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ENDOGENOUS CAPITAL-AUGMENTING TECHNOLOGICAL CHANGE

Gregory Casey
Ryo Horii

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The Institute of Social and Economic Research
Osaka University
6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan

Endogenous Capital-Augmenting Technological Change*

Gregory Casey† Ryo Horii‡

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Abstract

We construct a 3-factor, directed technical change growth model that exhibits capital-augmenting technical change on the balanced growth path (BGP), circumventing the issues usually caused by the 2-factor Uzawa growth theorem. We calibrate the model to the United States and consider a non-unitary elasticity of substitution between capital and labor. We show that the model converges to the BGP with capital-augmenting technical change from any initial condition. Our results indicate that natural resources and directed technical change play a central role in explaining balanced growth.

Keywords: Balanced Growth, Uzawa Steady-State Growth Theorem, Directed Technical Change (DTC), Natural Resources, Three-Factor Model

JEL Classification Codes E13, E22, O33, O41

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†Department of Economics, Williams College, Schapiro Hall, 24 Hopkins Hall Dr., Williamstown, MA, 01267. Email: gpc2@williams.edu.

‡Correspondence: Institute of Social and Economic Research (ISER), Osaka University, 6-1 Mihogaoka, Ibaraki city, Osaka 567-0047, Japan. Email: horii@iser.osaka-u.ac.jp.

1 Introduction

The two-factor neoclassical growth model is central to much of macroeconomics. As explained by Grossman et al. (2017), however, this model is inconsistent with stylized facts observed in the United States, including balanced growth, a non-unitary elasticity of substitution (EoS) between labor and reproducible capital, and the declining relative price of investment goods.[1] We present a three-factor neoclassical growth model that is consistent with all of these stylized facts. Our results suggest that natural resources and directed technical change are essential for making neoclassical growth models consistent with data.

The Uzawa (1961) steady state growth theorem explains that, on the balanced growth path (BGP) of a two-factor neoclassical growth model, all technological change must be labor-augmenting, unless the production function is Cobb-Douglas (Acemoglu, 2008; Jones and Scrimgeour, 2008). But, this is inconsistent with two well-documented pieces of evidence: the EoS between capital and labor is different than one (i.e., the production function is not Cobb-Douglas), and the relative price of investment has been falling on the balanced growth path (i.e., there is capital-augmenting technical change) (e.g., Greenwood et al., 1997; Antras et al., 2004; DiCecio, 2009; Oberfield and Raval, 2021).

Casey and Horii (forthcoming) argue that including three or more factors of production allows the neoclassical growth model to be consistent with data. They generalize the Uzawa growth theorem to show that a neoclassical growth model can be consistent with balanced growth and capital-augmenting technical change as long as capital has a unitary EoS with any *single* other factor of production. This leaves open the possibility that capital has a non-unitary elasticity with labor, but a unitary elasticity with some third factor. They argue that natural resources, like land and energy, could play the role of this third factor and discuss existing evidence consistent with this interpretation. When the EoS condition is satisfied, the generalized Uzawa growth theorem implies that there is a log-linear relationship between the rates of factor-augmenting technological change. The relationship guarantees stationary of factor shares in the long run. If technological change is exogenous, then this is a knife-edge condition. But, the theorem does not specify the economy mechanism

[1] See Jones (2016) for a recent review of these facts.

through which technology growth rates are determined.

Building on their results, we formulate a directed technological change model that has capital-augmenting technical change on the balanced growth path, despite a non-unitary elasticity of substitution between capital and labor. The direction of innovation is determined by profit-maximizing firms. The economy endogenously conforms to the log-linear condition identified in the generalized Uzawa growth theorem. We also calibrate the model to U.S. data and show that the BGP is both locally and globally stable. The transitional dynamics of our model are qualitatively different from the standard two-factor neoclassical model. The convergence to the BGP goes through two stages: capital and technological adjustments. The latter is much slower.

Our results contribute to the existing literature in two main ways. First, we show how neoclassical production functions can be made consistent with data on balanced growth and the EoS between capital and labor. Thus, our model will be useful in any macroeconomic setting where matching these data patterns is important. Second, we contribute to the existing literature on the Uzawa steady state theorem (e.g., Schlicht, 2006; Jones and Scrimgeour, 2008; Grossman et al., 2017). In particular, we present the economic mechanism by which the direction of technical change endogenously conforms to the log-linear technological condition in the generalized Uzawa growth theorem. Unlike the case of the exogenous technological change analyzed in Casey and Horii (forthcoming), we do not need to impose a parameter restriction on technological change. The sole restriction is the unitary elasticity between capital and a single other factor, like land or energy. In this way, our results indicate that including natural resources and directed technical change are promising ways to explain balanced growth in neoclassical growth models.

Related literature. Our work is part of a growing literature that examines the relationship between endogenous growth and the Uzawa steady state theorem. Several studies have used directed technical change to explain why the economy might endogenously conform to the two-factor version of the theorem. In particular, Acemoglu (2003) and Irmen and Tabaković (2017) provide models where capital-augmenting technical change disappears in the long run, while Jones (2005) and Leon-Ledesma and Satchi (2019) specify models that are Cobb-Douglas in the long run. To the best of our knowledge, Grossman et al. (2017) present the only other attempt to square balanced growth with the wider set of stylized facts observed in the United States. They specify a specific type of capital-skill complementarity that allows growth models with schooling to match the data. Casey and Horii (forthcoming) demonstrate how to interpret their findings in the context of the two-factor neoclassical growth model. We build on these works by presenting a directed technical change growth model that converges to a BGP that is consistent with data on both the existence of capital-augmenting technical change and the less-than-unitary EoS between capital and labor.

Our work is also related to the literature on directed technical change and natural resource use (e.g., Smulders and De Nooij, 2003; Hassler et al., 2021, 2022; Casey, forthcoming). This literature explains how technology endogenously evolves to ensure balanced growth when there are three or more factors of production in a neoclassical growth model. These models are usually used to study resource conservation or climate change. We show how to extend the models in this literature to be consistent with the additional stylized facts highlighted by Grossman et al. (2017). In doing so, we demonstrate that including directed technical change and natural resources in growth models is necessary to match data patterns unrelated to environmental questions, broadening the importance of this existing literature.

2 The Uzawa Steady-State Growth Theorem

In a neoclassical growth model, continued economic growth requires technological change. Theoretically, technological change in this context means time variation in the aggregate production function, which is defined as the mapping from aggregate production factors to aggregate output. Given that we observe sustained economic growth in most countries, it is essential to include technological change in a macroeconomic model. However, there are limitless ways in which the mapping could evolve over time. How should we specify technological change in the model so that the model is consistent with data? This poses a great challenge to economists, particularly when the mapping is not clearly observable.

The original Uzawa growth theorem provided a simple and convenient solution. On the balanced growth path, technological change in a 2-factor neoclassical growth model can always be represented as labor-augmenting technological progress.[2] This solution is widely used in the growth literature. However, recent studies found evidence that technological change is not purely labor-augmenting. The productivity of capital has been increasing, as reflected in the fall in the relative price of investment. To tackle this puzzle, Casey and Horii (forthcoming) provided a generalized version of the Uzawa theorem that incorporates the possibility of capital-augmenting technological change on the BGP. A key difference from the original Uzawa theorem is that the generalized version allows for more than two aggregate factors. Below, we present a simpler 3-factor version of their result, focusing on elements that are relevant to the directed technical change model in this paper.

Definition 1. *A* **3-factor neoclassical growth model** *is an economic environment that satisfies:*

1. *Output, Y_t, is produced from capital, K_t, labor L_t and another aggregate input X_t:*
$$Y_t = F_t(K_t, L_t, X_t). \tag{1}$$
In any $t \geq 0$, it has constant returns to scale (CRS) in each argument, and all inputs, K_t, L_t and X_t, have positive and diminishing marginal products.

2. *The amount of capital, K_t, evolves according to*
$$K_{t+1} = Y_t - C_t - R_t + (1-\delta)K_t,\ K_0 > 0, \tag{2}$$
where $C_t > 0$ is consumption, $R_t \geq 0$ is expenditure other than capital investment or consumption (e.g., R&D inputs), and $\delta \in [0,1]$ is the depreciation rate.

The focus of the theorem is how to express the evolution of production function $F_t(\cdot)$ over time. To accomplish this goal, we rely on the property that the economy has a BGP.

[2] An exception is when the production function is Cobb-Douglas. However, the estimates of the elasticity of substitution between labor and capital do not support the Cobb-Douglas specification (e.g., Oberfield and Raval, 2021).

Definition 2. *A* **balanced growth path (BGP)** *in a 3-factor neoclassical growth model is a path along which all quantities, $\{Y_t, K_t, L_t, X_t, C_t, R_t\}$, grow at constant exponential rates for all $t \geq 0$. We call it a* **non-degenerate** *BGP when the growth rate of K_t is larger than $-\delta$.*

On the BGP, we denote the growth factor of output by $g \equiv Y_t/Y_{t-1}$, and the growth factors of any variable Z_t by $g_Z \equiv Z_t/Z_{t-1}$. Unless otherwise noted, we focus on the non-generate BGP with positive investments, i.e., $g_K > 1-\delta$. We also need to define the elasticity of substitution, because the definition is not obvious when there are more than two factors.

Definition 3. *The* **Elasticity of Substitution** *between capital K_t and other factors in the 3-factor production function $F_t(K_t, L_t, X_t)$ is*

$$\sigma_{KL,t} = -\left.\frac{d\ln(K_t/L_t)}{d\ln\left(F_{K,t}/F_{L,t}\right)}\right|_{Y_t, X_t:\mathit{fixed}}, \quad \sigma_{KX,t} = -\left.\frac{d\ln(K_t/X_t)}{d\ln\left(F_{K,t}/F_{X,t}\right)}\right|_{Y_t, L_t:\mathit{fixed}}. \tag{3}$$

In (3), $F_{K,t}$ represents $\partial F_t(K_t, X_t, L_t)/\partial K_t$. $F_{L,t}$ and $F_{X,t}$ are similarly defined. When evaluating $\sigma_{KL,t}$, we consider small changes in K_t and L_t while keeping X_t and $Y_t = F_t(K_t, X_t, L_t)$ constant. When two factors have a unitary EoS between them, they can be represented with the usual Cobb-Douglas relationship. Now, we have all the definitions necessary to state a simpler version of the generalized Uzawa theorem.

Proposition 1 (Generalized Uzawa Growth Theorem for 3 Factors)**.** *Suppose that, on a BGP of a 3-factor neoclassical growth model, $\sigma_{KX,t} = 1$ holds and $\sigma_{KL,t} \neq 1$ is constant. Also, suppose that shares of factors s_K, s_L, and s_X are constant on the BGP, and let $\alpha = \frac{s_K}{s_K+s_X}$. Define factor-augmenting technologies by*

$$A_{K,t} = \gamma_K^t, A_{L,t} = \gamma_L^t, A_{X,t} = \gamma_X^t, \tag{4}$$

and choose growth factors of technologies γ_K, γ_L, and γ_X so that they satisfy

$$\gamma_K^{\alpha}(\gamma_X g_X)^{1-\alpha} = g^{1-\alpha} \text{ and } \gamma_L = g/g_L. \tag{5}$$

Then there exists a constant-returns-to-scale (CRS) function $\widehat{F}(\cdot,\cdot)$ that satisfies, on *the BGP,*

$$F(K_t, L_t, X_t) = \widehat{F}\left((A_{K,t}K_t)^{\alpha}(A_{X,t}X_t)^{1-\alpha}, A_{L,t}L_t\right) \tag{6}$$

$$\frac{\partial F(K_t, L_t, X_t)}{\partial K_t} = \frac{\partial \widehat{F}\left((A_{K,t}K_t)^{\alpha}(A_{X,t}X_t)^{1-\alpha}, A_{L,t}L_t\right)}{\partial K_t} \tag{7}$$

and similarly for the first partial derivatives with respect to X_t and L_t. Moreover, similarly to definition 3, define the elasticity of substitution in the RHS of (6) as $\widehat{\sigma}_{KL,t}$ and $\widehat{\sigma}_{KX,t}$. Then on the BGP,

$$\sigma_{KL,t} = \widehat{\sigma}_{KL,t}, \ \ \sigma_{KX,t} = \widehat{\sigma}_{KX,t}. \tag{8}$$

This proposition can be obtained as a special case of the generalized Uzawa theorem in Casey and Horii (forthcoming).[3] The proposition shows that technological change — equivalently, the evolution of F_t — can be represented by a fixed function $\widehat{F}$ along with factor augmenting terms $A_{K,t}$, $A_{L,t}$ and $A_{X,t}$. We call the RHS of (6) the *factor-augmenting representation* of technological change. This representation matches the level of the original production function in (6), first derivatives in (7), and the elasticity of substitution in (8) on the balanced growth path. Therefore, when the focus of economic analysis is on or around the BGP, the factor-augmenting representation can be used as an approximation of the true production function F_t, even when the economist does not know precisely how F_t evolves over time.

Unlike the original Uzawa theorem, the generalized Uzawa growth theorem allows positive capital-augmenting technological change on the BGP. Condition (5) implies there is freedom for an economist to choose $\gamma_K > 1$ to match the data.[4] Cancelling g in (5) and taking log gives

$$\log \gamma_K = \frac{1-\alpha}{\alpha}\left(\log \gamma_L + \log g_L - \log \gamma_X - \log g_X\right). \tag{9}$$

[3] Proposition 1 is obtained from Propositions 4, 5 and 6 in Casey and Horii (forthcoming) when the number of production factors is 3. They provide complete proof in the case of an arbitrary number of factors. They also explain how $\widehat{F}$ can be obtained when an economist has access to the shape of the true production function F_t at a point in time. Let this point $t = 0$. Then, in our 3-factor setting, function $\widehat{F}(M, N)$ is obtained as $F_0\left(X_0^{-1/\alpha} M^{1/\alpha}, N, X_0\right)$ if F_0, the true production function at time 0, is known.

[4] The rate of capital-augmenting technological change can be measured by the decline in the relative price of capital. By redefining the units of capital, it can also be interpreted as investment-specific technological change.

Thus, the BGP has positive capital-augmenting technological change ($\gamma_K > 1$) as long as the effective labor is growing faster than the effective input of the third factor ($\gamma_L g_L > \gamma_X g_X$). Moreover, the representation preserves the elasticity of substitution of the original production function, including $\sigma_{KL,t} \neq 1$. Therefore, the generalized Uzawa growth theorem provides an economist with a representation of the evolving production function that is simultaneously consistent with evidence on the non-unitary EoS between capital and labor and on capital-augmenting technological change. This is an improvement over the existing specifications of technological change, which can match only one of the two properties.

The aim of this paper is twofold. First, we demonstrate how the generalized Uzawa growth theorem can be utilized to build an endogenous growth model that exhibits capital-augmenting technological change in the long run. We do so starting in the next section. The second objective is to show technology condition (5), or equivalently its log-linear version (9), is naturally satisfied in the long-term equilibrium of the endogenous growth model. This condition means that effective inputs to function $\widehat{F}$ grows in a "balanced" way.[5] If the values of γ_K, γ_L and γ_X are exogenously given, then it is necessary to impose the log-linear relationship explained in Proposition 1. In other words, it is necessary to place an extra restriction on the model. This paper shows that when the rates of factor-augmenting technological change are endogenous, the log-linear relationship is endogenously satisfied in the long run, and the economy converges to a BGP with capital-augmenting technological progress.

Our result will contribute to a wide class of macroeconomic analysis. Even when an economist is not interested in building an endogenous growth model, our result provides a justification for assuming exogenous technological change that satisfies condition (5) and, therefore, using the factor-augmenting representation (6) as a good approximation of the true evolving production function in economic analysis.[6]

[5] The first part of Proposition 1 is directly derived from this balanced-ness. It can be shown that $g = g_K$ holds on any balanced growth path for a neoclassical model satisfying Definition 1. Then, technology condition (5) ensures that both arguments of $\widehat{F}$ grow at the same constant factor g. Since function $\widehat{F}$ has constant returns to scale, the RHS of (6) grow at the same rate of the output $Y_t = F(K_t, L_t, X_t)$. By appropriately defining the level of $\widehat{F}$, equation (6) holds for all t on the BGP.

[6] In this simplified version of the theorem, the assumption of $\sigma_{KX,t} = 1$ might also seem restrictive. However, in Casey and Horii (forthcoming), they showed that a similar theorem can be obtained in an environment with many input factors, and the requirement is that at least one factor

3 Model

In this section, we build an endogenous growth model in which firms undertake R&D investments to improve factor-augmenting technologies. From Proposition 1, we know that if there is a third production factor X such that $\sigma_{KX} = 1$, then technological change on the BGP can be approximated by a representation in the form of (6), which allows for capital-augmenting technological change. Since we want to build a model that is consistent with a BGP with capital-augmenting technological change, we provide a simple setting in which the aggregate production function takes the form of (6). In this model, we will later confirm that technology condition (5) is satisfied endogenously as a long-rem equilibrium outcome.

3.1 Structure

There are non-overlapping generations of representative firms, each of which exists for only one period. A representative firm performs two types of tasks, M-tasks and N-tasks.[7] The number of M-tasks, as well as that of N-tasks, determines the amount of final output. The M-tasks require effective capital $A_{K,t}K_t$ and effective natural resource $A_{X,t}X_t$ as inputs. X_t is composed of production factors that are not included in the conventional definition of (reproducible) capital K_t and labor L_t. Examples are land and energy. The number of M-tasks it can complete is given by

$$M_t = \left(A_{K,t}K_t\right)^{\alpha}\left(A_{X,t}X_t\right)^{1-\alpha}, \alpha \in (0,1). \tag{10}$$

We refer to the RHS as the aggregate amount of the *capital composite*, which combines effective capital and effective natural resources with unit elasticity. An N-task uses only effective labor, $A_{L,t}L_t$, where $A_{L,t}$ is the labor-augmenting technology of the representative firm. The number of N-tasks is simply

$$N_t = A_{L,t}L_t. \tag{11}$$

X_j has $\sigma_{KX_j} = 1$. They discuss the likely candidates for factor X, such as energy and land, referring to evidence found in empirical studies.

[7] This setting is first considered by Irmen (2017) and Irmen and Tabaković (2017), and we expanded it to incorporate three production factors. A benefit of using the model of tasks is that it can incorporate R&D activity within a perfectly competitive economy. Another benefit is that the model is scale independent.

By performing M_t and N_t tasks, the representative firm produces

$$Y_t = \widehat{F}(M_t, N_t) = \widehat{F}\Big(\big(A_{K,t}K_t\big)^{\alpha}\big(A_{X,t}X_t\big)^{1-\alpha}, A_{L,t}L_t\Big) \tag{12}$$

units of output, where $\widehat{F}(\cdot)$ is a standard neoclassical production function that has CRS and satisfies the Inada conditions.[8]

Now, we explain how the factor-augmenting technologies $\{A_{K,t}, A_{X,t}, A_{L,t}\}$ are determined. Technical knowledge can be kept within the firm for only one period, after which it becomes public. Thus, the representative firm at time t can freely use the technology of the period $t-1$ firm, $\{A_{K,t-1}, A_{X,t-1}, A_{L,t-1}\}$. In addition, the period t firm can improve each of factor-augmenting technologies through R&D. Tasks are differentiated and require separate R&D investments.[9] Specifically, for each factors $Z = K, X, L$, the technology follows $A_{Z,t} = (1 + a_Z(i_{Z,t}))A_{Z,t-1}$, where i_Z is the amount of investments in final goods. We assume that function $a_Z(\cdot)$ satisfies following properties: $a'_Z > 0$, $a''_Z < 0$, $a_Z(0) = 0$, $a_Z(\infty) = \infty$, and $a'_Z(0) = \infty$.[10]

It is convenient to define the R&D cost function by $i_Z(\gamma_Z) = a_Z^{(-1)}(\gamma_Z - 1)$, where $a_Z^{(-1)}(\cdot)$ is the inverse function of $a_Z(\cdot)$. To improve the Z-augmenting technology in a task by a factor of γ_Z, it costs $i_Z(\gamma_Z)$ in final goods. The properties of $a_Z(\cdot)$ imply

$$i'_Z > 0, i''_Z > 0, i_Z(1) = 0, i_Z(\infty) = \infty, i'_Z(1) = 0 \text{ for } Z = K, X, L. \tag{13}$$

Adding up the R&D costs for all tasks and technologies, the total R&D cost for the representative firm is

$$R_t = M_t \cdot \left(i_K\left(\frac{A_{K,t}}{A_{K,t-1}}\right) + i_X\left(\frac{A_{X,t}}{A_{X,t-1}}\right)\right) + N_t \cdot i_L\left(\frac{A_{L,t}}{A_{L,t-1}}\right). \tag{14}$$

[8] There are two ways to represent the production function in an intensive form: $f(M/N) = \widehat{F}(M/N, 1)$ and $h(N/M) = \widehat{F}(1, N/M)$. We assume that both $f(\cdot)$ and $h(\cdot)$ satisfy the Inada conditions.

[9] From the symmetry of tasks within each group (M or N) and from the convexity of the R&D cost functions as assumed in (13), it is always optimal to choose the same levels of $A_{K,t}$, $A_{X,t}$, and $A_{L,t}$ across individual tasks. Therefore, we omit subscripts for technologies for individual tasks.

[10] The declining marginal in the R&D function, $a''_Z < 0$, can be explained by congestion in R&D activities. When many researchers are devoted to improvements in the same task at the same time, some of them will end up inventing the same innovation. The risk of duplication becomes more prominent as R&D inputs increase. See Horii and Iwaisako (2007) for a simple micro foundation.

The objective of the representative firm is to maximize the single period profit net of R&D costs, because it lives only for one period and its knowledge will become public next period. By taking the output in each period as *numéraire*, the period profit is given by

$$\pi_t = \widehat{F}(M_t, N_t) - R_t - r_t K_t - \tau_t X_t - w_t L_t, \tag{15}$$

where r_t, τ_t, and w_t are interest rate, payment for a unit of natural resources (e.g., land rent), and wage rate, respectively.

The demand side of the economy is standard. There is a representative household. The size of the representative household (i.e., population) evolves according to[11]

$$L_t = L_0 g_L^t,\ L_0 > 0,\ g_L > 1 - \delta : \text{given}. \tag{16}$$

As in the Ramsey-Cass-Koopman model, the period utility of the household is given by the product of the number of household members and the per capita period felicity function:

$$u_t = L_t u(C_t/L_t), \tag{17}$$

where $C_t/L_t > 0$ is per capita consumption. We assume the felicity function $u(\cdot)$ takes the CRRA form. Then, the intertemporal objective function of the household can be written as

$$U = \sum_{t=0}^{\infty} L_t \beta^t \frac{(C_t/L_t)^{1-\theta} - 1}{1-\theta}, \tag{18}$$

where $\theta > 0$ is the degree of the relative risk aversion (i.e., the inverse of the intertemporal elasticity of substitution) and $\beta > 0$ is the discount factor. We later discuss the upper bound for β in Proposition 3.

The representative household owns capital, K_t, and natural resources, X_t, in addition to labor, L_t. The household also owns the representative firm and receives the profit, π_t, although, in equilibrium, profits will be zero due to perfect competition and free entry.[12] For simplicity, we assume that the supply of natural resources is

[11] $g_L > 1 - \delta$ is commonly assumed in neoclassical growth models to avoid a degenerate BGP.

[12] Note that R&D cost R_t is already subtracted from profit π_t. In addition, the firm can retain the rent from R&D only for one period. Therefore, the firms are indifferent to entering the market when $\pi_t = 0$.

exogenous:[13]

$$X_t = X_0 g_X^t,\ X_0,\ g_X > 0:\ \text{given.} \tag{19}$$

As in the case of population, its available quantity can be either constant $g_X = 1$, shrinking $g_X \in (0,1)$, or growing $g_X > 1$. The budget constraint of the household is

$$K_{t+1} = (r_t + 1 - \delta)K_t + \tau_t X_t + w_t L_t + \pi_t - C_t, \quad K_0 > 0 : \text{given,} \tag{20}$$

where physical capital accumulates through the savings of the household.[14] The household is subject to the no-Ponzi game condition. Specifically, the present value of its asset holding as $T \to \infty$ should not be negative:

$$\lim_{T\to\infty} \left(\prod_{t=1}^{T} (r_t + 1 - \delta) \right)^{-1} K_{T+1} \geq 0. \tag{21}$$

This completes the description of the model economy.

3.2 R&D by Firms and the Direction of Technological Progress

Now we examine the behavior of the representative firm, focusing on the role of R&D. The representative firm maximizes profit (15) subject to production function (12) and R&D cost (14) with respect to $\{K_t, X_t, L_t, A_{K,t}, A_{X,t}, A_{L,t}\}$, taking as given prices, $\{r_t,\ \tau_t,\ w_t\}$, and lagged technology levels, $\{A_{K,t-1},\ A_{X,t-1},\ A_{L,t-1}\}$. For convenience, we define $\mu_t \equiv M_t/N_t$, which is the relative task intensity in final good production. It also represents the ratio of effective capital composite to effective labor

[13] The factor share of natural resources is around 8-9% (Caselli and Feyrer, 2007), of which about 5% is from land (Valentinyi and Herrendorf, 2008). Since the supply of land is mostly constant, we assume X_t is exogenous in this baseline scenario. Our theory is also applicable to the case where X_t is depleted or expanded endogenously (See robustness scenario f in Section 4.1). Note that, although X_t is exogenous, its effective amount $A_{X,t}X_t$ as a production factor can be enhanced endogenously through R&D for A_X.

[14] Using (15), equation (20) becomes $K_{t+1} = \widehat{F}(M_t, N_t) - C_t - R_t + (1-\delta)K_t$. Since $Y_t = \widehat{F}(M_t, N_t)$, the evolution of capital in this model is exactly the same as (2) in Definition 1. The aggregate production function (12) also conforms to Definition 1. Therefore, the model in this section is a 3-factor neoclassical growth model. Moreover, the form of aggregate production (12) is exactly the same as the factor-augmenting representation (6) in Proposition 1. We did this modeling choice because we already know that a 3-factor production function (with $\sigma_{KX} = 1$) can be represented in the form of (6) whenever it has a BGP with capital-augmenting change, which is observed in the data.

$\mu_t = \left(A_{K,t}K_t\right)^{\alpha}\left(A_{X,t}X_t\right)^{1-\alpha}/A_{L,t}L_t$. Then, because $\widehat{F}(\cdot)$ in (12) is a CRS function, we can write it in intensive form, $\widehat{F}(M_t, N_t)/N_t = \widehat{F}(\mu_t, 1) \equiv f(\mu_t)$, $\widehat{F}_M(M_t, N_t) = f'(\mu_t)$, and $\widehat{F}_N(M_t, N_t) = f(\mu_t) - \mu_t f'(\mu_t)$.[15]

Using this notation, we can conveniently express the first-order conditions for factor demand. The firm demands capital, natural resources, and labor so as to satisfy[16,17]

$$r_t = (\alpha M_t/K_t)\left(f'(\mu_t) - i_K(\gamma_{K,t}) - i_X(\gamma_{X,t})\right), \tag{22}$$

$$\tau_t = ((1-\alpha)M_t/X_t)\left(f'(\mu_t) - i_K(\gamma_{K,t}) - i_X(\gamma_{X,t})\right), \tag{23}$$

$$w_t = A_{L,t}(f(\mu_t) - \mu_t f'(\mu_t) - i_L(\gamma_{L,t})). \tag{24}$$

Now, we turn to R&D, starting with the condition for improving the labor-augmenting technology $A_{L,t}$. The representative firm chooses $A_{L,t}$, or equivalently the speed of technological progress $\gamma_{L,t} \equiv A_{L,t}/A_{L,t-1} \geq 1$, to maximize the profit. The first order condition is [18]

$$\gamma_{L,t} i'_L(\gamma_{L,t}) + i_L(\gamma_{L,t}) = f(\mu_t) - \mu_t f'(\mu_t). \tag{25}$$

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[15] $\widehat{F}_M(\cdot)$ and $\widehat{F}_N(\cdot)$ represent the partial derivatives of function $\widehat{F}(\cdot)$ with respect to its first and second arguments, respectively.

[16] The RHS of (22) represents the (net) marginal product of K_t in producing output Y_t. It is given by the product of two parts. The first part, $\alpha M/K$, is the marginal product of K_t in increasing the number of M-tasks performed in the firm. The second part is the net marginal product of M_t in producing the final output. Note that, in the second part, the innovation cost for an M-task, $i_K(\gamma_{K,t}) + i_X(\gamma_{X,t})$, is subtracted from the "gross" marginal product of M_t, $f'(\mu_t)$. When the firm performs more M-tasks, it chooses to pay R&D costs to increase $A_{K,t}$ and $A_{X,t}$ in these tasks so as to keep up with other M-tasks. Similarly, in (23), $(1-\alpha)M/X$ is the marginal product of X_t in performing more M-tasks.

[17] By substituting (22), (23), and (24) into (15), it can be confirmed that the firm achieves zero profit, $\pi_t = 0$. This is due to the CRS property of the firm's problem.

[18] The firm's private benefit from improving technology $A_{L,t}$ is the ability to perform a larger number of N-tasks, which increases the final output $Y_t = \widehat{F}(M_t, N_t)$. The RHS of (25) shows the marginal benefit, $\widehat{F}_N(M_t, N_t) = f(\mu_t) - \mu_t f'(\mu_t)$. The LHS corresponds to the marginal cost of performing a larger number of N-tasks through augmenting labor efficiency $A_{L,t}$ (given labor employment L_t). This can be broken into two components. First, by intensifying the R&D efforts in existing N-tasks to raise labor efficiency, the representative firm can decrease labor inputs by just enough to perform one additional N-task. The cost associated with this activity is given by the first term $\gamma_{L,t} i'_L(\gamma_{L,t})$, which we call the *intensive* marginal R&D cost. The saved labor is then used to perform a new N-task, which means the representative firm needs to invest in R&D for one more N-task, which costs $i_L(\gamma_{L,t})$. This *extensive* marginal R&D cost is represented by the second term in the LHS.

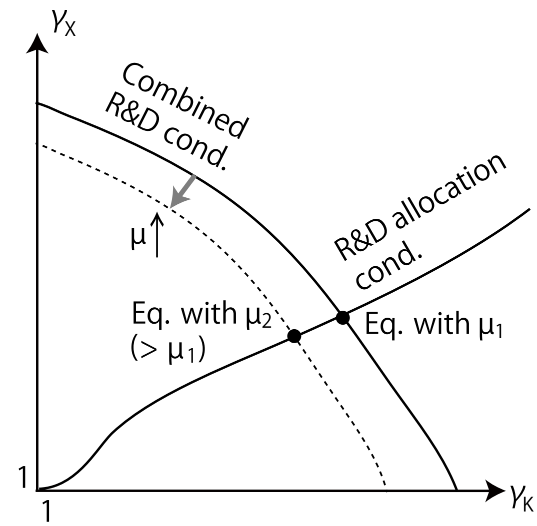


Figure 1: Equilibrium innovation in K- and X-augmenting technologies.
γ_K and γ_X are determined by the intersection of the combined R&D and R&D allocation conditions. When μ_t increases from μ_1 to μ_2, both γ_K and γ_X decrease.

As we formally prove in Proposition 2 below, condition (25) has a unique solution for $\gamma_{L,t}$ as a function of $\mu_t = M_t/N_t$, and it is strictly increasing in μ_t. Intuitively, a high value of $\mu_t \equiv M_t/N_t$ means that the resources to perform N-tasks (i.e., effective labor) are relatively scarce. Then, the marginal product of an N-task is higher, and therefore the benefit of improving $A_{L,t}$ to increase N_t is larger. Therefore, the firm chooses a larger $\gamma_{L,t}$ when μ_t is higher.

Next, the first-order conditions for $A_{K,t}$ and $A_{X,t}$ yield[19]

$$(\gamma_{K,t} i'_K(\gamma_{K,t}) + i_K(\gamma_{K,t})) + (\gamma_{X,t} i'_X(\gamma_{X,t}) + i_X(\gamma_{X,t})) = f'(\mu_t). \tag{26}$$

$$\frac{\gamma_{K,t} i'_K(\gamma_{K,t})}{\gamma_{X,t} i'_X(\gamma_{X,t})} = \frac{\alpha}{1-\alpha}, \tag{27}$$

Condition (26) specifies the optimal combined size of R&D investments.[20] Since

[19]The first order condition for $A_{K,t}$ yields $(\gamma_{K,t}/\alpha) i'_K(\gamma_{K,t}) + i_K(\gamma_{K,t}) + i_X(\gamma_{X,t}) = f'(\mu_t)$, whereas that for $A_{X,t}$ gives $(\gamma_{X,t}/1-\alpha) i'_X(\gamma_{X,t}) + i_K(\gamma_{K,t}) + i_X(\gamma_{X,t}) = f'(\mu_t)$. Condition (27) is obtained by subtracting the second equation from the first. Condition (26) is from adding α times the first equation and $(1-\alpha)$ times the second equation.

[20]Capital and natural resources are used in M-tasks, and therefore improving K- and X-augmenting technologies will enable the firm to perform more M-tasks. This marginal benefit is represented by the RHS of (26), $f'(\mu_t) = \widehat{F}_M(M_t, N_t)$. The LHS is the marginal cost of R&D, which has two parts, $\gamma_{K,t} i'_K(\gamma_{K,t}) + i_K(\gamma_{K,t})$ and $\gamma_{X,t} i'_X(\gamma_{X,t}) + i_X(\gamma_{X,t})$, because both K- and

the LHS is increasing both in γ_{K_t} and $\gamma_{X,t}$, the locus of $(\gamma_{K,t}, \gamma_{X,t})$ that satisfies this condition is depicted by the downward-sloping curve, as depicted in Figure 1. Condition (27) gives the optimal allocation of R&D investment between K- and X-augmenting technologies. Observe that $\gamma_{K,t} i'_K(\gamma_{K,t})$ and $\gamma_{X,t} i'_X(\gamma_{X,t})$ on the LHS are strictly increasing in γ_{K_t} and $\gamma_{X,t}$, respectively. Therefore, this condition can be expressed as an upward-sloping curve in the $(\gamma_{K,t}, \gamma_{X,t})$ space.[21] The intersection of the R&D allocation condition and the combined R&D condition gives the optimal rates of innovation for K- and X-augmenting technologies. When $\mu_t = M_t/N_t$ is higher, the combined R&D condition curve locates closer to the origin. Then the values of γ_K and γ_X at the intersection are smaller. Intuitively, When the resources for M-tasks, i.e., effective capital and effective natural resources, are relatively ample, the marginal product of an M-task is smaller. Then, the firm has less incentive to improve γ_K and γ_X. The proposition below summarises the results.

Proposition 2. (Direction of Technological Change)
In the endogenous growth model defined in Section 3.1, the growth factors of each of the factor augmenting technologies are uniquely determined as a function of $\mu_t = M_t/N_t \in (0, \infty)$*. Let us denote by them by* $\widehat{\gamma}_K(\mu_t)$*,* $\widehat{\gamma}_X(\mu_t)$*, and* $\widehat{\gamma}_L(\mu_t)$*. Then,*
(a) The signs of $(\widehat{\gamma}'_K(\mu), \widehat{\gamma}'_X(\mu), \widehat{\gamma}'_L(\mu))$ *are* $(-, -, +)$ *for all* $\mu \in (0, \infty)$*.*
(b) $\lim_{\mu \to 0} (\widehat{\gamma}_K(\mu), \widehat{\gamma}_X(\mu), \widehat{\gamma}_L(\mu)) = (\infty, \infty, 1)$*.*
(c) $\lim_{\mu \to \infty} (\widehat{\gamma}_K(\mu), \widehat{\gamma}_X(\mu), \widehat{\gamma}_L(\mu)) = (1, 1, \infty)$*.*

Proof. See Appendix A.2. □

Figure 2 illustrates the direction of the technological change in 3-dimensional space. The γ_K-γ_X plane depicted at the bottom of the figure is the same as Figure 1. The equilibrium direction is obtained by extending vertically from the intersecting

X-augmenting technologies receive some R&D according to the allocation condition (27). In each of the two parts, the first term represents the intensive marginal R&D cost, whereas the second term is the extensive marginal R&D cost, as in condition (25).

[21] As the RHS of condition (27) shows, the allocation should depend on the relative contribution of capital and natural resources in performing M-tasks. When capital's relative contribution is higher (i.e., when α is higher), more resources should be allocated to R&D for the capital-augmenting technology. In addition, the slope and convexity of the R&D cost function also affect the optimal allocation. For example, if it is relatively difficult to improve the efficiency of natural resources, i.e., if the marginal R&D cost $i'_X(\gamma_{X,t})$ increases more rapidly with its argument than $i'_K(\gamma_{K,t})$, then it is optimal not to improve $A_{X,t}$ as fast as $A_{K,t}$.

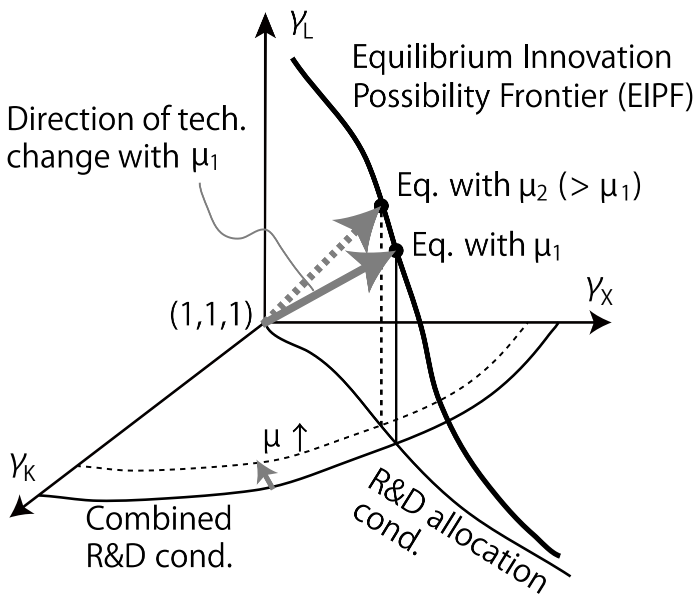


Figure 2: The direction of technological change and the Equilibrium Innovation Possibility Frontier (EIPF) curve.
The direction rotates toward the vertical axis along the EIPF curve as μ_t increases from μ_1 to μ_2.

point in the γ_K-γ_X plane by the distance of L-augmenting innovation, $\widehat{\gamma}_L(\mu_t) - 1$. As μ_t increases, the combined R&D locus shifts inward,[22] which lowers $\gamma_{K,t}$ and $\gamma_{X,t}$. At the same time $\gamma_{L,t}$ increases because $\widehat{\gamma}'_L(\mu_t) > 0$. This way, the direction of the technological change from the solid arrow to the dashed arrow in Figure 2. In general, property (a) in Proposition 2 says that when capital composite becomes more abundant relative to effective labor (i.e. when μ_t is higher), the direction of technological change becomes more upright, depressing improvements in technologies that enhance capital composite while enhancing improvements in labor-augmenting technology. In other words, firms are 'induced' to do more innovation that enhances the relatively scarce effective production factors.[23]

The thick downward-sloping curve in Figure 2 depicts the locus of all equilibrium points that correspond to various values of μ_t. This is the equilibrium innovation possibility frontier (EIPF).[24] As μ_t changes, the equilibrium direction of technological

[22]The shift occurs because the RHS of (26) is decreasing in μ_t.
[23]This notion of induced innovation was first introduced by Hicks (1932). See Acemoglu (2002) for further discussion.
[24]In this model, not only the direction within the EIPF, but also the EIPF itself is determined

changes moves along the EIPF curve. Property (b) in Proposition 2 says that when the capital composite is almost zero relative to effective labor, all the R&D efforts are directed to enhancing technologies that are related to capital composite. In this extreme case, the direction coincides with the vertical axis. On the contrary, property (c) implies that, when μ_t is almost 0, the direction rotates toward the R&D allocation condition curve and becomes almost flat. Either way, the economy moves away from the corner solution.

To summarize, when the direction of technological change is chosen by the representative firm in this model, the direction is adjusted so that the ratio of two inputs to function $\widehat{F}$, i.e., M_t and N_t is stabilized. This tendency provides a significant force to achieve balanced growth in the long run. Yet, we also need to solve the full equilibrium dynamics to see how (reproducible) capital is accumulated through saving and investment decisions. This is the theme of the next subsection.

3.3 Equilibrium Dynamics

The equilibrium path of this economy is given by the sequence of output, consumption, production factors, technologies, and R&D investments, $\{Y_t, C_t, K_t, X_t, L_t, A_{K,t}, \ldots A_{X,t}, A_{L,t}, R_t\}_{t=0}^{\infty}$, which satisfy the representative firm's optimization problem, the representative consumer's utility maximization problem, and the market clearing conditions for output and production factors. The economy is endowed with K_0, X_0 and L_0 at time 0, as well as the initial levels of publicly available technologies, $A_{K,-1}$,$A_{X,-1}$ and $A_{L,-1}$.

While the equilibrium involves many variables, we can analytically characterize its dynamic path in terms of only three: relative task intensity $\mu_t = M_t/N_t$, the amount of capital per effective labor $k_t \equiv K_t/A_{L,t}L_t$, and consumption per effective labor $c_t \equiv C_t/A_{L,t}L_t$. Below, we construct the equilibrium mapping from $\{\mu_t, k_t, c_t\}$ to $\{\mu_{t+1}, k_{t+1}, c_{t+1}\}$ for $t \geq 0$. The mapping and the initial conditions μ_0 and k_0, together with the transversality condition for c_t, will pin down the equilibrium path

endogenously from the firm's profit condition. In most models of the direction of technological change, it is assumed that innovation requires a certain type of exogenously given resource (e.g., scientists). In these cases, the innovation possibility frontier is derived from the resource constraint. On the contrary, in our model, the total amount of R&D input (R_t) is determined in equilibrium through profit maximization, and hence the frontier is called the 'equilibrium' innovation possibility frontier. Any innovation beyond this frontier is not profitable, although it might be feasible.

of $\{\mu_t, k_t, c_t\}$, from which the path of all variables in the model can be recovered.

Before so doing, it is convenient to define the net aggregate output in the economy as $V_t = \widehat{F}(M_t, N_t) - R_t$, which means the aggregate output minus the total R&D costs in the economy. The net output per effective labor can be written as a function of μ_t:

$$V_t/N_t = f(\mu_t) - \mu_t(i_K(\widehat{\gamma}_K(\mu_t)) + i_X(\widehat{\gamma}_X(\mu_t))) - i_L(\widehat{\gamma}_L(\mu_t)) \equiv v(\mu_t). \tag{28}$$

Then, substituting profits (15) into the budget constraint (20), we can express the growth of aggregate capital supply in terms of μ_t, k_t and c_t:

$$\frac{K_{t+1}}{K_t} = \frac{V_t + (1-\delta)K_t - C_t}{K_t} = \frac{v(\mu_t) - c_t}{k_t} + 1 - \delta. \tag{29}$$

Dynamics for $\boldsymbol{\mu_{t+1}}$. The growth factor of μ_{t+1} is defined by $\mu_{t+1}/\mu_t = (M_{t+1}/M_t)/(N_{t+1}/N_t)$. By using (10), (11), (16), (19) and (29), its value in equilibrium can be written as

$$\frac{\mu_{t+1}}{\mu_t} = \frac{(g_X \widehat{\gamma}_X(\mu_{t+1}))^{1-\alpha}}{g_L \widehat{\gamma}_L(\mu_{t+1})} \left(\widehat{\gamma}_K(\mu_{t+1}) \left(\frac{v(\mu_t) - c_t}{k_t} + 1 - \delta \right) \right)^{\alpha}, \tag{30}$$

where $\widehat{\gamma}_K(\mu_t)$, $\widehat{\gamma}_X(\mu_t)$, and $\widehat{\gamma}_L(\mu_t)$ are the rates of technological progress defined in Proposition 2. While equation (30) gives a relationship between the period-t variables $\{\mu_t, k_t, c_t\}$ and μ_{t+1}, it is not easy to understand how μ_{t+1} is determined since both sides of the equation depend on μ_{t+1}.

To obtain more straightforward dynamics, let us decompose the dynamic relationship in (30) into two steps. First, we define the pre-R&D relative factor intensity by

$$\mu_{t+1}^{\text{pre}} \equiv \frac{\left(A_{K,t} K_{t+1}\right)^{\alpha} \left(A_{X,t} X_{t+1}\right)^{1-\alpha}}{A_{L,t} L_{t+1}} = \frac{g_X^{1-\alpha}}{g_L} \left(\frac{v(\mu_t) - c_t}{k_t} + 1 - \delta \right)^{\alpha} \mu_t, \tag{31}$$

where the last equality is from (16), (19), (29) and the definition of μ_t. It is the value of μ_{t+1} before technologies are improved from their period-t state. Second, μ_{t+1}^{pre} and the post-R&D value of μ_{t+1} are related by the growth of technological levels $\widehat{\gamma}_K(\mu_t)$, $\widehat{\gamma}_X(\mu_t)$, and $\widehat{\gamma}_L(\mu_t)$ as follows:

$$\mu_{t+1}^{\text{pre}} = \frac{\widehat{\gamma}_L(\mu_{t+1})}{\widehat{\gamma}_K(\mu_{t+1})^{\alpha} \widehat{\gamma}_X(\mu_{t+1})^{1-\alpha}} \mu_{t+1} \equiv \Gamma(\mu_{t+1}). \tag{32}$$

Note that, Proposition 2 implies that function $\Gamma(\mu_{t+1})$ is a strictly increasing differentiable function with $\lim_{\mu \to 0} \Gamma(\mu) = 0$ and $\lim_{\mu \to \infty} \Gamma(\mu) = \infty$. Therefore, its inverse function $\mu_{t+1} = \Gamma^{(-1)}(\mu_{t+1}^{\text{pre}})$ is well-defined for all $\mu_{t+1}^{\text{pre}} > 0$, and is a strictly increasing differentiable function.

Using (31) and the inverse function of (32), the dynamic relationship (30) can be written as

$$\mu_{t+1} = \Gamma^{(-1)} \left(\frac{g_X^{1-\alpha}}{g_L} \left(\frac{v(\mu_t) - c_t}{k_t} + 1 - \delta \right)^{\alpha} \mu_t \right) \equiv \psi^{\mu}(\mu_t, k_t, c_t). \tag{33}$$

Function $\psi^{\mu}(\mu_t, k_t, c_t)$ gives a mapping from period-t variables μ_t, k_t, c_t to μ_{t+1}. It provides a natural 2-step interpretation of the equivalent equation (30). The argument of function $\Gamma^{(-1)}(\cdot)$ in (33) represents the pre-R&D relative task intensity, which is determined by the relative supply of production factors, as well as the period-t technology levels. Then, function $\Gamma^{(-1)}(\cdot)$ describes how R&D in period $t+1$ transforms the relative task intensity.

Dynamics for $\mathbf{k_{t+1}}$. From (16) and (29), the growth factor of $k_t \equiv K_t / A_{L,t} L_t$ is obtained as

$$\frac{k_{t+1}}{k_t} = \frac{1}{g_L \widehat{\gamma}_L(\mu_{t+1})} \left(\frac{v(\mu_t) - c_t}{k_t} + 1 - \delta \right). \tag{34}$$

While μ_{t+1} is present in the RHS, we can replace it with (33) so that the RHS depends only on the variables in period t.

$$k_{t+1} = \frac{1}{g_L \widehat{\gamma}_L(\psi^{\mu}(\mu_t, k_t, c_t))} \left(v(\mu_t) - c_t + (1-\delta) k_t \right) \equiv \psi^{k}(\mu_t, k_t, c_t). \tag{35}$$

This dynamic equation simply represents the process of capital accumulation per effective labor. The expression $(v(\mu_t) - c_t + (1-\delta)k_t)$ shows the sum of the net saving and the un-depreciated part of existing capital per effective labor in period t. It must be divided by $g_L \widehat{\gamma}_L$ because of the growth of effective labor between period t and $t+1$.

Dynamics for $\mathbf{c_{t+1}}$. The representative household maximizes the intertemporal utility function (18) subject to the budget constraint (20) and the non-Ponzi Game condition (21). The Euler equation for this problem is[25]

[25] Note that $\partial U / \partial C_t = \beta^t (C_t / L_t)^{-\theta}$ from From (18). From this, the Euler equation is $(C_t/L_t)^{-\theta} = (r_{t+1} + 1 - \delta) \beta (C_{t+1}/L_{t+1})^{-\theta}$, which simplifies to (36).

$$C_t^{-\theta} = (r_{t+1} + 1 - \delta)\beta g_L^\theta C_{t+1}^{-\theta}. \tag{36}$$

By substituting the market interest rate (22) into the Euler equation (36) and then applying it to the definition $c_t \equiv C_t/A_{L,t}L_t$, we obtain the growth factor of consumption per effective labor:

$$\frac{c_{t+1}}{c_t} = \frac{\beta^{1/\theta}}{\widehat{\gamma}_L(\mu_{t+1})}\left(\frac{\alpha\mu_{t+1}}{k_{t+1}}\left(f'(\mu_{t+1}) - i_K(\widehat{\gamma}_K(\mu_{t+1})) - i_X(\widehat{\gamma}_X(\mu_{t+1}))\right) + 1 - \delta\right)^{1/\theta}. \tag{37}$$

By replacing the period-$(t+1)$ variables in the RHS by (33) and (35), we can rewrite equation (37) as

$$\begin{aligned} c_{t+1} =& \frac{\beta^{1/\theta}c_t}{\widehat{\gamma}_L(\psi^\mu(\mu_t,k_t,c_t))}\left(\frac{\alpha\psi^\mu(\mu_t,k_t,c_t)}{\psi^k(\mu_t,k_t,c_t)}\Big(f'(\psi^\mu(\mu_t,k_t,c_t))\right.\\ &\left.- i_K(\widehat{\gamma}_K(\psi^\mu(\mu_t,k_t,c_t))) - i_X(\widehat{\gamma}_X(\psi^\mu(\mu_t,k_t,c_t)))\Big) + 1 - \delta\right)^{1/\theta} \equiv \psi^c(\mu_t,k_t,c_t). \end{aligned} \tag{38}$$

Equations (33), (35) and (38) constitute the equilibrium mapping from $\{\mu_t, k_t, c_t\}$ to $\{\mu_{t+1}, k_{t+1}, c_{t+1}\}$ for all $t \geq 0$.

Boundary Conditions. To obtain the equilibrium path of $\{\mu_t, k_t, c_t\}_{t=0}^\infty$, we need three boundary conditions. First, since K_0, X_0, L_0, $A_{K,-1}$,$A_{X,-1}$ and $A_{L,-1}$ are given, we can construct μ_0^{pre}, the pre-R&D relative task intensity for period 0. Using it with the inverse function of Γ from (32), we have the initial value of μ_t:

$$\mu_0 = \Gamma^{(-1)}\left(\frac{(A_{K,-1}K_0)^\alpha(A_{X,-1}X_0)^{1-\alpha}}{A_{L,-1}L_0}\right). \tag{39}$$

Second, using μ_0, the initial value of k_t is readily obtained by

$$k_0 = \frac{K_0}{\widehat{\gamma}_L(\mu_0)A_{L,-1}L_0}. \tag{40}$$

Finally, the initial value of c_t must be chosen so as to satisfy the non-Ponzi game condition (21) and the transversality condition

$$\lim_{T\to\infty} \beta^T \left(\frac{C_T}{L_T}\right)^{-\theta} K_{T+1} \leq 0. \tag{41}$$

Combining Euler equation (36) with (21) and (41) gives the unified terminal conditional

$$\lim_{T\to\infty} (\beta g_L)^T \left(\prod_{t=0}^{T} \widehat{\gamma}_L(\mu_t)\right)^{1-\theta} \widehat{\gamma}_L(\mu_{T+1}) c_T^{-\theta} k_{T+1} = 0. \tag{42}$$

The next subsection will show that the economy has a BGP that satisfies this terminal condition.

3.4 The Balanced Growth Path

Now, we are ready to characterize the BGP of this economy. We will show that the direction of technological progress is endogenously chosen so that in equilibrium there is a unique BGP with a positive rate of capital-augmenting technical change.

Lemma 1. *Define a BGP as an equilibrium path where the growth factors of* $\{Y_t, K_t, X_t, L_t, C_t, R_t, M_t, N_t\}$ *are all constant.*[26] *Then, on any BGP, the values of* μ_t, k_t *and* c_t *must be constant.*

Proof. See Appendix A.3. □

We denote the BGP values of μ_t, k_t and c_t by μ^*, k^* and c^*, respectively. Their values are obtained by substituting $\mu_{t+1} = \mu_t = \mu^*$, $k_{t+1} = k_t = k^*$ and $c_{t+1} = c_t = c^*$ into (33), (35), and (38).

First, from (33) and (35), the BGP value of $\mu_t \equiv M_t/N_t$ will satisfy

$$1 = \frac{(g_X \widehat{\gamma}_X(\mu^*))^{1-\alpha} (\widehat{\gamma}_K(\mu^*))^{\alpha}}{(g_L \widehat{\gamma}_L(\mu^*))^{1-\alpha}.} \equiv \Phi(\mu^*). \tag{43}$$

Proposition 2 implies $\Phi'(\mu^*) < 0$ with $\Phi(0) = \infty$ and $\Phi(\infty) = 0$. Therefore, there exists a unique value of $\mu^* > 0$ that satisfies $\Phi(\mu^*) = 1$, and hence condition (43).

[26] Here, we slightly extend Definition 2 by requiring constancy of the growth factors of M_t and N_t, i.e., the numbers of tasks performed in the economy.

An intuitive way to interpret (43) is to multiply both of its sides by $(g_L \widehat{\gamma}_L(\mu^*))^\alpha$.

$$(g_X \widehat{\gamma}_X(\mu^*))^{1-\alpha} (\widehat{\gamma}_K(\mu^*) g_L \widehat{\gamma}_L(\mu^*))^\alpha = g_L \widehat{\gamma}_L(\mu^*) \quad (= g^*). \tag{44}$$

The LHS represents the growth factor of M_t on the BGP, while the RHS is that for N_t. Therefore, this condition means that the relative factor intensity $\mu^* = M_t/N_t$ is determined so that M_t and N_t grow at the same speed. This condition singles out a point on the Equilibrium Innovation Possibility Frontier (recall Figure 2), which determines the direction of technological change on the BGP. Note that, due to the CRS property of production function $Y_t = \widehat{F}(M_t, N_t)$, the value of equation (44) also represents the economic growth factor $g^* \equiv Y_{t+1}/Y_t$.

Second, from the Euler equation (38), the BGP value of $k_t = K_t/(A_t L_t)$ is

$$k^* = \frac{\beta\alpha\mu^*(f'(\mu^*) - i_K(\widehat{\gamma}_K(\mu^*)) - i_X(\widehat{\gamma}_X(\mu^*)))}{\widehat{\gamma}_L(\mu^*)^\theta - \beta(1-\delta)}. \tag{45}$$

Intuitively, the capital-effective labor ratio on the BGP is determined from the interest rate r^* that yields constant consumption per effective labor on the BGP.[27] Third, from (35) and $g_L \widehat{\gamma}_L(\mu^*) = g^*$ in (44), the BGP value of $c^* = C_t/A_{L,t}L_t$ must satisfy

$$c^* = v(\mu^*) - (g^* - 1 + \delta)k^*. \tag{46}$$

These three equations describe the unique BGP in this economy. The following proposition shows that the BGP uniquely exists when the discount factor is sufficiently smaller than 1.[28]

[27] Using (22), condition (45) is shown to be equivalent to $r^* + 1 - \delta = \beta^{-1}\widehat{\gamma}_L(\mu^*)^\theta$. Here, the RHS is the marginal rate of intertemporal substitution given that consumption per effective labor is constant (which must be true on the BGP).

[28] There are two reasons why the existence of the BGP requires an upper bound for β (or, equivalently a lower bound for $\rho = (1-\beta)/\beta$). First, on the BGP, the amount of consumption for the household $C_t = A_{L,t}L_t c^*$ increases over time, causing the instantaneous utility to grow. Therefore, if β is too close to one, the intertemporal utility U in (18) becomes infinity, which means that the household's problem is not well defined. Second, as effective labor $A_{L,t}L_t$ grows, the household accumulates more capital K_t so as to prevent the dilution of capital per effective labor, k^*. However, when β is too large (i.e. when the discount rate ρ is too small), the BGP requires a too-low real interest rate, or a too-high level of k^*, to the extent that preventing the dilution is impossible even when all net output is invested in K_t. We rule out these extreme cases by assuming an upper bound for β.

Proposition 3. *There exists a value of $\overline{\beta} > 0$ such that whenever $\beta \in (0, \overline{\beta})$, there exists a unique BGP that satisfies $\mu^* > 0$, $k^* > 0$, $c^* > 0$, and the terminal condition (42). On this BGP, the long-term rate of capital-augmenting technological change is positive ($\gamma_K(\mu^*) > 1$).*

Proof. See Appendix A.4. The exact expression for the upper bound $\overline{\beta}$ is given by (A.8). $\square$

An important implication from this model is that the technology condition (9) in Section 2 is now an endogenous outcome. Specifically, the BGP condition (43) is equivalent to (9), except that the speed of technological progress is endogenously determined by profit-maximizing producers. This difference has important implications for the plausibility of capital-augmenting technological progress on the BGP. As discussed in Section 2, if the rates of innovation for the three factor-augmenting technologies are exogenously given, then (9) becomes a knife-edge condition. In contrast, this section has shown that, once we consider endogenous technical change, this condition is necessarily satisfied when the economy is on the BGP, which exists if discount factor β is sufficiently less than one.

A missing link is that we have not yet shown whether the economy actually converges to this BGP. If it is shown, we can conclude that condition (9) in the generalized Uzawa growth model is naturally satisfied in the long run, greatly widening the plausibility of the theorem as an explanation of the capital-augmenting technological change observed in data. We do so in the next section.

4 Numerical Analysis and Convergence to the BGP

In this section, we investigate the local and global stability of the three-factor endogenous growth model. Our primary objective is to show that the model economy converges to a BGP with capital-augmenting technical change, where log-linear relationship (9) is endogenously satisfied. We also illustrate how having multiple technologies (including K-augmenting technology) affects the transition dynamics. To accomplish these goals, we present a series of numerical examples for which we can check stability computationally. Whenever possible, we ensure that our numerical examples are consistent with macroeconomic data characterizing the BGP of the

United States. The data sources are provided in Appendix C. We stress, however, that this is not a complete calibration, and the results would be insufficient for a precise quantitative analysis.

4.1 Calibration

Functional Forms

We assume that the aggregate production function takes a CES form: $\widehat{F}(M_t, N_t) = (\eta M_t^{\frac{\epsilon-1}{\epsilon}} + N_t^{\frac{\epsilon-1}{\epsilon}})^{\frac{\epsilon}{1-\epsilon}}$, where $\epsilon > 0$ and $\eta \in (0,1)$. Output in the economy (12) can be written as

$$Y_t = \left\{\eta \left((A_{K,t}K_t)^{\alpha} (A_{X,t}X_t)^{1-\alpha}\right)^{\frac{\epsilon-1}{\epsilon}} + (1-\eta)(A_{L,t}L_t)^{\frac{\epsilon-1}{\epsilon}}\right\}^{\frac{\epsilon}{1-\epsilon}}. \tag{47}$$

Next, we assume that R&D costs are power functions[29]

$$i_Z(\gamma_Z) = \zeta_Z\big(\gamma_Z - 1\big)^{\lambda}, \quad \zeta_Z > 0,\ \lambda > 1,\ Z = K, X, L. \tag{48}$$

We allow R&D cost parameter ζ_Z to differ across types of technology. We normalize ζ_K to 1, and calibrate ζ_X and ζ_L. The degree of convexity, λ, is assumed to be the same across the three types of technology.

With these functional form assumptions, our model has 11 parameters, $\{\epsilon, \eta, \alpha, \lambda, \zeta_L, \zeta_X, \beta, \theta, \delta, g_L, g_X\}$. To calibrate the model, we also need to determine the period length, measured in years, denoted by χ. The period length in our model has an important economic meaning because it represents the duration for which a firm can monopolize the benefit from its R&D investments. Including χ, we have 12 parameters.

Exogenous Parameters

We set five parameters exogenously. Their values are given in Table 1. In the CES production function (47), we take $\epsilon = 0.7$ as the baseline value. This is a common estimate for the EoS between labor and reproducible capital (e.g., Antras et al., 2004; Oberfield and Raval, 2014). The mapping between these estimates and a structural parameter in our model is not exact, and we show robustness with $\epsilon = 0.9$ and

[29]Note that function (48) satisfies condition (13).

Parameter	Baseline	Alternate	Description	Explanation / Source
ϵ	0.7	0.9, 1.2	EoS b/w M and N	Oberfield and Raval (2014)
λ	2.0	1.5, 2.25	R&D Cost convexity	Quadratic
θ	1.0	0.5, 2	Inverse of IES	Log Preferences
$g_L^{1/\chi}$	1.01		Population growth	BEA 1960-2020 average
$g_X^{1/\chi}$	1.0	0.99	Growth of X	Fixed Supply of Land

Table 1: Exogenous Parameters

Target Moment (in annual values)	Value	Model Variable	Source
Capital output ratio	2.9	$K/(Y/\chi)$	BEA 1960-2020 average
Labor share of income	63%	$\kappa_L \equiv wL/Y$	BEA 1960-2020 average
Share of R&D payments in GDP	2.7%	$\kappa_R \equiv \frac{R}{Y}$	BEA 1960-2020 average
Consumption of fixed capital in GDP	14%	$\delta K/Y$	BEA 1960-2020 average
Growth rate of income per capita	1.9%	$\gamma_L^{1/\chi} - 1$	BEA 1960-2020 average
Decline in the relative price of capital	0.66%	$\gamma_K^{1/\chi} - 1$	BEA 1960-2020 average
Return on investment	4%	$(1 + r - \delta)^{1/\chi} - 1$	McGrattan *et al.* (2003)

Table 2: Target Moments for Calibration

$\epsilon = 1.2$.[30] In the baseline calibration, we assume the R&D cost function is quadratic ($\lambda = 2$) and also check robustness with $\lambda = 1.5$ and $\lambda = 2.25$. Quadratic cost is a common assumption, and it is consistent with existing empirical work in endogenous growth (Acemoglu et al., 2018; Akcigit and Kerr, 2018). As for utility function (18), we take log preferences ($\theta = 1$) as the baseline and also consider cases where the intertemporal EoS is higher or lower than 1 ($\theta = 0.5$ and 2). Population growth is set to the 1960-2020 average in the U.S. (1% per year). When one period in the model corresponds to χ years, this means $g_L^{1/\chi} = 1.01$. We do not have good data for the growth rate of factor X, which we interpret as natural resources, including land. Given that land is a major factor of production, we take $g_X = 1$ as a benchmark (i.e., constant X). We also consider the case where natural resources are depleted 1% per year ($g_X^{1/\chi} = 0.99$).

Data

We calibrate the remaining parameters so that the model variables on the BGP match data from the U.S. Table 2 reports the target moments and model variables in annualized values (e.g,. aggregate output per year is Y/χ, where one period in the model is χ years). For the capital-output ratio (2.9), labor share of income (63%), R&D share of income (2.7%), consumption of fixed capital as a share of GDP (14%), and real GDP per capita growth (1.9%), we use data from the Bureau of Economic Analysis (BEA) to calculate the arithmetic averages of the annual levels in the 1960-2020 period. To measure the growth rate of capital-augmenting technology, we calculate the annual decline in the relative price of all capital goods from 1960-2020 (0.66%). Finally, we set the rate of return on investment ($r^* - \delta$) equal to the return on bonds (4%) from McGrattan and Prescott (2003).

Calibration Results

There are seven remaining parameters to calibrate, $\{\delta, \beta, \alpha, \eta, \zeta_L, \zeta_X, \chi\}$, which we identify with the seven moments in Table 2. We do so in two steps. First, we use equilibrium conditions to derive four analytical relationships among these parameters. This leaves us with three undetermined parameters, $\{\zeta_L, \eta, \chi\}$. In the second step, we numerically pin them down so that the target moments in Table 2 match the corresponding model variables on the BGP. The details of the calibration procedure are presented in Appendix B.

Table 3 presents the results of the two-step calibration procedure with the baseline assumptions. Period length χ is 3.94 years, which is the time until the knowledge becomes public. At this point, someone else can freely use the knowledge to generate a new innovation. Discount factor β is 0.923 ($\beta^{\frac{1}{\chi}} = 0.98$/year). The depreciation rate δ is 0.19 per period, which is about 5% per year. The share parameter α is 0.76. This implies that the capital share in the GDP is 26.3%, whereas the natural resource share (including land) is 8%. Although these shares were not targeted in the calibration, they are consistent with the existing literature discussed in Casey and Horii (forthcoming).

[30] Karabarbounis and Neiman (2014) and Piketty (2014) estimate the EoS between reproducible capital and labor and find an elasticity that is greater than one.

Parameter	Calibrated	Annualized	Description
χ	3.94		Period Length (years)
β	0.923	0.980	Discount Factor
δ	0.190	5.21%	Depreciation Rate
α	0.767		Capital Share within K-X composite
η	0.685		CES Distribution parameter
ζ_X	0.279		Cost parameter for A_X R&D
ζ_L	20.8		Cost parameter for A_L R&D

Table 3: Calibrated Parameters for Baseline Scenario

Variable	Value	Description
κ_K	26.3%	Capital Share
κ_X	8.0%	Natural Resource Share (incl. Land)
$\gamma_X^{1/\chi} - 1$	0.72%	Tech. Change in A_X per year

Table 4: Untargeted Variables in Calibrated Model

Robustness

Changing the free parameters, we present calibration results with $\lambda \in \{1.5, 2.25\}$, $\theta \in \{0.5, 2\}$, and $\epsilon \in \{0.9, 1.2\}$. We also calibrated the model under the assumption that natural resources X are depleted by 1% per year; i.e., $g_X^{1/\chi} = 0.99$. In each case, we change one parameter from the baseline value and then re-calibrate the model. In all cases, we find the set of parameters with which the model matches all the target moments in Table 2. The results are reported in Table 5 as scenarios (a)–(g).

4.2 Local Stability

Using parameters calibrated for the baseline setting and alternative scenarios, we can now examine the local stability of the model. Recall that the dynamic system is characterized by three variables $\{\mu_t, k_t, c_t\}$, which evolve according to equations (33), (35) and (38). Also, note that the initial values of μ_0 and k_0 are pre-determined, whereas c_0 should be chosen endogenously so that the system satisfies the transversality condition (42). In this system, the BGP is saddle-stable and determinate if the Jacobian matrix evaluated at the BGP has two stable eigenvalues with absolute values less than one and one unstable eigenvalue with an absolute value greater than

Param-eters	Base-line	(a) λ =1.5	(b) λ =2.25	(c) ϵ =0.9	(d) ϵ =1.2	(e) θ =0.5	(f) θ =2	(g) $g_X^{1/\chi}$ =0.99
χ	3.94	2.92	4.20	3.94	3.94	3.94	3.94	3.72
$\beta^{1/\chi}$	0.98	0.98	0.98	0.98	0.98	0.97	0.998	0.98
$\delta^{1/\chi}$	5.21%	5.07%	5.25%	5.21%	5.21%	5.21%	5.21%	5.18%
α	0.77	0.76	0.77	0.77	0.77	0.77	0.77	0.77
η	0.69	0.38	0.79	0.41	0.27	0.69	0.69	0.70
ζ_L	20.82	1.88	61.74	10.05	7.64	20.82	20.82	25.69
ζ_X	0.28	0.29	0.28	0.28	0.28	0.28	0.28	0.11
κ_K	26.3%	26.1%	26.4%	26.3%	26.3%	26.3%	26.3%	26.2%
κ_X	8.00%	8.25%	7.93%	8.00%	8.00%	8.00%	8.00%	8.05%
$\gamma_X^{1/\chi} - 1$	0.72%	0.80%	0.69%	0.72%	0.72%	0.72%	0.72%	1.75%

Table 5: Calibrated Parameters for the Robustness Scenarios

one.

Table 6 summarizes the results of the local stability analysis. In all cases, we find that the BGP is saddle-stable and determinate: when state variables are near the BGP, they will converge to the BGP along the unique saddle path. In Subsection 3.4, we demonstrate that one of the conditions for balanced growth, (43), is equivalent to the technology condition (9). Therefore, the saddle stability of the BGP implies that the technology condition is endogenously satisfied as the economy converges to the BGP.

On this equilibrium path converging to the BGP, firms choose the intensities of three types of R&D, γ_K, γ_X, and γ_L, and hence the direction of the technological change, to maximize profits. The capital-augmenting technology A_K is still growing on the BGP, because firms always benefit from improving A_K. This naturally explains the observed long-term decline in the relative price of capital, which theoretically corresponds to capital-augmenting technological change.

The saddle stability is confirmed in all seven alternative scenarios. It demonstrates that our explanation of capital-augmenting technological change is robust to changes in parameters. It is particularly interesting to note that we find stability even when the EoS between labor and the capital-composite, ϵ, is greater than one (scenario d).

Scenario	Eigenvalues Stable		Unstable	BGP-Stability
Baseline	0.602	0.970	1.672	Saddle/Determinate
(a) $\lambda = 1.5$	0.667	0.957	1.479	Saddle/Determinate
(b) $\lambda = 2.25$	0.587	0.971	1.722	Saddle/Determinate
(c) $\epsilon = .9$	0.633	0.971	1.610	Saddle/Determinate
(d) $\epsilon = 1.2$	0.664	0.974	1.550	Saddle/Determinate
(e) $\theta = 0.5$	0.496	0.969	2.034	Saddle/Determinate
(f) $\theta = 2$	0.692	0.971	1.454	Saddle/Determinate
(g) $g_X = 0.99$	0.620	0.964	1.629	Saddle/Determinate

Table 6: Eigenvalues and Local Stability of the Calibrated Model

Most directed technical change growth models require a low elasticity to be stable, especially when allowing for the possibility of capital-augmenting technical change (e.g., Acemoglu, 2003; Grossman et al., 2017).[31]

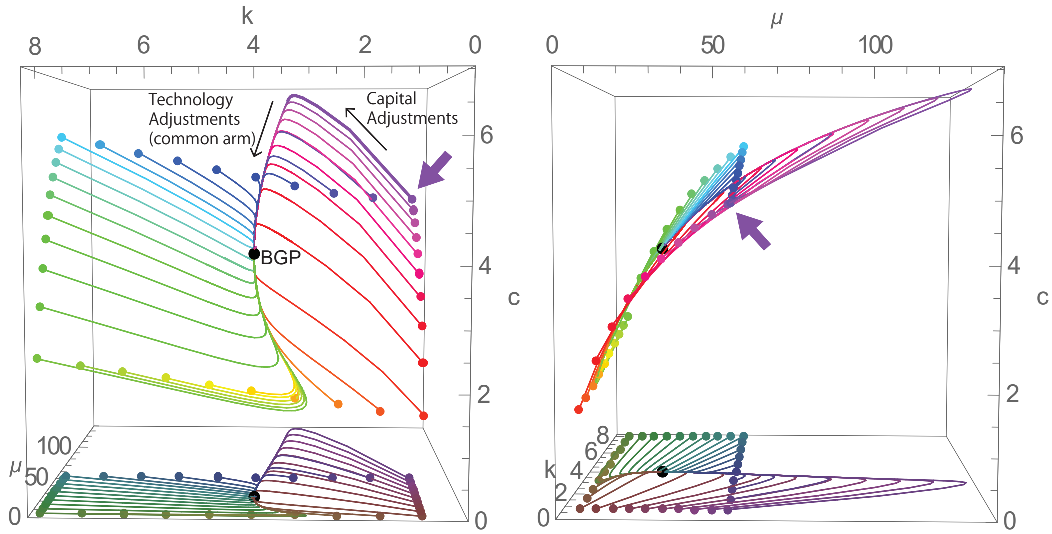
4.3 Transition Dynamics and Global Stability

Local stability only examines convergence within the neighborhood of the BGP. In this subsection, we go one step further and demonstrate that convergence to the BGP occurs even when the initial states are far away. We call this property global stability. With three factors of production, this is not a trivial exercise, because the transitional dynamics may take various patterns depending on the initial combination of μ_0 and k_0.[32] They are determined by initial stock of production factors K_0, X_0 and L_0, as well as initial technology levels $A_{K,-1}$, $A_{X,-1}$, $A_{L,-1}$. Depending on the initial state of technology or resources, μ_0 and k_0 will take a wide range of combinations.

To cover various possibilities, we consider a large rectangular area in μ-k plane surrounding the BGP: namely, $\mu_t \in [0.2\mu^*, 2\mu^*]$ and $k_t \in [0.2k^*, 2k^*]$. We choose 36 points on the border of a rectangular area and calculate the transition dynamics from each of them. We use a forward shooting method to determine the value of c_0

[31] An exception is a model by Irmen and Tabaković (2017), which has an elasticity greater than one. As explained above, their model has capital-augmenting technical change on the transition path, but not on the steady state.

[32] In typical macro models with two factors of production, the dynamics can be written in terms of k_t and c_t, where k_t is the only state variable. In this case, the transition dynamics only have two possible patterns, depending on whether k_0 is higher or lower than the steady-state value. In either case, k_t typically converges monotonically to the steady state.



(a) Dynamic equilibrium paths from various starting points, viewed from k-c plane

(b) The same figure as (a), viewed from μ-c plane

Figure 3: Global Stability of the Calibrated Model (Baseline Setting).

that eventually satisfies the transversality condition (42) as $t \to \infty$. The equilibrium path from each starting point is depicted in Figure 3, where the parameters are from baseline calibration in Section 4.1. Because the graph is three-dimensional, we depicted the same graph from two angles. We also provide the projection of the paths to the bottom μ-k plane in darker colors.

From each of the 36 starting pairs of μ_0 and k_0, we always find a unique level of c_0 such that the path from $\{\mu_0, k_0, c_0\}$ leads to the BGP (i.e., $\{\mu^*, k^*, c^*\}$). If c_0 is higher the resource constraint is eventually violated (k_t becomes negative), and if c_0 is lower the TVC is violated (c_t converges to zero). This means that convergence to the BGP is the only possible long-term outcome in equilibrium. These findings suggest that, as long as the initial μ_0 and k_0 are on or within the border of the rectangle, the economy necessarily converges to the BGP. Since the rectangle is reasonably large, we call it global stability.

There are a couple of properties worth observing from the figure. First, the convergence is not monotonic. To illustrate this, let us focus on the path that starts from the upper right corner in Figure 3(a), as indicated by a thick arrow ($\mu_0 = 2\mu^*$ and $k_0 = 0.2k^*$).[33] Although the initial level of μ_0 is double the steady-state level, μ_t

[33]In the color PDF version of the article, the path that we now focus on is depicted in purple.

initially increases further, going out of the rectangular area. This phenomenon can be interpreted as follows. At the initial state, the capital composite is abundant even though the reproducible capital is scarce. This happens when natural resources are so abundant that it more than offsets capital scarcity. In this setting, the consumption of reproducible capital (i.e., the depreciation of K_t) is small, and savings from ample production leads to more accumulation K_t, which increases the capital composite further. This process continues until the level of k_t comes close to the steady state level. This is the first stage of convergence. In the second stage, the ratio of capital composite to effective labor μ_t gradually falls to the steady state level. This is because a high μ_t means that effective labor is relatively scarce, and the firms have more incentives to improve A_L through R&D, rather than A_K or A_X. This tendency continues until μ_t reaches μ^*. Once μ_t comes to μ^*, firms have incentives to improve all types of technologies in a 'balanced' way such that the ratio of capital composite to effective labor does not change further. This illustrates how firms, in the long run, choose the direction of technological change that satisfies the BGP condition (43) or, equivalently, the technology condition (9).

The figure also shows that, even though the stable manifold[34] is two-dimensional, the equilibrium paths first converge to a common one-dimensional arm (or curve), and then converge to the BGP along the arm. This is because the system has two stable eigenvalues with significantly different magnitudes. In the baseline calibration, stable eigenvalues are 0.602 and 0.970. Given that one period χ is 3.94 years, those eigenvalues mean the speed of convergence is 12% and 0.7% per year, respectively. As we discussed in the above example (the path starting from the upper-right corner, indicated by a thick arrow), the convergence to the BGP typically goes through two stages, and each stage corresponds to a different eigenvalue. The initial adjustment towards the common arm is driven mainly by capital accumulation. It is relatively fast: the distance from the common arm declines by 12% every year. However, the second stage, along the common arm, is very slow. In the baseline example, the convergence speed is only 0.7% per year, which means it takes about 90 years to halve the distance. This adjustment takes much longer than capital accumulation because it is driven by the difference in the speed of technological change among

[34]The stable manifold is the set of points in the (μ, k, c) space that converges to the BGP. In Figure 3, all converging paths are on the (same) stable manifold.

A_K, A_X, and A_L. Note that these numbers are just for illustration, because the eigenvalues depend on the free parameters, as shown in Table 6. Still, this result suggests that, without considering the endogenous technological change for various production factors (including capital-augmenting technological change), neoclassical growth models may overestimate the speed of convergence to the steady state by large margins.

5 Conclusion

We build a neoclassical growth model that has capital-augmenting technical change on the BGP, despite a non-unitary EoS between capital and labor. As noted by Grossman et al. (2017), standard neoclassical growth models cannot incorporate these elements simultaneously, due to the Uzawa steady state theorem. This is a significant limitation, because each of the elements has strong empirical support. To overcome the restrictive nature of the theorem, we follow Casey and Horii (forthcoming) and add natural resources to the model as a third factor of production. We then add directed technical change and show that the model endogenously converges to the BGP with capital-augmenting technical change. By relaxing the constraints posed by the theorem, our model should be useful in a wide range of settings.

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Online Appendix

"Endogenous Capital-Augmenting Technological Change"

by Gregory Casey and Ryo Horii

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A Proofs of Propositions and Lemmas

A.1 Notation for derivatives

Unless otherwise mentioned, $F_K(\cdot;t)$ denotes the partial derivative of function $F(\cdot;t)$ with respect to its first argument, whereas $F_{X_j}(\cdot;t)$ denotes the partial derivative of $F(\cdot;t)$ with respect to its $1+j$th argument. The same applies to other functions, such as $\widetilde{F}(\cdot)$.

Following the convention in economics, $\frac{\partial}{\partial K_t}$ and $\frac{\partial}{\partial X_{j,t}}$ represent the partial derivatives with respect to variables K_t and $X_{j,t}$, respectively. For example, if $\widetilde{F}(\cdot)$ is the production function, $\frac{\partial}{\partial X_{j,t}}\widetilde{F}(\cdot)$ gives the marginal product of factor $X_{j,t}$.

Note that these two definitions are different when the argument of the function is not a single variable. For example, using the chain rule, we have

$$\frac{\partial}{\partial X_{j,t}}\widetilde{F}(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t}) = \widetilde{A}_{X_j,t}\widetilde{F}_{X_j}(K_t, \widetilde{A}_{X_1,t}X_{1,t}, ..., \widetilde{A}_{X_J,t}X_{J,t}). \quad \text{(A.1)}$$

A.2 Proof of Proposition 2

Properties of $\gamma_L(\mu_t)$

As explained in the main text, the representative firm chooses $\gamma_{L,t}$ so as to satisfy

$$\text{R\&D for N-tasks:} \quad \gamma_{L,t}i'_L(\gamma_{L,t}) + i_L(\gamma_{L,t}) = f(\mu_t) - \mu_t f'(\mu_t). \quad (25)$$

Let us denote the LHS of (25) by $\Psi_L(\gamma_{L,t})$ because it depends only on $\gamma_{L,t}$. Then, $\Psi'_L(\gamma_{L,t}) = \gamma_{L,t}i''_L(\gamma_{L,t})+2i'_L(\gamma_{L,t}) > 0$ for all $\gamma_{L,t} > 1$ from $i_L(\gamma_{L,t}) > 0$ and $i'_L(\gamma_{L,t}) > 0$ in (13). When $\gamma_{L,t} = 1$, the properties of $i_L(\cdot)$ imply $\Psi_L(1) = i'_L(1) + i_L(1) = 0$. Also,

$\Psi_L(\infty) \equiv \lim_{\gamma_{L,t}\to\infty} \Psi_L(\gamma_{L,t}) = \infty$ from $i_L(\infty) = \infty$ and $\gamma_{L,t} i_L'(\gamma_{L,t}) > 0$.[35] Then, since $\Psi_L(\cdot)$ is differentiable and strictly increasing, we can define its inverse function $\Psi_L^{(-1)}(\cdot)$, which is also differentiable and strictly increasing with $\Psi_L^{(-1)}(0) = 1$ and $\Psi_L^{(-1)}(\infty) = \infty$. Using this function, condition (25) can be solved for $\gamma_{L,t}$:

$$\gamma_{L,t} = \Psi_L^{(-1)}(f(\mu_t) - \mu_t f'(\mu_t)) \equiv \widehat{\gamma}_L(\mu_t). \tag{A.2}$$

Note that $f(\mu_t) - \mu_t f'(\mu_t)$ represents the marginal product of N_t in the production function, i.e., $\widehat{F}_N(\mu_t, 1)$. We can express the production function $Y_t = \widehat{F}(M_t, N_t)$ in an intensive form with respect to $N_t/M_t \equiv \nu_t$, instead of $\mu_t = M_t/N_t$. Namely, output per M_t can be expressed as $Y_t/M_t = \widehat{F}(M_t, N_t)/M_t = \widehat{F}(1, \nu_t) \equiv h(\nu_t)$. Since $\widehat{F}(\cdot)$ is CRS, its first derivative $\widehat{F}_N(\cdot)$ is homogeneous of degree 0. Using this property, $h'(\nu) = F_N(1, N_t/M_t) = F_N(M_t/N_t, 1) = F_N(\mu_t, 1) = f(\mu_t) - \mu_t f'(\mu_t)$. From the definition of the production function $\widehat{F}(\cdot)$, its alternate intensive form, $h(\nu_t)$, satisfies the Inada conditions. Therefore, $\lim_{\mu_t\to 0} f(\mu_t) - \mu_t f'(\mu_t) = \lim_{\nu_t\to\infty} h'(\nu_t) = 0$, and $\lim_{\mu_t\to\infty} f(\mu_t) - \mu_t f'(\mu_t) = \lim_{\nu_t\to 0} h'(\nu_t) = \infty$. Substituting these into (A.2) gives $\gamma_L(0) = \Psi_L^{(-1)}(0) = 1$ and $\gamma_L(\infty) = \Psi_L^{(-1)}(\infty) = \infty$.

Finally, we show $\gamma_L'(\mu_t) > 0$. The derivative of $f(\mu_t) - \mu_t f'(\mu_t)$ with respect to μ_t is $-\mu_t f''(\mu_t)$. It is positive for all $\mu_t > 0$ since the production function satisfies the Inada conditions, which include $f''(\mu_t) < 0$. Since $\Psi_L^{(-1)'}(\cdot) > 0$, this means $\gamma_L'(\mu_t) > 0$.

Properties of $\widehat{\gamma}_K(\mu_K)$ and $\widehat{\gamma}_X(\mu_X)$

The representative firm chooses $\gamma_{K,t}$ and $\gamma_{X,t}$ according to the following two conditions:

$$\text{R\&D allocation: } \frac{\gamma_{K,t} i_K'(\gamma_{K,t})}{\gamma_{X,t} i_X'(\gamma_{X,t})} = \frac{\alpha}{1-\alpha}, \quad \alpha \in (0,1), \tag{27}$$

$$\text{Combined R\&D: } (\gamma_{K,t} i_K'(\gamma_{K,t}) + i_K(\gamma_{K,t})) + (\gamma_{X,t} i_X'(\gamma_{X,t}) + i_X(\gamma_{X,t})) = f'(\mu_t). \tag{26}$$

[35] Similarly to the main text, we employ an abuse of notation by writing $i_L(\infty)$ to represent $\lim_{\gamma_L\to\infty} i_L(\gamma_L)$. We will employ similar abbreviations as long as they cause no confusion.

Let us define $\Omega_K(\gamma_{K,t}) \equiv \gamma_{K,t} i'_K(\gamma_{K,t})$ and similarly $\Omega_X(\gamma_{X,t}) \equiv \gamma_{X,t} i'_K(\gamma_{X,t})$. Then, from properties in (13), we can confirm $\Omega'_K(\gamma_{K,t}) > 0$ for $\gamma_{K,t} > 1$, $\Omega_K(1) = 0$ and $\Omega_K(\infty) = \infty$. Similar conditions also hold for $\Omega_X(\cdot)$. Then, since $\Omega_X(\cdot)$ is differentiable and strictly increasing, we can define its inverse function $\Omega_X^{(-1)}(\cdot)$, which is also differentiable and strictly increasing with $\Omega_X^{(-1)}(0) = 1$ and $\Omega_X^{(-1)}(\infty) = \infty$. Using this inverse function, condition (27) can be solved for $\gamma_{X,t}$ as

$$\gamma_{X,t} = \Omega_X^{(-1)} \left(\frac{\alpha}{1-\alpha} \Omega_K(\gamma_{K,t}) \right) \equiv \Omega(\gamma_{K,t}). \tag{A.3}$$

Now let us focus on condition (26). Let us define $\Psi_K(\gamma_{K,t}) \equiv \gamma_{K,t} i'_K(\gamma_{K,t}) + i_K(\gamma_{K,t})$ and likewise $\Psi_X(\gamma_{X,t}) \equiv \gamma_{X,t} i'_K(\gamma_{X,t}) + i_K(\gamma_{X,t})$. Using these and (A.3), the LHS of condition (26) can be expressed as a function only of $\gamma_{K,t}$:

$$\Psi_K(\gamma_{K,t}) + \Psi_X(\Omega(\gamma_{K,t})) \equiv \Psi(\gamma_{K,t}).$$

Note that the properties of $\Omega_K(\cdot)$ and $\Omega_X^{(-1)}(\cdot)$ imply that $\Omega(\gamma_{K,t}) > 0$ for all $\gamma_{K,t} > 1$, $\Omega(0) = 0$ and $\Omega(\infty) = \infty$. Also, in the same way that we derived the properties of $\Psi_L(\gamma_{L,t})$ earlier in this proof, we can confirm $\Psi_K(\gamma_{K,t}) > 0$ for all $\gamma_{K,t} > 1$, $\Psi_K(1) = 0$, $\Psi_K(\infty) = \infty$, and similar properties for $\Psi_X(\gamma_{X,t})$. From these, we have $\Psi(\gamma_{K,t}) > 0$ for all $\gamma_{K,t} > 1$, $\Psi(1) = 0$, $\Psi(\infty) = \infty$. On the RHS of (26), $f'(\mu_t)$ satisfies the usual Inada conditions. The results we have obtained so far can be summarized as

$\gamma_{K,t}$	1	$\cdots$	∞
$\Psi'(\gamma_{K,t})$		$+$	
$\Psi(\gamma_{K,t})$	0	$\nearrow$	∞

μ_t	0	$\cdots$	∞
$f''(\mu_t)$		$-$	
$f'(\mu_t)$	∞	$\searrow$	0

The tables above implies that condition (26), $\Psi(\gamma_{K,t}) = f'(\mu_t)$, gives a 1 to 1 correspondence between $\mu_t \in (0, \infty)$ and $\gamma_{K,t} \in (1, \infty)$ that satisfies property (a): $\widehat{\gamma}'_K(\mu_t) < 0$ for all $\mu_t > 0$, $\widehat{\gamma}_K(0) = \infty$, and $\widehat{\gamma}_K(\infty) = 1$.

Given $\widehat{\gamma}_K(\mu_t)$, equation (A.3) uniquely determines $\gamma_{X,t} = \Omega(\widehat{\gamma}_K(\mu_t)) \equiv \widehat{\gamma}_X(\mu_t)$. From the properties of $\Omega(\cdot)$ and $\widehat{\gamma}_K(\cdot)$ above, we can confirm that property (b) is satisfied: $\widehat{\gamma}'_X(\mu_t) < 0$ for all $\mu_t > 0$, $\widehat{\gamma}_X(0) = \infty$, and $\widehat{\gamma}_X(\infty) = 1$.

A.3 Proof of Lemma 1

Consider a BGP. We will show in turn that μ_t, k_t and c_t must be constant. First, from the definition of a BGP, $N_{t+1}/N_t = (A_{L,t+1}L_{t+1})/(A_{L,t}L_t) = \widehat{\gamma}_L(\mu_{t+1})g_L$ is constant. To keep the RHS of the latter equation constant, μ_t must also be constant, since $\widehat{\gamma}_L(\cdot)$ is a strictly increasing function from Proposition 2.

Second, since the growth factors of C_t and N_t are constant, the growth factor of $c_t = C_t/A_tL_t = C_t/N_t$ is also constant. This, in turn, means that the LHS of the Euler equation (37) is constant. Then, for the RHS of (37) to be constant, k_t must be constant, since we already know that μ_t is constant as shown above.

Third, the growth factor of $k_t = K_t/A_tL_t = K_t/N_t$ is constant on the BGP, which means the LHS of (34) is constant. For its RHS to be constant, given that μ_t and k_t are already shown to be constant, c_t also needs to be constant.

A.4 Proof of Proposition 3

Proof of $\mu^* > 0$

In the text, we have already shown that there exists a unique $\mu^* > 0$ such that $\Phi(\mu^*) = 1$ holds since Proposition 2 implies $\Phi'(\mu^*) < 0$ with $\Phi(0) = \infty$ and $\Phi(\infty) = 0$. Therefore, there exists a unique value of $\mu^* > 0$.

Proof of $k^* > 0$

The value of k^* is explicitly given by equation (45), shown again here:

$$k^* = \frac{\beta\alpha\mu^*(f'(\mu^*) - i_K(\widehat{\gamma}_K(\mu^*)) - i_X(\widehat{\gamma}_X(\mu^*)))}{\widehat{\gamma}_L(\mu^*)^\theta - \beta(1-\delta)}. \tag{45}$$

We now show that both the numerator and the denominator of the RHS are positive. Note that the combined R&D condition (26) is satisfied on the BGP. By rearranging terms, it gives

$$f'(\mu^*) - i_K(\widehat{\gamma}_K(\mu^*)) - i_X(\widehat{\gamma}_X(\mu^*)) = \widehat{\gamma}_K(\mu^*)i'_K(\widehat{\gamma}_K(\mu^*)) + \widehat{\gamma}_X(\mu^*)i'_X(\widehat{\gamma}_X(\mu^*)) > 0, \tag{A.4}$$

where the inequality follows from Proposition 2 and (13). Given $\beta \in (0,1)$, $\alpha \in (0,1)$, and $\mu^* > 0$, this means that the numerator of (45) is strictly positive. Now, note that $\widehat{\gamma}_L(\mu^*) > 1$ from Proposition 2. Combined with $\theta > 0$, $\beta \in (0,1)$ and $\delta \in [0,1]$, it turns out that the denominator of (45) is also strictly positive.

Proof of $c^* > 0$

The value of c^* is given by

$$c^* = v(\mu^*) - (g^* - 1 + \delta)k^*. \tag{46}$$

We first show $v(\mu^*) > 0$. Combining the R&D conditions (25) and (26), we have

$$\gamma_{L,t} i'_L(\gamma_{L,t}) + i_L(\gamma_{L,t}) + \mu_t \left((\gamma_{K,t} i'_K(\gamma_{K,t}) + i_K(\gamma_{K,t})) + (\gamma_{X,t} i'_X(\gamma_{X,t}) + i_X(\gamma_{X,t})) \right) = f(\mu_t).$$

Rearranging and then evaluating this condition at $\mu_t = \mu^*$ gives

$$\begin{aligned} v(\mu^*) &= f(\mu*) - i_L(\widehat{\gamma}_L(\mu^*)) - \mu^* \left(i_K(\widehat{\gamma}_K(\mu^*)) - i_X(\widehat{\gamma}_X(\mu^*)) \right) \\ &= \widehat{\gamma}_L(\mu^*) i'_L(\widehat{\gamma}_L(\mu^*)) + \mu^* \left(\widehat{\gamma}_K(\mu^*) i'_K(\widehat{\gamma}_K(\mu^*)) + \widehat{\gamma}_X(\mu^*) i'_X(\widehat{\gamma}_X(\mu^*)) \right) > 0, \end{aligned}$$

where the inequality follows from $\mu^* > 0$ and (13).

Note that $g^* = \widehat{\gamma}_L(\mu^*) g_L$ is greater than $1 - \delta$ because $\widehat{\gamma}_L(\mu^*) > 1$ and $g_L > 1 - \delta$ from (16). Therefore, $(g^* - 1 + \delta)$ in (46) is positive. From this, $c^* > 0$ is equivalent to

$$k^* < \frac{v(\mu^*)}{g^* - 1 + \delta}.$$

Using (45), we can rewrite this condition in terms of β:

$$\beta < \widehat{\gamma}_L(\mu^*)^\theta \left(\frac{\alpha \mu^*}{v(\mu^*)} \left(f'(\mu^*) - i_K(\widehat{\gamma}_K(\mu^*)) - i_X(\widehat{\gamma}_X(\mu^*)) \right) (g^* - 1 + \delta) + 1 - \delta \right)^{-1} \equiv \overline{\beta}_1. \tag{A.5}$$

Note that $\overline{\beta}_1 > 0$ from (A.4) and $g^* > 1 - \delta > 0$. Observe also that $\overline{\beta}_1$ does not depend on β itself since μ^* is determined entirely by the production side (see equation 43). Therefore, if $\beta > 0$ is sufficiently small, condition (A.5) holds and $c^* > 0$.

Terminal Condition

On the BGP, the terminal condition (42) becomes

$$\lim_{T\to\infty} \left(\beta g_L \widehat{\gamma}_L(\mu^*)^{1-\theta}\right)^T \widehat{\gamma}_L(\mu^*)(c^*)^{-\theta} k^* = 0. \tag{A.6}$$

Given that $\widehat{\gamma}_L(\mu^*) > 1$, $c^* > 0$ and $c^* > 0$, this condition is equivalent to

$$\beta < \frac{1}{g_L \widehat{\gamma}_L(\mu^*)^{1-\theta}} \equiv \overline{\beta}_2. \tag{A.7}$$

Note that $\overline{\beta}_2 > 0$ and that it does not depend on β since μ^* is determined entirely by the production side of the model. Therefore, if $\beta > 0$ is sufficiently small, condition (A.7) holds and the terminal condition (42) is satisfied.

Combining conditions (A.5) and (A.7), we have confirmed the unique existence of BGP with $\mu^* > 0$, $k^* > 0$, $c^* > 0$, and the terminal condition (42) whenever

$$\beta < \overline{\beta} \equiv \min\{\overline{\beta}_1, \overline{\beta}_2\}, \tag{A.8}$$

where $\overline{\beta} > 0$ is a constant that does not depend on β.

B Calibration Procedure

There are seven parameters to calibrate, $\{\delta, \beta, \alpha, \eta, \zeta_L, \zeta_X, \chi\}$, which we identify with the seven moments in Table 2. We do so in two steps.

Step 1: Analytical calibration. Given period length χ, exogenous parameters, and moments, we analytically derive the values of four parameters $\{\delta, \beta, \alpha, \eta\}$. The depreciation rate is determined by data on the consumption of fixed capital and the capital-output ratio:

$$\delta = \frac{\delta K/Y}{K/Y} = \frac{0.14}{2.9}\chi \equiv \overline{\delta}(\chi). \tag{B.1}$$

Evaluating the Euler equation (36) on the BGP gives the discount factor β:[36]

$$\beta = \frac{\gamma_L^\theta}{1+r-\delta} = \frac{(1.019^\chi)^{1.0}}{1.04^\chi} \equiv \overline{\beta}(\chi). \tag{B.2}$$

Similarly, the first-order conditions of the representative firm, (22) and (23), give the share parameter α:

$$\alpha = \frac{\kappa_K}{\kappa_K+\kappa_X} = \frac{(r-\delta)(K/Y)+\delta K/Y}{1-\kappa_L-\kappa_R} = \frac{(1.04^\chi-1)(2.9\chi)+0.14}{1-0.63-0.027} \equiv \overline{\alpha}(\chi), \tag{B.3}$$

where we used the identity $\kappa_K+\kappa_X+\kappa_L+\kappa_R=1$.

Next, BGP relationship (43), which is equivalent to the technology condition (9), gives the growth rate for the unobserved endogenous variable γ_X:

$$\gamma_X = \gamma_K^{-\frac{\alpha}{1-\alpha}} \frac{g_L\gamma_L}{g_X} = \left((1.066)^{-\frac{\overline{\alpha}(\chi)}{1-\overline{\alpha}(\chi)}} \frac{(1.01)(1.019)}{1.0}\right)^\chi \equiv \overline{\gamma}_X(\chi). \tag{B.4}$$

Using (B.4) and the R&D allocation condition (27), the R&D cost parameter ζ_X can be derived as follows.

$$\zeta_X = \frac{1-\alpha}{\alpha}\cdot\frac{\zeta_K\gamma_K(\gamma_K-1)^{\lambda-1}}{\gamma_X(\gamma_X-1)^{\lambda-1}} = \frac{1-\overline{\alpha}(\chi)}{\overline{\alpha}(\chi)}\cdot\frac{(1(1.0066)(0.0066)^{2.0-1})^\chi}{\overline{\gamma}_X(\chi)(\overline{\gamma}_X(\chi)-1)^{2.0-1}} \equiv \overline{\zeta}_X(\chi). \tag{B.5}$$

Step 2: Minimization. Among the 12 parameters of the model, five of them are given by Table 1, and four are given as functions of χ, in (B.1), (B.2), (B.3) and (B.5). This leaves us with three remaining parameters, $\{\zeta_L, \eta, \chi\}$. We calibrate them so as to minimize the squared sum of percent difference (error) between the target moments in Table 2 and the corresponding model variables on the BGP.

[36] Equation (B.2) assumes that parameter θ takes the baseline value of 1.0. When doing robustness checks with $\theta = 0.8$ and 1.2, the numbers in this equation are adjusted accordingly. The same applies for (B.3)-(B.6) when using alternative parameter values or calibration targets.

Let us define the squared sum of percent error as[37]

$$SSE = \left(\frac{K/(Y/\chi) - 2.9}{2.9}\right)^2 + \left(\frac{\kappa_L - 0.63}{0.63}\right)^2 + \left(\frac{\kappa_R - 0.027}{0.027}\right)^2 + \left(\frac{\gamma_L^{1/\chi} - 1 - 0.019}{0.019}\right)^2 + \left(\frac{\gamma_K^{1/\chi} - 1 - 0.0066}{0.0066}\right)^2. \tag{B.6}$$

In (B.6), endogenous variables K/Y, κ_L, κ_R, γ_L and γ_X represent their respective BGP values, when the model is solved given all 12 parameters. Using exogenous parameters and the results of analytical calibration, we determine the remaining three parameters as the solution to the following minimization problem:

$$\begin{aligned} \{\zeta_L, \eta, \chi\} = &\operatorname*{argmin}_{\zeta_L,\eta,\chi} SSE \quad s.t. \\ &\{\epsilon, \lambda, \theta, g_L, g_X\} : \text{given by Table 1}, \\ &\delta = \overline{\delta}(\chi),\ \beta = \overline{\beta}(\chi),\ \alpha = \overline{\alpha}(\chi),\ \zeta_X = \overline{\zeta}_X(\chi). \end{aligned} \tag{B.7}$$

We have done this minimization numerically utilizing 'FindMinimum' function of Mathematica. The minimized value of SSE is virtually zero (precisely, of the order of 10^{-22}), implying that we obtained the set of parameters that fits all the moments in Table 2.

Robustness Scenarios. To check the robustness of the result, we repeat the analytical calibration (B.1)-(B.5) with modified values for the exogenous parameters. In all robustness scenarios, the modified version of minimization problem (B.7) yields almost zero. This means that the model can match all the target moments in those scenarios.

C Data Sources

All data are originally from the Bureau of Economic Analysis (BEA) and retrieved from FRED, Federal Reserve Bank of St. Louis, `https://fred.stlouisfed.org`. We

[37] Among seven moments in Table 2, we use five moments to define the squared sum of errors. The other two moments, consumption of fixed capital in GDP and return on investments, always match the data given that other moments are correct, since we impose relationships (B.1) and (B.2). In numerical calibration, we confirmed that these two moments match the data exactly.

reference series by their codes in FRED. We use annual values of Real GDP (GDPCA), real investment (GPDICA), and real personal consumption expenditures (PCECCA). The real capital stock is calculated as the net stock of fixed assets at current cost (K1TTOTL1ES000) divided by the GDP price deflator (A191RD3A086NBEA). The relative price of investment is obtained by the price deflators for gross private investment (A006RD3A086NBEA) divided by the price deflator for personal consumption expenditures (DPCERD3A086NBEA). In addition to the variables listed above, the **calibration** utilizes data on nominal consumption of fixed capital (GDICONSPA), labor compensation, and R&D expenditure (Y694RC1A027NBEA) all relative to nominal GDP (GDPA) in **Table 2**, as well as population (B230RC0A052NBEA) in **Table 1**. Labor compensation is calculated as compensation of employees (A033RC1A027NBEA) plus proprietors' income with inventory valuation and capital consumption adjustments (A041RC1A027NBEA).