ISSN (Print) 0473-453X
ISSN (Online) 2435-0982

Discussion Paper No. 1212

Non-Exponential Growth Theory

Ryo Horii

September 2023

The Institute of Social and Economic Research
Osaka University
6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan

Non-Exponential Growth Theory*

Ryo Horii† (Osaka University)

September 2023

Abstract

The per capita real GDP growth rate has been remarkably stable for many decades in most developed countries. To explain the balanced growth, however, existing endogenous growth theories typically need to assume a knife-edge degree of externality, which is not yet confirmed by micro-level observations. We argue that this puzzle occurs because sustained growth has been commonly understood as exponential growth either in the quantity, quality, or variety of outputs, which is hard to explain without strong assumptions. By explicitly considering the movements of price and quantity of individual goods after their introduction, this paper shows that the observed stability of the real GDP growth rate can be explained under much weaker conditions without relying on the exponential growth of any variable. In particular, we develop an endogenous growth theory where a constant number (not exponentially many) of new goods are introduced per unit of time. Even without externality, a constant GDP growth rate is maintained when the expenditure for older goods shrinks over time so as not to inhibit the expenditure share given on newer goods.

Keywords: endogenous growth theory, knife-edge condition, externality, variety expansion. **JEL Classification Codes:** O41, O31

---

*The author is grateful to Daron Acemoglu, Koske Aoki, Nick Bloom, Angus Chu, Guido Cozzi, Koichi Futagami, Oded Galor, Peter Howitt, Chad Jones, Stelios Michalopoulos, David Weil, Hiroshi Yoshikawa, and seminar participants at Aix-Marseille School of Economics (Greqam), Brown University, GRIPS, Hitotsubashi University, University of Kent, Kobe University, Kyoto University, University of London, University of Macau, Nagoya City University, Osaka University, and Tokyo University for their valuable comments and suggestions. This study is financially supported by the JSPS Grant-in-Aid for Scientific Research (20H01477, 20H05633, 20H05631), the Behavioral Economics Research Center at ISER, the Fulbright Scholar Program (CIES), and the Abe Fellowship (SSRC) Needless to say, all remaining errors are my own.

†Correspondence: ISER, Osaka University, 6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan. E-mail: horii@econ.jpn.org

(a) Log per Capita Real GDP, USA (b) Per Capita Real GDP, Linear Scale

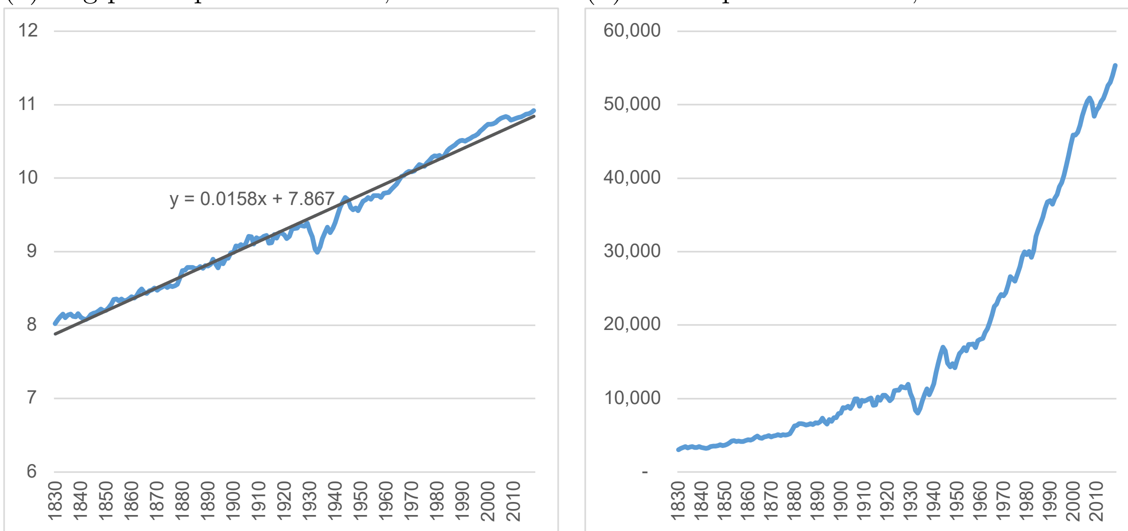


Figure 1: Long-term Evolution of Real GDP Per Capita in the United States from 1830 to 2018 (2011 International dollar). Source: Madison Project, Bolt and van Zanden (2020).

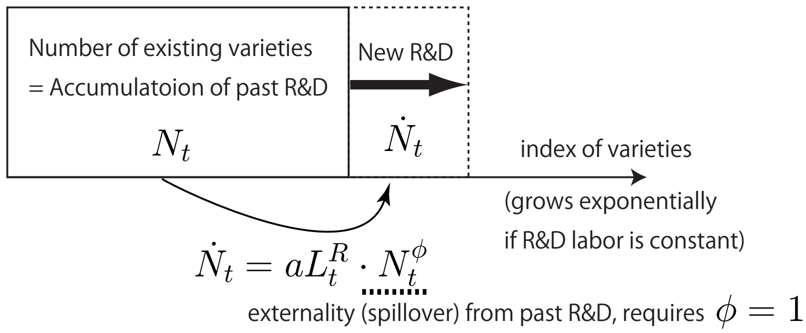
1 Introduction

Since around the time when the First Industrial Revolution was completed, the real GDP per capita growth in the United States has been surprisingly stable. Figure 1(a) depicts its time series on a log scale, where the slope of the series represents the growth rate. Although there were short- to mid-term fluctuations, the figure clearly shows the log real GDP per capita closely follows a linear trend, which implies the long-term rate of per capita GDP growth is almost constant. Figure 1(b) shows the time path of the U.S. real GDP per capita on a linear scale without taking logs. Given that the GDP growth rate is stable, the level of the real GDP per capita is increasing exponentially in the long run.

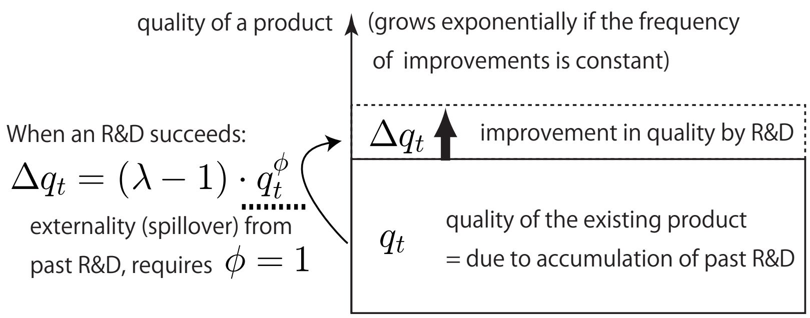
Given these findings, it was natural for existing studies on endogenous growth to explain the phenomenon of long-term growth by models in which the per capita output continues to grow exponentially. Initially, this was not an easy task because it was commonly understood that reproducible inputs are subject to diminishing returns, which implies that the accumulation of those factors cannot explain the exponential growth by themselves. The seminal studies in the endogenous growth theory thus overcame it by assuming strong intertemporal knowledge spillovers.

Figure 2 graphically explains the three representative formulations in endogenous growth

(a) Variety Expansion Models: e.g. Romer (1990), Grossman and Helpman (1991a)



(b) Quality Ladder models: e.g. Grossman-Helpman (1991b), Aghion and Howitt (1992)



(c) AK-type growth models: e.g. Romer (1986), Rebelo (1991)

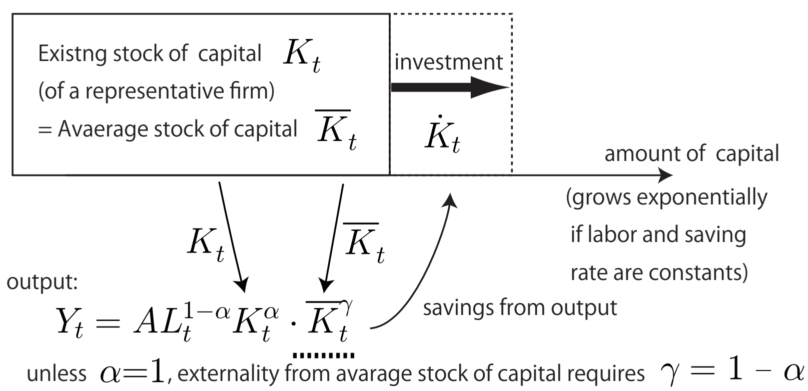


Figure 2: The Exact Degree of Externality Required in Three Types of Endogenous Growth Models

theory and their specific assumptions. In variety-expansion models, as illustrated in panel (a), there should exist a strong knowledge externality from past R&Ds to a new R&D, and the elasticity of this spillover ϕ need to equal exactly one. Similarly, in quality ladder models (b), the increment in the quality by a successful R&D depends on the quality of the existing good, which is a result of the past stock of R&Ds, and the elasticity of this relation should again be exactly one. Finally, in AK-type growth models (c), the elasticity of production with respect to all reproducible factors and the elasticity of their externality effects must add up exactly to one.[1] In almost all existing endogenous growth models, long-term growth can be sustained only when one of such knife-edge conditions is satisfied.[2]

Now, here is a puzzle. Surely, the externality and non-rivalry of knowledge play important roles in improving productivity (see, e.g., Griliches 1998). However, if we look at the process of spillover more precisely, to my knowledge, there is no concrete evidence that supports any of these exact assumptions. As for the elasticity of spillover ϕ in R&D-driven growth models, Jones (1995) clearly stated, *"$\phi = 1$ represents a completely arbitrary degree of increasing returns and, ... is inconsistent with a broad range of time-series data on R&D and TFP growth."* He convincingly stated that $\phi = 0$ is the most natural case, and while ϕ can either be negative by the "fishing out effect" or positive by the "better tools effect," it is reasonable to assume $\phi < 1$.[3] Bloom et al. (2020) found that it is getting increasingly

[1]When there are multiple sectors, a sector that produces a reproducible factor (typically physical or human capital) needs to satisfy this restriction. For example, Lucas (1998) initially introduced a human capital accumulation function $\dot{h}_t = h_t^{\zeta} G(1 - u_t)$ and then made an assumption of $\zeta = 1$ following Uzawa (1965). By so doing, he wrote "the feature that recommends his formulation to us, is that it exhibits sustained per-capita income growth," which gives a clear example of a case where such a knife-edge assumption is justified not by micro-level observations but by the aggregate outcome. Lucas also noted that "human capital accumulation is a *social* activity," which suggests that the elasticity $\zeta = 1$ includes the effect from externality.

[2]Growiec (2007, 2009) formally proved that, with any generalization in the functional forms, exponential growth cannot be explained without imposing at least one knife-edge assumption in the model.

[3]By assuming $\phi < 1$, Jones (1995) developed the *semi-endogenous* growth theory, where the long-term rate of growth is ultimately driven by population growth. By extending the theory to include the transitional increase in research intensity and educational attainments, Jones (2002) showed that it is possible to explain the observed *constant growth path* for a certain period of time. In his theory, future growth is predicted to slow down permanently and eventually will come to an end, given that there are upper limits in population, research intensity, and education attainments. Under a natural assumption of $\phi = 0$, this paper tries to present an alternative interpretation by developing a *full*-endogenous growth theory where the measured

harder to improve exponentially the quality of goods.[4] Klenow and Rodriguez-Clare (2005, Section 3) reviewed various AK-type models. They concluded that such models are empirically implausible based on the lack of a tight enough relationship between the investment rates and growth rates in cross-country data.

Why have the U.S. and other successful countries been able to grow quite steadily, even though existing theories of endogenous growth imply sustained growth is possible only under knife-edge conditions that seem very hard to be justified by data? This paper shows that this inconsistency occurs because we have been interpreting the observed sustained growth as an exponential process. If we consider a single final output (a scalar variable), sustained growth always means exponential growth. This is true even in most of the existing R&D-based growth models with many varieties, as they measure the growth rate after aggregating the varieties into a single final output.[5]

However, the real GDP growth rate in SNA statistics (e.g., the NIPA in the U.S.) is measured without aggregating all outputs into one variable. It is obtained by comparing quantities of various product groups in adjacent years, using the same set of prices for both years. Then, the aggregate level of real GDP is constructed by the chain rule. Conceptually, the SNA statistics construct the level of real GDP at a given year t from the measured real GDP growth rate g_t as follows:

$$\Big[\text{Real GDP at year } t\Big] = \Big[\text{Real GDP at reference year } \bar{t}\Big] \times \exp\left(\int_{\bar{t}}^{t} g_\tau d\tau\right),$$

where $\bar{t}$ is the reference year, which can be chosen freely. Real GDP at reference year $\bar{t}$ can also be set arbitrarily (because this is just an index), but for ease of interpretation, it is usually set to the nominal GDP in year $\bar{t}$. Therefore, even when we see the exponential growth in the time series of real GDP, it only says that the g_t is stationary over time.

---

economic growth can continue indefinitely with constant population.

[4]They reported that the number of researchers required today to achieve the famous doubling of computer chip density is more than 18 times larger than the number required in the early 1970s. This is not consistent with the typical quality ladder specification as illustrated in 2(b).

[5]A notable exception is Young (1991), who considered a learning-by-doing (LBD) model with many goods and calculated the growth rate before aggregation. In addition, he recognized that there is an upper bound in productivity gain from LBD for individual good. To obtain a constant rate of growth, however, he needed to assume that the upper bound itself improves exponentially through the aggregate LBD. Horii (2011) used the same disaggregated definition of growth rate, but it should be classified as a semi-endogenous growth model because it relied on an exponential increase in population (e.g., Jones 1995).

Focusing on this fact, this paper solves the puzzle by showing that steady state growth (constancy in the measured per capita real GDP growth rate) does not necessarily imply an exponential increase either in the quantity, quality, or variety of products. Because there is no need to explain the exponential increase in any variable, and we do not need to make knife-edge assumptions to support steady-state growth.

This paper shows that the measured GDP growth rate becomes a positive constant when (i) a constant number of new goods are developed per unit of time, (ii) the quality-adjusted price of each new good falls after introduction, and (iii) the expenditure share for the very old goods is limited. Condition (i) says that it is sufficient for the number of goods to increase linearly over time rather than exponentially, whereas conditions (ii) and (iii) state that the price and demand for each good should follow the well-observed pattern of product lifecycle. In the long run, it is less counter-intuitive for us to expect this type of economic movement to continue rather than to expect the output quantity, quality, or variety to be expanded by astronomical orders. Of course, knowledge externalities are nevertheless important for growth as they often work behind the quality improvements and cost reductions of existing goods. Still, we show shown that the fall in quality-adjusted prices needs not to occur at an exponential speed. A weaker externality is sufficient for sustaining growth.

The rest of the paper is constructed as follows. Section 2 presents a theory of sustained growth without exponential expansion. By explicitly focusing on the product lifecycle, we explain that the conventionally measured real GDP growth rate becomes asymptotically constant even when no variable grows exponentially. Section 3 develops a prototype endogenous growth model without knife-edge assumptions and examines how the long-term rate of economic growth is determined in equilibrium. Section 4 generalizes the theory and the prototype model in several directions so as to demonstrate that we can obtain a positive constant real GDP growth rate in wider (even less restrictive) situations. Section 5 concludes.

2 Theory

This section presents a theory that demonstrates the measured real GDP growth is sustained even when no underlying economic variable grows exponentially. More specifically, Subsection 2.1 discusses the evolution of the prices and outputs of individual goods in a setting where the number of varieties expands linearly over time rather than exponentially. Subsection 2.2 explains the definition of the real GDP growth rate when the quantity and price of goods change individually over their product life cycles. After deriving the real GDP growth rate in the steady state in Subsection 2.3, Subsection 2.4 offers graphical examples that illustrate the connection between the pattern of the product lifecycle and the measured real GDP growth rate. To retain the flexibility of the main result, this section does not specify the full model structure. It will be shown in Section 3 that such dynamics can be obtained as an equilibrium outcome in a general equilibrium model.

2.1 Steady-State Growth Dynamics with Product Life Cycle

Let us consider an economy with a constant population and many goods.[6] We follow a convention in the variety-expansion model by calling them goods, but it is more suitable to think of each good in the theory as a group of products that are based on the same technology. Each good is indexed by $i \in [0, N_t]$, where $i = 0$ is the oldest while $i = N_t$ is the most recently introduced good. Suppose that the number of goods N_t increases through R&D, and in the long run, it increases by a positive constant n per unit of time:

$$\dot{N}_t \to n > 0 \quad \text{as} \quad t \to \infty. \tag{1}$$

Recall that, as illustrated in Figure 2(a), existing variety-expansion models required a strong and exact degree of knowledge spillovers to maintain exponential expansion of varieties, where $\dot{N}_t / N_t$ is constant. In contrast, the linear increase of N_t in (1) does not require such strong knowledge spillovers within the R&D sector, as we will see in a general equilibrium model in Section 3.

However, because (1) implies $\dot{N}_t / N_t \to 0$, it is clear that the introduction of new goods alone cannot explain the sustained growth. Therefore, we consider the changes in the prices and outputs of individual goods after their introduction explicitly. Let $\widetilde{p}_t(i)$ and $\widetilde{x}_t(i)$ denote the price and quantity of each good i at time t and allow them to change over time. As in the SNA statistic, we define $\widetilde{p}_t(i)$ and $\widetilde{x}_t(i)$ as quality-adjusted. For example, if the quality of good i is doubled (so that consumers receive the same utility from half the quantity), then our measure of $\widetilde{x}_t(i)$ is doubled, while $\widetilde{p}_t(i)$ is halved.

[6] The assumption of constant population is relaxed in Subsection 4.2.

Let $s(i)$ denote the time when good i is developed. It is convenient to label each good by its age, $\tau \equiv t - s(i)$, i.e., time passed from its introduction. In the long run, where n new goods are introduced per unit of time, age τ good refers to the $n\tau$th newest good. This means that the index of a good i and its age τ have the following relationship.

$$i = N_t - n\tau, \text{ or equivalently, } \tau \equiv t - s(i) = \frac{N_t - i}{n}. \tag{2}$$

With this notation, let us say that the economy has come to a steady state if the price and quantity of every good follow the same time evolution against τ. Formally, the economy can be said to be converging to a steady state if there exist time-invariant functions $p(\tau)$ and $x(\tau)$ such that

$$\widetilde{p}_t(i) \to p(t - s(i)) \equiv p(\tau), \quad \widetilde{x}_t(i) \to x(t - s(i)) \equiv x(\tau) \quad \text{as} \quad t \to \infty. \tag{3}$$

Let $T > 0$ denote the age beyond which the product is never produced. In typical variety-expansion endogenous growth models, goods never retire from the market. In this case, $T = \infty$. However, in practice, we see many products disappear after some time. Our theory can be applied to both cases where T is finite or infinite.

We normalize the price level at each instant so that the nominal expenditure in the steady state is constant. We assume $p(\tau)$ and $x(\tau)$ satisfy the following conditions.

Assumption 1

(i) Both $p(\tau)$ and $x(\tau)$ are non-negative and differentiable for all $0 < \tau < T$, where T is such that $x(\tau) = 0$ for all $\tau \geq T$.

(ii) T can be infinite, but $x(\tau)$ does not grow exponentially: $\lim_{\tau \to \infty} x'(\tau)/x(\tau) \leq 0$.[7]

(iii) The newest good's price and quantity are strictly positive: $p(0) > 0$ *and* $x(0) > 0$.

Assumption 1(i) will simplify the exposition in this section. In principle, the theory can accommodate the cases where the price and output are discontinuous,[8] but the present paper focuses on the continuous setting because it is mathematically less demanding and does not sacrifice intuitions. Since $x(\tau)$ represents the quality-adjusted quantity, Assumption 1(ii), combined with (1), guarantees that neither quantity, quality, or variety grows

[7] Note that the time derivative of quantity in the steady state is $\dot{\widetilde{x}}_t(i) = \frac{d}{dt}x(t - s(i)) = x'(t - s(i)) = x'(\tau)$. Therefore, $x'(\tau)/x(\tau) = \dot{\widetilde{x}}_t(i)/\widetilde{x}_t(i)$ represents the growth rate of the quantity of age τ good, or equivalently, that of index $i = N_t - n\tau$ good.

[8] See the discussion in footnote 13.

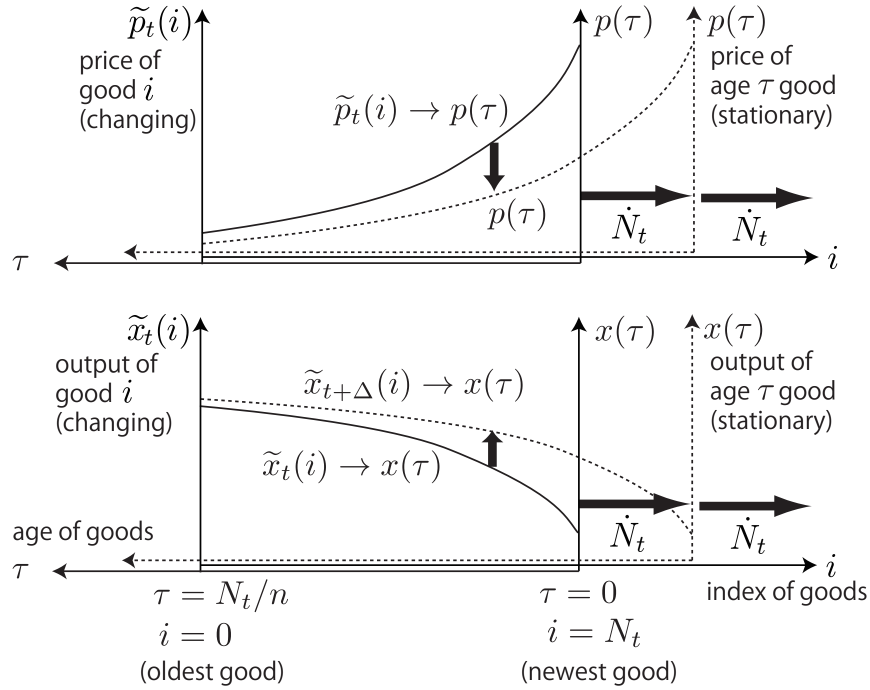


Figure 3: Evolution of Prices and Outputs of Goods in a Non-Exponential Steady-State.

exponentially in this economy. Assumption 1(iii) is an obvious one. When we say a new good appears in the market, it should imply that the expenditure for the good, $p(0)x(0)$, is positive.

Definition 1 *A non-exponential asymptotic steady-state is the situation wherein the paths of quality-adjusted prices and outputs of goods, $\widetilde{p}_t(i)$ and $\widetilde{x}_t(i)$, satisfy condition (3) and Assumption 1 as $T \to \infty$.*

In the remainder of the paper, we simply call it a steady state unless it is confusing.

Figure 3 intuitively explains the evolution of the quality-adjusted prices and outputs in the above definition of the steady state. The graphs can be viewed two ways: either drawn against the i-axis (index of goods) running from left to right, or drawn against the τ-axis (age of goods) running in the opposite direction. The two variables, i and τ, are related according to (2), but the relationship changes over time as N_t increases. At time t, the origin of the τ-axis coincides with the point of $i = N_t$ on the i-axis because the newest good $i = N_t$ is age $\tau = 0$ at time t. As time passes, the origin of the τ-axis moves to the right with the speed of the introduction of new goods, $\dot{N}_t = n$, and so does the position of the origin of the graph drawn against τ.

The upper panel of figure 3 illustrates a case where quality-adjusted price $p(\tau)$ is decreasing in age τ, either because a product becomes cheaper or has higher quality as it ages after its introduction. Since the newer goods have larger index i, it also means $\widetilde{p}_t(i)$ is increasing in i at any given time t. The figure also explains the movement of the price of each good $\widetilde{p}_t(i)$ over time. Even in the steady state where function $p(\tau)$ is stationary, the price of individual good $\widetilde{p}_t(i)$ shifts downward to the dotted curve because the position of function $p(\tau)$ continues to move to the right as new goods are developed.[9]

Similarly, the lower panel of Figure 3 explains the evolution of quality-adjusted outputs of goods over time. The panel shows the case where $x(\tau)$ is increasing in τ, which naturally matches our example that the older goods have a lower quality-adjusted price. In this case, the demand for each good $\widetilde{x}_t(i)$ increases over time as the $x(\tau)$ function shifts to the right. Note that, however, Assumption 1(ii) rules out exponential growth in the quantity of any good. Even when $T = \infty$, the growth rate of $x(\tau)$ must be either zero or negative as $\tau \to \infty$. Contrary to this particular example, we can also consider the possibility for the quantity to shrink with age, even when older goods are cheaper. Such a pattern will emerge when consumers do not like outdated goods or if newer goods replace parts of functions that are provided by older goods.

2.2 Measuring the GDP Growth Rate

When considering the growth rate in a model with many goods, an often employed practice is to consider an aggregation function to obtain a measure of the final output (a scalar variable) and then measure its growth rate. A problem with this methodology is that the resulting growth rate depends on the choice of the aggregation function. Because the final output is a virtual notion, there is no guarantee that the growth rate obtained in this way matches the numbers in the official statistics. Another way is to select a numeraire good, aggregate the outputs of various goods with the observed relative prices, and then calculate the growth rate of aggregated output. However, there is again a problem of

[9] Although this is a convenient way to explain the steady-state dynamics, note that the economic environment, such as technology, preference, and market structure, first determines the evolution of the price of individual goods $\widetilde{p}_t(i)$ in equilibrium. Then, the long-term pattern of movement in $\widetilde{p}_t(i)$ shapes the stationary $p(\tau)$ function as a result.

choosing the numeraire because the obtained growth rate depends on this choice since the relative price changes over time. Another substantial problem is that there is no good that is representative enough to be the numeraire for the long time because the expenditure shares across goods are always changing.

To avoid these problems, this paper directly looks at the changes in the prices and quantities of all goods without aggregating them into a single variable. We follow the standard procedure to construct the GDP growth rate that is often explained in undergraduate-level macroeconomics textbooks. In the SNA statistics, the real GDP growth rate is measured by comparing the value of outputs between two consecutive years, say year $t-1$ and year t. Their values are measured using the common set of prices, which usually is the set of observed prices in a given base year. Because the base year is frequently updated in official statistics and also because this paper is interested in long-term dynamics, we suppose that there is no gap between the base year and the year in which the growth rate is computed.[10] Then, using the notation from the previous subsection, the real GDP growth rate between years $t-1$ and t can be written as follows.[11]

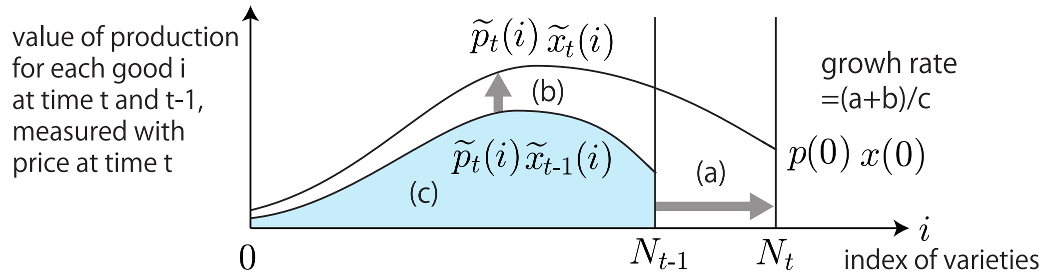
$$g_{t-1,t} = \frac{\int_{N_{t-1}}^{N_t} \widetilde{p}_t(i)\widetilde{x}_t(i)di + \int_0^{N_{t-1}} \widetilde{p}_t(i)\left(\widetilde{x}_t(i) - \widetilde{x}_{t-1}(i)\right) di}{\int_0^{N_t} \widetilde{p}_t(i)\widetilde{x}_{t-1}(i)di}. \tag{4}$$

This formula is made up from the integrals of two functions, $\widetilde{p}_t(i)\widetilde{x}_t(i)$ and $\widetilde{p}_t(i)\widetilde{x}_{t-1}(i)$. Figure 4 depicts the curves of these two functions against t, for the case where the demand for existing goods is always increasing over time (Case 1) and the case where demand for existing goods shrinks in some part of their life cycle (Case 2). In Case 1, observe that area (a) in the figure represents the sum of the values of new goods introduced between time $t-1$ and time t, evaluated by the prices at time t. Similarly, area (b) represents the value

[10]In the U.S., the NIPA computes the growth rate two ways, by setting the base year to t, and also by setting it to $t-1$. Then the agency calculates their geometric average. Here, we only calculate the growth rate in which the base year is t, but the difference disappears in the limit where the period length is brought to 0, as we do in the next subsection.

[11]To ease the understanding, here we employed a slight abuse of notation and treated $\widetilde{x}_t(i)$ as if it is a discrete-time variable. In the previous subsection, we defined $\widetilde{x}_t(i)$ as the instantaneous flow of output at time t per unit time. Since the SNA statistics use the cumulative output of good i for a given time period (e.g., a year or a quarter), we need to integrate $\widetilde{x}_t(i)$ for the duration of the time period to obtain the exact real GDP growth rate. As we take the limit where the duration of one period is almost zero, we confirmed that this exact GDP growth rate converges to the same expression as in (5).

Case 1: When $\widetilde{x}_t(i)$ is always increasing in t.



Case 2: When $\widetilde{x}_t(i)$ becomes decreasing in t sometime after introduction.

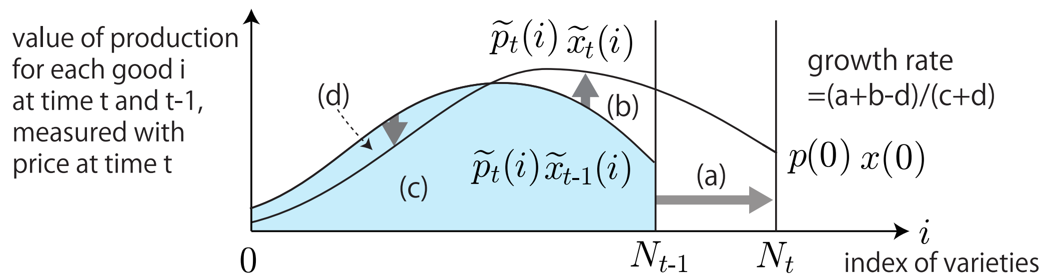


Figure 4: Calculation of the Real GDP Growth Rate: Two Cases

of the increased production of goods that already existed at time $t-1$. These two areas measure how economic activity has increased from time $t-1$ to time t and correspond to the two terms in the numerator of definition (4). Area (c) represents the value of total production in time $t-1$, evaluated again by the prices at time t. This area corresponds to the denominator of the definition (4). In this way, the real GDP growth rate can be understood as the ratio of area (a)+(b) to area (c), which measures the rate at which the economic activity at time t is increased from time $t-1$.

This procedure can be generalized to the case where output $\widetilde{x}_t(i)$ is not monotonic in t. Case 2 in Figure 3 illustrates an example where production of a certain range of goods declines between period $t-1$ and t. Then, a portion of curve $\widetilde{p}_t(i)\widetilde{x}_t(i)$ comes below curve $\widetilde{p}_t(i)\widetilde{x}_{t-1}(i)$. In this case, the real GDP growth rate is given by the ratio of area (a)+(b)−(d) to area (c)+(d).

2.3 Measured GDP growth rate in the Steady State

So far, Subsection 2.1 presented steady state dynamics with many goods where no variable grows exponentially, and Subsection 2.2 explained how GDP growth rate can be measured when there are many goods. Now we are ready to examine whether the non-exponential steady state implies a positive and constant real GDP growth rate.

Note that the conventional definition of real GDP growth in (4) gives the average growth rate between two discrete time periods. To map this definition to a continuous-time growth model, it is convenient to consider the instantaneous growth rate g_t at time t. This can be obtained by replacing $t-1$ in (4) by $t-\Delta$, and taking the limit of $\Delta \to 0$ in $g_{t-\Delta,t}/\Delta$.

$$g_t = \lim_{\Delta \to 0} \frac{g_{t-\Delta,t}}{\Delta} = \frac{\dot{N}_t \cdot \widetilde{p}_t(N_t)\widetilde{x}_t(N_t) + \int_0^{N_t} \widetilde{p}_t(i)\dot{\widetilde{x}}_t(i)di}{\int_0^{N_t} \widetilde{p}_t(i)\widetilde{x}_t(i)di}. \tag{5}$$

Suppose that the economy converges to a steady state, as defined in Definition 1. The number of goods grows linearly, and the evolution of prices and quantity in terms of age becomes stationary. Then, given that $\int_0^T p(\tau)x(\tau)d\tau$ is finite, the long-term growth rate can be obtained by substituting (1)-(3) into (5).[12]

$$g_t \to g \equiv \frac{np(0)x(0) + n\int_0^T p(\tau)x'(\tau)d\tau}{n\int_0^T p(\tau)x(\tau)d\tau} \quad \text{as} \quad t \to \infty. \tag{6}$$

The interpretation of growth rate (6) is essentially the same as in definition (4), except the fact that now growth is represented in terms of age and also in continuous time. In the numerator, $np(0)x(0)$ represents the value of newly introduced goods, whereas $n\int_0^T p(\tau)x'(\tau)d\tau$ gives the value of changes in production given price function $p(\tau)$. Both terms are multiplied by n because there are n goods per unit of age. The sum of these terms gives the speed of increase in economic activity. The denominator of (6), $n\int_0^T p(\tau)x(\tau)d\tau$, gives the value of existing production, i.e., the nominal GDP of the economy given prices $p(\tau)$. The ratio of the two gives the real GDP growth rate.

[12] Equation (6) can be obtained from (5) by changing the variable of integration from i to $\tau \equiv t - s(i)$. First, substitute $p(\tau)$ and $x(\tau)$ for $\widetilde{p}_t(i)$ and $\widetilde{x}_t(i)$. Similarly, $\dot{\widetilde{x}}_t(i)$ can be written as $x'(\tau)$ (See footnote 7). Next, we need to change the integration variable from di in (5) to $d\tau$. By differentiating (2) for given t, we obtain $di = nd\tau$. Finally, from (2), $i = 0$ and $i = N_t$ respectively correspond to $\tau = N_t/n$ and $\tau = 0$ as illustrated in Figure 3. As $t \to \infty$, N_t/n also approaches ∞. From these, $\lim_{t\to\infty} \int_0^{N_t} \widetilde{p}_t(i)\widetilde{x}_t(i)di = \lim_{t\to\infty} \int_{N_t/n}^{0} p(\tau)x(\tau)(-n)d\tau = n\int_0^\infty p(\tau)x(\tau)d\tau$. However, since $x(\tau) = 0$ for $\tau \geq T$, the latter expression becomes $n\int_0^T p(\tau)x(\tau)d\tau$. Likewise, the limit of the numerator of (5) becomes $np(0)x(0)+n\int_0^T p(\tau)x'(\tau)d\tau$.

In the steady state, the growth rate can also be interpreted in terms of the product lifecycle. By canceling out n and using integration by parts in (6), we obtain the following proposition.

Proposition 1 *Suppose that the economy converges to the asymptotic non-exponential steady-state, as defined by Definition 1. Given that*

$$-\int_0^T x(\tau)dp(\tau) \text{ is finite and strictly positive, and} \tag{7}$$

$$\int_0^T p(\tau)x(\tau)d\tau \text{ is finite,} \tag{8}$$

the real GDP growth rate asymptotes to a positive and finite constant

$$g = \frac{-\int_0^T x(\tau)dp(\tau)}{\int_0^T p(\tau)x(\tau)d\tau}. \tag{9}$$

The expression in (7), $-\int_0^T x(\tau)dp(\tau)$, appears in the numerator of (9).[13] It represents the cumulative reductions in the quality-adjusted price during the whole product lifecycle. When the quality-adjusted price of goods declines, consumers have more purchasing power, improving their utility. This income effect from price reductions is more significant when the quantity of the good is larger. Therefore, in (7), the price reduction $-dp(\tau)$ is weighted by quantity $x(\tau)$ and then integrated. This gives the total income effect that one product generates over its product lifecycle. The expression $\int_0^T p(\tau)x(\tau)d\tau$ in (8) is in the denominator of (9). It is the cumulative expenditure attracted by one product. Proposition 1 says that if every product follows the same lifecycle pattern, the real GDP growth rate in the economy is given by the ratio of the two. If both values are positive and finite in a non-exponential steady state, as defined in Definition 1, then it indicates that the real GDP growth rate can be sustained even when no variable grows exponentially. We will first provide three examples in the following subsection and then discuss conditions (7) and (8) more in detail in Subsection 2.5.

2.4 Graphical Examples

Proposition 1 shows that the real GDP growth rate depends only on functions $p(\tau)$ and $x(\tau)$. We can represent the real GDP growth rate graphically using the shapes of these

[13] When $p(\tau)$ is differentiable, $\int_0^T x(\tau)dp(\tau)$ is the same as $\int_0^T x(\tau)p'(\tau)d\tau$. A benefit of this notation is it allows the case where $p(\tau)$ is not differentiable. While we do not provide the formal proof here, Proposition 1 can be extended to this case.

two functions. Figure 5 presents three examples.

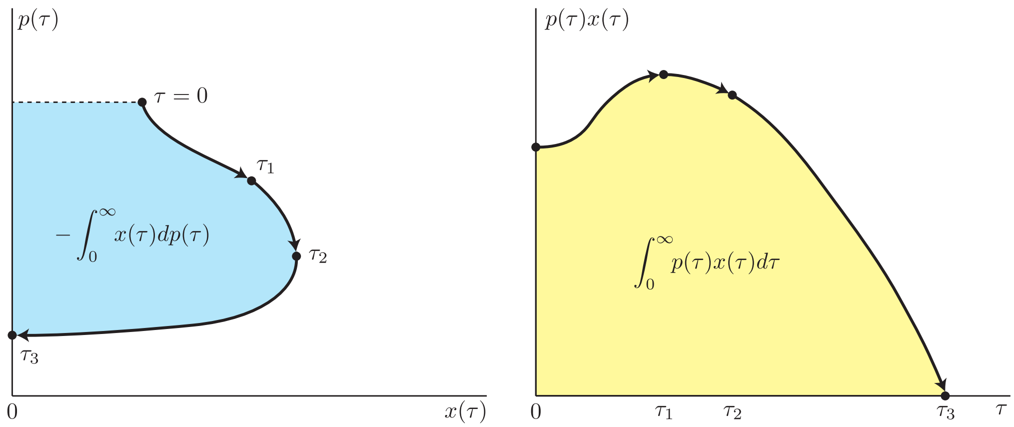
Example 1 shows the simplest case, where the quality-adjusted price weakly falls with age throughout the product life cycle. The left panel depicts the evolution of $\{x(\tau), p(\tau)\}$ in the p-x diagram. T is finite in this example. The good enters the market at the point of $\{x(0), p(0)\}$ and continues to be produced until its age reaches $T = \tau_3$. Then, the value of numerator (7) can be expressed by the area that is encompassed by the locus of $\{p(\tau), x(\tau)\}$ and the vertical axis in the p-x diagram (shown in the blue color). This can be interpreted as follows. Whenever the quality-adjusted price falls by $dp(\tau)$, either through cost reductions or through quality improvements, consumers can save the purchasing power by the amount of $-x(\tau)dp(\tau)$. The blue area shows the cumulative benefits throughout the life of this good. The area is positive and finite as long as $p(0) < p(T)$.[14]

The right panel plots the evolution of expenditure for a (representative) good against its age. The area below the curve gives the value of denominator (8) (shown in the yellow color). From Assumption 1, the expenditure for the good at the time of introduction ($p(0)x(0)$) is strictly positive, and evolves in the non-negative region during its lifetime. Since expenditure $p(\tau)x(\tau)$ falls to zero at finite $T = \tau_3$, this area is positive and finite. Proposition 1 says that the ratio of the blue area to the yellow area gives the real GDP growth rate. Therefore, we can conclude that the real GDP growth rate in this example is positive and finite.

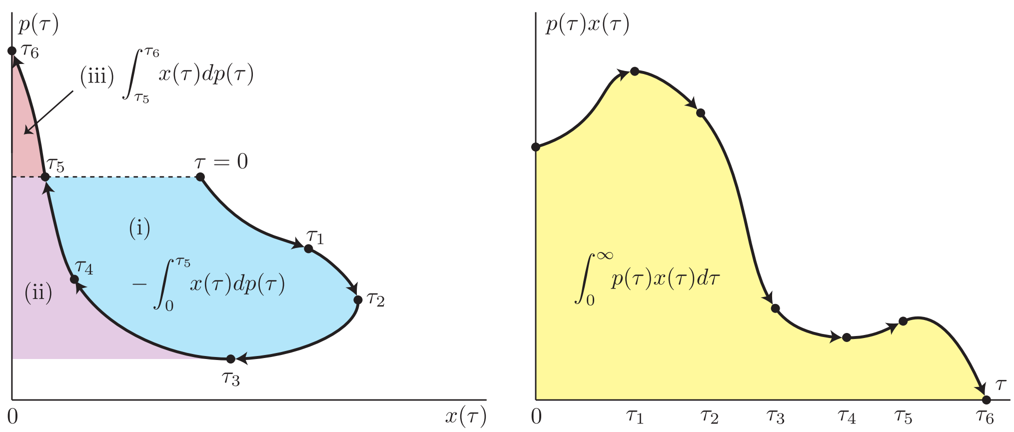
Next, Example 2 considers a case where $p(\tau)$ is not monotonic. Here, the quality-adjusted price begins to increase after τ_3 years. When the price of the good (relative to the newest good) rises in some stage of its life cycle, the area between this part of the p-x locus (from $\tau = \tau_3$ to τ_6) and the vertical axis represents the loss of the purchasing power of consumers. This area needs to be deducted from the benefits of the fall in quality-adjusted prices from $\tau = 0$ to τ_3. Therefore, the value of numerator (7) is given by area (i) minus area (iii) because area (ii) is canceled out. It can be either positive or negative but is always finite since $T = \tau_6$ is finite. Again the yellow area in the right panel gives the value of denominator (8), which is positive and finite. Therefore, the real GDP growth rate is finite, which is given by the ratio of the blue minus red area to the yellow area. Also, note that the growth rate becomes zero only by coincidence, only when the blue and red areas

[14] $p(0) < p(T)$ requires the price to fall strictly with age in some part of the good's life.

Example 1: When T is finite and $p(\tau)$ is weakly decreasing



Example 2: When T is finite and $p(\tau)$ is non-monotonic



Example 3: When $T = \infty$ and $p(\tau)$ is decreasing

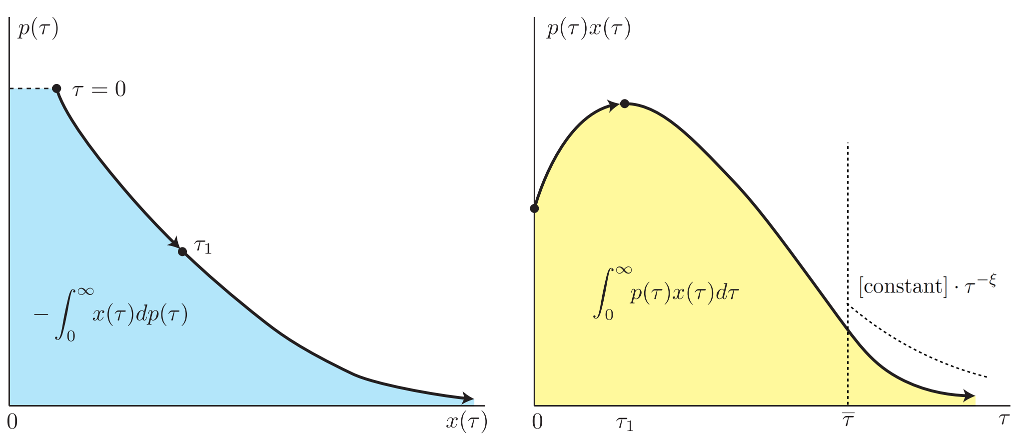


Figure 5: Graphical Representation of the Real GDP Growth Rate. The growth rate is measured by the ratio of the areas of the two panels.

are exactly the same size.[15]

Finally, Example 3 shows a case when the good stays in the market forever ($T = \infty$). The price $p(\tau)$ (relative to the newest good) falls throughout the life cycle, and the quantity $x(\tau)$ remains positive as $\tau \to \infty$. For the yellow area to be finite, the expenditure for very old goods has to shrink. More concretely, condition (8) is satisfied if expenditure for old goods is bounded by a polynomial function of age with the power of less than -1:[16]

$$p(\tau)x(\tau) \leq [\text{constant}] \cdot \tau^{-\xi} \text{ for all } \tau \geq \overline{\tau}, \tag{10}$$

for some $\xi > 1$ and $\overline{\tau} > 0$. The dotted curve in the right panel gives an example of such an upper bound. While we need a concrete model to see whether condition (10) is satisfied, let us note that the condition does not require an exponential fall in expenditure. The RHS of (10) falls with age at the rate of ξ/τ for $\tau > \overline{\tau}$, which can be arbitrarily close to zero when we choose a large $\overline{\tau}$. Therefore, there is no minimum rate at which the expenditure needs to fall.

Given that the yellow area is finite, a sufficient condition for the finiteness of the blue area is that the rate of change in the quality-adjusted price with respect to age is bounded, at least for older goods:[17]

$$\left| \frac{p'(\tau)}{p(\tau)} \right| < \overline{\gamma}_p \text{ for all } \tau \geq \widetilde{\tau}, \tag{11}$$

for some $\overline{\gamma}_p > 0$ and $\widetilde{\tau} > 0$. These two inequality conditions, (10) and (11), jointly guarantee that the GDP growth rate is positive and finite.

2.5 Discussion: Two Conditions for "Sustained Growth"

The previous three examples illustrated that the measured real GDP growth rate in the steady state becomes positive and finite in various scenarios. Here, we discuss the two

[15] Although Proposition 1 focuses on the case when the value of numerator (7) is positive, formula (9) holds even when (7) is negative.

[16] Suppose (10) is satisfied. Then, the denominator of (9) is $\int_0^\infty p(\tau)x(\tau)d\tau \leq \int_0^{\overline{\tau}} p(\tau)x(\tau)d\tau + \int_{\overline{\tau}}^\infty [\text{constant}] \cdot \tau^{\xi} d\tau$. The first term is finite, and the second term becomes $[\text{constant}] \cdot \overline{\tau}^{1-\xi}/(\xi - 1)$, which is also finite.

[17] Suppose that (11) is satisfied and $\int_0^\infty p(\tau)x(\tau)d\tau$ (the yellow area) is finite. Then, the numerator of (9) is $-\int_0^\infty x(\tau)dp(\tau) = -\int_0^{\widetilde{\tau}} x(\tau)dp(\tau) - \int_{\widetilde{\tau}}^\infty x(\tau)dp(\tau)$. The first term is finite. The absolute value of the second term satisfies $\left|\int_{\widetilde{\tau}}^\infty x(\tau)dp(\tau)\right| = \int_{\widetilde{\tau}}^\infty x(\tau)p(\tau)\left|p'(\tau)/p(\tau)\right| d\tau \leq \overline{\gamma}_p \int_{\widetilde{\tau}}^\infty p(\tau)x(\tau)d\tau$. Here, $\int_{\widetilde{\tau}}^\infty p(\tau)x(\tau)d\tau$ is finite since we assumed $\int_0^\infty p(\tau)x(\tau)d\tau$ is finite.

required conditions, (7) and (8), in Proposition 1 more generally.

Condition (7): quality-adjusted price falls during its product lifecycle

For this condition to be satisfied, $p(\tau)$ needs to fall with τ at least for a portion of its product lifecycle. Recall that we normalized the price level so that the nominal expenditure in the steady state is constant. This normalization also implies the price of the newest good is always the same. Therefore, condition (7) just requires the quality-adjusted prices of older goods to become lower relative to newer goods, and does not imply prices of individual goods measured in a currency to fall.[18]

With this definition, the quality-adjusted price of a good may fall with age for a number of reasons. For example, the cost of production falls through learning-by-doing and knowledge spillovers. In this case, time and production experiences will contribute to price reduction. Apart from cost reduction, changes in the form of competition may also lower prices because older goods are typically less protected from competition by patents and trade secrets than newer goods.

Price reductions also occur in the form of quality improvements. For example, the effective price of computers has been declining for decades, not only because the price of an average computer has become cheaper but also because the average performance of computers has drastically improved. The SNA statistics interpret such changes as a decline in the quality-adjusted price.

It is worth noting that our theory does not require an exponential fall in the quality-adjusted price. If the quality improvements are exponential, economic growth can easily be maintained as in usual quality-ladder models (See panel (b) of Figure 2). The quality of computers has been improving at a constant rate according to "Moore's Law," but this trend of exponential improvement is expected to slow down. In fact, computers are a remarkable exception in terms of continued improvements in performance. Most other products experience tapering in the rate of productivity improvements as they get mature. Our theory shows that slowdowns in productivity increases in individual goods are fully consistent with the positive rate of GDP growth, as long as a constant number of new

[18]Suppose that the nominal GDP growth rate in dollars (not in our price normalization) is $g^{\$}$. Then, the rate of price change of age τ good in dollars is $p'(\tau)/p(\tau) + g^{\$}$. If $g^{\$}$ is large, prices of individual goods will rise even when $p(\tau)$ is a decreasing function of τ.

products are introduced per unit time.

Lastly, let us discuss the case when the quality-adjusted price of the good rises for some part of its lifecycle, as we discussed in Example 2 of Figure 5. Although we need a concrete model to analyze whether condition (7) is satisfied, here we present two possibilities. One possibility is when products have antique or scarce value as they become very old. In this scenario, $p(\tau)$ will increase only when $x(\tau)$ has become considerably smaller than when they were newer. Another possibility is when it costs more to produce a good in small lots. It happens, for example, when a particular good continues to be produced to meet a niche demand, typically near the end of the product life cycle.

The value of numerator (7) is the weighted sum of the price changes, $dp(\tau)$, where the weights are the quantities, $x(\tau)$. Therefore, if the quantity $x(\tau)$ tends to be small when $p(\tau)$ increases, the negative effect of such movements on the GDP growth rate is likely to be limited. Therefore, even when the price at the end of the lifecycle $p(T)$ is higher than the initial price $p(0)$, the lifetime contribution of this representative good on the real GDP growth rate may well be positive, as in the case of Example 2.

Condition (8): The cumulative expenditure for the representative good is finite

This condition requires the expenditure for older goods $p(\tau)x(\tau)$ to fall so that they effectively retire the market in terms of expenditure share. It is always satisfied if the representative good ceases to be produced at finite age T. Even when the good stays in the market forever, the condition is satisfied if the expenditure falls with age reasonably fast (condition 10). Notably, the speed does not need to be exponential.

The expenditure for the good can shrink with age for several reasons. One possibility is that the price falls and the price elasticity of demand is less than one, at least for older goods. To illustrate this possibility simply, suppose that the demand for a good is determined solely by its own price $p(\tau)$, and that its price falls toward zero. Even when the good becomes almost free, it is not plausible for a consumer to demand an infinite amount of any particular product in reality. This consideration suggests that the price demand elasticity of a product will become small when the price has become sufficiently low, and the expenditure for the good will eventually vanish as $p(\tau) \to 0$.

More concretely, suppose that the price elasticity of demand becomes less than $\overline{\varepsilon} < 1$

for old goods with $\tau \geq \overline{\tau}$, for some $\overline{\tau} > 0$. Then, a sufficient condition for (8) is[19]

$$p(\tau) \leq p(\overline{\tau}) \left(\frac{\tau}{\overline{\tau}}\right)^{-\frac{\xi}{1-\overline{\varepsilon}}} \equiv \overline{p}(\tau) \text{ for all } \tau \geq \overline{\tau}, \tag{12}$$

for some $\xi > 1$. Condition (12) implies that the positive GDP growth rate can be maintained when the price of any individual goods continues to fall at least as fast as $\overline{p}(\tau)$. The upper bound function $\overline{p}(\tau)$ starts from $\overline{p}(\overline{\tau}) = p(\overline{\tau})$ at $\tau = \overline{\tau}$ and falls proportional to $\tau^{-\xi/1-\overline{\varepsilon}}$. Since this is a power function, the required rate of fall can be arbitrarily small if we choose a large $\overline{\tau}$. Therefore, the quality-adjusted price does not need to fall exponentially to maintain positive GDP growth. If the price change is driven by quality or productivity improvements, the improvements do not need to be exponential. In Section 3, we will present a full endogenous growth model based on this idea.

The expenditure for older goods can also fall for other reasons. The older goods may become obsolete, and consumers no longer demand them. Sometimes, consumers are attracted by the novelty of new goods but become less interested as time passes. Changes in the underlying economic environments may make older goods of no use. We will extend the theory to include those possibilities in Subsection 4.1.

2.6 Implications for Endogenous Growth Theory

We have established that a constant GDP growth rate can be maintained without exponential growth either in quantity, quality, or variety. Here, we discuss how this result allows for building a robust endogenous growth model without requiring knife-edge conditions.

In standard variety-expansion models, all goods are symmetric and receive the same share of expenditure. Therefore, as the number of goods increases, the share of the expenditure given to a single new good dilutes. Accordingly, the contribution of each new good to the economic growth rate also shrinks toward 0. This is why these models need to consider exponentially accelerating R&D so as to offset the dilution effects.

This requirement restricts the way an endogenous growth model could be built. If the expenditure share of a single new good tends to zero, profits obtained from a single suc-

[19]Suppose that $x(\tau)$ depends only on $p(\tau)$, its price elasticity is less than $\overline{\varepsilon}$ for $\tau \geq \tau$, and condition (12) is satisfied. Then, $x(\tau) \leq x(\overline{\tau}) \left(p(\tau)/p(\overline{\tau})\right)^{-\overline{\varepsilon}}$ since $p(\tau)/p(\overline{\tau}) \leq 1$ and the price elasticity is less than $\overline{\varepsilon}$. Therefore, expenditure at τ satisfies $p(\tau)x(\tau) \leq p(\tau)x(\overline{\tau}) \left(p(\tau)/p(\overline{\tau})\right)^{-\overline{\varepsilon}} = x(\overline{\tau})p(\overline{\tau})^{\overline{\varepsilon}}p(\tau)^{1-\overline{\varepsilon}} \leq x(\overline{\tau})p(\overline{\tau})^{\overline{\varepsilon}}\overline{p}(\tau)^{1-\overline{\varepsilon}} = p(\overline{\tau})x(\overline{\tau})\overline{\tau}^{\xi}\tau^{-\xi}$ for $\tau > \overline{\tau}$. Since $p(\overline{\tau})x(\overline{\tau})\overline{\tau}^{\xi}$ is a positive constant, the latter inequality implies (10). This completes the proof since (10) is a sufficient condition for (8).

cessful R&D also fall. Therefore, to give firms enough incentives to do R&D in equilibrium, those models require a strong degree of externality in the R&D process so that the cost of inventing new goods declines exponentially (See panel (a) in Figure 2).

By contrast, if the economy satisfies condition (8), the incentive to innovate can be maintained without such strong externalities. Under this condition, the expenditure share for one new good, $p(0)x(0)\,/n\int_0^\infty p(\tau)x(\tau)d\tau$, remains a positive constant even when more and more goods are introduced. In such a case, a constant flow of new goods, as well as improvements in production costs after the introduction, always constitutes a significant addition of economic activity relative to all existing activities. This enables the measured GDP growth rate to remain positive in the long run without accelerating R&D. In addition, since the expenditure share for one new good does not fall, firms obtain enough profits from R&D even when the cost of R&D does not fall as a result of externalities. In the next section, we build a general equilibrium endogenous model without R&D externality in creating new products.

3 A Prototype Non-Exponential Growth Model

In this section, we present a general equilibrium model that yields non-exponential steady-state dynamics. While the theory in the previous section suggests there are many ways to build a model that achieves non-exponential growth while capturing various aspects of reality, here we limit ourselves to presenting the simplest prototype model so as to convey the substance of the non-exponential growth theory as clearly as possible. Thereafter we will discuss the generalization of the prototype model to several directions.

3.1 Consumers and Market Demand

Consider an economy with infinitely-lived representative consumers of constant population L. At each point in time, each consumer supplies one unit of labor and obtains one unit of wage. The wage level is normalized to one.[20]

The lifetime utility function of the representative consumer is given by

$$\int_0^\infty \left[\int_0^{N_t} u(\widetilde{c}_t(i))di\right] e^{-\rho t}dt, \tag{13}$$

[20] In the steady state where the fraction of consumption out of labor income is constant, this normalization implies that the nominal expenditure is constant, consistent with the theory in the previous section.

which is separable both across time and goods. The sub-utility for individual goods takes the same form as the standard CRRA utility function.[21]

$$u(\widetilde{c}_t(i)) = \frac{\widetilde{c}_t(i)^{1-1/\varepsilon} - 1}{1 - 1/\varepsilon}, \quad 0 < \varepsilon < 1. \tag{14}$$

Similarly to Ngai and Pissarides (2007) and Acemoglu and Guerrieri (2008), we assume that the elasticity of substitution across goods, ε, is less than one.[22] We make this assumption to better capture the behavior of demand when the price becomes low while keeping the tractability of constant elasticity.[23]

The dynamic budget constraint of a representative consumer is given by

$$\dot{k}_t = r_t k_t + 1 - \int_0^{N_t} \widetilde{p}_t(i)\widetilde{c}_t(i)di, \tag{15}$$

where k_t is the asset holding of the consumer, and r_t is the interest rate under our price normalization. In equilibrium, Lk_t should equal the value of all firms in the economy. Consumers maximize their lifetime utility (13) subject to (15), given interest rate r_t, prices of goods $\widetilde{p}_t(i)$ for $i \in [0, N_t]$, the initial asset holding k_0, and the standard non-Ponzi-game condition.

From these, we obtain an isoelastic demand function for individual goods by a representative consumer.

$$\widetilde{c}_t(i) = \lambda_t^{-\varepsilon}\widetilde{p}_t(i)^{-\varepsilon}. \tag{16}$$

The shadow price of the budget constraint λ_t evolves according to Euler equation $\dot{\lambda}_t = (\rho - r_t)\lambda_t$, and its initial value is determined so that the transversality condition $\lim_{t\to\infty} e^{-\rho t}\lambda_t k_t = 0$ is satisfied given the evolution of k_t in (15).

[21]Note that the sub-utility function is symmetric across goods, which means that we do not model the obsolescence of older goods in this simplest prototype model. Subsection 4.1 introduces obsolescence and shows that a positive growth rate can be maintained under milder conditions (even when $\varepsilon > 1$).

[22]They developed multi-sector growth models where the elasticity of substitution between the goods of different sectors is less than one. When the productivity of a sector (e.g., agriculture) increases, the expenditure for this sector shrinks because the demand quantity does not rise as much as the price falls. While they examined a small and finite number of sectors, this paper considers a continuum of goods (or more precisely, product groups) and examines a steady-state shift of expenditure shares across these product groups.

[23]If $\varepsilon > 1$, the expenditure for a good will rise indefinitely as its price falls to zero. This would not be a good description of the behavior of consumers when our focus is on how goods effectively retire from the market at the end of their product lifecycle.

3.2 R&D and Production Technologies

Each consumer works either as a production worker or as a researcher. A researcher succeeds in developing a new good with a Poisson probability of a per unit of time. Let L_t^R denote the number of researchers in the economy, which is to be determined in equilibrium. Over time, the number of goods increases according to

$$\dot{N}_t = aL_t^R. \tag{17}$$

Equation (17) is similar to standard variety expansion models (recall panel (a) in Figure 2), except that there is no spillover term from the stock of past R&D.

Once developed, each individual good is produced with a linear production technology that requires only labor.

$$\widetilde{x}_t(i) = \widetilde{q}_t(i)\widetilde{l}_t(i), \tag{18}$$

where $\widetilde{x}_t(i)$ is the output of good i, $\widetilde{l}_t(i)$ is the labor input, and $\widetilde{q}_t(i)$ is the marginal product of labor in producing good i. Alternatively, we can interpret $\widetilde{x}_t(i)$ as the quality-adjusted output and $\widetilde{q}_t(i)$ as the quality of good i. In this case, one unit of labor produces one unit of good i with quality $\widetilde{q}_t(i)$. In either interpretation, we call $\widetilde{q}_t(i)$ the productivity for good i.

Suppose that, when any good is first developed, the productivity is 1. Then, as the production of this good proceeds, the productivity increases according to[24]

$$\dot{\widetilde{q}}_t(i) = I(\widetilde{x}_t(i)) \cdot \beta\widetilde{q}_t(i)^{\psi}, \quad 0 < \psi < 1. \tag{19}$$

$I(\widetilde{x}_t(i))$ is an indicator function that takes 1 when $\widetilde{x}_t(i) > 0$ and 0 otherwise. It means that productivity increases as long as production takes place. Observe that there is a similarity between the specification in (19) and those in the quality ladder models (see panel b in Figure 2). We assume that there are knowledge spillovers from the past productivity of the technology to today's increments in productivity. Parameter $\psi \in (0, 1)$ specifies the degree of such spillovers. While quality ladder models need to assume $\psi = 1$ to

[24]For simplicity, here we assume that experience in terms of time matters for productivity improvement. Alternatively, we can consider the experience in terms of cumulative production amount. Horii(2012) analyzed a model in the latter setting and derived the GDP growth rate defined similarly to (4), although it was a semi-endogenous growth model that required an exponentially growing population (c.f. Jones 1995).

achieve the exponential increase in productivity (or quality), we do not make this knife-edge assumption. For the moment, we consider the case of $\psi \in (0,1)$, and later compare the result to the case of $\psi = 1$. Parameter $\beta > 0$ represents other possible factors that affect the speed of productivity increases.

As long as $\widetilde{x}_t(i) > 0$, equation (19) is an autonomous differential equation in $\widetilde{q}_t(i)$. Similarly to Section 2, let $\tau \equiv t - s(i)$ denote the age of the good. Then the solution to the differential equation (19) can be written as

$$q(\tau) = \kappa_1 (\tau + \kappa_0)^{\theta}, \tag{20}$$

where $\theta \equiv 1/(1-\psi) > 1$, $\kappa_0 \equiv \theta/\beta > 0$, and $\kappa_1 \equiv (\beta/\theta)^{\theta} > 0$. Given that $\psi \in (0,1)$, the productivity increases less than exponentially. The rate of productivity increase is given by

$$g_q(\tau) = \frac{q'(\tau)}{q(\tau)} = \frac{\theta}{\tau + \kappa_0} = \frac{\beta}{(1-\psi)\beta\tau + 1}. \tag{21}$$

In this specification, $g_q(\tau)$ takes the highest value at the time of introduction, $g_q(0) = \beta$, and falls to 0 as a good gets older ($g_q(\infty) = 0$). This rules out the trivial possibility that the exponential increase in the productivity of individual goods explains the sustained growth.

3.3 Behavior of Firms

Let us now turn to the behavior of production firms. While any product is protected by a patent forever, we suppose that the patent *breadth* is limited (e.g., O'Donoghue, Scotchmer and Thisse 1998). This means that other producers are prohibited from using exactly the same technology as the original inventor, but they are allowed to produce similar products if they use a technology that is sufficiently different from the original. Alternatively, we may also think that a part of technology is kept secret by the inventor, and the outsiders need to rely on less efficient technologies. In either case, outsiders face lower productivity than the original firm.

To formalize this idea, let us assume that there are potentially many outside firms, and they have partial access to the technology of the original inventor $\widetilde{q}_t(i)$ to produce the same good i, but their productivity is $1/(1+\mu)$ times lower, where parameter μ represents the patent breadth or the strength of trade secret. Recall that the price elasticity of the demand function (16) is less than unity. In this case, the profit-maximizing strategy is to set the limit pricing, which is $(1+\mu)$ times higher than the marginal cost. Given the production function (18) and the fact that wage is normalized to one, the pricing by a firm that has τ years of experience is

$$p(\tau) = \frac{1+\mu}{q(\tau)}. \tag{22}$$

From (16) and (22), we obtain the output and the maximized profit of this firm.[25]

$$x(\tau) = Lc(\tau) = D_t q(\tau)^{\varepsilon}, \tag{23}$$

$$\pi(\tau) = L(p(\tau) - 1/q(\tau))c(\tau) = \mu D_t q(\tau)^{\varepsilon-1}, \tag{24}$$

where $D_t \equiv L(1+\mu)^{-\varepsilon}\lambda_t^{-\varepsilon}$ is a shifter of demand for all goods.

3.4 Steady-State Equilibrium

Now we derive the long-term growth property of the equilibrium dynamics in this prototype model. The following defines a notion of long-term equilibrium suitable to our model.

Definition 2 *An equilibrium path that satisfies the following properties as $t \to \infty$ is called Asymptotically Steady-State Equilibrium (ASSE).*

1. *The speed of the introduction of new goods converges to a positive constant: $\dot{N}_t \to n^* > 0$.*

2. *The shifter of the demand for all goods, $D_t = L(1+\mu)^{-\varepsilon}\lambda_t^{-\varepsilon}$, converges to a positive constant: $D_t \to D^* > 0$.*

Later, Subsection 3.6 confirms that the real aggregate variables observed in the ASSE obey the conventional notion of the balanced growth path.

In the following, we consider the state of the economy as $t \to \infty$ in the ASSE, and mention it simply "in the ASSE." The equilibrium values of n^* and D^* are determined by the free entry condition for R&D and the labor market clearing condition. Let us first focus on the R&D condition. Recall that the Euler equation is $\dot{\lambda}_t/\lambda_t = \rho - r_t$. Since $\lambda_t = (1+\mu)(L/D_t)^{1/\varepsilon}$ is stationary in the ASSE, the interest rate necessarily converges to $r_t \to \rho$. Using interest rate $r_t = \rho$ and the profit function (24), we can calculate the

[25] Strictly speaking, we need to put subscript t for $x(\tau)$ and $\pi(\tau)$ as they depend on D_t. In the steady state, D_t becomes constant, and then $x(\tau)$ and $\pi(\tau)$ do not depend on t.

present value of a new firm just after it has succeeded in developing a new good. The value of innovation in the ASSE becomes

$$V^* = \int_0^\infty \pi(\tau)e^{-\rho\tau}d\tau = \mu D^* \int_0^\infty q(\tau)^{\varepsilon-1}e^{-\rho\tau}d\tau. \tag{25}$$

From the R&D function (17), the expected cost of developing a new good is $1/a$. Therefore, given that there is a positive flow of R&D, $n > 0$, and that the financial market is complete, the value of the new firm (25) should be equalized to the expected cost of development: $V^* = 1/a$. This condition gives the equilibrium value of D^* in the ASSE.

$$D^* = \frac{1}{a\mu}\left(\int_0^\infty q(\tau)^{\varepsilon-1}e^{-\rho\tau}d\tau\right)^{-1}. \tag{26}$$

Substituting (20) into (26), we can obtain an explicit expression for D^*.[26]

Next, let us turn to the labor market. First, equation (17) implies that the number of research workers in the ASSE is $L^{R*} = n^*/a$. Second, from (18) and (23), the aggregate demand for production workers in the ASSE is[27]

$$\begin{aligned} L^{P*} &= \lim_{t\to\infty}\int_0^{N_t}\widetilde{l}_t(i)di \\ &\to n^*\int_0^\infty x(\tau)/q(\tau)d\tau = n^*D^*\int_0^\infty q(\tau)^{\varepsilon-1}d\tau. \end{aligned} \tag{28}$$

The labor supply is given by population L. Therefore, the labor market clearing condition is

$$L = L^{R*} + L^{P*} = \frac{n}{a} + nD^*\int_0^\infty q(\tau)^{\varepsilon-1}d\tau. \tag{29}$$

From this, we obtain the equilibrium research intensity in the ASSE

$$n^* = \frac{aL}{1 + aD^*\int_0^\infty q(\tau)^{\varepsilon-1}d\tau}. \tag{30}$$

From (30), $L^{R*} = n^*/a$ and $L^{P*} = L - L^{R*}$ are also obtained. The explicit solution for n^* can be obtained as follows. Using (26) and then (20), the ratio of two types of labor is

$$\left(\frac{L^P}{L^R}\right)^* = \frac{\int_0^\infty q(\tau)^{\varepsilon-1}d\tau}{\mu\int_0^\infty q(\tau)^{\varepsilon-1}e^{-\rho\tau}d\tau} \tag{31}$$

[26] Let $\Gamma(\cdot,\cdot)$ denote upper incomplete Gamma function, which is defined as $\Gamma(s,z) \equiv \int_z^\infty t^{s-1}e^{-t}dt$. By changing the variable of integration from τ to $\widetilde{\tau} = (\tau + \kappa_0)/\rho$, (26) becomes

$$D^* = \frac{\kappa_1^{1-\varepsilon}\rho^{\theta(1-\varepsilon)+1}}{a\mu e^{\rho\kappa_0}\Gamma(1-\theta(1-\varepsilon),\rho\kappa_0)} > 0. \tag{27}$$

$\Gamma(s,z)$ is positive and finite for all $s \in (-\infty,\infty)$ and $z \in (0,\infty)$, which implies $D^* > 0$. The values of $\Gamma(s,z)$ are available in most programming platforms, such as Python, Matlab, and Mathematica.

[27] The variable of integration is changed from i to τ in (28) using (2).

The equilibrium ratio of production and research workers $\left(L^P/L^R\right)^*$ in (32) becomes a positive finite constant if and only if $\theta(1-\varepsilon) > 1$.[28] Using definition $\theta \equiv 1/(1-\psi)$, the condition reduces to $\psi \in (\varepsilon, 1)$, where ψ is the degree of knowledge spillover from past productivity to its increments. Using this ratio, the ASSE research intensity can be written as

$$n^* = aL^{R*} = \frac{aL}{1 + (L^P/L^R)^*}, \tag{33}$$

which becomes a positive constant if $\psi \in (\varepsilon, 1)$, and 0 if $\psi < \varepsilon$.

The pair of D^* and n^* in (26) and (33) characterizes the long-term equilibrium of this economy. These equations also show how parameters affect long-term dynamics. For example, a larger μ means the breadth of patents is wider (or the trade secrets are better kept). A higher a means that R&D is less costly. In these cases, innovation intensity n^* rises because of higher profitability, whereas the output of each good, proportional to D^*, falls because there are more production firms to which the aggregate labor needs to be divided.[29] The opposite occurs when the time preference ρ is higher because it raises the interest rate and hence reduces the present value of profits.

When population L is larger, the research intensity n^* is multiplied proportionally to L. However, production of each good (proportional to D^*) does not change, because both the number of products introduced each year and the number of total production workers are multiplied by the same factor. This outcome resembles the mechanism of the second-generation endogenous growth models, where the horizontal number of sectors is adjusted proportionally to the total population.[30]

Before closing this subsection, let us briefly compare those results against the case of $\psi = 1$. When $\psi = 1$, the solution to differential equation (19) is exponential: $q(\tau) = e^{\beta\tau}$.

[28]Using (27), we obtain an explicit solution to (31) as follows. It becomes infinity if $\theta(1-\varepsilon) \leq 1$.

$$\left(\frac{L^P}{L^R}\right)^* = \frac{\kappa_0^{1-\theta(1-\varepsilon)}\rho^{\theta(1-\varepsilon)+1}}{\mu(\theta(1-\varepsilon)-1)e^{\rho\kappa_0}\Gamma(1-\theta(1-\varepsilon), \rho\kappa_0)} \text{ if } \theta(1-\varepsilon) > 1. \tag{32}$$

[29]The derivative of the upper incomplete Gamma function with respect to the second argument, $\partial\Gamma(s,z)/\partial z = -z^{s-1}e^{-z}$, is always negative. Using this, the properties in the text can be confirmed from (27), (32) and (33).

[30]However, note that the long-term growth in these models is typically maintained by the exponential increase in the productivity (or quality) in each sector, whereas this paper focuses on the case where such exponential improvements cannot be sustained ($\psi < 1$ in equation 19).

Then, we can calculate n^* and D^* in the ASSE as

$$n^* = \frac{\mu(1-\varepsilon)\beta a L}{(1+\mu)(1-\varepsilon)\beta + \rho}, \quad D^* = \frac{(1-\varepsilon)\beta + \rho}{a\mu}. \tag{34}$$

The comparative statistics properties with respect to μ, ρ, L and a are the same as the case of $\psi \in (\varepsilon, 1)$. Therefore, the case of exponential growth in productivity ($\psi = 1$) can be viewed as a special case of our model, although we do not focus on it because it is a knife-edge case.

3.5 Long-Term Real GDP Growth Rate

Now we are ready to examine the long-term rate of economic growth in this prototype model. The ASSE satisfies the definition of non-exponential asymptotic steady state in Definition 1. Using (20), (22) and (23), as well as definitions $\theta \equiv 1/(1-\psi) > 1$ and $\kappa_0 \equiv \theta/\beta > 0$, we can confirm that $p(\tau)$ and $x(\tau)$ satisfies conditions (7) and (8) as long as $\psi \in (\varepsilon, 1)$. Therefore, we can apply Proposition 1 to calculate the real GDP growth rate in the steady-state equilibrium. Combined with the result for the case of $\psi = 1$, we have

$$g^* = \frac{-\int_0^\infty p'(\tau)x(\tau)d\tau}{\int_0^\infty p(\tau)x(\tau)d\tau} = \begin{cases} \frac{\psi-\varepsilon}{1-\varepsilon}\beta & \text{for } \varepsilon < \psi \leq 1, \\ 0 & \text{for } \psi \leq \varepsilon. \end{cases} \tag{35}$$

Equation (35) shows that the measured growth rate takes a positive and finite value whenever $\psi \in (\varepsilon, 1]$. This requirement can be understood from the discussion in Subsection 2.5. There, we have seen that if the good remains in the market forever, the price of goods need to fall fast enough so the cumulative expenditure for old goods becomes finite. A sufficient condition is given by (12), which requires the power of function $p(\tau)$ to be less than $-1/(1-\varepsilon)$. Given that $\psi < 1$, the price of goods in the ASSE can be written as $p(\tau) = ((1+\mu)/\kappa_1)(\tau + \kappa_0)^{-\theta}$ from (20) and (22). The power of this price function is $-\theta = -1/(1-\psi)$, and it becomes lower than $-1/(1-\varepsilon)$ if and only if $\psi > \varepsilon$. In this particular model environment, sustained economic growth requires prices to fall fast enough, and for this to occur, the degree of intertemporal knowledge spillover in production ψ should be large enough. Still, compared to existing prototype models of endogenous growth (See Figure 2), it is remarkable that ψ does not need to be at a knife-edge level.

Given that the markup ratio μ is constant, the growth formula (9) in the ASSE can be represented as

$$g^* = \frac{\int_0^\infty g_q(\tau)e(\tau)d\tau}{\int_0^\infty e(\tau)d\tau}, \tag{36}$$

where $e(\tau) = p(\tau)x(\tau)$ is the expenditure for an age τ good and $g_q(\tau)$ is the rate of productivity increase for this product, defined in (21). The formula in this form clarifies that the real GDP growth is the weighted average of the rate of productivity increases among goods of various ages. Recall that, in our specification of the technology, the newest goods have the fastest rate of productivity improvements, β, while the rate of improvements is lower for the older goods because $g_q'(\tau) < 0$ (see equation 21). In particular, the rate of productivity improvement $g_q(\tau)$ is almost zero for very old goods with large τ. Therefore, it is natural that the aggregate GDP growth rate (35) is somewhere between zero and β because the economy consists of goods of all ages.

Now, it is clear why growth rate g^* in (35) is decreasing in ε. When the elasticity of substitution ε is high, consumers spend relatively more on old and low-priced goods and less on new and expensive goods. Since the rate of productivity increase in (21) is lower for older goods (with high age τ), the weighted average will also be low.

Equation (35) also shows that growth rate g^* in is increasing with ψ, the degree of knowledge spillover in production. When $\psi \le \varepsilon$, $(LP/LR)^*$ in (31) becomes infinity, which means $n^* = 0$. Without the introduction of new goods, the distribution of products ages simply moves up, and, the growth rate will fall to $g_q(\infty) = 0$. As ψ increases between ε and 1, the schedule of $g_q(\tau)$ in (21) moves up, and so does the real GDP growth rate. When ψ reaches 1, the long-term growth rate rises to β. This is an anticipated result; when $\psi = 1$, the productivity of all goods, both the new and the old, increases with a common constant exponential rate of β. Therefore, the case of $\psi = 1$ corresponds to the conventional growth theory where labor productivity increases exponentially and uniformly. However, our stress is that, even when the productivity of each product does not rise exponentially (i.e., with $\psi < 1$), the economy as a whole can exhibit a constant measured growth rate, although it is lower than β. Finally, $\psi > 1$ is unrealistic, because it means that the productivity of individual goods increases more than exponentially and $g_q(\infty) = \infty$.

It is worth noting that, in this simple prototype setting, the equilibrium long-term rate of growth (35) does not depend on equilibrium values of n^* and D^* as long as they are positive.[31] When the research intensity n^* is higher, there is more addition of economic

[31] Of course, this property depends on the simplistic settings in this prototype model. For example, when the aggregate R&D intensity n^* has some positive spillovers on the rate of productivity increases in individual goods $g_q(\tau)$, then n^* will affect g^*. Also, when the amount of production has some effects on

activity per unit of time, but at the same time, there is also proportionally more "stock" of existing activities. The real GDP growth rate expresses this ratio, which is unchanged.[32] Similarly, when D^* is larger, there is more demand for each good. This means the production of new goods, as well as the increment of production of other goods per unit of time, is higher. At the same time, however, the value of existing products is also higher, exactly canceling out the effects on g^*.[33] As a result, even when the changes in population L, R&D productivity a, or patent policy μ affect n^* and D^*, they do not affect the real GDP growth rate. This is a marked contrast to the implication of existing R&D-based growth models, where g^* follows directly from n^*. Although this result depends on the simplifying specification of the prototype model, it might provide a possible interpretation of why the measured GDP growth rates in the U.S. and some other developed countries have been quite stable, even though those underlying parameters seem to have changed greatly over long time periods.

3.6 Measured Real Aggregate Variables and Balanced Growth

The ASSE in this model works very differently from the balanced growth path (BGP) in existing growth models. Nonetheless, this section shows that when aggregate variables are measured in a conventional way, this model exhibits balanced growth in those measured aggregate variables.

Note that the total labor income for production is L^{P*} since the wage rate is normalized to one. All goods are sold at $(1+\mu)$ times the labor cost, as shown in (22), and therefore the aggregate value of production, which equals the aggregate value of consumption, is $C^* = (1+\mu)L^{P*}$. In our model, investments take the form of R&D, and the total value of R&D outputs is $I^* = n^*V^* = L^{R*}$. The GDP in our model can be calculated as the sum of the value of production and the value of investments: $Y^* = C^* + I^* = (1+\mu)L^{P*} + L^{R*}$.

$g_q(\tau)$, g^* will depend on D^*.

[32]Nonetheless, it is important that there is a positive flow of new innovations $n^* > 0$, since otherwise, g^* becomes 0.

[33]This can also be seen in Examples 3 of Figure 5. When D^* is increased, the left panel is stretched horizontally (along the $x(\tau)$ axis) while the right panel is stretched vertically (along the $p(\tau)x(\tau)$ axis) by the same magnification ratio. As a result, the growth rate, which is given by the ratio of the two areas, is unaffected.

The aggregate capital is defined as the value of all firms in the economy (knowledge capital). In Online Appendix, we derive $K^* = (\mu L^{P*} - L^{R*}) > 0$.

Note that those aggregate variables are measured under the price normalization of our model, in which the nominal wage is set to 1. We now calculate their real values, as defined in the SNA.[34] Let $\bar{t}$ be the reference year, and $Y^{\$}_{\bar{t}}$ be the dollar value of the GDP in year $\bar{t}$, which we assume is known to the researcher. Then, since the real GDP growth rate is constant at g^* in the ASSE, the real GDP level in t is $Y^{\text{real}}_t = Y^{\$}_{\bar{t}} e^{g^*(t-\bar{t})}$. Since the ratios among Y^*, C^*, I^* and K^* are constant, their dollar values grow in the same proportion:

$$C^{\text{real}}_t = \frac{C^*}{Y^*} Y^{\text{real}}_t = \frac{1+\mu}{1+\mu+(L^R/L^P)^*} Y^{\$}_{\bar{t}} e^{g^*(t-\bar{t})}, \tag{37}$$

$$I^{\text{real}}_t = \frac{I^*}{Y^*} Y^{\text{real}}_t = \frac{1}{(1+\mu)(L^P/L^R)^*+1} Y^{\$}_{\bar{t}} e^{g^*(t-\bar{t})}, \tag{38}$$

$$K^{\text{real}}_t = \frac{K^*}{Y^*} Y^{\text{real}}_t = \frac{\mu-(L^R/L^P)^*}{\rho(1+\mu+(L^R/L^P)^*)} Y^{\$}_{\bar{t}} e^{g^*(t-\bar{t})}, \tag{39}$$

where $(L^R/L^P)^*$ is given by (31).

Interest rate $r^* = \rho$ is also defined under our normalization of prices. Since the nominal GDP growth rate in the steady state is zero, the steady-state inflation rate is $-g^*$ in our price normalization. Then, the real interest rate in the steady state is $r^{\text{real}} = r^* + g^* = \rho + g^*$. We can also derive other real aggregate variables in similar ways, and the growth rates of all of them are constants. Therefore, if the statistical agency measures the aggregate variables in our model, they are observed to be exponentially growing along the BGP, even though neither quantity, quality, or variety of individual goods grow exponentially.

3.7 Welfare Improvements

Lastly, let us discuss the welfare of the representative consumer. From (13), the period utility is $U_t = \int_0^{N_t} u(\widetilde{c}_t(i))di$. In the ASSE, the improvements in welfare can be measured

[34] The NIPA publishes two series of real GDP. One is the quantity index, which takes 100 in the reference year (it is 2012 as of the time of writing). The values for other years are obtained by chaining the real GDP growth rate. The other is Chained (2012) dollar series. They are calculated as the product of the quantity index and the 2012 current-dollar value of the corresponding series, divided by 100. See U.S. Bureau of Economic Analysis, "Table 1.1.6. Real Gross Domestic Product, Chained Dollars." Here, We use the latter.

by[35]

$$\lim_{t\to\infty} \dot{U}_t = n^* \lim_{c\to\infty} u(c) = \frac{\varepsilon}{1-\varepsilon} n^*. \tag{40}$$

Given parameter $\varepsilon \in (0, 1)$, equation (40) shows that the speed of welfare improvements in the steady state is entirely determined by the speed of innovation n^*. This is in a stark contrast to the real GDP growth rate g^* in (35), which does not depend on n^*. At least in this simple prototype model, the non-exponential growth theory illustrates that the GDP growth rate is not a good measure of the speed of welfare improvements in the long run.

4 Generalizations

4.1 Obsolescence

In the prototype model of Section 3, sustained economic growth required the price elasticity of goods ε to be less than one, in the environment where goods stay in the market forever ($T = \infty$) and consumers have symmetric preference across goods (13). The condition $\varepsilon < 1$ was necessary to induce consumers to spend less on older (and cheaper) goods. However, even without such an assumption, consumers often spend more on new goods just because they like new goods. They may spend less on older goods because those are obsolete. Here, we show that condition $\varepsilon < 1$ can be relaxed once we include obsolescence.

Suppose now that utility function (13) is replaced by

$$\int_0^\infty \left[\int_0^{N_t} \delta(t - s(i)) u(\widetilde{c}_t(i)) di \right] e^{-\rho t} dt, \tag{41}$$

where $t - s(i) = \tau$ is the age of good i (time after its debut), and $\delta(\tau)$ is a decreasing function of τ with $\delta(0) = 1$ and $\delta(\infty) = 0$. The steepness of function $\delta(\tau)$ represents the speed of obsolescence, or equivalently, consumers' taste for newer goods. We keep all other settings in Section 3 except that now we allow any $\varepsilon > 0$. Then, the expenditure for an age τ good in the ASSE becomes $e(\tau) = p(\tau)x(\tau) = (1 + \mu)D^*\delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1}$, which illustrates that even when $\varepsilon > 1$, expenditure for older goods falls with age if obsolescence is

[35] Equation (40) gives the difference in the period utility between time $t+1$ and time t in the ASSE. Since the economy is in the ASSE, schedules for consumption against the age of goods are the same in those 2 time points, with the only difference that the economy in time $t+1$ has n^* more oldest goods than in time t. When $t \to \infty$, the quality-adjusted amount of consumption for each of those oldest goods approaches infinity. Therefore, the difference in the period utility is $n^* c(\infty)$. From (14), $c(\infty) = \varepsilon/(1-\varepsilon)$.

fast enough.[36] Proposition 1 continues to apply in an environment with obsolescence, and the formula for the GDP growth rate (36) shows that the growth rate becomes a positive constant if $\int_0^\infty e(\tau)d\tau$ is finite.

When the rate of obsolescence is constant $\overline{\delta} > 0$ per year, $\delta(\tau) = \exp(-\overline{\delta}\tau)$. In this case, the integration of $e(\tau) = (1+\mu)D^*\delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1}$ always becomes finite because $\delta(\tau)^\varepsilon$ is falling exponentially and no other variable is growing exponentially. Therefore, a constant rate of obsolescence always sustains positive GDP growth regardless of ε. Growth can be maintained with slower depreciation. Consider an example where $\delta(\tau)$ is a negative power function of τ: $\delta(\tau) = \delta_0^\omega(\tau+\delta_0)^{-\omega}$ where ω and δ_0 are positive constants. Then, $\int_0^\infty e(\tau)d\tau$ becomes finite if and only if[37]

$$\omega > \frac{\varepsilon - \psi}{\varepsilon(1-\psi)}. \tag{42}$$

The condition shows that growth can be sustained even when $\varepsilon > 1$ and obsolesce is not exponential. In a special case of $\delta_0 = \kappa_0$, where κ_0 is defined in (20), we obtain an explicit expression for the long-term GDP growth rate,

$$g^* = \frac{\psi - \varepsilon + (1-\psi)\varepsilon\omega}{1 - \varepsilon + (1-\psi)\varepsilon\omega}, \tag{43}$$

which is positive when (42) holds. Since $\psi < 1$, g^* is higher when the obsolescence is faster (ω is higher).

Intuitively, obsolescence skews expenditure toward newer goods. Since newer goods have more margins for productivity increases, the overall growth rate rises with obsolescence. This result has interesting policy implications. When the government tries to protect obsolete companies (or industries), it will reduce the GDP growth rate, not just because of efficiency loss but also because of the way the GDP growth rate is calculated. Conversely, advertisements and marketing practices that attract consumers from older goods to newer goods will enhance GDP growth, even when the attractiveness of the newer goods is illusionary.

[36] Similarly to the derivation of (26), we obtain $D^* = \left(a\mu \int_0^\infty \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1}e^{-\rho\tau}d\tau\right)^{-1}$, which is always positive and finite because of the $e^{-\rho\tau}$ term. Using this value of D^*, the speed of innovation is $n^* = aL\left(1 + aD^*\int_0^\infty \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1}d\tau\right)^{-1}$. n^* is strictly positive if and only if $\int_0^\infty \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1}d\tau$ is finite, which is equivalent to the finiteness of $\int_0^\infty e(\tau)d\tau$.

[37] Using (20), $\int_0^\infty e(\tau)d\tau = (1+\mu)\delta_0^{\varepsilon\omega}\kappa_1^{\varepsilon-1}D^*\int_0^\infty(\tau+\delta_0)^{-\omega\varepsilon}(\tau+\kappa_0)^{\theta(\varepsilon-1))}d\tau$. The integral becomes finite if and only if the sum of the powers of the integrand, $-\omega\varepsilon + \theta(\varepsilon-1)$ is less than one. Using $\theta = 1/(1-\psi)$, this condition is equivalent to (42).

4.2 Declining and Rising Population

The theory in Section 2 has explained that GDP growth can be maintained when the population is constant. Here, we extend the theory to declining and rising populations. Suppose that the population changes at the rate of g_L, which can be either negative or positive. We maintain the assumption that the creation of one new good requires a fixed number of R&D workers and consider a steady state where a constant fraction of the population is working in the R&D sector. Then, instead of (1), the number of goods will increase according to

$$\dot{N}_t = n_0 e^{g_L t}, \tag{44}$$

for some constant $n_0 > 0$. The relationship between the index of goods and their age in (2) changes to

$$i = N_t - \frac{n_0 e^{g_L t}}{g_L}\left(1 - e^{-g_L \tau}\right), \tag{45}$$

where the second term represents the number of goods created between $t - \tau$ and t. We also normalize prices so that the expenditure per capita is constant.[38] Except these, the environment of the economy in the steady state is the same as defined in Definition 1, including condition (3). The GDP growth rate in (5) still applies since it does not depend on the constant population. Therefore, using (3), (44), and (45) in (5), and then utilizing the integration by parts, we obtain a generalized version of Proposition 1.

Proposition 2 *Consider an economy where the population is changing at a constant rate g_L. Suppose that the economy converges to the asymptotic non-exponential steady-state, as defined by Definition 1. Given that*

$$-\int_0^T x(\tau)e^{-g_L\tau}dp(\tau) \text{ is finite and strictly positive, and} \tag{46}$$

$$\int_0^T p(\tau)x(\tau)e^{-g_L\tau}d\tau \text{ is finite,} \tag{47}$$

the growth rate of the real GDP per capita asymptotes to a positive and finite constant

$$g - g_L = \frac{-\int_0^T x(\tau)e^{-g_L\tau}dp(\tau)}{\int_0^T p(\tau)x(\tau)e^{-g_L\tau}d\tau}. \tag{48}$$

[38]The total expenditure in the steady state becomes $n_0 e^{g_L t}\int_0^T p(\tau)x(\tau)e^{-g_L\tau}d\tau$. Given that the integral is finite, the expenditure increases proportionally to the population.

Note that Proposition 2 differs from Proposition 1 only in that $x(\tau)$ is replaced by $x(\tau)e^{-g_L\tau}$. Therefore, We can graphically calculate the growth rate as we have done in Subsection 2.4 by replacing $x(\tau)$ by $x(\tau)e^{-g_L\tau}$.

Subsection 4.1 has shown that when there is a constant rate of obsolescence, $x(\tau)$ will be multiplied by $\delta(\tau)^\varepsilon = e^{-\varepsilon\overline{\delta}\tau}$. Therefore, the effect of population growth ($g_L > 0$) on the rate of GDP growth rate is similar to having a positive rate of obsolescence, $\overline{\delta} = g_L/\varepsilon$. This similarity can be interpreted as follows. When the speed of innovation accelerates according to (44), it means that there are relatively fewer old goods relative to new goods because R&D were slower in the past than now. In this sense, the situation is similar to the case where older goods disappear due to obsolescence. Therefore, a higher rate of population growth increases the fraction of the aggregate expenditure that goes to newer goods, and enhances the GDP growth rate, as we explained in Subsection 4.1.[39]

Conversely, the rate of GDP growth will be lower when the population is declining ($g_L < 0$). It is notable that real GDP growth may still be sustained in the long run, depending on parameters. For simplicity, suppose that newer goods are relatively more expensive than older ones, which guarantees that the expression in (46) is negative. Then, if goods retire from the market at a finite age $T < \infty$, both conditions (46) and (47) are satisfied, and Proposition 2 implies that the real GDP per capita will keep growing at a positive rate. When goods stay in the market forever ($T = \infty$), population decline makes positive growth more difficult. In this case, positive per capita GDP growth requires consumers to have a strong enough taste for new goods that will overcome the effect of the declining population. For example, when the consumers have utility function (41) and the rate of obsolescence is constant, $\delta(\tau) = e^{-\overline{\delta}\tau}$, positive per capita GDP growth can be sustained if $\varepsilon\overline{\delta} + g_L > 0$.

Note that the analysis above is about the measured per capita GDP growth, and not about welfare improvements. With a declining population, the speed of R&D ($\dot{N}_t$ in equation 44) will inevitably slow down to zero (Jones 2022). Therefore, our theory predicts stagnation of living standards despite the real GDP per capita growing at a positive rate. This is another example of the discrepancy between GDP growth and welfare improvements, as discussed in Subsection 3.7.

[39]In our theory, the rate of economic growth depends on the population growth rate, but not on the population size. In this sense, there is no scale effect.

5 Conclusion

The non-exponential growth theory provides an easier interpretation of the consequence of long-term growth. We have shown that economic growth does not necessarily mean "how many times" the economy is expanded. A constant rate of real GDP growth requires neither quantity, quality or variety to expand to an astronomical order in the future. Rather, the real GDP growth rate measures the new addition of the economic activity relative to the existing economic activity in terms of the market value, which should coincide with the marginal evaluation by consumers at that time.

If the marginal evaluation of each good is unchanged, a constant rate of economic growth cannot occur unless some component of production continues to expand exponentially. However, the marginal evaluation of each good, i.e., the relative market price of each good, changes endogenously as the economy grows. An increase in the production of a certain good may lower the evaluation of the value of the whole production of this good if the increased production is combined with a lower market price. Nonetheless, such a mechanism will work for sustaining growth, as it keeps the evaluation of existing production from exploding and maintains the relative evaluation of new goods high so that a constant flow of new goods can continue to constitute a positive GDP growth rate.

Our theory has several benefits when compared to conventional growth theories. First, it provides a robust theoretical basis on which economic analysis and empirical studies can be carried out without being constrained by knife-edge conditions. If older goods eventually disappear from the market, sustained growth is obtained as long as their quality-adjusted price falls somewhere in their lifecycle. Even in the least favorable setting where no good drops out from the market, the externality for productivity increase can be much weaker than usually assumed and is not necessarily exactly at a certain level.

Second, the non-exponential growth theory enables growth economists to focus on factors that were previously not viewed as a determinant of economic growth. Existing R&D-based models have shown that policies can influence the long-term growth rate only by affecting the rate of R&D.[40] The non-exponential growth theory, by contrast, shows the possibility that various factors affect the measured long-term rate of growth through the pattern of evolution in prices and quantities of individual products.

[40]See Aghion, Akcigit and Howitt (2014) for an excellent survey.

Third, existing endogenous growth models typically have only one engine of growth that drives the exponential growth of innovation. It is not very meaningful to consider several growth engines at the same time because the engine that realizes the highest rate of exponential expansion will dominate all the other factors. In contrast, our theory opens the door to combining various growth engines in one framework. All the factors that affect the evolution of the price or quantity of individual goods, as well as those that affect the age distribution of products, will show up in the long-term growth rate. We hope that it will eventually be used to quantify the composition of the source of economic growth.

References

Acemoglu, Daron and Veronica Guerrieri, 2008. "Capital Deepening and Nonbalanced Economic Growth," Journal of Political Economy, vol. 116(3), pages 467-498.

Aghion, Philippe; Akcigit, Ufuk, and Howitt, Peter, 2014. "What Do We Learn From Schumpeterian Growth Theory?," Handbook of Economic Growth, in: Handbook of Economic Growth, Volume 2B, chapter 1, pages 515-563 Elsevier.

Aghion, Philippe and Howitt, Peter, 1992. "A Model of Growth through Creative Destruction," Econometrica, vol. 60(2), pages 323-51.

Bolt, J., van Zanden, J.L., 2020. "Maddison style estimates of the evolution of the world economy. A new 2020 update" Maddison-Project Working Paper WP-15.

Bertola, Giuseppe; Reto Foellmi, and Josef Zweimuller, 2015. *Income Distribution in Macroeconomic Models*, Princeton University Press.

Bloom, Nicholas, Charles I. Jones, John Van Reenen, and Michael Webb. 2020. "Are Ideas Getting Harder to Find?" American Economic Review, 110 (4): 1104-44.

Griliches, Zvi, 1998. *R&D and Productivity: The Econometric Evidence*, NBER Books, National Bureau of Economic Research, Inc, number gril98-1, May.

Grossman, G.M., and Helpman, E., 1991a. "Expanding Product Variety," ch. 3 in G.M. Grossman and E. Helpman (ed.) *Innovation and Growth in the Global Economy*, Cambridge, MA: MIT Press.

Grossman, Gene M, and Helpman, Elhanan, 1991b. "Quality Ladders in the Theory of Growth," Review of Economic Studies, vol. 58(1), 43-61.

Growiec, Jakub , 2007. "Beyond the Linearity Critique: The Knife-edge Assumption of Steady-state Growth," Economic Theory, vol. 31(3), 489-499.

Jakub, Growiec, 2010. "Knife-edge conditions in the modeling of long-run growth regularities," Journal of Macroeconomics, vol. 32(4), 1143-1154.

Horii, Ryo, 2012. "Wants and past knowledge: Growth cycles with emerging industries," Journal of Economic Dynamics and Control, vol. 36(2), 220-238.

Jones, Charles I, 1995. "R&D-Based Models of Economic Growth," Journal of Political Economy, vol. 103(4), 759-84.

Jones, Charles I., 2002. "Sources of U.S. Economic Growth in a World of Ideas," American Economic Review, vol. 92(1), 220-239.

Jones, Charles I. 2022. "The End of Economic Growth? Unintended Consequences of a Declining Population." American Economic Review, 112 (11): 3489-3527.

Klenow, Peter J. and Rodriguez-Clare, Andres, 2005. "Externalities and Growth," in: Philippe Aghion and Steven Durlauf (ed.), Handbook of Economic Growth, volume 1, chapter 11, pages 817-861, Elsevier.

Lucas, R.E.JR., 1988. "On the mechanics of economic development," *Journal of Monetary Economics* 22, 3–42.

Ngai, L. Rachel, and Christopher A. Pissarides, 2007. "Structural Change in a Multisector Model of Growth," American Economic Review, vol. 97(1), 429-443.

O'donoghue, T., Scotchmer, S., and Thisse, J. F. 1998. "Patent breadth, patent life, and the pace of technological progress," Journal of Economics and Management Strategy, 7(1), 1-32.

Romer, Paul M, 1986. "Increasing Returns and Long-run Growth," Journal of Political Economy, vol. 94(5), 1002-37.

Romer, Paul M, 1990. "Endogenous Technological Change," Journal of Political Economy, vol. 98(5), S71-102.

Thompson, Peter. 2010. “Learning by Doing.” In Handbook of Economics of Innovation, edited by Bronwyn Hall and Nathan Rosenberg, 429–76, Elsevier.

Thompson, Peter. 2012. “The Relationship between Unit Cost and Cumulative Quantity and the Evidence for Organizational Learning-by-Doing.” Journal of Economic Perspectives, 26(3): 203-24.

Rebelo, S., 1991. “Long-run policy analysis and long-run growth,” *Journal of Political Economy* 99, 500–521.

Uzawa, Hirofumi, 1965. “Optimum technical change in an aggregative model of economic growth,” International Economic Review, 6(1), 18–31.

Young, Alwyn, 1991. “Learning by Doing and the Dynamic Effects of International Trade,” The Quarterly Journal of Economics, vol. 106(2), 369-405.