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CAREER CONCERNS AND INCENTIVE COMPATIBLE TASK DESIGN

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Career Concerns and Incentive Compatible Task Design*

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Abstract

This paper studies the optimal disclosure of information about an agent's talent when it consists of two components. The agent observes the first component of his talent as his private type, and reports it to a principal to perform a task which reveals the second component of his talent. Based on the report and performance, the principal discloses information to a firm who pays the agent the wage equal to his expected talent. We study incentive compatible disclosure rules that minimize the mismatch between the agent's true talent and his wage. The optimal rule entails full disclosure when the agent's talent is a supermodular function of the two components, but entails partial pooling when it is submodular. Under a mild degree of submodularity, we show that the optimal disclosure rule is obtained as a solution to a linear programming problem, and identify the number of messages required under the optimal rule. We relate it to the agent's incentive compatibility conditions, and show that each pooling message has binary support.

Key words: talent, mechanism, revelation, pooling, performance.
JEL Codes: C72, D47, D82.

1 Introduction

The labor market often evaluates the productivity of a worker on the basis of multiple skills some of which are recognized by the worker himself, whereas others are identified only through actual engagement in a task. For example, the market finds it important that a worker possess both "hard" and "soft" skills: While a worker is well aware of his hard skills, which represent expertise in a particular field, familiarity with a specific

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tool, fluency in a foreign language, and so on, he may need to place himself in the actual work environment to discover his soft skills, which represent ability in communication, teamwork, time management, leadership, and so on. While these two sets of skills may be correlated, they need to be separately identified for the correct evaluation of a worker's talent.

In this paper, we develop a model of career concerns where an agent's talent has two components. The agent observes the first component as his private type, but learns the second component only through his own performance at a task assigned by a principal. In light of the above discussion, the agent's type can be identified as the level of hard skills he possesses, whereas performance at a task can be identified as the level of his soft skills. The agent's talent then is a positive combination of these skills. The agent reports his type to the principal, and is assigned a task based on his report. The agent's performance at his task is measured by a stochastic score, which we assume is more likely to be higher when his type is higher in the sense of stochastic dominance. Based on the agent's reported type and his performance score, the principal discloses information to a firm, who then pays the agent the wage equal to the expected value of his talent conditional on the disclosed information. The principal's instruments consist of the disclosure rule as well as the cost of effort required to complete the task assigned to each agent type. The agent's utility function is quasi-linear and equals the wage payment from the firm minus the effort cost. The principal uses these instruments to provide the agent an incentive for truth-telling, and his objective is to minimize the loss function which equals the quadratic difference between the agent's true talent and his wage.

The model described above differs from the standard career concerns models (e.g., Holmström, 1999) in the sense that the agent's talent has two underlying components, and that the principal controls information disclosed to the labor market.[1] The principal in our model can be thought of as a public school system that accepts students to different programs and then sends them to the labor market with grade information, or a human resources department within a firm that offers prospective employees different probationary tasks based on their self-reported qualifications before a permanent contract is signed.[2,3] As an alternative interpretation of our model, we may view it as a model of product certification where the principal is a public agency who performs product tests for fees based on the quality classes of the products as submitted by the suppliers of those products, and then discloses product information to consumers. Also

[1]In career concerns models, an agent's talent (along with effort) stochastically determines his performance. Although it may appear that our assumption that the agent's talent is a function of his performance score reverses this relationship, no formal distinction exists between the two formulations of this point in the sense that the market infers the agent's talent from information about his performance in both cases.

[2]The principal may also be viewed as an autonomous certification agency that offers programs and courses whose participants can demonstrate competence in soft skills. These programs are gaining popularity: Examples include the Emotional Intelligence Certification (EIC), the Certified Professional in Learning and Performance (CPLP), the Project Management Professional (PMP), and the Certified Scrum Master (CSM).

[3]Recent literature in education sciences advocate the development of soft employability skills (Teng et al., 2019; Dolce et al., 2020).

related are stress tests by the banking authority which evaluates the test performance of a bank at the time of crisis.[4]

In the absence of the agent's incentive problem, full disclosure of both the agent's private type and his performance score minimizes the principal's loss function since it allows the wage to be equalized to the agent's talent. However, we show that full disclosure does not always induce the agent to report his type truthfully. Our main result highlights the importance of the functional form of the agent's talent in formulating the optimal disclosure mechanism. Specifically, we show that with proper adjustment of the effort cost of each task, full disclosure can be made incentive compatible if the agent's talent is a supermodular function of his type and the performance score. Intuitively, supermodularity implies that the higher is the agent's private type, the larger the marginal impact of the performance score on his talent. For example, the function is supermodular if the talent equals the product or sum of the agent's type and his performance score. Conversely, the talent function is submodular if the higher is the agent's private type, the smaller the marginal impact of the performance score on his talent. With a strictly submodular talent function, it is not possible to adjust the effort costs of different tasks to make full disclosure incentive compatible.

To understand the intuition behind the need for pooling, consider the simplest 2×2 environment in which the agent's types and his performance scores are both binary. By assumption, the distribution of the performance score of the high type agent stochastically dominates that of the low type agent. Since the high type agent is more confident in generating a higher performance score than the low type agent, the expected wage in the event of the high performance score is more important for the high type than for the low type. Conversely, the expected wage in the event of the low performance score is more important for the low type than for the high type. This suggests that a disclosure rule can be made incentive compatible if it creates a larger difference in the expected wages between low and high performance scores for the task assigned to the high type than for the task assigned to the low type. If the agent's talent function is supermodular, this is achieved by perfectly revealing the reported type and realized performance score. On the other hand, when it is submodular, full disclosure fails to achieve this, and there should be some message that does not fully reveal the agent's type or his performance score. One way to create such a message is to pool two performance scores at the task assigned to the low type while separating the two performance scores at the task assigned to the high type. With appropriate adjustment in the effort cost, then, the low type can be induced to choose the task intended for the low type since the expected wage differential is zero at the task, and the high type is induced to choose the task intended for the high type since the expected wage differential is high at the task. Another way to create an imperfect message is to pool the two types when their performance score is low while separating the two types when their performance

[4]In the stress test model of Goldstein and Leitner (2018), the regulator observes a bank's type and discloses information to the public so as to influence its subsequent interaction with the market. According to the stress test interpretation of the present model, the regulator first collects information about a bank's type and then observes its test performance before disclosing information.

score is high. This also makes the expected wage differential at the task intended for the high type higher than that at the task intended for the low type. In other words, if we define the *ex post expected wage function* to be the agent's expected talent as a function of his reported type and the performance score, then the disclosure rule can be made incentive compatible if and only if it renders the ex post expected wage function supermodular. Although the argument so far assumes pure disclosure rules that entail complete pooling depending on the realization of the underlying type and score, the optimal disclosure rule typically entails partial pooling in order to reduce the mismatch loss from pooling. This implies that each combination of the type and performance score is perfectly revealed with positive probability, and one pooling message is sent with positive probability based on particular realizations of the type and performance score. In other words, in the 2×2 environment under consideration, the optimal rule typically sends four perfectly revealing messages and one pooling message. Note that the above discussion suggests that the pooling message has binary support in the sense that it is sent only after the realization of certain combinations of the type and performance score.

In the 2×2 environment described above, supermodularity of the ex post expected wage function is expressed by a single condition: The difference in the expected talents of the agent with the low and high performance scores is higher when his type is high than when it is low. The number of pooling messages under the optimal disclosure rule hence equals the number of the conditions required for the supermodularity of the ex post expected wage function. The same holds for the environment in which the agent has $K \geq 3$ types, but the performance score is binary. In this case, a disclosure rule can be made incentive compatible with the adjustment in the effort cost if and only if it leads to a supermodular ex post expected wage function as in the 2×2 environment. Since the performance score is binary, the supermodularity of the ex post expected wage function reduces to the $K - 1$ local conditions.[5] When the degree of submodularity of the talent function is mild, we show that $K - 1$ gives an upper bound on the number of pooling messages in the optimal disclosure rule.

When the performance score takes more than two values, however, we show that the disclosure rule can be made incentive compatible under a weaker requirement than the supermodularity of the ex post expected wage function.[6] This condition, which we call cyclical supermodularity, pertains to the agent's *interim expected wage function,* which equals his expected talent as a function of his true and reported types, and summarizes the conditions that make the incentive compatibility conditions feasible with the adjustment in the effort costs of different tasks. Again under a mild degree of submodularity of the talent function, we show that the number of conditions required for cyclical supermodularity gives an upper bound on the number of pooling messages under the optimal disclosure rule.

We approach the optimization problem by expressing the disclosure rule by the prob-

[5] That is, it is supermodular if and only if for any reported type s, the difference in the expected talents of the agent with the low and high performance scores at the task intended for type s is larger than that at the task intended for the type right below type s.

[6] Compared with the case of binary performance scores, supermodularity is a much stronger requirement when the preformance score is more than binary.

ability that each pooling message is sent after the realization of each pair of the agent type and the performance score. The key step in the analysis is to show that both the feasibility constraints (i.e., the supermodularity of the ex post expected wage function or the cyclical modularity of the interim expected wage function) as well as the principal's quadratic loss function can be written as linear functions of a certain transformation of these probabilities. As such, this optimization problem has a corner solution and this corresponds to the optimal disclosure rule if the solution can indeed be replicated by the probabilities of pooling messages. In general, this last step requires that the degree of submodularity of the talent function be not too large.

The paper is organized as follows: Section 2 discusses the related literature. We formulate our model in Section 3 and discuss the implementability of a disclosure rule in Section 4. Section 5 establishes the optimality of full disclosure under the supermodularity of the talent function, and also presents a description of the general disclosure rule. Assuming a submodular talent function, we proceed to the analysis of the optimal disclosure rule: Section 6 studies the environment with binary performance scores, and Section 7 studies the problem when the performance score can take three or more values. In Section 8, we consider an extension of the baseline model in which the agent can make an ex ante action choice that determines the distribution of his private type. We characterize the optimal disclosure rule that induces the agent to take action that stochastically enhances his type. We conclude with a discussion in Section 9.

2 Related Literature

The present paper is related to a few strands of the literature. First, it is related to the literature on career concerns (see, e.g., Holmström, 1999; Dewatripont et al., 1999*a*,*b*; Bonatti and Hörner, 2017). As mentioned in the Introduction, however, our model marks a few important departures from this literature. On the one hand, in standard models of career concerns, an agent's performance is stochastically determined by his effort and talent, and the agent faces moral hazard when choosing effort. Our model assumes away moral hazard and instead introduces adverse selection by assuming that the agent reports his private type.[7] We hence take the mechanism design approach and consider an incentive compatible task-assignment mechanism that specifies the agent's effort cost required to perform each task.[8] On the other hand, an implicit but important assumption in the standard career concerns models is that the agent's performance is publicly observable by the labor market.[9] In contrast, the principal in our model filters

[7]Effort provision in the presence of asymmetric information is studied by several authors in different contexts. Cisternas (2018) assumes that an agent's effort improves his skills, and Board and Meyer-ter-Vehn (2013) assume that an agent exerts effort to control the evolution of his type. In Li and Li (2021), agents privately observe how much they care about their future careers.

[8]In contrast, career concerns models assume that the tasks are exogenously specified. This is so even in multitasking environments (see, e.g., Dewatripont et al., 1999*b*; Kaarbøe and Olsen, 2006; Alesina and Tabellini, 2008). Siemsen (2008) supposes that agents choose their tasks from a (prefixed) menu.

[9]As an exception, Rodina (2020) studies a career concerns model in which the principal engages in information disclosure with the objective of maximizing an agent's effort.

information that is released to the labor market. As mentioned in the Introduction, we show that whether or not full disclosure is optimal depends on how the agent's type and performance score interactively determine his talent.

Second, our model is related to the extensive literature on certification design.[10] In canonical models of certification design, a supplier privately informed about the quality of his good chooses whether or not to have it certified, and the certifier optimizes over certification schemes by controlling information disclosed to the public. The key difference hence is that our model has the task performance stage which would correspond to product testing in the certification framework.[11,12] One salient conclusion in the certification design literature is that information revealed by the optimal disclosure rule is coarse. For example, it is often found that the binary "pass-fail" scheme is optimal. However, the optimality of coarse information in the certification literature is based on a different logic from the optimality of partial obscurity found here since it arises when the talent is submodular in the agent's type and his performance score. On this point, it is instructive to compare our model with the certification design model of Harbaugh and Rasmusen (2018). Harbaugh and Rasmusen (2018) show that the certifier finds it optimal to disclose coarse information when the supplier's reporting incentive is taken into account.[13] Since the certifier's objective function in Harbaugh and Rasmusen (2018) is given by the quadratic loss function as in the present model, their model closely resembles the one considered in this paper if the agent's talent is independent of his performance score. The supermodularity of the talent function holds under independence and our conclusion implies that full disclosure would be optimal. The difference in the conclusions mainly arises from the fact that in Harbaugh and Rasmusen (2018), there is no monetary transfer between the certifier and the supplier, whereas the effort cost chosen for each task by the principal serves the role of transfer in our model.

One interpretation of the present model is that it combines the models of certification, where an agent reports his private information to a certifier but performs no task, and those of career concerns, where an agent has no private information but performs a task to signal his quality.

Third, our model can also be related to the literature on school design which discusses the grading system of a school as a way to disclose information to a potential employer of its students. Among them, Bizzotto and Vigier (2021) present a model that includes types and performance as in this paper. In their model, a planner allocates a population of students to schools based on their types, and designs a grading system with the objective of maximizing the students' performance as measured by the rate at which they acquire competency through their effort. One major difference is that the planner

[10] See Dranove and Jin (2010) for a comprehensive survey.

[11] There is also difference in the certifier's objective function: In many models, certifiers either maximize their own profit (see, e.g., Lizzeri, 1999), or the senders' benefit as in the case of Ostrovsky and Schwarz (2010) where colleges maximize the students' job prospects.

[12] Some recent literature on certification considers the combination of reporting of private information and testing by a certifier. See for example Bizzotto et al. (2020).

[13] Harbaugh and Rasmusen (2018) assume that the certifier perfectly observes the agent's type when the agent choose to receive certification, and also uses pure disclosure rules.

observes the students' types in Bizzotto and Vigier (2021) so that allocation and grading are determined so as to mitigate the moral hazard problem.

Finally, with the principal being the sender of information about the agent's private type and performance score, our model is one of information design as pioneered by Kamenica and Gentzkow (2011).[14] The major departure from that literature is the assumption that the principal must collect part of his information from the agent through the provision of proper incentives. This is in contrast with the standard assumption in the literature that the sender has free access to information that he discloses. The standard concavification argument cannot be used because of the incentive constraints required for truthful reporting by the agent.[15]

3 Model

There exist an agent, a principal, and a firm. The agent's talent θ consists of two components s and ω. The first component s is observed privately by the agent, and is distributed over a finite set $S \equiv \{s_1, \ldots, s_K\}$ where $s_1 < \cdots < s_K$ and $K \geq 2$. On the other hand, the second component ω of the talent θ is the agent's performance at a task offered by the principal, and is distributed over a finite set $\Omega \equiv \{\omega_1, \ldots, \omega_L\}$. The agent's talent θ is a non-negative increasing function of s and ω:

$$\theta = \theta(s, \omega).$$

The pair (s, ω) is referred to as a *profile* and denoted by v. For any profile $v = (s, \omega)$, write $\theta_v = \theta_{sw} = \theta(s, \omega)$. The probability of every profile $v = (s, \omega)$ is positive

$$p_v \equiv \Pr(v) > 0 \quad \text{for every } v = (s, \omega) \in V \equiv S \times \Omega,$$

and the conditional distribution $g_s(\omega) \equiv \Pr(\omega \mid s)$ is ordered by stochastic dominance: For any s, $t \in S$ such that $s < t$,

$$\Pr(\omega \leq \omega_\ell \mid s) > \Pr(\omega \leq \omega_\ell \mid t) \quad \text{for } \ell = 1, \ldots, L-1. \tag{1}$$

In other words, the agent with a higher type s is more likely to generate a higher performance score ω.

The principal elicits from the agent his private type s and then assigns him a task. Each task is characterized by its difficulty, which is measured in terms of the effort cost y incurred by the agent to complete it. The agent faces no moral hazard and needs to bear the cost of the assigned task. Let $y : S \to \boldsymbol{R}$ be a *cost assignment rule* which specifies the effort cost for each reported type. The principal observes the agent's performance score at the assigned task, but the firm observes neither the reported type nor the performance score. At the completion of the task, the principal chooses a message as a function of both s and ω and sends it to the firm. The functional relationship between the profile

[14]The principal in our model minimizes the expected posterior variance of the talent.

[15]See Section 9.

$v = (s, \omega)$ and the message is described by a *disclosure rule* (Z, f): Z is the set of possible messages and $f : V \to \Delta Z$ maps each profile $v = (s, \omega)$ to a probability distribution over Z. Specifically, $f(z \mid v) \in [0, 1]$ is the probability that message z is sent when the profile $v \in V$ is realized. The firm then offers to the agent wage w equal to his expected talent:

$$w = E_v[\theta_v \mid z]. \tag{2}$$

The agent's utility equals the wage minus the effort cost of his assigned task: $w - y$. A *task assignment mechanism* $\Gamma = (y, Z, f)$ is a pair of the cost assignment rule y and the disclosure rule (Z, f). The timing of events is summarized as follows:

1. The principal chooses and publicly announces the mechanism $\Gamma = (y, Z, f)$.
2. The agent reports to the principal his type s.
3. The principal assigns to the agent a task with the effort cost $y(s)$.
4. The agent performs the task and the principal observes the performance score ω.
5. The principal sends a message z to the firm according to the disclosure rule (Z, f).
6. The firm pays to the agent wage equal to his expected talent $E_v[\theta_v \mid z]$.

We now describe the conditions that incentivize the agent to report his type truthfully to the principal. Let a disclosure rule (Z, f) be given, and define the *ex post expected wage function* $\phi : V \to \boldsymbol{R}_+$ by

$$\phi(s, \omega) = \sum_{z \in Z} E_v[\theta_v \mid z] \, f(z \mid s, \omega) \quad \text{for } v = (s, \omega) \in V.$$

$\phi(s, \omega)$ is the expected talent of the agent (and hence his expected wage) conditional on his report s and the performance score ω. Define also the *interim expected wage function* $H : S^2 \to \boldsymbol{R}_+$ by

$$H(s, t) = \sum_{\omega \in \Omega} g_s(\omega) \, \phi(t, \omega) \quad \text{for } s,\, t \in S.$$

$H(s, t)$ is the expected talent of the agent before the realization of the performance score ω but after the agent learns his type s and reports t to the principal.

The mechanism $\Gamma = (y, Z, f)$ is *incentive compatible* (IC) if the agent has incentive to report his private type truthfully:

$$H(s, s) - y(s) \geq H(s, t) - y(t) \quad \text{for any } s,\, t \in S, \tag{3}$$

and is *individually rational* (IR) if his utility from participation in the mechanism with truth-telling is greater than or equal to that of his outside option, which is normalized to zero:

$$H(s, s) - y(s) \geq 0 \quad \text{for any } s \in S.$$

The principal chooses a mechanism to best inform the firm about the agent's talent. Specifically, the principal aims to minimize the quadratic difference between the agent's talent as revealed from (s, ω), and the market expectation of his talent formed from the principal's announcement:[16]

$$\mathcal{L}(\Gamma) = E_{v,z}\big[\big(\theta_v - E_v[\theta_v \mid z]\big)^2\big]. \tag{4}$$

Note that (4) can equivalently be written as $\mathcal{L}(\Gamma) = E_z\big[\mathrm{Var}(\theta_v \mid z)\big]$, where $\mathrm{Var}(\theta_v \mid z)$ is the posterior variance of θ_v given $z \in Z$. Hence, we can alternatively state that the principal's objective is minimization of the expected value of the posterior variance.[17]

The mechanism $\Gamma^* = (y, Z, f)$ is *optimal* if it minimizes $\mathcal{L}(\Gamma)$ in the class of incentive compatible and individually rational mechanisms:

$$\Gamma^* \in \operatorname{argmin}\{\mathcal{L}(\Gamma) : \Gamma \text{ satisfies (IC) and (IR)}\}.$$

Since the cost of the task assigned to the agent does not enter the principal's objective function, it is used solely for the purpose of controlling the agent's incentive in the reporting stage. Our primary focus is hence on the disclosure rule which constitutes an incentive compatible and individually rational mechanism when coupled with *some* cost assignment rule. Specifically, a disclosure rule (Z, f) is *implementable* if there exists a cost assignment rule y such that the mechanism $\Gamma = (y, Z, f)$ is incentive compatible (IC) and individually rational (IR).

4 Implementable disclosure rules

We begin with the characterization of implementable disclosure rules in terms of the interim expected wage function H. The function $H : S^2 \to \boldsymbol{R}$ is *cyclically supermodular* if, for any $n = 2, \dots, K$ and any $k_1, \dots, k_n \in \{1, \dots, K\}$ which are all distinct, $k_1 = \min_i k_i$, and $k_{n+1} = k_1$,

$$\sum_{i=1}^{n} \{H(s_{k_i}, s_{k_i}) - H(s_{k_i}, s_{k_{i+1}})\} \geq 0. \tag{5}$$

Cyclical supermodularity is illustrated in Figure 1, where $a = H(s_1, s_2) - H(s_1, s_1)$, $b = H(s_2, s_4) - H(s_2, s_2)$, $c = H(s_3, s_3) - H(s_3, s_1)$, and $d = H(s_4, s_4) - H(s_4, s_3)$. The inequality (5) for the sequence $(k_1, k_2, k_3, k_4) = (1, 2, 4, 3)$ requires that $c + d \geq a + b$. In general, the graphical interpretation of (5) is that when we draw a series of (non-overlapping) right triangles with their hypotenuses on the diagonal both above and below it, the sum of the changes in the value of H along the vertical line segments of those triangles above the diagonal ($a + b$ in the example) is no larger than the sum of the corresponding changes below the diagonal ($c + d$ in the example). Different sequences

[16] This loss function encompasses a preference for conveying accurate information, a reputational incentive if the principal relies on an external certifier agency, or a concern for the welfare of the firm if the principal relies on a specific department within the firm.

[17] This is in contrast with a common assumption in the information design literature that the sender's objective is a function of the posterior mean of the state (e.g., Dworczak and Martini, 2019; Kolotilin, 2018; Arieli et al., 2023).

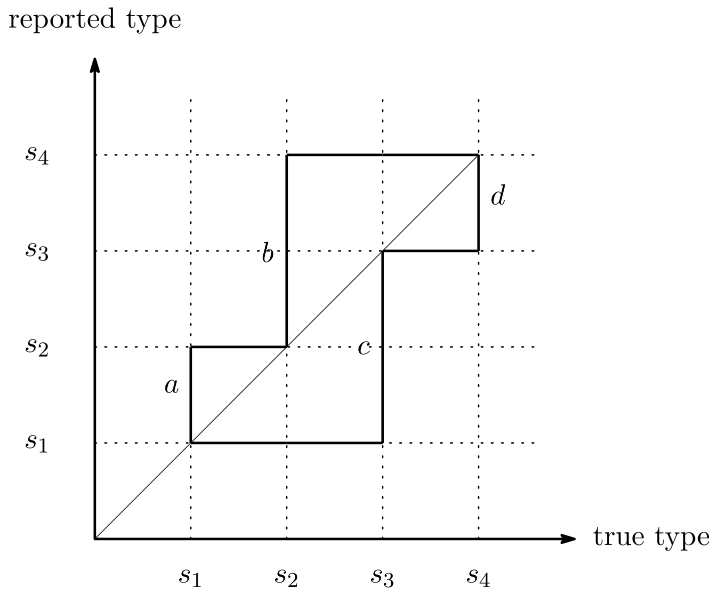

Figure 1: Cyclical supermodularity of H requires $a + b \leq c + d$ when $(k_1, k_2, k_3, k_4) = (1, 2, 4, 3)$ in (5)

$(k_1, \ldots, k_n)$ correspond to different collections of such triangles. Since k_1 is chosen to be the smallest among $k_1, \ldots, k_n$, there exist $(n-1)!$ such sequences for a fixed n. The total number of inequalities in (5) for $n = 2, \ldots, K$ hence equals

$$N = \sum_{n=2}^{K} \binom{K}{n} (n-1)!. \tag{6}$$

Importantly, cyclical supermodularity is weaker than supermodularity, which requires that the change in the value of H along each vertical line segment is no larger than the corresponding change along the vertical line segment to the right. The following lemma is a formal statement of this observation.

Lemma 1 *If H is supermodular, then it is cyclically supermodular.*

To understand the relationship between cyclical supermodularity of H and implementability of a mechanism, suppose first that the agent has two types: $S = \{s_1, s_2\}$. In this case, the unique relevant sequence is $(k_1, k_2) = (1, 2)$, and (5) is written as

$$H(s_1, s_2) - H(s_1, s_1) \leq H(s_2, s_2) - H(s_2, s_1).$$

We can then choose the cost assignment rule y to satisfy

$$H(s_1, s_2) - H(s_1, s_1) \leq y(s_2) - y(s_1) \leq H(s_2, s_2) - H(s_2, s_1).$$

It can be readily verified that these inequalities correspond to the (IC) conditions (3) for s_1 and s_2. Suppose next that the agent has three types $S = \{s_1, s_2, s_3\}$. Under Γ,

s_1 should have no incentive to misrepresent himself as s_2, s_2 as s_3, and s_3 as s_1. These conditions can be respectively written as:

$$\begin{aligned} y(s_2) - y(s_1) &\geq H(s_1, s_2) - H(s_1, s_1), \\ y(s_3) - y(s_2) &\geq H(s_2, s_3) - H(s_2, s_2), \\ H(s_3, s_3) - H(s_3, s_1) &\geq y(s_3) - y(s_1). \end{aligned}$$

Adding these inequalities side by side, we obtain

$$H(s_3, s_3) - H(s_3, s_1) \geq H(s_1, s_2) - H(s_1, s_1) + H(s_2, s_3) - H(s_2, s_2),$$

which is equivalent to (5) for the sequence $(k_1, k_2, k_3) = (1, 2, 3)$. Different sequences appearing in (5) likewise correspond to the feasibility of different combinations of incentive conditions. The following proposition shows that cyclical supermodularity is not only necessary but also sufficient for the existence of a cost assignment rule y that makes Γ incentive compatible and individually rational.

Proposition 2 *The disclosure rule (Z, f) is implementable if and only if H is cyclically supermodular.*

In general, a cost assignment rule y that makes Γ incentive compatible and individually rational is defined only implicitly. When H is supermodular, however, an explicit characterization is available as follows.

Proposition 3 *Suppose that (Z, f) is such that H is supermodular. Then $\Gamma = (y, Z, f)$ is incentive compatible and individually rational if the effort cost y is defined recursively by*

$$\begin{aligned} &y(s_1) = 0, \\ &y(s_{k+1}) = y(s_k) + H(s_k, s_{k+1}) - H(s_k, s_k) \quad \textit{for } k = 1, \ldots, K-1. \end{aligned} \tag{7}$$

Note that y defined in (7) is increasing if and only if

$$H(s_k, s_{k+1}) - H(s_k, s_k) \geq 0 \quad \text{for every } k = 1, \ldots, K-1.$$

That is, if type s_k expects a higher wage by reporting s_{k+1} than by reporting s_k, then the agent should be assigned a task of a higher effort cost when he reports s_{k+1}. However, it does not follow from the general specification of the disclosure rule that reporting a higher type indeed leads to a higher expected wage.

Despite the difference between interim and ex post expectations, the following lemma shows that the cyclical supermodularity of H reduces to the supermodularity of the ex post expected wage function ϕ when the performance score is binary ($\Omega = \{\omega_1, \omega_2\}$).

Lemma 4 *When the performance score is binary $\Omega = \{\omega_1, \omega_2\}$, the disclosure rule (Z, f) is implementable if and only if ϕ is supermodular:*

$$\phi(t, \omega_2) + \phi(s, \omega_1) \geq \phi(t, \omega_1) + \phi(s, \omega_2) \quad \textit{if } s < t. \tag{8}$$

Furthermore, ϕ is supermodular if and only if

$$\phi(s_{k+1}, \omega_2) + \phi(s_k, \omega_1) \geq \phi(s_{k+1}, \omega_1) + \phi(s_k, \omega_2) \quad \textit{for } k = 1, \ldots, K-1. \tag{9}$$

Based on this observation, we below present a few examples that illustrate the implementability of some disclosure rules in the simple 2×2 environment: $S = \{s_1, s_2\}$ and $\Omega = \{\omega_1, \omega_2\}$. Denote the four profiles by

$$v_1 = (s_1, \omega_1), \quad v_2 = (s_1, \omega_2), \quad v_3 = (s_2, \omega_1), \quad \text{and} \quad v_4 = (s_2, \omega_2), \tag{10}$$

and let

$$p_m = p(v_m), \quad \theta_m = \theta(v_m), \quad \phi_m = \phi(v_m) \quad \text{for } m = 1, \ldots, 4. \tag{11}$$

Example 1 Let $Z = V$ and

$$f(v_i \mid v_i) = 1 \quad \text{for } i = 1, \ldots, 4.$$

This is the "full" disclosure rule that perfectly reveals every profile (Figure 2). Since $\phi(v_i) = E_v[\theta_v \mid v_i] = \theta_i$ under this rule, it is implementable if and only if θ is supermodular by Lemma 4.

Since the full disclosure rule eliminates any loss arising from the difference between the true and expected talents, it is clearly optimal for the principal if it is implementable. In the following examples, then, suppose that θ is not supermodular:

$$\Delta \equiv \theta_2 + \theta_3 - \theta_1 - \theta_4 > 0. \tag{12}$$

Example 2 Let $Z = \{z_1, z_2\}$, and

$$f(z_1 \mid v) = \begin{cases} 1 & \text{if } v = v_1, \\ 0 & \text{otherwise,} \end{cases} \qquad f(z_2 \mid v) = \begin{cases} 0 & \text{if } v = v_1, \\ 1 & \text{otherwise.} \end{cases}$$

This is the rule where the agent obtains the "Fail" grade z_1 when the profile $v_1 = (s_1, \omega_1)$ realizes and the "Pass" grade z_2 otherwise (Figure 2). In this case,

$$E_v[\theta_v \mid z] = \begin{cases} \theta_1 & \text{if } z = z_1, \\ \mu_2 & \text{otherwise,} \end{cases}$$

where $\mu_2 = E_\theta[\theta \mid v \neq v_1]$. Verify also that

$$\phi_m = \begin{cases} \theta_1 & \text{if } m = 1, \\ \mu_2 & \text{otherwise.} \end{cases}$$

Since $\mu_2 > \theta_1$, ϕ is not supermodular:

$$\phi_4 - \phi_3 - \phi_2 + \phi_1 = \theta_1 - \mu_2 < 0.$$

It follows that this disclosure rule is *not* implementable.

Example 3 Let $Z = \{z_1, z_2\}$, and

$$f(z_1 \mid v) = \begin{cases} 0 & \text{if } v = v_4, \\ 1 & \text{otherwise,} \end{cases} \qquad f(z_2 \mid v) = \begin{cases} 1 & \text{if } v = v_4, \\ 0 & \text{otherwise.} \end{cases}$$

This is the rule where the agent obtains the "Pass" grade z_2 when the profile $v_4 = (s_2, \omega_2)$ realizes and the "Fail" grade otherwise (Figure 2). In this case,

$$E_v[\theta_v \mid z] = \begin{cases} \theta_4 & \text{if } z = z_2, \\ \mu_1 & \text{otherwise,} \end{cases}$$

where $\mu_1 = E_v[\theta_v \mid v \neq v_4]$. Verify also that

$$\phi_m = \begin{cases} \theta_4 & \text{if } m = 4, \\ \mu_1 & \text{otherwise.} \end{cases}$$

Since $\mu_1 < \theta_4$, ϕ is supermodular:

$$\phi_4 - \phi_3 - \phi_2 + \phi_1 = \theta_4 - \mu_1 > 0.$$

This disclosure rule is hence implementable.

The above examples suggest the following intuition: In order to make ϕ supermodular when θ is not, one needs to either "shrink" the distance $\phi_2 - \phi_1$ or $\phi_3 - \phi_1$ by "pushing up" ϕ_1 and at the same time "pulling down" ϕ_2 or ϕ_3 by pooling v_1 with a higher realization v_2 or v_3. The disclosure rule in Example 3 above does this and hence is implementable. On the other hand, the disclosure rule in Example 2 pools the highest profile v_4 with lower realizations. Such a rule is not implementable since it will shrink the distance $\phi_4 - \phi_3$ and $\phi_4 - \phi_2$ and hence lead to an even severer violation of the supermodularity condition.

Example 4 Let $Z = \{z_1, z_2, z_3\}$, and

$$f(z_1 \mid v) = \begin{cases} 1 & \text{if } v = v_1, \\ 0 & \text{otherwise,} \end{cases} \qquad f(z_2 \mid v) = \begin{cases} 1 & \text{if } v = v_2 \text{ or } v_3, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f(z_3 \mid s, \omega) = \begin{cases} 1 & \text{if } v = v_4, \\ 0 & \text{otherwise.} \end{cases}$$

This is the rule where the agent receives the "high" grade z_3 if $v = (s_2, \omega_2)$, the "low" grade z_1 if $v = (s_1, \omega_1)$, and the "medium" grade z_2 otherwise (Figure 2). In this case,

$$E_v[\theta_v \mid z] = \begin{cases} \theta_4 & \text{if } z = z_3, \\ \theta_1 & \text{if } z = z_1, \\ \mu_2 & \text{otherwise,} \end{cases}$$

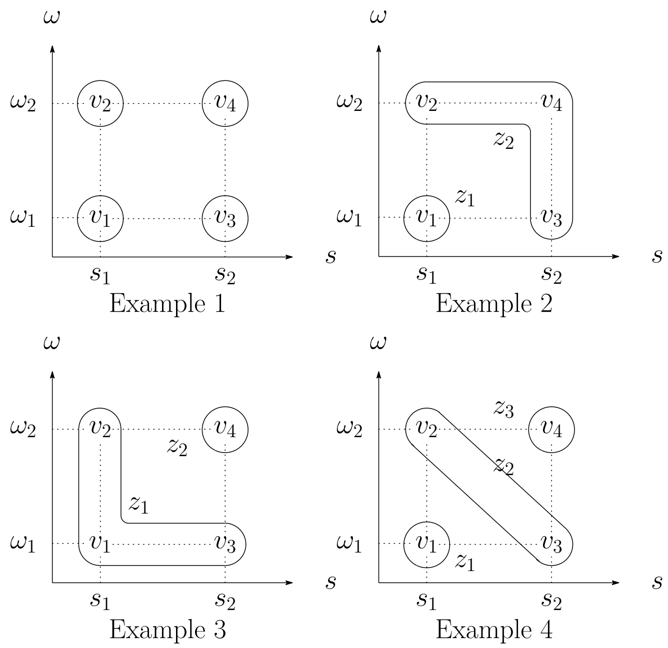

Figure 2: Disclosure rules in the 2×2 environment

Connected profiles are pooled.

Example	implementable when $\Delta > 0$?
1:	No;
2:	No;
3:	Yes;
4:	Yes iff $\Delta \leq \frac{(\theta_3-\theta_2)(p_2-p_3)}{p_2+p_3}$.

where

$$\mu_2 = E_v[\theta_v \mid v \in \{v_2, v_3\}] = \frac{p_2\theta_2 + p_3\theta_3}{p_2 + p_3}.$$

We also have

$$\phi_m = \begin{cases} \theta_4 & \text{if } m = 4, \\ \theta_1 & \text{if } m = 1, \\ \mu_2 & \text{otherwise.} \end{cases}$$

Since $\theta_1 < \mu_2 < \theta_4$, this disclosure rule is implementable if and only if

$$\begin{aligned} & \phi_4 - \phi_3 - \phi_2 + \phi_1 = \theta_4 - 2\mu_2 + \theta_1 \geq 0 \\ \Leftrightarrow \quad & \Delta \leq \frac{(p_2 - p_3)(\theta_3 - \theta_2)}{p_2 + p_3}, \end{aligned} \tag{13}$$

where $\Delta = \theta_2 + \theta_3 - \theta_1 - \theta_4$. Hence, if

$$(p_2 - p_3)(\theta_3 - \theta_2) > 0, \tag{14}$$

then (13) holds if $\frac{\Delta}{|\theta_3 - \theta_2|} > 0$ is small compared with $|p_2 - p_3|$. We will return to this last observation in Section 6.

5 Full and partial disclosure rules

As noted in Section 4, when there is no incentive issue in the reporting stage, perfectly disclosing information about the realized profile $v = (s, \omega)$ clearly minimizes the principal's loss function L. Specifically, (Z, f) is a *full disclosure* rule if $Z = S \times \Omega$, and

$$f(z \mid s, \omega) = \begin{cases} 1 & \text{if } z = (s, \omega), \\ 0 & \text{otherwise.} \end{cases}$$

The full disclosure rule however may not induce truth-telling from the agent when his type s is private. To see when full disclosure induces truth-telling, note that the ex post expected talent ϕ equals the true talent θ under full disclosure, and hence that the interim expected wage function H is given by $H(s, t) = \sum_\omega g_s(\omega)\, \theta(t, \omega)$. Even when type s is private, hence, full disclosure is implementable if and only if this function is cyclically supermodular. The following proposition presents a sufficient condition for this as the first main result on the optimal disclosure rule.

Proposition 5 *Suppose that the talent function θ is supermodular. Then the optimal mechanism Γ entails full disclosure.*

For example, the talent function θ is supermodular if for $\delta \geq 0$,

$$\theta(s, \omega) = s + \omega + \delta\, s\omega. \tag{15}$$

Note that the supermodularity of θ does not imply complementarity between the private type and performance in the determination of talent since they are perfect substitutes if $\delta = 0$.

In view of Proposition 5, we assume in what follows that the talent function θ is not supermodular, and more concretely, that it is *strictly submodular*: For any s, $t \in S$ and ω, $\hat{\omega} \in \Omega$ such that $t > s$ and $\hat{\omega} > \omega$,

$$\theta(t, \hat{\omega}) + \theta(s, \omega) < \theta(s, \hat{\omega}) + \theta(t, \omega). \tag{16}$$

For example, θ is strictly submodular if $\delta < 0$ in (15), or if there exists a strictly concave and increasing function $u : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ such that

$$\theta(s, \omega) = u(s + \omega).$$

When considering a class of talent functions in our analysis, we fix the relative ranking of their values at different profiles. Specifically, we consider a total ordering $\succcurlyeq$ over the set $V = S \times \Omega$ of profiles that are *consistent* with the value of ϕ in the sense that

$$v \prec w \quad \Leftrightarrow \quad \theta(v) < \theta(w).$$

Since we assume that θ is increasing, $\succcurlyeq$ satisfies

$$(s, \omega) \leq (\hat{s}, \hat{\omega}) \text{ and } (s, \omega) \neq (\hat{s}, \hat{\omega}) \quad \Rightarrow \quad (s, \omega) \prec (\hat{s}, \hat{\omega}).$$

When $\hat{s} > s$ and $\hat{\omega} > \omega$, any of $(s, \hat{\omega}) \succ (\hat{s}, \omega)$, $(\hat{s}, \omega) \succ (s, \hat{\omega})$, and $(s, \hat{\omega}) \sim (\hat{s}, \omega)$ is possible.

We now turn to the description of a general disclosure rule. Any disclosure rule (Z, f) with a finite message set Z can be expressed as:

$$\begin{aligned} &Z = V \cup \{z_1, \ldots, z_R\}, \quad V \cap \{z_1, \ldots, z_R\} = \emptyset, \\ &f(v \mid v) + \sum_{r=1}^{R} f(z_r \mid v) = 1 \text{ for any } v \in V. \end{aligned} \tag{17}$$

In other words, when the profile v is realized, one of the $R + 1$ messages v, $z_1, \ldots, z_R$ is potentially chosen. Since the message $v \in V$ is sent only after its realization ($f(\hat{v} \mid v) = 0$ if v, $\hat{v} \in V$ and $v \neq \hat{v}$), each $v \in V$ is a perfectly revealing message of its realization.[18] In contrast, each $z_r \in Z$ is a *pooling message* that is sent with positive probability after the realization of multiple profiles. Given any disclosure rule (Z, f) represented as in (17), define for each v, $w \in V$ and $r = 1, \ldots, R$,

$$\alpha_v^r = f(z^r \mid v), \quad \sigma^r = \sum_{v \in V} p_v \alpha_v^r, \quad \mu^r = E_v[\theta_v \mid z^r] = \frac{1}{\sigma^r} \sum_{v \in V} p_v \alpha_v^r \theta_v. \tag{18}$$

As seen, α_v^r is the probability that message z^r is sent when the profile v is realized, σ^r is the marginal probability that message z^r is sent, and μ^r is the expected talent of the

[18] Inclusion of such a message v in Z is without loss of generality since $f(v \mid v) = 0$ is also allowed.

agent conditional on the observation of message z^r. For each message $z \in Z$, let $\text{supp}(z)$ denote the *support* of z:

$$\text{supp}(z) = \{v : \alpha_v^r > 0\}.$$

Define now

$$x_{vw} = |\theta_v - \theta_w| \sum_{r=1}^{R} \frac{\alpha_v^r \alpha_w^r}{\sigma^r}. \tag{19}$$

By definition, $x_{vw} = x_{wv}$ for any v, $w \in V$ and $x_{vw} = x_{wv} = 0$ if $v = w$. Although it is not easy to give an intuitive meaning to x_{vw}, they play the key role in the analysis in what follows since the expected wage functions ϕ and H as well as the quadratic loss function $\mathcal{L}$ are all linear functions of $x = (x_{vw})_{v,w}$ as shown in the following lemma.

Lemma 6 *Suppose that the disclosure rule (Z, f) is given by (17) and is implementable. Then*[19]

$$\phi(s,\omega) = \theta_{s\omega} + \sum_v p_v\, x_{v,s\omega}\, (-1)^{\mathbf{1}\{v \prec s\omega\}}, \tag{20}$$

$$H(s,t) = \sum_\omega g_s(\omega) \Big[\theta_{t\omega} + \sum_v p_v\, x_{v,t\omega}\, (-1)^{\mathbf{1}\{v \prec t\omega\}}\Big], \tag{21}$$

$$\mathcal{L}(\Gamma) = \sum_{\{(v,w):\, v \prec w\}} p_v p_w\, x_{vw}\, |\theta_w - \theta_v|. \tag{22}$$

The rest of this section introduces some conditions on the talent function. Given $\varepsilon > 0$, we say that a submodular talent function θ is ε-linear if there exists $h > 0$ such that for any $s < t$ and ω,

$$\left| \frac{\theta(t,\omega) - \theta(s,\omega)}{t - s} - h \right| < \varepsilon. \tag{23}$$

When θ is submodular, its degree of submodularity is small if it is also ε-linear since for any $s < t$ and $\omega < \hat{\omega}$,

$$\begin{aligned} \big\{\theta(t,\hat{\omega}) + \theta(s,\omega)\big\} - \big\{\theta(s,\hat{\omega}) + \theta(t,\omega)\big\} &\geq (h - \varepsilon)(t - s) - (h + \varepsilon)(t - s) \\ &= -2\varepsilon(t - s) \\ &\geq -2\varepsilon(s_K - s_1). \end{aligned}$$

[19] $\mathbf{1}$ is the indicator function so that

$$(-1)^{\mathbf{1}\{v \prec s\omega\}} = \begin{cases} -1 & \text{if } \theta_v < \theta_{s\omega}, \\ 1 & \text{if } \theta_v \geq \theta_{s\omega}. \end{cases}$$

Given $\eta > 0$, a talent function θ has a *value margin* η if, when θ changes its values from one profile v to another profile w, it does so at least by margin η: There exists $\eta > 0$ such that for any v, $w \in V$,

$$\theta(v) \neq \theta(w) \quad \Rightarrow \quad |\theta(v) - \theta(w)| > \eta.$$

Given $\succcurlyeq$ and $\eta > 0$, let $\Theta_{\succcurlyeq,\eta}$ be the class of talent functions such that

$$\Theta_{\succcurlyeq,\eta} = \{\theta : \theta \text{ is consistent with } \succcurlyeq, \text{ and has a value margin } \eta\}. \tag{24}$$

6 Models with binary performance scores

A binary performance score is relevant in many situations where performance can only be judged either good or poor because of physical limitation in precise measurement. When the performance score is binary ($L = 2$), a disclosure rule is implementable if and only if the ex post expected wage function ϕ is supermodular by Lemma 4. Lemma 4 further shows that submodularity of ϕ reduces to the $(K - 1)$ local conditions (9), implying that implementability of a disclosure rule is expressed by $(K - 1)$ inequalities. We begin with the 2×2 model where $K = L = 2$ (i.e, $S = \{s_1, s_2\}$ and $\Omega = \{\omega_1, \omega_2\}$), and then consider the case where $K \geq 3$.

6.1 2×2 Model

The discussion of implementable disclosure rules in Section 4 (Examples 1-4) already furnishes the key intuitions developed in this section. We use the notation in (10) and (11) in Section 4 while noting that the submodularity of θ in (16) is equivalent to (12). Although none of the disclosure rules in Examples 2-4 involve randomization, the principal may also benefit from randomization if pooling multiple profiles without randomization results in the slackness in the supermodularity of ϕ. This can be seen in the following example.

Example 5 Suppose that the disclosure rule (Z, f) is such that

- $Z = V \cup \{z\}$ for $z \notin V$;
- $f(z \mid v_1) = f(z \mid v_2) = \lambda$ for some $\lambda \in (0, 1)$;
- $f(v_1 \mid v_1) = f(v_2 \mid v_2) = 1 - \lambda$ and $f(v_3 \mid v_3) = f(v_4 \mid v_4) = 1$.

Figure 3 illustrates this disclosure rule, which pools v_1 and v_2 with probability λ, but perfectly reveals them with probability $1 - \lambda$. It also perfectly reveals both v_3 and v_4. ϕ is supermodular if and only if

$$\begin{aligned}\phi_1 + \phi_4 - \phi_2 - \phi_3 &= \{\lambda\theta_1 + (1 - \lambda)\mu_1\} + \theta_4 - \{\lambda\theta_2 + (1 - \lambda)\,\mu_1\} - \theta_3 \\ &= (\theta_4 - \theta_3) - \lambda(\theta_2 - \theta_1) \\ &\geq 0.\end{aligned}$$

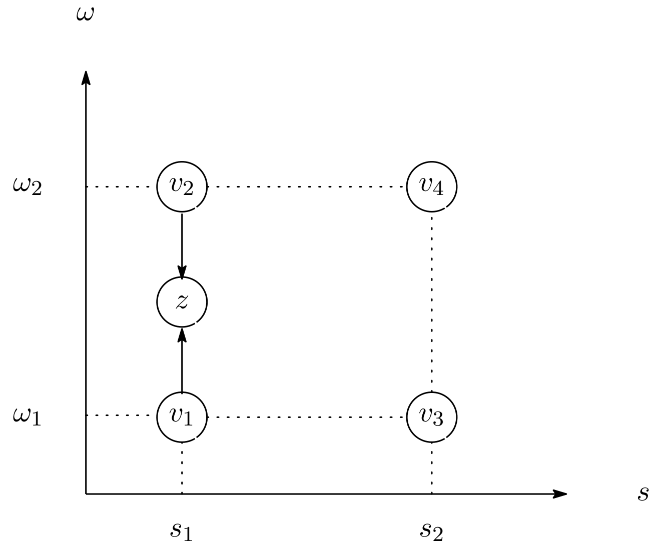

Figure 3: Disclosure rule of Example 5 in the 2×2 environment
Each circle represents a message. z is an pooling message that is sent when either v_1 or v_2 is realized.

When $\lambda = 1$, (Z, f) is the full disclosure rule and not implementable if θ is submodular ($\frac{\theta_4-\theta_3}{\theta_2-\theta_1} < 1$). On the other hand, when $\lambda = 0$, it is implementable but always generates a loss equal to $(\theta_1 - \mu_1)^2$ when the profile is (s_1, ω_1) and $(\theta_2 - \mu_1)^2$ when the profile is (s_1, ω_2). One can minimize the probability of such a loss while maintaining implementability by setting $\lambda = \frac{\theta_4-\theta_3}{\theta_2-\theta_1}$.

Proposition 7 *Suppose that the talent function θ is submodular ($\Delta > 0$) and that (θ, p) satisfies either one of (25), (26), and (27) below:*[20]

$$(p_2 - p_3)(\theta_3 - \theta_2) \leq 0, \tag{25}$$

$$(p_2 - p_3)\left(\frac{\theta_2 - \theta_1}{\theta_3 - \theta_1} - \frac{p_3(p_1 + p_2)}{p_2(p_1 + p_3)}\right) \leq 0, \tag{26}$$

$$\Delta \leq \frac{(p_2 - p_3)(\theta_3 - \theta_2)}{p_2 + p_3}. \tag{27}$$

Then there exists an optimal disclosure rule (Z, f) with exactly one pooling message z:

$$Z = V \cup \{z\} \text{ for } z \notin V.$$

[20] Since (26) implies $(p_2 - p_3)(\theta_3 - \theta_2) > 0$, (25) and (26) are mutually exclusive, and so are (25) and (27). On the other hand, (26) and (27) have an overlap.

The support of the pooling message z is binary and is given by

$$\mathrm{supp}(z) = \begin{cases} \{v_1,\, v_2\} & \textit{if } \frac{\theta_2-\theta_1}{\theta_3-\theta_1} \leq \frac{p_3(p_1+p_2)}{p_2(p_1+p_3)} \textit{ and} \\ & \textit{either } p_2 > p_3 \textit{ or } (p_2-p_3)(\theta_3-\theta_2) \leq 0, \\ \{v_1,\, v_3\} & \textit{if } \frac{\theta_2-\theta_1}{\theta_3-\theta_1} \geq \frac{p_3(p_1+p_2)}{p_2(p_1+p_3)} \textit{ and} \\ & \textit{either } p_2 < p_3 \textit{ or } (p_2-p_3)(\theta_3-\theta_2) \leq 0 \\ \{v_2,\, v_3\} & \textit{if (27) holds but (26) fails.} \end{cases}$$

Proposition 7 shows that the unique pooling message is sent only after the realization of a particular pair of profiles.[21] In line with the intuition provided in Examples 2-5, the pooling shrinks the difference between the expected talents at v_1 and at v_2 or between those at v_1 and at v_3. The disclosure rule of Example 5 (in Figure 3) is indeed one of the rules described in Proposition 7. Although the highest profile v_4 is never pooled with other profiles, v_2 and v_3 are pooled with each other in some cases.[22]

We also note that the probability $f(z \mid v)$ that the pooling message z is sent increases with Δ: The proof of Proposition 7 shows that the probability of pooling under the optimal disclosure rule can be taken as:

$$\begin{array}{ll} f(z \mid v_1) = f(z \mid v_2) = \frac{\Delta}{\theta_2-\theta_1} & \text{if } \mathrm{supp}(z) = \{v_1, v_2\}, \\ f(z \mid v_1) = f(z \mid v_3) = \frac{\Delta}{\theta_3-\theta_1} & \text{if } \mathrm{supp}(z) = \{v_1, v_3\}, \\ f(z \mid v_2) = f(z \mid v_3) = \frac{p_2+p_3}{p_2-p_3}\, \frac{\Delta}{\theta_3-\theta_2} & \text{if } \mathrm{supp}(z) = \{v_2, v_3\}. \end{array} \tag{28}$$

The interpretation is that the higher degree of submodularity Δ requires a higher probability of pooling in order to make the ex post expected wage function ϕ supermodular.

Graphical illustration of Proposition 7 is possible with the introduction of some structure on p. Let $q = \Pr(s_1) \in (0,1)$ be the probability that the agent has the low type, and suppose that

$$\gamma \equiv \Pr(\omega_1 \mid s_1) = \Pr(\omega_2 \mid s_2) > \tfrac{1}{2}.$$

γ is the probability that the performance score is low when the agent has the low type or that it is high when the agent has the high type. $\gamma > \frac{1}{2}$ ensures first-order stochastic dominance.[23] Denote

$$\beta = \frac{\theta_2-\theta_1}{\theta_3-\theta_1}.$$

[21]In other words, the posterior belief given the pooling message is the convex combination of two degenerate posteriors. Kolotilin and Wolitzky (2020) call such posteriors *pairwise*.

[22]Since (27) is equivalent to (13), the disclosure rule that pools v_2 and v_3 with positive probability is optimal only if the disclosure rule that pools these profiles with probability one (in Example 5) is implementable.

[23]The joint distribution p is hence given by

$$p_1 = q\gamma, \quad p_2 = q(1-\gamma), \quad p_3 = (1-q)(1-\gamma), \quad p_4 = (1-q)\gamma.$$

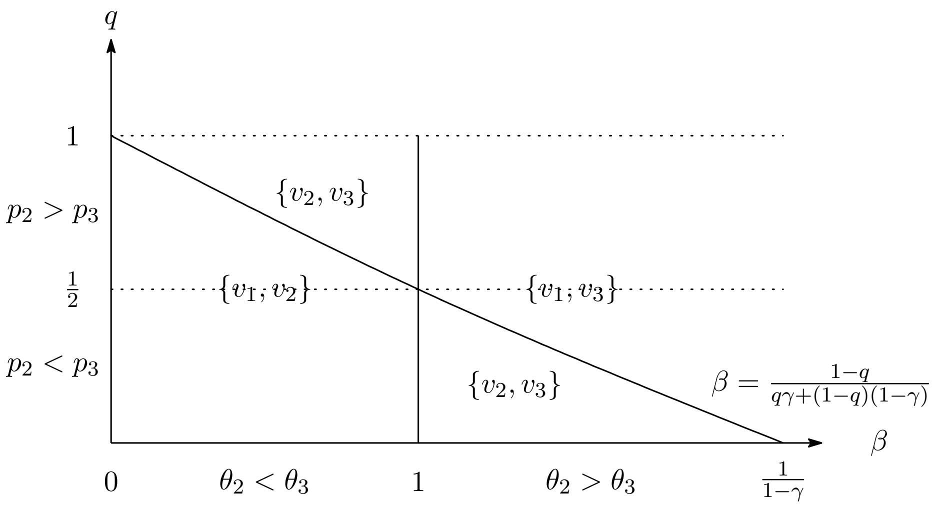

Figure 4: Support of the pooling message z

$q = \Pr(s_1)$, $\beta = \frac{\theta_2-\theta_1}{\theta_3-\theta_1}$, and $\gamma = \Pr(\omega_1 \mid s_1) = \Pr(\omega_2 \mid s_2)$.
$q \lesseqgtr \frac{1}{2} \Leftrightarrow p_2 \lesseqgtr p_3$ and $\beta \lesseqgtr 1 \Leftrightarrow \theta_2 \lesseqgtr \theta_3$.
Pooling with support $\{v_2, v_3\}$ is optimal only if Δ satisfies (27).

The sufficient conditions of Proposition 7 are then written as:

$$\begin{aligned}
(25):& \quad (1-\beta)(2q-1) \le 0, \\
(26):& \quad (2q-1)\left(\beta - \frac{1-q}{q\gamma + (1-q)(1-\gamma)}\right) \le 0, \\
(27):& \quad \Delta \le (1-\beta)(2q-1).
\end{aligned}$$

Figure 4 describes the optimal disclosure rule for each combination of (β, q). As seen, pooling v_1 and v_2 is optimal when $\beta < 1$ ($\Leftrightarrow \theta_2 < \theta_3$) and $q = \Pr(s_1)$ is not high, and pooling v_1 and v_3 is optimal when $\beta > 1$ ($\Leftrightarrow \theta_2 > \theta_3$) and q is not low. The disclosure rule that pools v_2 and v_3 is feasible only when Δ satisfies (27).

The intuition behind Proposition 7 is as follows: Using Lemma 6, the proof of the proposition converts the optimization problem with respect to the probabilities $\alpha_v^r = f(z^r \mid v)$ of message z^r given the profile v into a linear problem with respect to the variables $x = (x_{vw})_{v \succ w}$: Since the objective function $\mathcal{L}$ and the inequality expressing the supermodularity of the ex post talent function ϕ are both linear in x, there exists a corner solution x^* to this problem. Specifically, corresponding to the single inequality constraint, there exists a single coordinate (v, w) such that $x_{vw}^* > 0$. By the definition of x, this implies that $\sum_{r=1}^R \alpha_v^r \alpha_w^r > 0$ for exactly one pair of profiles (v, w). Given this, we can find the probabilities α_v^1, $\alpha_w^1 \in (0, 1]$ that replicate the solution x_{vw}^*. In other words, we can take the number R of pooling messages equal to one, and let this pooling message z^1 be sent only when the realized profile is either v or w. This last construction requires (27) when the pooling message z has support $\{2, 3\}$ so that the probabilities

α_{v_2} and α_{v_3} would not exceed one.[24]

If $x^*_{23} > 0$ but (27) fails, replication of x^*_{23} using α^1_2 and α^1_3 is not possible since their values would then exceed one. We conjecture that the optimal disclosure rule in this case involves four fully revealing messages $v_1, \ldots, v_4$ along with a pooling message z with support $\text{supp}(z) = \{v_1, v_2, v_3\}$ as in Example 3. Characterization of the optimal rule however is difficult and remains an open question. On the other hand, Proposition 7 holds whenever the talent function θ is mildly submodular (and satisfies (27)) since the conditions (25) and (26) do not involve Δ. This observation leads to the following corollary to Proposition 7.

Corollary 8 *Let $\succcurlyeq$ and $\eta > 0$ be given, and suppose that $\theta \in \Theta_{\succcurlyeq,\eta}$ is submodular and ε-linear for ε satisfying $\frac{4\varepsilon}{\eta} \leq \frac{|p_2-p_3|}{p_2+p_3}$. Then there exists an optimal disclosure rule as described in Proposition 7.*

In the analysis of a more general environment below, we generalize Corollary 8 by assuming that $\Delta > 0$ is not large while fixing the probability distribution p.

6.2 $K \times 2$ Model

We now suppose that the number $K = |S|$ of the agent's types s can be greater than two but continue to assume that his performance score ω is binary.

Example 6 Consider the following generalization of the disclosure rule discussed in Example 5: (Z, f) is such that for $z_1, \ldots, z_K \notin V$ and $\lambda_1, \ldots, \lambda_K \in [0, 1]$,

- $Z = V \cup \{z_1, \ldots, z_K\}$;
- $f(v_{k\ell} \mid v_{k\ell}) = \lambda_k$ for every k, ℓ;
- $f(z_k \mid v) = \begin{cases} 1 - \lambda_k & \text{if } v = v_{k1} \text{ or } v_{k2}, \\ 0 & \text{otherwise,} \end{cases}$

Figure 5 illustrates this disclosure rule, which either perfectly reveals the realized profile or pools the two profiles v_{k1} and v_{k2} if either of them occurs. Define

$$\mu_k = E_v[\theta_v \mid z_k] = \frac{p_{k1}\theta_{k1} + p_{k2}\theta_{k2}}{p_{k1} + p_{k2}}.$$

Then

$$\phi_{k\ell} = \lambda_k \theta_{k\ell} + (1 - \lambda_k)\,\mu_k,$$

so that

$$\phi_{k1} + \phi_{k+1,2} - \phi_{k2} - \phi_{k+1,1} = -\lambda_k(\theta_{k2} - \theta_{k1}) + \lambda_{k+1}(\theta_{k+1,2} - \theta_{k+1,1}).$$

[24] This can be seen from the fact that $f(z \mid v_2) = f(z \mid v_3) \leq 1$ in the third line of (28) if and only if (27) holds.

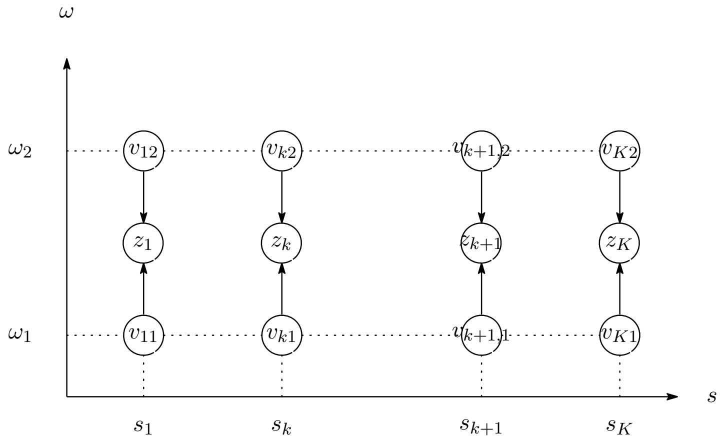

Figure 5: Disclosure rule of Example 6 in the $K \times 2$ environment

It follows that ϕ is supermodular if and only if

$$\frac{\lambda_{k+1}}{\lambda_k} \geq \psi_k \equiv \frac{\theta_{k2} - \theta_{k1}}{\theta_{k+1,2} - \theta_{k+1,1}} \quad \text{for } k = 1, \ldots, K-1. \tag{29}$$

Since θ is assumed submodular, $\psi_k \geq 1$, suggesting that the probability of perfect revelation should increase with k. In particular, we may take $\lambda_K = 1$ so that full disclosure takes place when the agent reports the highest type s_K.

Proposition 9 *Let p, $\succcurlyeq$ and $\eta > 0$ be given, and suppose that the talent function $\theta \in \Theta_{\succcurlyeq,\eta}$ is submodular. Then there exists $\varepsilon > 0$ such that if θ is ε-linear, there exists an optimal disclosure rule (Z, f) such that*

- *$Z = V \cup \{z_1, \ldots, z_R\}$ for some $R \leq K-1$ and $z_1, \ldots, z_R \notin V$.*
- *$|\operatorname{supp}(z_r)| = 2$ for every $r = 1, \ldots, R$.*
- *$\operatorname{supp}(z_r) \neq \{v_{12}, v_{K2}\}$, $\{v_{K1}, v_{K2}\}$ for any $r = 1, \ldots, R$.*

Proposition 9 shows that if the talent function θ is mildly submodular, there exists an optimal disclosure policy with at most $(K-1)$ pooling messages each of which pools no more than two profiles. Furthermore, the pooled pair of profiles is never the combination of the extreme upper-left profile and the extreme upper-right profile (v_{12}, v_{K2}) or the extreme lower-right profile and the extreme upper-right profile (v_{K1}, v_{K2}).

Although Proposition 9 establishes the existence of an optimal disclosure rule with at most $K-1$ pooling messages, not every optimal disclosure rule needs to have such a property. First, as mentioned in Section 6.1, the argument is based on the existence of a corner solution x^* to the linear problem that has at most $K-1$ strictly positive coordinates. For a non-generic specification of (p, θ), the linear problem may have multiple (non-corner) solutions which would correspond to more than $K-1$ pairs of profiles being pooled. Second, the proof of the proposition replicates x^* using $(\alpha_v^r)_{v,r}$ such that for each $r = 1, \ldots, K-1$, α_v^r, $\alpha_w^r > 0$ for a single pair (v, w). There may as well be other ways to replicate x^*. The number of pooling messages can also be strictly less than $K-1$. To see this point, return to Example 6 and assume $K = 4$. The disclosure rule in this example is in line with the statement of Proposition 9 since it has $K-1 = 3$ pooling messages each of which has binary support. On the other hand, if a disclosure rule has just one pooling message z_1 which has support $\{v, \hat{v}, \tilde{v}\}$, then this message pools $\binom{3}{2} = 3$ pairs of profiles with each other ($(v, \hat{v})$, $(v, \tilde{v})$ and $(\hat{v}, \tilde{v})$), and hence would imply $x^*_{vw} > 0$ for three pairs of profiles (v, w). Such a rule would also be consistent with the proof of the proposition.

7 General model

We now consider the most general framework where the number of the agent's types can be greater than two ($K \geq 2$) and the number of performance scores is greater than or equal to three ($L \geq 3$). Unlike when $L = 2$, the ex post expected wage function ϕ is not required to be supermodular when $L \geq 3$. Instead, as seen in Proposition 2, the disclosure rule (Z, f) is implementable if and only if the interim talent function $H : S^2 \to \boldsymbol{R}$ is cyclically supermodular. Recall from (6) that $N \equiv \sum_{n=2}^{K} \binom{K}{n} (n-1)!$ equals the number of inequalities in the definitions of cyclical supermodularity of H.

Example 7 Further generalize the disclosure rule discussed in Example 6 as follows. (Z, f) is such that for $z_1, \ldots, z_K \notin V$ and $\lambda_1, \ldots, \lambda_K \in [0, 1]$,

- $Z = V \cup \{z_1, \ldots, z_K\}$;
- $f(v_{k\ell} \mid v_{k\ell}) = \lambda_k$ for every k, ℓ;
- $f(z_k \mid v) = \begin{cases} 1 - \lambda_k & \text{if } v \in \{v_{k1}, \ldots, v_{kL}\}, \\ 0 & \text{otherwise.} \end{cases}$

In other words, when the profile $v_{k\ell}$ is realized, it is either perfectly revealed or is pooled with *all other* profiles with the same s-coordinates. For any s, $t \in S$, define

$$\nu(s, t) = \sum_{\omega \in \Omega} g_s(\omega)\, \theta_{t\omega}.$$

$\nu(s, t)$ can be interpreted as the interim expected talent under full disclosure when the agent has true type s but reports t. When $s < t$, $g_t(\cdot)$ stochastically dominates $g_s(\cdot)$ by (1). Since $\theta_{t\omega}$ is increasing in ω, we have for any $\hat{t}$,

$$\nu(t, \hat{t}) > \nu(s, \hat{t}) \quad \text{if } s < t. \tag{30}$$

Since $E_v[\theta_v \mid z_k] = \nu(s_k, s_k)$, the function H can be written in terms of λ_t as:

$$H(s, t) = \lambda_t\, \nu(s, t) + (1 - \lambda_t)\, \nu(t, t).$$

We look for the conditions under which the function H is supermodular, which by Lemma 1 ensures that (Z, f) is implementable. Take any s, $\hat{s}$, t, $\hat{t} \in S$ such that $s < \hat{s}$ and $t < \hat{t}$.

$$\begin{aligned}
& H(\hat{s},\hat{t}) - H(s,\hat{t}) \geq H(\hat{s},t) - H(s,t) \\
\Leftrightarrow \quad & \left\{\lambda_{\hat{t}}\, \nu(\hat{s},\hat{t}) + (1-\lambda_{\hat{t}})\, \nu(\hat{t},\hat{t})\right\} - \left\{\lambda_{\hat{t}}\, \nu(s,\hat{t}) + (1-\lambda_{\hat{t}})\, \nu(\hat{t},\hat{t})\right\} \\
& \geq \left\{\lambda_t\, \nu(\hat{s},t) + (1-\lambda_t)\, \nu(t,t)\right\} - \left\{\lambda_t\, \nu(s,t) + (1-\lambda_t)\, \nu(t,t)\right\}.
\end{aligned}$$

Since $\nu(\hat{s},\hat{t}) - \nu(s,\hat{t}) > 0$ when $\hat{s} > s$ as noted above, H is supermodular if and only if

$$\frac{\lambda_{\hat{t}}}{\lambda_t} \geq \frac{\nu(\hat{s},t) - \nu(s,t)}{\nu(\hat{s},\hat{t}) - \nu(s,\hat{t})} \quad \text{if } s < \hat{s} \text{ and } t < \hat{t}. \tag{31}$$

By assumption, θ is submodular so that $\theta_{\hat{t}\omega} - \theta_{t\omega}$ is decreasing in ω when $t < \hat{t}$. By the stochastic dominance (1) of $g_{\hat{s}}$ over g_s, we then have

$$\nu(\hat{s},\hat{t}) - \nu(\hat{s},t) = \sum_{\omega} g_{\hat{s}}(\omega)\,(\theta_{\hat{t}\omega} - \theta_{t\omega}) \leq \sum_{\omega} g_s(\omega)\,(\theta_{\hat{t}\omega} - \theta_{t\omega}) = \nu(s,\hat{t}) - \nu(s,t).$$

It follows that the right-hand side of (31) satisfies

$$\frac{\nu(\hat{s},t) - \nu(s,t)}{\nu(\hat{s},\hat{t}) - \nu(s,\hat{t})} \geq 1.$$

Hence, (Z, f) is implementable when λ_t is chosen to be increasing in t to satisfy the inequalities in (31). In particular, for $k < \ell$, denote

$$\psi_{k\ell} = \min_{s<\hat{s}} \frac{\nu(\hat{s},s_k) - \nu(s,s_k)}{\nu(\hat{s},s_\ell) - \nu(s,s_\ell)} \geq 1.$$

Then (31) holds if for any $\lambda_{s_K} > 0$, $\lambda_{s_1}, \ldots, \lambda_{s_{K-1}}$ satisfy

$$\lambda_{s_k} = \min \frac{\lambda_{s_K}}{\psi_{k_1k_2} \cdots \psi_{k_{m-1}k_m}} \quad \text{for each } k = 1, \ldots, K-1,$$

where minimization is taken over all sequences $(k_1, \ldots, k_m)$ such that $k_1 = k < k_2 < \cdots < k_{m-1} < k_m = K$.

Proposition 10 *Let p, $\succcurlyeq$ and $\eta > 0$ be given, and suppose that the talent function $\theta \in \Theta_{\succcurlyeq,\eta}$ is submodular and defined over $V = S \times \Omega$ with $L = |\Omega| \geq 3$. Then for N defined in (6), there exists $\varepsilon > 0$ such that if θ is ε-linear, there exists an optimal disclosure rule (Z, f) such that*

- *$Z = V \cup \{z_1, \ldots, z_R\}$ for some $R \leq N$ and $z_1, \ldots, z_R \notin V$.*
- *$|\operatorname{supp}(z_r)| = 2$ for every $r = 1, \ldots, R$.*

Although the disclosure rule described in Example 7 is implementable as long as $\lambda_{s_1}, \ldots, \lambda_{s_K}$ satisfy (31) and the number of pooling messages K satisfies $K < N$, Proposition 10 implies that it is *not* optimal when L is large compared with K provided that the talent function θ is mildly submodular: To see this, note that the disclosure rule in Example 7 has at least $(K-1) \times \binom{L}{2}$ pairs of profiles (v, w) that are pooled with each other.[25] On the other hand, Proposition 10 shows that the number of pairs of profiles that are pooled together is a function of only K. Suppose for example that the types are binary ($K = 2$) and that the performance score can take three values ($L = 3$). In this case, the disclosure rule in Example 7 with $\lambda_{s_2} = 1$ sends no pooling message when $s = s_2$, and sends one pooling message when $s = s_1$. This pooling message has support $\{v_{11}, v_{12}, v_{13}\}$ and hence pools three $(= \binom{3}{2})$ pairs of profiles: $\{v_{11}, v_{12}\}$, $\{v_{12}, v_{13}\}$, and $\{v_{11}, v_{13}\}$. The variable x that corresponds to this rule hence has three strictly positive coordinates. Proposition 10, on the other hand, states that there exists a unique pair of profiles (v, w) for which $x^*_{vw} > 0$. This implies that the disclosure rule in Example 7 is not optimal for a generic specification of (θ, p).[26] We should interpret N as an upper bound on the number of inequalities for the cyclical supermodularity of H since some of those inequalities may be redundant in some cases. This is most evident when the performance score is binary ($L = 2$). As seen in Section 6.2, the number of inequalities for the supermodularity of ϕ is just $K - 1$ so that there exist $N - (K - 1)$ redundant inequalities.[27]

Proposition 10 has the following intuitive implication: When a performance score takes many values, it is commonly observed that those scores are bundled into a few categories. For example, scores above some threshold are bundled as a good score and those below it are bundled as a bad score, and disclosure is based on the binary categorization. Proposition 10 implies that disclosure based on such bundling not only aggravates the quadratic loss, but also involves excessive coarsening of information from the point of view of inducing truth-telling from the agent.[28]

8 Mechanism inducing skill acquisition

Although the only decision that the agent makes in our baseline model concerns reporting of the private component of his talent to the principal, it is conceivable in some situations that the agent also engages himself in the enhancement of his skills before reporting to the principal. For example, a student may make effort to acquire software skills before attending college, or a company may invest in the improvement of its product before

[25]Assume that $\lambda_{s_K} = 1$ so that the performance score is perfectly revealed when the agent reports the highest type s_K.

[26]Under a generic specification of (θ, p), the linear programming problem in terms of x has a unique corner solution.

[27]When $K = 3$, for example, $N = 5$ and $K - 1 = 2$ so that three inequalities are redundant. The redundant inequalities are those corresponding to: $(k_1, k_2) = (1, 3)$, $(k_1, k_2, k_3) = (1, 2, 3)$, and $(k_1, k_2, k_3) = (1, 3, 2)$.

[28]There may exist other motives for bundling such as simplification of the process which we do not address.

applying for a certification program. This section considers one such extension: The agent makes an ex ante choice over costly action e which determines the probability distribution of his private type s. For simplicity, we assume that the set of action choices is binary $\{0,1\}$: $e=0$ and $e=1$ are interpreted as making no effort and making effort, respectively. Let q_e denote the probability distributions of the agent's private type when his action is e. We assume that q_1 stochastically dominates q_0 so that the agent is more likely to have a higher private type when he chooses $e=1$ than when he chooses $e=0$. The cost of action e equals c_e and satisfies $c_1-c_0>0$. We assume that conditional on his private type s, the agent's performance score ω is independent of his action choice, and given by $g_s(\omega)$. Since the agent's incentive in the reporting stage is solely guided by the distribution of the performance score conditional on his private type, the conditional independence assumption implies that the incentive compatibility of the task assignment mechanism is independent of the agent's action choice. The following lemma records this observation.

Lemma 11 *The mechanism $\Gamma=(y,Z,f)$ is incentive compatible when the agent's private type s is distributed according to $q_1\in\Delta S$ if and only if it is incentive compatible when his type is distributed according to $q_0\in\Delta S$.*

Lemma 11 in particular implies that given any incentive compatible mechanism, no combinatorial deviation, where the agent first deviates in his action choice and then misreports his type, is profitable. Let U_e denote the agent's ex ante utility when he chooses action e:

$$U_e=\sum_s\big\{\sum_\omega \phi(s,\omega)\,g_s(\omega)-y(s)\big\}\,q_e(s)-c_e.$$

An incentive compatible task-assignment mechanism $\Gamma=(y,Z,f)$ is *effort-inducing* if the agent finds it optimal to choose $e=1$, or equivalently,

$$U_1\geq U_0. \tag{32}$$

An effort-inducing incentive compatible mechanism is *optimal* if it minimizes the quadratic loss function (4) among the class of such mechanisms.[29] A disclosure rule (Z,f) is *effort-inducing* if there exists a cost-assignment rule y for which $\Gamma=(y,Z,f)$ is effort-inducing, and is optimal if it corresponds to an optimal effort-inducing incentive compatible mechanism.

For simplicity, the analysis in what follows restricts attention to the 2×2 environment of Section 6.1 where both the private type and performance score are binary. The four profiles are named as in (10). Stochastic dominance of q_1 over q_0 is equivalent to $q_1(s_2)>q_0(s_2)$, and (32) can be simplified as

$$y(s_2)-y(s_1)\leq\sum_\omega\big\{\phi(s_2,\omega)\,g_{s_2}(\omega)-\phi(s_1,\omega)\,g_{s_1}(\omega)\big\}-\frac{c_1-c_0}{q_1(s_2)-q_0(s_2)}. \tag{33}$$

[29]With the presence of ex ante action choice, the appropriate participation constraint would be ex ante individual rationality. This, however, is ignored in this section for simplicity.

Clearly, it is not possible to induce the agent to choose $e = 1$ if it is very costly compared with the corresponding increase in the expected wage. We introduce the following measure to quantify the effect of the cost of $e = 1$:

$$C = \frac{c_1 - c_0}{\{g_{s_2}(\omega_2) - g_{s_1}(\omega_2)\}\{q_1(s_2) - q_0(s_2)\}} - (\theta_4 - \theta_3). \tag{34}$$

As seen, C measures the difference between the marginal cost of $e = 1$ ($c_1 - c_0$, adjusted by the probabilities) and the maximum wage differential ($\theta_4 - \theta_3$) when the agent has the high type s_2. The following proposition describes a sufficient condition in terms of C for the feasibility of an effort-inducing mechanism. In line with our intuition developed in the previous sections, the number of pooling messages required under the optimal mechanism is related to the number of incentive conditions faced by the agent. Since (32) places one additional constraint compared with the baseline 2×2 model, it follows that the optimal mechanism entails at most two pooling messages as shown in the following proposition.

Proposition 12 *Suppose that the talent function θ is submodular ($\Delta > 0$) and that (θ, p) satisfies either one of (25), (26), and (27) of Proposition 7. Furthermore, suppose that C satisfies*

$$\begin{aligned} &C \leq \tfrac{p_1(\theta_3-\theta_1)}{p_1+p_3}, && \\ &C \leq \tfrac{p_2|\theta_3-\theta_2|}{p_2+p_3} && \text{if } \theta_2 \neq \theta_3, \\ &C \leq \tfrac{p_2(\theta_4-\theta_3)(\theta_3-\theta_2)}{(p_3-p_2)(\theta_3-\theta_2)+(p_2+p_3)(\theta_2-\theta_1)} && \text{if } \theta_2 < \theta_3 \text{ and } p_2 < p_3. \end{aligned} \tag{35}$$

Then there exists an optimal effort-inducing disclosure rule (Z, f) with at most two pooling messages:

$$Z = V \cup \{z_1, z_2\} \quad \text{for } z_1,\ z_2 \notin V.$$

The support of each pooling message is binary and the combination of the support of z_1 and z_2 is given by

$$\big(\mathrm{supp}(z_1),\, \mathrm{supp}(z_2)\big) \in \big\{(\{1,2\},\{1,3\}),\ (\{1,2\},\{2,3\}),\ (\{1,3\},\{2,3\})\big\}.$$

9 Conclusion

We have formulated a model in which an agent with career concerns reports his private type to a principal and then performs a task assigned to him. The analysis highlights the intricacy of managing the agent's incentive and minimizing the loss in such a framework. We show that whether the talent function is supermodular or not determines if any pooling is required under optimal disclosure. It is interesting to note that the conclusion is independent of whether the two components of the agent's talent – his private type and performance – are complements or substitutes. While the model is one of information

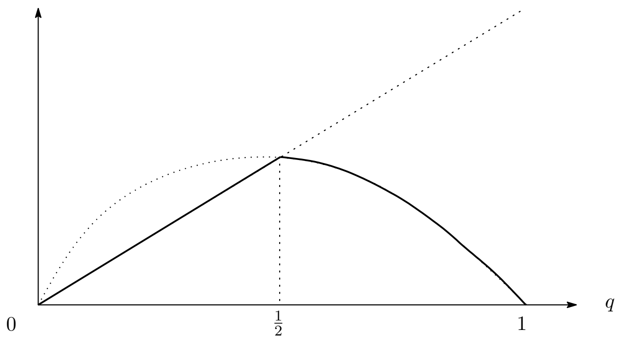

Figure 6: Profile of the minimized loss L^*

The graph depicts L^* along the vertical line segment $\beta = 1$ ($\Leftrightarrow \theta_2 = \theta_3$) in Figure 4.

$$L^* = \begin{cases} \gamma(1-\gamma)q\Delta(\theta_2 - \theta_1) & \text{if } q \leq \frac{1}{2}, \\ \frac{\gamma(1-\gamma)q(1-q)}{\gamma q + (1-\gamma)(1-q)}\,\Delta(\theta_3 - \theta_1) & \text{if } q \geq \frac{1}{2}. \end{cases}$$

design in the sense that the principal commits to a disclosure rule, the standard concavification argument cannot be applied because of the presence of the incentive constraints required for truthful reporting by the agent. To see this, redefine a disclosure rule to be a probability distribution of posteriors: $\tau \in \Delta(\Delta(V))$ satisfying $\sum_\zeta \tau(\zeta)\,\zeta = p$, where p is the prior over the set V of profiles $v = (s, \omega)$. Take two priors p and $\tilde{p}$ over V, and let τ and $\tilde{\tau}$ be implementable disclosure rules for p and $\tilde{p}$, respectively. The concavification argument would hold if their convex combination $(1-\eta)\,\tau + \eta\,\tilde{\tau}$ ($\eta \in (0,1)$) is also an implementable disclosure rule. Unfortunately, this does not hold. Figure 6 illustrates this point by depicting the quadratic loss under the optimal disclosure rule in the 2×2 example of Section 6.1 corresponding to Figure 4. Since the principal's objective is to *minimize* the loss, cancavification in the current context would imply a convex function. This however is clearly not the case: Full disclosure is optimal and hence entails no loss at both ends ($q = \Pr(s_1) = 0, 1$) since the agent's type is unique at these points and submodularity is irrelevant. On the other hand, full disclosure is not implementable at interior points where submodularity creates an incentive issue.

One key assumption of the present model is that the firm as an employer of the agent does not directly observe the task assigned to the agent. It is rather the object of disclosure by the principal since the task assignment reflects the agent's private type. Requiring the principal to fully disclose the task assignment amounts to considering the subset of disclosure rules where no two profiles can be pooled when they correspond to different tasks. This alternative assumption hence leads to a simpler characterization of the optimal disclosure. In the 2×2 environment, for example, the optimal rule under the requirement is one that partially pools profiles $v_1 = (s_1, \omega_1)$ and $v_2 = (s_1, \omega_2)$ as in Example 5 (Figure 3).

One main lesson from the career concerns models is that even if an agent chooses his effort in private, the market forms an expectation of his talent correctly anticipating his effort choice. Despite this, we assume away moral hazard in our model: Introducing moral hazard creates a major complication when the principal does not necessarily disclose the task assignment. In such a case, the expectation is a function of how tasks of different difficulty levels are pooled in disclosure, and the disclosure rule must take into account its impact on both the market expectation and the agent's incentive.

We suppose that the principal is concerned only with the quadratic difference between the agent's true talent and the expected talent of the agent conditional on the disclosed information. This yields the key observation that his objective function can be expressed as a linear function of the variable x, which is the transformation of the probabilities of the pooling messages. One may want to model a situation where the principal has additional objectives. For example, he may profit from the agent's high performance. If the agent's objective continues to be the maximization of his expected wage, his incentive constraints are the same as in the present model and a disclosure rule is implementable if and only if the interim expected wage function H is cyclically supermodular. Since implementability then is expressed as a system of linear inequalities of x, the conclusion of the paper would continue to hold if the principal's objective function is linear, or more generally convex, in the variable x. The problem then would reduce to verifying when the principal's objective can be expressed as a convex function of x.

This paper focuses on a model without any formal contract with monetary transfer. Alternatively, we may assume that the principal sells information to the firm and also compensates the agent for his participation in the mechanism. Examining such a framework, encompassing formal contracts between the agent and principal and between the firm and the principal, along with informal contracts between the agent and firm, could be an interesting avenue for future research.

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Appendix A Proofs

Proof of Lemma 1. When $n = 2$, (5) holds under supermodularity since

$$H(s_1, s_1) - H(s_1, s_2) + H(s_2, s_2) - H(s_2, s_1) \geq 0.$$

As an induction hypothesis, suppose that supermodularity implies (5) for $n = m - 1$. Suppose that $n = m$, and assume without loss of generality that $k_m > \max_{i<m} k_i$. Then,

$$\begin{aligned}
&\sum_{i=1}^{m} \{H(s_{k_i}, s_{k_i}) - H(s_{k_i}, s_{k_{i+1}})\} \\
&= \sum_{i=1}^{m-1} \{H(s_{k_i}, s_{k_i}) - H(s_{k_i}, s_{k_{i+1}})\} + H(s_{k_m}, s_{k_m}) - H(s_{k_m}, s_{k_1}) \\
&\geq \sum_{i=1}^{m-1} \{H(s_{k_i}, s_{k_i}) - H(s_{k_i}, s_{k_{i+1}})\} + H(s_{k_{m-1}}, s_{k_m}) - H(s_{k_{m-1}}, s_{k_1}) \\
&= \sum_{i=1}^{m-2} \{H(s_{k_i}, s_{k_i}) - H(s_{k_i}, s_{k_{i+1}})\} + H(s_{k_{m-1}}, s_{k_{m-1}}) - H(s_{k_{m-1}}, s_{k_1}) \\
&= \sum_{i=1}^{m-1} \{H(s_{k_i}, s_{k_i}) - H(s_{k_i}, s_{k_{i+1}})\} \\
&\geq 0,
\end{aligned}$$

where the first inequality follows from the supermodularity of H and the second inequality from the induction hypothesis. ■

Proof of Proposition 2. When the agent with true type s_k reports s_ℓ, we have from the (IC) conditions (3),

$$y(s_k) - y(s_\ell) \leq H(s_k, s_k) - H(s_k, s_\ell). \tag{36}$$

For k, $\ell = 1, \ldots, K$ such that $k \neq \ell$, let $a^{k\ell} = (a_i^{k,\ell})_{i=1}^{K}$ be a K-dimensional row vector such that

$$a_k(k,\ell) = 1, \quad a_\ell(k,\ell) = -1, \quad \text{and} \quad a_j(k,\ell) = 0 \quad \text{for } j \neq k,\ \ell,$$

and define A to be an $K(K-1) \times K$ matrix such that

$$A = \begin{bmatrix} a(1,2) \\ a(2,1) \\ \vdots \\ a(K-1,K) \\ a(K,K-1) \end{bmatrix} = \begin{bmatrix} 1 & -1 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}.$$

We also define y to be an K-dimensional column vector, and α to be an $K(K-1)$-dimensional column vector such that

$$y = \begin{bmatrix} y(s_1) \\ \vdots \\ y(s_K) \end{bmatrix}, \quad \text{and} \quad \alpha = \begin{bmatrix} H(s_1, s_1) - H(s_1, s_2) \\ H(s_2, s_2) - H(s_2, s_1) \\ \vdots \\ H(s_{K-1}, s_{K-1}) - H(s_{K-1}, s_K) \\ H(s_K, s_K) - H(s_K, s_{K-1}) \end{bmatrix}.$$

The incentive conditions (36) for every k, $\ell \in \{1, \dots, K\}$ with $k \neq \ell$ can be expressed in matrix form as:

$$A\,y \leq \alpha. \tag{37}$$

It follows that a disclosure rule (Z, f) is implementable if and only if (37) has a solution y. By the theorem of the alternatives (Theorem 22.2, p.198, Rockafellar, 1997), (37) has a solution if and only if for any $K(K-1)$-dimensional row vector

$$\lambda = (\lambda_{1,2}, \lambda_{2,1}, \dots, \lambda_{K-1,K}, \lambda_{K,K-1}),$$

$$\lambda \geq 0 \quad \text{and} \quad \lambda A = 0 \quad \Rightarrow \quad \lambda \cdot \alpha \geq 0. \tag{38}$$

For any $m \in \{2, \dots, K\}$, consider the set of ordered pairs of indices

$$\{(k_1, \ell_1), \dots, (k_m, \ell_m)\}.$$

We say that such a set is *cyclical* if $k_1, \dots, k_m$ are all distinct, and there exists a permutation π over $\{1, \dots, m\}$ such that

$$k_1 = \ell_{\pi(1)},\ k_2 = \ell_{\pi(2)},\ \dots,\ k_m = \ell_{\pi(m)}.$$

For example, the sets $\{(1,2),(2,1)\}$, $\{(1,2),(2,3),(3,1)\}$, $\{(1,3),(3,2),(2,1)\}$ are all cyclical. A $K(K-1)$-dimensional vector $\lambda = (\lambda_{1,2}, \lambda_{2,1}, \dots, \lambda_{K-1,K}, \lambda_{K,K-1})$ is *cyclical* if $\lambda_{k,\ell} \in \{0,1\}$ for every (k,ℓ), and

$$\{(k,\ell) : \lambda_{k,\ell} = 1\} \text{ is cyclical.}$$

It can be readily verified that $\lambda A = 0$ if λ is cyclical. For example, if $\{(k, \ell) : \lambda_{k,\ell} = 1\} = \{(1, 2), (2, 1)\}$, then the first two rows of A cancel out each other so that $\lambda A = a(1, 2) + a(2, 1) = 0$, and if $\{(k, \ell) : \lambda_{k,\ell} = 1\} = \{(1, 2), (2, 3), (3, 1)\}$, then $\lambda A = a(1, 2) + a(2, 3) + a(3, 1) = 0$. Let $\Lambda = \{\lambda^{c_1}, \ldots, \lambda^{c_N}\}$ be the set of all cyclical vectors.We can then show that any non-negative solution of $\lambda A = 0$ is expressed as a positive combination of $\lambda^{c_1}, \ldots, \lambda^{c_N}$ as shown by Lemma 13 below. It follows that (38) holds if

$$\lambda^{c_n} \cdot \alpha \geq 0 \quad \text{for } n = 1, \ldots, N,$$

which is equivalent to (5) by the definition of α.

We now show that if there exists y such that $\Gamma = (y, Z, f)$ is incentive compatible, then there exists a cost assignment rule $\hat{y}$ for which $(\hat{y}, Z, f)$ is both incentive compatible and individually rational. Specifically, let y be a cost assignment rule such that $\Gamma = (y, Z, f)$ is incentive compatible, and denote by $U(s_k)$ the interim expected utility of each type under Γ:

$$U(s_k) = H(s_k, s_k) - y(s_k) \quad \text{for } k = 1, \ldots, K.$$

If $\underline{U} \equiv \min_k U(s_k) \geq 0$, then $\Gamma = (y, Z, f)$ is individually rational. If $\underline{U} < 0$, then define an alternative cost assignment rule $\hat{y} : S \to \boldsymbol{R}$ by

$$\hat{y}(s_k) = y(s_k) - \underline{U}.$$

Since shifting the cost by a constant does not affect the agent's incentive in the reporting stage, it is clear that $(\hat{y}, Z, f)$ is both incentive compatible and individually rational. ■

Lemma 13 *Suppose* $\lambda \geq 0$. *Then*

$$\lambda A = 0 \quad \Leftrightarrow \quad \lambda = \sum_{j=1}^{n} \delta_j \lambda^{c_j} \text{ for some } \delta = (\delta_j)_{j=1}^{n} \geq 0.$$

Proof. ($\Leftarrow$) This readily follows from the discussion above.
($\Rightarrow$) We show that $\mu A \neq 0$ if μ is not a positive combination of λ^{c_j}'s. Without loss of generality, we can take μ to be such that

$$J \equiv \{(k, \ell) : \mu_{k,\ell} > 0\} \text{ is not a superset of any cyclical set.} \tag{39}$$

since if the set J is a superset of some cyclical set C, then for the cyclical vector λ^C such that $\{(k, \ell) : \lambda^C_{k,\ell} = 1\} = C$,

$$\mu' \equiv \mu - \lambda^C \min\{\mu_{k,\ell} : (k, \ell) \in C\}$$

is not a positive combination of λ^{c_j}'s either. Denote

$$J = \{(k_1, \ell_1), \ldots, (k_n, \ell_n)\}.$$

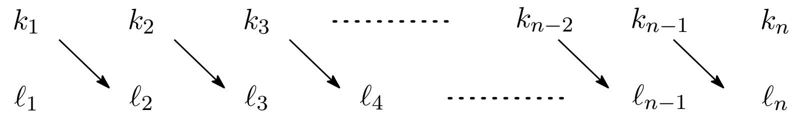

Figure 7: Illustration of the proof of Lemma 13

If $k_1 \notin \{\ell_1, \ell_2, \ldots, \ell_n\}$, then we have $\mu A \neq 0$. Otherwise, since $k_1 \neq \ell_1$ by definition, suppose without loss of generality that $k_1 = \ell_2$. If $k_2 \notin \{\ell_1, \ldots, \ell_n\}$, then $\mu A \neq 0$. Otherwise, $k_2 \neq \ell_2$ by definition and also $k_2 \neq \ell_1$ since $k_2 = \ell_1$ would imply the violation of (39): $\{(k_1, \ell_1), (k_2, \ell_2)\} = \{(k_1, \ell_1), (\ell_1, k_1)\}$ is cyclical. Suppose then without loss of generality that $k_2 = \ell_3$. We cannot have $k_3 = \ell_1$ or $k_3 = \ell_2$ since either of them would imply the violation of (39): $k_3 = \ell_1$ would imply that $\{(k_1, \ell_1), (k_2, \ell_2), (k_3, \ell_3)\} = \{(k_1, \ell_1), (k_2, k_1), (\ell_1, k_2)\}$ is cyclical, and $k_3 = \ell_2$ would imply that $\{(k_2, \ell_2), (k_3, \ell_3)\} = \{(k_2, \ell_2), (\ell_2, k_2)\}$ is cyclical. Continuing in the same way, we have either $\mu A \neq 0$ or $k_{n-1} = \ell_n$ (Figure 7). Since we have a violation of (39) if $k_n \in \{\ell_1, \ldots, \ell_{n-1}\}$ and also $k_n \neq \ell_n$ by definition, $k_n \notin \{\ell_1, \ldots, \ell_n\}$ so that $\mu A \neq 0$. ■

Proof of Proposition 3.

We first show that Γ is incentive compatible. Take any k and ℓ such that $k < \ell$. Since (7) implies that

$$y(s_\ell) - y(s_k) = \sum_{m=k}^{\ell-1} \big\{ H(s_m, s_{m+1}) - H(s_m, s_m) \big\},$$

neither type s_k nor type s_ℓ has incentive to misrepresent himself as the other type if and only if

$$H(s_k, s_\ell) - H(s_k, s_k) \leq \sum_{m=k}^{\ell-1} \big\{ H(s_m, s_{m+1}) - H(s_m, s_m) \big\} \leq H(s_\ell, s_\ell) - H(s_\ell, s_k).$$

The right inequality is implied by the cyclical supermodularity of H, and hence by the supermodularity of H. We can use an induction argument to show that the left-inequality is also implied by the supermodularity of H: When $\ell = k + 1$, the inequality reduces to

$$H(s_k, s_{k+1}) - H(s_k, s_k) \leq H(s_k, s_{k+1}) - H(s_k, s_k),$$

which always holds. As an induction hypothesis, suppose that the inequality holds for some ℓ. Then

$$
\begin{aligned}
&\sum_{m=k}^{\ell} \big\{H(s_m, s_{m+1}) - H(s_m, s_m)\big\} \\
&= \sum_{m=k}^{\ell-1} \big\{H(s_m, s_{m+1}) - H(s_m, s_m)\big\} + \big\{H(s_\ell, s_{\ell+1}) - H(s_\ell, s_\ell)\big\} \\
&\geq \big\{H(s_k, s_\ell) - H(s_k, s_k)\big\} + \big\{H(s_k, s_{\ell+1}) - H(s_k, s_\ell)\big\} \\
&= H(s_k, s_{\ell+1}) - H(s_k, s_k),
\end{aligned}
$$

where the inequality follows from the induction hypothesis and from the supermodularity of H. This shows that the inequality holds for $\ell + 1$.

We next show that Γ is individually rational. Since $y(s_1) = 0$, $H(s_1, s_1) - y(s_1) \geq 0$. As an induction hypothesis, suppose that $H(s_\ell, s_\ell) - y(s_\ell) \geq 0$. Then

$$
\begin{aligned}
H(s_{\ell+1}, s_{\ell+1}) - y(s_{\ell+1}) &= H(s_{\ell+1}, s_{\ell+1}) - \{H(s_\ell, s_{\ell+1}) - H(s_\ell, s_\ell)\} - y(s_\ell) \\
&\geq H(s_{\ell+1}, s_{\ell+1}) - \{H(s_{\ell+1}, s_{\ell+1}) - H(s_{\ell+1}, s_\ell)\} - y(s_\ell) \\
&= H(s_{\ell+1}, s_\ell) - y(s_\ell),
\end{aligned}
$$

where the inequality follows from supermodularity. However, since stochastic dominance of $g_{s_{\ell+1}}$ over g_{s_ℓ} implies

$$H(s_{\ell+1}, s_\ell) = \sum_{\omega} g_{s_{\ell+1}}(\omega)\, \phi(s_\ell, \omega) \geq g_{s_\ell}(\omega)\, \phi(s_\ell, \omega) = H(s_\ell, s_\ell),$$

we have by the induction hypothesis,

$$H(s_{\ell+1}, s_{\ell+1}) - y(s_{\ell+1}) \geq H(s_{\ell+1}, s_\ell) - y(s_\ell) \geq H(s_\ell, s_\ell) - y(s_\ell) \geq 0.$$

This advances the induction step. ■

Proof of Proposition 5. Let (Z, f) be the full disclosure rule so that $\phi(s, \omega) = \theta(s, \omega)$ for every (s, ω). We then have

$$H(s, t) = \sum_{\omega \in \Omega} g_s(\omega)\, \theta(t, \omega).$$

Take any s, $\hat{s}$, t, and $\hat{t} \in S$ such that $s < \hat{s}$ and $t < \hat{t}$. Since θ is supermodular, $\theta(\hat{t}, \omega) - \theta(t, \omega)$ is an increasing function of ω, and since $g_{\hat{s}}(\omega)$ (first-order) stochastically dominates $g_s(\omega)$ by (1),

$$
\begin{aligned}
H(\hat{s}, \hat{t}) - H(\hat{s}, t) &= \sum_{\omega \in \Omega} g_{\hat{s}}(\omega)\, \{\theta(\hat{t}, \omega) - \theta(t, \omega)\} \\
&\geq \sum_{\omega \in \Omega} g_s(\omega)\, \{\theta(\hat{t}, \omega) - \theta(t, \omega)\} \\
&= H(s, \hat{t}) - H(s, t).
\end{aligned}
$$

It follows that H is supermodular, and hence is cyclically supermodular by Lemma 1. It follows that (Z, f) is implementable by Proposition 2. ■

Proof of Lemma 4. It suffices to show that (5) $\Leftrightarrow$ (8).
(5) $\Rightarrow$ (8). When $m = 2$, $s_{k_1} = s$ and $s_{k_2} = t$, (5) is written as:

$$H(s,t) - H(s,s) \leq H(t,t) - H(t,s),$$

which is equivalent to

$$\sum_{\omega} g_s(\omega) \{\phi(t,\omega) - \phi(s,\omega)\} \leq \sum_{\omega} g_t(\omega) \{\phi(t,\omega) - \phi(s,\omega)\}. \tag{40}$$

Substituting $g_s(\omega_1) = 1 - g_s(\omega_2)$ and $g_t(\omega_1) = 1 - g_t(\omega_2)$ and then simplifying, we see that (40) is equivalent to

$$\{g_t(\omega_2) - g_s(\omega_2)\} \left\{\phi(t,\omega_2) - \phi(s,\omega_2) - \phi(t,\omega_1) + \phi(s,\omega_1)\right\} \geq 0.$$

When $s < t$, $g_t(\omega_2) - g_s(\omega_2) > 0$ by stochastic dominance (1) so that (8) holds.
(8) $\Rightarrow$ (5). Take any s, $\hat{s}$, t, $\hat{t} \in S$ such that $s < \hat{s}$ and $t < \hat{t}$. By (8), $\phi(\hat{t},\omega) - \phi(t,\omega)$ is an increasing function of ω. Since $g_{\hat{s}}$ stochastically dominates g_s by (1), we have

$$\sum_{\omega} g_s(\omega) \{\phi(\hat{t},\omega) - \phi(t,\omega)\} \leq \sum_{\omega} g_{\hat{s}}(\omega) \{\phi(\hat{t},\omega) - \phi(t,\omega)\}.$$

By the definition of H, this is equivalent to

$$H(s,\hat{t}) - H(s,t) \leq H(\hat{s},\hat{t}) - H(\hat{s},t),$$

which shows that H is supermodular. It then follows from Lemma 1 that H is cyclically supermodular (5).

We next show (9) $\Rightarrow$ (8) since the implication (8) $\Rightarrow$ (9) is clear. Take any s_m, $s_n \in S$ with $m < n$. Since (9) holds for $k = m, \ldots, n-1$, we have

$$\begin{aligned}
\phi(s_n,\omega_2) - \phi(s_n,\omega_1) &\geq \phi(s_{n-1},\omega_2) - \phi(s_{n-1},\omega_1) \\
&\geq \cdots \\
&\geq \phi(s_{m+1},\omega_2) - \phi(s_{m+1},\omega_1) \\
&\geq \phi(s_m,\omega_2) - \phi(s_m,\omega_1),
\end{aligned}$$

which implies (8). ■

Proof of Lemma 6. For any $s \in S$ and $\omega \in \Omega$,

$$\begin{aligned}
\phi(s,\omega) &= \sum_r f(z \mid s,\omega)\, E_\theta[\theta \mid z_r] \\
&= f(v_{s\omega} \mid v_{s\omega})\, \theta_{s\omega} + \sum_r f(z_r \mid v_{s\omega})\, \mu_r \\
&= \big(1 - \sum_r \alpha^r_{s\omega}\big)\, \theta_{s\omega} + \sum_r \alpha^r_{s\omega}\, \mu_r \\
&= \theta_{s\omega} + \sum_r \alpha^r_{s\omega} \Big(\frac{\sum_v p_v \alpha^r_v \theta_v}{\sigma^r} - \theta_{s\omega} \Big) \\
&= \theta_{s\omega} + \sum_r \frac{\alpha^r_{s\omega}}{\sigma^r} \Big(\sum_v p_v \alpha^r_v \,(\theta_v - \theta_{s\omega}) \Big) \\
&= \theta_{s\omega} + \sum_v p_v\, x_{v,s\omega}\, (-1)^{\mathbf{1}\{v \prec s\omega\}},
\end{aligned}$$

and hence for any $s,\, t \in S$,

$$\begin{aligned}
H(s,t) &= \sum_\omega g_s(\omega)\, \phi(t,\omega) \\
&= \sum_\omega g_s(\omega) \Big[\theta_{t\omega} + \sum_v p_v\, x_{v,t\omega}\, (-1)^{\mathbf{1}\{v \prec t\omega\}} \Big].
\end{aligned}$$

The loss function is given by

$$\mathcal{L}(\Gamma) = \sum_{v \in V} p_v \sum_r \alpha^r_v\, (\mu_r - \theta_v)^2.$$

Substitution of (18) and (19) into L yields

$$\begin{aligned}
\mathcal{L}(\Gamma) &= \sum_r \sum_v p_v\, \alpha^r_v \Big\{ \mu_r^2 - 2\mu_r \theta_v + \theta_v^2 \Big\} \\
&= \sum_r \Big\{ \mu_r^2 \sum_v p_v\, \alpha^r_v - 2\mu_r \sum_v p_v\, \alpha^r_v\, \theta_v + \sum_v p_v\, \alpha^r_v\, \theta_v^2 \Big\} \\
&= \sum_r \Big\{ -\frac{1}{\sigma^r} \big(\sum_v p_v\, \alpha^r_v\, \theta_v \big)^2 + \sum_v p_v\, \alpha^r_v\, \theta_v^2 \Big\} \\
&= \sum_r \frac{1}{\sigma^r} \Big\{ -\big(\sum_v p_v\, \alpha^r_v\, \theta_v \big)^2 + \big(\sum_v p_v\, \alpha^r_v \big) \big(\sum_v p_v\, \alpha^r_v\, \theta_v^2 \big) \Big\} \\
&= \sum_r \sum_{\{(v,w):\, v \prec w\}} \frac{\alpha^r_v \alpha^r_w}{\sigma^r}\, p_v p_w (\theta_w - \theta_v)^2 \\
&= \sum_{\{(v,w):\, v \prec w\}} p_v p_w x_{vw}\, |\theta_w - \theta_v|
\end{aligned}$$

Substituting the definition of x_{vw} in (19), we obtain (22). ■

Proof of Proposition 7. Let (Z, f) be any disclosure rule as described in (17). By Lemma 4, (Z, f) is implementable if and only if ϕ is supermodular:

$$\phi_1 + \phi_4 - \phi_2 - \phi_3 \geq 0. \tag{41}$$

Using Lemma 6, we can rewrite (41) as

$$\sum_v p_v x_{v1} (-1)^{\mathbf{1}\{v \prec 1\}} + \sum_v p_v x_{v4} (-1)^{\mathbf{1}\{v \prec 4\}} \\ - \sum_v p_v x_{v2} (-1)^{\mathbf{1}\{v \prec 2\}} - \sum_v p_v x_{v3} (-1)^{\mathbf{1}\{v \prec 3\}} \geq \Delta,$$

where Δ is as defined in (12) and $\Delta > 0$ when θ is submodular. Collecting terms while noting $1 \prec 2$, $1 \prec 3$, $1 \prec 4$, $2 \prec 4$, and $3 \prec 4$, we obtain

$$\begin{aligned} & x_{12} (p_1 + p_2) + x_{13} (p_1 + p_3) - x_{24} (p_2 + p_4) - x_{34} (p_3 + p_4) \\ & + x_{14} (p_4 - p_1) - (-1)^{\mathbf{1}\{2 \prec 3\}} x_{23} (p_2 - p_3) \geq \Delta. \end{aligned} \tag{42}$$

To summarize, the principal's problem is to minimize the objective function (22) with respect to $x = (x_{mn})_{m \prec n}$ subject to (42). Suppose for the moment that each x_{mn} can take any non-negative values. Since both the objective function and the constraint are linear in x, there exists an optimal solution x^* such that $x^*_{mn} > 0$ for a unique pair (m, n) $(m \prec n)$ and $x^*_{mn} = 0$ for any other pair. Furthermore, $x^*_{24} = x^*_{34} = 0$ since their coefficients in (42) are unambiguously negative. It follows that the optimal solution x^* to this linear problem and the corresponding optimum L^* are given by one of the following:

i) $x^*_{12} = \frac{\Delta}{p_1+p_2} \Rightarrow L^*_{12} = \frac{p_1 p_2}{p_1+p_2} \Delta (\theta_2 - \theta_1)$.
ii) $x^*_{13} = \frac{\Delta}{p_1+p_3} \Rightarrow L^*_{13} = \frac{p_1 p_3}{p_1+p_3} \Delta (\theta_3 - \theta_1)$.
iii) $x^*_{23} = \frac{\Delta}{|p_2-p_3|} \Rightarrow L^*_{23} = \frac{p_2 p_3}{p_2-p_3} \Delta (\theta_3 - \theta_2)$ if $(p_2 - p_3)(\theta_3 - \theta_2) > 0$.
iv) $x^*_{14} = \frac{\Delta}{p_4-p_1} \Rightarrow L^*_{14} = \frac{p_1 p_4}{p_4-p_1} \Delta (\theta_4 - \theta_1)$ if $p_4 > p_1$.

Among these, we see that case (iv) is dominated by case (i): $L^*_{14} < L^*_{12}$ for any (θ, p).[30] We proceed by separately considering conditions (25)-(27).

1. (θ, p) satisfies (25): Since case (iii) is irrelevant in this case, either case (i) or case (ii) is optimal. We note that $L^*_{12} \leq L^*_{13}$ if and only if

$$\frac{p_2}{p_1 + p_2} (\theta_2 - \theta_1) \leq \frac{p_3}{p_1 + p_3} (\theta_3 - \theta_1) \quad \Leftrightarrow \quad \frac{\theta_2 - \theta_1}{\theta_3 - \theta_1} \leq \frac{p_3(p_1 + p_2)}{p_2(p_1 + p_3)}.$$

2. (θ, p) satisfies (26): $(p_2 - p_3) \left(\frac{\theta_2-\theta_1}{\theta_3-\theta_1} - \frac{p_3(p_1+p_2)}{p_2(p_1+p_3)} \right) \leq 0$.[31] We first show that (26) is equivalent to both

$$L^*_{12} \leq L^*_{23} \quad \text{and} \quad L^*_{13} \leq L^*_{23}.$$

[30] When $p_4 > p_1$, $\frac{p_1 p_2}{p_1+p_2} \Delta (\theta_2 - \theta_1) < \frac{p_1 p_4}{p_4-p_1} \Delta (\theta_4 - \theta_1)$.
[31] Note that (26) implies $(p_2 - p_3)(\theta_3 - \theta_2) > 0$.

Suppose that $p_2 - p_3 > 0$. From cases (i) and (iii), we see that $L_{12}^* \leq L_{23}^*$ if and only if

$$\begin{aligned} & \frac{p_3}{p_2 - p_3} (\theta_3 - \theta_2) \geq \frac{p_1}{p_1 + p_2} (\theta_2 - \theta_1) \\ \Leftrightarrow \quad & p_3(p_1 + p_2)(\theta_3 - \theta_2) \geq p_1(p_2 - p_3)(\theta_2 - \theta_1) \\ \Leftrightarrow \quad & p_3(p_1 + p_2)(\theta_3 - \theta_1) \geq p_2(p_1 + p_3)(\theta_2 - \theta_1) \\ \Leftrightarrow \quad & \frac{\theta_2 - \theta_1}{\theta_3 - \theta_1} \leq \frac{p_3(p_1 + p_2)}{p_2(p_1 + p_3)}. \end{aligned}$$

Likewise, from cases (ii) and (iii), we see that $L_{13}^* \leq L_{23}^*$ if and only if

$$\begin{aligned} & \frac{p_2}{p_2 - p_3} (\theta_3 - \theta_2) \geq \frac{p_1}{p_1 + p_3} (\theta_3 - \theta_1) \\ \Leftrightarrow \quad & p_2(p_1 + p_3)(\theta_3 - \theta_2) \geq p_1(p_2 - p_3)(\theta_3 - \theta_1) \\ \Leftrightarrow \quad & p_3(p_1 + p_2)(\theta_3 - \theta_1) \geq p_2(p_1 + p_3)(\theta_2 - \theta_1) \\ \Leftrightarrow \quad & \frac{\theta_2 - \theta_1}{\theta_3 - \theta_1} \leq \frac{p_3(p_1 + p_2)}{p_2(p_1 + p_3)}. \end{aligned}$$

When $p_2 - p_3 < 0$, we can likewise show that both $L_{12}^* \leq L_{23}^*$ and $L_{13}^* \leq L_{23}^*$ are equivalent to

$$\frac{\theta_2 - \theta_1}{\theta_3 - \theta_1} \geq \frac{p_3(p_1 + p_2)}{p_2(p_1 + p_3)}.$$

When $p_2 - p_3 = 0$, (26) always holds and so do $L_{12}^* \leq L_{23}^*$ and $L_{13}^* \leq L_{23}^*$.
We next show that in each of case (i) and case (ii), there exists $\alpha = (\alpha_m^r)_{m,r}$ that generates x^*. In case (i), let

$$R = 1, \quad \text{and} \quad \alpha_m^1 = \begin{cases} \frac{\Delta}{\theta_2 - \theta_1} & \text{if } m = 1 \text{ or } 2, \\ 0 & \text{otherwise.} \end{cases}$$

α then satisfies $\alpha_1^1 = \alpha_2^1 \in (0,1)$ since $\frac{\Delta}{\theta_2 - \theta_1} < 1$, and generates from (19) x^* in case (i): $x_{mn}^* = 0$ for any $(m,n) \neq (1,2)$ and

$$x_{12}^* = \frac{\Delta}{p_1 + p_2}.$$

In case (ii), let

$$R = 1, \quad \text{and} \quad \alpha_m^1 = \begin{cases} \frac{\Delta}{\theta_3 - \theta_1} & \text{if } m = 1 \text{ or } 3, \\ 0 & \text{otherwise.} \end{cases}$$

α then satisfies $\alpha_1^1 = \alpha_3^1 \in (0,1)$ since $\frac{\Delta}{\theta_3 - \theta_1} < 1$, and generates from (19) x^* in case (ii): $x_{mn}^* = 0$ for any $(m,n) \neq (1,3)$ and

$$x_{13}^* = \frac{\Delta}{p_1 + p_3}.$$

3. (θ, p) satisfies (27) but violates (26).
   Since (27) implies $(p_2 - p_3)(\theta_3 - \theta_2) > 0$, case (iii) is relevant, and indeed optimal since the violation of (26) is equivalent to $L^*_{23} < L^*_{12}$ and $L^*_{23} < L^*_{13}$ as seen above. Let

$$R = 1, \quad \text{and} \quad \alpha^1_m = \begin{cases} \frac{(p_2+p_3)\,\Delta}{(p_2-p_3)(\theta_3-\theta_2)} & \text{if } m = 2 \text{ or } 3, \\ 0 & \text{otherwise.} \end{cases}$$

   α then satisfies $\alpha^1_2 = \alpha^1_3 \in (0, 1)$ by (27), and generates x^* in case (iii): $x^*_{mn} = 0$ for any $(m, n) \neq (2, 3)$, and

$$x^*_{23} = \frac{\alpha^1_2 \alpha^1_3}{p_2\alpha^1_2 + p_3\alpha^1_3} |\theta_3 - \theta_2| = \frac{\Delta}{|p_2 - p_3|}.$$

   It follows that $L^* = L^*_{23}$.

This completes the proof. ■

Proof of Corollary 8. The conclusion is immediate if (θ, p) satisfies the sufficient condition (26) of Proposition 7. Suppose then that (θ, p) violates (26). This implies that $(p_2 - p_3)(\theta_3 - \theta_2) > 0$. Since θ is ε-linear, we have

$$\Delta = 2\Big(\frac{\theta_3 - \theta_1}{2} - \frac{\theta_4 - \theta_2}{2}\Big) \leq 2\big((h + \varepsilon) - (h - \varepsilon)\big) = 4\varepsilon.$$

Hence, for ε as given, $\theta \in \Theta_{\succcurlyeq,\eta}$ implies

$$\Delta \leq 4\varepsilon \leq \eta\frac{|p_2 - p_3|}{p_2 + p_3} < \frac{(p_2 - p_3)(\theta_3 - \theta_2)}{p_2 + p_3},$$

which again shows that the sufficient condition (27) of Proposition 7 holds. ■

Proof of Proposition 9. Since θ is assumed to be submodular,

$$\Delta_k \equiv \theta_{k2} + \theta_{k+1,1} - \theta_{k1} - \theta_{k+1,2} > 0 \quad \text{for } k = 1, \ldots, K - 1.$$

Note that ϕ is supermodular if and only if

$$\phi_{k1} + \phi_{k+1,2} - \phi_{k2} - \phi_{k+1,1} \geq 0 \quad \text{for } k = 1, \ldots, K - 1. \tag{43}$$

Using (20), we can rewrite (43) as

$$\begin{aligned} \sum_v p_v \big[& x_{v,s_k\omega_1} (-1)^{\mathbf{1}_{\{v \prec k1\}}} + x_{v,s_{k+1}\omega_2} (-1)^{\mathbf{1}_{\{v \prec k+1,2\}}} \\ & - x_{v,s_k\omega_2} (-1)^{\mathbf{1}_{\{v \prec k2\}}} - x_{v,s_{k+1}\omega_1} (-1)^{\mathbf{1}_{\{v \prec k+1,1\}}} \big] \geq \Delta_k \\ & \text{for } k = 1, \ldots, K - 1. \end{aligned} \tag{44}$$

When θ is ε-linear, $\Delta_k < 4\varepsilon$ so that $\Delta_k \to 0$ for each k as $\varepsilon \to 0$. We proceed in the following steps. In step 1, we consider minimization of L with respect to x, and show that there exists a solution x^* which has at most $K - 1$ positive entries. In step 2, we show that when Δ is small, the solution x^* is close to 0. In step 3, we show that when x^* is small, there exists α that generates it, and corresponds to $K - 1$ imperfect messages each with support consisting of two profiles.

1. Consider minimizing (22) with respect to $x = (x_{vw})_{v \prec w}$ subject to (44) as well as the non-negativity constraints $x_{vw} \geq 0$. The set of solutions is non-empty since x that corresponds to the implementable disclosure rule in Example 6 satisfies the feasibility constraints. Let $q = \binom{2K}{2}$ denote the dimension of $x = (x_{vw})_{v \prec w}$. Since (44) involves $K - 1$ inequalities, if we denote by A the $(K-1) \times q$ matrix of coefficients on x, then (44) can be expressed in matrix form as

$$Ax \geq \Delta \equiv \begin{bmatrix} \Delta_1 \\ \vdots \\ \Delta_{K-1} \end{bmatrix}.$$

The optimization problem with respect to x is hence written as:

$$\begin{aligned} &\min \sum_{\{(v,w):\, v \prec w\}} p_v p_w \, x_{vw} \, |\theta_w - \theta_v| \\ &\text{subject to: } x \in P \equiv \{x : \; Ax \geq \Delta, \, x \geq 0\}. \end{aligned} \tag{45}$$

Since the objective function is also linear in x, there exists a solution x^* to (45) which is an extreme point of the polyhedron P. Let $J = \{j : \, j = (v, w), \, x^*_{vw} > 0\}$ be the indices of strictly positive entries of x^*.
We will show that $|J| \leq K - 1$. Suppose to the contrary that $|J| \geq K$, and consider the collection $(e_j)_{j \in J}$, where e_j is the jth unit vector, which has 1 in the jth entry and zero in all other entries. Denote also by $(\zeta_i)_{i=1}^d$ the base of the null space of A: $\{x : \; Ax = 0\}$. Since the dimension d of this null space satisfies $d = q - \text{rank}(A) \geq q - (K-1)$, the collection of $d + |J| \geq q - K + 1 + K = q + 1$ vectors $((e_j)_{j \in J}, (\zeta_i)_{i=1}^d)$ is linearly dependent. There then exist $(\lambda_i)_{i=1}^d$ and $(\mu_j)_{j \in J}$ such that $\sum_i \lambda_i \zeta_i + \sum_j \mu_j e_j = 0$, where λ_i's are not all equal to zero, and μ_j's are not all equal to zero. For $\kappa > 0$, consider $\hat{x} \equiv x^* + \kappa \sum_i \lambda_i \zeta_i$ and $\tilde{x} \equiv x^* - \kappa \sum_i \lambda_i \zeta_i$. Since $A\zeta_1 = \cdots = A\zeta_d = 0$,

$$A\hat{x} = A\tilde{x} = Ax^*.$$

Furthermore, since $\sum_i \lambda_i \zeta_i = -\sum_j \mu_j e_j$, if $x^*_{vw} = 0$ for any (v, w) (*i.e.*, $j = (v, w) \notin J$), then $\hat{x}_{vw} = \tilde{x}_{vw} = 0$ as well. It follows that both $\hat{x}$ and $\tilde{x}$ belong to the polyhedron P provided that κ is sufficiently small. Since $x^* = (\hat{x} + \tilde{x})/2$, this contradicts the fact that x^* is an extreme point of P.
2. When $\Delta = 0$, it is clear that (45) has a solution $x = 0$. We show that when Δ is small, any solution to (45) is close to zero using the theorem of the maximum. For this, take $B > 0$ large enough and consider the following maximization problem:

$$\begin{aligned} &\max_x \, (-1) \sum_{\{(v,w):\, v \prec w\}} p_v p_w \, x_{vw} \, |\theta_w - \theta_v| \\ &\text{subject to } x \in \Lambda(\delta), \end{aligned} \tag{46}$$

where

$$\Lambda(\delta) = \{x = (x_{vw})_{v \prec w} : Ax \geq \delta,\ 0 \leq x \leq (B, \dots, B)\}.$$

The objective function is linear in x and hence continuous. Note also that the matrix A in the constraints is a function only of p and $\succcurlyeq$, and is independent of the choice of $\theta \in \Theta_{\succcurlyeq,\eta}$.

The correspondence $\Lambda : \boldsymbol{R}_+^{K-1} \to \boldsymbol{R}^q$ is continuous at $\delta = 0$ and compact-valued: To see that it is upper hemi-continuous at $\delta = 0$, note that for any open set $G \supset \Lambda(0)$ and any $\delta \geq 0$, $\Lambda(0) \supset \Lambda(\delta)$ so that there exists a neighborhood $U \subset \boldsymbol{R}_+^{K-1}$ of $\delta = 0$ such that $\delta \in U$ implies $\Lambda(\delta) \subset G$. To see that Λ is lower hemi-continuous at $\delta = 0$, take any open set $G \subset \boldsymbol{R}^q$ such that $G \cap \Lambda(0) \neq \emptyset$. Let x^0 be an element of this intersection, and $\bar{x} \geq 0$ be the value of x corresponding to the disclosure rule in Example 6 for some fixed $\Delta = \bar{\Delta} \gg 0$ so that $A\bar{x} \geq \bar{\Delta}$. Take $\varepsilon > 0$ small enough so that $\varepsilon\bar{x} + (1-\varepsilon)x^0 \in G$, and take $U = [0, \varepsilon\bar{\Delta}) \subset \boldsymbol{R}_+^{K-1}$ as a neighborhood of $0 \in \boldsymbol{R}_+^{K-1}$. Then for any $\delta \in U$, we have

$$A\big(\varepsilon\bar{x} + (1-\varepsilon)x^0\big) \geq \varepsilon A\bar{x} \geq \varepsilon\bar{\Delta} \gg \delta,$$

and hence $\varepsilon\bar{x} + (1-\varepsilon)x^0 \in \Lambda(\delta)$. It follows that $\delta \in U$ implies that $G \cap \Lambda(\delta) \neq \emptyset$, showing that Λ is lower hemi-continuous at $\delta = 0$.

We note that the original optimization problem is equivalent to (46) for $\delta = \Delta$ since the upper bound B on x_{vw} can be ignored if B is taken large enough. Define $G^*(\delta)$ to be the set of solutions to (46) for each $\delta \geq 0$. Note that $G^*(0) = \{0\}$ since $x = 0$ is the unique solution to (46) when $\delta = 0$. If we take an open ball around 0 of radius $\frac{\eta}{K-1}$, then $G^*(0) = \{0\} \subset O$. Since the correspondence $G^* : \boldsymbol{R}_+^{K-1} \to \boldsymbol{R}^q$ is upper hemi-continuous by Berge's theorem of the maximum, there exists a neighborhood $U \subset \boldsymbol{R}_+^{K-1}$ of $\delta = 0$ such that $\delta \in U$ implies $G(\delta) \subset O$, or equivalently, $\delta \in U$ implies $x^*_{vw}(\delta) < \frac{\eta}{K-1}$ for every (v, w) with $v \prec w$.

3. Take $\delta > 0$ as above and write $x^* \equiv x^*(\delta)$. Let J be the set of indices of non-zero entries of x^*. By Step 1, $|J| \leq K - 1$. We show that when $x^*_{vw} < \frac{\eta}{K-1}$ for every $v \prec w$, there exist the set of imperfect messages $\{z_1, \dots, z_R\}$ and their probabilities $\alpha = (\alpha^r_v)_{v,r}$ that replicate x^*. First, let $R = |J|$ and define

$$\{z_1, \dots, z_R\} = \{vw : v \prec w \text{ and } x^*_{vw} > 0\}.$$

For each (v, w) with $v \prec w$, define α by

$$\alpha^{vw}_{\hat{v}} = \begin{cases} \frac{p_v + p_w}{|\theta_w - \theta_v|}\, x^*_{vw} & \text{if } \hat{v} \in \{v, w\}, \\ 0 & \text{otherwise.} \end{cases} \tag{47}$$

In other words, the imperfect message $z = vw$ is sent only when either v or w is realized. To see that α are well-defined probabilities, note that $x^*_{vw} > 0$ for at most $K - 1$ pairs (v, w) with $v \prec w$ and $x^*_{vw} \leq \frac{\eta}{K-1}$ for any such pair. Hence, for each

$v \in V$, the sum of probabilities that imperfect messages is sent at v is given by

$$\sum_{\substack{w\in V\\ v\prec w}} \alpha_v^{vw} + \sum_{\substack{w\in V\\ v\succ w}} \alpha_v^{wv} = \sum_{\substack{w\in V\\ v\prec w\\ x_{vw}^*>0}} \alpha_v^{vw} + \sum_{\substack{w\in V\\ w\prec v\\ x_{wv}^*>0}} \alpha_v^{wv}$$
$$= \sum_{\substack{w\in V\\ v\prec w\\ x_{vw}^*>0}} \frac{p_v+p_w}{|\theta_w-\theta_v|} x_{vw}^* + \sum_{\substack{w\in V\\ w\prec v\\ x_{wv}^*>0}} \frac{p_v+p_w}{|\theta_v-\theta_w|} x_{wv}^*$$
$$\leq \frac{1}{\eta}\Big\{\sum_{\substack{w\in V\\ v\prec w}} x_{vw}^* + \sum_{\substack{w\in V\\ w\prec v}} x_{wv}^*\Big\} \leq 1.$$

Finally, α replicates x^* since for the imperfect message $z = vw$,

$$|\theta_w-\theta_v| \frac{\alpha_v^{vw}\alpha_w^{vw}}{\sum_{\hat{v}} \alpha_{\hat{v}}^{vw}} = |\theta_w-\theta_v| \frac{\alpha_v^{vw}\alpha_w^{vw}}{p_v\alpha_v^{vw}+p_w\alpha_w^{vw}}$$
$$= \frac{|\theta_w-\theta_v|\,\alpha_v^{vw}}{p_v+p_w} = x_{vw}^* \quad \text{for any } (v,w) \text{ with } v \prec w.$$

We finally show that

$$\{v:\ f(z_r \mid v) > 0\} \neq \{v_{12}, v_{K2}\} \text{ and } \{v_{K1}, v_{K2}\} \text{ for any } r = 1,\dots,R.$$

Note that these are equivalent to $x_{12,K2}^* = x_{K1,K2}^* = 0$. First, $x_{12,K2} = x_{12,s_K\omega_2}$ (or $x_{K2,12} = x_{s_K\omega_2,12}$) appears when $k = 1$ and when $k = K-1$ in (44), and appears once in each of them: When $k = 1$ and $v = s_K\omega_2$, the coefficient of $x_{v,s_1\omega_2}$ is $-p_v(-1)^{\mathbf{1}\{v\prec 12\}} = -p_v$, and when $k = K-1$ and $v = s_1\omega_2$, the coefficient of $x_{v,s_K\omega_2}$ is $p_v(-1)^{\mathbf{1}\{v\prec K2\}} = -p_v$. Since both coefficients are negative, and the optimal solution must have $x_{12,K2}^* = 0$. Next, $x_{K1,K2} = x_{s_K\omega_1,s_K\omega_2}$ (or $x_{K2,K1} = x_{s_K\omega_2,s_K\omega_1}$) appears only when $k = K-1$ in (44), and appears twice in it: When $v = s_K\omega_1$, the coefficient of $x_{v,s_K\omega_2}$ is $p_v(-1)^{\mathbf{1}\{v\prec K2\}} = -p_v$, and when $v = s_K\omega_2$, the coefficient of $x_{v,s_K\omega_1}$ is $-p_v(-1)^{\mathbf{1}\{v\prec K1\}} = -p_v$. Again, both coefficients are negative and the optimal solution must have $x_{K1,K2}^* = 0$. ■

Proof of Proposition 10.

Recall from (21) that we can express the function H in terms of x defined in (19) as

$$H(s,t) = \sum_{\omega} g_s(\omega)\Big[\theta_{t\omega} + \sum_v p_v x_{v,t\omega}\,(-1)^{\mathbf{1}\{v\prec t\omega\}}\Big].$$

The conditions (5) for cyclical supermodularity is then given by

$$\sum_{i=1}^{n}\Big\{\sum_{\omega} g_{s_{k_i}}(\omega)\Big[\theta_{s_{k_i}\omega} + \sum_v p_v x_{v,s_{k_i}\omega}\,(-1)^{\mathbf{1}\{v\prec s_{k_i}\omega\}}\Big]$$
$$-\sum_{\omega} g_{s_{k_i}}(\omega)\Big[\theta_{s_{k_{i+1}}\omega} + \sum_v p_v x_{v,s_{k_{i+1}}\omega}\,(-1)^{\mathbf{1}\{v\prec s_{k_{i+1}}\omega\}}\Big]\Big\} \geq 0,$$

which can be rewritten as

$$\sum_{i=1}^{n} \sum_{\omega} g_{s_{k_i}}(\omega) \Big[\sum_{v} p_v x_{v,s_{k_i}\omega} \, (-1)^{\mathbf{1}\{v \prec s_{k_i}\omega\}} - \sum_{v} p_v x_{v,s_{k_{i+1}}\omega} \, (-1)^{\mathbf{1}\{v \prec s_{k_{i+1}}\omega\}} \Big] \geq \sum_{i=1}^{n} \sum_{\omega} g_{s_{k_i}}(\omega) \left(\theta_{s_{k_{i+1}}\omega} - \theta_{s_{k_i}\omega} \right) \quad \text{for } (k_1, \ldots, k_n) \text{ and } n = 2, \ldots, K. \tag{48}$$

As noted in (6), (48) is a collection of $N = \sum_{n=2}^{K} \binom{K}{n} (n-1)!$ inequalities which are linear in $x = (x_{vw})_{v \prec w}$. Since x is itself $\binom{KL}{2}$-dimensional, we can express (48) in matrix form as

$$Ax \geq \Delta,$$

where Δ is an N-dimensional vector whose entries are indexed by $(k_1, \ldots, k_n)$ for $n = 2, \ldots, K$ and given by the right-hand side of (48):

$$\Delta_{(k_1,\ldots,k_n)} = \sum_{i=1}^{n} \sum_{\omega} g_{s_{k_i}}(\omega) \left(\theta_{s_{k_{i+1}}\omega} - \theta_{s_{k_i}\omega} \right).$$

For any $h > 0$, when the talent function θ is ε-linear, we see that $\Delta_{(k_1,\ldots,k_n)} \to 0$ as $\varepsilon \to 0$ since by the definition of $(k_1, \ldots, k_n)$,

$$\sum_{i=1}^{n} \left(k_{i+1} - k_i \right) = 0.$$

Note also that the matrix A is again a function only of p and $\succcurlyeq$, and independent of the particular choice of $\theta \in \Theta_{\succcurlyeq,\eta}$. Consider now the following optimization problem with respect to x:

$$\min \mathcal{L}(\Gamma) = \sum_{\{(v,w):\, v \prec w\}} x_{vw} \, p_v p_w \, |\theta_w - \theta_v| \quad \text{subject to: } Ax \geq \Delta, \; x \geq 0. \tag{49}$$

This problem is identical to the problem (45) in the proof of Proposition 9 for the $K \times 2$ case except for the number of inequalities in the constraint set. It follows that the conclusion of the proposition follows if we repeat the argument in the proof of Proposition 9 once we note that the existence of $\bar{x} \geq 0$ with $A\bar{x} \geq \bar{\Delta}$ is now given by Example 7. ■

Proof of Proposition 12. By Lemma 11, incentive compatibility of a task-assignment mechanism is independent of the agent's action choice e. When the private type is binary, the mechanism $\Gamma = (y, Z, f)$ is incentive compatible if and only if

$$\sum_{\omega} \left\{\phi(s_2, \omega) - \phi(s_1, \omega)\right\} g_{s_1}(\omega) \leq y(s_2) - y(s_1) \\ \leq \sum_{\omega} \left\{\phi(s_2, \omega) - \phi(s_1, \omega)\right\} g_{s_2}(\omega). \tag{50}$$

It follows that an incentive compatible effort-inducing mechanism $\Gamma = (y, Z, f)$ exists if and only if (33) and (50) hold, or equivalently, ϕ is supermodular, and

$$\sum_{\omega} \left\{\phi(s_2, \omega) - \phi(s_1, \omega)\right\} g_{s_1}(\omega) \\ \leq \sum_{\omega} \left\{\phi(s_2, \omega)\, g_{s_2}(\omega) - \phi(s_1, \omega)\, g_{s_1}(\omega)\right\} - \frac{c_1 - c_0}{q_1(s_2) - q_0(s_2)}. \tag{51}$$

Simplifying (51) using Lemma 6 as well as the fact that the performance signal ω is binary, we see that the disclosure rule (Z, f) is effort-inducing if and only if

$$x_{12}\,(p_1 + p_2) + x_{13}\,(p_1 + p_3) - x_{24}\,(p_2 + p_4) - x_{34}\,(p_3 + p_4) \\ + x_{14}\,(p_4 - p_1) - (-1)^{\mathbf{1}_{\{2 \prec 3\}}} x_{23}\,(p_2 - p_3) \geq \Delta. \tag{52}$$

and

$$-p_1 x_{14} - p_2 x_{24} - (p_3 + p_4) x_{34} + p_1 x_{13} - (-1)^{\mathbf{1}_{\{2 \prec 3\}}}\, p_2 x_{23} \geq C, \tag{53}$$

where C is as defined in (34).[32]

Let x^* denote the optimal solution. Clearly, $x^*_{24} = x^*_{34} = 0$ since their coefficients are negative in both (52) and (53). It follows that only x^*_{12}, x^*_{13}, x^*_{14}, and x^*_{23} can be strictly positive, and by the same logic as in the proof of Proposition 9, we can take x^* so that at most two of them are strictly positive.

We begin by showing that $x^*_{14} = 0$. Suppose to the contrary that $p_4 - p_1 > 0$ and $x^*_{14} > 0$. For (53) to hold, it must be the case that either $x^*_{13} > 0$ or $x^*_{23} > 0$. If $x^*_{14} > 0$, then, $x^*_{vw} > 0$ for two coordinates (v, w). We can then assume that x^* satisfies both (52) and (53) with equality since otherwise, there would exist a solution x^* such that $x^*_{vw} > 0$ only for a single coordinate (v, w).

- $x^*_{13} > 0$, $x^*_{14} > 0$, and $x^*_{vw} = 0$ for $(v, w) \neq (1, 3)$, $(1, 4)$.

  Let $\hat{x}$ and $\tilde{x}$ be such that

$$\hat{x}_{13} = \tfrac{\Delta}{p_1 + p_3}, \text{ and } \hat{x}_{vw} = 0 \text{ for } (v, w) \neq (1, 3),$$
$$\tilde{x}_{14} = \tfrac{\Delta}{p_4 - p_1}, \text{ and } \tilde{x}_{vw} = 0 \text{ for } (v, w) \neq (1, 4).$$

[32] Note that (52) is identical to (42) in the proof of Proposition 7.

Note that both $\hat{x}$ and $\tilde{x}$ satisfy (52) with equality. Since x^* also satisfies (52) with equality, x^* is a linear combination of $\hat{x}$ and $\tilde{x}$. Furthermore, $(p_4 - p_1)\, x^*_{14} = \Delta - (p_1 + p_3)\, x^*_{13} > 0$ so that $x^*_{13} < \frac{\Delta}{p_1+p_3}$, which implies that

$$p_1 \hat{x}_{13} - p_1 \hat{x}_{14} = p_1 \frac{\Delta}{p_1 + p_3} > p_1 x^*_{13} - p_1 x^*_{14} = C.$$

In other words, $\hat{x}$ satisfies (53). The inequalities $\frac{p_3}{p_1+p_3} < \frac{p_4}{p_4-p_1}$ and $\theta_3 < \theta_4$ together imply

$$\begin{aligned} p_1 p_3 \left(\theta_3 - \theta_1\right) \hat{x}_{13} &= p_1 p_3 \left(\theta_3 - \theta_1\right) \frac{\Delta}{p_1 + p_3} \\ &< p_1 p_4 \left(\theta_4 - \theta_1\right) \frac{\Delta}{p_4 - p_1} \\ &= p_1 p_4 \left(\theta_4 - \theta_1\right) \tilde{x}_{14}, \end{aligned}$$

and hence $\mathcal{L}(\hat{x}) < \mathcal{L}(\tilde{x})$. Since x^* is a convex combination of $\hat{x}$ and $\tilde{x}$ as noted above and since $\mathcal{L}$ is linear, $\mathcal{L}(x^*)$ is also a convex combination of $\mathcal{L}(\hat{x})$ and $\mathcal{L}(\tilde{x})$, and satisfies $\mathcal{L}(x^*) > \mathcal{L}(\hat{x})$. Given that $\hat{x}$ satisfies both (52) and (53), this is a contradiction to the optimality of x^*.

- $x^*_{14} > 0$, $x^*_{23} > 0$, and $x^*_{vw} = 0$ for $(v, w) \neq (2, 3)$, $(1, 4)$.

  Let $\hat{x}$ be such that $\hat{x}_{13} = \frac{p_4-p_1}{p_1+p_3}\, x^*_{14}$, $\hat{x}_{14} = 0$, and $\hat{x}_{vw} = x^*_{vw}$ for $(v, w) \neq (1, 3)$, $(1, 4)$. $\hat{x}$ then satisfies both (52) and (53). Since $\frac{p_3}{p_1+p_3} < \frac{p_4}{p_4-p_1}$ and $\theta_3 < \theta_4$, we have

  $$p_1 p_3 (\theta_3 - \theta_1)\, \hat{x}_{13} = p_1 p_3 (\theta_3 - \theta_1) \frac{p_4 - p_1}{p_1 + p_3}\, x^*_{14} < p_1 p_4 (\theta_4 - \theta_1)\, x^*_{14},$$

  which leads to the contradiction that $\mathcal{L}(\hat{x}) < \mathcal{L}(x^*)$.

We hence conclude that at most two of x^*_{12}, x^*_{13}, and x^*_{23} can be strictly positive, and proceed by examining the following three cases separately.

1) If $x^*_{12} = 0$, then (52) and (53) reduce to

$$x^*_{13} \left(p_1 + p_3\right) - (-1)^{\mathbf{1}_{\{2 \prec 3\}}} x^*_{23} \left(p_2 - p_3\right) \geq \Delta, \tag{54}$$

$$p_1 x^*_{13} - (-1)^{\mathbf{1}_{\{2 \prec 3\}}} p_2 x^*_{23} \geq C. \tag{55}$$

   At least one of these two inequalities with equality.

   (a) Suppose first that (55) holds with equality.

   - If $2 \prec 3$ ($\Leftrightarrow \theta_2 < \theta_3$), then (55) becomes

     $$p_1 x^*_{13} + p_2 x^*_{23} = C.$$

By (35), both $(x_{13}, x_{23}) = (\frac{C}{p_1}, 0)$ and $(x_{13}, x_{23}) = (0, \frac{C}{p_2})$ satisfy

$$\frac{p_1 + p_3}{\theta_3 - \theta_1} x_{13} + \frac{p_2 + p_3}{|\theta_3 - \theta_2|} x_{23} \leq 1. \tag{56}$$

(x_{13}^*, x_{23}^*) is a convex combination of these two points, and hence satisfies (56) as well.

- If $2 \succcurlyeq 3$ ($\Leftrightarrow \theta_2 \geq \theta_3$), then (55) becomes

$$p_1 x_{13}^* - p_2 x_{23}^* = C.$$

If $p_2 \geq p_3$ so that (25) holds, then $x_{23}^* = 0$ and $x_{13}^* = \max\{\frac{C}{p_1}, \frac{\Delta}{p_1+p_3}\}$. By (35), x^* satisfies (56). Assume then in what follows that (25) does not hold so that $\theta_2 > \theta_3$ and $p_2 < p_3$. If x^* satisfies (54) with strict inequality, then $(x_{13}^*, x_{23}^*) = (\frac{C}{p_1}, 0)$, which satisfies (56) by (35). On the other hand, suppose x^* satisfies (54) also with inequality. If (26) holds, then $(x_{13}^*, x_{23}^*) = (\frac{\Delta}{p_1+p_3}, 0)$, which satisfies (56). If (26) does not hold, then (27) holds by assumption. Note that x^* in this case is a convex combination of $(x_{13}, x_{23}) = (\frac{\Delta}{p_1+p_3}, 0)$ and $(x_{13}, x_{23}) = (0, \frac{\Delta}{p_3-p_2})$, both of which satisfy (56) under (27). It follows that x^* also satisfies (56).

(b) If (55) holds with strict inequality, then x^* is identical to the optimal solution in the proof of Proposition 7, and satisfies (56) under (25), (26) or (27).

Take the number of pooling messages $R = 2$ and let the probability $\alpha_v^r = f(z^r \mid v)$ of message z^r given profile $v \in V$ ($r = 1, 2$ and $v = 1, 2, 3$) be defined by

$$\alpha_2^1 = 0, \quad \alpha_1^1 = \alpha_3^1 = \frac{p_1 + p_3}{\theta_3 - \theta_1} x_{13}^*, \quad \text{and} \quad \alpha_1^2 = 0, \quad \alpha_2^2 = \alpha_3^2 = \frac{p_2 + p_3}{\theta_3 - \theta_2} x_{23}^*.$$

Then α_v^r is well-defined since $\alpha_3^1 + \alpha_3^2 \leq 1$ by (56), and hence also $\alpha_1^1 + \alpha_1^2$, $\alpha_2^1 + \alpha_2^2 \leq 1$. Furthermore, it replicates x^* since

$$\sum_{r=1}^{2} \frac{\alpha_1^r \alpha_3^r}{p_1 \alpha_1^r + p_3 \alpha_3^r} = \frac{\alpha_1^1}{p_1 + p_3} = x_{13}^*, \quad \sum_{r=1}^{2} \frac{\alpha_2^r \alpha_3^r}{p_2 \alpha_2^r + p_3 \alpha_3^r} = \frac{\alpha_2^2}{p_2 + p_3} = x_{23}^*.$$

2) If $x_{13}^* = 0$, then (52) and (53) reduce to

$$x_{12}^* (p_1 + p_2) - (-1)^{\mathbf{1}_{\{2 \prec 3\}}} x_{23}^* (p_2 - p_3) \geq \Delta, \tag{57}$$
$$- (-1)^{\mathbf{1}_{\{2 \prec 3\}}} p_2 x_{23}^* \geq C. \tag{58}$$

Again, at least one of these two inequalities with equality. Note that (58) requires that $2 \prec 3$ ($\Leftrightarrow \theta_2 < \theta_3$).

(a) Suppose first that (58) holds with equality:

$$p_2 x_{23}^* = C.$$

If (57) holds with strict inequality, then $(x_{12}^*, x_{23}^*) = (0, \frac{C}{p_2})$. By (35), x^* satisfies

$$\frac{p_1 + p_2}{\theta_2 - \theta_1}\, x_{12}^* + \frac{p_2 + p_3}{|\theta_3 - \theta_2|}\, x_{23}^* \leq 1. \tag{59}$$

Suppose now that (57) also holds with equality.

- $p_2 > p_3$. In this case, (25) does not hold. If (26) holds, then $(x_{12}^*, x_{23}^*) = (\frac{\Delta}{p_1+p_2}, 0)$. If (26) does not hold either, then (27) holds by assumption and (x_{12}^*, x_{23}^*) is a convex combination of $(x_{12}, x_{23}) = (0, \frac{\Delta}{p_2-p_3})$ and $(x_{12}, x_{23}) = (\frac{\Delta}{p_1+p_2}, 0)$, and satisfies (56) since both these points satisfy (59) under (27).
- $p_2 \leq p_3$. In this case,

$$(x_{12}^*, x_{23}^*) = \Big(\frac{1}{p_1 + p_2}\,\{\Delta + \frac{p_3 - p_2}{p_2}\, C\},\, \frac{C}{p_2}\Big).$$

  x^* then satisfies (59) because of the third condition in (35).

(b) If (58) holds with strict inequality, then x^* is identical to the optimal solution in the proof of Proposition 7 and satisfies (59) under (25), (26) or (27).

If we let $R = 2$ and define $\alpha_v^r = f(z^r \mid v)$ $(r = 1\ ,2)$ by

$$\alpha_3^1 = 0, \quad \alpha_1^1 = \alpha_2^1 = \frac{p_1 + p_2}{\theta_2 - \theta_1}\, x_{12}^*, \quad \text{and} \quad \alpha_1^2 = 0, \quad \alpha_2^2 = \alpha_3^2 = \frac{p_2 + p_3}{|\theta_3 - \theta_2|}\, x_{23}^*,$$

then α_v^r is a well-defined probability by (59) and replicates x^* as above.

3) If $x_{23}^* = 0$, then (52) and (53) reduce to

$$x_{12}^*\,(p_1 + p_2) + x_{13}^*\,(p_1 + p_3) \geq \Delta, \tag{60}$$

$$p_1 x_{13}^* \geq C. \tag{61}$$

Again, the optimal $x = x^*$ satisfies at least one of these two inequalities with equality.

(a) If (61) holds with equality and (60) holds with strict inequality, then $(x_{12}^*, x_{13}^*) = (0, \frac{C}{p_1})$. By (35), x^* satisfies

$$\frac{p_1 + p_2}{\theta_2 - \theta_1}\, x_{12}^* + \frac{p_1 + p_3}{\theta_3 - \theta_1}\, x_{13}^* \leq 1. \tag{62}$$

(b) If x^* satisfies both (61) and (60) with equality, then it satisfies (62) because it is a convex combination of $(x_{12}, x_{13}) = (\frac{\Delta}{p_1+p_2}, 0)$ and $(x_{12}, x_{13}) = (0, \frac{\Delta}{p_1+p_3})$, both of which satisfy (62).

(c) If x^* satisfies (61) with strict inequality, then x^* is identical to the optimal solution in the proof of Proposition 7 and satisfies (62) under (25), (26) or (27).

If we let $R = 2$ and define $\alpha_v^r = f(z^r \mid v)$ $(r = 1, 2)$ by

$$\alpha_3^1 = 0, \quad \alpha_1^1 = \alpha_2^1 = \frac{p_1 + p_2}{\theta_2 - \theta_1} x_{12}^*, \quad \text{and} \quad \alpha_2^2 = 0, \quad \alpha_1^2 = \alpha_3^2 = \frac{p_1 + p_3}{\theta_3 - \theta_1} x_{13}^*,$$

then α_v^r is a well-defined probability by (62) and replicates x^* as above.

In all the three cases above, hence, we can take $R = 2$ and α_v^r to be positive for at most two profiles for each r. This completes the proof. ■