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GROWTH PROMOTION POLICIES WHEN TAXES CANNOT BE RAISED

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Growth Promotion Policies when Taxes cannot be Raised*

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Abstract

This paper examines the effect of growth-enhancing policies in an R&D-based endogenous growth model when the government does not have the ability to raise taxes to finance the required expenses. We show that the government can increase the economic growth rate by debt-financed R&D subsidies while perpetually rolling over the debt, if the productivity of research workers in product development is higher than a threshold. If the condition is not met, the government debt becomes unsustainable, or the growth rate is reduced by subsidy.

Keywords: Economic Growth, Endogenous Growth, Public Debt, Ponzi Scheme, Research Subsidy

JEL Classification Codes: O38, O41

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1 Introduction

Once one starts to think about [growth], it is hard to think about anything else
— Robert Lucas Jr. (1988)

As Lucas (1988) mentioned, economic growth is one of the most important objectives in economics. How can we achieve faster economic growth? Modern theories of endogenous growth provide a surprisingly simple answer. Economic growth, in the long run, is determined by technological change. Technological change is realized by R&D. Therefore, by promoting R&D, e.g., by government subsidies on R&D, economic growth can be accelerated. Still, many economies are struggling with slower economic growth than they desire. Even when they know that R&D promotion policies will enhance growth, there are often insufficient funds to implement those policies. Constraints on the governmental budget limit the ability of the government to promote R&D and, therefore, economic growth.

Recent literature on political economy suggests that it is difficult for the government to raise tax rates because of political constraints. For example, Jiang, Sargent, Wang and Yang (2022) assume there is an upper bound for the tax rate based on Keynes (1923)'s political considerations.[1] However, existing studies of R&D-based growth typically disregard such constraints by implicitly assuming that the government can levy lump-sum taxes to implement policies. (e.g. Grossman and Helpman, 1991a,b; Jones and Williams, 2000). Some studies include the distortions caused by factor-income taxation but still assume that the government can set any tax rate (Grossmann, Steger and Trimborn, 2013).

In contrast, this paper considers an extreme situation where the government cannot raise (extra) taxes at all. If the government wants to support R&D, it must be financed entirely by public debt. It also does not have the ability to raise taxes in the future to repay the debt. Therefore, the debt must be rolled over infinitely. In other words, we consider the environment in which any growth promotion policies must be financed by a Ponzi scheme. While common sense suggests such policies would not be sustainable, O'Connell and Zeldes (1988) show that the government indeed can run a "rational" Ponzi scheme if the rate of

[1] In an overlapping generations political economy model, Song, Storesletten and Zilibotti (2012) shows the possibility that the inter-generational conflict causes the government to raise no tax if the political power of the old is stronger than the young. See also Alesina and Passalacqua (2016) and Yared (2019) for the literature review.

economic growth is higher than the interest rate on the governmental debt; $g > r$ for short.

Mehrotra and Sergeyev (2021) report that, in the 1946-2006 period, the median value of $g - r$ is 1.0 for the United States and 0.8 for the average of 17 advanced countries. Blanchard (2019) also mentions that $g > r$ have been more historical the rule rather than the exception in the U.S. since 1950. Mauro and Zhou (2020) analyzed data on average effective borrowing costs for 55 countries over up to 200 years and found that $g > r$ prevails for both advanced and emerging economies.

Given these findings, we examine whether the growth rate can be enhanced when the government finances the R&D subsidies entirely by perpetually rolled-over debts. The result is not obvious because such policies affect both g and r. The direct effect of R&D subsidies is to induce private firms to do more R&D, which will speed up technological progress. However, the government's debt may crowd out private R&D investments by raising the equilibrium interest rate in the financial markets. More precisely, a higher interest rate implies that the present value of future profits realized by R&D is lower, thereby discouraging R&D. Additionally, if the increase in r is larger than g, then the $g-r$ gap will shrink with the R&D promotion policies. When g falls below r, the public debt becomes unsustainable, given that the government has no ability to raise taxes.

We find that the overall effects of such policies crucially depend on the productivity of R&D; i.e., how many innovations can be realized per R&D workers. We show that the $g > r$ holds in equilibrium if the productivity of R&D is high enough. However, this does not always mean that the growth rate can be enhanced by debt-financed R&D promotion policies. For these policies to raise the long-run growth rate, the productivity of R&D has to be even higher. Even in such a case, there is a maximum rate of economic growth that can be achieved because when the rate of subsidies becomes too high, its effect on raising r dominates the positive effect on g. Moreover, there is a level of subsidies above which no balanced growth exists; i.e., the Ponzi scheme becomes unsustainable. We also calculate the maximal debt-GDP ratio that can be supported without raising taxes. We find that while debt-financed R&D subsidies may increase the long-run growth rate, they reduce the fiscal space in the sense the highest level of the debt-GDP ratio from which the economy can come back to a steady state is now lower. Those results indicate that the government's ability to promote growth is not limitless if it is unable to raise taxes, even when $g > r$.

There is a strand of literature that examines the sustainability of government debt in an environment where $g > r$ holds in equilibrium (e.g. Blanchard, 2019; Reis, 2021; Ball and Mankiw, 2023; Barro, 2023). Similarly to some of those studies, we use a continuous-time overlapping generations model to explain why the interest rate can be lower than the growth rate in the long run. However, in existing studies, the long-run growth rate is exogenous, and the focus is mostly on the effect of policies on the safe interest rate. The exceptions are Saint-Paul (1992) and King and Ferguson (1993), who developed AK-type endogenous growth models in which $g > r$ holds in equilibrium. They have shown that the economy is dynamically efficient even when $g > r$ because, in endogenous growth models, the social return on investment is higher than the interest rate, which also holds true in our model. The most important difference from those studies is that we consider an R&D-based variety-expansion growth model rather than an AK model, where long-term growth is determined solely by capital accumulation. By explicitly modeling the R&D process, we are able to examine the effect of debt-financed R&D promotion policies on the incentives for technological innovations, which are unarguably an important source of economic growth. This paper is also related to Angeletos, Lian and Wolf (2024) in that both show that future tax increases are not necessary after debt-financed policies are implemented. While they consider a short-term stimulus policy in a New Keynesian setting, we consider a long-term growth promotion policy in an R&D-based endogenous growth model.

2 Model

2.1 Individuals

We consider a variant of the continuous-time overlapping generations model by Blanchard (1985) and Yaari (1965). The economy is populated by individuals who face a constant Poisson death rate of $\mu > 0$. For simplicity, we assume that the birth rate per population is also μ so that the total population is unchanged, which we denote by L. Let us denote by $N_{s,t}$ the number of individuals who are born at time s and still alive at time t. Then,

$$N_{s,t} = \mu L e^{-\mu(t-s)}, \tag{1}$$

which adds up to L by integrating over s from $-\infty$ to t. To capture aging and their lifetime motive of savings, we also assume that the productivity of individuals decreases with age at the rate of $\delta > 0$. Specifically, each individual of generation s inelastically supplies

$$\ell_{s,t} = \frac{\delta + \mu}{\mu} e^{-\delta(t-s)} \tag{2}$$

units of labor. The productivity at age 0 is normalized to $(\delta + \mu)/\mu$ so that the aggregate supply of labor is L.[2]

The expected utility of a generation s individual, taking into account their mortality, is given by

$$U_s = \int_s^\infty (\ln c_{s,t}) e^{-(\rho+\mu)(t-s)} dt, \tag{3}$$

where $c_{s,t}$ is the amount of consumption by a generation-s individual at time t, and $\rho > 0$ is the discount rate. Observe that they further discount the future by their survival probability, $e^{-\mu(t-s)}$. We assume that the discount rate is not too high. In particular, we assume that $\rho < \delta$, so that the individuals have incentives to save for their later age. (See, Rachel and Summers, 2019).

Let $k_{s,t}$ be the real asset holding by a generation s, and r_t be the real interest rate on bonds. Following Blanchard (1985), we assume that there is a perfect market for annuities. Then, the return from the annuities is $r_t + \mu$ for survivors. Since $r_t + \mu > r_t$, individuals hold all their assets in the form of annuities. We also normalize the price of the final good to be one. Then, the budget constraint is given by

$$\dot{k}_{s,t} = (r_t + \mu)\, k_{s,t} + \ell_{s,t} w_t - c_{s,t}, \tag{4}$$

where w_t is the real wage per unit of labor. The newborn generation has zero financial assets, which means $k_{t,t} = 0$. Given those, each individual maximizes the expedited utility (3) subject to the budget constraint (4), initial condition $k_{t,t} = 0$, and the usual non-Ponzi-game condition.

Let us define aggregate consumption and the aggregate asset of individuals by

$$C_t = \int_{-\infty}^t c_{s,t} N_{s,t} ds, \quad K_t = \int_{-\infty}^t k_{s,t} N_{s,t} ds. \tag{5}$$

In Appendix A.1, we show that C_t evolves according to

$$\dot{C}_t = (r_t - \rho + \delta)\, C_t - (\rho + \mu)(\delta + \mu) K_t. \tag{6}$$

[2] This can be shown from $\int_{-\infty}^t \ell_{s,t} N_{s,t} ds = \frac{\delta+\mu}{\mu} L \int_{-\infty}^t e^{-(\mu+\delta)(t-s)} ds = L$.

The $(r_t - \rho)C_t$ part comes directly from the Euler equation for individuals. In addition, generational change affects the growth rate of aggregate consumption both positively and negatively. The newly-born generation has a higher productivity than the population that it replaces, and the difference in the productivity between the young and the old is more pronounced when δ is larger. This effect allows the newborns to consume more than those replaced. Therefore, δ in the first term positively affects $\dot{C}_t$. However, the newborns do not have financial assets, while those who pass away on average have K_t. This effect lets the newborns consume less than those replaced. Therefore, K_t negatively affects $\dot{C}_t$.[3]

2.2 Supply Side

The supply side of this economy is purposefully close to that of the standard variety-expansion model of Grossman and Helpman (1991a). It consists of three sectors: the final goods sector, the intermediate goods sector, and the R&D sector. In the final goods sector, a representative firm competitively produces final goods X_t from a continuum of varieties of intermediate goods $x_t(i)$. The production function is given by

$$X_t = \left[\int_0^{n_t} x_t(i)^\alpha di\right]^{1/\alpha}, \tag{7}$$

where n_t is the number of intermediate goods available at time t and $\alpha \in (0,1)$ is a production parameter. Let $p_t(i)$ be the price of intermediate good i. The representative final-good firm maximizes its profit,

$$X_t - \int_0^{n_t} p_t(i)x_t(i)di. \tag{8}$$

The first-order condition for profit maximization implies that the demand function of the intermediate goods is

$$x_t(i) = p_t(i)^{-\frac{1}{1-\alpha}} X_t. \tag{9}$$

In the intermediate goods sector, there are n_t intermediate-good firms, each of which produces its own variety of goods, $x_t(i)$. The production of one unit of $x_t(i)$ requires one unit of labor, and therefore its profit is given by $\pi_t(i) = (p_t(i) - w_t)x_t(i)$. Given that $x_t(i)$

[3] The second effect depends on the assumption that all individuals fully invest their assets in annuities and leave no bequests for new generations.

is determined by the demand function (9), the profit-maximizing pricing implies

$$p_t(i) = \frac{w_t}{\alpha}, \quad x_t(i) = \left(\frac{\alpha}{w_t}\right)^{\frac{1}{1-\alpha}} X_t, \tag{10}$$

$$\pi_t(i) = (1-\alpha)\left(\frac{\alpha}{w_t}\right)^{\frac{\alpha}{1-\alpha}} X_t. \tag{11}$$

The above result shows that all intermediate firms produce the same amount of output. The output of each intermediate firm can be written as $x_t(i) = L_t^P/n_t$ for all i, where L_t^P is the total amount of labor employed in this sector. By substituting it to the final goods production function (7), we obtain

$$X_t = n_t^{\frac{1-\alpha}{\alpha}} L_t^P. \tag{12}$$

The R&D sector has a representative R&D firm, which competitively creates new goods according to

$$\dot{n}_t = a n_t L_t^R, \tag{13}$$

where L_t^R is the amount of labor used for R&D and $a > 0$ is a parameter that specifies the efficiency of R&D. We follow a standard setting in the variety-expansion model and assume that there is an externality from the past R&D to the current one. The term n_t in the RHS of (13) reflects this externality.

Equation (13) implies that the creation of a new intermediate good requires $1/an_t$ units of labor. We assume that the government subsidizes the fraction $\theta \in [0,1)$ of the R&D cost. Then, the private cost of developing a new intermediate good is $(1-\theta)w_t/an_t$. Now, let v_t be the value of an intermediate-good firm. Then, the free entry condition for the R&D is

$$v_t \le (1-\theta)\frac{w_t}{an_t} \text{ with equality if } \dot{n}_t > 0. \tag{14}$$

2.3 Government

As explained above, the government subsidizes the fraction θ of the cost of R&D. Since the pre-subsidy aggregate cost of R&D is $w_t L_t^R$, the amount of government expenditure is $\theta w_t L_t^R$. We assume that the government cannot collect taxes and that all expenditure is financed by government debts. Then, the amount of the government debts, B_t, evolves according to

$$\dot{B}_t = r_t B_t + \theta w_t L_t^R. \tag{15}$$

Since the government bond is never repaid (or repaid entirely by issuing new bonds), the government is running a Ponzi scheme.[4] We investigate the possibility that the government can run a rational Ponzi game, similar to the one examined by O'Connell and Zeldes (1988), and use the revenue to promote economic growth.

3 Analysis of the Model

3.1 Growth Rate and Interest Rate

Given the economy described in the previous section, here we derive the relationship between the growth rate and the interest rate in equilibrium. First, we derive the real interest rate. The consumers hold all their assets in the form of annuities, and the annuity company invests the assets in government bonds and the shares of intermediate firms. Therefore, the equilibrium in the asset market is

$$K_t = B_t + n_t v_t. \tag{16}$$

Since risks are fully diversified, the expected return on holding the share of one intermediate-good firm should be equal to the interest rate on bonds. This no-arbitrage condition can be written as

$$r_t = \frac{\pi_t + \dot{v}_t}{v_t}, \tag{17}$$

where π_t is given by (11). In the main text, we focus on the case where the amount of R&D is positive, leaving the discussion of the case of $\dot{n}_t = 0$ for Appendix A.3. Then, v_t is given by (14) with equality. Both π_t and v_t depend on the wage level, which we now derive. Since the final-good sector is competitive, the maximized profit of the final good firm in equation (8) should be zero. By substituting $x_t(i)$ and $p_t(i)$ into (8), this condition determines the market wage,

$$w_t = \alpha n_t^{\frac{1-\alpha}{\alpha}}. \tag{18}$$

[4]Note that this is in contrast to individuals, who maximize their lifetime utility subject to the no-Ponzi game condition. The difference in the ability to borrow between the government and individuals reflects the reality. Without collateral, individuals usually cannot borrow large amounts of money for various reasons (e.g., the risk of running away). We can rewrite the model with a borrowing constraint for individuals, which yields the same result as the present setting in the steady state.

Substituting the values of π_t and v_t into (17) yields the real interest rate in equilibrium.[5]

$$r_t = \frac{a}{\alpha}\left(\frac{1-\alpha}{1-\theta}L_t^P - (2\alpha - 1)L_t^R\right). \tag{19}$$

Next, we turn to the growth rate. The GDP of this economy is defined by the sum of consumption expenditure C_t, private investment expenditure for R&D, $(1-\theta)w_t L_t^R$, and government expenditure, $\theta w_t L_t^R$. Note that the final output is used only for consumption, and therefore the equilibrium in the goods market means $C_t = X_t$. Then, using (12) and (18), the GDP can be written as

$$\text{GDP}_t = n_t^{\frac{1-\alpha}{\alpha}}(L_t^P + \alpha L_t^R). \tag{20}$$

Since $\alpha < 1$, the GDP is higher when more labor is used for production, given the value of n_t. This is because there is a positive markup in the intermediate goods sector, whereas none in the R&D sector. Let us also define the potential GDP, Y_t, as the level of GDP when all labor is used for production. With $L_t^P = L$ and $L_t^R = 0$, (20) reduces to

$$Y_t = n_t^{\frac{1-\alpha}{\alpha}} L. \tag{21}$$

This is the upper bound for GDP_t given n_t. Y_t can also be viewed as the supply capacity of the economy. From (13), the growth rate of the potential GDP in (21) is given by

$$g_t \equiv \frac{\dot{Y}_t}{Y_t} = \overline{g}\frac{L_t^R}{L}, \tag{22}$$

where $\overline{g} \equiv (1-\alpha)aL/\alpha$ is the maximum growth rate of the potential GDP that is realized when all labor is used for R&D. Note that when g_t becomes constant in a steady state, g_t also represents the growth rate of the GDP.[6] We simply call g_t the growth rate unless otherwise noted.

Now, we are ready to show the relationship between the interest rate and the growth rate. The labor market equilibrium condition is

$$L_t^P + L_t^R = L. \tag{23}$$

[5] Using (18), the values of π_t and v_t can be obtained as follows. From (12) and (18), equation (11) gives $\pi_t = (1-\alpha)n_t^{(1-2\alpha)/\alpha}L_t^P$. From (14) with equality and (18), the value of firm is $v_t = (1-\theta)w_t/en_t = ((1-\theta)\alpha/a)n_t^{(1-2\alpha)/\alpha}$. Using (13), its derivative is $\dot{v}_t = -((2\alpha-1)/\alpha)aL_t^R v_t$. Substituting these results into (17) yields (19).

[6] When g_t is constant, (22) means L_t^R is constant and the same for $L_t^P = L - L_t^R$. Then, from (20) and (21), the growth rate of GDP_t coincides with $\dot{Y}_t/Y_t$.

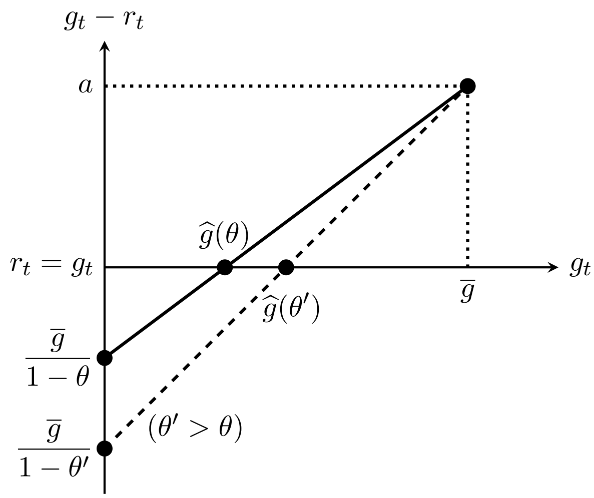


Figure 1: Relationship between $g_t - r_t$ and the growth rate for a given level of research subsidy. The dashed line shows the relationship when the rate of research subsidy is increased from θ to θ'.

By eliminating L_t^P and L_t^R from (19) using (22) and (23), we obtain the gap between the growth rate and the interest rate as a function of g_t.

$$g_t - r_t = s(\theta)(g_t - \widehat{g}(\theta)), \text{ where} \tag{24}$$
$$s(\theta) = \frac{1-\alpha\theta}{(1-\alpha)(1-\theta)} > 1, \quad \widehat{g}(\theta) = \frac{1-\alpha}{1-\alpha\theta}\overline{g} \in (0, \overline{g}].$$

As shown in Figure 1, given parameters, $g_t - r_t$ is positively and linearly related to g_t, with the slope of $s(\theta) > 1$. In particular, g_t is higher than r_t when g_t is greater than the threshold at $\widehat{g}(\theta)$. This means that keeping the growth rate high is crucial to maintain $g_t > r_t$, and hence to run the government's Ponzi scheme.

Observe that, when the rate of R&D subsidy, θ, is increased, the thick line in Figure 1 rotates counter-clockwise, which makes $g_t - r_t$ lower for a given growth rate. Accordingly, the threshold, $\widehat{g}(\theta)$ in (24), increases with θ. While research subsidy may raise the growth rate, g_t needs to be even higher to maintain $g_t > r_t$. In the following subsections, we analyze the equilibrium dynamics that determine the path of g_t, first without the research subsidy and then with it.

3.2 Equilibrium Dynamics

The dynamics of this economy can be examined by focusing on two variables, g_t and $D_t \equiv B_t/Y_t$. Here, D_t is the ratio of the government debt to the potential GDP.[7] We simply call it the debt-GDP ratio. Using (15), (18), (22) and (24), its time derivative is given by

$$\dot{D}_t = -s(\theta)\left(g_t - \widehat{g}(\theta)\right) D_t + \frac{\theta \alpha L}{\overline{g}} g_t. \tag{25}$$

The first term of the RHS represents $(r_t - g_t)D_t$. With a balanced budget, the debt-GDP ratio would expand or shrink at the rate of $r_t - g_t$. The second term is the ratio of the government spending on subsidies to the potential GDP.[8] It accelerates the increase in D_t.

Next, we derive the time evolution of g_t. In this model, final goods are used only for consumption. Therefore, $X_t = C_t$ from the equilibrium of the goods market. Then, using (6), (12), (14), (16), (18), (21), (22) and (24), we obtain[9]

$$\dot{g}_t = (\overline{g} - g_t)\big(s(\theta)\left(g_t - \widehat{g}(\theta)\right) - \delta + \rho\big) + \overline{g}(\rho + \mu)(\delta + \mu)\left(D_t + \frac{\alpha(1-\theta)}{aL}\right). \tag{26}$$

In the first term, $s(\theta)\left(g_t - \widehat{g}(\theta)\right) - \delta + \rho$ represents $g_t - (r_t - \rho + \delta)$. There is a positive effect of g_t on $\dot{g}_t$ because growth in Y_t means that fewer production workers are required to produce a given amount of C_t. Also, as explained in Section 2.1, the Euler equation for aggregate consumption (equation 6) implies that a rise in $r_t - \rho + \delta$ increases $\dot{C}_t$, which means less labor is allocated to R&D, reducing $\dot{g}_t$. In the second term, $D_t + \alpha(1-\theta)/aL$ represents the sum of government debt and the value of all firms, divided by the potential

[7] A benefit of focusing on $D_t \equiv B_t/Y_t$ rather than B_t/GDP_t is that Y_t depends only on state variable n_t, and therefore D_t is predetermined. It means that we can use D_t as an initial condition for the equilibrium dynamics. In contrast, since GDP_t depends on jump variable L_t^P and L_t^R, it is not possible to use B_t/GDP_t as an initial condition.

[8] The ratio of the government spending on subsidies to the potential GDP is $\theta w_t L_t^R/Y_t$. Note that $w_t = \alpha Y_t$ from (18) and (21), and that $L_t^R = g_t/\overline{g}$ from (22). Using these, $\theta w_t L_t^R/Y_t = \theta\alpha g_t/\overline{g}$.

[9] From (22) and (23), $\dot{g}_t = -\overline{g}(\dot{L}_t^P/L)$. Also, from (12), (21) and $X_t = C_t$, we have $L_t^P = LX_t/Y_t = LC_t/Y_t$. Since L is constant, the rate of change of this equation is $\dot{L}_t^P/L_t^P = \dot{C}_t/C_t - \dot{Y}_t/Y_t$. Therefore, $\dot{g}_t = -\overline{g}(L_t^P/L)(\dot{L}_t^P/L_t^P) = -\overline{g}(L_t^P/L)(\dot{C}_t/C_t - \dot{Y}_t/Y_t)$. In the RHS, $\dot{C}_t/C_t$ is given by the Euler equation (6) and $\dot{Y}_t/Y_t = g_t$. The Euler equation depends on aggregate asset K_t. From (14), (18), and (21), $n_t v_t = \alpha(1-\theta)Y_t/aL$. Substituting this into (16) yields $K_t/Y_t = D_t + \alpha(1-\theta)/aL$. Substituting these results and (24) into $\dot{g}_t = -\overline{g}(L_t^P/L)((\dot{C}_t/C_t - g_t)$ yields (26).

GDP (i.e., K_t/Y_t).[10] A larger amount of the aggregate asset negatively affects $\dot{C}_t$ since those who pass away, on average, have an asset of K_t and they are replaced by newborns who do not have financial assets. Then, $\dot{g}_t$ increases since more labor will be allocated to R&D.

3.3 Dynamics without Government Expenditure

While we are interested in the effect of research subsidies, it is informative to start the analysis of the phase diagram with the case of no research subsidy ($\theta = 0$). We first look at the $\dot{D}_t = 0$ locus. With $\theta = 0$, $\dot{D}_t$ in (25) becomes zero when either $g_t = \widehat{g}(0)$ or $D_t = 0$ holds. On the $g_t = \widehat{g}(0)$ line, $r_t = g_t$ holds, which means that the government debt is growing at the same rate as potential GDP, and therefore the debt-GDP ratio is stationary. D_t is also stationary on the $D_t = 0$ line because there is no income or expenditure by the government. Those lines are drawn in red in Figure 2(a)-(c).

Next, we turn to the $\dot{g}_t = 0$ locus. With $\theta = 0$, $\dot{g}_t$ in (26) becomes zero when

$$D_t = \frac{\overline{g} - g_t}{\overline{g}(\rho + \mu)(\delta + \mu)} \left(\delta - \rho - s(0)\left(g_t - \widehat{g}(0)\right)\right) - \frac{\alpha}{aL}. \tag{27}$$

As depicted by a blue curve, the $\dot{g} = 0$ locus is a parabola that opens toward the right. Recall that g_t can only take the values between $[0, \overline{g}]$, where $\overline{g} \equiv (1-\alpha)aL/\alpha$ is the growth rate of potential GDP when all labor is used for R&D. Therefore, we limit the attention to the area of $g_t \in [0, \overline{g}]$.

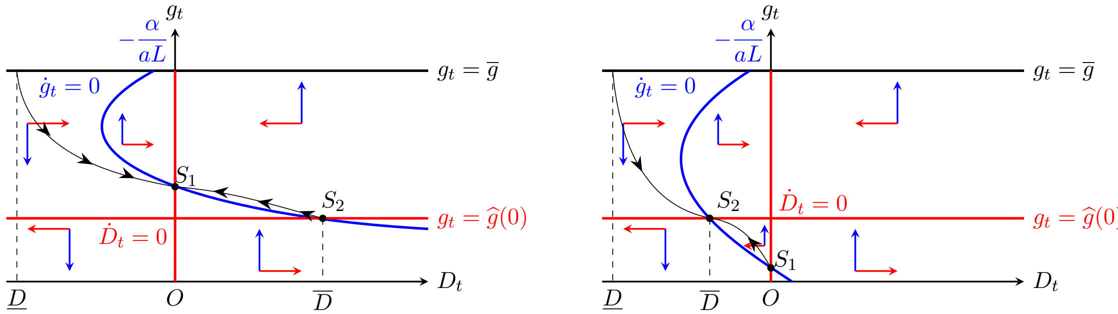
At the upper end of $g_t = \overline{g}$, the $\dot{g} = 0$ parabola starts from $D_t = -\alpha/aL$, as shown in Figure 2(a)-(c). Since this value is negative, the parabola intersects with the $D_t = 0$ line at most once. When the intersection exists (as in Figure 2(a)-(b)), it is a steady state with $\dot{D}_t = \dot{g}_t = 0$, which we label S_1 and denote its coordinate as $D_1^* = 0$ and $g_1^* \in [0, \overline{g})$. Also, the parabola crosses the $g_t = \widehat{g}(0)$ line exactly once before reaching the lower end of 0. We call this crossing point S_2. The coordinate of S_2 is $D_2^* = \overline{D}$ and $g_2^* = \widehat{g}(0)$, where

$$\overline{D} \equiv \frac{\alpha(\delta - \rho)}{(\rho + \mu)(\delta + \mu)} - \frac{\alpha}{aL}. \tag{28}$$

The pattern of the dynamics changes depending on whether $\overline{D}$ is positive or negative and whether S_1 exists or not. We explain three cases in turn.

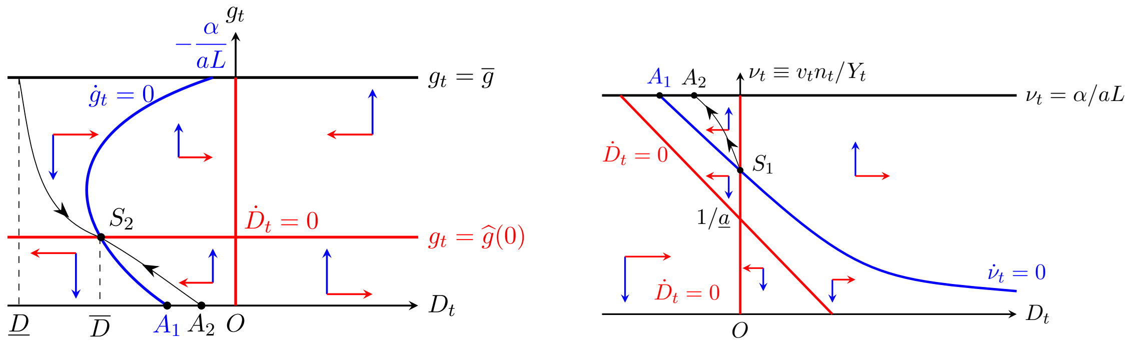
Case 1: Saddle-stable steady state with $g_t > r_t$ ($a > \underline{a}$)

[10] See footnote 9.



(a) When $a > \underline{a}$, S_1 is a saddle with $g_1^* > r_1^*$ and S_2 is a source with $g_2^* = r_2^*$.

(b) When $a \in (\underline{\underline{a}}, \underline{a})$, S_2 is a saddle with $r_2^* = g_2^*$, and S_1 is a source with $g_1^* < r_1^*$.



(c) When $a \in (0, \underline{\underline{a}})$, S_2 is a saddle with $r_2^* = g_2^*$, which is the only steady state in the $g_t > 0$ region.

(d) When $a \in (0, \underline{\underline{a}})$, there is another unstable steady state in the $g_t = 0$ region.

Figure 2: Phase diagram when there is no government expenditure ($\theta = 0$). The $\dot{D}_t = 0$ loci are depicted in red, while $\dot{g}_t = 0$ loci are in blue.

Since we assumed $\delta > \rho$, $D_2^* = \overline{D}$ is positive if and only if the research productivity parameter a is higher than[11]

$$\underline{a} \equiv \frac{(\rho + \mu)(\delta + \mu)}{(\delta - \rho)L} > 0. \tag{29}$$

When $a > \underline{a}$, as shown in panel (a) of Figure 2, Steady state S_1 is above the $g_t = \widehat{g}(0)$ line (i.e., $g_1^* \in (\widehat{g}(0), \overline{g})$). From (24), this implies that the growth rate in the steady state is higher than the interest rate. In Appendix A.2, we show that S_1 is saddle-stable, while S_2 is totally unstable (a source). Therefore, there is a stable arm that originated from S_2 and converges to S_1.[12] Therefore, whenever the initial value of the debt-GDP ratio D_0 is

[11] Note that $\underline{a} > 0$ because $\delta > \rho$.

[12] Strictly speaking, the economy can converge to the steady state only when $D_0 \in [\underline{D}, \overline{D}]$, where $\underline{D} < 0$ is

less than D_2^*, there is an equilibrium path that converges to S_1, where the debt-GDP ratio is zero in the long run.[13] This implies that, even when the government has no revenue, its debt-GDP ratio can be stabilized as long as the initial amount is not too large, given that $a > \underline{a}$.

Case 2: Saddle-stable steady state with $g_t = r_t$ ($a < \underline{a}$)

When $a < \underline{a}$, steady state S_2 is located in the $D_t < 0$ region.[14] Still, if the $\dot{g}_t = 0$ parabola crosses the $D_t = 0$ line, another steady state S_1 exists, with $g_1^* \in (0, \widehat{g}(0))$, as depicted in Figure 2(b). This happens when $\underline{\underline{a}} < a < \underline{a}$, where the threshold is given by

$$\underline{\underline{a}} \equiv \frac{-(\delta - \rho) + \sqrt{(\delta - \rho)^2 + 4(1-\alpha)(\rho + \mu)(\delta + \mu)}}{2((1-\alpha)/\alpha)L} > 0. \tag{30}$$

In this case, as formally discussed in Appendix A.2, S_2 is saddle-stable and S_1 totally unstable (a source). Therefore, the stability property is the opposite of Case 1. There is a stable arm originating from S_1 and converging to S_2. Therefore, given that the economy starts from a positive government net asset ($D_t < 0$), then there is an equilibrium path that stabilizes the net asset-GDP ratio in the long run.[15] In this steady state, the government asset grows at the same rate as the potential GDP ($r_t = g_t$), hence stabilizing the ratio. However, there is no equilibrium path converging to the saddle-stable steady state if the amount of initial net debt is positive. The government will go bankrupt when starting from $D_t > 0$.

When $a < \underline{\underline{a}}$, there is only one steady state in the phase diagram shown in Figure 2(c). Similarly to Case 2, S_2 is saddle-stable, and there is a stable arm converging to it. However, the stable arm starts from point A_2, which is located to the left of the origin.

This means that the stable arm exists only in the region where D_t is significantly negative, at least in this diagram. Then what happens if the economy starts from an

the point where the downward-sloping stable arm crosses the $g_t = \overline{g}$ line. Intuitively, if the initial asset/GDP ratio of the government is too large, and given that it does not use the asset at all, the asset/GDP explodes, and there is no steady state. Numerically, we find that the absolute value of $\underline{D}$ is very large under various parameter values, so it is not realistic to consider the case of $D_t < \underline{D}$. Therefore, in the main text, we disregard this possibility.

[13]Note that $g_t = \overline{g} L_t^R$ is a jump variable from (22).

[14]We ignore the border case of $a = \underline{a}$ because the case has a measure zero possibility.

[15]Strictly speaking, the economy converges to the stable steady state when $D_0 \in [\underline{D}, 0]$. See the discussion in footnote 12.

initial debt that is around zero? Note that the phase diagram in Figure 2 is drawn under the assumption that the free entry condition (14) holds with equality. In fact, this model economy has another phase diagram in the D_t and $\nu_t = n_t v_t / Y_t$ space that applies when the amount of R&D is zero (i.e., $g_t = 0$), where the free entry condition does not need to hold with equality. In Appendix A.3, we show that there is an unstable steady state in this region if $a < \underline{\underline{a}}$. Also, there is a stable arm that originates from this steady state and connects to point A_2, as shown in Figure 2(d). Therefore, if the economy starts from a slightly negative D_t, it will experience a period of zero growth before arriving at point A_2, and then g_t gradually increases until the economy reaches the saddle-stable steady state S_2.[16]

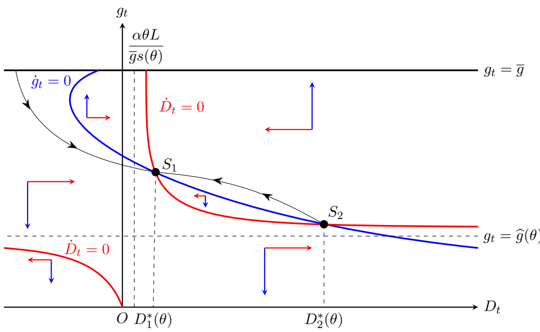
The following proposition summarizes the results from the three cases.

Proposition 1 *Suppose that that $\theta = 0$. The growth rate in the saddle-stable steady state is higher than the interest rate if and only if $a > \underline{a}$. There is an equilibrium path converging to this steady state if the amount of initial debt is less than $\overline{D} > 0$. If $a < \underline{a}$, the growth rate in the saddle-stable steady state is the same as the interest rate. There is an equilibrium path converging to this steady state only when the initial debt is less than zero.*

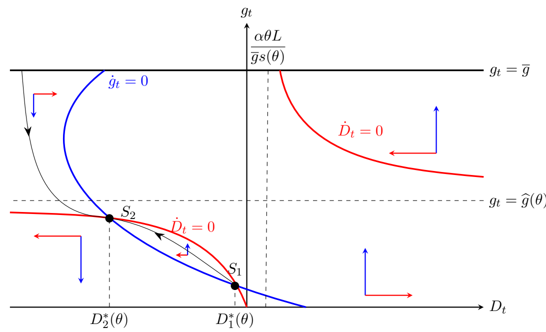
In a simplified setting where there is no revenue or expenditure by the government, the proposition shows that the productivity of R&D, a, is critical for keeping the growth rate higher than the interest rate. If it is below the threshold $\underline{a}$, the economy can reach a steady state only when the government holds a net positive asset, and the growth rate will be equalized to the interest rate.

The proposition implies that the government is able to roll over the debt infinitely if the productivity of R&D is higher than $\underline{a}$ and the initial debt is less than $\overline{D}$. A natural question, then, is whether it can use some government money without collecting taxes. Better yet, the revenue could be used to enhance growth. The following subsection considers the effects of R&D subsidies not backed by any revenue.

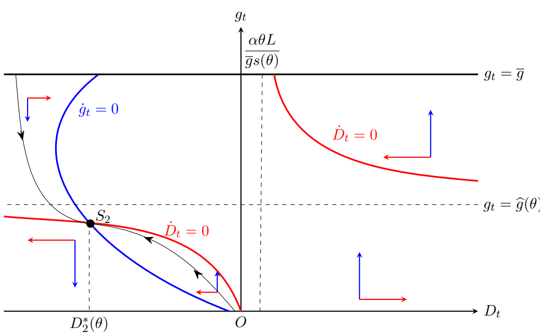
[16]Therefore, the economy converges to the stable steady state if $D_0 \in [\underline{D}, 0]$ similarly to the case of $\underline{\underline{a}} < a < \underline{a}$.



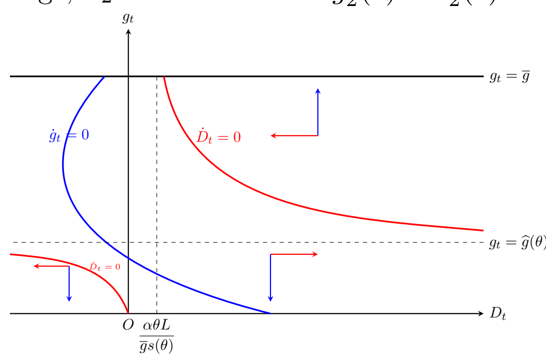
(a) Case 1: When $a > \underline{a}$ and θ is not large, S_1 is a saddle with $g_1^*(\theta) > r_1^*(\theta)$.



(b) Case 2-1: When $a < \underline{a}$ and θ is not large, S_2 is a saddle with $g_2^*(\theta) < r_2^*(\theta)$.



(c) Case 2-2: Similar to (b), but there is no S_1 in the $g_t > 0$ region.



(d) Case 3: No equilibrium

Figure 3: Phase diagram when $\theta \in (0, 1)$

3.4 Research Subsidy Financed by Perpetually Rolled-Over Debt

Now we consider the effect of the research subsidy on the dynamics of the economy while keeping the assumption that the government does not have any revenue. With the rate of research subsidy of $\theta \in [0, 1)$, equation (26) implies the $\dot{g} = 0$ locus is

$$D_t = \frac{g_t - \overline{g}}{\overline{g}(\rho + \mu)(\delta + \mu)}\big(s(\theta)\left(g_t - \widehat{g}(\theta)\right) - (\delta - \rho)\big) - \frac{\alpha(1-\theta)}{aL}. \tag{31}$$

From (25), the $\dot{D} = 0$ locus is

$$D_t = \frac{\alpha\theta L}{\overline{g}s(\theta)}\frac{g_t}{g_t - \widehat{g}(\theta)}. \tag{32}$$

The $\dot{g} = 0$ locus is a parabola, and the $\dot{D} = 0$ is a rectangular hyperbola, with its asymptotes are $D_t = \alpha\theta L/\overline{g}s(\theta)$ and $g_t = \widehat{g}(\theta)$. They may or may not intersect with each other depending on parameter values, in particular on θ. Figure 3 shows three possible cases.

Case 1: Saddle-stable steady state with $g_t > r_t$ ($a > \underline{a}$ and θ is not too large)

Recall that, if $a > \underline{a}$, the economy has a saddle-stable steady state with $g^* > r^*$ when $\theta = 0$. (See Figure 2(a).) Because the phase diagram moves continuously with θ, the economy still has a saddle-stable steady state with $g_t > r_t$ if $a > \underline{a}$ and θ is not too large, as shown by S_1 in Figure 3(a). We denote the coordinate of S_1 in the D_t-g_t space by $(D_1^*(\theta), g_1^*(\theta))$ since it changes with θ. As long as the $\dot{g}_t = 0$ locus intersects with the upper right portion of the $\dot{D}_t = 0$ locus, the value of g_t in the steady state is always higher than the horizontal asymptote $g_t = \widehat{g}(\theta)$. Then, (24) implies g_t is larger than r_t.

Also, note that the long-term level of the debt-GDP ratio, $D_1^*(\theta)$, is positive since the stable steady state S_1 is to the right of the asymptote line at $D_t = \alpha\theta/\overline{g}s(\theta) > 0$. Intuitively, the government issues a new bond each year to finance the research subsidy and pays the interest on it forever. Still, the debt-GDP ratio converges to a positive constant because the economy is growing faster than the interest rate.

Another steady state S_2 at $(D_2^*(\theta), g_2^*(\theta))$ is a source (totally unstable). There is a saddle path originating from S_2 and converging to S_1. Therefore, the debt-GDP ratio can be stabilized in the long run if the initial debt-GDP ratio if less than $D_2^*(\theta)$.

Case 2: Saddle-stable steady state with $g_t < r_t$ ($a < \underline{a}$ and θ is not too large)

Contrary to Case 1, if $a < \underline{a}$ and θ is not too large, the $\dot{g}_t = 0$ locus intersects with the lower left portion of the $\dot{D}_t = 0$ locus, as shown in Figure 2(b)-(c). In this case, S_2 at $(D_2^*(\theta), g_2^*(\theta))$is a saddle-stable steady state, and there may or may not exist an unstable steady state (S_1) in the $g_t > 0$ region depending on parameters, as discussed in Case 2 of Section 3.4.

Observe that the value of $g_2^*(\theta)$ is always lower than the horizontal asymptote at $\widehat{g}(\theta)$. Therefore, from (24), the interest rate in this steady state, denoted by $r_2^*(\theta)$, is higher than the growth rate $g_2^*(\theta)$. Also, $D_2^*(\theta)$ is negative because the hyperbola is downward sloping and goes through the origin. It means that, in steady state S_2, the government holds a positive net asset that grows at the rate of economic growth so that the asset-GDP ratio is constant at $|D_2^*(\theta)|$. Given that $r_2^*(\theta) > g_2^*(\theta)$, the government can use $(r_2^*(\theta) - g_2^*(\theta))\,|D_2^*(\theta)|\,Y_t$ portion of out of the interest revenue $r_2^*(\theta)\,|D_2^*(\theta)|\,Y_t$, while keeping the asset-GDP ratio constant. In the steady state, the amount of this surplus is just enough to finance the expenditure for the research subsidy.

While the situation might seem desirable, it might be difficult to reach this steady state.

Suppose that the unstable steady state S_1 exists in the $g_t > 0$ region, as depicted in Figure 3(b). Then the stable steady state S_2 can be reached only when the initial value of D_t is less than $D_1^*(\theta)$, the debt-GDP ratio of S_1. Since $D_1^*(\theta) < 0$, the government needs to start by holding a net asset, and the asset-GDP ratio must be greater than $|D_1^*(\theta)|$. When $a < \underline{a}$, it is not possible for the government to subsidize R&D while not collecting taxes unless it has enough initial assets.

Case 3: No steady state (θ *is larger than a certain threshold*)

If θ is too large, there is no intersection between the $\dot{g}_t = 0$ locus and the $\dot{D}_t = 0$ locus in the phase diagram, as shown Figure 3(d). Therefore, regardless of the initial value of D_t, the economy cannot reach a steady state. Such a large debt-financed subsidy is not sustainable regardless of the initial GDP-debt or GDP-asset ratio.

3.5 The Growth Effect of Research Subsidy

In Case 1 of the previous subsection, we have shown that the government can provide research subsidies for firms while rolling over its debts if the research productivity a is higher than $\underline{a}$ and the initial GDP-debt ratio is less than $D_2^*(\theta)$. Here, we examine whether such a policy can actually enhance the long-term growth rate.

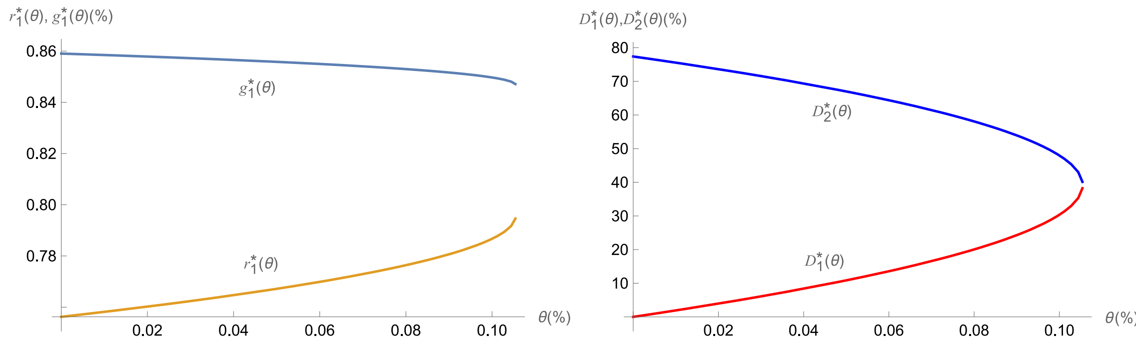
The research subsidy has two opposing effects on the growth rate. First, it promotes the research activity by reducing the R&D cost. Second, the government expense for the subsidy will raise the long-term debt-GDP ratio $D_1^*(\theta)$, which increases the equilibrium interest rate and reduces the value of firms. The following proposition shows that the first effect dominates if a is sufficiently large.

Proposition 2 *A marginal increase in* θ *from* $\theta = 0$ *raises the growth rate in the saddle-stable steady state if and only if*

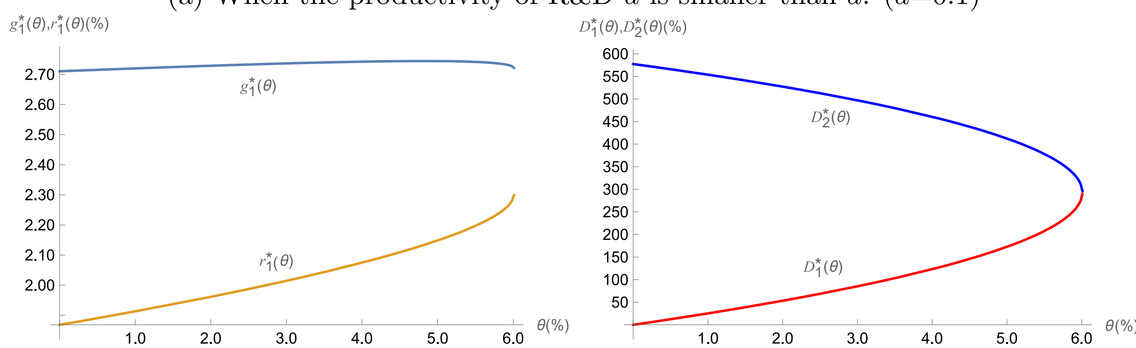
$$a > 2\underline{a} + \frac{\delta - \rho}{2L} \equiv \hat{a}. \tag{33}$$

Proof: in Appendix A.4.

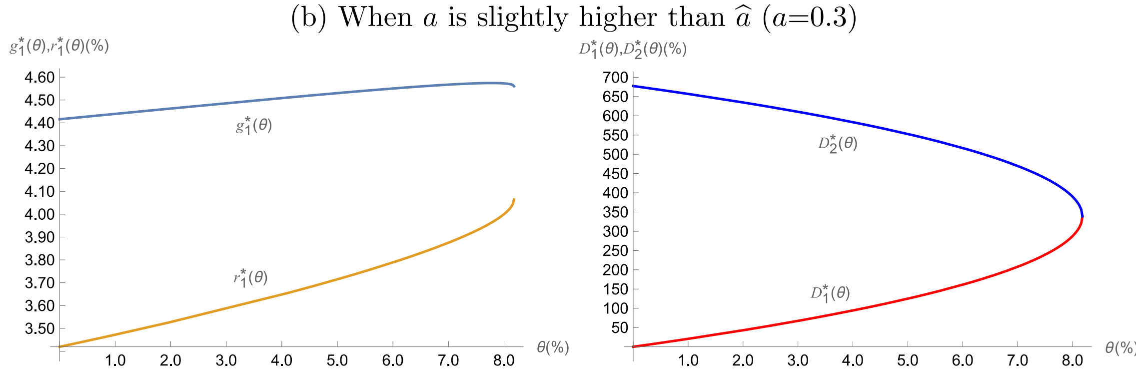
Figure 4 numerically plot how the growth rate, the interest rate and the debt-GDP ratio in the steady state respond to the rate of research subsidy θ when the parameters



(a) When the productivity of R&D a is smaller than $\widehat{a}$. (a=0.1)



(b) When a is slightly higher than $\widehat{a}$ (a=0.3)



(c) When a is significantly higher than $\widehat{a}$ (a=0.5)

Figure 4: The effect of research subsidy rate θ on the growth rate $D_1^*(\theta)$, the interest rate $r_1^*(\theta)$ and the debt-GDP ratio $D_1^*(\theta)$ in the saddle-stable steady state.

$D_2^*(\theta)$ indicates the maximum amount of the debt-GDP ratio from which the economy can converge to the saddle-stable steady state.

satisfy the condition for Proposition 2.[17] The results are shown for three different values of a. We can confirm the result of Proposition 2 in the left panel of Figure 4(a), which shows that $g_1 * (\theta)$ is always decreasing with θ when $a < \widehat{a}$. In contrast, in Figure 4(b)-(c) shows the case of $a > \widehat{a}$. The value of $g_1^*(\theta)$ is increasing in θ when θ is small, but further increases in θ will reduce $g_1^*(\theta)$. This is because, as shown in the right panels, the steady-state debt-GDP ratio, $D_1^*(\theta)$, sharply increases with θ when θ is high, and therefore the interest rate in the steady state, $r_1^*(\theta)$, follows the same pattern.[18] Eventually, when the subsidy rate θ reaches a threshold, which we call $\overline{\theta}$, the phase diagram changes from Figure 3(a) to (c), and the economy does not have a steady state under such a policy. Observe also that, as long as the saddle steady state exists, $g_1^*(\theta)$ is always higher than $\widehat{g}(\theta)$, and therefore, from (24), $g_1^*(\theta) - r_1^*(\theta)$ is always positive. In other words, the government can roll over its debt only when its policy allows the existence of a steady state with $g_t > r_t$.

In the right panel of Figure 4, $D_2^*(\theta)$ gives the maximum value of the debt-GDP ratio from which the economy can converge to the saddle steady state (See the phase diagram in Figure 3). Observe that $D_2^*(\theta)$ is decreasing in θ, and it eventually connects to $D_1^*(\theta)$ just before the steady state disappears. Therefore, even though a higher θ may increase the growth rate, it entails two kinds of costs: it increases the steady state level of the debt GDP ratio, $D_1^*(\theta)$, and reduces the maintainable debt-GDP ratio, $D_2^*(\theta)$. When θ reaches $\overline{\theta}$, $D_1^*(\theta)$ and $D_2^*(\theta)$ coincide, which means that the steady state is not maintainable beyond this point.

Figure 5 contour-plots the long-term growth rate, $g_1^*(\theta)$, against a and θ. The growth rate is higher when the color is lighter. The thick black curve delineates the value of $\overline{\theta}$ as a function of a. Since it is upward-sloping, a higher rate of research subsidy is maintainable when the research productivity in the first place is higher. The area above the $\overline{\theta}$ curve, shown in dark blue, indicates that there is no steady state with $g_t > r_t$ for the given combination of a and θ; i.e., such a policy is not sustainable. Recall that, from Proposition

[17] In the numerical examples of Figures 4 and 5, we set population $L = 1$, discount rate $\rho = 0.01$, mortality rate $\mu = 1/(76.4 - 20)$, where 76.4 is the average life expectancy in the U.S, and 20 is the age from which we assume the agents starts economic activities. The deterioration rate of labor productivity is set to 1/45, and the inverse of markup is 3/4. Given these, we can calculate that the value of $\widehat{a}$ in Proposition 2 is 0.19.

[18] From (24), the interest rate in steady state S_1 is obtained by $r_1(\theta) = s(\theta)\widehat{g}(\theta) - (s(\theta) - 1)g_1^*(\theta)$.

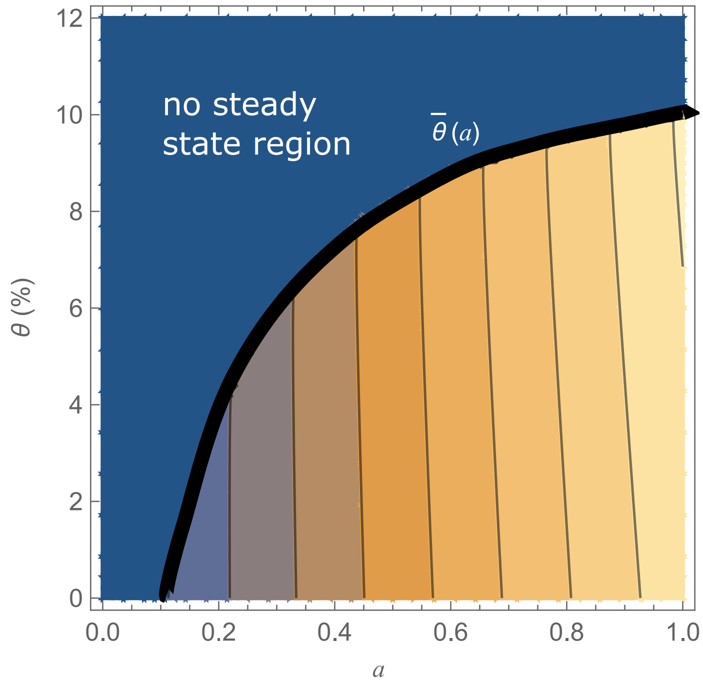


Figure 5: Contour plot of the growth rate in the steady state $g_1^*(\theta)$ against the productivity of R&D a and the R&D subsidy rate θ

1, the government can roll over the debt only when $a > \overline{a}$.[19] Therefore, the $\overline{\theta}(a)$ curve intersects with the horizontal axis to the right of the origin at $a = \overline{a}$.

The results so far demonstrate the importance of research productivity, a. When the productivity of R&D is not sufficiently high, the research subsidy financed by perpetually rolled-over debt is neither effective for growth nor maintainable. While subsidizing R&D is a standard policy for enhancing growth, the results suggest that we may need to focus on other policies that directly improve research productivity in the long run, such as enhancing education and public-funded basic research.

4 Conclusion and Future Research

This paper has examined the effect of R&D subsidies in an R&D-based endogenous growth model, assuming that the government cannot raise taxes. Such a policy is sustainable only when the interest rate is lower than the growth rate. We find that $g > r$ is realized in the steady state if the productivity of R&D is high enough. Given that the findings in

[19]With the parameter values explained in footnote 17, $\overline{a}$ is 0.091.

the recent literature that $g > r$ is more historical the rule rather than the exception, we focused on the case where the above condition is satisfied.

Given this, we find that the government can subsidize R&D by perpetually rolling over the government debt. As long as the subsidy rate and the initial debt-GDP ratio are not too high, the economy converges to a saddle-stable steady state with $g > r$. However, such a policy does not necessarily raise the growth rate. When a is high enough to support $g > r$ but is lower than another higher threshold, the R&D subsidy actually lowers the growth rate because the effect of the increased government debt on the interest rate dominates the positive direct effect of the subsidy. When a is higher than this threshold, the policy can indeed enhance the growth rate. Still, it is to be reminded that such a policy will reduce the upper bound in the debt-GDP ratio (or fiscal space) from which the economy can recover to a steady state. Once the subsidy rate exceeds a certain value, the economy has no equilibrium path. It means that such a policy cannot be implemented in the rational expectation equilibrium; the government will go bankrupt.

This paper has shown the importance of the productivity of private R&D, a, which represents how many innovations can be realized per R&D worker. While we treat this parameter as exogenous, the government may be able to influence this parameter by other policies. For example, when private R&D is reliant on science and basic research, public funding for these will enhance the value of a. Also, education (particularly higher education) will improve the ability of R&D workers, and therefore a. Both of those policies will need time to take effect, but if the initial level of a is low, it will be more effective than subsidizing private R&D directly. Those policies could also be financed by perpetually rolled-over debts. Future work is required to determine the feasibility and the effectiveness of those alternative policies.

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A Appendix

A.1 Derivation of the Dynamics for Aggregate Consumption

In this section, we derive the dynamics of aggregate consumption in (6). As explained in the main text, each individual maximizes the expedited utility (3) subject to the budget constraint (4), initial condition $k_{t,t} = 0$, and the non-Ponzi-game condition,

$$\lim_{T\to\infty} e^{-\int_t^T (r_m+\mu)dm} k_{s,T} \leq 0. \tag{A.1}$$

The Euler equation of individuals is $dc_{s,t}/dt = (r_t - \rho)c_{s,t}$. From (4), (A.1), and this Euler equation, we obtain

$$c_{s,t} = (\rho + \mu)(k_{s,t} + h_{s,t}), \tag{A.2}$$

where $h_{s,t}$ is the present value of future labor income, or "human wealth" of an individual of generation s evaluated at time t. It is defined by

$$h_{s,t} \equiv \int_t^\infty \ell_{s,t'} w_{t'} e^{-\int_t^{t'} (r_m+\mu)dm} dt'. \tag{A.3}$$

Equation (A.2) implies a relationship between aggregate variables

$$C_t = (\rho + \mu)\left(K_t + H_t\right), \tag{A.4}$$

where C_t and K_t are defined by (5), and

$$H_t \equiv \int_{-\infty}^t N_{s,t} h_{s,t} ds \tag{A.5}$$

is the aggregate present value of future labor income of currently living individuals. Equation (A.4) means that the aggregate consumption is a constant fraction of the total wealth of currently living individuals. Therefore, the movement of aggregate consumption is given by

$$\dot{C}_t = (\rho + \mu)\left(\dot{K}_t + \dot{H}_t\right). \tag{A.6}$$

Now, we want to express the RHS of (A.6) in terms of C_t and K_t. Differentiating the definition of K_t in (5) with respect to t and using individual budget constraint (4) and initial condition $k_{t,t} = 0$, we obtain the dynamics for the aggregate capital:

$$\dot{K}_t = r_t K_t + w_t - C_t. \tag{A.7}$$

Next, we consider $\dot{H}$. Note that equation (1) implies that $\dot{N}_{s,t} = -\mu N_{s,t}$. Differentiating the definition of H_t in (A.5) with t and using $\dot{N}_{s,t} = -\mu N_{s,t}$ yields

$$\dot{H}_t = \int_{-\infty}^{t} \dot{h}_{s,t} N_{s,t} ds + \mu(h_{t,t} - H_t). \tag{A.8}$$

In the RHS of (A.8), $h_{t,t}$ is the human wealth of a newborn at time t. Equation (2) implies that the human wealth of an age $t-s$ individual is $e^{-(t-s)\delta} \in (0,1)$ times that of a newborn, $h_{t,t}$. Substituting this relation, $h_{s,t} = e^{-\delta(t-s)} h_{t,t}$, into the definition of H_t in (A.5) gives $H_t = (\mu/(\delta+\mu))h_{t,t}$. Also, by differentiating (A.3) with respect to t, we have $\dot{h}_{s,t} = (r_t + \mu)h_{s,t} - \ell_{s,t} w_t$. Using these, we can eliminate $\dot{h}_{s,t}$ and $h_{t,t}$ from (A.8),

$$\dot{H}_t = (r_t + \mu + \delta)H_t - w_t. \tag{A.9}$$

Note that, using (A.4), H_t in the RHS of (A.9) can be represented in terms of C_t and K_t. Substituting this result, (A.7), and (A.9) into (A.6) yields (6).

A.2 Stability of the Steady State when There is no Subsidy

Substituting $\theta = 0$ into (25) and (26) and linearlizing the system around steady states (S_1 and S_2) yields

$$\begin{bmatrix} \dot{D}_t \\ \dot{g}_t \end{bmatrix} = \boldsymbol{J} \begin{bmatrix} D_t - D^* \\ g_t - g^* \end{bmatrix},$$

where the Jacobian matrix is

$$\boldsymbol{J} \equiv \begin{bmatrix} -s(0)(g^* - \widehat{g}(0)) & -s(0)D^* \\ \overline{g}(\rho+\mu)(\delta+\mu) & s(0)(\widehat{g}(0) + \overline{g} - 2g^*) + \delta - \rho \end{bmatrix}, \tag{A.10}$$

where $s(0) = 1/(1-\alpha)$, $\widehat{g}(0) = (1-\alpha)\overline{g}$, and $\overline{g} = ((1-\alpha)/\alpha)aL$. Since $D_1^* = 0$ in steady state S_1, the determinant and the trace of of $\boldsymbol{J}$ in S_1 are

$$\det \boldsymbol{J}|_{S_1} = -s(0)(g_1^* - \widehat{g}(0)) \left\{ s(0)[(2-\alpha)\overline{g} - 2g_1^*] + \delta - \rho \right\}, \tag{A.11}$$

$$\operatorname{tr} \boldsymbol{J}|_{S_1} = \delta - \rho + s(0)\big(2\widehat{g}(0) + \overline{g} - 3g_1^*\big). \tag{A.12}$$

In steady state S_2, $D_2^* = \overline{D}$, $g_2^* = \widehat{g}(0)$, and therefore

$$\det \boldsymbol{J}|_{S_2} = \overline{g}(\rho+\mu)(\delta+\mu)\overline{D}, \tag{A.13}$$

$$\operatorname{tr} \boldsymbol{J}|_{S_2} = aL + \delta - \rho > 0. \tag{A.14}$$

In the following, we examine the stability of the steady states (S_1 and S_2) in the three cases as discussed in the main text.

Case 1: $\boldsymbol{a > \underline{a}}$. First, we show that S_1 is saddle stable. As explained in the main text, $\dot{g}_t = 0$ parabola cuts the $g_t = \overline{g}$ line at $D_t = -\alpha/aL < 0$. It implies that the vertex of the parabola is in the $D_t < 0$ region, and given that S_1 exists, the g_1^* is necessarily lower than the g-coordinate of the vertex of the parabola, $g_{\text{vertex}} = [(\delta-\rho)(1-\alpha)+(2-\alpha)\overline{g}]/2$ is the g-coordinate of the $\dot{g}_t = 0$ parabola. In the right-hand side (RHS) of (A.11), the term $s(0)[(2-\alpha)\overline{g} - 2g_1^*] + \delta - \rho$ is a decreasing function of g_1^* and becomes 0 when $g_1^* = g_{\text{vertex}}$. Since $g_1^* < g_{\text{vertex}}$ as shown above, this term is positive. Also the term $g_1^* - \widehat{g}(0)$ is positive since since $g_1^* > \widehat{g}(0)$ in Case 1. From these, $\det \boldsymbol{J}|_{S_1}$ in (A.11) is negative. It means that only one eigenvalue is negative, and hence S_1 is saddle stable.

Since $\overline{D} > 0$ in Case 1, (A.13) implies $\det \boldsymbol{J}|_{S_2} > 0$ in S_2. Together with $\operatorname{tr} \boldsymbol{J}|_{S_2} > 0$ in (A.14), S_2 has two positive eigenvalues and is hence totally unstable.

Case 2: $\boldsymbol{\underline{\underline{a}} < a < \underline{a}}$. Similarly to Case 1, $s(0)[(2-\alpha)\overline{g} - 2g_1^*] + \delta - \rho > 0$ holds from $g_1^* < g_{\text{vertex}}$. However, since $g_1^* < \widehat{g}(0)$, $\det|_{S_1} > 0$ from (A.11). Also, substituting $g_1^* < \widehat{g}(0)$ into (A.12), which is an decreasing function of g_1^*, we have $\operatorname{tr} \boldsymbol{J}|_{S_1} \geq \delta-\rho+\alpha s(0)\overline{g} > 0$. From $\det \boldsymbol{J}|_{S_1} > 0$ and $\operatorname{tr} \boldsymbol{J}|_{S_1} > 0$, S_1 has two positive eigenvalues and is therefore totally unstable.

Since $\overline{D} < 0$ in Case 2, (A.13) implies $\det \boldsymbol{J}|_{S_2} < 0$ in S_2. Therefore, S_2 has only one negative eigenvalue and is hence saddle stable.

Case 3: $\boldsymbol{a < \underline{\underline{a}}}$. Similarly to Case 2, S_2 is saddle stable. There is no S_1 steady state in the $g_t > 0$ region. However, we show in Appendix A.3 that there is an unstable steady state (S_1) in the dynamics where $g_t = 0$.

A.3 Dynamics in the $g_t = 0$ region

This section explains the dynamics of the economy when g_t becomes 0, which means $\dot{n}_t = 0$, $L^P = L$, and $L^R = 0$. For simplicity, we focus on the case of $\theta = 0$ in the model, but the analysis can be extended to the case of $\theta > 0$.

The aggregate consumption C_t is also constant from the equilibrium of the goods market $C_t = X_t (= Y_t)$. Let us define $\nu_t \equiv v_t n_t / Y_t$, which represents the ratio of the total market value of firms to the potential GDP. Then, when $\dot{n}_t = 0$, the free entry condition (14) holds whenever $\nu_t \leq \alpha/aL$. Substituting $\dot{C}_t = 0$ and $K_t = B_t + v_t n_t$ from the equilibrium of the asset market yields into the Euler equation (6) the interest rate.

$$r_t = (\delta - \rho)\big(\underline{a}L(D_t + \nu_t) - 1\big) \tag{A.15}$$

The dynamics of the economy when $g_t = 0$ can be examined by ν_t and D_t.[20] Since the government has no revenue or expense in the $g_t = 0$ region, its debt B_t grows at the rate of r_t. Also, since Y_t is constant, $\dot{D}_t/D_t = \dot{B}_t/B_t = r_t$. Therefore,

$$\dot{D}_t = (\delta - \rho)\big(\underline{a}L(D_t + \nu_t) - 1\big)D_t. \tag{A.16}$$

Again, since Y_t and n_t are constant in $\nu_t \equiv n_t v_t / Y_t$, we get $\dot{\nu}_t/\nu_t = \dot{v}_t/v_t$. Also from $L^P = L$, $\pi_t = (1-\alpha)Y_t/n_t$.[21] Then the no-arbitrage condition (17) implies

$$\dot{\nu}_t = (\delta - \rho)\big(\underline{a}L(D_t + \nu_t) - 1\big)\nu_t - (1 - \alpha). \tag{A.17}$$

From (A.16) and (A.17), the phase diagram of the economy when $g_t = 0$ can be represented as follows:

$$\text{The } \dot{D}_t = 0 \text{ locus:} \quad D_t + \nu_t = \frac{1}{\underline{a}L}, \text{ and} \tag{A.18}$$

$$D_t = 0. \tag{A.19}$$

$$\text{The } \dot{\nu}_t = 0\text{: locus} \quad D_t + \nu_t = \frac{1}{\underline{a}L} + \frac{1-\alpha}{(\rho+\mu)(\delta+\mu)}\frac{L}{\nu_t}. \tag{A.20}$$

Recall from the free entry condition that the diagram for the case of $g_t = 0$ is defined only for $\nu \in [0, \alpha/aL]$. The phase diagram is shown in Figure (A.1) for the case when the $\dot{D}_t = 0$ locus intersects with the ν_t axis ($a < \underline{\underline{a}}$) and when it does not ($a > \underline{\underline{a}}$). From (A.20), the D_t coordinate of the $\dot{\nu}_t = 0$ locus at $\nu_t = \alpha/aL$ (denoted as point A_1 in Figure A.1) is

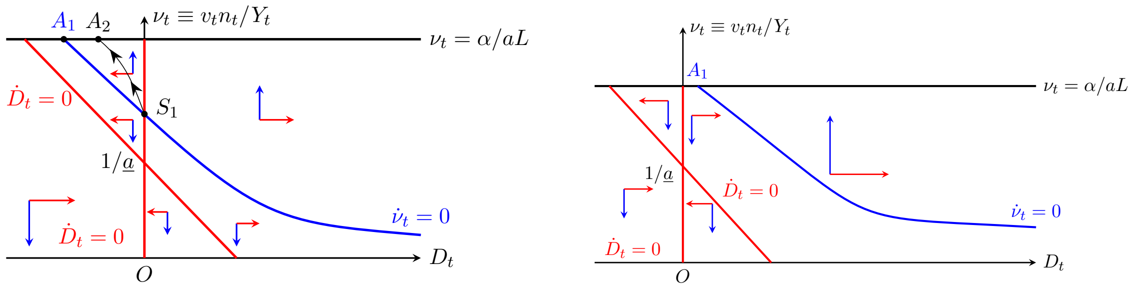
$$\frac{1-\alpha}{(\rho+\mu)(\delta+\mu)}\frac{aL}{\alpha} - \frac{\alpha}{aL} + \frac{1}{\underline{a}L}, \tag{A.21}$$

which is positive if and only if $a > \underline{\underline{a}}$. Moreover, (A.21) coincides with the D_t-intercept of the $\dot{g}_t = 0$ locus in Figure 2(b) and 2(c).

We now consider the steady state. From (A.20), we see that the $\dot{\nu}_t = 0$ locus is downward sloping, is always above the sloping part of $\dot{D}_t = 0$ locus (A.18), and D_t tends

[20] Depending on whether the free entry condition (14) holds with equality or not, either g_t or n_t can move, while the other is fixed.

[21] See footnote 5.



(a) When $a < \underline{\underline{a}}$, S_1 exists in the $D_t - \nu_t$ plane.

(b) When $a > \underline{\underline{a}}$, no S_1 in the $D_t - \nu_t$ plane.

Figure A.1: Phase diagram when $g_t = 0$

to infinity as $\nu_t \to 0$. Therefore, the $\dot{\nu}_t$ locus intersects with the vertical portion of the $\dot{D}_t = 0$ locus given that $a < \underline{\underline{a}}$; i.e., when point A_1 is to the left of the vertical axis, as shown in Figure A.1(a). We call the steady state S_1 in this case and let $\nu^* > 1/\underline{a}L$ denote its ν_t value.[22] If $a > \underline{\underline{a}}$, there is no steady state as shown in Figure A.1(b).

Next, we show that S_1 is unstable. Linearlizing (A.16) and (A.17) around $(0, \nu^*)$ yields

$$\begin{bmatrix} \dot{D}_t \\ \dot{\nu}_t \end{bmatrix} = \mathbf{J}_1 \begin{bmatrix} D_t \\ \nu_t - \nu^* \end{bmatrix}, \text{ where,}$$

$$\mathbf{J}_1 \equiv \begin{bmatrix} L(\rho+\mu)(\delta+\mu)\nu^* - (\delta-\rho) & 0 \\ L(\rho+\mu)(\delta+\mu)\nu^* & 2L(\rho+\mu)(\delta+\mu)\nu^* - (\delta-\rho) \end{bmatrix}. \tag{A.22}$$

Its determinant is

$$\det \mathbf{J}_1 = 2L^2(\delta-\rho)^2\underline{a}^2 \left(\nu^* - \frac{1}{\underline{a}L}\right)\left(\nu^* - \frac{1}{2\underline{a}L}\right), \tag{A.23}$$

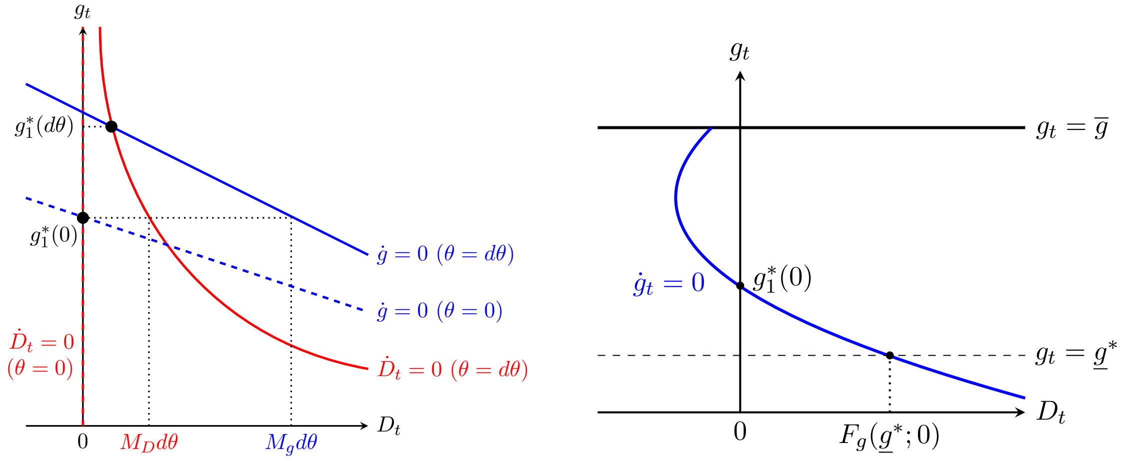
which is positive since $\nu^* > 1/\underline{a}L$. The trace of J_1 is positive:

$$\operatorname{tr} \mathbf{J}_1 = 3L(\rho+\mu)(\delta+\mu)\nu^* - 2(\delta-\rho) > \delta - \rho > 0, \tag{A.24}$$

where the first inequality comes from $\nu^* > 1/\underline{a}L$. From (A.23) and (A.24), we can conclude that both eigenvalues are positive.

Finally, we explain there is a saddle path that originates from S_1 and connects to the saddle-stable steady state S_2 in Figure 2(c). As shown in 2(c), there is a saddle path that originates from a point on the horizontal axis (i.e., $g_t = 0$). This point is located

[22] We can confirm $\nu^* > 1/\underline{a}$ by substituting $D_t = 0$ in (A.20).



(a) The shifts of the $\dot{g}_t = 0$ and $\dot{D}_t = 0$ loci and the movement of the steady state.

(b) Condition $g_1^* > \underline{g}^*$ holds if and only if $F_g(\underline{g}^*; 0) > 0$.

Figure A.2: Proof of Proposition 2

between the origin and the intercept of the $\dot{g}_t = 0$ locus (point A_1), and we call it point A_2. Note that point A_2 also belongs to Figure A.1(a) because A_2 satisfies both $g_t = 0$ and $\nu_t = v_t n_t / Y_t$. Since steady state S_1 is completely unstable, there exists a point in the neighborhood from which the path leads to A_2. This is the saddle path of this economy. Once the economy reaches A_2 in Figure A.1(a), it experiences a phase transition to Figure 2(c).

The case of $\boldsymbol{\theta > 0}$: So far, we focused on the case of $\theta = 0$. Even when θ is positive, the phase diagram within the $g_t = 0$ region is almost the same as the case of $\theta = 0$ because $g_t = 0$ means no firms do R&D, and therefore the government expenditure is zero. The only difference is that the border between the phase diagrams. With $\theta > 0$, the $\nu_t = (1-\theta)\alpha/aL$ line in the diagram of the $g_t = 0$ region will be connected to the $g_t = 0$ line in the phase diagram of the $g_t > 0$ region. Intuitively, with $\theta > 0$, firms are more eager to start R&D even when the index of firm values, ν_t, is not as high as in the case of $\theta = 0$. Therefore, the economy transitions from the $g_t = 0$ region to the $g_t > 0$ region with a smaller ν_t.

A.4 Proof of Proposition 2

Before the increase in the subsidy, the economy is in a saddle-stable steady state. As illustrated as S_1 in Figure 2(a), this steady state is given by an intersection between the $\dot{g}_t = 0$ locus and the $\dot{D}_t = 0$ locus.

We consider the situation where the rate of the R&D subsidy, θ, is increased from 0 marginally by $d\theta$. Then, both the $\dot{g}_t = 0$ locus and the $\dot{D}_t = 0$ locus shift to the right. The location of the steady state always moves to the right, which means that D_t in the new steady state is higher than before. Whether g_t in the SS rises or falls depends on the relative magnitude of the size of the shifts of the two loci. As illustrated in panel (a) of A.2, g_t in the new steady state should be higher than before if and only if the size of the shift of the $\dot{g} = 0$ locus ($M_g d\theta$), measured at the point of the saddle-stable steady state ($D_t = 0,\ g_t = g_1^*$), is larger than that of the $\dot{D} = 0$ locus ($M_D d\theta$).

Let $F_g(g_t;\theta)$ be the RHS of (31), and $F_D(g_t;\theta)$ the RHS of (32). Then, $D_t = F_g(g_t;\theta)$ and $D_t = F_D(g_t;\theta)$ respectively represent the $\dot{g} = 0$ locus and the $\dot{D} = 0$ locus. Then, M_g and M_D are calculated as

$$M_g \equiv \frac{\partial F_g(g_1^*;0)}{\partial \theta} = \frac{\alpha}{aL} + \frac{(g_1^* - \overline{g})^2}{\overline{g}(\rho+\mu)(\delta+\mu)}, \tag{A.25}$$

$$M_D \equiv \frac{\partial F_D(g_1^*;0)}{\partial \theta} = \frac{\alpha^2 g_1^*}{aL}\frac{1}{g_1^* - \widehat{g}(0)}. \tag{A.26}$$

From (A.25), (A.26) and $\widehat{g}(0) < g_1^* < \overline{g}$, we find $M_g - M_D > 0$ holds if and only if

$$(\overline{g} - g_1^*)(g_1^* - \widehat{g}(0)) > (1-\alpha)^2(\rho+\mu)(\delta+\mu). \tag{A.27}$$

The fact that the steady state before the shift ($D_t = 0,\ g_t = g_1^*$) is on the $\dot{g}_t = 0$ locus means $F_g(g_1^*;0) = 0$. This equation can be rearranged to

$$(1-\alpha)^2(\rho+\mu)(\delta+\mu) = (\overline{g} - g_1^*)\left((1-\alpha)(\delta-\rho) + \widehat{g}(0) - g_1^*\right). \tag{A.28}$$

Substituting (A.28) to the RHS of (A.27) and rearranging yields

$$g_1^* > \frac{(1-\alpha)(\delta-\rho)}{2} + \widehat{g}(0) \equiv \underline{g}^*. \tag{A.29}$$

This inequality shows that the growth rate on the saddle-stable steady state will increase when θ is slightly increased from 0 if and only if $g_1^* > \underline{g}^*$.

As illustrated in panel (b) of Figure A.2, g_1^* is higher than $\underline{g}^*$ if and only if the intersection of the $\dot{g}_t = 0$ locus and the $g_t = \underline{g}^*$ line. Because the $\dot{g}_t = 0$ locus is downward sloping, this condition holds if and only if $F_g(\underline{g}^*;0) > 0$. When we solve this condition for a, we can confirm it is equivalent to (30).