2', R_{-2}) = 0$. By Lemma 1, $f_1(R_2', R_{-2}) = (0,0)$. By Lemma 5 and $x_1^f(R) = 1$, $f_1(R) = (1, p_1^f)$. By $v_1(0) > p_1^f$, $f_1(R) = (1, p_1^f)\ P_1\ (0,0) = f_1(R_2', R_{-2})$. By $x_1^f(R) = 1$, $x_2^f(R) = 0$. Thus, by Lemma 1, $f_2(R) = (0,0)$. By $v_2'(0) \leq p_2^f$, Lemmas 1 and 5 imply $f_2(R) = (0,0)\ R_2'\ f_2(R_2', R_{-2})$. Thus, $\{1,2\}$ benefits by misrepresenting $R_{1,2}$ at (R_2', R_{-2}). By *strategy-proofness*, for each $R_1' \in \mathcal{R}$, $f_1(R)\ R_1\ f_1(R_1', R_{-1})$. By $v_2'(0) \leq p_2^f$, Lemmas 1 and 5 imply that for each $R_2'' \in \mathcal{R}$, $f_2(R) = (0,0)\ R_2'\ f_2(R_2'', R_{-2})$, which contradicts *effectively pairwise strategy-proofness*.. $\square$

The next lemma shows the existence of a priority that satisfies the first condition of a priority rule. The proof strategy basically follows that of Proposition 3 of Juarez (2013), but we cannot apply it directly because he assumes *group strategy-proofness* and *ontoness* (and *individual rationality* and *nonnegative payments*), while we only assume weaker *effectively pairwise strategy-proofness* (and weaker *non-imposition*). The previous lemmas and the additional steps in the proof of the following lemma enable us to follow his proof strategy.

Lemma 10. *There is a priority $\succ^f$ over N_+^f such that for each $R \in \mathcal{R}^n$ and each $i \in N_+^f$, if $v_j(0) \leq p_j^f$ for each $j \succ^f i$ and $v_i(0) > p_i^f$, then $f_i(R) = (1, p_i^f)$.*

Proof. W.l.o.g., let $N^f_+ = \{1, \ldots, k\}$. The proof is in three steps.

STEP 1. We show that there is $i_1 \in N^f_+$ such that for each $R \in \mathcal{R}^n$, if $v_{i_1}(0) > p^f_{i_1}$, then $x^f_{i_1}(R) = 1$. Suppose not. Then, for each $i \in N^f_+$, there is $R^i \in \mathcal{R}^n$ such that $v^i_i(0) > p^f_i$ and $x^f_i(R^i) = 0$. For each $i \in N^f_+$, there is $\tilde{R}^i \in \mathcal{R}^n$ such that $x^f_i(\tilde{R}^i) = 1$ for each $i \in N^f_+$. By Lemma 5, $f_i(\tilde{R}^i) = (1, p^f_i)$.

By richness, we can choose $R_{N^f_+} \in \mathcal{R}^{|N^f_+|}$ such that $v_i(0) > p^f_i$ for each $i \in N^f_+$. Recall that $x^f_1(\tilde{R}^1) = 1$. By Lemma 6, there is $i \in N^f_+$ such that $x^f_i(R_{N^f_+}, \tilde{R}^1_{-N^f_+}) = 1$. By Lemma 7, $x^f_i(R_{N^f_+}, R^i_{-N^f_+}) = 1$. By $v_j(0) > p^f_j$ for each $j \in N^f_+ \backslash \{i\}$, Lemma 8 implies $x^f_i(R_i, R^i_{-i}) = 1$. Thus, by Lemmas 1 and 5 and $v^i_i(0) > p^f_i$, *strategy-proofness* implies $x^f_i(R^i) = 1$, which contradicts $x^f_i(R^i) = 0$.

STEP 2. By Step 1, there is $i_1 \in N^f_+$ such that for each $R \in \mathcal{R}^n$, if $v_{i_1}(0) > p^f_{i_1}$, then $x^f_{i_1}(R) = 1$. W.l.o.g., let $i_1 = 1$. In this step, we show that there is $i_2 \in N^f_+ \backslash \{1\}$ such that for each $R \in \mathcal{R}^n$, if $v_1(0) \leq p^f_1$ and $v_{i_2}(0) > p^f_{i_2}$, then $x^f_{i_2}(R) = 1$. Suppose not. Let $i \in N^f_+ \backslash \{1\}$. Then, there is $R^i \in \mathcal{R}^n$ such that $v^i_1(0) \leq p^f_1$, $v_i(0) > p^f_i$, and $x^f_i(R^i) = 0$. By $i \in N^f_+$, there is $\tilde{R}^i \in \mathcal{R}^n$ such that $x^f_i(\tilde{R}^i) = 1$.

By richness, we can choose $R_{N^f_+ \backslash \{1\}} \in \mathcal{R}^{|N^f_+|-1}$ such that $v_i(0) > p^f_i$ for each $i \in N^f_+ \backslash \{1\}$. Note that $x^f_2(\tilde{R}^2) = 1$. By Lemma 6, there is $i \in N^f_+ \backslash \{1\}$ such that $x^f_i(R_{N^f_+ \backslash \{1\}}, \tilde{R}^2_{-N^f_+ \backslash \{1\}}) = 1$. By $v^i_1(0) \leq p^f_1$ and $v_i(0) > p^f_i$, Lemma 9 gives $x^f_i(R^i_1, R_{N^f_+ \backslash \{1\}}, R^2_{N^f_+}) = 1$. By Lemma 7, $x^f_i(R^i_1, R_{N^f_+ \backslash \{i\}}, R^i_{-N^f_+}) = 1$. By $v_j(0) > p^f_j$ for each $j \in N^f_+ \backslash \{1, i\}$, Lemma 8 implies $x^f_i(R_i, R^i_{-i}) = 1$. By Lemmas 1 and 5 and $v^i_i(0) > p^f_i$, *strategy-proofness* implies $x^f_i(R^i) = 1$, which contradicts $x^f_i(R^i) = 0$.

STEP 3. Repeating the same arguments as in Step 2 for the remaining agents in N^f_+ inductively, we obtain a sequence $(i_j)^k_{j=1}$ such that for each $j \in \{1, \ldots, k\}$, if $v_{i_l} \leq p^f_{i_l}$ for each $l \leq j$ and $v_{i_j}(0) > p^f_{i_j}$, then $x^f_{i_j}(R) = 1$. Let $\succ^f$ be a strict order over N^f_+ such that

$$i_1 \succ^f i_2 \succ^f \cdots \succ^f i_{k-1} \succ^f i_k.$$

Let $R \in \mathcal{R}^n$ and $i \in N^f_+$ be such that $v_j(0) \leq p^f_j$ for each $j \succ^f i$ and $v_i(0) > p^f_i$. Then, there is $j \in \{1, \ldots, k\}$ such that $i = i_j$. By $v_l(0) \leq p^f_l$ for each $l \succ^f i$, $v_{i_l}(0) \leq p^f_{i_l}$ for each $l \leq j$. Thus, by $v_{i_j}(0) > p^f_{i_j}$, $x^f_i(R) = x^f_{i_j}(R) = 1$. $\square$

Finally, the next lemmas shows the second condition of a priority rule.

Lemma 11. *Let $R \in \mathcal{R}^n$ and $i \in N^f_+$. If $x^f_i(R) = 1$, then $v_j(0) \leq p^f_j$ for each $j \succ^f i$ and $v_i(0) \geq p^f_i$.*

Proof. Suppose $x_i^f(R) = 1$. First, we show $v_j(0) \leq p_j^f$ for each $j \succ^f i$. Suppose by contradiction that there is $j \succ^f i$ such that $v_j(0) > p_j^f$. W.l.o.g., let j be the first agent among the agents in N_+^f according to $\succ^f$ such that $k \succ^f i$ and $v_k(0) < p_k^f$. Then, for each $k \succ^f j$, $v_k(0) \leq p_k^f$. Thus, by $v_j(0) > p_j^f$, Lemma 10 implies $x_j^f(R) = 1$, which contradicts $x_i^f(R) = 1$ and $j \neq i$.

Next, we show $v_i(0) \geq p_i^f$. If there is $R_i' \in \mathcal{R}$ such that $x_i(R_i', R_{-i}) = 0$, then *strategy-proofness* implies $v_i(0) \geq p_i^f$. If there is no $R_i' \in \mathcal{R}$ such that $x_i(R_i', R_{-i}) = 0$, then $x_i(R_i^0, R_{-i}) = 1$. By *non-imposition*, $p_i^f = t_i(R^0, R_{-i}) = 0$. Thus, $v_i(0) \geq 0 = p_i^f$. □

By Lemmas 1, 5, 10 and 11, f is a priority rule. ■

B Proof of Proposition 2

In this section, we prove Proposition 2.

By Theorem 1, both a dictatorial rule and the no-trade rule satisfies any of our group incentive properties and *non-imposition*. The no-trade rule trivially satisfies *welfare continuity*. It is straightforward to verify that a dictatorial rule satisfies *welfare continuity*. Thus, we here show that if a rule satisfies any of our group incentive properties, *welfare continuity*, and *non-imposition*, then it is either a dictatorial rule or the no-trade rule. Let f satisfy the three properties. By Theorem 1, it is a priority rule. It suffices to show $|N_+^f| \leq 1$. By contradiction, suppose $|N_+^f| > 1$. Then, there is a distinct pair $i, j \in N_+^f$. W.l.o.g., let $i = 1$ and $j = 2$ and $1 \succ^f 2 \succ^f k$ for each $k \in N_+^f \setminus \{1, 2\}$. With a slight abuse of notation, let f denote the restriction of f to $(\mathcal{R}^Q)^n$. For each $n \in \mathbb{N}$, let $R^n \in (\mathcal{R}^Q)^n$ be such that $v^n = (p_1^f + \frac{1}{n}, p_2^f + 1, 0 \ldots, 0)$. Then, for each $n \in \mathbb{N}$, $f_2(R^n) = (0, 0)$, and so $v_2^n \cdot x_2^f(R^n) - t_2^f(R^n) = 0$. Let $R \in (\mathcal{R}^Q)^n$ be such that $v = (p_1^f, p_2^f + 1, 0, \ldots, 0)$. Then, $v^n \to v$ as $n \to \infty$. By $f_2(R) = (1, p_2^f)$, $v_2 \cdot x_2^f(R) - t_2^f(R) = p_2^f + 1 - p_2^f = 1$. Thus, as $n \to \infty$, $v_2^n \cdot x_2^f(R^n) - t_2^f(R^n) \to 0 \neq 1 = v_2 \cdot x_2^f(R) - t_2^f(R)$. By Remark 2, this contradicts *welfare continuity*. ■

C Proof of Proposition 4

In this section, we prove Proposition 4.

C.1 The "if" part

We show the "if" part. Let f be a priority rule such that $N_+^f = N$ and $p_i^f = 0$ for each $i \in N$. By Theorem 1, it satisfies any of our group incentive properties and *non-imposition*. Let g be a rule satisfying both the properties. By Theorem 1, it is a priority rule. By contradiction, suppose that g Pareto dominates f. Then, for each

$R \in \mathcal{R}^n$ and each $i \in N$, $g_i(R)\ R_i\ f_i(R)$, and there are $R' \in \mathcal{R}^n$ and $j \in N$ such that $g_j(R')\ P'_j\ f_j(R')$. We claim that $x_j^f(R') = 0$ and $x_j^g(R') = 1$. If $x_j^f(R') = x_j^g(R') = 0$, then $g_j(R') = f_j(R') = (0,0)$, a contradiction. If $x_j^f(R') = x_j^g(R') = 1$, then by $p_j^f = 0 \leq p_j^g$, $f_j(R') = (1, p_j^f)\ R'_j\ (1, p_j^g) = g_j(R')$, a contradiction. If $x_j^f(R') = 1$ and $x_j^g(R') = 0$, then by $p_j^f = 0$, $f_j(R') = (1,0)\ R'_j\ (0,0) = g_j(R')$, a contradiction. Thus, $x_j^f(R') = 0$ and $x_j^g(R') = 1$. Then, $(1, p_j^g) = g_j(R')\ P'_j\ f_j(R') = (0,0)$, which implies $v'_j(0) > p_j^g \geq 0 = p_j^f$. Thus, by $x_j^f(R') = 0$, the definition of a priority rule implies that there is $k \succ^f j$ such that $v'_k(0) > p_k^f$. Let k be the first such agent. Then, for each $i \succ^f k$, $v'_i(0) \leq p_i^f$, and $v'_k(0) > p_k^f$. Thus, $f_k(R') = (1, p_k^f) = (1,0)$. By $j \neq k$ and $x_j^g(R') = 1$, $x_k^g(R') = 0$. Thus, $g_k(R') = (0,0)$. By $(0,0) = g_k(R')\ R'_k\ f_k(R') = (1,0)$, $v'_k(0) = 0$. However, this contradicts $v'_k(0) > p_k^f = 0$. ■

C.2 The "only if" part

We show the "only if" part. Let f be *constrained efficient* within the class of rules satisfying any of our group incentive properties and *non-imposition*. By Theorem 1, it is a priority rule. The proof is in two steps.

Step 1. We claim $N_+^f = N$. Suppose by contradiction that $N_+^f \subsetneq N$. Then, there is $i \in N \backslash N_+^f$. Let g be a priority rule such that $N_+^g = N_+^f \cup \{i\}$, $j \succ^g i$ for each $j \in N_+^f$, $p_i^g = 0$, $p_j^g = p_j^f$ for each $j \in N_+^f$, and for each $R \in \mathcal{R}^n$ and each $j \in N_+^f$, if $x_j^f(R) = 1$, then $x_j^g(R) = 1$. Note that by Theorem 1, it satisfies the properties. Clearly, for each $R \in \mathcal{R}^n$ and each $j \in N$, $g_j(R)\ R_j\ f_j(R)$. By richness, we can find $R \in \mathcal{R}^n$ such that $v_i(0) > 0 = p_i^g$, and $v_j(0) \leq p_j^g$ for each $j \in N_+^f$. By the definition of a priority rule, $g_i(R) = (1, p_i^g) = (1,0)$. By $i \notin N_+^f$, $f_i(R) = (0,0)$. By $v_i(0) > 0$, $g_i(R) = (1,0)\ P_i\ (0,0) = f_i(R)$. Thus, g Pareto dominates f, a contradiction.

Step 2. We show that for each $i \in N$, $p_i^f = 0$. Suppose by contradiction that there is $i \in N$ such that $p_i^f > 0$. W.l.o.g., let i be the lowest priority such agent according to $\succ^f$. Thus, for each $j \in N$ with $i \succ^f j$, $p_j^f = 0$.

Let g be a priority rule such that $N_+^g = N$, $\succ^g = \succ^f$, $p_i^g < p_i^f$, $p_j^g = p_j^f$ for each $j \in N \backslash \{i\}$, and for each $R \in \mathcal{R}^n$ and each $j \in N$ with $j \succ^f i$, if $x_j^f(R) = 1$, then $x_j^g(R) = 1$. By Theorem 1, it satisfies the properties.

Let $R \in \mathcal{R}^n$. We show that for each $j \in N$, $g_j(R)\ R_j\ f_j(R)$. We consider the following four cases.

Case 1. For each $j \in N$, $x_j^f(R) = 0$.

Let $j \in N$ be such that $x_j^g(R) = 1$ (if it exists). By the definition of a priority rule, $v_j(0) \geq p_j^g$, which implies $g_j(R) = (1, p_j^g)\ R_j\ (0,0) = f_j(R)$. For each $k \in N$ with $x_k^g(R) = 0$, $g_k(R) = f_k(R) = (0,0)$.

Case 2. $x_j^f(R) = 1$ for some $j \in N$ with $j \succ^f i$.

Then, by the definition of the rule g, $g_j(R) = f_j(R) = (1, p_j^f)$. For each $k \in N \backslash \{j\}$, $g_k(R) = f_k(R) = (0,0)$.

Case 3. $x_i^f(R) = 1$.

By the definition of a priority rule, $v_j(0) \leq p_j^f$ for each $j \succ^f i$, and $v_i(0) \geq p_i^f$. By $p_i^g < p_i^f$, $v_i(0) > p_i^g$. For each $j \succ^g i$, by $\succ^g = \succ^f$ and $p_j^g = p_j^f$, $v_j(0) \leq p_j^g$. Thus, $x_i^g(R) = 1$. By $p_i^g < p_i^f$, $g_i(R) = (1, p_i^g)\ P_i\ (1, p_i^f) = f_i(R)$. For each $j \in N \backslash \{i\}$, $g_j(R) = f_j(R) = (0,0)$.

Case 4. $x_j^f(R) = 1$ for some $j \in N$ with $i \succ^f j$.

By $i \succ^f j$, $p_j^f = 0$. If $x_j^g(R) = 1$, then by $p_j^g = p_j^f$, $g_j(R) = f_j(R) = (1, p_j^f)$, and $g_k(R) = f_k(R) = (0,0)$ for each $k \in N \backslash \{j\}$. Thus, suppose $x_j^g(R) = 0$. Then, either $v_j(0) \leq p_j^g$ or $v_k(0) > p_k^g$ for some $k \succ^g j$. We show $v_j(0) \leq p_j^g$. Suppose not. Then, $v_k(0) > p_k^g$ for some $k \succ^g j$. By $\succ^g = \succ^f$ and $p_k^g = p_k^f$, $v_k(0) > p_k^f$ and $k \succ^f j$. Let k be the first such agent according to $\succ^f$. Then, for each $l \succ^f k$, $v_l(0) \leq p_l^f$. Thus, by the definition of a priority rule, $x_k^f(R) = 1$, which contradicts $x_j^f(R) = 1$ and $k \neq j$.

Thus, $v_j(0) \leq p_j^g$. By $p_j^g = p_j^f = 0$, $0 \leq v_j(0) \leq p_j^g = 0$, so that $v_j(0) = 0$. Thus, $g_j(R) = (0,0)\ I_j\ (1,0) = f_j(R)$. If $x_k^g(R) = 0$ for each $k \in N \backslash \{j\}$, then $g_k(R) = f_k(R) = (0,0)$ for each $k \in N \backslash \{j\}$. Suppose there is $k \in N \backslash \{j\}$ such that $x_k^g(R) = 1$. By the definition of a priority rule, $v_k(0) \geq p_k^g$. By $x_j^f(R) = 1$ and $j \neq k$, $x_k^f(R) = 0$. Thus, by $v_k(0) \geq p_k^g$, $g_k(R) = (1, p_k^g)\ R_k\ (0,0) = f_k(R)$. For each $l \in N \backslash \{j,k\}$, $g_l(R) = f_l(R) = (0,0)$.

Thus, in any case, $g_j(R)\ R_j\ f_j(R)$ for each $j \in N$. By $i \in N_+^f$, there is $R \in \mathcal{R}^n$ such that $x_i^g(R) = 1$. We have shown (in Case 3) that $g_i(R)\ P_i\ f_i(R)$. Thus, g Pareto dominates f, a contradiction. ∎

D Proof of Proposition 5

In this section, we prove Proposition 5.

D.1 The "if" part

First, we show the "if" part. Let f be a priority rule such that $N_+^f = N$, $p_i^f > 0$ for each $i \in N$, and it satisfies the two conditions of a priority rule in Proposition 5. Let g be a rule satisfying any of our group incentive properties and *non-imposition* such that for each $R \in \mathcal{R}^n$, $\sum_{i \in N} t_i^g(R) \geq \sum_{i \in N} t_i^f(R)$. By Theorem 1, it is a priority rule. The proof is in three steps.

STEP 1. We show that $N_+^g = N$. Suppose $N_+^g \subsetneq N$. Then, there is $i \in N \backslash N_+^g$. By richness, we can choose $R \in \mathcal{R}^n$ such that $v_i(0) > p_i^f$, and for each $j \in N \backslash \{i\}$, $v_j(0) = 0$. Then, by the definition of a priority rule, $f_i(R) = (1, p_i^f)$, and for each $j \in N \backslash \{i\}$, $f_j(R) = (0,0)$. Further, by $i \notin N_i^g$, $g_i(R) = (0,0)$. For each $j \in N \backslash \{i\}$, by $v_j(0) = 0$, $t_j^g(R) = 0$. Thus,

$$\sum_{j \in N} t_j^f(R) = p_i^f > 0 = \sum_{j \in N} t_j^g(R),$$

a contradiction.

STEP 2. Next, we show that for each $i \in N$, $p_i^f = p_i^g$. Let $i \in N$. By contradiction, suppose $p_i^f \neq p_i^g$. We consider the following two cases.

CASE 1. $p_i^f < p_i^g$.

By richness, we can choose $R \in \mathcal{R}^n$ such that $p_i^f < v_i(0) < p_i^g$, and for each $j \in N \backslash \{i\}$, $v_j(0) = 0$. Then, by the definition of a priority rule, $f_i(R) = (1, p_i^f)$, and for each $j \in N \backslash \{i\}$, $f_j(R) = (0,0)$. By $v_i(0) < p_i^g$, the definition of a priority rule gives $g_i(R) = (0,0)$. For each $j \in N \backslash \{i\}$, by $v_j(0) = 0$, $t_j^g(R) = 0$. Thus,

$$\sum_{j \in N} t_i^f(R) = p_i^f > 0 = \sum_{j \in N} t_j^g(R),$$

a contradiction.

CASE 2. $p_i^f > p_i^g$.

By richness, we can find $R \in \mathcal{R}^n$ such that $v_i(0) > p_i^f > p_i^g$, and for each $j \in N \backslash \{i\}$, $v_j(0) \leq \min\{p_j^f, p_j^g\}$. Then, by the definition of a priority rule, $f_i(R) = (1, p_i^f)$, $g_i(R) = (1, p_i^g)$, and for each $j \in N \backslash \{i\}$, $f_j(R) = g_j(R) = (0,0)$. By $p_i^f > p_i^g$,

$$\sum_{j \in N} t_j^f(R) = p_i^f > p_i^g = \sum_{j \in N} t_j^g(R),$$

a contradiction.

STEP 3. Now, we conclude the proof of the "if" part. Suppose by contradiction that there is $R \in \mathcal{R}^n$ such that $\sum_{i \in N} t_i^g(R) > \sum_{i \in N} t_i^f(R)$. Then, there are $i \in N_+^f$ and $j \in N_+^g$ such that $x_i^f(R) = 1$, $x_j^g(R) = 1$, and $t_i^f(R) = p_i^f < p_j^g = t_j^g(R)$. By Step 1, $i, j \in N = N_+^f = N_+^g$. By Step 2 and $p_i^f \neq p_j^g$, $i \neq j$. By the definition of a priority rule, $v_i(0) \geq p_i^f$ and $v_j(0) \geq p_j^g$. We consider the following three cases, and derive a contradiction in each of the cases.

CASE 1. $v_j(0) > p_j^g$.

By $v_j(0) > p_j^g$ and Step 2, $v_j(0) > p_j^f$. Thus, by $x_i^f(R) = 1$ and $i \neq j$, the definition of a priority rule implies $i \succ^f j$. By assumption, $i \succ^f j$ implies $p_i^f \geq p_j^f$. By Step 2, $p_i^f \geq p_j^g$, which contradicts $p_j^g > p_i^f$.

CASE 2. $v_i(0) > p_i^f$ and $v_j(0) = p_j^g$.

By $v_i(0) > p_i^f$ and Step 2, $v_i(0) > p_i^g$. By $x_j^g(R) = 1$ and $i \neq j$, the definition of a priority rule implies $j \succ^g i$. Note that for each $k \in N$ with $k \succ^g j$, by the definition of a priority rule, $v_k(0) \leq p_k^g$.

We show that for each $k \in N$ with $j \succ^g k \succ^g i$, $v_k(0) \leq p_k^g$. Let k be the immediate successor of j according to $\succ^g$, i.e., there is no $l \in N$ such that $j \succ^g l \succ^g k$. We claim that $v_k(0) \leq p_k^g$. Suppose by contradiction that $v_k(0) > p_k^g$. Then, by $v_l(0) \leq p_l^g$ for each $l \succ^g k$, $x_k^g(R) = 1$, which contradicts $x_j^g(R) = 1$ and $k \neq j$.

Repeating the same arguments for the remaining agents such that $j \succ^g l \succ^g i$ inductively, we get $v_l(0) \leq p_l^g$ for each $l \in N$ such that $j \succ^g l \succ^g i$. Thus, for each $l \in N$ with $l \succ^g i$, $v_l(0) \leq p_l^g$, and so by $v_i(0) > p_i^f$, $x_i^g(R) = 1$, which contradicts $x_j^g(R) = 1$ and $i \neq j$.

CASE 3. $v_i(0) = p_i^f$ and $v_j(0) = p_j^g$.

By Step 2 and $v_j(0) = p_j^g$, $v_j(0) = p_j^f$. If there is $k \in N \backslash \{i, j\}$ such that $v_k(0) > p_k^f$, then by the definition of a priority rule, the agent with the highest priority according to $\succ^f$ among the agents in N_+^f with $v_k(0) > p_k^f$ wins the object. Let k be such an agent. Then, $x_k^f(R) = 1$ and $v_k(0) > p_k^f$. By $v_i(0) = p_i^f$, $k \neq i$. However, this contradicts $x_i^f(R) = 1$. Thus, for each $k \in N \backslash \{i, j\}$, $v_k(0) \leq p_k^f$. Then, by assumption, $x_i^f(R) = 1$ implies that $v_i(0) = p_i^f \geq p_k^f$ for each $k \in N$ with $v_k(0) = p_k^f$. However, this contradicts $v_j(0) = p_j^f$ and $p_j^f = p_j^g > p_i^f$, where $p_j^f = p_j^g$ follows from Step 2. ■

D.2 The "only if" part

Next, we show the "only if" part. Let f be a rule that is *revenue undominated* within the class of rules satisfying any of our group incentive properties and *non-imposition*. Then, by Theorem 1, it is a priority rule. The proof is in four steps.

STEP 1. We show $N^f_+ = N$. Suppose by contradiction that $N^f_+ \subsetneq N$. Then, there is $i \in N \backslash N^f_+$. Let g be a priority rule such that $N^g_+ = N^f_+ \cup \{i\}$, $p^g_i > 0$, $p^g_j = p^f_j$ for each $j \in N^f_+$, $j \succ^g i$ for each $j \in N^f_+$, and for each each $R \in \mathcal{R}^n$ and each $j \in N^f_+$, if $x^f_j(R) = 1$, then $f_j(R) = g_j(R)$. By Theorem 1, it satisfies both the properties. Clearly, for each $R \in \mathcal{R}^n$, $\sum_{j\in N} t^g_j(R) \geq \sum_{j\in N} t^f_j(R)$. By richness, we can choose $R \in \mathcal{R}^n$ such that $v_i(0) > p^g_i$, and for each $j \in N\backslash\{i\}$, $v_j(0) = 0$. Then, by the definition of a priority rule, $g_i(R) = (1, p^g_i)$, and for each $j \in N\backslash\{i\}$, $g_j(R) = (0, 0)$. By $i \notin N^f_+$, $f_i(R) = (0, 0)$. For each $j \in N\backslash\{i\}$, by $v_j(0) = 0$, $t^f_j(R) = 0$. Thus, by $p^g_i > 0$,

$$\sum_{j\in N} t^g_j(R) = p^g_i > 0 = \sum_{j\in N} t^f_j(R).$$

Thus, g revenue dominates f, a contradiction.

STEP 2. We show that for each $i \in N$, $p^f_i > 0$. Suppose by contradiction that there is $i \in N$ such that $p^f_i = 0$. Let g be a priority rule such that $N^g_+ = N$, $\succ^g = \succ^f$, $p^g_i > 0$, $p^g_j = p^f_j$ for each $j \in N\backslash\{i\}$, and for each $R \in \mathcal{R}^n$ and each $j \in N\backslash\{i\}$, if $x^f_j(R) = 1$, then $x^g_j(R) = 1$.[11] By Theorem 1, it satisfies the properties.

Let $R \in \mathcal{R}$. If $x^f_j(R) = 0$ for each $j \in N$, then

$$\sum_{j\in N} t^g_j(R) \geq 0 = \sum_{j\in N} t^f_j(R).$$

If $x^f_i(R) = 1$, then by $p^f_i = 0$,

$$\sum_{j\in N} t^g_j(R) \geq 0 = \sum_{j\in N} t^f_j(R).$$

Finally, if $x^f_j(R) = 1$ for some $j \in N\backslash\{i\}$, then $x^g_j(R) = 1$. Thus, by $p^g_j = p^f_j$,

$$\sum_{k\in N} t^g_k(R) = p^g_j = p^f_j = \sum_{k\in N} t^f_k(R).$$

[11]Note that if $x^f_j(R) = 1$ for some $j \in N$ with $i \succ^f j$, then by $x^f_i(R) = 0$ and $p^f_i = 0 < p^g_i$, $v_i(0) \leq p^f_i < p^g_i$. Thus, $x^g_i(R) = 0$. This, together with $p^g_j = p^f_j$ for each $j \in N\backslash\{i\}$, enables a priority rule g to satisfy the last condition.

Thus, for each $R \in \mathcal{R}^n$,

$$\sum_{j \in N} t_j^g(R) \geq \sum_{j \in N} t_j^f(R).$$

By richness, we can find $R \in \mathcal{R}^n$ such that $v_i(0) > p_i^g$ and $v_j(0) \leq p_j^f = p_g^f$ for each $j \in N \backslash \{i\}$. By the definition of a priority rule, $f_i(R) = (1, 0)$, $g_i(R) = (1, p_i^g)$, and $f_j(R) = g_j(R) = (0, 0)$ for each $j \in N \backslash \{i\}$. Then, by $p_i^g > 0$,

$$\sum_{j \in N} t_j^g(R) = p_i^g > 0 = \sum_{j \in N} t_j^f(R).$$

Thus, g revenue dominates f, a contradiction.

Step 3. Let $R \in \mathcal{R}^n$ be such that $v_i(0) \leq p_i^f$ for each $i \in N$, and $v_j(0) = p_j^f$ for some $j \in N$. We show that $x_i^f(R) = 1$ for $i \in N$ such that $v_i(0) = p_i^f \geq p_j^f$ for each $j \in N$ with $v_j(0) = p_j^f$. Suppose not. Note that by the definition of a priority rule and Step 2, for each $i \in N$, if $x_i^f(R) = 1$, then $v_i(0) = p_i^f = p_i^g$. Then, there are two cases.

Case 1. For each $i \in N$, $x_i^f(R) = 0$.

Let g be a priority rule such that $N_+^g = N$, $\succ^g = \succ^f$, $p_i^g = p_i^f$ for each $i \in N$, for each $R' \in \mathcal{R}^n \backslash \{R\}$, $g(R') = f(R')$, and $g_i(R) = (1, p_i^g)$ for some $i \in N$ such that $v_i(0) = p_i^g$. By Theorem 1, it satisfies the properties. For each $R' \in \mathcal{R}^n \backslash \{R\}$, by $g(R') = f(R')$, $\sum_{j \in N} t_j^g(R') = \sum_{j \in N} t_j^f(R')$. By Step 2, $p_i^f > 0$. Thus, by $p_i^g = p_i^f$,

$$\sum_{j \in N} t_j^g(R) = p_i^g = p_i^f > 0 = \sum_{j \in N} t_j^f(R).$$

Thus, g revenue dominates f, a contradiction.

Case 2. There is $i \in N$ such that $x_i^f(R) = 1$, and for some $j \in N$ with $v_j(0) = p_j^f$, $p_j^f > p_i^f$.

Let g be a priority rule such that $N_+^g = N$, $\succ^g = \succ^f$, $p_k^g = p_k^f$ for each $k \in N$, for each $R' \in \mathcal{R}^n \backslash \{R\}$, $g(R') = f(R')$, and $g_j(R) = (1, p_j^g)$.[12] By Theorem 1, it satisfies the properties. For each $R' \in \mathcal{R}^n \backslash \{R\}$, by $g(R') = f(R')$, $\sum_{k \in N} t_k^g(R') = \sum_{k \in N} t_k^f(R')$. Further, by $p_j^g = p_j^f$ and $p_j^f > p_i^f$,

$$\sum_{k \in N} t_k^g(R) = p_j^g = p_j^f > p_i^f = \sum_{k \in N} t_k^f(R).$$

[12] Note that by $v_i(0) = p_i^g$, a priority rule g can satisfy the last condition.

$$\succ^f: i_1, \ldots, i_{k_1}, i, i'_1, \ldots, i'_{k_2}, j, i''_1, \ldots, i''_{k_3}$$

$$\succ^g: i_1, \ldots, i_{k_1}, j, i, i'_1, \ldots, i'_{k_2}, i''_1, \ldots, i''_{k_3}$$

Figure 1: An illustration of the priorities $\succ^f$ and $\succ^g$.

Thus, g revenue dominates f, a contradiction.

STEP 4. Finally, we show that for each pair $i, j \in N$, if $i \succ^f j$, then $p_i^f \geq p_j^f$. Suppose by contradiction that there is a pair $i, j \in N$ such that $i \succ^f j$ and $p_i^f < p_j^f$. W.l.o.g., assume that for each $k \in N$ with $i \succ^f k \succ^f j$, $p_k^f \leq p_i^f$.

Let g be a priority rule such that

- $N_+^g = N$.
- $p_k^g = p_k^f$ for each $k \in N$.
- $\succ^g$ is a priority over N that follows the priority $\succ^f$ except that j is the immediate predecessor of i according to $\succ^g$ (see Figure 1 above). Formally, $\succ^g$ is a priority over N such that (i) $j \succ^g i$, (ii) there is no $k \in N$ such that $j \succ^g k \succ^g i$, and for each pair $k, l \in N \backslash \{i, j\}$, (iii) $k \succ^g l \succ^g j$ if and only if $k \succ^f l \succ^f i$, and (iv) $i \succ^g k \succ^g l$ if and only if $i \succ^f k \succ^f l$.
- For each $R \in \mathcal{R}^n$ and each $k \in N$ with $k \succ^f i$, if $x_k^f(R) = 1$, then $x_k^g(R) = 1$.
- For each $R \in \mathcal{R}^n$ and each $k \in N$ with $i \succeq^f k \succ^f j$, if $x_k^f(R) = 1$ and $v_j(0) \geq p_j^f$, then $x_j^g(R) = 1$.[13]
- For each $R \in \mathcal{R}^n$ and each $k \in N$ with $i \succeq^f k \succ^f j$, if $x_k^f(R) = 1$ and $v_j(0) < p_j^f$, then $x_k^g(R) = 1$.
- For each $R \in \mathcal{R}^n$ and each $k \in N$ with $j \succ^f k$, if $x_k^f(R) = 1$, then $x_k^g(R) = 1$.

By Theorem 1, g satisfies the properties.

Let $R \in \mathcal{R}^n$. We show $\sum_{k \in N} t_k^g(R) \geq \sum_{k \in N} t_k^f(R)$. If there is no $k \in N$ such that $x_k^f(R) = 1$, then $f_k(R) = (0, 0)$ for each $k \in N$, and so

$$\sum_{k \in N} t_k^g(R) \geq 0 = \sum_{k \in N} t_k^f(R).$$

[13]For a given pair $i, j \in N_+^f$ and a priority $\succ^f$ over N_+^f, $i \succeq^f j$ if and only if $i = j$ or $i \succ^f j$.

Suppose $x_k^f(R) = 1$ for some $k \in N$. If $k \succ^f i$, $j \succ^f k$, or $i \succeq^f k \succ^f j$ and $v_j(0) < p_j^f$, then $g(R) = f(R)$, and so

$$\sum_{l \in N} t_l^g(R) = \sum_{l \in N} t_l^f(R).$$

Thus, suppose $i \succeq^f k \succ^f j$ and $v_j(0) \geq p_j^f$. Then, $x_j^g(R) = 1$. Recall that $p_l^f \leq p_i^f$ for each $l \in N$ with $i \succ^f l \succ^f j$. Thus, by $p_j^f > p_i^f$, $p_j^f > p_l^f$ for each $l \in N$ with $i \succeq^f l \succ^f j$. Thus, by $i \succeq^f k \succ^f j$, $p_j^f > p_k^f$. Note that by $x_j^g(R) = 1$, $g_l(R) = (0,0)$ for each $l \in N \backslash \{j\}$, and by $x_k^f(R) = 1$, $f_l(R) = (0,0)$ for each $l \in N \backslash \{k\}$. Thus, by $p_j^g = p_j^f$ and $p_j^f > p_k^f$,

$$\sum_{l \in N} t_l^g(R) = p_j^g = p_j^f > p_k^f = \sum_{l \in N} t_l^f(R).$$

By richness, we can choose $R \in \mathcal{R}^n$ such that $v_i(0) > p_i^f$, $v_j(0) \geq p_j^f$, and for each $k \in N \backslash \{i,j\}$, $v_k(0) \leq p_k^f = p_k^g$. Then, by the definition of a priority rule and $i \succ^f j$, $x_i^f(R) = 1$. Thus, by $v_j(0) \geq p_j^f$, the last case of the above discussion applies, and we obtain

$$\sum_{k \in N} t_k^g(R) > \sum_{k \in N} t_k^f(R).$$

Thus, g revenue dominates f, a contradiction. ■

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