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# GENERALIZED UZAWA GROWTH THEOREM

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# A Generalized Uzawa Growth Theorem*

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**Abstract**

We prove a generalized, multi-factor version of the Uzawa steady-state growth theorem. Balanced growth with capital-augmenting technical change is possible when capital has a unitary elasticity of substitution with at least one other factor of production. Thus, a neoclassical growth model with three or more factors of production can be consistent with empirical evidence on both the capital-labor elasticity of substitution and the declining price of investment relative to consumption. In a three-factor model calibrated to US data, medium-run fluctuations in the investment price explain labor share movements from 1960-2000, but not the subsequent fall in the labor share.

**Keywords**: Balanced Growth, Uzawa Steady-State Growth Theorem, Technological Change, Land, Natural Resources

**JEL Classification Codes** E13, E22, O33, O41

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# 1 Introduction

The neoclassical growth model was developed to explain a set of stylized macroeconomic facts that can be classified under the umbrella of balanced growth (Solow, 1956, 1994). As conventionally understood, the Uzawa (1961) steady-state growth theorem says that on the balanced growth path (BGP) of a neoclassical growth model, all technological change must be labor-augmenting, unless the aggregate production function is Cobb-Douglas (Jones and Scrimgeour, 2008). This creates a significant problem for the neoclassical growth model, because data from the United States strongly suggest that (i) there is capital-augmenting technical change on the BGP and (ii) the aggregate production function is not Cobb-Douglas (see, e.g., Antras et al., 2004; Grossman et al., 2017; Oberfield and Raval, 2021).

The standard neoclassical growth model assumes that there are only two factors of production, labor and reproducible capital. In reality, there are many other factors of production, including various types of land, energy, and other natural resources. These factors do not fit well in the notion of capital in the neoclassical growth model in that they cannot be readily accumulated (or reproduced) through savings. In this paper, we derive conditions under which it is possible to make neoclassical models consistent with the data by adding additional factors of production. We then show how these results can be used to model the behavior of the economy along the BGP.

We start by proving a multi-factor version of the Uzawa (1961) steady-state growth theorem. When building macroeconomic models, researchers have incomplete knowledge of how the aggregate production function evolves over time due to technological change. We show that, if an economy has a BGP, the Uzawa theorem provides guidance on how to choose a simple *representation* of the ever-changing production function. We call this the *Uzawa Representation*. The Uzawa Representation gives the correct relationship between aggregate inputs and aggregate output on the BGP, while capturing steady-state technological change through factor-augmenting terms on inputs other than reproducible capital. The Uzawa Representation has the same derivatives and elasticity of substitution (EoS) as the true production function. However, this representation implies that there is no capital-augmenting technical change, which is at odds with the evidence.

To identify other possible representations that are a better match with data, we prove a generalized version of the multi-factor Uzawa theorem. The generalized theorem demonstrates that there are a *continuum* of representations with capital-augmenting technical change, as long as reproducible capital has a unitary EoS with at least one other factor. From this broader class of *Factor-Augmenting Representations*, it is possible to choose a representation that matches the empirically-observed speed of capital-augmenting technological progress. When we explicitly consider three or more production factors, the factor-augmenting representations can be simultaneously consistent with balanced growth, a non-unitary EoS between capital and labor, and capital-augmenting technical change. We also provide conditions under which these Factor-Augmenting Representations have the same derivatives and EoS as the true production function. Therefore, even when the exact pattern of technological change cannot be observed, factor-augmenting representations can be used as a production function in economic analysis, as long as the economy follows a BGP in the long run.

We then discuss existing evidence on the elasticity of substitution between capital and other factors of production. We highlight that long-run elasticities are relevant for understanding balanced growth. While inconclusive, existing evidence suggests that energy may have a unitary long-run EoS with capital (e.g., Koetse et al., 2008; Van der Werf, 2008). To the best of our knowledge, evidence on capital-land substitution is limited to sector-specific studies, where the EoS between land and structures is often reported to be close to one (e.g., Epple et al., 2010; Ahlfeldt et al., 2015). We hope that our theoretical results will spur further work estimating these elasticities, which could make a significant contribution to the understanding of balanced growth.

We also demonstrate how to calibrate a three-factor production function to simultaneously match U.S. data on balanced growth, the negative trend in the relative price of capital, and the non-unitary elasticity of substitution between capital and labor. It is not possible to identify the true production function and the pattern of technological change just from those data. However, given that we know that the economy has a BGP, our propositions imply that the factor-augmenting representation constructed from the data matches the marginal properties of the true production function near the BGP and can therefore be used in place of the production function for a many of economic analyses.

As an example, we use our calibrated model to study how medium-run fluctuations in capital-augmenting technical change have influenced the labor share of income over the last sixty years. Movements in capital-augmenting technical change are a common explanation for labor share fluctuations (Grossman and Oberfield, 2022). Our model provides a unique perspective on how the two are related in the vicinity of the BGP. The generalized Uzawa theorem implies that the labor share is constant on a BGP with a constant rate of capital-augmenting technological change, but the labor share fluctuates when technology deviates from its BGP trend. This prediction differs from those generated by two-factor production functions, which must imply either that labor shares are always constant or that the labor share changes whenever there is non-zero capital-augmenting technical change. We find that medium-run deviations of capital-augmenting technology from the long-term trend explain labor share movements from 1960-2000, but not the subsequent fall in the labor share.

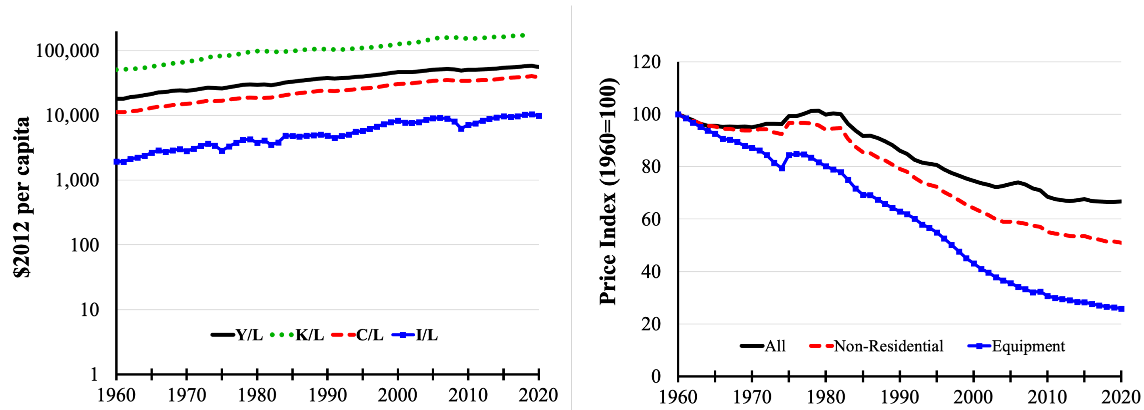
**Related Literature.** This paper is related to a long literature on balanced growth and the Uzawa steady-state growth theorem. Although the theorem is well known, Uzawa (1961) does not provide a clear statement or proof of the theorem. A simple and intuitive proof was proposed by Schlicht (2006) and updated by Jones and Scrimgeour (2008), Acemoglu (2008), Irmen (2016), and Grossman et al. (2017). With the exception of Acemoglu (2008), the literature has been concerned only with whether a particular production function can match the level of output on the BGP. We contribute to this literature in several ways. First, we extend the theorem to multiple factors of production. Second, we prove a generalized version of the theorem that provides a set of representations from which an economist can choose the one that matches the data on capital-augmenting technological progress. Third, we derive conditions under which representations have the same first-order derivatives and EoS as the true production function.

As noted above, the existing literature has treated the Uzawa theorem as a restrictive condition. As a result, many studies have tried to explain why the economy might endogenously conform to the two-factor version of the theorem. Acemoglu (2003) and Irmen and Tabaković (2017) provide models where capital-augmenting technical change disappears in the long run, while Jones (2005) and Leon-Ledesma and Satchi (2019) specify models that are Cobb-Douglas in the long-run. We build on these works by presenting a model that is consistent with data on both the existence of capital-augmenting technical change and the less-than-unitary long-run EoS between capital and labor.

To the best of our knowledge, Grossman et al. (2017, 2021) provide the only other attempt to square the Uzawa steady-state growth theorem with data on the EoS and the capital-augmenting technical change. In their model, schooling is both labor-augmenting and capital-dis-augmenting. In this setting, they show that there is a scope for additional capital-augmenting technological change. Our results indicate that there is a wider scope for ways in which the neoclassical growth model can be made to be consistent with the data. Indeed, their results can be understood as a particular case of the two-factor Uzawa theorem (see subsection 5.2).

Our results also stress the importance of natural resources for understanding macroeconomic outcomes. Historically, natural resources were only included in aggregate macroeconomic analyses when the research question under study was explicitly about those natural resources. For example, energy is generally only included in growth models when studying the depletion of finite resources (e.g., Hotelling, 1931; Heal, 1976; Hassler et al., 2021) or climate change (e.g., Nordhaus and Boyer, 2003; Golosov et al., 2014). Our results suggest a much broader importance of non-reproducible factors. The neoclassical growth model was originally developed to explain the balanced growth facts (Solow, 1956, 1994). In order to explain a wider set of stylized facts, including the existence of capital-augmenting technological change, the model must incorporate factors beyond reproducible capital and labor.

**Roadmap.** The remainder of the paper proceeds as follows. Section 2 discusses the evidence motivating this study. Section 3 proves a multi-factor version of the Uzawa steady-state growth theorem. In Section 4, we generalize the theorem, proving that neoclassical models can have a positive rate of capital-augmenting on the BGP. Section 5 presents three applications of these results, focusing on simple cases and existing literature. Section 6 discusses existing evidence on the EoS between capital and natural resources. Section 7 presents a simple calibration exercise and an application for labor share. Section 8 extends our results to investment-specific technological change. Section 9 concludes.




(a) Real Aggregate Variables per Capita

(b) Relative Price of Investment

Figure 1: Balanced Growth with Capital-Augmenting Technical Change

Note: See Appendix Section D.1 for data sources.

## 2 Motivation and Stylized Facts

The neoclassical growth model, first developed by Solow (1956) and Swan (1956), serves as a basis for much contemporary research in macroeconomics. Such models are designed to explain a set of stylized facts, known as 'balanced' or 'steady' growth (Jones, 2016). The main stylized fact is that income per capita has grown at a constant rate over long periods of time. Panel (a) in figure 1 presents U.S. data from 1960-2020, which clearly demonstrates this fact. In addition, it shows that real investment, consumption and capital per capita have grown at roughly the same rate as real income per capita over this period, capturing the notion of 'balance'.[1]

The neoclassical growth model is founded on two building blocks. The first is the process of capital accumulation: capital is accumulated linearly from saved output. This specification implies that capital 'inherits' the growth rate of output (Jones and Scrimgeour, 2008). The second is a neoclassical aggregate production function that has constant returns to scale (CRS) in all production factors. With a CRS production function, balanced growth is achieved when all effective factors grow at the same rate

[1]See Papell and Prodan (2014), Jones (2016), and others for longer time series and data from other countries.

| Factor | Share | Source |
|---|---|---|
| Natural Resources (incl. Land) | 8% | Caselli and Feyrer (2007) |
| Land | 5% | Valentinyi and Herrendorf (2008) |
| Energy | 4% | Golosov et al. (2014) |

Table 1: This table presents some estimates of U.S. factor shares for inputs other than reproducible capital and labor. Definitions and methodologies vary. *Natural Resources* in Caselli and Feyrer (2007) encompasses land, energy, and other resources.

as output. Since capital grows at the same rate as output, this implies that there is no capital-augmenting technological progress. In the two-factor neoclassical growth model, the only exception is when the aggregate production function is Cobb-Douglas. In this case, effective factors can grow at different speeds while keeping the growth rate of output constant. These results are widely known as the Uzawa theorem.

A long literature has estimated the elasticity between capital and labor in a two-factor production function and rejected the Cobb-Douglas specification. Most of the papers in the literature argue that the elasticity is less than one (e.g., Antras et al., 2004; Chirinko and Mallick, 2017; Oberfield and Raval, 2021). In addition, there is evidence that capital-augmenting technical change has been occurring even as the economy has exhibited signs of balanced growth (Grossman et al., 2017). As shown in Appendix C.1, the speed of capital-augmenting technological change in the neoclassical growth model can be measured by the rate of decline in the relative price of investment, calculated as the ratio of the implicit price deflator for investment to that for personal consumption expenditures in the NIPA statistics, with a one-period lag. Panel (b) of Figure 1 shows that the price of investment goods, and equipment in particular, has been falling relative to the price of consumption goods in the United States over at least the last half a century. A long literature demonstrates that declining investment price (equivalently, capital-augmenting technological change) explains a quantitatively significant portion of economic growth in the United States (e.g., Greenwood et al., 1997; Krusell et al., 2000).[2]

These findings create a puzzle. Given that the EoS between capital and labor is not equal to one, the Uzawa theorem implies that any two-factor neoclassical growth

[2]See He et al. (2008) and Maliar and Maliar (2011) for discussions of the Uzawa steady-state growth theorem in this context. Karabarbounis and Neiman (2014) show that declining investment prices are a widespread phenomenon in cross-country data.

model that is consistent with balanced growth is necessarily at odds with evidence on capital-augmenting technical change. Put differently, the standard neoclassical growth model cannot explain the broader set of stylized growth facts that we observe in the United States.

In this paper, we examine production functions with additional inputs, beyond reproducible capital and labor. It is obvious that other factors – such as land, energy, and other natural resources – exist in the production process. Table 1 collects some evidence on the importance of these factors in the United States. Broadly speaking, estimates suggest that non-reproducible factors other than labor account for about 8-9% of total factor payments. In standard two-factor neoclassical growth models, those factors are often implicitly included in capital. However, this is not an adequate treatment unless they can be linearly accumulated with saved output. This paper shows that explicitly separating these factors from capital is key to solving the puzzle raised by the Uzawa theorem.

# 3 A Multi-factor Uzawa Theorem

In this section, we prove that the steady-state growth theorem by Uzawa (1961) (hereafter, the Uzawa theorem) extends to multi-factor environments that explicitly consider inputs beyond labor and reproducible capital. As shown by Solow (1956), sustained economic growth requires the shape of the production function to change over time, which economists usually call technological change. Given the existence of a BGP, the Uzawa theorem provides a convenient representation of the evolution of the production function. We stress the importance of making a clear distinction between this representation and the true production function, for which we often have limited information. In particular, we also prove a new set of propositions that clarify the conditions under which the representation given by the Uzawa theorem matches important properties of the true production function, implying that the representation serves as a good approximation of the true production function in economic analysis.

## 3.1 Neoclassical Growth Model

The Uzawa theorem depends on two assumptions: (a) the economy is described by a neoclassical growth model, and (b) the economy has a balanced growth path (BGP). We start with a description of a neoclassical growth model, which is defined broadly to incorporate a wide range of dynamic macroeconomic models. We consider a discrete-time setting, where $t = 0, 1, 2, \ldots$, but it is straightforward to consider the continuous-time equivalents of the results.

**Definition 1.** *A* **multi-factor neoclassical growth model** *is an economic environment that satisfies:*

1. *Output, $Y_t$, is produced from capital, $K_t$, and $J \geq 1$ kinds of other inputs, $\{X_{j,t}\}_{j=1}^{J}$:*[3,4]

$$Y_t = F(K_t, X_{1,t}, ..., X_{J,t}; t). \tag{1}$$

   *In any $t \geq 0$, it has constant returns to scale (CRS) in all inputs, $K_t, X_{1,t}, ..., X_{J,t}$, and each input has a positive and diminishing marginal product.*

2. *Capital, $K_t$, accumulates linearly with the saved output*

$$K_{t+1} = Y_t - C_t - R_t + (1 - \delta)K_t, \ \ K_0 > 0, \tag{2}$$

   *where $C_t > 0$ is consumption, $R_t \geq 0$ is expenditure other than capital investment or consumption, and $\delta \in [0, 1]$ is the depreciation rate. The term $Y_t - C_t - R_t$ on the RHS represents the amount of saved output, or equivalently, physical capital investment.*

There are five points to note regarding Definition 1. First, production function $F(\cdot; t)$ in (1) depends on $t$, capturing technological progress. Importantly, we place no restrictions on how the shape of $F(\cdot; t)$ changes over time. As discussed below,

[3] If we allow $J = 0$, the only constant-returns-to-scale production function is in the form of $Y_t = A_{K,t}K_t$. Although it cannot satisfy decreasing marginal products of its input ($K_t$), this $AK$ functional form is also subject to the Uzawa theorem, in the sense that $A_{K,t}$ must be constant on the BGP (i.e., there may not be any technological change).

[4] We follow Uzawa (1961) and Jones and Scrimgeour (2008) by including $t$ as an argument in $F$. Alternatively, We can write equation (1) as $Y_t = F_t(K_t, X_{1,t}, ..., X_{J,t})$ to highlight that $F_t$ changes with $t$.

the Uzawa theorem provides insight into how to approximate the time dependence of $F(\cdot;t)$ with standard factor-augmenting terms.

Second, if $J$ equals 1 and $X_{1,t}$ is interpreted as labor, $L_t$, then equation (1) reduces to a familiar two-factor neoclassical production function, $Y_t = F(K_t, L_t; t)$. In addition, if we assume $L_t$ grows exogenously, Definition 1 essentially coincides with the definition of a neoclassical growth model in Schlicht (2006) and Jones and Scrimgeour (2008), who provide a simple statement and proof of the two-factor Uzawa theorem.

Third, the only reason why capital, $K_t$, is distinguished from other production factors $X_{1,t}, ..., X_{J,t}$ is that we explicitly specify its linear accumulation process (2). From a theoretical viewpoint, $K_t$ needs not to be limited to physical capital. $K_t$ can be any combination of factors that can be accumulated linearly with saved output.[5]

Fourth, the Uzawa theorem holds regardless of the evolution process for other inputs. The $X_{j,t}$'s can be either endogenous or exogenous. If some factors are endogenous, $R_t$ term in (2) may include costs to enhance them. For example, the future growth of labor may be dependent on child-raising costs, which could be included in $R_t$.[6] Technological change, represented by the last $t$ term in the production function (1), can also be exogenous or endogenous. If R&D expenditures enhance technologies, such expenditures would also be included in $R_t$ in (2).

Fifth, equation (2) implies that period $t$ final output and period $t + 1$ capital are measured in the same units. Since this is merely a choice of units, it does not limit the applicability of our results. In models with investment-specific technological change (ISTC), one unit of output can be converted to increasingly many units of capital as technology improves. Section 8 shows that all of our results can be translated to a model with ISTC by a change of variables.

## 3.2 Balanced Growth Path

Now, we turn to the second requirement of the Uzawa theorem, the BGP.

**Definition 2.** *A* **balanced growth path (BGP)** *in a multi-factor neoclassical growth model is a path along which all quantities,* $\{Y_t, K_t, X_{1,t}, ..., X_{J,t}, C_t, R_t\}$*, grow*

[5] For example, in the pre-industrial Malthusian economy where population was proportional to output (e.g., Galor, 2011; Li et al., 2016), labor could be included in $K_t$, not in $X_{j,t}$.

[6] As highlighted in footnote 5, child-rearing costs could be part of investment in a Malthusian economy.

*at constant exponential rates for all $t \geq 0$. On the BGP, we denote the growth factor of output by $g \equiv Y_t/Y_{t-1}$, and the growth factors of any variable $Z_t \in \{ K_t, X_{1,t}, ..., X_{J,t}, C_t, R_t \}$ by $g_Z \equiv Z_t/Z_{t-1}$. A* **non-degenerate** *balanced growth path is a BGP with $g_K > 1 - \delta$.*

From (2), condition $g_K > 1-\delta$ means that physical capital investment $Y_t - C_t - R_t$ is strictly positive along the BGP. The rest of the paper focuses on this non-trivial case. We call it a non-degenerate BGP and simply mention it as a BGP when there is no risk of confusion. Note that, while a BGP requires variables to grow at constant rates, it does not require them to grow at the same rate. Still, the following lemma confirms that capital and consumption need to grow at the same speed as output to maintain a BGP.

**Lemma 1.** *On any non-degenerate BGP in a multi-factor neoclassical growth model, the capital-output ratio $K_t/Y_t$ and the consumption-output ratio $C_t/Y_t$ are constant and strictly positive.*

*Proof.* See Appendix A.2. □

The proof utilizes the assumption of $C_0 > 0$ from Definition 1. If $R_0 > 0$, we can similarly show that $R_t/Y_t$ is constant.

## 3.3 Uzawa Representation and Its Properties

Having defined the neoclassical growth model and the BGP, we are ready to present a multi-factor version of the Uzawa theorem.

**Proposition 1. (A Multi-Factor Uzawa Theorem)** *Consider a non-degenerate BGP in a multi-factor neoclassical growth model, and define $\widetilde{A}_{X_j,t} \equiv (g/g_{X_j})^t$ where $j = 1, ..., J$. Then, on the BGP,*

$$Y_t = \widetilde{F}(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t}) \text{ holds for all } t \geq 0, \tag{3}$$

*where $\widetilde{F}(\cdot) \equiv F(\cdot; 0)$.*

*Proof.* From the definition of $\widetilde{A}_{X_j,t} \equiv (g/g_{X_j})^t$, the growth factor of $\widetilde{A}_{X_j,t}X_{j,t}$ is $g$ for all $j$. The growth factor of $K_t$ is also $g$ from Lemma 1. Therefore, all the arguments in function $\widetilde{F}(\cdot)$ are multiplied by $g$ each period. This means that the RHS of (3) is multiplied by $g$ each period since $\widetilde{F}(\cdot) \equiv F(\cdot;0)$ has CRS. Note that in period 0, equation (3) holds because it is identical with (1). Therefore, (3) holds for all $t \geq 0$, where both sides are multiplied by $g$ in every period. $\square$

It is important to understand what the theorem does and does not imply. Recall that the neoclassical production function $F(\cdot;t)$ in (1) is a time-varying function that potentially depends on $t$ in complex ways. If the economy is on the BGP, the Uzawa theorem says that there should be a simple *representation* of this dependence of function $F(\cdot;t)$ on $t$, which holds at least along this particular BGP. We call this representation, which is given by (3), the *Uzawa representation*. It consists of a time-invariant function $\widetilde{F}(\cdot)$ and exponentially growing $\widetilde{A}_{X_j,t}$ terms. At $t = 0$, equation (3) coincides with the true production function (1). The Uzawa representation illustrates how the production function evolves from there as $t$ changes.

However, caution is needed when interpreting $\widetilde{F}(\cdot)$ as a production function beyond $t = 0$, because Proposition 1 only guarantees that the value of $\widetilde{F}(\cdot)$ coincides with that of the true production function $F(\cdot;t)$ exactly on a particular BGP. As is clear from the proof of the proposition, function $\widetilde{F}(\cdot)$ contains no information about what will happen when inputs deviate even slightly from the BGP. As a result, there is no guarantee that the derivatives of function $\widetilde{F}(\cdot)$, even on the BGP, are equal to the derivatives of the production function $F(\cdot;t)$, apart from time $t = 0$. Without further information, therefore, the Uzawa theorem has little use in economic analysis.

In the following two propositions, we extend the theorem by focusing on the conditions under which the Uzawa representation has the 'correct' marginal properties. We start by looking at first-order derivatives.

**Proposition 2. (Derivatives of the Uzawa representation)** *Let $F_Z(\cdot;t)$ denote the partial derivative of function $F(\cdot;t)$ with respect to its argument $Z \in \{K_t, X_{1,t}, ..., X_{J,t}\}$.*[7] *If the share of factor $Z$, i.e., $s_{Z,t} = F_Z(\cdot;t)Z_t/Y_t$, is constant on a non-degenerate BGP of a multi-factor neoclassical growth model, then the following holds on the BGP:*

$$\frac{\partial}{\partial Z_t}\widetilde{F}(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t}) = F_Z(K_t, X_{1,t}, ..., X_{J,t};t) \text{ for all } t \geq 0. \quad (4)$$

[7] Appendix A.1 discusses the details regarding notation for derivatives.

*Proof.* See Appendix A.3. □

If the factor shares are constant on the BGP, equation (4) says that $\widetilde{F}(\cdot)$ has the same derivatives as the true production function $F(\cdot;t)$ on the BGP. We can also show that the elasticity of substitution (EoS) between capital and other production factors in the Uzawa representation $\widetilde{F}(\cdot)$ coincides with the EoS in the true production function $F(\cdot;t)$ on the BGP, if the latter does not change over time.[8] Let us first define the EoS when there are more than two inputs.[9]

**Definition 3.** *The* **Elasticity of Substitution** *between capital $K_t$ and input $X_j$ in multi-factor neoclassical production function $F(K, X_1, ..., X_J; t)$ in (1) is defined by*

$$\sigma_{KX_j,t} = -\left.\frac{d\ln(K_t/X_{j,t})}{d\ln\left(F_K(K_t, X_{1,t}, ..., X_{J,t}; t)/F_{X_j}(K_t, X_{1,t}, ..., X_{J,t}; t)\right)}\right|_{Y_t, \mathbf{X}_{-j,t}: const}, \quad (5)$$

*where $\mathbf{X}_{-j,t} \equiv \{X_{1,t}, ..., X_{J,t}\} \backslash X_{j,t}$ represents the inputs other than $K_t$ and $X_{j,t}$.*

Using this definition, we can show that the Uzawa representation has the correct EoS if the true EoS is stationary on the BGP.

**Proposition 3. (Elasticity of Substitution in the Uzawa Representation)** *Let $\widetilde{\sigma}_{KX_j,t}$ denote the EoS in the Uzawa representation, as in Definition 3. If the EoS of the true production function, $\sigma_{KX_j,t}$ for some $j \in \{1, ..., J\}$, is constant over time on the BGP, then $\widetilde{\sigma}_{KX_j,t} = \sigma_{KX_j,t}$ holds for all $t \geq 0$ on the BGP.*

*Proof.* See Appendix A.4. □

# 4 A Generalized Uzawa Growth Theorem

By viewing $\widetilde{A}_{X_j,t}$ as the factor $X_{j,t}$-augmenting technology term, Proposition 1 implies that it is always possible to interpret the time variation of the true production function

[8] To the best of our knowledge, Acemoglu (2008) is the only example of previous work considering first-order properties implied by the Uzawa theorem. He looks at first-order conditions in the two-factor case, providing a special case of Proposition 2.

[9] When there are more than two production factors, there are various ways to define the elasticity of technical substitution. See Stern (2011) for a concise taxonomy. The elasticity in (5) is calculated using the inverse of the symmetric elasticity of complementarity (SEC), defined in Stern (2010), which has a desirable property of symmetry between the two variables.

$F(\cdot;t)$ on the BGP in terms of exponential augmentation of production factors. In addition, Propositions 2 and 3 show that the Uzawa representation (3) is 'correct' in terms of its marginal properties. It is tempting to conclude that there should be no technological change that enhances the productivity of capital on the BGP, because there is no $\widetilde{A}_{K,t}$ term in (3). This reasoning is insufficient because Proposition 1 does not establish uniqueness. As a result, it does not rule out the existence of better representations of the true production function.

In this section, we prove a further generalized version of the Uzawa theorem that allows for representations with capital-augmenting technical change. We explore the possibility that the true production function has more than one *factor-augmenting representation* and identify a condition under which there will be a representation that matches the data on the capital-augmenting technological change shown in Section 2. To satisfy this condition while remaining consistent with empirical evidence, it is essential to include factors of production beyond labor and reproducible capital in the aggregate production function.

## 4.1 Factor-Augmenting Representation and Factor Substitution

We start by defining a factor-augmenting representation.

**Definition 4.** *A* **Factor-Augmenting Representation** *of the true production function (1) is a combination of a time-invariant constant-returns-to-scale function $\overline{F}(\cdot)$ and the growth factors of factor-augmenting technologies $\gamma_K > 0$ and $\gamma_{X_j} > 0$, $j \in \{1, ..., J\}$, such that the paths of output and inputs on a BGP satisfy*

$$Y_t = \overline{F}(A_{K,t}K_t, A_{X_1,t}X_{1,t}, ..., A_{X_J,t}X_{J,t}) \text{ holds for all } t \geq 0, \tag{6}$$

*where $A_{K,t} = (\gamma_K)^t$ and $A_{X_j,t} = (\gamma_{X_j})^t$.*

Our objective is to find a factor-augmenting representation $\overline{F}(\cdot)$ that matches a wider set of properties of the BGP, including the evidence of capital-augmenting technological change. By comparing (3) with (6), it is clear that the Uzawa representation is a special case of a factor-augmenting representation. As we explain below,

(3) assumes that all effective factors grow at the same rate of $g$, while (6) permits different growth rates among different effective factors. In other words, the Uzawa representation hypothesizes that there is no factor substitution taking place when the economy grows along the BGP. The homothetic expansion of every effective input is the simplest interpretation of a steadily growing economy, but it does not necessarily constitute the best description of reality.

To see this, suppose that every effective input, including effective capital, grows at the same speed as the output. Recall that physical capital is already growing at the same speed as output on the BGP (Lemma 1). Then, there is no room for additional capital-augmenting technological progress to further augment its effectiveness. As discussed in Section 2, however, there is clear evidence that the productivity of capital, measured in terms of output as in our model, has steadily been increasing on the BGP. Thus, the interpretation of the BGP as being a homothetic expansion of every input is at odds with a well-established stylized fact.

Motivated by this contradiction, we now consider a broader range of possibilities in which effective inputs grow at different constant rates. To have balanced growth with non-homothetic expansion of production factors, it is necessary to further restrict the possible functional forms of the factor-augmenting representation. Before moving to formal propositions, we provide a heuristic discussion that highlights the key intuition. Suppose that the true production function can be represented in the factor-augmenting way (6) on the BGP with the correct derivatives and EoS. Then, the growth rate of output can approximately be written as follows:[10]

$$g \equiv Y_{t+1}/Y_t \approx s_{k,t}\gamma_K g_K + \sum_{j=1}^{J} s_{X_j,t}\gamma_{X_j} g_{X_j}, \tag{7}$$

where $s_{k,t} \equiv F_K(K_t, X_{1,t}, ..., X_{J,t}; t)K_t/Y_t$ is the share of capital at time $t$ and similarly for $s_{X_j,t}$.

Equation (7) says that the growth rate of the output is the weighted average of the growth rates of different effective factors, where the weights are factor shares. When

[10]To obtain this decomposition, we performed a Taylor-expansion on the RHS of (6) for $t+1$ with respect to every effective factor around the period $t$ values. Then, we divided the result by the RHS of (6) for $t$. The Taylor expansion is exact when the variables in $t$ and $t+1$ are sufficiently close, or equivalently, in continuous time.

the effective factors grow at different speeds, $\gamma_K g_K$ and the $\gamma_{X_j} g_{X_j}$'s are different. Specifically, let us assume that effective capital grows faster than output due to K-augmenting technological change ($\gamma_K g_K > g$). Then there must be at least one effective factor that is growing slower than output. Let us say that this factor is $X_1$ (i.e.,$\gamma_{X_1} g_{X_1} < g$) and that all the other effective factors are growing at the same rate as output. Then, dividing the factor augmenting representation (6) by $Y_t$ gives

$$1 = \overline{F}\left(\frac{A_{K,t}K_t}{Y_t}, \frac{A_{X_1,t}X_{1,t}}{Y_t}, constants\right). \tag{8}$$

In this form, it is evident the growing effective capital-output ratio $A_{K,t}K_t/Y_t$ permits the production of unit output with the shrinking effective $X_1$-output ratio $A_{X_1,t}X_{1,t}/Y_t$. In other words, factor substitution is occurring.

Now, let us check if this ongoing factor substitution is consistent with the definition of the BGP. On the BGP, output grows at a constant rate, $g$, which means that the RHS of (7) must also be constant. Given $\gamma_K g_K > \gamma_{X_1} g_{X_1}$, the RHS of (7) only remains constant when the factor shares, $s_{k,t}$ and $s_{X_1,t}$, do not change over time. This happens if and only if the EoS of $\overline{F}(\cdot)$ between $K$ and $X_1$, defined in Definition 3, is equal to one. To summarize, for K-augmenting technological change to happen on the BGP in a factor-augmenting representation, the functional form of $\overline{F}(\cdot)$ needs to have a unitary EoS between capital and some other factor.[11] In this case, it is possible to have balanced growth even when effective capital grows faster than output.

Once we obtain a factor-augmenting representation, we hope to use it as an approximation of the true production function. In particular, as in Proposition 3, the representation is especially useful if the EoS of $\overline{F}(\cdot)$ matches that of the true production function. This is only possible when the true production function $F(\cdot;t)$ has a unitary EoS between capital and some other factor, because we already know that $\overline{F}(\cdot)$ must have a unitary EoS. As discussed in section 2, there is a great deal of evidence suggesting that the EoS between capital and labor is different than one. However, our definition of the neoclassical growth model allows for any number of

[11] In the Uzawa representation, $\gamma_K g_K = \gamma_{X_j} g_{X_j}$ holds for all $j$. Because the production function is assumed to have CRS (which guarantees $s_{K,t} + \sum_{j=1}^{J} s_{X_j,g} = 1$), the RHS is always constant. Therefore, we can use $F(\cdot;0)$ as the Uzawa representation without checking its EoS properties (see Proposition 1) at the cost that it cannot accommodate the possibility of K-augmenting technological change.

inputs. Once we consider the realistic case with more than two factors of production, it becomes more likely that at least one input has a unitary EoS with capital.

## 4.2 A Steady-State Growth Theorem with $K$-augmenting Technical Change

Here, we formally construct a function that can be used as a basis for a factor-augmenting representation. Consider factors of production other than capital, $\{X_{1,t}, ..., X_{J,t}\}$, and suppose that some of them are substitutable with capital, $K_t$, with unitary elasticity in the period 0 production function, $F(\cdot; 0)$. Without loss of generality, we reorder these factors so that the first $j^* \in \{1, ..., J\}$ of them can be substituted with capital with the unitary EoS.

If capital is substitutable with $j^*$ other factors with a unitary elasticity, we can interpret them as if they are combined together in the Cobb-Douglas fashion to form an intermediate input. The intermediate input, which we call the *capital composite*, will then be one argument in the final production function. Using the share of factors in period 0, $s_{K,0} \equiv F_K(K_0, X_{1,0}, ..., X_{J,0}; 0)K_t/Y_t$ and $s_{X_j,0} \equiv F_K(K_0, X_{1,0}, ..., X_{J,0}; 0)X_{j,t}/Y_t$, we define period-0 relative shares within the capital composite:

$$\alpha = s_{K,0}/(s_{K,0} + \sum_{j=1}^{j^*} s_{X_j,0}), \quad \xi_j = s_{X_j,0}/(s_{K,0} + \sum_{j=1}^{j^*} s_{X_j,0}). \tag{9}$$

Using these relative shares, we can represent the production function in a nested form:[12]

$$\overline{F}(k, x_1, ..., x_J) \equiv \widehat{F}\left(k^\alpha \prod\nolimits_{j=1}^{j^*} x_j^{\xi_j}, x_{j^*+1}, ..., x_J\right). \tag{10}$$

The first argument of the $\widehat{F}(\cdot)$, $m = k^\alpha \prod_{j=1}^{j^*} x_j^{\xi_j}$, represents the capital composite, which combines capital and the other $j^*$ factors that have a unitary EoS with capital. Capital composite $m$ is an argument in the outside function $\widehat{F}(\cdot)$, along with other

[12] In this section, we use lowercase letters $k, x_1, ..., x_J$ to denote variables, while uppercase letters $K_t, X_{1,t}, ..., X_{J,t}$ are the BGP values, unless otherwise noted. Also, with a slight abuse of notation, here we define function $\overline{F}(\cdot)$ by (10), while $\overline{F}(\cdot)$ was previously used in a factor-augmenting representation. This abuse will be resolved in Proposition 4, which shows $\overline{F}(\cdot)$ in (10) actually constitutes a factor-augmenting representation.

factors $x_{j^*+1}, ..., x_J$. The shape of the outside function $\widehat{F}(\cdot)$ is defined using the period-0 production function $F(\cdot; 0)$:[13]

$$\widehat{F}(m, x_{j^*+1}, ..., x_J) \equiv F\left(\left(\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha}, X_{1,0}, ..., X_{j^*,0}, x_{j^*+1}, ..., x_J; 0\right). \tag{11}$$

The first argument of $\widehat{F}(\cdot)$, $m$, collects the $j^*$ relevant inputs and combines them with capital in the first argument. As a result, function $\widehat{F}(\cdot)$ has $j^*$ fewer arguments than $F(\cdot; 0)$. Note that the RHS of (11) includes the BGP values $X_{j,0}$, $J = 1, \ldots, j^*$, which are treated as constants. Changes in the $x_{j,0}$ terms only matter through $m$.

As the following lemma shows, the nested representation, $\overline{F}(\cdot)$ with $\widehat{F}(\cdot)$, approximates the true production function around the BGP in period 0.

**Lemma 2. (Nested representation of the production function at $t = 0$)**

*a.* $\overline{F}(K_0, X_{1,0}, ..., X_{J,0}) = F(K_0, X_{1,0}, ..., X_{J,0}; 0)$.

*b.* *For any* $Z \in \{K, X_1, ..., X_J\}$, $\overline{F}_Z(K_0, X_{1,0}, ..., X_{J,0}) = F_Z(K_0, X_{1,0}, ..., X_{J,0}; 0)$.

*c.* *For any* $j = 1, ..., j^*$, $\overline{\sigma}_{KX_j,0} = \sigma_{KX_j,0}$, *where* $\overline{\sigma}_{KX_j,0}$ *is the EoS of function* $\overline{F}(k, x_1, ..., x_J)$ *between* $k$ *and* $x_j$, *evaluated at the period-0 BGP.*

*d.* *Functions* $\widehat{F}(m, x_{j^*+1}, ..., x_J)$ *and* $\overline{F}(k, x_1, ..., x_J)$ *have constant returns to scale.*

*Proof.* See Appendix A.5. □

Properties $a$, $b$, and $c$ respectively confirm that the nested representation $\overline{F}(\cdot)$ matches the period-0 true production function, $F(\cdot; 0)$, in terms of the level of inputs and output, the first derivatives for any input, and the EoS between $K$ and any other input $X_j$, when the function is evaluated around the period-0 BGP.[14] Property $d$ confirms the CRS property.

Thanks to the CRS property, the nested representation can be used not only for period 0, but also for representing how the production function evolves from there along the BGP. The following proposition establishes that, with the nested

[13] $F(\cdot; 0)$ needs to satisfy $\sigma_{KX_j,0} = 1$ for $j = 1, ..., j^*$. Other than that, the following analysis only requires that the local properties of $F(\cdot; 0)$ are known to researchers. See Remarks 1-3 for related discussions in the context of the multi-factor Uzawa theorem.

[14] Namely, when $\{k, x_1, ..., x_J\}$ are at the period-0 BGP values $\{K_0, X_{1,0}, ..., X_{J,0}\}$.

representation $\overline{F}(\cdot)$, there are multiple ways to represent the technological change in a factor-augmenting fashion.

**Proposition 4. (A Generalized Uzawa Growth Theorem)** *Suppose that* $\sigma_{KX_j,0} = 1$ *for* $j = 1, ..., j^*$. *On a non-degenerate BGP, let* $\gamma_K > 0$ *and* $\gamma_{X_j} > 0$, $j \in \{1, ..., j^*\}$, *be* **any** *combination that satisfies the technology condition*

$$(\gamma_K g)^\alpha \prod_{j=1}^{j^*} (\gamma_{X_j} g_{X_j})^{\xi_j} = g. \tag{12}$$

*For* $j \in \{j^* + 1, ..., J\}$, *let* $\gamma_{X_j} = g/g_{X_j}$. *With* $\gamma_K$ *and each* $\gamma_{X_j}$, *define* $A_{K,t} = (\gamma_K)^t$ *and* $A_{X_j,t} = (\gamma_{X_j})^t$. *Also, define function* $\overline{F}(\cdot)$ *by (9) and (10). Then, on the BGP,*

$$Y_t = \overline{F}\left(A_{K,t}K_t, A_{X_1,t}X_{j,t}, ..., A_{X_J,t}X_{J,t}\right) \text{ for all } t \geq 0. \tag{13}$$

*Proof.* See Appendix A.6. □

Note that (13) constitutes a factor augmenting representation, as defined by Definition 4.[15] Thus, Proposition 4 characterizes the set of factor-augmenting representations of the true production function along the BGP. When there is no factor that is substitutable with capital with a unitary elasticity at time 0 (i.e., $j^* = 0$), then Proposition 4 becomes identical to Proposition 1.[16] However, given that there are many factors of production in reality, it seems plausible that at least one of them is substitutable with capital with a unitary elasticity ($j^* \geq 1$). In this case, there are several aspects of the proposition that warrant further discussion.

First, unlike Proposition 1, the generalized theorem implies that there is a continuum of representations. Factor-augmenting terms, $\gamma_K$ and $\gamma_{X_j}$ for $j = 1, ..., j^*$, can be any combination that satisfy condition (12). This enables applied researchers to pick the representation that is most consistent with data on technical change. The Uzawa representation is a special case of the factor-augmenting representation with $\gamma_K = 0$.

[15] Recall that function $\overline{F}(\cdot)$ has CRS from Lemma 2.

[16] If $j^* = 0$, condition $\alpha + \sum_{j=1}^{j^*} \xi_j = 1$ in Lemma 2 implies $\alpha = 1$. Then, condition (12) reduces to $\gamma_K = 1$, which means $A_{K,t} = 1$ for all $t$. Then, (13) becomes identical to (3).

Second, condition (12) implies that the amount of *effective* capital composite,

$$M_t = (A_{K,t}K_t)^{\alpha} \prod_{j=1}^{j^*} (A_{X_j,t}X_{j,t})^{\xi_j},$$

must grow at the same speed of output, $g$. This result can be expressed in a log-linear form:

$$\alpha \log \gamma_K + \sum_{j=1}^{j^*} \xi_j \log \gamma_{X_j} = (1-\alpha) \log g - \sum_{j=1}^{j^*} \xi_j \log g_{X_j}. \tag{14}$$

When the growth rates of the factor-augmenting technologies are exogenous, this log-linear condition may seem restrictive. In a model where the direction of technical change is endogenous, however, this condition can be endogenously satisfied as the economy converges to the BGP. If the speed of capital-augmenting technological change is slower than required in (14), then the effective capital composite gradually becomes scarcer relative to the other effective factors. The scarcity will induce firms to do more R&D to enhance the capital-augmenting technology rather than enhancing other factors. As a result, condition (14) will necessarily be satisfied in the long run as long as the economy converges to a BGP. We build such a model in a companion paper (Casey and Horii, 2023). The results suggest that the log-linear condition does not impose any extra restrictions.

Third, similar to the original Uzawa theorem (Proposition 1), equation (13) is not a functional relationship. It only states that the level of inputs and outputs in this representation match those of the true production function on the BGP. The following propositions establish that, under conditions similar to Propositions 2 and 3, the factor-augmenting representation (13) gives the correct first derivatives and the correct EoS between capital and other factors around the BGP.

**Proposition 5. (Derivatives of the Factor-Augmenting Representation)** *Suppose that $\sigma_{KX_j,0} = 1$ for $j = 1, ..., j^*$. If the share of factor $Z_t \in \{K_t, X_{1,t}, ..., X_{J,t}\}$, i.e., $s_{Z,t} = F_Z(\cdot;t)Z_t/Y_t$, is constant on a non-degenerate BGP of a multi-factor neoclassical growth model, the following holds on the BGP:*

$$\frac{\partial}{\partial Z_t}\overline{F}\left(A_{K,t}K_t, A_{X_1,t}X_{1,t}, ..., A_{X_J,t}X_{J,t}\right) = F_Z(K_t, X_{1,t}, ..., X_{J,t}; t) \text{ for all } t \geq 0. \tag{15}$$

*Proof.* See Appendix A.7. □

**Proposition 6. (The EoS of the Factor-Augmenting Representation)** *Suppose that $\sigma_{KX_j,0} = 1$ for $j = 1, ..., j^*$ and let $\overline{\sigma}_{KX_j,t}$ denote the EoS in the factor-augmenting representation*
*$\overline{F}(A_{K,t}K_t, A_{X_1,t}X_{1,t}, ..., A_{X_J,t}X_{J,t})$. If the EoS of the true production function, $\sigma_{KX_j,t}$ for some $j \in \{1, ..., J\}$, is constant over time on the BGP, then $\overline{\sigma}_{KX_j,t} = \sigma_{KX_j,t}$ holds for all $t \geq 0$ on the BGP.*

*Proof.* See Appendix A.8. □

## 4.3 Practical Use and Comparison to Uzawa Theorem

Propositions 4-6 demonstrate that the factor-augmenting representation captures key elements of the true production function and is potentially useful for economic analysis. When developing a dynamic macroeconomic model, researchers need to take a stand on how to represent technical change. In other words, they need to decide how the shape of the production function will evolve over time. This is a challenging task that can influence the results, especially in a quantitative setting.

Given the requirement that the model should have a BGP, the propositions provide guidance in choosing a suitable representation of the evolution of the production function. As shown in equation (1), the definition of the neoclassical growth model allows the aggregate production function to evolve in any way. The factor-augmenting representation captures this evolution only through factor-augmenting terms. Proposition 4 demonstrates that the representation matches the level of all the key variables on the BGP, recreating an important set of stylized facts. Proposition 5 implies that the representation has the correct derivatives, and therefore factor shares, as long as factor shares are constant on the BGP. Relatedly, Proposition 6 says that the Uzawa representation has the correct EoS between capital and other variables, as long as that elasticity is constant. Thus, the Uzawa representation can be useful as a local approximation of the true function around the BGP.

These properties echo that of the Uzawa representation, as previously shown in Propositions 1-3. However, the original Uzawa theorem explains the balanced growth by homothetic expansion of every effective production factor. In other words, the Uzawa representation hypothesizes that no factor substitution is taking place along the BGP. As a result, it requires the productivity of capital to stay constant. Our generalized theorem in Proposition 4 clarifies that the Uzawa theorem is only a single possibility out of a continuum of possible factor-augmenting representations, as long as the production function allows factor substitution on the BGP (i.e., at least one factor of production has a unitary EoS with capital).[17] Every candidate representation can explain the observed quantities on the BGP, but they differ in the rates of factor-augmenting technological progress among different production factors. Given that other properties are the same, it is useful to choose the candidate representation that matches the rate of capital-augmenting technological progress observed in data.

In Propositions 4-6, we construct the factor-augmenting representation based on the period-0 production function. Period 0 can be chosen freely by the researcher as long as the economy is on the BGP in that period. Still, it may suggest that the complete shape of the production function at some point in time must be known. However, this assumption is made for the sake of clarity and is not necessary. The following remarks state that we can apply the propositions even when only the local properties of the period-0 production function are known.[18]

**Remark 1.** *The proof of Proposition 4 holds as long as $\widehat{F}(\cdot)$ in (10) is any CRS function that matches the level of inputs and output in period 0: i.e.,* $\widehat{F}\left(K_0^{\alpha}\prod_{j=1}^{j^*}X_{j,0}^{\xi_j}, X_{j^*+1,0}, ..., X_{J,0}\right) = Y_0$.

**Remark 2.** *The proof of Proposition 5 hold as long as $\widehat{F}(\cdot)$ matches the first derivative of $F(\cdot;0)$ in period 0, i.e.,* $\frac{\partial}{\partial Z_0}\widehat{F}\left(K_0^{\alpha}\prod_{j=1}^{j^*}X_{j,0}^{\xi_j}, X_{j^*+1,0}, ..., X_{J,0}\right) = F_Z(K_0, X_{1,0}, ..., X_{J,0}; 0)$ *for $Z = K_t, X_1, ..., X_J$, in addition to the condition in Remark 1.*

**Remark 3.** *The proof of Proposition 6 holds as long as the EoS of $\widehat{F}(\cdot)$ between capital and all other inputs evaluated in period 0 match that of $F(\cdot;0)$, in addition to the conditions in Remark 1.*

Thus, when building models for macroeconomic research, economists can pick a CRS production function (e.g., a CES production function) and calibrate its parameters to match the level, derivatives (or factor shares), and the EoS from the data at a particular point in time, which can be regarded as time-0 BGP values. Then, the representation will continue to match these moments on the BGP, as long as the factor

[17] The Uzawa representation, where $\gamma_K = 1$ and $\gamma_{X_j} = g/g_{X_j}$ for all $j$, satisfies condition (14). Therefore, it is a special case of the factor-augmenting representation.

[18] Similar remarks apply to Propositions 1-3.

shares and the EoS are stationary. In addition, according to the data on technological change, the economist can calibrate the speed of factor-augmenting technological change, $\gamma_K$ and $\gamma_{X_j}$'s, using constraint (12) or, equivalently, its log-linear version (14).

# 5 Three Examples

In this section, we explain the use and implications of the generalized Uzawa theorem in three concrete settings. We choose examples to explore the simplest way to make neoclassical models consistent with aggregate data on the relative price of capital and the EoS between capital and labor. In subsection 5.1, we explain why a standard neoclassical economy with only two factors cannot accomplish this goal. Then, subsection 5.2 discusses the approach taken by Grossman et al. (2017) as a special case of the 2-factor neoclassical environment. Finally, subsection 5.3 shows that the conflict between data and neoclassical models can be resolved when including factors of production beyond labor and reproducible capital. These examples will also illustrate how the theorem can be applied to other settings.

## 5.1 Standard 2-Factor Neoclassical Growth Model

Suppose that the true production function uses only two kinds of inputs, capital, $K_t$, and labor, $L_t$, i.e., $Y_t = F(K_t, L_t; t)$. The production function $F(\cdot; t)$ depends on time due to the technological change. Then, Proposition 1 says that, on any BGP with positive investment, technological change can always be represented as $Y_t = \widetilde{F}(K_t, A_{L,t}L_t)$. If these two factors are substitutable with a unitary elasticity ($\sigma_{KL} = 1$), Proposition 4 shows there are other possible factor-augmenting representations of the same BGP:[19]

$$Y_t = \overline{A}(A_{K,t}K_t)^{\alpha}(A_{L,t}L_t)^{1-\alpha}, \text{where } \overline{A} > 0 \text{ is a constant,} \tag{16}$$

which includes an Uzawa representation $Y_t = \overline{A}K_t^{\alpha}(\widetilde{A}_{L,t}L_t)^{1-\alpha}$ as a special case. Given the growth factors of output and labor on the BGP, condition (12) implies that any

[19]When there are two factors ($J = 1$) and they are substitutable with a unitary elasticity ($j^* = 1$), equation (13) in Proposition 4 implies that $Y_t = \widehat{F}\left((A_{K,t}K_t)^{\alpha}(A_{L,t}L_t)^{1-\alpha}\right)$. Because function $\widehat{F}(\cdot)$ has CRS and has only one argument, we can write $\widehat{F}(x) = \overline{A}x$ for some $\overline{A} > 0$, which gives (16).

combination of $\gamma_K = A_{K,t+1}/A_{K,t}$ and $\gamma_L = A_{L,t+1}/A_{L,t}$ is consistent with the BGP as long as they satisfy $\gamma_K^{\alpha}(\gamma_L g_L)^{1-\alpha} = g^{1-\alpha}$. By rewriting (16) as $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$, where total factor productivity (TFP), $A_t$, is given by $A_t \equiv \overline{A} A_{K,t}^{\alpha} A_{L,t}^{1-\alpha}$, it is clear that various combinations of capital- and labor-augmenting technological changes give the same rate of growth for TFP and, therefore, output.

This result confirms the widely understood version of the Uzawa theorem: on a BGP, all technological progress must be labor-augmenting, unless the production function is Cobb-Douglas. As we have seen in Section 2, this theoretical result is in contradiction with two stylized facts: (i) the productivity of capital has been steadily increasing, and (ii) the EoS between capital and labor is less than one, ruling out the Cobb-Douglas production function. No standard two-factor production function can reconcile these two stylized facts on a BGP.

## 5.2 Inclusion of Schooling in a Two-Factor model

Grossman et al. (2017) propose a possible solution to this contradiction by including schooling, $s_t \geq 0$, in a standard two-factor production function. Their result can be understood intuitively in terms of our analytical framework. While they start their analysis from a factor-augmenting representation, it is worthwhile to consider an underlying time-varying true production function in the form of (1):[20]

$$Y_t = F(K_t, L_t; t) = F^s(D(s_t)^a K_t, D(s_t)^{-b} L_t; t), \tag{17}$$

where $a > 0$, $b > 0$, $D(\cdot) \in [0, 1]$, and $D'(\cdot) < 0$. With $D(s_t)$ terms, the RHS of (17) specifies the production function beyond the general form $F(K_t, L_t; t)$. When $s_t$ increases, the multiplier $D(s_t)^a$ on $K_t$ shrinks, raising the marginal product of capital. The opposite holds for labor. In this way, Grossman et al. (2017) specify a certain type of complementarity between schooling and capital.

Note that $s_t$ is not a production factor in the neoclassical sense, because the production function has CRS only in capital and labor. Still, as the $D(s_t)$ term

[20] They considered not only factor-augmenting technological progress, but also investment-specific technological change. Definition 1 can include both cases as we will show in Section 8. In a subsequent study (Grossman et al., 2021), they used human capital instead of schooling as an argument in the production function otherwise similar to (17). Our interpretation also applies to this paper.

changes over time, it affects the amount of output produced from given quantities of $K_t$ and $L_t$. This is a particular form of technological change, and we can consider $D(s_t)$ as being included in the $t$ term of $F(K_t, L_t; t)$, as in the middle part of (17). Therefore, it falls within the definition of a two-factor neoclassical growth model (i.e., Definition 1 with $J = 1$).

From Proposition 1, this production function has an Uzawa representation $Y_t = \widetilde{F}(K_t, \widetilde{A}_{L,t} L_t)$ with $\widetilde{A}_L = (g/g_L)^t$ on a BGP, where both effective factors $K_t$ and $\widetilde{A}_{L,t} L_t$ grow at the same speed as output. The production function in Grossman et al. (2017) can be interpreted in the following way, keeping the multiplier $D(s_t)$ term in the expression:

$$Y_t = \widetilde{F}(K_t, \widetilde{A}_{L,t} L_t) = \widetilde{F}(A_{K,t} D(s_t)^a K_t, A_{L,t} D(s_t)^{-b} L_t). \tag{18}$$

Comparing the arguments in the RHS to those in the middle, we immediately obtain $A_{K,t} = D(s_t)^{-a}$ and $A_{L,t} = \widetilde{A}_{L,t} D(s_t)^b$ on the BGP. Because the multiplier $D(s_t)^a$ shrinks as $s_t$ increases, the capital-augmenting technology $A_{K,t}$ must grow so as to exactly offset the shrinking $D(s_t)^a$ term. Conversely, the labor-augmenting term $A_{L,t}$ should grow slower than that in the Uzawa representation $\widetilde{A}_{L,t}$, because the multiplier $D(s_t)^{-b}$ is also augmenting labor.[21] In this sense, there is no overall growth in capital productivity in the Grossman et al. (2017) formulation.

Within the limits of the two-factor Uzawa theorem, Grossman et al. (2017) propose a new interpretation of the production function, which provides the first possible solution to the contradiction raised by the Uzawa theorem. In their formulation, it is important that schooling enters the production function precisely in the form of (18), where the same function $D(s_t)$ appears both before capital and labor, with powers of opposite signs. In addition, the functional form of $D(s_t)$ and the dynamic path $s_t$ in equilibrium must be specified such that $D(s_t)$ shrinks exponentially over time.

Future empirical work could inform the understanding of long-run economic growth

[21] From these observations, the main result of (Grossman et al., 2017, proposition 2) can easily be obtained as follows. Taking the growth factor of the both sides of $A_K = D(s_t)^{-a}$ gives $\gamma_K = g_D^{-a}$. From this, we obtain a discrete-time equivalent of their Proposition 2(ii): $g_D = \gamma_K^{-1/a}$. Note that Grossman et al. (2017) assumed $L_t = D(s_t)N_t$, which means $g_L = g_D g_N$. Because effective labor $A_{L,t} D(s_t)^{-b} L_t$ in (18) must grow at the same rate as output, $g = \gamma_L g_D^{-b} g_L = \gamma_L g_D^{1-b} g_N = \gamma_L \gamma_K^{(b-1)/a} g_N$, which is a discrete time equivalent of their proposition 2(i).

by testing whether the formulation (17) is consistent with data. In this paper, we propose a wider class of functions that are consistent with balanced growth. The next subsection discusses a particularly simple example.

### 5.3 A Simple Three-Factor Model with Natural Resources

As shown in Section 2, a significant portion of GDP is paid to production factors that do not fit well in the notion of $K_t$ or $L_t$. Thus, it is natural to consider production functions with more than two factors. Adding these additional factors makes it possible to reconcile neoclassical models with the data. While labor cannot be substituted by capital with unitary elasticity ($\sigma_{KL} \neq 1$), Proposition 4 only requires that there is a *single* production factor satisfies this requirement. In this case, there exist factor-augmenting representations of the production function that have capital-augmenting technological change ($\gamma_K > 1$).

Let us consider the simplest extension of the standard neoclassical production function,

$$Y_t = F_t(K_t, L_t, X_t; t), \text{ where } X_t = X_0 g_X^t \text{ for all } t, X_0 > 0, g_X > 0. \tag{19}$$

Here, we have a third production factor $X_t$, which is either growing ($g_X > 1$), shrinking ($g_X \in (0,1)$), or constant ($g_X = 1$). One example of such a factor is land. In that case, $g_X$ represents the growth factor of the available land space. If the total area of available land is constant, $g_X$ would be one on the BGP. Another example is fossil fuels. Any kind of natural resource, or a collection of natural resources (including land), is a candidate for $X_t$.

Among many candidates for the third production factor, we focus on those that have a unitary EoS with capital: $\sigma_{KX} = 1$. Then, Proposition 4 implies that, along a non-degenerate BGP, technological change can be represented in a factor-augmenting fashion:

$$Y_t = \widehat{F}\left(\left(A_{K,t}K_t\right)^{\alpha}\left(A_{X,t}X_t\right)^{1-\alpha}, A_{L,t}L_t\right), \ \alpha \in (0,1), \tag{20}$$

where the growth factor of technology variables must satisfy $\gamma_L = g/g_L$ and $\gamma_K^{\alpha}(\gamma_X g_X)^{1-\alpha} =$

$g^{1-\alpha}$. As in (14), the latter condition can be written in a log-linear form:

$$\log \gamma_K = \frac{1-\alpha}{\alpha} \left( \log \gamma_L + \log g_L - \log \gamma_X - \log g_X \right). \tag{21}$$

Thus, there must be a positive capital-augmenting technological change on a BGP ($\gamma_K > 1$), as long as the economy is growing faster than the effective input of the third factor ($g = \gamma_L g_L > \gamma_X g_X$). We will calibrate this simple three-factor model to U.S. data in Section 7. Before doing so, we discuss empirical evidence on the elasticity of substitution between capital and the third factor $X_t$.

# 6 The EoS: Linking Theory and Empirics

In this section, we discuss existing empirical evidence on the elasticity of substitution between capital and other factors, and the link between these estimates and economic theory. We start by highlighting the difference between short- and long-run elasticity and then discuss existing empirical work on capital-labor, capital-energy, and capital-land substitution.

## 6.1 Short- and Long-run EoS: Theory

When the relative factor prices change, firms try to adjust relative inputs to the production function, substituting one factor with another. The EoS measures the extent of this activity. The ability of firms to substitute between production factors depends on the time horizon. In the short run, the ability of factor substitution is typically limited, because firms face various constraints that are alleviated over time. Put differently, there is a difference between the short-run EoS, which is affected by these constraints, and the long-run EoS, which is not.

Leon-Ledesma and Satchi (2019) show that it is the long-run EoS that is relevant for the Uzawa theorem, because the theorem is concerned only with outcomes along the balanced growth path. They consider an aggregate production function in the form of $Y_t = F^{\text{short}}(A_{K,t}K_t, A_{L,t}L_t; \theta)$, where technology parameter $\theta > 0$ is taken as given in the short run. In this formulation, the EoS of function $F^{\text{short}}(\cdot; \theta)$, given the value of $\theta$, provides the short-term (or constrained) EoS between $K_t$ and $L_t$. In the long run, firms will choose $\theta$ optimally. Therefore, the long-run production function is given as the solution to $F^{\text{long}}(A_{K,t}K_t, A_{L,t}L_t) \equiv \max_\theta F^{\text{short}}(A_{K,t}K_t, A_{L,t}L_t, \theta)$. They have theoretically shown that the EoS of $F^{\text{long}}(\cdot)$ is higher than that of $F^{\text{short}}(\cdot;\theta)$, consistent with the finding in the empirical literature. They also show that, to obtain the BGP with capital-augmenting technological change, the long-term EoS between capital and labor needs to be one, but there is no restriction on the short-run EoS.[22] In other words, it is the long-run EoS that determines compliance with the two-factor Uzawa growth theorem.

In our theory, we consider the neoclassical production function (1) without short-term restrictions, such as $\theta$ above. Therefore, $F(K_t, X_{1,t}, ..., X_{J,t}; t)$ in (1) and its factor-augmenting representation $Y_t = \overline{F}\left(A_{K,t}K_t, A_{X_1,t}X_{j,t}, ..., A_{X_J,t}X_{J,t}\right)$ in (13) are unconstrained, long-run production functions. As with most neoclassical growth models, we implicitly assume that production function $F$ includes all substitution possibilities, given the currently available set of technologies in the economy.[23] Differently from Leon-Ledesma and Satchi (2019), we focus on the case where the long-term (or unconstrained) EoS between $K_t$ and $L_t$ is less than one, which fits better the findings in the empirical literature. To do so, we look for some other factor $X_t$ such that EoS between $K_t$ and $X_t$ in this long-term aggregate production function is one.

## 6.2 Long-run EoS: Empirical evidence

The distinction between the short-run and long-run EoS is important for interpreting empirical estimates. Analyses with time-series and panel data looking at the immediate reaction of factor use to relative price changes will estimate the short-run (constrained) elasticity, which may be very different than the long-run (unconstrained) elasticity. To estimate long-run elasticities, it is often necessary to use cross-sectional

[22]To be consistent with the BGP, they have shown that production function $Y_t = F^{\text{short}}(A_{K,t}K_t, A_{L,t}L_t; \theta)$ needs to take the form of $\widetilde{F}(A_t\theta^{\alpha-1}K_t, A_t\theta^{\alpha}L_t)$, where $A_t$ is the general level of technology. Since $\widetilde{F}(\cdot)$ is CRS, we can show that $Y_t = A_tK_t^{\alpha}L_t^{1-\alpha}\ \max_{\theta>0}\ \widetilde{F}((\theta L_t/K_t)^{\alpha-1}, (\theta L_t/K_t)^{\alpha})$. The maximized value in the latter expression, $a^* = \max_{\theta>0}\ \widetilde{F}((\theta L_t/K_t)^{\alpha-1},\ (\theta L_t/K_t)^{\alpha})$ becomes constant because the maximand depends only on the technology-adjusted factor intensity $\theta L_t/K_t$, which can be set to any positive value adjusting $\theta$. Therefore, the long-term (or unconstrained) production function is Cobb-Douglas: $Y_t = a^*A_tK_t^{\alpha}L_t^{1-\alpha}$.

[23]If $\Theta_t$ is the available set of technology in time $t$ and $F^{\text{short}}(K_t, X_{1,t}, ..., X_{J,t}; t, \theta)$ is the production function given technology $\theta$, then $F(K_t, X_{1,t}, ..., X_{J,t}; t) = \max_{\theta\in\Theta_t} F^{\text{short}}(K_t, X_{1,t}, ..., X_{J,t}; t, \theta)$.

data or adopt other approaches that separate short- and long-run variation. In this case, it is possible to observe the degree of factor substitution after short-run constraints have dissipated. In the following sections, we briefly review the existing evidence on substitution between capital and labor (Section 6.2.1), capital and energy (Section 6.2.2), capital and land (Section 6.2.3).

#### 6.2.1 Capital and Labor

In Section 2 and other places, we mentioned that existing evidence suggests that capital and labor have an EoS that is different than one. This is true for the long-run EoS. Chirinko (2008) provides a helpful summary of the literature. He stresses the distinction between the short- and long-run EoS and the fact that empirical evidence suggests that the long-run EoS is less than one. This result is confirmed by more recent work (e.g., Chirinko and Mallick, 2017; Oberfield and Raval, 2021).[24]

#### 6.2.2 Capital and Energy

There is a long literature on the economic implications of changes in energy prices. A key finding in this literature is that the long-run EoS between energy and non-energy inputs is significantly higher than the short-run EoS, which is near zero (e.g., Berndt and Wood, 1975; Griffin and Gregory, 1976; Pindyck and Rotemberg, 1983; Atkeson and Kehoe, 1999). Indeed, the existing evidence is consistent with a long-run EoS between capital and energy of one.

Koetse et al. (2008) perform a meta-study of existing literature on substitution between capital and energy. They find country-level, cross-sectional elasticities that are very close to one in North America. The estimates for Europe are around 0.8, but do not reject unitary elasticities. This is the most direct test of the long-run elasticity, as highlighted in the previous section.

Van der Werf (2008) estimates nested CES production functions. When he adopts a nesting structure consistent with the existence of a *capital composite*, he finds elasticities very close to one, although alternate nesting structures yield higher goodness-of-fit measures. This estimation is done in panel data and does not distinguish between

[24] Karabarbounis and Neiman (2014) is an interesting exception. Their estimate of the long-run elasticity is above one.

short- and long-run elasticities.

Hassler et al. (2021) study a structural model of fossil fuel energy use and directed technical change. Similarly to Leon-Ledesma and Satchi (2019), they stress the role that endogenous technology plays in creating a difference between short- and long-run elasticities and derive similar conditions under which the long-run elasticity between energy and other factors is equal to one. In their estimated model, they assume a nesting structure that is different from that implied by the generalized Uzawa theorem and stress the lack of precision in their long-run estimates. Still, their estimates suggest only a modest change in the energy expenditure share following a significant change in trend energy prices, implying that the implied elasticity is not far from one.

Our reading of the empirical evidence is as follows. Energy use (or perhaps some subset of energy) is a reasonable candidate for the third factor $X$ that has a unitary EoS with capital. However, more research is needed, both to estimate the relevant elasticity and to determine the appropriate nesting structure. We hope that our theoretical results will spur new empirical work, which could significantly improve our understanding of balanced growth.

### 6.2.3 Capital and Land

Land is an intuitive candidate for the third factor, because it has a relatively large factor share. Unfortunately, the existing literature on capital-land substitution is more limited than the evidence regarding energy. In the urban economics literature, it is common to estimate the EoS between land and other inputs from cross-sectional data on the production of housing services, and the estimated elasticities are often close to one (e.g., Epple et al., 2010; Ahlfeldt and McMillen, 2014; Ahlfeldt et al., 2015). While this result is suggestive, the EoS estimated in the urban literature does not necessarily coincide with the macro EoS, because the urban literature only examines the elasticity in housing, while the macro elasticity is affected by the elasticity in all sectors and the reallocation between sectors induced by changes in relative factor prices (Oberfield and Raval, 2021). Markandya and Pedroso-Galinato (2007) use aggregate data to examine many different nesting structures that include land. In several specifications, they find an EoS between capital and land that is close to one.[25] Our interpretation of the evidence on the capital-land EoS is similar to our interpretation for the capital-energy EoS. It seems plausible that the capital-land EoS is close to one, but further research is needed. We hope that our theoretical results will spur this empirical work.

# 7 Quantitative Exercise

## 7.1 Calibration

In this section, we show how the generalized Uzawa theorem can be used to calibrate the three-factor neoclassical growth model from Section 5.3, assuming perfect competition. We will obtain a factor-augmenting representation that is simultaneously consistent with the data on balanced growth, capital-augmenting technical change, and the elasticity of substitution between labor and capital. Our calibration uses annual U.S. data from 1960-2020. Appendix D.2 contains details on data sources. Table 2 lists the calibration targets and the parameters that are taken directly from the data.

This calibration exercise assumes that the third factor $X$ is a Cobb-Douglas combination of land and fossil fuel energy, which we refer to as 'natural resources.' We also assume that the EoS between $K$ and $X$ is one. As highlighted in Definition 1, the common structure of the neoclassical growth model only specifies the accumulation process for capital (2). The determinants of the supply of other factors, $L_t$ and $X_t$, depend on the application. We calibrate the model without relying on specifications for the supply of other factors, demonstrating that our approach could be used in a wide range of settings.[26]

The generalized Uzawa theorem (Proposition 4) implies that, along a non-degenerate BGP, technological change can always be represented as in equation (20). The functional form of $\widehat{F}(\cdot)$ in this equation is given by (11), which depends on the shape of

[25]Some of their estimates for the capital-energy EoS are close to one, while others are lower.

[26]For example, energy could be available in finite supply and costlessly extracted from the environment as in Hotelling (1931) and Hassler et al. (2021), or it could be extracted from the environment using the final good as in Casey (forthcoming). In the latter case, extraction costs would be a component of $R_t$ in the Definition 1. It is common to model land as an endowment that is renewed every period, but it would also be reasonable to take into account the fact that some expenditure is necessary to keep land in a usable state.

Table 2: Calibration Targets and Parameters Taken from Data

| Target | Value | Description | Source |
|---|---|---|---|
| $s_L$ | 62.5% | Labor share | NIPA |
| $s_X$ | 9% | Natural resource share | NIPA |
| $K_0/Y_0$ | 2.9 | Capital-output ratio | NIPA |
| $\sigma_{KL}$ | 0.6 | EoS between L and K | Oberfield and Raval (2021) |
| $g_L - 1$ | 1.02% | Population growth | NIPA |
| $g_{\text{fossil}} - 1$ | 0.85% | Fossil fuel energy growth | EIA |
| $\gamma_L - 1$ | 1.93% | Labor-augmenting tech. change | NIPA |
| $\gamma_K - 1$ | 0.80% | Capital-augmenting tech. change | NIPA |
| $Y_0, L_0, X_0$ | 1 | Initial variables | Normalization |

the true aggregate production function on the BGP at some point in time, $F(\cdot;0)$. Even when we do not know the exact shape of $F(\cdot;0)$, Remarks 1-3 indicate that we can use a CES production function instead of $F(\cdot;0)$, as long as it matches the levels, first derivatives (or shares), and the EoS. Therefore, in this exercise, we consider a CES function for $\widehat{F}(\cdot,\cdot)$ and calibrate its parameters to match the U.S. long-run data. Equation (20) now can be written as

$$\begin{aligned} Y_t &= \left\{\eta_{KX}\left((A_{K,t}K_t)^{\alpha}(A_{X,t}X_t)^{1-\alpha}\right)^{\frac{\epsilon-1}{\epsilon}} + \eta_L (A_{L,t}L_t)^{\frac{\epsilon-1}{\epsilon}}\right\}^{\frac{\epsilon}{\epsilon-1}}, \\ A_{K,t} &= \gamma_K^t,\ A_{X,t} = \gamma_X^t,\ A_{L,t} = \gamma_L^t, \end{aligned} \tag{22}$$

where $\alpha \in (0,1)$ is the share of physical capital in the capital composite, $\epsilon > 0$ is the elasticity of substitution between labor and the capital composite, and $\eta_{KX}$ and $\eta_L$ are distribution parameters. Here, we assume $\epsilon \neq 1$, since $\epsilon = 1$ would imply that the entire production function is Cobb-Douglas, which is inconsistent with the observed long-term elasticity of substitution between capital and labor.

Our data indicate that $s_L = 62.5\%$ was the average labor share of income over this period. For the share of natural resources, $s_X$, we use the results from Table 1, which reports that the shares of land and energy are 4% and 5%, respectively. Thus, $s_X = 4\% + 5\% = 9\%$. This pins down the share of capital, $s_K = 1 - s_L - s_X = 28.5\%$, and the relative weights inside the capital composite, $\alpha = s_K/(s_K + s_X) = 0.76$.

Lemma 1 implies the BGP growth rate of capital, $g_K$, always coincides with the growth factor of output, $g$. The definition of balanced growth (Definition 2) implies that the growth factors of the other inputs, $g_L$ and $g_X$, as well as the growth factors technologies, $\gamma_L, \gamma_K$, and $\gamma_X$, are constant. In this environment, the capital composite, $(A_{K,t}K_t)^\alpha (A_{X,t}X_t)^{1-\alpha}$, grows at a factor of $(\gamma_K g)^\alpha(\gamma_X g_X)^{1-\alpha}$, and effective labor, $A_{L,t}L_t$, grows at factor of $\gamma_L g_L$. The generalized Uzawa theorem (Proposition 4 with $j^* = 1$) requires that these two growth factors coincide with $g$ on the BGP:

$$g = (\gamma_K g)^\alpha(\gamma_X g_X)^{1-\alpha} = \gamma_L g_L, \tag{23}$$

which is equivalent to equation (21) in Section 5.3. From the U.S. data, we take the annual growth rate of real GDP per capita to be $(\gamma_L - 1) = 1.93\%$ and the population growth rate to be $(g_L - 1) = 1.02\%$. This gives a GDP growth rate of $(g - 1) = \gamma_L g_L - 1 = 2.95\%$. Average growth rates are measured by fitting an exponential trend through the data using ordinary least squares.

The growth rate of natural resources, $g_X$, is a geometric mean of the growth factors of land and fossil energy, where the weights are the expenditure shares. We assume that land is constant and take the growth rate of fossil energy to be $(g_{\text{fossil}} - 1) = 0.85\%$ per year, based on data from the Energy Information Administration (2023). Thus, $(g_X - 1) = 1^{\frac{5}{9}}(g_{\text{fossil}})^{\frac{4}{9}} - 1 = 0.38\%$.

With this information, we can use equation (23) to pick the growth rates of capital- and natural resource-augmenting technological change, $\{\gamma_K, \gamma_X\}$. It is customary to choose $\gamma_K = 1$ (constant $A_{K,t}$) — because this is the only choice consistent with balanced growth in two-factor neoclassical growth models without Cobb-Douglas production (see Section 5.1) — and to exclude the third variable $X$. In our setting, we can choose $\gamma_K$ to be consistent with data. In particular, the growth rate of capital-augmenting technical change can be chosen to match the fall in the relative price of capital, yielding $(\gamma_K - 1) = 0.80\%$. Then, from the technology condition (23), we obtain $(\gamma_X - 1) = 0.04\%$.[27]

Next, we show how to calibrate $\epsilon$, assuming that a researcher has a target value

[27] As explained in Appendix C, the fall in the relative price of investment can be used to measure the growth rate of capital-augmenting technological change, because the neoclassical growth model (Definition 1) is built on the assumption that final goods in period $t$ can be converted to $K_{t+1}$ linearly. This property does not hold for other production factors, implying that we can not use price data to calibrate $\gamma_X$ directly. For example, a standard Hotelling (1931) model would imply that changes in the price of energy reflect changes in scarcity rents, rather than energy-augmenting technical change. Therefore, we use condition (23) to calibrate $\gamma_X$.

for the elasticity of substitution between capital and labor. In Appendix B.1, we calculate the elasticity of substitution between $K$ and $L$ in function (22) according to Definition 3, which yields

$$\sigma_{KL} = \frac{s_K/s_L + 1}{(s_K/s_L + \alpha)\epsilon^{-1} + (1-\alpha)} \equiv \Sigma(\epsilon, \alpha, s_K/s_L). \tag{24}$$

Equation (24) shows that $\sigma_{KL} = \Sigma(\epsilon, \alpha, s_K/s_L)$ is an increasing function $\epsilon$, with $\Sigma(0, \alpha, s_K/s_L) = 0$, $\Sigma(1, \alpha, s_K/s_L) = 1$, and $\lim_{\epsilon\to\infty} \Sigma(\epsilon, \alpha, s_K/s_L) = (s_K/s_L+1)/(1-\alpha)$. This means the researcher can pick $\epsilon > 0$ to match any target value of $\sigma_{KL}$ between 0 and $(s_K/s_L + 1)/(1-\alpha)$. Specifically,

$$\epsilon = \frac{s_K/s_L + \alpha}{(s_K/s_L + 1)\sigma_{KL}^{-1} - (1-\alpha)} \quad \text{if} \quad 0 \le \sigma_{KL} < \frac{s_K/s_L + 1}{1-\alpha}. \tag{25}$$

We use $\sigma_{KL} = 0.6$ (see, e.g., Chirinko, 2008; Oberfield and Raval, 2021), which gives[28] $\epsilon = 0.56$.

Lastly, we discuss how to calibrate distribution parameters. We consider some period $t = 0$ in which the economy is on a BGP. Appendix B.2 shows that the factor shares and the distribution parameters on the BGP are related by

$$\eta_{KX} = \frac{s_K}{\alpha}\left(\frac{K_0^{\alpha} X_0^{1-\alpha}}{Y_0}\right)^{\frac{1-\epsilon}{\epsilon}}, \quad \eta_L = s_L\left(\frac{L_0}{Y_0}\right)^{\frac{1-\epsilon}{\epsilon}}. \tag{26}$$

The values of $\eta_{KX}$ and $\eta_L$ in the above equations reflect the choice of units. To set units for labor and natural resources, we normalize $L_0 = X_0 = 1$ without loss of generality. Since the process of capital accumulation is given by (2), capital and output must be measured in the same units.[29] In the data, the capital-to-output ratio is $K_0/Y_0 = 2.9$. So, we normalize $Y_0 = 1$ and $K_0 = 2.9$. This gives $\eta_{KX} = 0.71$ and $\eta_L = 0.63$.

[28] While we think that this is a reasonable calibration of $\epsilon$, it should be noted that most estimates of $\sigma_{KL}$ assume a two-factor production function. Our results suggest that there is considerable value in re-estimating substitution elasticities in multi-factor settings that include natural resources.

[29] Strictly speaking, the amount of capital in a given period, $K_t$, should be measured in the unit of the previous period's output, $Y_{t-1}$. However, there is no simple way to do this. The standard convention in the calibration of the neoclassical growth model is to measure simply the nominal value of the capital stock and output in the same year to attain the capital-output ratio. We follow this convention. This compromise only affects the CES distribution parameters, $\eta_{KX}$ and $\eta_L$.

Table 3: Calibrated Parameters

| Parameter | Value | Description |
|---|---|---|
| $\epsilon$ | 0.56 | EoS b/w capital composite and L |
| $\alpha$ | 0.76 | Capital share in capital composite |
| $\eta_L$ | 0.63 | CES distribution parameter |
| $\eta_{KX}$ | 0.71 | CES distribution parameter |
| $g_X - 1$ | 0.38% | Natural resource ($X$) growth |
| $\gamma_X - 1$ | 0.04% | $X$-augmenting tech. change |

This completes the simple calibration exercise. The calibrated parameter values are shown in Table 3. We explained how a researcher could obtain a representation of the evolution of production function in the form of (22), where all the parameters, $\{\alpha, \epsilon, \eta_{KX}, \eta_L, \gamma_K, \gamma_X, \gamma_L\}$ are specified. This representation can be used as an approximation of the true production function around the BGP. Unlike standard two-factor production functions, it simultaneously matches the declining price of capital on the BGP ($\gamma_K > 1$) and the non-unitary elasticity of substitution between labor and capital ($\sigma_{KL} \neq 1$).

## 7.2 Explaining movements in the labor share

Here, we present a simple application of the calibrated model from the previous section. We use it to study how fluctuations in capital-augmenting technical change (as reflected in the relative price of investment) have influenced the labor share over the last sixty years. Our model provides a new perspective on this question. In a standard CES production function, the labor share is constant only when $A_{K,t}$ is constant, and a long-term increase in $A_{K,t}$ would imply that the labor share will continue to rise (if $\sigma_{KL} < 1$) or fall (if $\sigma_{KL} > 1$). In a standard Cobb-Douglas production function, the labor share is always constant. In our model, by contrast, the labor share is constant when $A_{K,t}$ grows along the BGP at a rate determined by the log-linear technology condition (21). Any deviation of capital-augmenting technology from its BGP path will generate fluctuations in the labor share.

In the following, let $A_{K,t}$ represent the BGP value of capital-augmenting technical change at time $t$, and $a_{K,t}$ denote its actual value, whether or not it is on the BGP. To focus on the impacts of a deviation of $a_{K,t}$ from $A_{K,t}$, we assume that all other technologies and production factors follow their respective balanced growth paths. In Appendix B.3, we show that the labor share in period $t$ implied by our calibrated model is

$$s_{L,t} = \left( \left( \frac{1}{s_L} - 1 \right) \left( \frac{a_{K,t}}{A_{K,t}} \right)^{\alpha(\epsilon-1)/\epsilon} + 1 \right)^{-1}, \tag{27}$$

where $s_L$ (without a timescript) is the labor share on the BGP.

It is straightforward to take this expression to the data. In our calibration, $\alpha = s_K/(1-s_L)$ and $\epsilon$ in (25) depend only on factors shares and the estimate of $\sigma_{KL}$. Therefore, the calculation of (27) does not require much information other than the deviation of capital-augmenting technology from its BGP path, $a_{K,t}/A_{K,t}$. The relative price of investment can be used to measure this deviation. In Appendix C.1, we derive

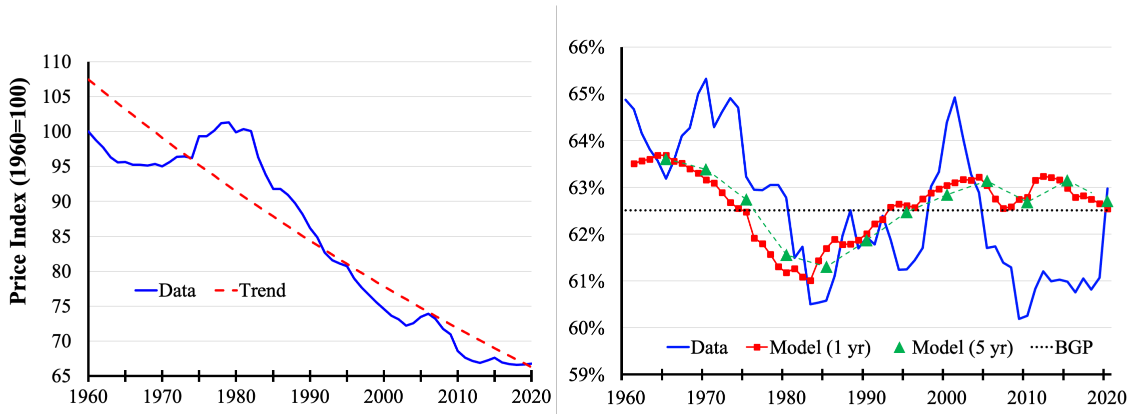
$$\frac{a_{K,t}}{A_{K,t}} = \left( \frac{rpi_{t-1}}{RPI_{t-1}} \right)^{-1}, \tag{28}$$

where $rpi_{t-1}$ is the relative price of investment and consumption in the NIPA statistics at time $t-1$, and $RPI_{t-1}$ is the BGP value. The lag between the measurement of capital-augmenting technical change and the relative price of investment reflects the fact that the efficiency and quantity of period $t$ capital depends on investment in period $t-1$. These results imply that the period length matters for our results. We calculate results using period lengths of one year and five years.[30]

The results are shown in Figure 2. Panel (a) shows the relative price of investment from the NIPA data and its exponential trend estimated via ordinary least squares, which we take to be the BGP level. From (28), comparing the data and trend yields a measure of how much capital-augmenting technical change has deviated from its BGP. As noted in the previous section, the relative price of investment fell at an average rate of 0.80% over this period. The actual data lie below the trend prior to 1975 and spike well above the trend in the early 1980s. Toward the end of the period, the data lie close to the long-run trend.

Panel (b) shows the movements in the labor share in the BEA data and those

[30]For the calculation with five-year periods, we first compute the deviation of the actual relative price of investment from its BGP value for each year. Then, we take averages over five-year periods, starting with 1960-1964. For each of these five-year periods, we compute (28) using data from the previous period. Finally, we plug back into (27). The markers in the figure show the first year of each five-year period.




(a) Investment Price

(b) Labor Share

Figure 2: The Relative Price of Investment and the Labor Share.

Note: See Appendix D.3 for data sources.

implied by the model. When the relative price of investment is above its BGP level, the ratio of the effective capital composite to the effective labor is below its BGP level. Note that $\epsilon < 1$ means that the labor share declines when labor becomes abundant relative to the effective capital composite. Therefore, an upward deviation of the relative price of investment from its BGP level implies a lower labor share in our model. The opposite occurs when the relative price of investment is below its BGP level. Thus, consistent with the data, the model correctly predicts that the labor share was above its long-run level early in the 1960s and 1970s and fell below its long-run level in the 1980s. Towards the end of the period, however, the data and model diverge. There was a fall in the labor share starting in the early 2000s even though the relative price of investment has been near its long-run trend.

Our results imply that changes in the relative price of investment can explain swings in the labor share prior to 2000, but not the precipitous drop in this century. These findings suggest that other potential explanations for the changing labor share, like market power or trade, are more likely explanations (Grossman and Oberfield, 2022).

# 8 Investment-Specific Technological Change

Throughout the paper, we have considered standard neoclassical models where one unit of final output in period $t$ can be transformed into one unit of capital at time $t+1$, as specified by (2). The following lemma shows that our theory can accommodate models where $q_t > 0$ units capital can be produced from a unit of final output, where $q_t$ is investment-specific technology (IST). The growth of $q_t$ over time is called investment-specific technical change (ISTC).

**Lemma 3. (Investment-Specific Technological Change)** *Consider an economic environment where output $Y_t$ and capital $K_t^{\text{IST}}$ are determined by*

$$Y_t = F^{\text{IST}}(K_t^{\text{IST}}, X_{1,t}, ..., X_{J,t}; t), \tag{29}$$

$$K_{t+1}^{\text{IST}} = (Y_t - C_t - R_t)q_t + (1 - \delta^{\text{IST}})K_t^{\text{IST}}, \tag{30}$$

*where $F^{\text{IST}}(\cdot)$ has CRS and positive and diminishing returns to all inputs. If the growth factor of IST, $g_q = q_t/q_{t-1}$, is constant,*[31] *this environment can be transformed into (1) and (2) in Definition 1 through a change of variables of $K_t \equiv K_t^{\text{IST}}/q_{t-1}$ and $\delta = (\delta^{\text{IST}} + g_q - 1)/g_q$, as well as a redefinition of the production function,*

$$F(K_t, X_{1,t}, ..., X_{J,t}; t) = F^{\text{IST}}(q_{t-1}K_t, X_{1,t}, ..., X_{J,t}; t). \tag{31}$$

*Proof.* See Appendix A.9. □

Note that the linearity of the accumulation function equation is still preserved in (30), which is a key assumption of the neoclassical growth model in Definition 1. In Lemma 3, the change of variables effectively normalizes the unit of capital so that capital $K_t \equiv K_t^{\text{IST}}/q_{t-1}$ is always measured in terms of the previous period's final good, as in (2). The depreciation rate after this normalization, $\delta$, should be higher than $\delta^{\text{IST}}$, because positive investment-specific technological change decreases the value of older capital.

All the results in this paper can be applied to models with ISTC by re-interpreting the variables and the production function using Lemma 3. When the transformed

[31] We assume that $g_q$ is constant for simplicity. This condition is not necessary if we extend Definition 1 to allow the depreciation rate to change over time. As long as we focus on the BGP, the results will not be affected.

production function (31) is used in Definition 1, it already includes a type of capital-augmenting technological change, because $q_{t-1}$, which multiplies $K_t$ on the RHS, grows over time. This component of capital-augmenting technological change is often called *embodied technological change*. Even when the model has ISTC, the aggregate production function (29) may exhibit further capital-augmenting technological change that is not included in the growth of $q_t$. Such technological change is called *disembodied technological change*. When we convert the variables and the production function using Lemma 3, the sum of embodied and disembodied technological change will show up as overall capital-augmenting technological change in production function (1). Therefore, the original two-factor Uzawa theorem in the presence of ISTC says that the sum of these technological changes must be zero on the BGP, unless the production function is Cobb-Douglas.

We also note how the sum of these technological changes can be obtained from data. Appendix C.2 shows that the relative price of investment in the NIPA statistics measures the sum of the rates of embodied and disembodied technological changes when capital accumulation is given by (30).[32] Given that the ratio is falling, the original two-factor Uzawa theorem requires any model with positive ISTC to adopt a Cobb-Douglas aggregate production function (e.g., Greenwood et al., 1997, 2000). This paper presents a way to relax this limitation.

# 9 Conclusion

The relative price of investment has been falling in the U.S. for long periods of time, indicating the existence of capital-augmenting technological change on the balanced growth path. Due to the Uzawa steady-state theorem, however, this fact could not be incorporated into macroeconomic models that use empirically relevant values for the elasticity of substitution between capital and labor. This paper presents a generalized Uzawa theorem, which demonstrates how this limitation can be overcome by taking the realistic step of adding additional inputs, such as land and energy, into the

[32]Since the NIPA data only gives $\gamma_K \cdot g_q$, we need another source of information either on the embodied technological change, $g_q$, or disembodied technological change, $\gamma_K$, to calibrate the model with ISTC. Accordingly, we would have obtained a different value for $\gamma_K$ in the calibration of Subsection 7.1 if we used accumulation equation (30), rather than (2). This difference demonstrates that the result of the calibration depends on the specification of the accumulation process.

production function.

On a BGP, all effective factors of production grow at the same rate as output, unless there is a unitary elasticity between two or more factors. As noted by Jones and Scrimgeour (2008), the linear accumulation process implies that "capital inherits the trend in output." On a BGP where all effective factors grow proportionally to output, there should be no capital-augmenting technological change, because capital is already growing at the same rate as output. The generalized Uzawa growth theorem considers a hybrid of capital and other non-accumulable inputs, which we call the capital composite. There is a unit EoS between the components of the capital composite. If the capital composite would grow slower than output in the absence of technological change, capital composite-augmenting technical change is necessary for a BGP. Thus, capital-augmenting technical change, which is one component of composite-augmenting technical change, is compatible with balanced growth and a non-unitary EoS between capital and labor.

*What if there is no $X_j$ with $\sigma_{KX_j} = 1$?*

For a production factor to be a part of the capital composite, the long-run EoS between capital and this factor must be unity, since otherwise factor shares and the growth rate cannot be kept constant simultaneously. We discussed energy and land as candidates in Section 6.2, and there are a wide range of other factors used in production. However, it is still an open question whether there is a production factor that has unit-elastic substitution with capital. If future research determines there is no such factor, the generalized Uzawa Growth theorem would then imply that the speed of capital-augmenting technological change must be exactly zero on any BGP and the puzzle would persist.

In this case, it would be necessary to question the assumptions of the theorem. A remarkable property of the generalized Uzawa theorem is that it depends on very few assumptions: (i) the economy can be expressed by a neoclassical growth model, and (ii) there is a BGP. The NIPA data strongly suggests the existence of the BGP. Therefore, we can narrow down the concern to the assumptions in the neoclassical growth model, as given in Definition 1. The definition consists of two parts, the aggregate production function (1) and the capital accumulation equation (2). In the

Uzawa theorem, the latter is only utilized in Lemma 1, which shows that $K/Y$ must be constant in the BGP. The result of this lemma is clearly visible in the NIPA data depicted in Figure 1(a). Therefore, this accumulation equation seems to do no additional harm.

The remaining suspect is the aggregate production function: $Y_t = F(K_t, X_{1,t}, ..., X_{J,t}; t)$. It assumes that there is a mapping from aggregate inputs to the aggregate output. This is not a weak assumption. While the NIPA statistics show that the price of investment is declining relative to consumption, it does not mean that various kinds of equipment are becoming cheaper proportionally. If newer capital goods have more margins for cost reductions and quality improvements, their quality-adjusted prices will fall faster. Moreover, new kinds of capital goods are continually introduced through R&D, whereas old goods disappear from the market. This cycle also contributes to the fall in the quality-adjusted price of investment in the NIPA statistics. However, in the framework of the neoclassical growth model, we need to map the statistics to the model assuming that all kinds of capital goods can be aggregated into one variable. The same can be said for the left-hand side of the production function, i.e., aggregated output $Y_t$. When the composition of inputs and outputs evolves over time, it is not obvious whether we can define a functional relation over the aggregate variables. If there is no $X_j$ with $\sigma_{KX_j} = 1$, it might suggest that aggregation created the problem. Exploring disaggregated models seems important in this respect.[33]

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[33] In a separate study, Horii (2023) develops a prototype model of endogenous growth in this spirit.

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# Online Appendix

"A Generalized Uzawa Growth Theorem "

by Gregory Casey and Ryo Horii

October 2023

# A Proofs of Propositions and Lemmas

## A.1 Notation for derivatives

Unless mentioned otherwise, $F_K(\cdot;t)$ denotes the partial derivative of function $F(\cdot;t)$ with respect to its first argument, whereas $F_{X_j}(\cdot;t)$ denotes the partial derivative of $F(\cdot;t)$ with respect to its $(1+j)$th argument. The same applies to other functions, such as $\widetilde{F}(\cdot)$.

Following the convention in economics, $\frac{\partial}{\partial K_t}$ and $\frac{\partial}{\partial X_{j,t}}$ represent the partial derivatives with respect to variables $K_t$ and $X_{j,t}$, respectively. For example, if $\widetilde{F}(\cdot)$ is the production function, $\frac{\partial}{\partial X_{j,t}}\widetilde{F}(\cdot)$ gives the marginal product of factor $X_{j,t}$.

Note that these two definitions are different when the argument of function is not a single variable. For example, using the chain rule, we have

$$\frac{\partial}{\partial X_{j,t}}\widetilde{F}(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t}) = \widetilde{A}_{X_j,t}\widetilde{F}_{X_j}(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t}). \quad \text{(A.1)}$$

## A.2 Proof of Lemma 1

Using the notation in Definition 2, equation (2) can be written as $K_0 g_K^{t+1} = Y_0 g^t - C_0 g_C^t - R_0 g_R^t + (1-\delta)K_0 g_K^t$. Dividing all terms by $g^t$ and rearranging them gives

$$Y_0 = C_0(g_C/g)^t + R_0(g_R/g)^t + K_0(g_K + \delta - 1)(g_K/g)^t. \quad \text{(A.2)}$$

Because all three terms on the right hand side (RHS) of (A.2) are non-negative exponential functions of $t$, every one of them needs to be constant for the sum of all the terms to become constant ($Y_0$). For the first term $C_0(g_C/g)^t$ to be constant, $g_C = g$ must hold since $C_0 > 0$ from Definition 1. This means $C_t/Y_t = C_0/Y_0 > 0$. For the third term $(g_K+\delta-1)(g_K/g)^t$ to be constant, $g_K = g$ must hold since $K_0 > 0$

and $g_K > 1-\delta$. This implies $K_t/Y_t = K_0/Y_0 > 0$. If $R_0 > 0$, $g_R = R$ must hold since otherwise the second term cannot be constant.

## A.3 Proof of Proposition 2

Because the production function in period 0 is $F(\cdot;0) \equiv \widetilde{F}(\cdot)$, we can write the share of factor $Z$ in period 0 as

$$s_{Z,0} = \widetilde{F}_Z(K_0, X_{1,0}, ..., X_{J,0})\frac{Z_0}{Y_0}, \tag{A.3}$$

where $\widetilde{F}_Z(\cdot)$ represents the derivative of function $\widetilde{F}(\cdot)$ with respect to its argument (see Appendix Section A.1). Note that, since function $\widetilde{F}(\cdot)$ has constant returns to scale, its partial derivative function $\widetilde{F}_Z(\cdot)$ must be homogeneous of degree 0 (See Theorem M.B.1 in Mas-Colell et al., 1995). Therefore, the value of $\widetilde{F}_Z(\cdot)$ will be unchanged when all of its arguments are multiplied by the same factor $g^t = Y_t/Y_0 = K_t/K_0 = \widetilde{A}_{X_j,t}X_{j,t}/X_{j,0}$. (Here we used $g_K = g$ from Lemma 1.) Applying this for (A.3) gives

$$s_{Z,0} = \widetilde{F}_Z(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t})\frac{Z_0}{Y_0}.$$

In addition, because the effective amount of production factors and the output grow at the same speed, $Z_0/Y_0 = \widetilde{A}_{Z,t}Z_t/Y_t$ holds on the BGP. (In the case of $Z_t = K_t$, we define $\widetilde{A}_{K,t} \equiv 1$.) Therefore,

$$s_{Z,0} = \widetilde{F}_Z(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t})\widetilde{A}_{Z,t}\frac{Z_t}{Y_t} = \frac{\partial \widetilde{F}(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t})}{\partial Z_t}\frac{Z_t}{Y_t}, \tag{A.4}$$

where the validity of the second equality is guaranteed by the chain rule.[34] Recall that we assumed that the share is constant over time, which means

$$s_{Z,0} = s_{Z,t} = F_Z(K_t, X_{1,t}, ..., X_{J,t}; t)\frac{Z_t}{Y_t}. \tag{A.5}$$

By comparing (A.4) and (A.5), we obtain (4).

[34] See Appendix A.1.

## A.4 Proof of Proposition 3

As in Definition 3, the EoS between $K_t$ and $X_j$, $j \in \{1, ..., J\}$, in the Uzawa Representation $\widetilde{F}(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t})$ is defined as

$$\widetilde{\sigma}_{KX_j,t} = - \left. \frac{d \ln(K_t/X_{j,t})}{d \ln \left( \frac{\widetilde{F}_K(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t})}{\widetilde{A}_{X_j,t}\widetilde{F}_{X_j}(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t})} \right)} \right|_{Y_t, \mathbf{X}_{-j,t}: \text{const}} . \tag{A.6}$$

We used (A.1) for calculating the marginal product of $X_j$ in the denominator. Note that, in addition to output $Y_t$ and other production factors $\mathbf{X}_{-j,t}$, we keep technologies $\widetilde{A}_{X_1,t}, ..., \widetilde{A}_{X_J,t}$ fixed when calculating the EoS.

In this proof, we evaluate the value of (A.6) on the BGP. This means $Y_t$ and $\mathbf{X}_{-j,t}$ are their BGP values, but we still need to consider (infinitesimally) small perturbations of $K_t$ and $X_{j,t}$ from these BGP values. To make this distinction, let $Y_t, K_t, X_{1,t}, ..., X_{J,t}$ denote the specific BGP values, and $k$ and $x_j$ the variables to be perturbed. Then, (A.6) can be written as[35]

$$\widetilde{\sigma}_{KX_j,t} = - \left. \frac{d \ln(k/x_j)}{d \ln \left( \frac{\widetilde{F}_K(k, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_j,t}x_j, ..., \widetilde{A}_{X_J,t}X_{J,t})}{\widetilde{A}_{X_j,t}\widetilde{F}_{X_j}(k, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_j,t}x_j, ..., \widetilde{A}_{X_J,t}X_{J,t})} \right)} \right|_{\substack{\widetilde{F}(k, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_j,t}x_j, ..., \widetilde{A}_{X_J,t}X_{J,t}) = Y_t \\ k = K_t, x_j = X_{j,t}}} . \tag{A.7}$$

Condition $\widetilde{F}(k, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_j,t}x_j, ..., \widetilde{A}_{X_J,t}X_{J,t}) = Y_t$ says that $k$ and $x_j$ need to move to ensure that this equality is satisfied. The other conditions $k = K_t, x_j = X_{j,t}$ say that, after the differentiation is complete, the EoS is evaluated at the BGP values.

Now, consider a change of variables: $k' = g^{-t}k$ and $x'_j = g^{-t}\widetilde{A}_{X_j,t}x_j$. Then, $k$ in (A.7) is replaced by $k = g^t k'$ and $x_j$ is by $(g^t/\widetilde{A}_{X_j,t})x'_j$. Specifically, $k/x_j$ in the numerator becomes $\widetilde{A}_{X_j,t}k'/x'_j$. In the denominator,

$$\begin{aligned} \widetilde{F}_K(k, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_j,t}x_j, ..., \widetilde{A}_{X_J,t}X_{J,t}) &= \widetilde{F}_K(g^t k', g^t X_{1,0}, ..., g^t x'_j, ..., g^t X_{J,0}) \\ &= \widetilde{F}_K(k', X_{1,0}, ..., x'_j, ..., X_{J,0}), \end{aligned}$$

[35]We omit condition "$\mathbf{X}_{-j,t}$: const" because $X_{j,t}$'s BGP values, not variables.

where we used the definition of $\widetilde{A}_{X_j,t} \equiv g^t X_{j,0}/X_{j,t}$ and the homogeneity of degree 0 property of the $\widetilde{F}_K(\cdot)$ function.[36] Similarly,

$$\widetilde{F}_{X_j}(k, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_j,t}x_j, ..., \widetilde{A}_{X_J,t}X_{J,t}) = \widetilde{F}_{X_j}(k', X_{1,0}, ..., x'_j, ..., X_{J,0}).$$

Note that, using the CRS property of $\widetilde{F}(\cdot)$, condition $\widetilde{F}(k, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_j,t}x_j, ..., \widetilde{A}_{X_J,t}X_{J,t}) = Y_t$ can be simplified as

$$\widetilde{F}(g^t k', g^t X_{1,0}, ..., g^t x'_j, ..., g^t X_{J,0}) = g^t \widetilde{F}(k', X_{1,0}, ..., x'_j, ..., X_{J,0}) = Y_t.$$

Since $Y_t = g^t Y_0$, the condition reduces to $\widetilde{F}(k', X_{1,0}, ..., x'_j, ..., X_{J,0}) = Y_0$. The point of evaluation, $k = K_t$, becomes $g^t k' = K_t$, or $k' = g^{-t} K_t = K_0$. Similarly, $x_j = X_{j,t}$ becomes $x'_j = g^{-t}\widetilde{A}_{X_j,t}X_{j,t} = X_{j,0}$. Therefore, (A.7) can be expressed in terms of $k'$ and $x'_j$ as follows:

$$\widetilde{\sigma}_{KX_j,t} = - \left. \frac{d\ln(\widetilde{A}_{X_j,t}k'/x'_j)}{d\ln\left(\frac{\widetilde{F}_K(k',X_{1,0},...,x'_j,...,X_{J,0})}{\widetilde{A}_{X_j,t}\widetilde{F}_{X_j}(k',X_{1,0},...,x'_j,...,X_{J,0})}\right)} \right|_{\substack{\widetilde{F}(k',X_{1,0},...,x'_j,...,X_{J,0})=Y_0 \\ k'=K_0, x'_j=X_{j,0}}} . \tag{A.8}$$

Recall that we keep technology $\widetilde{A}_{X_j,t}$ fixed when calculating the EoS. We can eliminate $\widetilde{A}_{X_j,t}$ from the numerator from $d\ln(\widetilde{A}_{X_j,t}k'/x'_j) = d(\ln(k'/x'_j) + \ln\widetilde{A}_{X_j,t}) = d\ln(k'/x'_j)$. In the same way, $\widetilde{A}_{X_j,t}$ in the denominator can also be eliminated (or replaced by $A_{X_j,0} \equiv 1$). Finally, using $\widetilde{F}(\cdot) \equiv F(\cdot;0)$, (A.8) can be written as

$$\widetilde{\sigma}_{KX_j,t} = - \left. \frac{d\ln(k'/x'_j)}{d\ln\left(\frac{F_K(k',X_{1,0},...,x'_j,...,X_{J,0};0)}{F_{X_j}(k',X_{1,0},...,x'_j,...,X_{J,0};0)}\right)} \right|_{\substack{F(k',X_{1,0},...,x'_j,...,X_{J,0};0)=Y_0 \\ k'=K_0, x'_j=X_{j,0}}} . \tag{A.9}$$

Then, comparing with Definition 3, it turns out that the RHS of (A.9) exactly matches the definition of $\sigma_{KX_j,0}$, evaluated at the period-0 BGP. Since it is assumed that $\sigma_{KX_j,t}$ does not change over time, we have $\widetilde{\sigma}_{KX_j,t} = \sigma_{KX_j,0} = \sigma_{KX_j,t}$.

[36] For the homogeneity of degree 0 property, see the proof of Proposition 2 in appendix A.3.

## A.5 Proof of Lemma 2

**Proof of part** $a$

By substituting $K_0, X_{1,0}, ..., X_{J,0}$ into (10) and then using (11),

$$\begin{aligned}\overline{F}(K_0, X_{1,0}, ..., X_{J,0}) &= \widehat{F}\left(K_0^{\alpha}\prod\nolimits_{j=1}^{j^*} X_{j,0}^{\xi_j}, X_{j^*+1,0}, ..., X_{J,0}\right)\\ &= F\left(\left(\prod\nolimits_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha}\left(K_0^{\alpha}\prod\nolimits_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{1/\alpha}, X_{1,0}, ..., X_{J,0}; 0\right)\\ &= F\left(K_0, X_{1,0}, ..., X_{J,0}; 0\right).\end{aligned}$$

**Proof of part** $b$

Let $M_0 = K_0^{\alpha}\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}$ denote the amount of capital composite $m$ in period 0, and $\widehat{F}_M(\cdot)$ denote the derivative of function $\widehat{F}(\cdot)$ with respect to its first argument. By differentiating both sides of (11) by $m$ with the chain rule and substituting the period-0 BGP values for $k, x_{j^*+1}, ..., x_J$,

$$\begin{aligned}\widehat{F}_M\left(M_0, X_{j^*+1,0}, ..., X_{J,0}\right) &= F_K(K_0, X_{1,0}, ..., X_{J,0}; 0)\left(\prod\nolimits_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha}\frac{1}{\alpha}M_0^{(1-\alpha)/\alpha}\\ &= F_K(K_0, X_{1,0}, ..., X_{J,0}; 0)\frac{K_0}{\alpha M_0},\end{aligned} \tag{A.10}$$

where the last equality follows from the definition of $M_0 = K_0^{\alpha}\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}$. Now, consider the case of $Z = K$. By differentiating both sides of (10) by $k$ with the chain rule and substituting the period-0 BGP values,

$$\begin{aligned}\overline{F}_K(K_0, X_{1,0}, ..., X_{J,0}) &= \widehat{F}_M\left(M_0, X_{j^*+1,0}, ..., X_{J,0}\right)\alpha M_0/K_0\\ &= F_K(K_0, X_{1,0}, ..., X_{J,0}; 0),\end{aligned}$$

where the last equality is from (A.10). Similarly, for the case of $Z = X_j$, where $j \in \{1, ..., j^*\}$,

$$\begin{aligned}\overline{F}_{X_j}(K_0, X_{1,0}, ..., X_{J,0}) &= \widehat{F}_M\left(M_0, X_{j^*+1,0}, ..., X_{J,0}\right)\xi_j M_0/X_{j,0} \\ &= F_K(K_0, X_{1,0}, ..., X_{J,0}; 0)\frac{\xi_j}{\alpha}\frac{K_0}{X_{j,0}}.\end{aligned} \tag{A.11}$$

Note that, from the definitions of $\alpha$ and $\xi_j$ in (9), $\xi_j/\alpha = s_{X_j,0}/s_{K,0}$. Therefore, (A.11) becomes

$$F_K(K_0, X_{1,0}, ..., X_{J,0}; 0)\frac{F_{X_j}(K_0, X_{1,0}, ..., X_{J,0}; 0)X_{j,0}}{F_K(K_0, X_{1,0}, ..., X_{J,0}; 0)K_0}\frac{K_0}{X_{j,0}} = F_{X_j}(K_0, X_{1,0}, ..., X_{J,0}; 0).$$

Finally, consider the case of $Z = X_j$, where $j \in \{j^* + 1, ..., J\}$. Similarly to the proof of part $a$, we can confirm that $\overline{F}(K_0, X_{1,0}, ..., X_{j^*,0}, x_{j^*+1}, ...x_J) = F\left(K_0, X_{1,0}, ..., X_{j^*,0}, x_{j^*+1}, ..., x_J\right.$ for any $x_{j^*+1}, ..., x_J$. This means that they are identical functions of $x_{j^*+1}, ..., x_J$, and have the same derivatives with respect to these variables. Therefore, for $j \in \{j^* + 1, ..., J\}$, we have $\overline{F}_{X_j}(K_0, X_{1,0}, ..., X_{J,0}) = F_{X_j}(K_0, X_{1,0}, ..., X_{J,0}; 0)$.

**Proof of part $c$**

The EoS for function $\overline{F}(\cdot)$ between capital and factor $j$, evaluated at the period-0 BGP, is defined as

$$\overline{\sigma}_{KX_j,0} = -\left.\frac{d\ln(k/x_j)}{d\ln\left(\frac{\overline{F}_K(k,X_{1,0},...,x_j,...,X_{J,0})}{\overline{F}_{X_j}(k,X_{1,0},...,x_j,...,X_{J,0})}\right)}\right|_{\substack{\overline{F}(k,X_{1,0},...,x_j,...,X_{J,0})=Y_0 \\ k=K_0,x_j=X_{j,0}}}, \tag{A.12}$$

where $k$ and $x_j$ are variables to be perturbed and $Y_0, K_0, X_{1,0}, ..., X_{J,0}$ are the period-0 BGP values.

Let us first examine $\overline{\sigma}_{KX_j,0}$ for the case of $j \in \{1, ..., j^*\}$. In this case, factors $X_{j^*+1,0}, ..., X_{J,0}$ are fixed at the BGP values. Using (10), function $\overline{F}(k, X_{1,0}, ..., x_j, ..., X_{J,0})$ can be written as $\widehat{F}(m, X_{j^*+1,0}, ..., X_{J,0})$, where $m$ is the amount of capital composite, defined as $m = k^\alpha x_j^{\xi_j}\prod_{j'\in\{1,...,j^*\}\backslash j} X_{j',0}^{\xi_{j'}}$. Using the chain rule, its derivative with respect to $k$ becomes

$$\begin{aligned}\overline{F}_K(k, X_{1,0}, ..., x_j, ..., X_{J,0}) &= \frac{\partial}{\partial k}\widehat{F}(m, X_{j^*+1,0}, ..., X_{J,0}) \\ &= \widehat{F}_M(m, X_{j^*+1,0}, ..., X_{J,0})\frac{\partial m}{\partial k} \\ &= \widehat{F}_M(m, X_{j^*+1,0}, ..., X_{J,0})\alpha\frac{m}{k}.\end{aligned}$$

Similarly, $F_{X_j}(k, X_{1,0}, ..., x_j, ..., X_{J,0}) = \widehat{F}_M(m, X_{j^*+1,0}, ..., X_{J,0})\xi_j \frac{m}{x_j}$. Substituting these into (A.12) gives

$$\overline{\sigma}_{KX_j,0} = -\left.\frac{d\ln(k/x_j)}{d\ln\left(\frac{\alpha}{\xi_j}\frac{x_j}{k}\right)}\right|_{\substack{\overline{F}(k,X_{1,0},...,x_j,...,X_{J,0})=Y_0 \\ k=K_0,x_j=X_{j,0}}}. \tag{A.13}$$

Since $\alpha$ and $\xi_j$ are constant parameters, the denominator can be simplified as $d\ln((\alpha/\xi_j)(x_j/k)) = d(\ln(\alpha/\xi_j) + \ln(x_j/k)) = d\ln(x_j/k)$. Using this, (A.13) gives $\overline{\sigma}_{KX_j,0} = 1$. Recall that $\sigma_{KX_j,0} = 1$ because $j \in \{1, ..., j^*\}$. Therefore, $\overline{\sigma}_{KX_j,0} = \sigma_{KX_j,0}$ holds.

Next, we examine $\overline{\sigma}_{KX_j,0}$ for the case of $j \in \{j^* + 1, ..., J\}$. In this case, equations (10) and (11) imply

$$\overline{F}(k, X_{1,0}, ..., x_j, ..., X_{J,0}) = F(k, X_{1,0}, ..., x_j, ..., X_{J,0}; 0),$$

for any $k > 0$ and $x_j > 0$. Therefore, the EoS of function $\overline{F}(k, X_{1,0}, ..., x_j, ..., X_{J,0})$ between $k$ and $x_j$ is identical with that of function $F(k, X_{1,0}, ..., x_j, ..., X_{J,0}; 0)$. This means $\overline{\sigma}_{KX_j,0} = \sigma_{KX_j,0}$.

**Proof of part** $d$

Let us first consider the CRS property of function $\widehat{F}(m, x_{j^*+1}, ..., x_J)$. We multiply every argument by an arbitrary factor of $\lambda > 0$. From (11),

$$\begin{aligned}
&\widehat{F}(\lambda m, \lambda x_{j^*+1}, ..., \lambda x_J) \\
&= F\left(\lambda^{1/\alpha}\left(\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha}, X_{1,0}, ..., X_{j^*,0}, \lambda x_{j^*+1}, ..., \lambda x_J; 0\right) \\
&= \lambda F\left(\lambda^{(1-\alpha)/\alpha}\left(\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha}, \frac{X_{1,0}}{\lambda}, ..., \frac{X_{j^*,0}}{\lambda}, x_{j^*+1}, ..., x_J; 0\right),
\end{aligned} \tag{A.14}$$

where the last equality comes from the CRS property of the period-0 true production function $F(\cdot;0)$. (All the arguments are divided by $\lambda$.) Our objective it to show that the last line of (A.14) coincides with $\lambda\widehat{F}(m, x_{j^*+1}, ..., x_J)$. Using (11), this desired condition can be written as

$$\begin{aligned}
&F\left(\left(\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha}, X_{1,0}, ..., X_{j^*,0}, x_{j^*+1}, ..., x_J; 0\right) \\
&= F\left(\lambda^{(1-\alpha)/\alpha}\left(\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha}, \frac{X_{1,0}}{\lambda}, ..., \frac{X_{j^*,0}}{\lambda}, x_{j^*+1}, ..., x_J; 0\right).
\end{aligned} \tag{A.15}$$

In the following, we establish this equality by focusing on the isoquants of function $F(\cdot;0)$.

Recall that we defined $j^*$ such that the period-0 true production function $F(k, x_1, x_2, \ldots, x_J; 0)$ satisfies $\sigma_{KX_j} = 1$ for $j = 1, ..., j^*$. For concreteness, let us focus on capital $k$ and $x_1$. From Definition 3, $\sigma_{KX_1} = 1$ means that equation $d\ln(F_K/F_{X_1})/d\ln(k/x_1) = -1$ holds when the output and other inputs are kept constant.[37] In other words, this differential equation is satisfied on the isoquant curve in the $k$-$x_1$ space. Integrating equation $d\ln(F_K/F_{X_1})/d\ln(k/x_1) = -1$ gives $\ln(F_K/F_{X_1}) = -\ln(k/x_1) + \widetilde{\xi}_1$, where $\widetilde{\xi}_1$ is a constant of integration. Taking the exponential of the both sides gives

$$F_K/F_{X_1} = (\exp\widetilde{\xi}_1)(x_1/k). \tag{A.16}$$

[37] To minimize notation we omit the arguments of the functions $F_K(k, x_1, ..., x_J; 0)$ and $F_{X_1}(k, x_1, ..., x_J; 0)$.

From the definition of the isoquant curve, the amount of output must be constant: $dY = F_K dk + F_{X_1} dx_1 = 0$. Rearranging and using (A.16), we have the slope of the isoquant curve as $dx_1/dk = -F_K/F_{X_1} = -(\exp \widetilde{\xi}_1)(x_1/k)$. Integrating this differential equation by separation of variables gives $\ln k = -(1/\exp \widetilde{\xi}_1) \ln x_1 + \widetilde{y}_1$, where $\widetilde{y}_1$ is another constant of integration.[38] By taking the exponential,

$$k = (\exp \widetilde{y}_1) x_1^{-1/\exp \widetilde{\xi}_1}. \tag{A.17}$$

Equation (A.17) defines an isoquant curve with two parameters, $\widetilde{y}_1$ and $\widetilde{\xi}_1$. The value $\widetilde{\xi}_1$ can be pinned down by the factor share. Using (A.16), the relative share between $k$ and $x_1$ is written as $kF_K/x_1F_{X_1} = \exp \widetilde{\xi}_1$. The result does not depend on $k$ or $x_1$, which means that the relative share is constant on the isoquant curve. Also, notice that the value of $\widetilde{\xi}_1$ must be the same across all isoquant curves, since otherwise they intersect with each other, which is impossible by the definition of the isoquant curve. From (9), we know that the relative share in period 0 is $\alpha/\xi_1$. Using these, the isoquant curve (A.17) can be written as

$$k = (\exp \widetilde{y}_1) x_1^{-\xi_1/\alpha}. \tag{A.18}$$

The remaining parameter $\widetilde{y}_1$ specifies the location of the isoquant curve. Now, consider a particular isoquant curve that goes through $k = \left(\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha}$ and $x_1 = X_{1,0}$, which means $\exp \widetilde{y}_1 = \left(\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha} X_{1,0}^{\xi_1/\alpha}$. From (A.18), we can confirm that this isoquant curve also goes through $k' = \lambda^{\xi_1/\alpha} \left(\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha}$

[38] This integration can be done by separation of variables. Rearranging the equation $dx_1/dk = -(\exp \widetilde{\xi}_1)(x_1/k)$, we have $(1/k)dk = -(1/\exp \widetilde{\xi}_1)(1/x_1)dx_1$. Integrating both sides of this equation separately gives $\int (1/k)dk = -(1/\exp \widetilde{\xi}_1) \int (1/x_1)dx_1$. Since $\int (1/k)dk = \ln k + \mathit{constant}$ and $\int (1/x_1)dx = \ln x_1 + \mathit{constant}$, we obtain the result in the text.

and $x_1' = X_{1,0}/\lambda$.[39] Since the output is the same on an isoquant curve, we have

$$\begin{aligned} &F\left(\left(\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha}, X_{1,0}, ..., X_{j^*,0}, x_{j^*+1}, ..., x_J; 0\right) \\ &= F\left(\lambda^{\xi_1/\alpha}\left(\prod_{j=1}^{j^*} X_{j,0}^{\xi_j}\right)^{-1/\alpha} m^{1/\alpha}, \frac{X_{1,0}}{\lambda}, ..., X_{j^*,0}, x_{j^*+1}, ..., x_J; 0\right). \end{aligned} \tag{A.19}$$

By repeating this operation for $j = 2, ..., j^*$ and using $\sum_{j=1}^{j^*} \xi_j = 1 - \alpha$ from (9), we obtain (A.15). This establishes the CRS property of function $F(m, x_{j^*+1}, ..., x_J)$.

Next, we prove the CRS property of function $\overline{F}(k, x_1, ..., x_J)$. From (10),

$$\begin{aligned} \overline{F}(\lambda k, \lambda x_1, ..., \lambda x_J) &= \widehat{F}\left((\lambda k)^\alpha \prod_{j=1}^{j^*} (\lambda x_j)^{\xi_j}, \lambda x_{j^*+1,0}, ..., \lambda x_{J,0}\right) \\ &= \widehat{F}\left(\lambda k^\alpha \prod_{j=1}^{j^*} x_j^{\xi_j}, \lambda x_{j^*+1,0}, ..., \lambda x_{J,0}\right) \\ &= \lambda \widehat{F}\left(k^\alpha \prod_{j=1}^{j^*} x_j^{\xi_j}, x_{j^*+1,0}, ..., x_{J,0}\right) \\ &= \lambda \overline{F}(k, x_1, ..., x_J). \end{aligned}$$

The second equality utilizes $\alpha + \sum_{j=1}^{j*} \xi_j = 1$ from (9), whereas the third equality is from the CRS property of function $\widehat{F}(\cdot)$.

## A.6 Proof of Proposition 4

Using (10), the RHS of equation (13) can be written as

$$\widehat{F}\left((A_{K,t}K_t)^\alpha \prod_{j=1}^{j^*} (A_{X_j,t}X_{j,t})^{\xi_j}, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_J,t}X_{J,t}\right). \tag{A.20}$$

The first argument of function $\widehat{F}(\cdot)$ represent the effective amount of capital composite on the BGP. It is multiplied by $g$ each period from condition (12). Also, all the other arguments of $\widehat{F}(\cdot)$ are multiplied by $g$ each period because it is assumed that $\gamma_{X_j} = g/g_{X_j}$ for $j \in \{j^* + 1, ..., J\}$. Since $\widehat{F}(\cdot)$ has CRS from property $d$ of Lemma 2, (A.20) is multiplied by $g$ each period.

[39]This can be confirmed by substituting $k'$ and $x_1'$ into (A.18). It yields the same $\exp \widetilde{y}_1$ as in the previous sentence.

Also, the LHS of (13), $Y_t$, is multiplied by $g$ every period by the definition of the BGP. In period 0, (13) holds from property $a$ of Lemma 2. Therefore, (13) holds for all $t \geq 0$.

## A.7 Proof of Proposition 5

The proof relies on Lemma 2, but otherwise it proceeds similarly to the proof for Proposition 2. Let us first consider the case of $Z_t = K_t$. Using property $b$ of Lemma 2 and (10), the share of factor $K$ in period 0 can be written as (10),

$$\begin{aligned} s_{K,0} &= F_K(K_0, X_{1,0}, ..., X_{J,0}; 0)\frac{K_0}{Y_0} \\ &= \overline{F}_K(K_0, X_{1,0}, ..., X_{J,0})\frac{K_0}{Y_0} \\ &= \frac{\partial}{\partial K_0}\widehat{F}\left(K_0^\alpha \prod\nolimits_{j=1}^{j^*} X_{j,0}^{\xi_j}, X_{j^*+1,0}, ..., X_{J,0}\right)\frac{K_0}{Y_0}. \end{aligned} \tag{A.21}$$

Let $\widehat{F}_M(\cdot)$ be the derivative of function $\widehat{F}(\cdot)$ with respect to its first argument. Note that, in (A.21), the first argument is the capital composite in period 0, $M_0 = K_0^\alpha \prod_{j=1}^{j^*} X_{j,0}^{\xi_j}$. Using the chain rule, (A.21) becomes

$$s_{K,0} = \widehat{F}_M\left(M_0, X_{j^*+1,0}, ..., X_{J,0}\right)\frac{dM_0}{dK_0}\frac{K_0}{Y_0} = \widehat{F}_M\left(M_0, X_{j^*+1,0}, ..., X_{J,0}\right)\frac{\alpha M_0}{Y_0}, \tag{A.22}$$

where the second equality follows from $dM_0/dK_0 = \alpha M_0/K_0$.

Recall that $\widehat{F}(\cdot)$ has CRS from Lemma 2, and therefore its derivative $\widehat{F}_M(\cdot)$ is a homogeneous function of degree 0. Let $M_t = (A_{K,t}K_t)^\alpha \prod_{j=1}^{j^*}(A_{X_j,t}X_{j,t})$ denote the effective amount of capital composite in period $t$. From condition (12), $M_t$ grows by a factor of $g$ every period. The same applies to the effective amounts of factors not in the capital composite: $A_{X_J,t}X_{j,t}$ for $j = j^*+1, ..., J$. Therefore, when we consider function $\widehat{F}_M\left(M_t, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_J,t}X_{J,t}\right)$, every argument is multiplied by $g$ every period, which does not change the value of $F_M(\cdot)$ over time due to homogeneity of degree 0. Therefore, (A.22) can be written as

$$s_{K,0} = \widehat{F}_M\left(M_t, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_J,t}X_{J,t}\right)\frac{\alpha M_0}{Y_0}. \tag{A.23}$$

Note that, because $M_t$ and $Y_t$ grow at the same speed, the last term can be transformed as $\alpha M_0/Y_0 = \alpha M_t/Y_t = (\alpha M_t/K_t)(K_t/Y_t)$. In addition, $\alpha M_t/K_t$ in the latter expression represents $dM_t/dK_t$, which can be confirmed by differentiating $M_t = (A_{K,t}K_t)^\alpha \prod_{j=1}^{j^*}(A_{X_j,t}X_{j,t})$ by $K_t$. Therefore, (A.23) becomes

$$\begin{aligned} s_{K,0} &= \widehat{F}_M\left(M_t, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_J,t}X_{J,t}\right)\frac{dM_t}{dK_t}\frac{K_t}{Y_t} \\ &= \frac{\partial}{\partial K_t}\widehat{F}\left((A_{K,t}K_t)^\alpha \prod\nolimits_{j=1}^{j^*}(A_{X_j,t}X_{j,t}), A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_J,t}X_{J,t}\right)\frac{K_t}{Y_t} \\ &= \frac{\partial}{\partial K_t}\overline{F}_K\left(A_{K,t}K_t, A_{X_1,t}X_{1,t}, ..., A_{X_J,t}X_{J,t}\right)\frac{K_t}{Y_t}, \end{aligned} \tag{A.24}$$

where the second equality uses the chain rule, and the third is from the definition of function $\overline{F}(\cdot)$ in (10). Note that the share of capital is the same in period $t$ and 0, which implies

$$s_{K,0} = s_{K,t} = F_K(K_t, X_{1,t}, ..., X_{J,t}; t)\frac{K_t}{Y_t}. \tag{A.25}$$

By comparing (A.24) with (A.25), we obtain (15) for the case of $Z_t = K_t$. The proof of the proposition for the case of $Z_t = X_{j,t}$, $j \in \{1, ..., j^*\}$ proceeds exactly the same way as above, with only the modification that $K_t$ is replaced by $X_{j,t}$ and $\alpha$ by $\xi_j$.

Finally, the case of $Z_t = X_{j,t}$, $j \in \{j^* + 1, ..., J\}$, can be confirmed in a similar way as in Proposition 2, because the value of $A_{X_j,t}$ is the same as $\widetilde{A}_{X_j,t}$ in the Uzawa theorem. In particular, we use $\widehat{F}(M_t, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_J,t}X_{J,t})$ instead of $\widetilde{F}(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t})$, and define $\widehat{F}_{X_j}(\cdot)$, $j \in \{j^*, ..., J\}$, as the derivative of function $\widehat{F}(\cdot)$ with respect to its $(j - j^* + 1)$th argument.[40] Except for these slight modifications, the proof proceeds exactly as in Appendix A.3.

[40] $\widehat{F}_{X_j}(\cdot)$ needs to be defined this way because $j^*$ arguments are eliminated from function $\widehat{F}(\cdot)$ in definition (11). Also, note that similarly to function $\widetilde{F}(\cdot)$, function $\widehat{F}(\cdot)$ has a CRS property from Lemma 2.

## A.8 Proof of Proposition 6

Similarly to Definition 3, the EoS $\overline{\sigma}_{KX_j,t}$ on the BGP is defined as

$$\overline{\sigma}_{KX_j,t} = - \left. \frac{d\ln(k/x_j)}{d\ln\left(\frac{A_{K,t}\overline{F}_K(A_{K,t}k, A_{X_1,t}X_{1,t},...,A_{X_j,t}x_j,...,A_{X_J,t}X_{J,t})}{A_{X_j,t}\overline{F}_{X_j}(A_{K,t}k, A_{X_1,t}X_{1,t},...,A_{X_j,t}x_j,...,A_{X_J,t}X_{J,t})}\right)} \right|_{\substack{\overline{F}(A_{K,t}k, A_{X_1,t}X_{1,t},...,A_{X_j,t}x_j,...,A_{X_J,t}X_{J,t})=Y_t \\ k=K_t, x_j=X_{j,t}}} . \tag{A.26}$$

where $Y_t, K_t, X_{1,t}, ..., X_{J,t}$ indicate the BGP values, and $k$ and $x_j$ are the variables to be perturbed.[41]

Let us first consider the case of $j \in \{1, ..., j^*\}$. In this case, factors $X_{j^*+1,t}, ..., X_{J,t}$ are fixed at the BGP values. Now, we simplify the denominator of (A.26), particularly focusing on the fraction inside $\ln(\cdot)$. Using (10), function $\overline{F}(A_{K,t}k, A_{X_1,t}X_{1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t})$ can be written as $\widehat{F}(m, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_J,t}X_{J,t})$, where $m$ is the effective amount of capital composite, $m = (A_{K,t}k)^\alpha (A_{X_j,t}x_j)^{\xi_j} \prod_{j' \in \{1,...,j^*\} \setminus j} (A_{X_{j'},t}X_{j',0})^{\xi_{j'}}$. Note that $dm/dk = \alpha m/k$. Using these properties and the chain rule, we have

$$\begin{aligned} &A_{K,t}\overline{F}_K(A_{K,t}k, A_{X_1,t}X_{1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t}) \\ &\quad = \frac{\partial}{\partial k}\overline{F}(A_{K,t}k, A_{X_1,t}X_{1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t}) \\ &\quad = \frac{\partial}{\partial k}\widehat{F}(m, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_J,t}X_{J,t}) \\ &\quad = \widehat{F}_M(m, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_J,t}X_{J,t})\alpha\frac{m}{k}. \end{aligned} \tag{A.27}$$

Similarly,

$$\begin{aligned} &A_{X_j,t}\overline{F}_{X_j}(A_{K,t}k, A_{X_1,t}X_{1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t}) \\ &\quad = \widehat{F}_M(m, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_J,t}X_{J,t})\xi_j\frac{m}{x_j}. \end{aligned} \tag{A.28}$$

[41] In definition (A.26), condition $\overline{F}(A_{K,t}k, A_{X_1,t}X_{1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t}) = Y_t$ means that $k$ and $x_j$ are perturbed so that output $Y_t$ is unchanged from the BGP value. Condition $k = K_t, x_j = X_{j,t}$ says that the EoS is evaluated at the BGP values. It is also important to keep in mind the notation for derivatives: $A_{K,t}\overline{F}_K(A_{K,t}k, A_{X_1,t}X_{1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t})$ is the partial derivative of $\overline{F}(\cdot)$ with respect to $k$, which corresponds to $F_K(\cdot;t)$ in Definition 3.

Substituting (A.27) and (A.28) into (A.26) gives

$$\overline{\sigma}_{KX_j,t} = - \left. \frac{d\ln(k/x_j)}{d\ln\left(\frac{\alpha}{\xi_j}\frac{x_j}{k}\right)} \right|_{\substack{\overline{F}(A_{K,t}k, A_{X_1,t}X_{1,t},...,A_{X_j,t}x_j,...,A_{X_J,t}X_{J,t})=Y_t \\ k=K_t, x_j=X_{j,t}}} . \tag{A.29}$$

Since $\alpha$ and $\xi_j$ are constant parameters, the denominator can be simplified as $d\ln\left((\alpha/\xi_j)(x_j/k)\right) = d\left(\ln(\alpha/\xi_j) + \ln(x_j/k)\right) = d\ln(x_j/k)$. Using this, (A.29) gives $\overline{\sigma}_{KX_j,t} = 1$. Recall that $\sigma_{KX_j,0} = 1$ because $j \in \{1, ..., j^*\}$, and that $\sigma_{KX_j,t}$ does not change over time on the BGP. Therefore, $\sigma_{KX_j,t} = \sigma_{KX_j,0} = 1 = \overline{\sigma}_{KX_j,t}$ holds.

Next, we examine $\overline{\sigma}_{KX_j,t}$ for the case of $j \in \{j^*+1, ..., J\}$. Similarly to the proof of Proposition 3, consider a change of variables: $k' = g^{-t}k$ and $x'_j = g^{-t}A_{X_j,t}x_j$. Then, $k$ in (A.26) is replaced by $k = g^t k'$ and $x_j$ is replaced by $(g^t/A_{X_j,t})x'_j$. In the numerator, $k/x_j$ becomes $A_{X_j,t}k'/x'_j$. In the denominator, by the same operations as in (A.27), $A_{K,t}\overline{F}_K(\cdot)$ can be written as

$$\widehat{F}_M(m, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t})\alpha\frac{m}{k}, \tag{A.30}$$

where $m = (A_{K,t}k)^\alpha \prod_{j'=1}^{j^*}(A_{X_{j'},t}X_{j',0})^{\xi_{j'}}$. The definition of $m$ does not include $x_j$ because $j \in \{j^*+1, ..., J\}$ means that $x_j$ is not a part of capital composite. Instead, $A_{X_j,t}x_j$ appears in (A.30) as the $(j - j^* + 1)$th argument of the $\widehat{F}(\cdot)$ function. Using $x_j = (g^t/A_{X_j,t})x'_j$, $A_{X_j,t}x_j$ can be written as $g^t x'_j$. Since $M_t = (A_{K,t}K_t)^\alpha \prod_{j=1}^{j^*}(A_{X_j,t}X_{j,t})$ grows by a factor of $g$ every period, the capital composite $m$ can also be written as

$$m = \left(g^t k'/K_t\right)^\alpha M_t = (k'/K_0)^\alpha g^t M_0 = g^t m',$$

where $m' = (k')^\alpha \prod_{j'=1}^{j^*}(A_{X'_j,0}X_{j',0})$. Other effective factors also grow by a factor of

$g$: $A_{X_{j'},t}X_{j',t} = g^t X_{j',0}$ for $j' \in \{j^*+1, ..., J\}\backslash j$. Using these, (A.30) becomes

$$\begin{aligned}
&\widehat{F}_M(g^t m', g^t X_{j^*+1,0}, ..., g^t x_j', ..., g^t X_{J,0})\alpha \frac{g^t m'}{g^t k'} \\
&= \widehat{F}_M(m', X_{j^*+1,0}, ..., x_j', ..., X_{J,0})\alpha \frac{m'}{k'} \\
&= \frac{\partial}{\partial k'}\widehat{F}(m', X_{j^*+1,0}, ..., x_j', ..., X_{J,0}) \\
&= \overline{F}_K(k', X_{1,0}, ..., x_j', ..., X_{J,0}),
\end{aligned}$$

where the first equality is from the homogeneity of degree 0 property of the $\widehat{F}_M(\cdot)$ function, the second equality is from the chain rule and $dm'/dk' = \alpha m'/k'$, and the last equality is from the definition of $\overline{F}(\cdot)$ in (10).

Likewise, $A_{X_j,t}\overline{F}_{X_j}(\cdot)$ in the denominator of (A.26) can be expressed in terms of $k'$ and $x_j'$ as

$$\begin{aligned}
&\frac{\partial}{\partial x_j}\overline{F}(A_{K,t}k, A_{X_{1,t},t}X_{1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t}) \\
&= \frac{\partial}{\partial x_j}\widehat{F}(m, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t}) \\
&= \widehat{F}_{X_j}(m, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t})\frac{dA_{X_j,t}x_j}{dx_j} \\
&= \widehat{F}_{X_j}(g^t m', g^t X_{j^*+1,0}, ..., g^t x_j', ..., g^t X_{J,0})A_{X_j,t} \\
&= \widehat{F}_{X_j}(m', X_{j^*+1,0}, ..., x_j', ..., X_{J,0})A_{X_j,t} \\
&= \frac{\partial}{\partial x_j'}\widehat{F}(m', X_{j^*+1,0}, ..., x_j', ..., X_{J,0})A_{X_j,t} \\
&= A_{X_j,t}\overline{F}_{X_j}(k', X_{1,0}, ..., x_j', ..., X_{J,0}).
\end{aligned}$$

The definition in (A.26) evaluates $\bar{F}(\cdot)$ and its arguments at their BGP values.[42]

[42] I.e., the conditions that are written to the right of "|".

We also need to re-write these conditions in terms of their period-0 values. Note that

$$\begin{aligned}
&\overline{F}(A_{K,t}k, A_{X_1,t}X_{1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t}) \\
&\qquad = \widehat{F}(m, A_{X_{j^*+1},t}X_{j^*+1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t}) \\
&\qquad = \widehat{F}(g^t m', g^t X_{j^*+1,0}, ..., g^t x'_j, ..., g^t X_{J,0})A_{X_j,t} \\
&\qquad = g^t \widehat{F}(m', X_{j^*+1,0}, ..., x'_j, ..., X_{J,0})A_{X_j,t} \\
&\qquad = g^t \overline{F}(k', X_{1,0}, ..., x'_j, ..., X_{J,0}).
\end{aligned}$$

Therefore, condition $\overline{F}(A_{K,t}k, A_{X_1,t}X_{1,t}, ..., A_{X_j,t}x_j, ..., A_{X_J,t}X_{J,t}) = Y_t$ can be substituted by

$$\overline{F}(k', X_{1,0}, ..., x'_j, ..., X_{J,0}) = Y_0.$$

The point of evaluation, $k = K_t$, becomes $g^t k' = K_t$, or $k' = g^{-t}K_t = K_0$. Similarly, $x_j = X_{j,t}$ becomes $x'_j = g^{-t}A_{X_j,t}X_{j,t} = X_{j,0}$. Using all these results, (A.26) can be expressed in terms of $k'$ and $x'_j$ as follows:

$$\overline{\sigma}_{KX_j,t} = -\left.\frac{d\ln(A_{X_j,t}k'/x'_j)}{d\ln\left(\frac{\overline{F}_K(k', X_{1,0}, ..., x'_j, ..., X_{J,0})}{A_{X_j,t}\overline{F}_{X_j}(k', X_{1,0}, ..., x'_j, ..., X_{J,0})}\right)}\right|_{\substack{\overline{F}(k', X_{1,0}, ..., x'_j, ..., X_{J,0}) = Y_0 \\ k' = K_0, x'_j = X_{j,0}}}. \tag{A.31}$$

We can eliminate constant $A_{X_j,t}$ from the numerator because $d\ln(A_{X_j,t}k'/x'_j) = d(\ln(k'/x'_j) + \ln A_{X_j,t}) = d\ln(k'/x'_j)$. In the same way, $A_{X_j,t}$ in the denominator can also be eliminated. Also, recall that $A_{K,0} = A_{X_j,0} = 1$. Then, comparing (A.31) with (A.26), it turns out that the RHS of (A.31) coincides with $\overline{\sigma}_{KX_j,0}$. From Lemma 2, $\overline{\sigma}_{KX_j,0} = \sigma_{KX_j,0}$ holds in period 0. In addition, it is assumed that $\sigma_{KX_j,0}$ does not change over time. Therefore, $\overline{\sigma}_{KX_j,t} = \overline{\sigma}_{KX_j,0} = \sigma_{KX_j,0} = \sigma_{KX_j,t}$.

## A.9 Proof of Lemma 3

Note that $\delta = (\delta^{\text{IST}} + g_q - 1)/g_q$ means $1 - \delta^{\text{IST}} = (1-\delta)g_q = (1-\delta)q_t/q_{t-1}$. Dividing equation (30) by $q_t$ and using the above result, we have $K_{t+1} = K^{\text{IST}}_{t+1}/q_t = (Y_t - C_t - R_t) + (1 - \delta^{\text{IST}})K^{\text{IST}}_t/q_t = (Y_t - C_t - R_t) + (1-\delta)(q_t/q_{t-1})K^{\text{IST}}_t/q_t = (Y_t - C_t - R_t) + (1-\delta)K_t$, which coincides with (2).

Using $K_t^{\text{IST}} = q_{t-1}K_t$, production function (29) can be written as $Y_t = F^{\text{IST}}(q_{t-1}K_t,\ X_{1,t}, ..., X_{J,t}; t)$. It is a CRS function of $K_t, X_{1,t}, ..., X_{J,t}$ and depends on time both through the shape of $F^{\text{IST}}(\cdot; t)$ and through the growth of $q_t$. Therefore, we can define a new production function (31), $F(K_t, X_{1,t}, ..., X_{J,t}; t) \equiv F^{\text{IST}}(q_{t-1}K_t, X_{1,t}, ..., X_{J,t}; t)$, where dependence of $F(\cdot; t)$ on $t$ includes the effect from $q_{t-1}$. From the assumptions on $F^{\text{IST}}(\cdot)$, function $F(\cdot; t)$ obviously satisfies the required marginal product properties in (1).

# B Calibration Details

## B.1 Deriving the Elasticity of Substitution in a Three-Factor CES function (24)

In this proof, we write (22) as $Y_t = F_t(K_t, L_t, X_t)$ and drop time subscripts for convenience.[43] Also, in Appendix B, we use $F_K$, $F_L$ and $F_X$ to denote the derivative of $F(K, L, X)$ with respect to $K$, $L$ and $X$, respectively. From Definition 3, the elasticity of substitution (EoS) between capital and labor for function $F(K, L, X)$ is defined as

$$\sigma_{KL} = -\left.\frac{d\ln(K/L)}{d\ln(F_K/F_L)}\right|_{Y,X:\,const}. \tag{B.1}$$

Total differentiation of $Y = F(K, L, X)$ gives

$$F_K dK + F_X dX + F_L dL = dY.$$

Noting that $\sigma_{KL}$ is defined with the constraint that $X$ and $Y$ are kept constant ($dX = dY = 0$), the latter can be re-written as

$$dL = -\frac{F_K}{F_L}dK. \tag{B.2}$$

[43]Function (22) depends also on technologies $A_{K,t}$, $A_{L,t}$, and $A_{X,t}$. This dependence is included in $F_t$, which changes with time. However, because we are calculating the elasticity of substitution of production function at a given $t$, technologies are fixed in this calculation. Therefore, we omit $t$ from $F_t$ for simplicity.

Therefore, the numerator of (B.1) can be written as

$$
\begin{aligned}
d\ln\left(\frac{K}{L}\right) &= d\left(\ln K - \ln L\right) \\
&= \frac{1}{K} + \frac{1}{L}\frac{F_K}{F_L}dK \\
&= \left(1 + \frac{KF_K}{LF_L}\right)\frac{dK}{K}.
\end{aligned}
$$

Since we assume perfect competition, $KF_K/LF_L$ is the ratio of capital share $s_K$ to labor share $s_L$. Therefore,

$$
d\ln\left(\frac{K}{L}\right) = \left(\frac{s_K}{s_L} + 1\right)\frac{dK}{K}. \tag{B.3}
$$

Next, we calculate the denominator. By taking the partial derivatives in (22), the ratio of marginal products turns out to be

$$
\frac{F_K}{F_L} = \frac{\eta_{KX}\alpha\left(A_L L\right)^{\frac{1}{\epsilon}}\left(A_K K\right)^{\alpha\frac{\epsilon-1}{\epsilon}}\left(A_X X\right)^{(1-\alpha)\frac{\epsilon-1}{\epsilon}}}{\eta_L A_L K}. \tag{B.4}
$$

Again noting that the EoS is defined for $dX = dY = 0$, the total differentiation of the log of (B.4) becomes

$$
\begin{aligned}
d\ln\left(\frac{F_K}{F_L}\right) &= d\left\{\ln L^{\frac{1}{\epsilon}} + \ln K^{\alpha\frac{\epsilon-1}{\epsilon}-1}\right\} \\
&= \frac{1}{\epsilon}\frac{dL}{L} - \left(1 - \alpha\frac{\epsilon-1}{\epsilon}\right)\frac{dK}{K} \\
&= -\frac{1}{\epsilon L}\frac{F_K}{F_L}dK - \left(1 - \alpha\frac{\epsilon-1}{\epsilon}\right)\frac{dK}{K} \\
&= -\left[\frac{1}{\epsilon}\frac{KF_K}{LF_L} + 1 - \alpha\frac{\epsilon-1}{\epsilon}\right]\frac{dK}{K} \\
&= -\left[\left(\frac{s_K}{s_L} + \alpha\right)\frac{1}{\epsilon} + (1-\alpha)\right]\frac{dK}{K}.
\end{aligned} \tag{B.5}
$$

Substituting (B.3) and (B.5) into (B.1) gives (24) in the main text.

## B.2 Factor shares and distribution parameters (26)

The partial derivatives of (22) with respect to $L_t$ and $K_t$ are:

$$F_{L,t} = \eta_L \left(A_{L,t}L_t\right)^{\frac{\epsilon-1}{\epsilon}} L_t^{-1} Y_t^{\frac{1}{\epsilon}} \tag{B.6}$$

$$F_{K,t} = \eta_{KX} \alpha \left(\left(A_{K,t}K_t\right)^{\alpha} \left(A_{X,t}X\right)^{1-\alpha}\right) K_t^{-1} Y_t^{\frac{1}{\epsilon}}. \tag{B.7}$$

With perfect competition, factors shares in period $t$ are given by $s_{L,t} = L_t F_{L,t}/Y_t$ and $s_{K,t} = K_t F_{K,t}/Y_t$. First order conditions (B.6) and (B.7) implies that their values are

$$s_{L,t} = \eta_L \left[\frac{A_{L,t}L_t}{Y_t}\right]^{\frac{\epsilon-1}{\epsilon}} \tag{B.8}$$

$$s_{K,t} = \alpha \eta_{KX} \left[\frac{\left(A_{K,t}K_t\right)^{\alpha} \left(A_{X,t}X\right)^{1-\alpha}}{Y_t}\right]^{\frac{\epsilon-1}{\epsilon}}. \tag{B.9}$$

Note that the above result holds even when the economy is not on the BGP ($s_{L,t}$ can be different from its BGP value $s_L$).

We calibrate the model to a starting period $t = 0$ when the economy is on a BGP. From (22), $A_{K,0} = A_{X,0} = A_{L,0} = 1$. Also, $s_{L,0} = s_L$ and $s_{K,0} = s_K$. Then, evaluating (B.8) and (B.9) at $t = 0$ gives equation (26) in the main text.

## B.3 Deviation of the labor share from the BGP value

In this section, we derive (27), the labor share when the capital-augmenting technology $a_{K,t}$ deviates from its BGP path $A_{K,t}$. As explained in Subsection 7.2, we assume that other technologies and inputs ($A_{X,t}$, $A_{L,t}$, $K_t$, $X_t$ and $L_t$) follow the BGP path. Then, from (22), the output deviates from the BGP value $Y_t$ and now becomes

$$y_t = \left\{\eta_{KX} \left(\left(a_{K,t}K_t\right)^{\alpha} \left(A_{X,t}X_t\right)^{1-\alpha}\right)^{\frac{\epsilon-1}{\epsilon}} + \eta_L (A_{L,t}L_t)^{\frac{\epsilon-1}{\epsilon}}\right\}^{\frac{\epsilon}{\epsilon-1}}. \tag{B.10}$$

Substituting (B.10) into (B.8) gives the labor share when $a_{K,t}$ deviates from $A_{K,t}$.

$$\begin{aligned} s_{L,t} =& \frac{\eta_L (A_{L,t}L_t)^{\frac{\epsilon-1}{\epsilon}}}{\eta_{KX}\left((a_{K,t}K_t)^{\alpha}(A_{X,t}X_t)^{1-\alpha}\right)^{\frac{\epsilon-1}{\epsilon}} + \eta_L (A_{L,t}L_t)^{\frac{\epsilon-1}{\epsilon}}} \\ =& \left(\frac{\eta_{KX}}{\eta_L}\left(\frac{(A_{K,t}K_t)^{\alpha}(A_{X,t}X_t)^{1-\alpha}}{A_{L,t}L_t}\right)^{\frac{\epsilon-1}{\epsilon}}\left(\frac{a_{K,t}}{A_{K,t}}\right)^{\alpha\frac{\epsilon-1}{\epsilon}} + 1\right)^{-1}. \end{aligned} \tag{B.11}$$

In contrast, if all variables, including $A_{K,t}$, follow the BGP, dividing (B.9) by (B.8) gives

$$\frac{\eta_{KX}}{\eta_L}\left(\frac{(A_{K,t}K_t)^{\alpha}(A_{X,t}X_t)^{1-\alpha}}{A_{L,t}L_t}\right)^{\frac{\epsilon-1}{\epsilon}} = \frac{s_K/\alpha}{s_L}, \tag{B.12}$$

where $s_K$ and $s_L$ are the shares on the BGP. Note that the LHS of (B.12) is the same as the first term in the last line of (B.11). Note also that $s_K/\alpha = 1 - s_L$ since $\alpha$ is defined as $s_K/(s_K + s_X) = s_K/(1 - s_L)$. Hence, the RHS of (B.12) is $(1/s_L) - 1$. Therefore, by substituting (B.12) into (B.11), we obtain (27) in Section 7.2.

# C Mapping data on investment prices to K-augmenting technical change in the model

In this appendix, we show how changes in the relative price of investment in the NIPA statistics can be used to measure capital-augmenting technical change in the neoclassical growth model. In Section C.1, we explain the mapping between the NIPA data and the standard neoclassical growth model, as defined by Definition 1. In Section C.2, we explain how the result changes when the data is mapped to a neoclassical growth model with investment-specific technical change, as discussed in Section 8. Both results follow from the property that capital can be accumulated linearly from the previous period's output, specified either by (2) or (30).

## C.1 Relative Price of Investment in the NIPA Statistics Interpreted in the Standard Neoclassical Growth Model

In this section, we prove that the speed of capital-augmenting technological change in the neoclassical growth model can be measured by the rate of decline in the relative investment price in the NIPA statistics with a one-period lag. This result will be given in equation (C.4), which is a basis for the discussion in Section 2. We then use this result to show how to measure the deviation of the capital-augmenting technological change from its BGP path, which leads to equation (28) in Section 7.2.

We start by discussing the relative prices of investment and consumption in the model. The unit of the consumption good is defined so that the period utility is a time-invariant function of $C_t/L_t$. This implies that one unit of consumption in the model always increases the argument of the utility function by one. The process of capital accumulation (2) implies that one unit of the final good can be used as either one unit of the consumption good or one unit of the investment good, implying that the price of the investment good is also equal to one. In addition, one unit of the investment good in period $t$ becomes one unit of capital in period $t+1$. Therefore, $K_{t+1}$ is measured in the same units as $C_t$. One unit of investment in period $t$ will increase effective capital in $t+1$ by $A_{K,t+1}$ efficiency units.

Now, we explain the mapping from relative prices in the NIPA statistics to variables in the model. First, we consider the mapping for the consumption good. In the NIPA statistics, the prices and quantities are adjusted for quality changes. Therefore, any improvement in the quality of consumption goods shows up as an increase in quantity. When the statistics are mapped to the model, this means that one unit of consumption in the NIPA statistics always increases the argument of the utility function by the same amount. Let this amount be $\chi_C$. Since one unit of consumption in the model increases the argument of the same utility function by one, one unit of consumption in the NIPA statistics should equal $\chi_C$ units of consumption in the neoclassical growth model. Next, let $P_{C,t}^{\mathrm{NIPA}}$ be the dollar price of one unit of consumption in the NIPA statistics, and $P_t^{\$}$ be the dollar price of one unit of consumption in the neoclassical growth model. Then, since one unit of consumption in the NIPA is $\chi_C$ times that in the model, we have

$$P_{C,t}^{\mathrm{NIPA}} = P_t^{\$}\chi_C. \tag{C.1}$$

Similarly, we can derive the mapping for the investment good. The NIPA statistics are quality-adjusted also for investment goods. So, as before, any quality improvements will appear as quantity increases. As a result, one unit of investment in the statistics should increase the next period's effective capital by a time-invariant amount, which we denote with $\chi_I$. Recall that one unit of the investment good in period $t$ in the model increases the next period's effective capital by $A_{K,t+1}$. Therefore, one unit of investment in NIPA should be the same as $\chi_I/A_{K,t+1}$ units of investment in the model. Since the price of investment is the same as the price of consumption in the model, the dollar price of investment in the model is given by $P_t^{\$}$. Then, since one unit of investment in NIPA is $\chi_I/A_{K,t+1}$ times that in the model, the dollar price of one unit of investment in the NIPA statistics is

$$P_{I,t}^{\text{NIPA}} = \frac{P_t^{\$}\chi_I}{A_{K,t+1}}. \tag{C.2}$$

Combining (C.1) and (C.2), the relative price of investment from the NIPA data, denoted by $RPI_t$, can be mapped to the model by

$$RPI_t = \frac{P_{C,t}^{\text{NIPA}}}{P_{I,t}^{\text{NIPA}}} = \frac{\chi_C}{\chi_I A_{K,t+1}}. \tag{C.3}$$

Because $\chi_C$ and $\chi_I$ do not depend on time,

$$\frac{A_{K,t+1}}{A_{K,t}} = \left(\frac{RPI_t}{RPI_{t-1}}\right)^{-1}. \tag{C.4}$$

The above equation means that the speed of capital-augmenting technological change in the neoclassical growth model is given by the speed of the fall in the relative investment price in the NIPA statistic with a one-period lag. This property is mentioned in Section 2. If the economy is on a BGP, equation (C.4) implies that $RPI_t$ changes at a constant rate $g_{RPI} = 1/\gamma_K$. This is the BGP path for $RPI_t$.

Now, let us consider the situation in which capital-augmenting technical change deviates from its BGP value. Below, we use $A_{K,t}$ to represent the BGP value, and $a_{K,t}$ to denote the value of capital-augmenting technical change at time $t$, whether or not it is on the BGP. Our analysis that leads to (C.3) does not assume that the economy is on the BGP. Therefore, the relative price of investment in this case can also be given by

$$rpi_t = (\chi_C/\chi_I)\frac{1}{a_{K,t+1}}, \tag{C.5}$$

where $rpi_t$ represents the relative price of investment that would be observed even if the level of capital augmenting technical change, $a_{K,t}$, deviates from its balanced growth level, $A_{K,t}$. Dividing (C.5) by (C.3) and substituting $t$ by $t-1$ gives (28).

## C.2 Relative Price of Investment Interpreted in the Model with Investment-Specific Technological Change

In the neoclassical growth model with IST, as shown in (30), one unit of consumption good $C_t$ in period $t$ can be converted to $q_t$ units of capital $K_{t+1}^{\mathrm{IST}}$ in period $t+1$. (Similarly to Lemma 3, we put superscript IST on capital to show that it is now measured in different units than in the standard neoclassical growth model). So, $q_t$ represents investment-specific technology in period $t$, which may improve over time. Then, the price of investing in one unit of $K_{t+1}^{\mathrm{IST}}$ at period $t$ is $1/q_t$. Suppose that, in addition the IST, there is also capital-augmenting technology $A_{K,t+1}^{\mathrm{IST}}$ in period $t+1$ production. Then, one unit of investment in period $t$ will increase effective capital in $t+1$ by $A_{K,t+1}^{\mathrm{IST}}$ efficiency units.

Now, we explain how the relative price of investment in the NIPA statistics can be represented by variables in the model with IST. Since there is no change in the definition of consumption goods, the dollar price of consumption goods in the NIPA can still be represented by (C.1). As noted in the previous subsection, the NIPA statistics are also quality-adjusted for investment goods, and one unit of investment in the statistics should increase the next period's effective capital by a time-invariant amount. Let us denote this by $\chi_I^{\mathrm{IST}}$. As shown above, one unit of the investment good in period $t$ in the model increases the next period's effective capital by $A_{K,t+1}^{\mathrm{IST}}$. Therefore, one unit of investment in NIPA should be the same as $\chi_I^{\mathrm{IST}}/A_{K,t+1}^{\mathrm{IST}}$ units of investment in the model. Since the price of investment is $1/q_t$ times that of the consumption in the model, the dollar price of investment in the model is given by $P_t^{\$}/q_t$. Then, since one unit of investment in the NIPA is $\chi_I^{\mathrm{IST}}/A_{K,t+1}^{\mathrm{IST}}$ times that in the model, the dollar price of one unit of investment in the NIPA statistics can be represented as

$$P_{I,t}^{\mathrm{NIPA}} = \frac{\chi_I^{\mathrm{IST}}}{A_{K,t+1}^{\mathrm{IST}}} \cdot \frac{P_t^{\$}}{q_t} = \frac{\chi_I^{\mathrm{IST}} P_t^{\$}}{q_t A_{K,t+1}}. \tag{C.6}$$

Combining (C.1) and (C.6), the relative price of investment in the NIPA now becomes

$$RPI_t = \frac{P_{C,t}^{\text{NIPA}}}{P_{I,t}^{\text{NIPA}}} = \frac{\chi_C}{\chi_I^{\text{IST}} q_t A_{K,t+1}^{\text{IST}}}. \tag{C.7}$$

Because $\chi_C$ and $\chi_I^{\text{IST}}$ do not depend on time,

$$\frac{q_t}{q_{t-1}} \cdot \frac{A_{K,t+1}^{IST}}{A_{K,t}^{IST}} = \left(\frac{RPI_t}{RPI_{t-1}}\right)^{-1}. \tag{C.8}$$

When the model includes IST, the rate of decline in the relative price of investment in the NIPA statistics in period $t$ is the sum of the rate of investment-specific technological change in period $t$ and the rate of capital-augmenting technological change in period $t+1$. In the literature on IST, $q_t/q_{t-1}$ is called the rate of *embodied* technological change, while $A_{K,t+1}^{IST}/A_{K,t}^{IST}$ is that of *disembodied* technological change. Both technologies improve the process in which saved output in period $t$ contributes to the production $t+1$. The sum of both improvements shows up as the fall in the relative price of investment in the NIPA statistic. In effect, the model with IST divides this process into two parts. When we define the combination of the two technology as $A_{K,t+1} = q_t A_{K,t+1}^{\text{IST}}$ in (C.7) and (C.8), then we are back to the standard neoclassical growth model and obtain (C.3) and (C.4).

# D Data Sources and Definitions

Most data are originally from the Bureau of Economic Analysis (BEA) and retrieved from FRED, Federal Reserve Bank of St. Louis, `https://fred.stlouisfed.org`. We reference these series by their codes in FRED. Energy use data are from the Energy Information Administration (2023).

## D.1 Figure 1

Panel (a) show several real aggregate variables per capita for the United States.

- **Real GDP** (GDPCA), **real investment** (GPDICA), and **real personal consumption expenditures** (PCECCA) are originally from the National Income and Product Accounts (NIPA) collected by Bureau of Economic Analysis (BEA).

- The **nominal capital stock** is the current cost net stock of fixed assets (K1TTOTL1ES000). It is taken from the fixed asset tables collected by the BEA.

- The **GDP deflator** (A191RD3A086NBEA) is set to 100 in 2012. It is originally from NIPA.

- The **real capital stock** is the ratio of the nominal capital stock and the GDP deflator.

- **Population** (B230RC0A052NBEA) includes resident population and armed force overseas. It is originally from NIPA.

- **Real variables per capita** are calculated by dividing the relevant aggregate quantity by population.

Panel (b) shows the relative price of investment and consumption. All data are originally from NIPA and indexed to be equal to 100 in 2012.

- We use three different **price deflators for gross private investment**: all (A006RD3A086NBEA), non-residential (A008RD3A086NBEA), and equipment (Y033RD3A086NBEA).

- The **price deflator for consumption** (DPCERD3A086NBEA) covers all personal consumption expenditures.

- To measure the **relative price of investment**, we divide the price deflator for investment by the price deflator for consumption.

## D.2 Calibration in Section 7

- The data for **real GDP**, **population** and the **relative price of investment** are the same as shown in Section D.1.

- **Nominal GDP** (GDPA) is from NIPA.

- The **relative price of investment** comes from Figure 1 and cover all investment.

- **Labor compensation** compensation of employees (A033RC1A027NBEA) plus proprietors' income with inventory valuation and capital consumption adjustments (A041RC1A027NBEA). Both series are originally from NIPA.
- The **labor share of income** is the ratio of labor compensation and nominal GDP.
- The **capital-output ratio** is the ratio of nominal capital and nominal GDP.
- **Total fossil fuel energy use** is the total consumption of fossil fuels measured in BTUs. It is taken from the Energy Information Administration (EIA)'s Annual Review. See 'Total Fossil Fuels Consumption' in Section 1.1 of https://www.eia.gov/totalenergy/data/annual/.

## D.3 Figure 2

- Panel (a) uses data on the **relative price of investment** from Section D.2.
- Panel (b) uses the **labor share of income** measure from Section D.2.