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Non-Exponential Growth Theory

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Abstract

Existing endogenous growth theories typically need a knife-edge degree of externality to explain long-term growth. However, micro-level observations do not confirm such an exact degree of externality. This puzzle occurs because sustained growth has been commonly understood as exponential growth in quantity, quality, or variety of outputs. By explicitly considering the movements of price and quantity of individual goods during the product lifecycle, this paper shows that the observed stability of the long-term real GDP growth can be explained under much weaker conditions without relying on the exponential growth of any variable. In particular, we develop a new endogenous growth theory where a constant number (not exponentially many) of new goods are introduced per unit of time. Even without externality, a positive and finite GDP growth rate is maintained when the expenditure for older goods shrinks over time so as not to inhibit the expenditure share given on newer goods.

Keywords: endogenous growth theory, knife-edge condition, externality, variety expansion, product lifecycle, balanced growth.

JEL Classification Codes: O41, O31.

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(a) Log per Capita Real GDP, USA (b) Per Capita Real GDP, Linear Scale

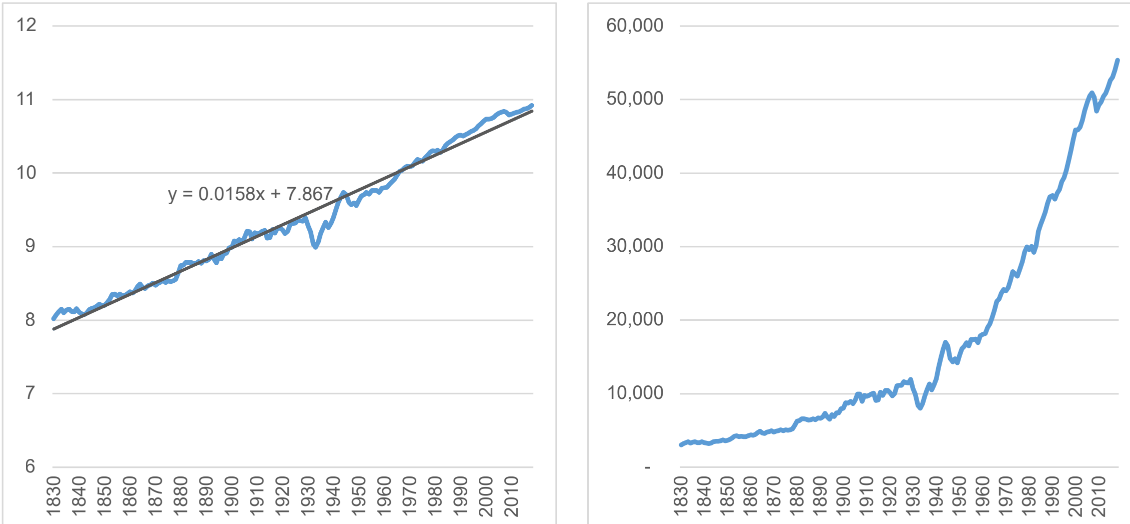


Figure 1: Long-term Evolution of Real GDP Per Capita in the United States from 1830 to 2018 (2011 International dollar). Source: Madison Project, Bolt and van Zanden (2020).

1 Introduction

Since around the time when the First Industrial Revolution was completed, the real GDP per capita growth in the United States has been surprisingly stable. Figure 1(a) depicts its time series on a log scale, where the slope of the series represents the growth rate. Although there were short- to mid-term fluctuations, the figure clearly shows the log of the real GDP per capita closely follows a linear trend, implying the long-term rate of per capita GDP growth is almost constant. Figure 1(b) shows the time path of the U.S. real GDP per capita on a linear scale without taking logs. Given that the GDP growth rate is stable, the level of the real GDP per capita is increasing exponentially in the long run.

Given these findings, it was natural for existing studies on endogenous growth to explain the phenomenon of long-term growth by models in which the per capita output continues to grow exponentially. Initially, this was an extremely challenging task because it was commonly understood that reproducible inputs are subject to diminishing returns, which implies that the accumulation of those factors cannot explain the exponential growth by themselves. The seminal studies in the endogenous growth theory thus overcame it by assuming strong intertemporal knowledge spillovers.

Figure 2 graphically explains the three representative formulations in endogenous growth theory and their specific assumptions. In variety-expansion models, as illustrated in panel (a), there should exist a strong knowledge externality from past R&D to new R&D, and the elasticity of this spillover ϕ needs to equal exactly one. Similarly, in quality ladder models (b), the increment in the quality by a successful R&D depends on the quality of the existing good, which is a result of the past stock of R&D, and the elasticity of this relation should again be exactly one. Finally, in AK-type growth models (c), the elasticity of production with respect to all reproducible factors and the elasticity of their externality effects must add up precisely to one.[1] In almost all endogenous growth models, long-term growth can only by sustained when one of such knife-edge conditions is satisfied.[2]

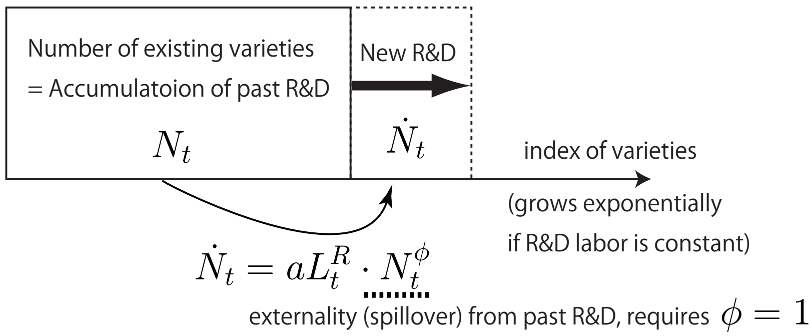
Now, here is a puzzle. Indeed, the externality and non-rivalry of knowledge play essential roles in improving productivity (e.g. Griliches, 1998). However, if we look at the process of spillover more precisely, no concrete evidence supports any of these exact assumptions. As for the elasticity of spillover ϕ in R&D-driven growth models, Jones (1995) clearly stated, *"$\phi = 1$ represents a completely arbitrary degree of increasing returns and, ... is inconsistent with a broad range of time-series data on R&D and TFP growth."* He convincingly stated that $\phi = 0$ is the most natural case, and while ϕ can either be negative by the "fishing out effect" or positive by the "better tools effect," it is reasonable to assume $\phi < 1$.[3] Bloom

[1]When there are multiple sectors, a sector that produces a reproducible factor (typically physical or human capital) must satisfy this restriction. For example, Lucas (1988) initially introduced a human capital accumulation function $\dot{h}_t = h_t^{\zeta} G(1 - u_t)$ and then made an assumption of $\zeta = 1$ following Uzawa (1965). By so doing, he wrote, "the feature that recommends his formulation to us, is that it exhibits sustained per-capita income growth," which gives a clear example of a case where such a knife-edge assumption is justified not by micro-level observations but by the aggregate outcome. Lucas also noted that "human capital accumulation is a *social* activity," which suggests that the elasticity $\zeta = 1$ includes the effect of externality.

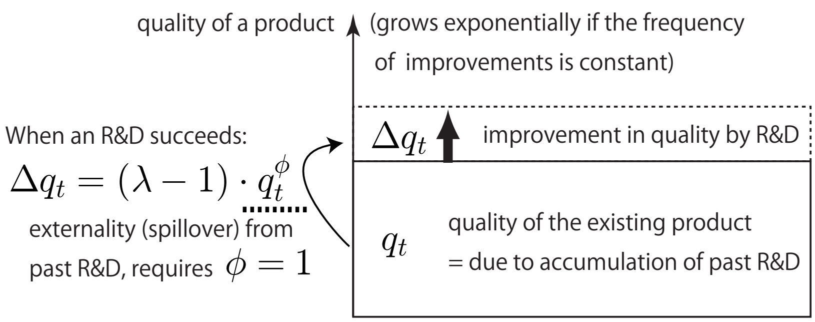
[2]Growiec (2007, 2010) formally proved that, with any generalization in the functional forms, exponential growth cannot be explained without imposing at least one knife-edge assumption in the model.

[3]By assuming $\phi < 1$, Jones (1995) developed the *semi-endogenous* growth theory, where the long-term rate of growth is ultimately driven by population growth. By extending the theory to include the transitional increase in research intensity and educational attainments, Jones (2002) showed that it is possible to explain

(a) Variety Expansion Models: e.g., Romer (1990), Grossman and Helpman (1991a)



(b) Quality Ladder models: e.g.,Grossman and Helpman (1991b) , Aghion and Howitt (1992)



(c) AK-type growth models: e.g., Romer (1986), Rebelo (1991)

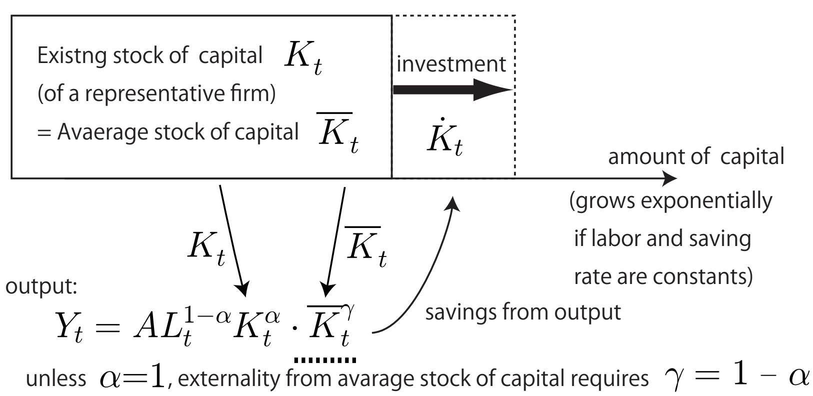


Figure 2: The Exact Degree of Externality Required in Three Types of Endogenous Growth Models

et al. (2020) found that it is getting increasingly harder to improve the quality of goods at a constant exponential rate.[4] Klenow and Rodriguez-Clare (2005, Section 3) reviewed various AK-type models. They concluded that such models are empirically implausible based on the lack of a tight enough relationship between the investment rates and growth rates in cross-country data.

Why have the U.S. and other prosperous countries grown quite steadily, even though existing endogenous growth theories imply sustained growth is possible only under knife-edge conditions that seem very hard to justify by data? This paper shows that this inconsistency occurs because we have interpreted the observed sustained growth as an exponential process. If we consider a single final output (a scalar variable), sustained growth always means exponential growth. This is true even in most of the existing R&D-based growth models with many varieties, as they measure the growth rate after aggregating the varieties into a single final output.[5]

However, the real GDP growth rate in SNA statistics (e.g., the NIPA in the U.S.) is measured without aggregating all outputs into one variable. It is obtained by comparing quantities of various product groups in adjacent years, using the same set of prices for both years. Then, the aggregate level of real GDP is constructed by the chain rule. Conceptually, the SNA statistics construct the level of real GDP at a given year t from the measured real

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the observed *constant growth path* for a certain period of time. In his theory, future growth is predicted to slow down permanently and eventually will come to an end, given that there are upper limits in population, research intensity, and education attainments (Jones, 2022). Under a natural assumption of $\phi = 0$, this paper tries to present an alternative interpretation by developing a *full*-endogenous growth theory where the measured economic growth can continue indefinitely with a constant population.

[4]They reported that the number of researchers required today to achieve the famous doubling of computer chip density is more than 18 times larger than the number required in the early 1970s. This is inconsistent with the typical quality ladder specification, as illustrated in 2(b).

[5]A notable exception is Young (1991), who considered a learning-by-doing (LBD) model with many goods and calculated the growth rate before aggregation. In addition, he recognized that there is an upper bound in productivity gain from LBD for individual good. To obtain a constant rate of growth, however, he needed to assume that the upper bound itself improves exponentially through the aggregate LBD. Horii (2012) used the same disaggregated definition of growth rate, but it is a semi-endogenous growth model as it relied on an exponential increase in population.

GDP growth rate g_t as follows:

$$\Big[\text{Real GDP at year } t\Big] = \Big[\text{Real GDP at reference year } \bar{t}\Big] \times \exp\left(\int_{\bar{t}}^{t} g_\tau d\tau\right),$$

where $\bar{t}$ is the reference year, which can be chosen freely. Real GDP at reference year $\bar{t}$ can also be set arbitrarily (because this is just an index), but for ease of interpretation, it is usually set to the nominal GDP in year $\bar{t}$. Therefore, even when we see the exponential growth in the time series of real GDP, it only says that the g_t is stationary over time.

Focusing on this fact, this paper solves the puzzle by showing that steady state growth (constancy in the measured per capita real GDP growth rate) does not necessarily imply an exponential increase either in quantity, quality, or variety of products. Because there is no need to explain the exponential increase in any variable, and we do not need to make knife-edge assumptions to support steady-state growth.

This paper shows that the measured GDP growth rate becomes a positive constant when (i) new goods (or services) are continually introduced to the market, (ii) the quality-adjusted price of each good falls as they get older when compared to newer goods, and (iii) the expenditure share for the very old goods is limited. Condition (i) does not require the number of goods to increase exponentially. The baseline theory shows that growth can be sustained when a constant number of new goods are introduced every year.. Conditions (ii) and (iii) state that the price and demand for each good should follow the well-observed pattern of the product lifecycle. In the long run, it is less counter-intuitive to expect this type of economic movement to continue rather than to expect the output quantity, quality, or variety to be expanded by astronomical orders. Of course, knowledge externalities are nevertheless crucial for growth as they often work behind the quality improvements and cost reductions of existing goods, and our prototype endogenous model incorporates these. Still, we have shown that the fall in quality-adjusted prices does not need to occur at an exponential speed. A weaker externality is sufficient for sustaining growth.

Studies recent studies view long-term growth differently than an exponential increase in final output at the rate of the measured GDP growth rate. León-Ledesma and Moro (2020) consider a two-sector model and calculate the growth rate using the methodology employed in NIPA. They showed that the shift in the expenditure share from goods to services explains cross-country growth facts. This paper shows not just from goods to services, but continual shifts in expenditure shares from older goods and services to newer ones are crucial in sustaining growth. Aghion et al. (2019) examine the possibility that the measured GDP growth rate underestimates the welfare gains from creative destruction. Complementarily to their study, this paper shows another fundamental reason why the measured GDP growth may not represent an increase in welfare in the long run. Philippon (2022) suggests that a linear trend fits the TFP data better than an exponential trend for periods ranging between several decades to a few centuries. In his theory, long-term growth can be sustained only when there are occasional changes in the linear trend (e.g., by the arrival of general-purpose technologies), and the slope of the linear trend needs to jump up exponentially. This paper explores the mechanism of sustained growth that does not require exponential increases and knife-edge degree of externalities, even in the very long run.

The rest of the paper is constructed as follows. Section 2 presents a theory of sustained growth without exponential expansion. By explicitly focusing on the product lifecycle, we explain that the conventionally measured real GDP growth rate becomes asymptotically constant even when no variable grows exponentially. Section 3 develops a prototype endogenous growth model without knife-edge assumptions and examines how the long-term rate of economic growth is determined in equilibrium. Section 4 generalizes the theory and the prototype model in several directions so as to demonstrate that we can obtain a positive constant real GDP growth rate in wider (even less restrictive) situations. Section 5 concludes.

2 Theory

This section presents a theory that demonstrates the measured real GDP growth is sustained even when no underlying economic variable grows exponentially. More specifically, Subsection 2.1 discusses the evolution of the prices and quantities of individual goods in a setting where the number of varieties expands linearly over time rather than exponentially. Subsection 2.2 explains the definition of the real GDP growth rate when the quantity and price of goods change individually over their product lifecycles. After deriving the real GDP growth rate in the steady state in Subsection 2.3, Subsection 2.4 offers graphical examples that illustrate the connection between the pattern of the product lifecycle and the measured real GDP growth rate. To retain the flexibility of the main result, this section does not specify the entire model structure. It will be shown in Section 3 that such dynamics can be obtained as an equilibrium outcome in a general equilibrium model.

2.1 Steady-State Growth Dynamics with Product Life Cycle

Let us consider an economy with a constant population and many goods. We follow a convention in the variety-expansion model by calling them goods, but it is more suitable to think of each good in the theory as a group of products based on the same technology. Each good is indexed by $i \in [0, N_t]$, where $i = 0$ is the oldest while $i = N_t$ is the most recently introduced good. Suppose that the number of goods N_t increases through R&D, and in the long run, it increases by a positive constant n per unit of time:

$$\dot{N}_t \to n > 0 \quad \text{as} \quad t \to \infty. \tag{1}$$

Recall that, as illustrated in Figure 2(a), existing variety-expansion models required a strong and exact degree of knowledge spillovers to maintain the exponential expansion of varieties, where $\dot{N}_t/N_t$ is constant. In contrast, the linear increase of N_t in (1) does not require such strong knowledge spillovers within the R&D sector, as we will see in a general equilibrium model in Section 3.

However, because (1) implies $\dot{N}_t/N_t \to 0$, it is clear that introducing new goods alone cannot explain sustained growth. Therefore, we explicitly consider the changes in the prices and quantities of individual goods after their introduction. Let $\widetilde{p}_t(i)$ and $\widetilde{x}_t(i)$ denote the price and quantity of each good i at time t and allow them to change over time. We normalize the price level at each instant to keep the nominal expenditure (per capita) constant in the long run. As in the SNA statistic, we define $\widetilde{p}_t(i)$ and $\widetilde{x}_t(i)$ as quality-adjusted. For example, if the quality of good i is doubled (so that consumers receive the same utility from half the quantity), then our measure of $\widetilde{x}_t(i)$ is doubled, while $\widetilde{p}_t(i)$ is halved.

Let $s(i)$ denote the time when good i is developed. It is convenient to label each good by its age, $\tau \equiv t - s(i)$, i.e., time passed from its introduction. In the long run, where n

new goods are introduced per unit of time, age τ good refers to the $n\tau$th newest good. This means that the index of a good i and its age τ have the following relationship.

$$i = N_t - n\tau, \text{ or equivalently, } \tau \equiv t - s(i) = \frac{N_t - i}{n}. \tag{2}$$

With this notation, let us say that the economy has come to a steady state if every good's price and quantity follow the same time evolution against τ. Formally, the economy can be said to be converging to a steady state if there exist time-invariant functions $p(\tau)$ and $x(\tau)$ such that

$$\widetilde{p}_t(i) \to p(t - s(i)) \equiv p(\tau), \quad \widetilde{x}_t(i) \to x(t - s(i)) \equiv x(\tau) \quad \text{as} \quad t \to \infty. \tag{3}$$

Let $T > 0$ denote the age beyond which the product is never produced. In typical variety-expansion endogenous growth models, goods never retire from the market. In this case, $T = \infty$. However, in practice, we see many products disappear after some time. Our theory can be applied to both cases where T is finite or infinite.

We assume $p(\tau)$ and $x(\tau)$ satisfy the following conditions.

Assumption 1.
(i) Both $p(\tau)$ and $x(\tau)$ are non-negative and differentiable for all $0 < \tau < T$, where T is such that $x(\tau) = 0$ for all $\tau \geq T$.
(ii) T can be infinite, but $p(\tau)$ and $x(\tau)$ does not grow exponentially: $\lim_{\tau\to\infty} p'(\tau)/p(\tau) \leq 0$ *and* $\lim_{\tau\to\infty} x'(\tau)/x(\tau) \leq 0$ *if $T = \infty$.*[6]
(iii) The newest good's price and quantity are strictly positive: $p(0) > 0$ and $x(0) > 0$.

By Assumption 1(i), the present paper focuses on the continuous setting because it is mathematically less demanding and does not sacrifice intuitions. Since $x(\tau)$ represents the quality-adjusted quantity, Assumption 1(ii), combined with (1), guarantees that neither quantity, quality, or variety grows exponentially in this economy. Assumption 1(iii) is an obvious one. When we say a new good appears in the market, it should imply that the expenditure for the good, $p(0)x(0)$, is positive.

[6] Note that the time derivative of quantity in the steady state is $\dot{\widetilde{x}}_t(i) = \frac{d}{dt}x(t-s(i)) = x'(t-s(i)) = x'(\tau)$. Therefore, $x'(\tau)/x(\tau) = \dot{\widetilde{x}}_t(i)/\widetilde{x}_t(i)$ represents the growth rate of the quantity of age τ good, or equivalently, that of index $i = N_t - n\tau$ good. Similarly, $p'(\tau)/p(\tau) = \dot{\widetilde{p}}_t(i)/\widetilde{p}_t(i)$ in the steady state.

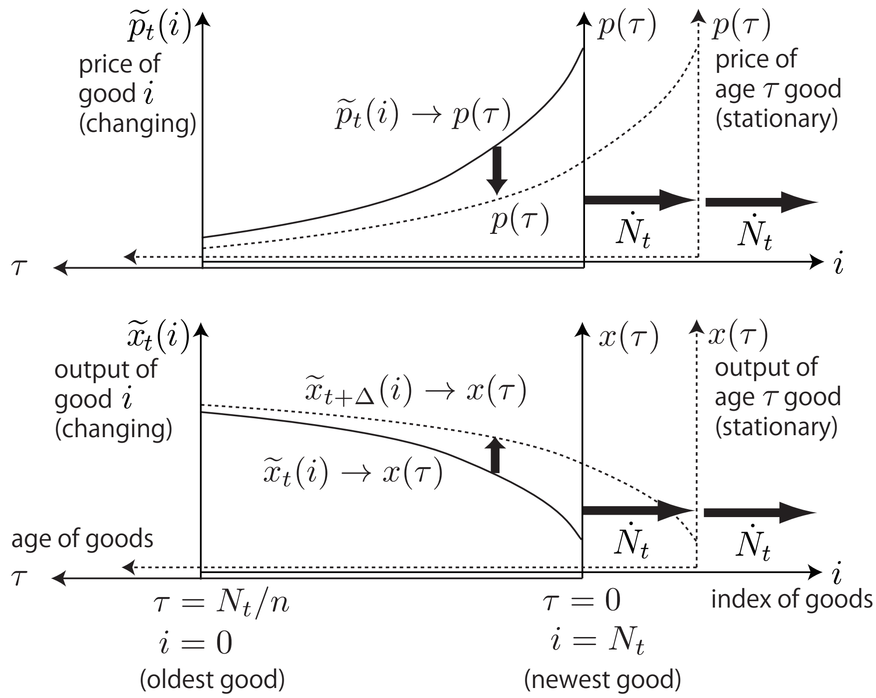


Figure 3: Evolution of Prices and Outputs of Goods in a Non-Exponential Steady-State.

Definition 1. *A non-exponential asymptotic steady-state is the situation wherein the number of goods follows (1) and the paths of quality-adjusted prices and quantities of goods,* $\widetilde{p}_t(i)$ *and* $\widetilde{x}_t(i)$*, satisfy condition (3) and Assumption 1 as* $t \to \infty$*.*

In the remainder of the paper, we simply call it a steady state unless it is confusing. Figure 3 intuitively depicts the evolution of the quality-adjusted prices and quantities in the above definition of the steady state. The graphs can be viewed two ways: either drawn against the i-axis (index of goods) running from left to right, or drawn against the τ-axis (age of goods) running in the opposite direction. The two variables, i and τ, are related according to (2), but the relationship changes over time as N_t increases. At time t, the origin of the τ-axis coincides with the point of $i = N_t$ on the i-axis because the newest good $i = N_t$ is age $\tau = 0$ at time t. As time passes, the origin of the τ-axis moves to the right with the speed of the introduction of new goods, $\dot{N}_t = n$, and so does the position of the origin of the graph drawn against τ.

The upper panel of Figure 3 illustrates a case where quality-adjusted price $p(\tau)$ is decreasing in age τ, either because a product becomes cheaper or has higher quality as it ages after its introduction. Since the newer goods have a larger index i, it also means $\widetilde{p}_t(i)$ is increasing in i at any given time t. The figure also explains the movement of the price of each good $\widetilde{p}_t(i)$ over time. Even in the steady state where function $p(\tau)$ is stationary, the price of individual good $\widetilde{p}_t(i)$ shifts downward to the dotted curve because the position of function $p(\tau)$ continues to move to the right as new goods are developed.[7]

Similarly, the lower panel of Figure 3 explains the evolution of quality-adjusted quantities of goods over time. The panel shows the case where $x(\tau)$ is increasing in τ, which naturally matches our example that the older goods have a lower quality-adjusted price. In this case, the demand for each good $\widetilde{x}_t(i)$ increases over time as the $x(\tau)$ function shifts to the right. Note that, however, Assumption 1(ii) rules out exponential growth in the quantity of any good. Even when $T = \infty$, the growth rate of $x(\tau)$ must be either zero or negative as $\tau \to \infty$. Contrary to this particular example, we can also consider the possibility of the quantity shrinking with age, even when older goods are cheaper. Such a pattern will emerge when consumers do not like outdated goods or if newer goods replace parts of functions that are provided by older goods, as we discuss later in Subsection 4.1.

2.2 Measuring the GDP Growth Rate

When considering the growth rate in a model with many goods, an often employed practice is to consider an aggregation function to obtain a measure of the final output (a scalar variable) and then measure its growth rate. A problem with this methodology is that the resulting growth rate depends on the choice of the aggregation function. Because the final output is a virtual notion, there is no guarantee that the growth rate obtained in this way matches the numbers in the official statistics. Another way is to select a numéraire good, aggregate the outputs of various goods with the observed relative prices, and then calculate the growth rate of aggregated output. However, there is again a problem of choosing the

[7] Although this is a convenient way to explain the steady-state dynamics, note that the economic environment, such as technology, preference, and market structure, first determines the evolution of the price of individual goods $\widetilde{p}_t(i)$ in equilibrium. Then, the long-term pattern of movement in $\widetilde{p}_t(i)$ shapes the stationary $p(\tau)$ function as a result.

numéraire because the obtained growth rate depends on this choice since the relative price changes over time. Another substantial problem is that no good is representative enough to be the numéraire for a long time because the expenditure shares across goods are always changing.

To avoid these problems, this paper directly looks at the changes in the prices and quantities of all goods without aggregating them into a single variable. We follow the standard procedure to construct the GDP growth rate that is often explained in undergraduate-level macroeconomics textbooks. In the SNA statistics, the real GDP growth rate is measured by comparing the value of outputs between two consecutive years, say year $t-1$ and year t. Their values are measured using the common set of prices, which usually is the set of observed prices in a given base year. Because the base year is frequently updated in official statistics and also because this paper is interested in long-term dynamics, we suppose that there is no gap between the base year and the year in which the growth rate is computed.[8] Then, using the notation from the previous subsection, the real GDP growth rate between years $t-1$ and t can be written as follows.[9]

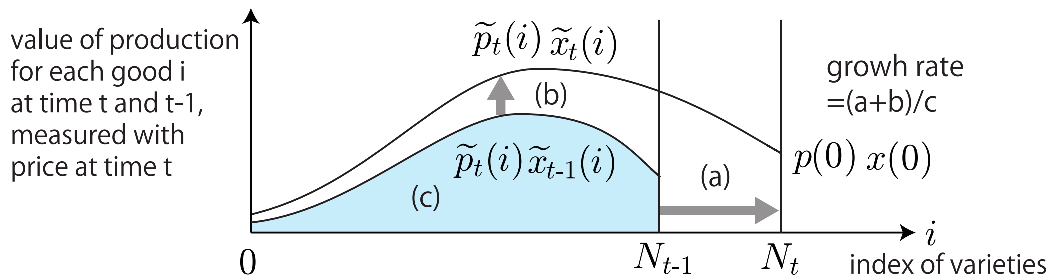
$$g_{t-1,t} = \frac{\int_{N_{t-1}}^{N_t} \widetilde{p}_t(i)\widetilde{x}_t(i)di + \int_0^{N_{t-1}} \widetilde{p}_t(i)\left(\widetilde{x}_t(i) - \widetilde{x}_{t-1}(i)\right)di}{\int_0^{N_t} \widetilde{p}_t(i)\widetilde{x}_{t-1}(i)di}. \tag{4}$$

This formula is made up of the integrals of two functions, $\widetilde{p}_t(i)\widetilde{x}_t(i)$ and $\widetilde{p}_t(i)\widetilde{x}_{t-1}(i)$. Figure 4 depicts the curves of these two functions against t for two cases, where the demand for existing goods always increases with time (Case 1) and where demand for existing goods shrinks in some part of their lifecycle (Case 2). In Case 1, observe that area (a) represents

[8]In the U.S., the NIPA computes the growth rate in two ways: by setting the base year to t and also by setting it to $t-1$. Then, the agency calculates their geometric average. Here, we only calculate the growth rate in which the base year is t, but the difference disappears in the limit where the period length is brought to 0, as we do in the next subsection.

[9]To ease the understanding, here we employed a slight abuse of notation and treated $\widetilde{x}_t(i)$ as if it is a discrete-time variable. In the previous subsection, we defined $\widetilde{x}_t(i)$ as the instantaneous flow of output quantity at time t per unit time. Since the SNA statistics use the cumulative output of good i for a given time period (e.g., a year or a quarter), we need to integrate $\widetilde{x}_t(i)$ for the duration of the time period to obtain the exact real GDP growth rate. As we take the limit where the duration of one period is almost zero, we confirmed that this exact GDP growth rate converges to the same expression as in (5).

Case 1: When $\widetilde{x}_t(i)$ is always increasing in t.



Case 2: When $\widetilde{x}_t(i)$ becomes decreasing in t sometime after introduction.

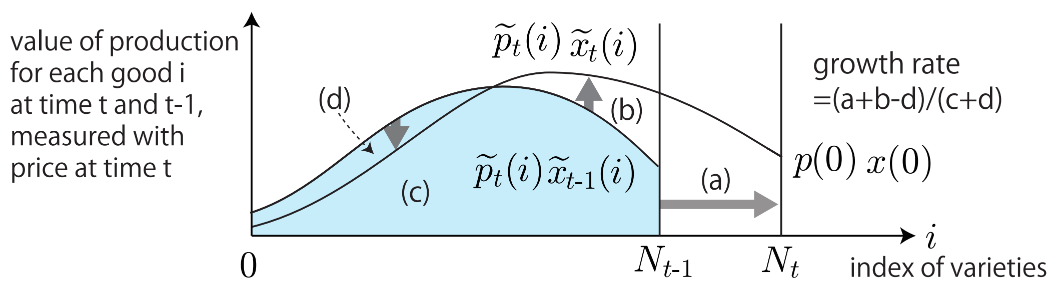


Figure 4: Calculation of the Real GDP Growth Rate: Two Cases

the sum of the values of new goods introduced between time $t-1$ and time t, evaluated by the prices at time t. Similarly, area (b) represents the value of the increased production of goods that already existed at time $t-1$. These two areas measure how economic activity has increased from time $t-1$ to time t and correspond to the two terms in the numerator of definition (4). Area (c) represents the value of total production in time $t-1$, evaluated again by the prices at time t. This area corresponds to the denominator of the definition (4). In this way, the real GDP growth rate can be understood as the ratio of area (a)+(b) to area (c), which measures the rate at which the economic activity at time t is increased from time $t-1$.

This procedure can be generalized to the case where output quantity $\widetilde{x}_t(i)$ is not monotonic in t. Case 2 in Figure 3 illustrates an example where production of a certain range of goods declines between period $t-1$ and t. Then, a portion of curve $\widetilde{p}_t(i)\widetilde{x}_t(i)$ comes below curve $\widetilde{p}_t(i)\widetilde{x}_{t-1}(i)$. In this case, the real GDP growth rate is given by the ratio of area (a)+(b)−(d) to area (c)+(d).

2.3 Measured GDP growth rate in the Steady State

So far, Subsection 2.1 presented steady state dynamics with many goods where no variable grows exponentially, and Subsection 2.2 explained how the GDP growth rate could be measured when there are many goods. Now, we are ready to examine whether the non-exponential steady state implies a positive and constant real GDP growth rate.

Note that the conventional definition of real GDP growth in (4) gives the average growth rate between two discrete time periods. To map this definition to a continuous-time growth model, it is convenient to consider the instantaneous growth rate g_t at time t. The instantaneous growth rate can be obtained by replacing $t-1$ in (4) by $t-\Delta$, and taking the limit of $\Delta \to 0$ in $g_{t-\Delta,t}/\Delta$.

$$g_t = \lim_{\Delta \to 0} \frac{g_{t-\Delta,t}}{\Delta} = \frac{\dot{N}_t \cdot \widetilde{p}_t(N_t)\widetilde{x}_t(N_t) + \int_0^{N_t} \widetilde{p}_t(i)\dot{\widetilde{x}}_t(i)di}{\int_0^{N_t} \widetilde{p}_t(i)\widetilde{x}_t(i)di}. \tag{5}$$

Suppose that the economy converges to a steady state, as defined in Definition 1. The number of goods grows linearly, and the evolution of prices and quantity in terms of age becomes stationary. Then, given that $\int_0^T p(\tau)x(\tau)d\tau$ is finite, the long-term growth rate can be obtained by substituting (1)-(3) into (5).[10]

$$g_t \to g \equiv \lim_{\overline{T} \to T} \frac{np(0)x(0) + n\int_0^{\overline{T}} p(\tau)x'(\tau)d\tau}{n\int_0^{\overline{T}} p(\tau)x(\tau)d\tau} \quad \text{as} \quad t \to \infty. \tag{6}$$

The interpretation of growth rate (6) is essentially the same as in definition (4), except for the fact that now growth is represented in terms of age and also in continuous time. In the numerator, $np(0)x(0)$ represents the value of newly introduced goods, whereas

[10] Equation (6) can be obtained from (5) as follows. First, substitute $p(\tau)$ and $x(\tau)$ for $\widetilde{p}_t(i)$ and $\widetilde{x}_t(i)$. Similarly, $\dot{\widetilde{x}}_t(i)$ can be written as $x'(\tau)$ (See footnote 6). Next, we change the integration variable from di in (5) to $d\tau$. By differentiating (2) for given t, we obtain $di = nd\tau$. We also need to change the integration interval. From (2), $i = 0$ and $i = N_t$ respectively correspond to $\tau = N_t/n$ and $\tau = 0$ as illustrated in Figure 3. As $t \to \infty$, N_t/n also approaches ∞. From these, $\lim_{t\to\infty} \int_0^{N_t} \widetilde{p}_t(i)\widetilde{x}_t(i)di = \lim_{t\to\infty} \int_{N_t/n}^0 p(\tau)x(\tau)(-n)d\tau \to n\int_0^\infty p(\tau)x(\tau)d\tau$. However, since $x(\tau) = 0$ for $\tau \geq T$, the limit becomes $n\int_0^T p(\tau)x(\tau)d\tau$. Likewise, the limit of the numerator of (5) is $np(0)x(0) + n\int_0^T p(\tau)x'(\tau)d\tau$. If T is finite, both limits are finite, and therefore we can substitute these into (5). However, when $T = \infty$, the limits may be infinite. In this case, we use a large but finite $\overline{T}$ in place of T before substituting them into (5), and take the limit of $\overline{T} \to T = \infty$ for the whole fraction, as shown in (6).

$n \int_0^T p(\tau) x'(\tau) d\tau$ gives the value of changes in quantities of existing goods given price function $p(\tau)$. Both terms are multiplied by n because there are n goods per unit of age. The sum of these terms gives the speed of increase in economic activity. The denominator of (6), $n \int_0^T p(\tau) x(\tau) d\tau$, gives the value of existing production, i.e., the nominal GDP of the economy given prices $p(\tau)$. The ratio of the two gives the real GDP growth rate.

In the steady state, the following proposition expresses the long-term GDP growth rate using the pattern of the evolution of prices and quantities in the product lifecycle.

Proposition 1. *Suppose that the economy converges to the asymptotic non-exponential steady-state, as defined by Definition 1. Then the real GDP growth rate g_t asymptotes to g in the long run, where g is given as follows:*
(i) If $\int_0^T p(\tau) x(\tau) d\tau$ is finite (which is always true when T is finite), then[11]

$$g = \frac{-\int_0^T x(\tau) dp(\tau)}{\int_0^T p(\tau) x(\tau) d\tau}. \tag{7}$$

(ii) If $\int_0^T p(\tau) x(\tau) d\tau = \infty$, then $g = 0$.

Proof. (i) When $\int_0^T p(\tau) x(\tau) d\tau$ is finite, we can take out $\lim_{\overline{T} \to T}$ in the RHS of (6) and replace $\overline{T}$ with T. In its numerator, integration by parts implies $\int_0^T p(\tau) x'(\tau) d\tau = p(T)x(T) - p(0)x(0) - \int_0^T p'(\tau) x(\tau) d\tau$, where $p(0)x(0)$ cancels out. When T is finite, $p(T)x(T) = 0$. When $T = \infty$, the finiteness of $\int_0^T p(\tau) x(\tau) d\tau$ implies $\lim_{\tau \to \infty} p(\tau) x(\tau) = 0$ (i.e. $p(T)x(T) = 0$). Therefore, we obtain (7).
(ii) In this case, T is necessarily ∞. If $\int_0^\infty p(\tau) x'(\tau) d\tau$ is finite, the result directly follows from (6). Now suppose $\int_0^\infty p(\tau) x'(\tau) d\tau$ is either $+\infty$ or $-\infty$. Since both the numerator and the denominator in (6) are infinite, we apply L'Hospital's rule to (6) to get

$$g = \lim_{\overline{T} \to \infty} \frac{p\left(\overline{T}\right) x'\left(\overline{T}\right)}{p\left(\overline{T}\right) x\left(\overline{T}\right)} = \lim_{\overline{T} \to \infty} \frac{x'\left(\overline{T}\right)}{x\left(\overline{T}\right)} \leq 0, \tag{8}$$

where the last inequality follows from Assumption 1(ii). In the following, we show that $g < 0$ does not happen by contradiction. For g to be strictly negative, $x(\tau)$ needs to shrink exponentially, which also means that $x'(\tau)$ must shrink exponentially. However,

[11] Note that $\int_0^T x(\tau) dp(\tau)$ is equivalent to $\int_0^T p'(\tau) x(\tau) d\tau$ given that $p'(\tau)$ exists.

from $\lim_{\tau\to\infty} p'(\tau)/p(\tau) \leq 0$ in Assumption 1(ii), it would imply $\int_0^T p(\tau)x'(\tau)d\tau$ is finite since $p(\tau)x'(\tau)$ should be shrinking exponentially. It contradicts the initial assumption.

□

Proposition 1 immediately implies the requirements for positive long-term GDP growth.

Corollary 1. *The long-term real GDP growth rate g is a positive and finite constant if and only if the following two conditions are satisfied.*[12]

$$-\int_0^T x(\tau)dp(\tau) \text{ is finite and strictly positive, and} \tag{9}$$

$$\int_0^T p(\tau)x(\tau)d\tau \text{ is finite.} \tag{10}$$

The expression in (9), $-\int_0^T x(\tau)dp(\tau)$, is the numerator of (7). It represents the cumulative reductions in the quality-adjusted price during the whole product lifecycle. When the quality-adjusted price of goods declines, consumers have more purchasing power, improving their utility. This income effect from price reductions is more significant when the quantity of the good is larger. Therefore, in (9), the price reduction $-dp(\tau)$ is weighted by quantity $x(\tau)$ and then integrated. The integrated sum gives the total income effect that one product generates over its product lifecycle. The expression $\int_0^T p(\tau)x(\tau)d\tau$ in (10) is the denominator of (7). It is the cumulative expenditure attracted by one product. Proposition 1 says that if every product follows the same lifecycle pattern, the real GDP growth rate in the economy is given by the ratio of the two. If both values are positive and finite in a non-exponential steady state, as defined in Definition 1, then it indicates that the real GDP growth rate can be sustained even when no variable grows exponentially. We will first provide three examples in the following subsection and then discuss the implications of conditions (9) and (10) in more detail in Subsection 2.5.

2.4 Graphical Examples

Proposition 1 shows that the real GDP growth rate depends only on functions $p(\tau)$ and $x(\tau)$. We can represent the real GDP growth rate graphically using the shapes of these two

[12]Note that $\int_0^T p(\tau)x(\tau)d\tau$ is always strictly positive from Assumption 1, and therefore we require only finiteness in (10).

functions. Figure 5 provides three examples.

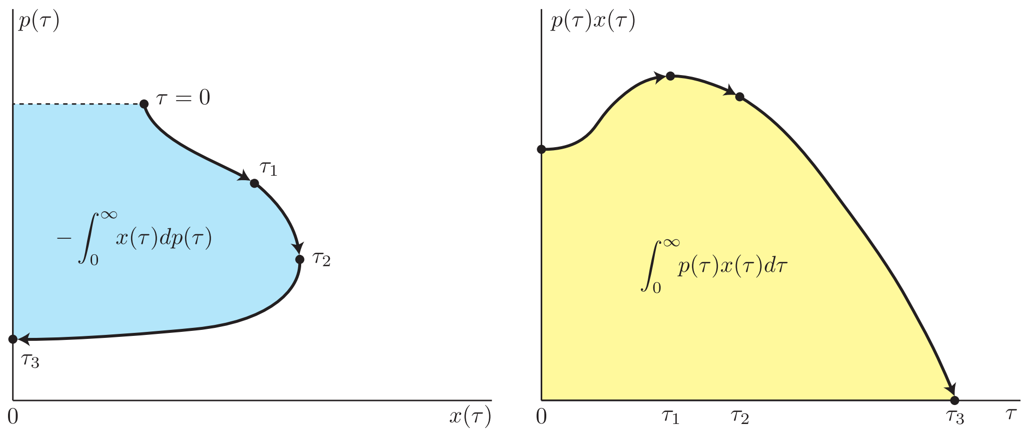
Example 1 shows the simplest case, where the quality-adjusted price weakly falls with age throughout the product lifecycle. The left panel depicts the evolution of $\{x(\tau), p(\tau)\}$ in the x-p diagram. T is finite in this example. The good enters the market at the point of $\{x(0), p(0)\}$ and continues to be produced until its age reaches $T = \tau_3$. Then, the numerator, $-\int_0^T x(\tau)dp(\tau)$, can be expressed by the area that is encompassed by the locus of $\{p(\tau), x(\tau)\}$ and the vertical axis in the x-p diagram (shown in blue). This graphical representation can be interpreted as follows. Whenever the quality-adjusted price falls by $dp(\tau)$, either through cost reductions or through quality improvements, consumers can save the purchasing power by the amount of $-x(\tau)dp(\tau)$. The blue area shows the cumulative benefits of this good throughout its lifetime. The area is positive and finite as long as $p(0) < p(T)$.[13]

The right panel plots the evolution of expenditure for a single good against its age, $p(\tau)x(\tau)$. The area below the curve (shown in yellow) gives the denominator, $\int_0^T p(\tau)x(\tau)d\tau$. From Assumption 1, the expenditure for the good is strictly positive at the time of introduction, and it evolves in the non-negative region during its lifetime. Since expenditure $p(\tau)x(\tau)$ falls to zero at finite $T = \tau_3$, this area is positive and finite. Proposition 1 says that the ratio of the blue area to the yellow area gives the real GDP growth rate. Therefore, we can conclude that the real GDP growth rate in this example is positive and finite.

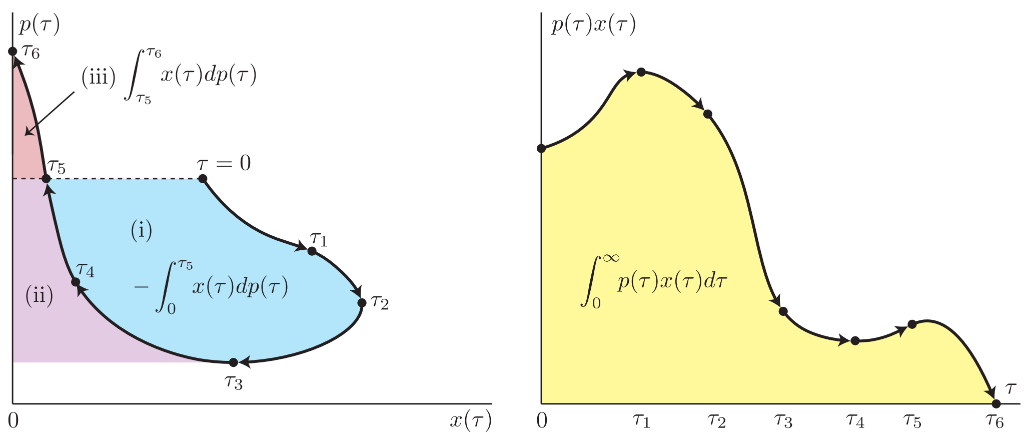
Next, Example 2 considers a case where $p(\tau)$ is not monotonic. Here, the quality-adjusted price begins to increase after τ_3 years. When the price of the good (relative to the newest good) rises in some stage of its lifecycle, the area between this part of the locus in the x-p diagram (from $\tau = \tau_3$ to τ_6) and the vertical axis represents the loss of the purchasing power of consumers. This area needs to be deducted from the benefits of the fall in quality-adjusted prices from $\tau = 0$ to τ_3. Therefore, the numerator, $-\int_0^T x(\tau)dp(\tau)$, is given by area (i) minus area (iii) because area (ii) cancels out. It can be either positive or negative but is always finite since $T = \tau_6$ is finite. Again, the yellow area in the right panel gives the denominator, $\int_0^T p(\tau)x(\tau)d\tau$, which is positive and finite. Therefore, the real GDP growth rate is finite, which is given by the ratio of the blue minus red area to the

[13] $p(0) < p(T)$ requires the price to fall strictly with age in some part of the good's life.

Example 1: When T is finite and $p(\tau)$ is weakly decreasing



Example 2: When T is finite and $p(\tau)$ is non-monotonic



Example 3: When $T = \infty$ and $p(\tau)$ is decreasing

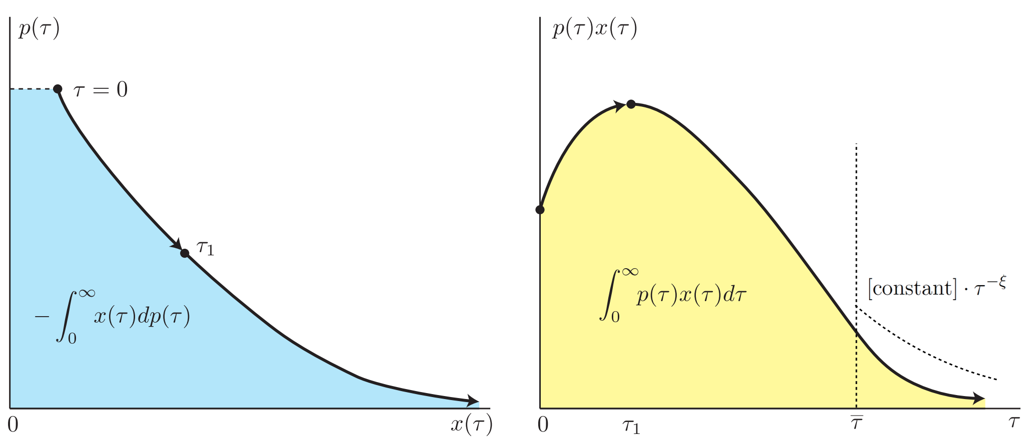


Figure 5: Graphical Representation of the Real GDP Growth Rate. The growth rate is measured by the ratio of the areas of the two panels.

yellow area. Also, note that the growth rate becomes zero only by coincidence, only when the blue and red areas are the same size.

Finally, Example 3 shows a case when the good stays in the market forever ($T = \infty$). The price $p(\tau)$ (relative to the newest good) falls throughout the lifecycle, and the quantity $x(\tau)$ remains positive as $\tau \to \infty$. For the yellow area to be finite, the expenditure for very old goods has to shrink. More concretely, condition (10) is satisfied if expenditure for old goods is bounded by a polynomial function of age with the power of less than -1:[14]

$$p(\tau)x(\tau) \leq [\text{constant}] \cdot \tau^{-\xi} \text{ for all } \tau \geq \overline{\tau}, \tag{11}$$

for some $\xi > 1$ and $\overline{\tau} > 0$. The dotted curve in the right panel gives an example of such an upper bound. While we need a concrete model to see whether condition (11) is satisfied, let us note that the condition does not require an exponential fall in expenditure. The RHS of (11) falls with age at the rate of ξ/τ for $\tau > \overline{\tau}$. The rate of decline in the quality-adjusted price, ξ/τ, can be arbitrarily close to zero when we choose a large $\overline{\tau}$. Therefore, there is no minimum rate at which the expenditure needs to fall.

The blue area is strictly positive, given that the quality-adjusted price falls throughout the product lifecycle. Combined with (11), the GDP growth rate is also strictly positive. The growth rate is finite if $p(\tau)$ bounded away from 0 as $\tau \to \infty$.[15] If $p(\tau)$ falls to 0 as $\tau \to \infty$, the finiteness depends on the relationship between $p(\tau)$ and $x(\tau)$. Specifically, if the quantity depends only on price, the area becomes finite if the price elasticity of the demand is less than one as the price approaches 0 from above.[16]

[14] Suppose (11) is satisfied. Then, the denominator of (7) is $\int_0^\infty p(\tau)x(\tau)d\tau \leq \int_0^{\overline{\tau}} p(\tau)x(\tau)d\tau + \int_{\overline{\tau}}^\infty [\text{constant}] \cdot \tau^{\xi} d\tau$. The first term is finite, and the second term becomes $[\text{constant}] \cdot \overline{\tau}^{1-\xi}/(\xi - 1)$, which is also finite.

[15] In this case, $x(\tau)$ stays finite as $\tau \to \infty$ since otherwise $p(\tau)x(\tau)$ becomes infinite, contradicting (11). Given this, the blue area is finite.

[16] Suppose that we can define a static inverse demand function $P(x)$. Focusing on the case of $x \to \infty$ and $P(x) \to 0$, the blue area can be written as $p(0)x(0) + \int_{x(0)}^\infty P(x)dx$. If the price elasticity of the demand as $p \to 0$ is less than one, the elasticity of $P(x)$ with respect to x as $x \to \infty$ is larger than one. This means that $P(x)$ is bounded by $[\text{constant}] \cdot x^{-\xi' x}$ for some $\xi' > 1$ for large x. Therefore, the integral is finite.

2.5 Discussion: Two Conditions for Sustained Growth

The previous three examples illustrated that the measured real GDP growth rate in the steady state becomes strictly positive and finite in various scenarios. Here, we discuss more generally when the two required conditions in Corollary 1 hold.

Condition (9): quality-adjusted price falls during its product lifecycle

For this condition to be satisfied, $p(\tau)$ must fall with τ at least for a portion of its product lifecycle. Recall that we normalized the price level so that the nominal expenditure in the steady state is constant. This normalization also implies the price of the newest goods when they appear does not change over time in the steady state. Therefore, condition (9) just requires the quality-adjusted prices of older goods to become lower relative to newer goods, and it is not essential for prices of individual goods measured in a currency to fall. In terms of actual currencies, we can determine that $p(\tau)$ is falling if the quality-adjusted currency prices of individual goods lag behind the growth of the nominal per capita GDP.[17]

With this definition, the quality-adjusted price of a good may fall with age for a number of reasons. For example, the cost of production falls through learning-by-doing and knowledge spillovers. In this case, time and production experiences will contribute to price reduction. Apart from cost reduction, changes in the form of competition may also lower prices because older goods are typically less protected from competition by patents and trade secrets than newer goods.

Price reductions also occur in the form of quality improvements. For example, the effective price of computers has been declining for decades, not only because the price of an average computer has become cheaper but also because the average performance of computers has drastically improved. The SNA statistics record such changes as a decline in the quality-adjusted price.

[17] Suppose that the per capita nominal GDP growth rate in dollars is $g^{\$}$. Note that in the price normalization in our theory, nominal per capita expenditure is constant, which means that there is a $g^{\$}$ difference in the inflation rate between the prices in theory and in dollars. Then, in dollars, the rate of price change of age τ good is $p'(\tau)/p(\tau) + g^{\$}$. Therefore, we can determine $p'(\tau)$ is negative if the quality-adjusted dollar prices of individual goods are increasing slower than $g^{\$}$.

It is worth noting that our theory does not require an exponential fall in the quality-adjusted price. If the quality improvements are exponential, economic growth can easily be maintained as in usual quality-ladder models (See panel (b) of Figure 2). According to "Moore's Law," the quality of computers has been improving at a constant rate, but this trend of exponential improvement is expected to slow down. In fact, computers are a remarkable exception in terms of continued improvements in performance. Most other products experience tapering in the rate of productivity improvements as they mature. Our theory shows that slowdowns in productivity increases in individual goods are consistent with a sustained rate of GDP growth, as long as a constant number of new products are introduced per unit time.

Lastly, let us discuss the case when the quality-adjusted price of the good rises for some part of its lifecycle, as we discussed in Example 2 of Figure 5. Although we need a concrete model to analyze how this happens and whether condition (9) is satisfied, here we discuss two possibilities. One possibility is when products have antique or scarce value as they become very old. In this scenario, $p(\tau)$ will increase only when $x(\tau)$ has become considerably smaller than when they were newer. Another possibility is that producing a good in small lots costs more. It happens, for example, when a particular good continues to be produced to meet a niche demand, typically near the end of the product lifecycle.

The numerator of the formula, $-\int_0^T x(\tau)dp(\tau)$, is the weighted sum of the price changes, $dp(\tau)$, where the weights are the quantities, $x(\tau)$. Therefore, if the quantity $x(\tau)$ tends to be small when $p(\tau)$ increases, the negative effect of such movements on the GDP growth rate is likely to be limited. Therefore, even when the price at the end of the lifecycle $p(T)$ is higher than the initial price $p(0)$, the lifetime contribution of this good to the real GDP growth rate may well be positive, as in the case of Example 2.

Condition (10): The cumulative expenditure for a single good is finite

This condition requires the expenditure for older goods $p(\tau)x(\tau)$ to fall so that they effectively retire from the market in terms of expenditure share. The condition is always satisfied if the representative good ceases to be produced at finite age T. Even when the good stays in the market forever ($T = \infty$), the condition is satisfied if the expenditure falls with age reasonably fast (condition 11). Notably, the decline in the speed of expenditure does not need to be exponential.

The expenditure for the good can shrink with age for several reasons. One possibility is that the price falls and the price elasticity of demand is less than one, at least for older goods. To illustrate this possibility, suppose that the demand for a good is determined solely by its price $p(\tau)$, and the price falls toward zero. Even when the good becomes almost free, it is unrealistic to expect consumers to demand an infinite amount of any particular product. This consideration suggests that the price demand elasticity of a product will become small when the price has become sufficiently low, and the expenditure for the good will eventually vanish as $p(\tau) \to 0$. Section 3 presents a full endogenous growth model based on this idea.

The expenditure for older goods can also fall for other reasons. Sometimes, consumers are attracted by the novelty of new goods, but they become less interested as time passes. Advertisements for newer goods enhance the speed of obsolescence of older goods. Changes in the underlying economic environments may also make older goods useless. When these effects are present, condition (10) may be satisfied regardless of the elasticity of demand. We will extend the model to include obsolescence in Subsection 4.1.

2.6 Implications for Endogenous Growth Theory

We have established that a constant GDP growth rate can be maintained without exponential growth in quantity, quality, or variety. Here, we discuss how this result allows building a robust endogenous growth model without requiring knife-edge conditions.

In standard variety-expansion models, all goods are symmetric and receive the same expenditure share. Therefore, as the number of goods increases, the share of the expenditure given to a single new good dilutes. Accordingly, the contribution of each new good to the economic growth rate also shrinks toward 0. This is why these models need to consider exponentially accelerating R&D to offset the dilution effects.

This requirement restricts the way an endogenous growth model could be built. If the expenditure share of a single new good tends to zero, profits obtained from a single successful R&D also fall. Therefore, to give firms enough incentives to do R&D in equilibrium, those models require a strong degree of externality in the R&D process so that the cost of inventing new goods declines exponentially (See panel (a) in Figure 2).

By contrast, if the economy satisfies condition (10), the incentive to innovate can be maintained without such strong externalities. Under this condition, the expenditure share for one new good, $p(0)x(0)\,/n\int_0^\infty p(\tau)x(\tau)d\tau$, remains a positive constant even when more and more goods are introduced. In such a case, a constant flow of new goods, as well as improvements in production costs after the introduction, always constitutes a significant addition of economic activity relative to all existing activities. This enables the measured GDP growth rate to remain positive in the long run without accelerating R&D. In addition, since the expenditure share for one new good does not fall, firms obtain enough profits from R&D even when no externalities reduce the cost of R&D. In the next section, we build a general equilibrium endogenous model without R&D externality in creating new products.

3 A Prototype Non-Exponential Growth Model

This section presents a general equilibrium model that yields non-exponential steady-state dynamics. While the theory in the previous section suggests there are many ways to build a model that achieves non-exponential growth while capturing various aspects of reality, here we limit ourselves to presenting the simplest prototype model so as to convey the substance of the non-exponential growth theory as clearly as possible. Thereafter, we will discuss the generalization of the prototype model in the following section.

3.1 Consumers and Market Demand

Consider an economy with infinitely-lived representative consumers of constant population L. At each point in time, each consumer supplies one unit of labor and obtains one unit of wage. The wage level is normalized to one.[18]

The lifetime utility function of the representative consumer is given by

$$\int_0^\infty \left[\int_0^{N_t} u(\widetilde{c}_t(i))di\right] e^{-\rho t}dt, \tag{12}$$

[18] In the steady state where the fraction of consumption out of labor income is constant, this normalization implies that the nominal expenditure is constant, consistent with the theory in the previous section.

which is separable both across time and goods. Note that the sub-utility function is symmetric across goods, so we do not model the obsolescence of older goods in this simplest prototype model. The sub-utility for individual goods takes the same form as the standard CRRA utility function with the relative risk aversion of greater than one ($1/\varepsilon > 1$).

$$u(\widetilde{c}_t(i)) = \frac{\widetilde{c}_t(i)^{1-1/\varepsilon} - 1}{1 - 1/\varepsilon}, \quad 0 < \varepsilon < 1. \tag{13}$$

Here, ε also represents the elasticity of substitution (EoS) across goods. Similarly to Ngai and Pissarides (2007) and Acemoglu and Guerrieri (2008), the EoS is less than one.[19] This setting suitably captures the demand behavior when the price becomes low while keeping the tractability of constant elasticity.[20]

The dynamic budget constraint of a representative consumer is given by

$$\dot{k}_t = r_t k_t + 1 - \int_0^{N_t} \widetilde{p}_t(i)\widetilde{c}_t(i)di, \tag{14}$$

In equilibrium, Lk_t should equal the value of all firms in the economy. Consumers maximize their lifetime utility (12) subject to (14), given interest rate r_t, prices of goods $\widetilde{p}_t(i)$ for $i \in [0, N_t]$, the initial asset holding k_0, and the standard non-Ponzi-game condition.

From these, we obtain an isoelastic demand function for individual goods by a representative consumer.

$$\widetilde{c}_t(i) = \lambda_t^{-\varepsilon}\widetilde{p}_t(i)^{-\varepsilon}. \tag{15}$$

The shadow price of the budget constraint λ_t evolves according to Euler equation $\dot{\lambda}_t = (\rho - r_t)\lambda_t$, and its initial value is determined so that the transversality condition $\lim_{t\to\infty} e^{-\rho t}\lambda_t k_t = 0$ is satisfied given the evolution of k_t in (14).

[19] They developed multi-sector growth models where the elasticity of substitution between the goods of different sectors is less than one. When the productivity of a sector (e.g., agriculture) increases, the expenditure for this sector shrinks because the demand quantity does not rise as much as the price falls. While they examined a small and finite number of sectors, this paper considers a continuum of goods (or, more precisely, product groups) and examines a steady-state shift of expenditure shares across these product groups.

[20] Given that we do not consider obsolescence here, $\varepsilon > 1$ would mean that the expenditure for a good will rise indefinitely as its price falls to zero. This would not be a good description of the behavior of consumers when our focus is on how goods effectively retire from the market at the end of their product lifecycle.

3.2 R&D and Production Technologies

Each consumer works either as a production worker or as a researcher. A researcher succeeds in developing a new good with a Poisson probability of a per unit of time. Let L_t^R denote the number of researchers in the economy, which is to be determined in equilibrium. Over time, the number of goods increases according to

$$\dot{N}_t = aL_t^R. \tag{16}$$

Equation (16) is similar to standard variety expansion models (recall panel (a) in Figure 2), except that there is no spillover term from the stock of past R&D.

Once developed, each individual good is produced with a linear production technology that requires only labor. The output of good i is given by

$$\widetilde{x}_t(i) = \widetilde{q}_t(i)\widetilde{l}_t(i), \tag{17}$$

where $\widetilde{l}_t(i)$ is the labor input, and $\widetilde{q}_t(i)$ is the marginal product of labor in producing good i. Alternatively, we can interpret $\widetilde{x}_t(i)$ as the quality-adjusted output and $\widetilde{q}_t(i)$ as the quality of good i. In this case, one unit of labor produces one unit of good i with quality $\widetilde{q}_t(i)$. In either interpretation, we call $\widetilde{q}_t(i)$ the productivity for good i.

When any good is first developed, the productivity is normalized to 1. Then, as the production of this good proceeds, the productivity increases according to[21]

$$\dot{\widetilde{q}}_t(i) = I(\widetilde{x}_t(i)) \cdot \beta\widetilde{q}_t(i)^{\psi}, \quad 0 < \psi < 1. \tag{18}$$

$I(\widetilde{x}_t(i))$ is an indicator function that takes 1 when $\widetilde{x}_t(i) > 0$ and 0 otherwise. It means that productivity increases as long as production takes place. Observe a similarity between the specifications in (18) and quality ladder models (see panel b in Figure 2). We assume that there are knowledge spillovers from the past productivity of technology to today's productivity increments. Parameter $\psi \in (0, 1)$ specifies the degree of such spillovers. While quality

[21] For simplicity, here we assume that only experience in terms of time matters for productivity improvement. Alternatively, we can consider the experience in terms of cumulative production amount. Horii (2012) analyzed a model in the latter setting and derived the GDP growth rate defined similarly to (4), although it was a semi-endogenous growth model that required an exponentially growing population (c.f. Jones, 1995).

ladder models need to assume $\psi = 1$ to achieve the exponential increase in productivity (or quality), we do not make this knife-edge assumption. For the moment, we consider the case of $\psi \in (0, 1)$ and later compare the result to the case of $\psi = 1$. Parameter $\beta > 0$ represents other possible factors that affect the speed of productivity increases.

As long as $\widetilde{x}_t(i) > 0$, equation (18) is an autonomous differential equation in $\widetilde{q}_t(i)$. Similarly to Section 2, let $\tau \equiv t - s(i)$ denote the age of the good. Then, the solution to the differential equation (18) can be written as

$$q(\tau) = \kappa_1 \left(\tau + \kappa_0\right)^{\theta}, \tag{19}$$

where $\theta \equiv 1/(1-\psi) > 1$, $\kappa_0 \equiv \theta/\beta > 0$, and $\kappa_1 \equiv (\beta/\theta)^{\theta} > 0$. Given that $\psi \in (0, 1)$, the productivity improvement is less than exponential. The rate of productivity increase is given by

$$g_q(\tau) = \frac{q'(\tau)}{q(\tau)} = \frac{\theta}{\tau + \kappa_0} = \frac{\beta}{(1-\psi)\beta\tau + 1}. \tag{20}$$

In this specification, $g_q(\tau)$ takes the highest value at the time of introduction, $g_q(0) = \beta$, and falls to 0 as a good gets older ($g_q(\infty) = 0$). This rules out the trivial possibility that the exponential increase in the productivity of individual goods explains the sustained growth.

3.3 Behavior of Firms

Let us now turn to the behavior of production firms. While any product is protected by a patent forever, we the patent *breadth* is limited (e.g. O'Donoghue, Scotchmer, and Thisse, 1998). This means that other producers are prohibited from using the identical technology as the original inventor, but they are allowed to produce similar products if they use a technology that is sufficiently different from the original. Alternatively, we may also think that a part of technology is kept secret by the inventor, and the outsiders need to rely on less efficient technologies. In either case, outsiders face lower productivity than the original firm.

To formalize this idea, let us assume that there are potentially many outside firms. They have partial access to the technology of the original inventor $\widetilde{q}_t(i)$ to produce the same good i. However, their productivity is $1/(1+\mu)$ times lower, where parameter μ represents the patent breadth or the strength of the trade secret. Recall that the price elasticity of the demand function (15) is less than unity. In this case, the profit-maximizing strategy is to set the limit price, which is $(1+\mu)$ times higher than the marginal cost. Given the production function (17) and the fact that wage is normalized to one, the pricing by a firm that has τ years of experience is

$$p(\tau) = \frac{1+\mu}{q(\tau)}. \tag{21}$$

From (15) and (21), the equilibrium output is $x_t(\tau) = D_t q(\tau)^{\varepsilon}$, and the maximized profit is $x_t(\tau) = D_t q(\tau)^{\varepsilon}$, where $D_t \equiv L(1+\mu)^{-\varepsilon}\lambda_t^{-\varepsilon}$ is a shifter of demand for individual goods.

3.4 Steady-State Equilibrium

Now, we derive the long-term growth property of the equilibrium dynamics in this prototype model. The following defines a notion of long-term equilibrium suitable to our model.

Definition 2. *An equilibrium path that satisfies the following properties as $t \to \infty$ is called Asymptotically Steady-State Equilibrium (ASSE).*

1. *The speed of the introduction of new goods converges to a positive constant: $\dot{N}_t \to n^* > 0$.*
2. *The shifter of the demand for individual goods, $D_t = L(1+\mu)^{-\varepsilon}\lambda_t^{-\varepsilon}$, converges to a positive constant: $D_t \to D^* > 0$.*

When $D_t \to D^*$, $x_t(\tau)$ and $\pi_t(\tau)$ also converge to time-invariant functions:

$$x(\tau) = D^* q(\tau)^{\varepsilon}, \tag{22}$$

$$\pi(\tau) = \mu D^* q(\tau)^{\varepsilon-1}. \tag{23}$$

In the following, we consider the state of the economy as $t \to \infty$ in the ASSE and mention it simply "in the ASSE." Later, Subsection 3.6 confirms that the real aggregate variables observed in the ASSE obey the conventional notion of the balanced growth path.

The equilibrium values of n^* and D^* are determined by the free entry condition for R&D and the labor market clearing condition. Let us first focus on the R&D condition. Recall that the Euler equation is $\dot{\lambda}_t/\lambda_t = \rho - r_t$. Since $\lambda_t = (1+\mu)(L/D_t)^{1/\varepsilon}$ is stationary in the ASSE, the interest rate necessarily converges to $r_t \to \rho$. Using interest rate $r_t = \rho$ and the profit function (23), we can calculate the present value of a new firm just after it has succeeded in developing a new good. The value of innovation in the ASSE becomes

$$V^* = \int_0^\infty \pi(\tau) e^{-\rho\tau} d\tau = \mu D^* \int_0^\infty q(\tau)^{\varepsilon-1} e^{-\rho\tau} d\tau. \tag{24}$$

From the R&D function (16), the expected cost of developing a new good is $1/a$. Therefore, given that there is a positive flow of R&D, $n > 0$, and that the financial market is complete, the value of the new firm (24) should be equalized to the expected cost of development: $V^* = 1/a$. This condition gives the equilibrium value of D^* in the ASSE.

$$D^* = \frac{1}{a\mu} \left(\int_0^\infty q(\tau)^{\varepsilon-1} e^{-\rho\tau} d\tau \right)^{-1}. \tag{25}$$

Substituting (19) into (25), we can obtain an explicit expression for D^*.[22]

Next, let us turn to the labor market. First, equation (16) implies that the number of research workers in the ASSE is $L^{R*} = n^*/a$. Second, from (17) and (22), the aggregate demand for production workers in the ASSE is[23]

$$\begin{aligned} L^{P*} &= \lim_{t\to\infty} \int_0^{N_t} \widetilde{l}_t(i) di \\ &\to n^* \int_0^\infty x(\tau)/q(\tau) d\tau = n^* D^* \int_0^\infty q(\tau)^{\varepsilon-1} d\tau. \end{aligned} \tag{27}$$

The labor supply is given by population L. Therefore, the labor market clearing condition is

$$L = L^{R*} + L^{P*} = \frac{n}{a} + n D^* \int_0^\infty q(\tau)^{\varepsilon-1} d\tau. \tag{28}$$

From this, we obtain the equilibrium research intensity in the ASSE

$$n^* = \frac{aL}{1 + aD^* \int_0^\infty q(\tau)^{\varepsilon-1} d\tau}. \tag{29}$$

[22] Let $\Gamma(\cdot,\cdot)$ denote upper incomplete Gamma function, defined as $\Gamma(s,z) \equiv \int_z^\infty t^{s-1} e^{-t} dt$. Function $\Gamma(s,z)$ is positive and finite for all $s \in (-\infty,\infty)$ and $z \in (0,\infty)$. By changing the variable of integration from τ to $\widetilde{\tau} = (\tau + \kappa_0)/\rho$ and utilizing (19), equation (25) becomes

$$D^* = \frac{\kappa_1^{1-\varepsilon} \rho^{1+\eta}}{a\mu e^{\rho\kappa_0} \Gamma(1-\eta, \rho\kappa_0)} > 0, \tag{26}$$

where $\eta \equiv \theta(1-\varepsilon) \equiv (1-\varepsilon)/(1-\psi)$. The values of $\Gamma(s,z)$ are available in most programming platforms.

[23] The variable of integration is changed from i to τ in (27) using (2).

From (29), $L^{R*} = n^*/a$ and $L^{P*} = L - L^{R*}$ are also obtained. The explicit solution for n^* can be obtained as follows. Using (25) and then (19), the ratio of two types of labor is

$$\left(\frac{L^P}{L^R}\right)^* = \frac{\int_0^\infty q(\tau)^{\varepsilon-1} d\tau}{\mu \int_0^\infty q(\tau)^{\varepsilon-1} e^{-\rho\tau} d\tau} \tag{30}$$

The equilibrium ratio of production and research workers $\left(L^P/L^R\right)^*$ in (31) becomes a positive finite constant if and only if $\theta(1-\varepsilon) > 1$.[24] Using definition $\theta \equiv 1/(1-\psi)$, the condition reduces to $\psi \in (\varepsilon, 1)$, where ψ is the degree of knowledge spillover from past productivity to its increments. Using $\left(L^P/L^R\right)^*$ in (30), the ASSE research intensity can be written as

$$n^* = aL^{R*} = \frac{aL}{1 + (L^P/L^R)^*}, \tag{32}$$

which becomes a positive constant if $\psi \in (\varepsilon, 1)$, and 0 if $\psi < \varepsilon$.

The pair of D^* and n^* in (25) and (32) characterizes the long-term equilibrium of this economy. These equations also show how parameters affect long-term dynamics. For example, a larger μ means the breadth of patents is wider (or the trade secrets are better kept). A higher value of a means that R&D requires less labor. In these cases, innovation intensity n^* rises because of higher profitability, whereas the output of each good, proportional to D^*, falls because there are more production firms to which the aggregate labor needs to be divided.[25] The opposite occurs when the time preference ρ is higher because it raises the interest rate, reducing the present value of profits.

When population L is larger, the research intensity n^* is multiplied proportionally to L. However, the production of each good (proportional to D^*) does not change, because both the number of products introduced each year and the number of total production workers are multiplied by the same factor. This outcome resembles the mechanism of the second-

[24]Using (26), we obtain an explicit solution to (30) as follows.

$$\left(\frac{L^P}{L^R}\right)^* = \frac{\kappa_0^{1-\eta}\rho^{1+\eta}}{\mu(\eta-1)e^{\rho\kappa_0}\Gamma(1-\eta, \rho\kappa_0)} \text{ if } \psi > \varepsilon, \quad \left(\frac{L^P}{L^R}\right)^* = \infty \text{ otherwise.} \tag{31}$$

[25]The derivative of the upper incomplete Gamma function with respect to the second argument, $\partial\Gamma(s,z)/\partial z = -z^{s-1}e^{-z}$, is always negative. Using this, the properties in the text can be confirmed from (26), (31) and (32).

generation endogenous growth models, where the horizontal number of sectors is adjusted proportionally to the total population.[26]

Before closing this subsection, let us briefly compare those results against the case of $\psi = 1$. When $\psi = 1$, the solution to the differential equation (18) is exponential: $q(\tau) = e^{\beta\tau}$. Then, we can calculate n^* and D^* in the ASSE as

$$n^* = \frac{\mu(1-\varepsilon)\beta a L}{(1+\mu)(1-\varepsilon)\beta + \rho}, \quad D^* = \frac{(1-\varepsilon)\beta + \rho}{a\mu}. \tag{33}$$

The comparative statistics properties with respect to μ, ρ, L and a are the same as the case of $\psi \in (\varepsilon, 1)$. Therefore, the exponential growth in productivity ($\psi = 1$) can be viewed as a particular case of our model, although we do not focus on it because it is a knife-edge case.

3.5 Long-Term Real GDP Growth Rate

Now, we are ready to examine the long-term economic growth rate in this prototype model. In this subsection, we assume $\psi \in (\varepsilon, 1)$, so that the economy has an ASSE with $n^* > 0$ and $D^* > 0$. This ASSE satisfies the definition of non-exponential asymptotic steady state in Definition 1. Also, we can confirm that prices satisfy the condition that the per capita expenditure (per capita) is constant, as we assumed in Section 2.[27] Using (19), (21) and (22), we can confirm that $p(\tau)$ and $x(\tau)$ satisfies conditions (9) and (10) as long as $\psi \in (\varepsilon, 1)$.[28] Therefore, we can apply Proposition 1 to calculate the real GDP growth rate

[26] However, note that the long-term growth in these models is typically maintained by the exponential increase in the productivity (or quality) in each sector, whereas this paper focuses on the case where such exponential improvements cannot be sustained ($\psi < 1$ in equation 18).

[27] Per capita expenditure in the ASSE is $\int_0^\infty \widetilde{p}_t(i)\widetilde{c}_t(i)di \to (n^*/L)\int_0^\infty p(\tau)x(\tau)d\tau = (n^*D^*/L)(1+\mu)$ $\int_0^\infty q(\tau)^{\varepsilon-1}d\tau$. Using the definition of $q(\tau)$ in (19) and $\theta \equiv 1/(1-\psi) > 1$, it becomes $(n^*D^*/L)(1+\mu)(1-\psi)$ $\kappa_0^{1-(\psi-\varepsilon)/(1-\psi)}/(\psi-\varepsilon)$, which is a positive and finite constant given that $\psi \in (\varepsilon, 1)$.

[28] Similarly to the calculation in footnote 27, we find that the denominator of the formula is $\int_0^\infty p(\tau)x(\tau)d\tau = (1+\mu)(1-\psi)\kappa_0^{1-(\psi-\varepsilon)/(1-\psi)}/(\psi-\varepsilon)$. Using $p'(\tau) = -(1+\mu)g_q(\tau)/q(\tau)$ and (20), we also find that the value of the numerator is $-\int_0^\infty p'(\tau)x(\tau)d\tau = D^*(1+\mu)\kappa_0^{-(\psi-\varepsilon)/(1-\psi)}/(1-\varepsilon)$. Both are positive and finite given that $\psi \in (\varepsilon, 1)$.

in the steady-state equilibrium.[29]

$$g^* = \frac{-\int_0^\infty p'(\tau)x(\tau)d\tau}{\int_0^\infty p(\tau)x(\tau)d\tau} = \frac{\psi - \varepsilon}{1-\varepsilon}\beta \quad \text{for } \varepsilon < \psi \leq 1. \tag{34}$$

Note that (34) also applies the special case of $\psi = 1$, where the output of all goods increases exponentially at the rate of β ($g^* = \beta$).

Equation (34) shows that the measured growth rate takes a positive and finite value whenever $\psi \in (\varepsilon, 1]$. The requirement $\psi > \varepsilon$ can be understood in terms of condition (10) in Corollary 1. Given that $\psi < 1$, the expenditure for an age-τ good in the ASSE can be written as $p(\tau)x(\tau) = [\text{constant}] \cdot (\tau + \kappa_0)^{-(1-\varepsilon)\theta}$. For $\int_0^\infty p(\tau)x(\tau)d\tau$ to be finite, the power of $(\tau+\kappa_0)^{-(1-\varepsilon)\theta}$ have to be less than -1.[30] Since $\theta = 1/(1-\psi)$, this condition is equivalent to $\psi > \varepsilon$. Intuitively, for the expenditure on existing goods to be finite, the expenditure for a single good must decline reasonably fast with age. In this prototype model environment, the condition is accomplished if the degree of spillover in the productivity increase, ψ, is larger than ε. Otherwise, $\int_0^\infty p(\tau)x(\tau)d\tau$ becomes infinite, and Proposition 1 implies that the long-term GDP growth rate is zero.

Given that the markup ratio μ is constant and that condition (10) is satisfied, the growth formula (7) in the ASSE can also be represented as

$$g^* = \int_0^\infty g_q(\tau)\sigma(\tau)d\tau, \text{ where } \sigma(\tau) = \frac{p(\tau)x(\tau)}{\int_0^\infty p(\tau')x(\tau')d\tau'} \tag{35}$$

is the expenditure share for age τ goods, and $g_q(\tau)$ is the rate of productivity increase for those products, defined in (20). The growth formula in this form clarifies that the real GDP growth is the weighted average of the rate of productivity increases among goods of various ages. Recall that, in our specification of the technology, the newest goods have the fastest rate of productivity improvements, β, while the rate of improvements is lower for the older goods because $g_q'(\tau) < 0$ (see equation 20). In particular, the rate of productivity improvement $g_q(\tau)$ is almost zero for very old goods with large τ. Therefore, it is natural that the aggregate GDP growth rate (34) is somewhere between zero and β because the economy consists of goods of all ages.

[29]The calculations from footnote 28 implies $-\int_0^\infty p'(\tau)x(\tau)d\tau / \int_0^\infty p(\tau)x(\tau)d\tau = (\psi-\varepsilon)/(1-\varepsilon)(1-\psi)\kappa_0$. Using definitions $\kappa_0 \equiv \theta/\beta$ and $\theta \equiv 1/(1-\psi)$, we obtain (34).

[30]This is a particular case of condition (11).

Now, it is clear why the growth rate g^* in (34) is decreasing in price elasticity of demand, ε. Recall that ε also represents the elasticity of substitution across goods. With a higher ε, consumers spend more on old and low-priced goods and less on new and expensive goods. Since the rate of productivity increase in (20) is lower for older goods (with high age τ), the weighted average will also be low.

Equation (34) also shows that the growth rate g^* is increasing with ψ, the degree of knowledge spillover in production. When $\psi \le \varepsilon$, $\left(L^P/L^R\right)^*$ in (30) becomes infinity, which means $n^* = 0$. Without introducing new goods, the distribution of products ages simply moves up, and the growth rate will fall to $g_q(\infty) = 0$. As ψ increases between ε and 1, the schedule of the $g_q(\tau)$ function in (20) moves up, and so does the real GDP growth rate. When ψ reaches 1, the long-term growth rate rises to β. This is an anticipated result; when $\psi = 1$, the productivity of all goods, both the new and the old, increases with a common constant exponential rate of β. Therefore, the case of $\psi = 1$ corresponds to the conventional growth theory, where labor productivity increases exponentially and uniformly. However, the main finding is that even when the productivity of each product does not rise exponentially (i.e., with $\psi < 1$), the economy as a whole can exhibit a constant measured growth rate, although it is lower than β. Finally, $\psi > 1$ is unrealistic because it means that the productivity of individual goods increases more than exponentially and $g_q(\infty) = \infty$.

It is worth noting that, in this simple prototype setting, the equilibrium long-term rate of growth (34) does not depend on equilibrium values of n^* and D^* as long as they are positive.[31] When the research intensity n^* is higher, more economic activity is added per unit of time. However, at the same time, there is also proportionally more "stock" of existing activities. The real GDP growth rate expresses the ratio between the two, which is unchanged.[32] Similarly, when D^* is larger, each good has more demand. This means the production of new goods, as well as the increment of production of other goods per

[31] Of course, this property depends on the simplistic settings in this prototype model. For example, when the aggregate R&D intensity n^* has some positive spillovers on the rate of productivity increases in individual goods $g_q(\tau)$, then n^* will affect g^*. Also, when the amount of production has some effects on $g_q(\tau)$, g^* will depend on D^*.

[32] Nonetheless, it is essential that there is a positive flow of new innovations $n^* > 0$, since otherwise, g^* becomes 0.

unit of time, is higher. At the same time, however, the value of existing products is also higher, exactly canceling out the effects on g^*.[33] As a result, even when the changes in population L, R&D productivity a, or patent policy μ affect n^* and D^*, they do not affect the real GDP growth rate. This result contrasts with the implication of existing R&D-based growth models, where g^* follows directly from n^*. Although this result depends on the simplifying specification of the prototype model, it might provide a possible interpretation of why the measured GDP growth rates in the U.S. and some other developed countries have been relatively stable, even though those underlying parameters seem to have significantly changed over long periods.

3.6 Measured Real Aggregate Variables and Balanced Growth

The ASSE in this model works very differently from the balanced growth path (BGP) in existing growth models. Nonetheless, here we show that when aggregate variables are measured in a conventional way, this model exhibits balanced growth in those measured aggregate variables.

Note that the total labor income for production is L^{P*} since the wage rate is normalized to one. All goods are sold at $(1+\mu)$ times the labor cost, as shown in (21). Therefore, the aggregate value of production, which equals the aggregate value of consumption, is $C^* = (1+\mu)L^{P*}$. In our model, investments take the form of R&D, and the total value of R&D outputs is $I^* = n^*V^* = L^{R*}$. The GDP in our model can be calculated as the sum of the value of production and the value of investments: $Y^* = C^* + I^* = (1+\mu)L^{P*} + L^{R*}$. Similarly, we can derive the steady-state value of aggregate capital K^*, defined as the value of all firms in the economy (knowledge capital).[34]

[33] This can also be seen in Example 3 of Figure 5. When D^* is increased, the left panel is stretched horizontally (along the $x(\tau)$ axis) while the right panel is stretched vertically (along the $p(\tau)x(\tau)$ axis) by the same magnification ratio. As a result, the growth rate, given by the ratio of the two areas, is unaffected.

[34] K^* can be calculated as the sum of the present value of the future profits from all firms that exist today. In v years from now, the present value of the profit from those firms will be $e^{-\rho v}\int_v^\infty \pi(\tau)n^* d\tau$, since the profits from firms of less than v years at that time will not be part of the value of today's firm. By aggregating all v and using (23), we have $K^* = \mu n^* D^* \int_0^\infty e^{-\rho v}\int_v^\infty q(\tau)^{\varepsilon-1} d\tau dv$, which is constant in the model price normalization.

Note that those aggregate variables are measured under the price normalization of our model, in which the nominal wage is set to 1. We now calculate their real values in the same spirit as the SNA.[35] Let $\bar{t}$ be the reference year, and $Y_{\bar{t}}^{\$}$ be the dollar value of the GDP in year $\bar{t}$, which we assume is known to the researcher. Then, since the real GDP growth rate is constant at g^* in the ASSE, the real GDP level in t is $Y_t^{\text{real}} = Y_{\bar{t}}^{\$} e^{g^*(t-\bar{t})}$. Since the ratios among Y^*, C^*, I^* and K^* are constant, their real values grow in the same proportion. Specifically,

$$C_t^{\text{real}} = \frac{C^*}{Y^*} Y_t^{\text{real}} = \frac{1+\mu}{1+\mu+\left(L^R/L^P\right)^*} Y_{\bar{t}}^{\$} e^{g^*(t-\bar{t})}, \tag{36}$$

$$I_t^{\text{real}} = \frac{I^*}{Y^*} Y_t^{\text{real}} = \frac{1}{(1+\mu)\left(L^P/L^R\right)^* + 1} Y_{\bar{t}}^{\$} e^{g^*(t-\bar{t})}, \tag{37}$$

where $\left(L^R/L^P\right)^*$ is given by the inverse of (30).

Interest rate $r^* = \rho$ is also defined under our normalization of prices. Since the nominal GDP growth rate in the steady state is zero, the steady-state inflation rate is $-g^*$ in our price normalization. Then, the real interest rate in the steady state is $r^{\text{real}} = r^* + g^* = \rho + g^*$. We can also derive other real aggregate variables in similar ways, and their growth rates are constants. Therefore, if the statistical agency measured the aggregate variables in our model economy, those observed variables would grow exponentially along the BGP, even though neither quantity, quality, or variety of individual goods were growing exponentially.

3.7 Welfare Improvements

Lastly, let us discuss the welfare of the representative consumer. From (12), the period utility is $U_t = \int_0^{N_t} u(\widetilde{c}_t(i))di$. In the ASSE, the improvements in welfare can be measured by[36]

$$\lim_{t\to\infty} \dot{U}_t = n^* \lim_{c\to\infty} u(c) = \frac{\varepsilon}{1-\varepsilon} n^*. \tag{38}$$

[35] The NIPA publishes two series of real GDP. One is the quantity index, which takes 100 in the reference year (2012 as of the time of writing). The values for other years are obtained by chaining the real GDP growth rate. The other is Chained (2012) dollar series. They are calculated as the product of the quantity index and the 2012 current-dollar value of the corresponding series divided by 100. See U.S. Bureau of Economic Analysis, "Table 1.1.6. Real Gross Domestic Product, Chained Dollars." Here, We use the latter.

[36] Equation (38) gives the difference in the period utility between time $t+1$ and time t in the ASSE. Since the economy is in the ASSE, schedules for consumption against the age of goods are the same in t and

Given $\varepsilon \in (0,1)$, equation (38) shows that the speed of welfare improvements in the steady state is entirely determined by the speed of innovation n^*. In this sense, R&D and innovations are important for the economy, but they do not show up in the long-term real GDP growth rate g^* in (34). At least in this simple prototype model, the GDP growth rate is not a good measure of the speed of welfare improvements in the long run.

4 Generalizations

Sections 2 and 3 have shown the theory and a prototype model in the most straightforward possible setting to convey essential intuitions. This section generalizes the theory and the prototype model to show that the positive GDP growth rate can be explained under more relaxed conditions. In the first subsection, we introduce obsolescence to the prototype non-exponential growth model in Section 3 and show that the GDP growth rate can be positive even when $\varepsilon > 1$. In the second section, we extend the non-exponential growth theory of Section 2 to include multiple types of goods that follow different patterns of $p(\tau)$ and $x(\tau)$.

4.1 Obsolescence

In the prototype model of Section 3, we considered an environment where goods stay in the market forever ($T = \infty$) and consumers have symmetric preference across goods (12). Then, sustained economic growth required the price elasticity of demand ε to be less than one. The condition $\varepsilon < 1$ was necessary to induce consumers to spend less on older (and cheaper) goods. However, even without such an assumption, consumers may spend more on new goods simply because they prefer them to older ones. They may spend less on older goods because those are obsolete. Here, we show that condition $\varepsilon < 1$ can be relaxed once we include obsolescence.

Suppose now that utility function (12) is replaced by

$$\int_0^\infty \left[\int_0^{N_t} \delta(t - s(i)) u(\widetilde{c}_t(i)) di \right] e^{-\rho t} dt, \tag{39}$$

$t+1$, with the only difference being that the economy in time $t+1$ has n^* more oldest goods than in time t. When $t \to \infty$, the quality-adjusted amount of consumption for each of those oldest goods approaches infinity. Therefore, the difference in the period utility is $n^* c(\infty)$. From (13), $c(\infty) = \varepsilon/(1-\varepsilon)$.

where $t - s(i) = \tau$ is the age of good i (time after its debut), and $\delta(\tau)$ is a decreasing function of τ with $\delta(0) = 1$ and $\delta(\infty) = 0$. The steepness of function $\delta(\tau)$ represents the speed of obsolescence, or equivalently, consumers' taste for newer goods. We keep all other settings in Section 3 except that now we allow any $\varepsilon > 0$. Then, the expenditure for an age τ good in the ASSE becomes $e(\tau) = p(\tau)x(\tau) = (1+\mu)D^*\delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1}$, which illustrates that even when $\varepsilon > 1$, expenditure for older goods falls with age if obsolescence is fast enough.[37] Proposition 1 continues to apply in an environment with obsolescence, and the formula for the GDP growth rate (35) shows that the growth rate becomes a positive constant if $\int_0^\infty e(\tau)d\tau \equiv \int_0^\infty p(\tau)x(\tau)d\tau$ is finite.

When the rate of obsolescence is constant $\overline{\delta} > 0$ per year, function $\delta(\tau)$ can be expressed as $\exp(-\overline{\delta}\tau)$. In this case, the integration of $e(\tau) = (1+\mu)D^*\delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1}$ always becomes finite because $\delta(\tau)^\varepsilon$ is falling exponentially and no other variable is growing exponentially. Therefore, a constant rate of obsolescence always sustains positive GDP growth regardless of ε. Growth can also be maintained with slower, non-exponential obsolescence. Consider an example where $\delta(\tau)$ is a negative power function of τ: $\delta(\tau) = \delta_0^\omega(\tau+\delta_0)^{-\omega}$ where ω and δ_0 are positive constants.[38] Then, $\int_0^\infty e(\tau)d\tau$ becomes finite if and only if[39]

$$\varepsilon < \begin{cases} \frac{\psi}{1-\omega(1-\psi)} & \text{if } \omega < \frac{1}{1-\psi} \ (\equiv \theta) \\ \infty & \text{if } \omega \geq \frac{1}{1-\psi}. \end{cases} \tag{40}$$

In a particular case of $\delta_0 = \kappa_0$, where κ_0 is defined in (19), we obtain an explicit expression for the long-term GDP growth rate,

$$g^* = \frac{\psi - \varepsilon + (1-\psi)\varepsilon\omega}{1 - \varepsilon + (1-\psi)\varepsilon\omega}\beta, \tag{41}$$

[37] Similarly to the derivation of (25), we obtain $D^* = \left(a\mu \int_0^\infty \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} e^{-\rho\tau} d\tau\right)^{-1}$, which is always positive and finite because of the $e^{-\rho\tau}$ term. Using this value of D^*, the speed of innovation is $n^* = aL\left(1 + aD^* \int_0^\infty \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} d\tau\right)^{-1}$. n^* is strictly positive if and only if $\int_0^\infty \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} d\tau$ is finite, which is equivalent to the finiteness of $\int_0^\infty e(\tau)d\tau$.

[38] We need constant $\delta_0 > 0$ because otherwise $\tau^{-\omega}$ cannot be defined when $\tau = 0$ and $\omega > 0$. The δ_0^ω term normalizes the $\delta(\tau)$ function so that $\delta(0) = 1$.

[39] Using (19), $\int_0^\infty e(\tau)d\tau = (1+\mu)\delta_0^{\varepsilon\omega}\kappa_1^{\varepsilon-1} D^* \int_0^\infty (\tau+\delta_0)^{-\omega\varepsilon}(\tau+\kappa_0)^{\theta(\varepsilon-1)} d\tau$. The integral becomes finite if and only if the sum of the powers of the integrand, $-\omega\varepsilon + \theta(\varepsilon-1)$, is less than minus one. From $\theta = 1/(1-\psi)$, this condition is equivalent to (40).

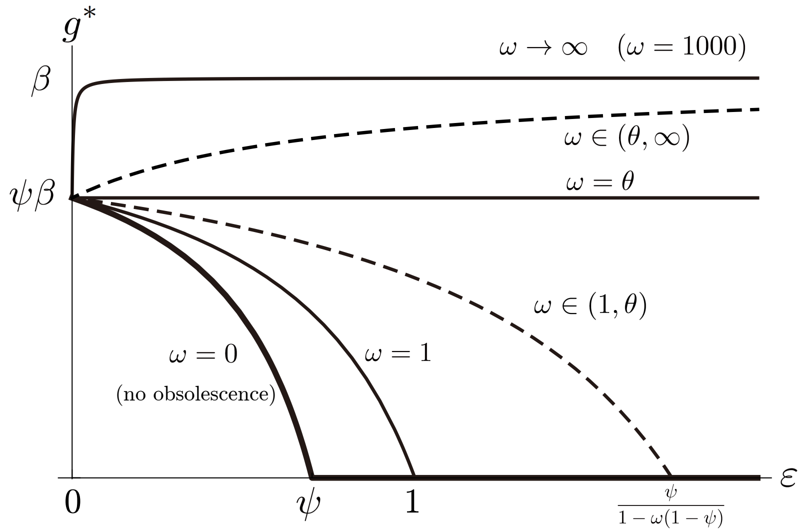


Figure 6: Price elasticity of goods and the long-term GDP growth rate under different degrees of obsolescence

which is positive when (40) holds. Figure 6 depicts the relationship between ε and g^* for various values of ω. As we have seen in Section 3, sustained GDP growth requires $\varepsilon < \psi$ without obsolescence. This is a special case of (40) with $\omega = 0$. As ω increases, the required upper bound for the elasticity, $\psi/(1-\omega(1-\psi))$, becomes higher. In particular, when $\omega > 1$, the upper bound is larger than one, which means that $\varepsilon < 1$ is not necessary for $g^* > 0$. When the power of the obsolescence function, ω, equals or is higher than $1/(1-\psi)$, the long-term GDP growth rate g^* is positive regardless of ε.[40]

Figure 6 also shows that when ω is increased, the entire curve for g^* moves (weakly) upward. Faster obsolescence not only makes sustained growth more likely but also accelerates the measured rate of economic growth. Intuitively, obsolescence skews expenditure toward newer goods. Since newer goods have more margins for productivity increases, the overall growth rate rises with obsolescence. This result has important policy implications. When the government tries to protect obsolete companies (or industries), it will reduce

[40]Interestingly, the measured GDP growth rate will increase with ε when $\omega > 1/(1-\psi)$. A higher ε means that consumers are more willing to move from old and obsolete goods to newer goods, thus enhancing the positive effect of obsolescence on growth.

the GDP growth rate, not just because of efficiency loss but also because of the way the GDP growth rate is calculated. Conversely, advertisements and marketing practices that attract consumers from older goods to newer goods will enhance GDP growth, even when the attractiveness of the newer goods is illusionary.

4.2 Multiple sectors

In the non-exponential growth theory, we defined the steady state as the situation wherein the paths of quality-adjusted prices and quantities, $p(\tau)$ and $x(\tau)$, follow the same pattern in terms of their age (See Definition 1 in Section 2.1). This definition allows the prices and quantities of individual goods at a given time to differ depending on their age. In this sense, our definition of the steady state is more flexible than in most endogenous growth models where goods are symmetric in the steady state. Still, once we look at the data, it is immediately apparent that goods in different categories follow distinct lifecycle patterns. For example, while the product lifecycle is relatively fast in electronics, some basic goods (e.g., grains) show little sign of lifecycle movements.

In this subsection, we further extend the notion of the steady state by allowing $p(\tau)$ and $x(\tau)$ to follow different patterns. We categorize goods into groups (which we call sectors) so that goods in a sector have the same pattern of movements in the quality-adjusted price and quantity with respect to their age, at least in the long run. More specifically, suppose that there are $J > 0$ sectors (or categories) of goods, and label each by $j \in \{1, \ldots, J\}$. $N_{j,t}$ denotes the index of the newest good in sector $j \in \{1, \ldots, J\}$. The number of new goods introduced per unit time, $\dot{N}_{j,t} \geq 0$, can differ across sectors. The quality-adjusted price of the ith good in sector j and its quality-adjusted quantity are denoted by $\widetilde{p}_{j,t}(i)$ and $\widetilde{x}_{j,t}(i)$. In this setting, we define the asymptotic steady state as follows.

Definition 3. *A non-exponential asymptotic steady state with multiple sectors is the situation where $\dot{N}_{j,t}$, $\widetilde{p}_{j,t}(i)$ and $\widetilde{x}_{j,t}(i)$, for all $j \in \{1, \ldots, J\}$, satisfy the following conditions:*
(a) $\dot{N}_{j,t}$ converges to a constant: $\dot{N}_{j,t} \to n_j \geq 0$.
(b) $\widetilde{p}_{j,t}(i)$ and $\widetilde{x}_{j,t}(i)$ converge to time-invariant functions of $\tau = t - s(i)$: $p_j(\tau)$ and $x_j(\tau)$.
(c) Assumption 1 holds, where $p(\tau)$, $x(\tau)$ and T are replaced by $p_j(\tau)$, $x_j(\tau)$ and T_j.

(d) The expenditure share of the sector, defined by

$$\alpha_{j,t} = \frac{\int_0^{N_{j,t}} \widetilde{p}_{j,t}(i)\widetilde{x}_{j,t}(i)di}{\sum_{j'=1}^{J}\int_0^{N_{j',t}} \widetilde{p}_{j',t}(i)\widetilde{x}_{j',t}(i)di} \tag{42}$$

converges to a constant value: $\alpha_{j,t} \to \alpha_j \geq 0$.

Definition 3 says that the economy is in a steady state if the composition of sectors in terms of expenditure share is stationary, and each sector satisfies the requirement for the steady state in Definition 1. In addition, observe that Definition 3 does not require n_j to be positive, thus including the possibility where the introduction of goods eventually stops in some sectors. Also, α_j may be zero for some j, allowing for the possibility that some sectors disappear in the long run.

Similarly to (5), the instantaneous GDP growth rate in this multi-sector economy at any given time t can be defined by

$$g_t = \frac{\sum_{j=1}^{J} \dot{N}_{j,t}\widetilde{p}_{j,t}(N_{j,t})\widetilde{x}_{j,t}(N_{j,t}) + \sum_{j=1}^{J}\int_0^{N_{j,t}} \widetilde{p}_{j,t}(i)\dot{\widetilde{x}}_{j,t}(i)di}{\sum_{j=1}^{J}\int_0^{N_{j,t}} \widetilde{p}_{j,t}(i)\widetilde{x}_{j,t}(i)di}. \tag{43}$$

Here, the denominator gives the expenditure for all goods, the first term in the numerator is the value of all new goods introduced at time t, and the second term is the value of the changes in the production of existing goods. Using the sectoral expenditure share defined by (42), equation (43) can be expressed as the share-weighted average of the sectoral GDP growth rate.

$$\begin{aligned} g_t &= \sum_{j=1}^{J} \alpha_{j,t} g_{j,t}, \text{ where,} \\ g_{j,t} &= \frac{\dot{N}_{j,t}\widetilde{p}_{j,t}(N_{j,t})\widetilde{x}_{j,t}(N_{j,t}) + \int_0^{N_{j,t}} \widetilde{p}_{j,t}(i)\dot{\widetilde{x}}_{j,t}(i)di}{\int_0^{N_{j,t}} \widetilde{p}_{j,t}(i)\widetilde{x}_{j,t}(i)di}. \end{aligned} \tag{44}$$

Since (44) takes the same form as (5), we can utilize Proposition 1 to obtain the long-term GDP growth rate in a steady state.

Proposition 2. *Suppose that the multi-sector economy converges to the asymptotic steady-state, as defined by Definition 3. Then, the real GDP growth rate* g_t *asymptotes to*

$$g = \sum_{j=1}^{J} \alpha_j g_j, \tag{45}$$

where g_j is given by Proposition 1 where $p(\tau)$, $x(\tau)$ and g are replaced by $p_j(\tau)$, $x_j(\tau)$ and g_j, respectively.

Proposition 2, combined with Proposition 1, implies that if there is a category of goods (a sector) with a positive GDP share where conditions (9) and (10) hold in the long run, the economy-wide long-term GDP growth rate can be strictly positive and finite. Similarly to Figure 5 in Section 2.4, we can draw the evolution of $\{x_j(\tau), p_j(\tau)\}$ in the quantity-price space, and also the evolution of $p_j(\tau)x_j(\tau)$ against τ. Then, the numerator and the denominator of g_j are graphically represented as the blue and yellow areas. If $\alpha_j > 0$ and both areas are finite and positive, then sector j contributes strictly positively to the long-term GDP growth rate. As in Example 2 of Figure 5, g_j could be negative if the prices of older and disappearing goods in that sector are higher than the new goods in the same sector. Nonetheless, the aggregate GDP growth becomes zero only by coincidence, and therefore, non-zero long-term growth rates will be the norm rather than the exception. This result contrasts with existing endogenous growth models, where the growth rate can be non-zero only under strict knife-edge conditions.

As a final note, observe that g_j's in Proposition 2 are the sectoral output growth rates measured in their own sectoral price indices. They do not coincide with the sectoral output growth calculated using the general price levels (e.g., the GDP deflator). In the long run, the expenditure to all the surviving sectors (those with positive α_j's) will grow at the same rate. Even the sectors with $g_j = 0$ will record real income growth of g.

5 Concluding Remarks

The non-exponential growth theory provides a novel interpretation of long-term growth by focusing on the movement of quantities and prices of individual goods and calculating the GDP growth rate based on its definition in the SNA statistics (e.g., the NIPA). The theory part of the paper (Section 2) has shown that the real GDP growth rate can be sustained at a positive level when the expenditure for older goods shrinks so that the newly introduced goods can receive a constant proportion of the total expenditure. The theory opened up the possibility that an endogenous growth model can be built without knife-edge assumptions.

Based on it, we presented a simple prototype model where the GDP growth rate can be sustained if the price elasticity of demand for older goods is less than one and the price declines with age (Section 3). If goods become obsolete as they age, the above condition can be significantly relaxed (Subsection 4.1). We further extended the theory (Subsection 4.2) to show that if a group of goods with a non-zero expenditure share satisfies a required condition, the long-term GDP growth rate can be positive. Therefore, positive long-term growth is consistent even when some goods do not lose expenditure forever (e.g., some kinds of food).

This paper suggests that an endogenous growth theory can be applied to data with much weaker restrictions than before. Still, we made simplifying assumptions for expositional simplicity and ease of understanding. Notably, while existing variety-expansion endogenous growth models assume the elasticity of spillover from the R&D activity is exactly at $\phi = 1$ (see Figure 2), we assume that there is none, i.e., $\phi = 0$. In a working paper,[41] we confirmed that the intuitions from the non-exponential growth theory continue to hold when $\phi < 1$, although the analysis becomes significantly intricate because the number of new goods introduced per unit time is no longer constant. Also, this paper abstracted from population growth and decline by assuming constant population. While it is standard to make this assumption in existing (full-)endogenous growth models, studies by Jones (1995, 2002, 2022) have shown that the rate of population growth is crucial in sustaining growth when $\phi < 1$. The non-exponential growth theory also implies interactions between ϕ and population growth. An important difference from the semi-endogenous growth theory is that sustained GDP growth does not require positive population growth even when $\phi < 1$. It concerns a grave question of whether humanity can continue growth in the future. This paper, with $\phi = 0$ and zero population growth, provides a possible answer to this question in a straightforward setting.

[41] See Horii (2024), where the theory is extended to include $\phi \in (-\infty, 1)$ and non-zero population growth.

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