General Equilibrium with Multiple Liquid Assets[*]

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Abstract

This paper studies, analytically and numerically, economies with multiple liquid assets: fiat currency; fixed-supply real assets; and reproducible capital. Cases are considered where assets provide direct liquidity, and indirect liquidity via over-the-counter trade. The results shed new light on how monetary policy affects asset markets and investment. We also provide novel results on endogenous fluctuations (self-fulfilling prophecies), including coexistence of multiple equilibria with very different correlation and volatility patterns. Then we investigate if monetary policy can eliminate multiplicity. A calibration exercise assesses the impact on asset markets and on welfare of different policies, including changing inflation, and eliminating currency altogether.

Keywords: Liquidity, dynamics, asset returns, monetary policy

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Much of the empirical evidence about the impact of liquidity on asset prices is cross-sectional... Comparative statics of single-asset models are not appropriate to interpret such evidence: instead, one needs to formulate models in which multiple assets are traded. Weill (2020).

1 Introduction

This project explores dynamic general equilibrium models where multiple assets convey liquidity by facilitating transactions in decentralized exchange: fiat money M; a real fixed-supply asset A; and reproducible capital K. We analyze cases where assets provide direct liquidity – they can be used to acquire something, say q – and where they provide indirect liquidity – only M can be used to acquire q, but A or K can be traded for M in OTC (over-the-counter) markets. Also, the theory applies whether assets serve as media of exchange or as collateral. Together these ingredients allow us to extend many previous results in monetary economics, and to apply the findings, as well as the general methods, to issues in financial economics.

It is not only for realism or generality that it is useful to have M, A and K all in one model; it also affects substantive results. Consider, e.g., the classic Mundell-Tobin effect (the literature on this and other substantive issues mentioned in these introductory remarks is discussed in Section 2). That effect says higher inflation increases investment in K. We prove this happens here if M and K are the only liquid assets, but with all three higher inflation can decrease investment in K, depending on conditions relating to how assets interact in the payment process.

The conditions are made precise below, but here is the intuition: Suppose M and K but not A are used to make payments. Then higher inflation decreases demand for M and increases demand for its substitute K – our version of the Mundell-Tobin effect. Now let A also be used for these payments. Then inflation decreases demand for M and increases it for A or K, but may increase it so much for A that it decreases demand for K if A and K are substitutes in some transactions. Symmetric results are provided for the stock market: the price of A goes up with inflation if A and M are the only liquid assets, but can fall if K is also liquid.

More generally, we describe how different types of monetary policy affect the returns to all assets. First, in steady state, it is equivalent to peg the growth rate of M, the inflation rate, some (but not just any) interest rate, or some measure of liquidity. In this context we provide quantitative results, showing how changes in policy can move real returns, revisiting venerable questions about the cost of inflation, and considering more recent questions about eliminating cash altogether. Quantitative answers to these questions depend on having multiple assets.

We then go beyond steady state to study equilibria where endogenous variables fluctuate, deterministically or stochastically, as a self-fulfilling prophecy.[1] When M is the only liquid asset we prove that a policy of pegging its growth rate allows many such equilibria, while pegging a certain nominal interest rate ι implies uniqueness. Then we show this does not generalize: with multiple liquid assets, pegging ι does not eliminate multiple dynamic equilibria. Intuitively, ι captures the opportunity cost of holding cash, so pegging it pins that cost down, and if M is the only liquid asset all other endogenous variables follow immediately. But suppose, e.g., that A is also liquid. Its value depends on its price path, which is not pinned down by ι. Hence A's value can fluctuate based on beliefs, and then M's value will too, because although the cost of holding M is still pinned down by ι the benefit varies when M and A are substitutes in some transactions.

Other results on dynamics show how over the cycle asset returns move with each other, with trading volume, with investment and with output. We also demonstrate how cyclic correlations can provide misleading predictions for the effects of policy.[2] Then we demonstrate there can coexist equilibria with very different dynamic patterns: for any two assets, either may be volatile while the other is relatively stable,

[1] To be clear from the start, we are interested in these endogenous dynamics not because we think fluctuations in the actual data are driven exclusively by self-fulfilling prophecies; rather, they might be a piece of the puzzle. Moreover, when one sees how simple models generate outcomes that vary over time in complicated ways based on beliefs, one may be more inclined to think the same can happen in the real world.

[2] To expand on this, by way of example, we construct cyclic equilibria where the real values of M and K move together. Seeing this, one might recommend lowering inflation to raise the real value of M, thinking that will stimulate investment in K; but in the example, while lowering inflation does increase the value of M it actually reduces investment. While one might say this is "only" another example of the Lucas critique, we think it is a nice one.

or both may be volatile, and they may be positively or negatively correlated. Yet another result on dynamics shows how there can be belief-based cycles in K if it interacts with M or A in payments, but not otherwise, which is interesting since it is not easy to get endogenous cycles in K in models without liquidity considerations.

Further in terms of quantitative results, first we find the welfare cost of inflation is *higher* with multiple assets. This may be surprising based on partial equilibrium reasoning, because any individual is better off with access to alternative liquid assets, since when the inflation tax is high one can substitute from M to A or K; but in equilibrium, if all agents do so, it leads to bigger distortions, consistent with the principle of public finance that it is more inefficient to tax things that are more elastic. We then consider a policy of eliminating currency altogether, as championed by some people recently. The finding is that the welfare cost of this can be very large, although to be fair we do not incorporate any of the potential benefits, like reducing criminal activity. Another result shows how welfare results can change when the acceptability of different assets is made endogenous.

Finally, consider an economy where assets provide indirect liquidity: only M buys q, while agents trade A or K for M in an OTC market. At least under some natural conditions it is shown that many of the results go through, basically as stated, from models of direct liquidity, where A and K as well as M can be used to acquire q. This seems important in the sense that, in reality, households mostly use M, or claims on M deposited in their banks, to buy goods, not A or K (although one can argue that firms and financial institutions regularly use such assets to facilitate trade). In this version of the framework, agents wanting to buy q first need to sell other assets to get M, and it is good to know that at least in some cases that makes them *as if* they can use other assets to get q directly.

The rest of the paper is organized as follows, Section 2 reviews the literature. Sections 3 and 4 present the theory and study steady state. Sections 5, 6 and 7 analyze dynamics in various settings. Sections 8 and 9 discuss policy and OTC markets. Section 10 concludes. A few technical results and variations on the model are contained in a series of Appendices.

2 Literature

We now discuss how the project relates to past work.[3] In terms of the big picture, studying inflation's effects on investment and growth goes back to Mundell (1963), Tobin (1965) and Sidrauski (1967). Studying its effects on equity markets goes back to Fama (1981) and Geske and Roll (1983), while the way we model these markets is based on Lucas (1978), with the metaphor that the simplest firm is like a tree that bears fruit (dividends) with no additional inputs. Our thinking, if not our model, about the impact of monetary policy on the equilibrium asset-return distribution comes from Wallace (1980). As the literature on the welfare cost of inflation is huge, we simply direct readers to Bethune et al. (2020) for references. The idea of eliminating currency comes from Rogoff (2017). Our approach to credit frictions is related to work based on Kiyotaki and Moore (1997), although most of those papers do not have money. The point about the equivalence between using assets as collateral or as media of exchange is discussed in various places, e.g., Lagos (2010).

In terms of endogenous dynamics in monetary economics, Azariadis (1993) surveys OLG (overlapping-generations), CIA (cash-in-advance) and MUF (money-in-utility-function) models of fiat currency, while Rocheteau and Wright (2013) treat models more like the one here, which is based on Lagos and Wright (2005). That framework, which has been used extensively in studies of liquidity (see Lagos et al. 2017 and Rocheteau and Nosal 2017 for surveys), combines some centralized and some decentralized trade, a formulation that is ideal for our purposes. Somewhat relatedly, Gu et al. (2013) get similar dynamics in markets without money, but with endogenous debt limits, as in Kehoe and Levine (1993). While these papers are just examples, they provide references to many others, but it is fair to say that typically they all study models with a single payment instrument.

There are some papers based on Lagos and Wright (2005) with multiple payment

[3] A referee asked for an extensive literature review since "After all, if one of the goals of the paper is to exhaustively catalogue... results that follow from the various assumptions... it is inevitable that many of them have been separately studied, and the references should be similarly exhaustive (moreso than would be asked of a paper that confines itself to just a single big result)." However, readers that prefer to see the theory first can skip ahead without loss of continuity.

instruments: Geromichalos et al. (2007) have money and fixed-supply real assets; Ferraris and Watanabe (2008, 2011) and Lagos and Rocheteau (2008) have money and capital; He et al. (2008) have money and bank liabilities; Lagos (2010) has equity and bonds while Lagos (2011) has these plus money; Williamson (2012) and Rocheteau et al. (2018) have money and bonds; Zhang (2014) and Gomis-Porqueras et al. (2017) have home and foreign currencies; Keister and Sanches (2020) and Chiu et al. (2022) have currency plus e money; and Venkateswaran and Wright (2014) consider various payment instruments. These papers all focus on steady state, or transitions to steady state, not the limit-cycle and sunspot equilibria we analyze.

In particular, Herrenbrueck (2019) has money and bonds, but analyzes only transitions after one-time policy changes, not endogenous fluctuations. Geromichalos and Herrenbrueck (2021) have money, bonds and capital, but different from the way it is modeled here, in that setup K is an input into decentralized market production, as in Aruoba et al. (2011), and again they study transitions after parameter changes, not endogenous fluctuations. Like us they get an ambiguous effect of inflation on investment, but the channel is different: they have a standard Mundell-Tobin effect; plus, since K is an input to decentralized trade, higher inflation makes sellers want less of it because it makes buyers bring lower real balances to the market, which is not what is going on here.[4]

Other related work includes Berentsen and Waller (2011), who study price-level targeting by a central bank, and show that it can control inflation expectations and improve welfare by stabilizing short-run shocks. Their model is similar to ours in some ways, although all markets are centralized, and they do not consider endogenous dynamics. Andolfatto and Williamson (2015) is also similar, although again all markets are centralized, and they only consider equilibria that converge to steady states, not persistent endogenous fluctuations. Andolfatto and Martin (2018) is also similar, and they discuss the multiplicity of stationary equilibrium under an interest rate peg, but again not endogenous dynamics. Domínguez and

[4]See Altermatt and Wipf (2023), Cui et al. (2022) and references therein for other papers with ambiguous effects of inflation on investment, but for different reasons than this paper, where there is only a liquidity channel but multiple assets compete in terms of providing liquidity services.

Gomis-Porqueras (2019) also discuss the multiplicity of steady states but not limit cycles or sunspots.

In Ferraris and Watanabe (2008) capital is used as collateral to borrow cash. Heuristically, that makes M and K complements, while in our baseline model they are substitutes. They also get an ambiguous effect of inflation on capital accumulation depending on factors that are not driving our results. Then Ferraris and Watanabe (2011) use the model to explore endogenous cycles in K, and seem to be the first to do so in this framework. Relatedly, Lagos and Zhang (2015) explore sunspot equilibria in a model with M and A. Again, having A, M and K allows us to generalize earlier results and provide new ones. Further, even with only two assets, in Section 6.1 we provide a novel method for constructing cyclic equilibria. Our particular interest in cycles where K fluctuates is motivated by Boldrin and Rustichini (1994), who show this is hard to get without liquidity considerations.

A model with government fiat money plus asset-backed private money is studied by van Buggenum (2022). He gets multiple equilibria due to endogenous search intensity: if buyers' search effort is high, firms make high profits, which means private money has a high value, and that rationalizes high search effort. This feedback is not operational here. Of course, there are many search-based models with endogenous dynamics based on increasing returns in matching (e.g., Diamond and Fudenberg 1989) or production (e.g., Mortensen 1999). Although we use methods from search theory, we do not need endogenous search intensity or increasing returns to generate interesting dynamics, or multiplicity, when liquidity is endogenous, as is by now well known (an early paper making this point is Johri 1999).

Motivated by Weill (2020), we consider cases where M is needed to acquire q while A and K trade for M in OTC exchange.[5] For some parameters we prove this is equivalent to having A and K trade directly for q, so the results on comparative

[5] The original idea of modeling OTC asset markets using search theory is in Duffie et al. (2005) (see also, e.g., Weill 2007 and Lagos et al. 2011). However, they model "liquidity shocks" in a reduced-form way, and restrict asset holdings to the set $\{0, 1\}$, as in early search-based monetary models – e.g., Kiyotaki and Wright (1989, 1993), Aiyagari and Wallace (1991), Shi (1995) or Trejos and Wright (1995). That restriction on asset holdings is relaxed by Lagos and Rocheteau (2009) and Lester et al. (2015), but the approach here is different, and, we think, more tractable.

statics, dynamics and policy with direct liquidity apply without modification to (arguably more realistic) models with indirect liquidity. Similar studies include Berentsen et al. (2014), Mattesini and Nosal (2016), Geromichalos and Herrenbrueck (2016, 2017), Lagos and Zhang (2019) and Geromichalos et al. (2021). These models are all related to the banking model in Berentsen et al. (2007), since they all let agents reallocate liquidity prior to decentralized trade, but again the papers do not have M, A and K, nor do they analyze dynamics, the way we do.

3 The Baseline Model

3.1 Environment

At each date t in discrete time, a continuum of infinitely-lived agents interact in two distinct ways. In the first subperiod there convenes a decentralized market, or DM, with frictions detailed momentarily; then in the second subperiod there convenes a frictionless centralized market, or CM. This induces an asynchronicity of expenditures and receipts crucial to any analysis of money or credit – agents sometimes want to make purchases in the DM while their incomes accrue in the CM, so they must pay for those purchases either using assets acquired in the past or credit settled in the future. As is well understood, when frictions hinder credit, fiat money can be valued for liquidity (Kocherlakota 1998), and so can other assets, as discussed in many of the papers summarized in Section 2.

There is a measure 1 of agents called *buyers*, and a measure n called *sellers*, as in many applications of this framework that amend the original version to have a two-sided DM, rather than letting all agents act as buyers or sellers depending on who they meet. Buyers or sellers meet bilaterally and at random in the DM, where the former want something the latter can provide, denoted q, where $\alpha \leq \min\{1, n\}$ is the probability a buyer meets a seller while α/n is the probability a seller meets a buyer. In the CM, firms produce output $y_t = F(h_t, k_t)$ using labor and capital, where F is monotone and concave. As is standard, y_t can be used for consumption by households, denoted x_t, or government, denoted G_t, or it can be invested in new capital that becomes productive at $t+1$ and depreciates at rate δ. Factor payments to h_t and k_t are ω_t and κ_t. Usually we assume CRS (constant returns to scale), in which case firm optimization implies[6]

$$\omega_t = F_h(h_t, k_t) \text{ and } \kappa_t = F_k(h_t, k_t). \tag{1}$$

In addition to k_t, there are two other storable objects. One is a real asset A_t with dividend $\rho > 0$ in units of y_t in each CM, where often in the literature A is called a tree and ρ fruit, as mention in Section 2. The other is fiat money M_t. In the baseline model a_t lives forever in fixed supply normalized to 1, but Appendix A1 also considers N-period trees and bonds. The supply of M_t is given by $M_{t+1} = (1+\mu)M_t$, which increases at rate μ via lump-sum transfers, or taxes if $\mu < 0$, in the CM. For now, think of policy as a money supply rule fixing μ, but alternatives are discussed in detail later. CM asset prices are ϕ^m, ϕ^a and, since k_t is the same physical object as numeraire, $\phi^k = 1$. A portfolio is denoted $\zeta_t = (m_t, a_t, k_t)$.

As a special case we sometimes use $F(h_t, k_t) = Ch_t + f(k_t)$, and sometimes $f(k_t) = \varepsilon k_t$, which can be interpreted as a pure storage technology. This may not be the best specification for studying growth, but it suits our needs for several reasons: (i) It facilitates comparison to related papers with storage (Wallace 1980; Andolfatto 2015). (ii) It pins down wages by $\omega_t = C$, neutralizing general equilibrium effects on asset markets coming from labor markets. (iii) It is a simple way to add a third asset K that, like M and A, is liquid, but unlike M is real, not nominal, and unlike A is reproducible, not in fixed supply. (iv) It implies there are no cycles in K if it is not used in payments, or is used but independently of M and A, and there are cycles in K if it interacts with M or A in payments, making clear how investment dynamics emerge from liquidity considerations. Still, F is kept general for other purposes, including the analyses in Sections 8 and 9.

To hinder credit, assume no commitment, so buyers can renege on promised payments, and anonymity, so it is impossible to punish renegers as in Kehoe and Levine (1993). Then assets have a role in facilitating trade. To make this more

[6]Some examples below use DRS (decreasing returns to scale) to illustrate a few points; in principle, the general theory can handle CRS or DRS.

interesting, let sellers differ in what they accept: a buyer meets a DM seller that only accepts M with probability $\alpha_m \geq 0$; meets one that only accepts A with probability $\alpha_a \geq 0$; and so on for α_k, α_{ma}, α_{mk}, α_{ak}, α_e where e stands for everything. Also, sellers who accept something may not accept arbitrarily large amounts: χ_m, χ_a and χ_k are the fractions of buyers' M, A and K they accept, if they accept any.

Thus the α's capture liquidity on the extensive margin – how many sellers accept an asset – while the χ's capture it on the intensive margin – how much they accept. One can endogenize these as in Lester et al. (2012) or Li et al. (2012) using private information on asset quality, but in the baseline model the α's and χ's are fixed parameters. One reason is that endogenizing the α's or χ's can lead to multiple steady states, making comparative statics too difficult (i.e., mostly indeterminate) and endogenous dynamics too easy for our purposes (we are interested in multiple dynamic equilibria with a unique unique steady state). However, we do endogenize acceptability in Section 8.4 to show how it matters for policy analysis.[7]

Let q_t^j be the quantity and p_t^j the payment in a type j meeting, with $j = m, ma, ...e$ indicating which assets are accepted. Then $Q_t = \sum_j \alpha_j q_t^j$ is DM trading volume, a key variable since low volume is a common measure of frictions in decentralized markets, both in theory (Weill 2008; Lagos and Rocheteau 2009) and empirical work (Brennan et al. 1998). The period payoff for a buyer is

$$u(q_t) + U(x_t) - h_t, \tag{2}$$

where $\beta \in (0,1)$, $u(q_t)$ is the payoff from q_t, $U(x_t)$ is the utility of x_t, and $-h_t$ is the disutility of labor, with $u(0) = 0$, $u', U' > 0$, and $u'', U'' < 0$. Sellers have similar CM utility, but with a cost $c(q_t)$ rather than a benefit from DM trade, where $c(0) = 0$, $c' > 0$, $c'' \geq 0$ and $c(\bar{q}) = u(\bar{q})$ for some $\bar{q} > 0$.

While one interpretation is that households trade goods among themselves in the DM, with $u(q)$ the utility from consuming q, all the results below hold if instead

[7] Special cases of this specification include: Geromichalos et al. (2007) have $\alpha_j > 0$ iff $j = ma$; Lagos and Rocheteau (2008) have $\alpha_j > 0$ iff $j = mk$; CIA models have $\alpha_j > 0$ iff $j = m$; many macro-finance papers use $\chi_k \in (0,1)$; setting $\chi_k = 0$ is like standard growth theory; and setting $\chi_m = 0$ effectively eliminates cash. Also note that even when we assume $\alpha_j > 0$ iff $j = e$ and $\chi_j = 1 \forall j$, so all assets are fully accepted by all sellers, m can be valued if a's dividend and k's productivity are low, as then real liquidity is tight, and buyers may want to top it up with cash.

$u(q_t)$ is output from producing the numeraire good using q_t as an input. Hence some applications of this framework think of producers trading inputs in the DM. Others have households acquiring goods from firms or retailers, financial institutions acquiring assets from each other, or firms acquiring funding from financial institutions. Our results apply to all of these interpretations.

3.2 Equilibrium

Let $W_t(\zeta)$ and $V_t(\zeta)$ be buyers' CM and DM value functions at t. Then

$$W_t(\zeta) = \max_{x,h,\hat{\zeta}} \{U(x) - h + \beta V_{t+1}(\hat{\zeta})\} \text{ st } x = \Omega + \omega h - \phi^m \hat{m} - \phi^a \hat{a} - \hat{k}, \tag{3}$$

where $\hat{\zeta} = (\hat{m}, \hat{a}, \hat{k})$ is the revised portfolio, $\Omega \equiv \phi^m m + (\phi^a + \rho)a + (1 + \kappa - \delta)k - T$ is wealth and T is the tax.[8] Eliminating h using the budget equation, (3) becomes

$$W_t(\zeta) = \frac{\Omega}{\omega} + \max_x \left\{U(x) - \frac{x}{\omega}\right\} + \max_{\hat{\zeta}} \left\{-\frac{\phi^m \hat{m} + \phi^a \hat{a} + \hat{k}}{\omega} + \beta V_{t+1}(\hat{\zeta})\right\}.$$

Given interior solutions, FOC's are $U'(x) = 1/\omega$ and:

$$\phi^m/\omega = \beta \partial V_{t+1}/\partial \hat{m} \tag{4}$$

$$\phi^a/\omega = \beta \partial V_{t+1}/\partial \hat{a} \tag{5}$$

$$1/\omega = \beta \partial V_{t+1}/\partial \hat{k} \tag{6}$$

These imply $\hat{\zeta}$ is independent of ζ, while $\partial W_t/\partial m = \phi^m/\omega$, $\partial W_t/\partial a = (\phi^a + \rho)/\omega$ and $\partial W_t/\partial k = (1 + \kappa - \delta)/\omega$ imply W is linear in wealth with slope $1/\omega$.[9]

Since W is linear in wealth with slope $1/\omega$,

$$V(\zeta) = W(\zeta) + \sum_j \alpha_j [u(q^j) - p^j/\omega] \tag{7}$$

where q^j is the quantity and p^j the payment, in numeraire, in a type j meeting. Thus, $V(\zeta)$ is the payoff to carrying ζ into the CM, $W(\zeta)$, plus the probability of a type j

[8] Our convention is to not index variables by time t when there is no risk of confusion. Also, we use subscripts for parameters, like α_j, but superscripts for endogenous variables, like ϕ^j.

[9] These standard results, which greatly simplify the analysis, follow immediately from quasi-linear utility, but they also hold in some more general settings (Rocheteau et al. 2008; Wong 2016).

meeting times the surplus $u(q^j) - p^j/\omega$, summed over j, where ω converts numeraire into time, and time (leisure) is utility by (2). Importantly, p^j is constrained by buyers' liquidity in any meeting, $p^j \leq z^j$, where in type m, a and k meetings

$$z^m = \chi_m \phi^m m,\ z^a = \chi_a(\phi^a + \rho)a \text{ and } z^k = \chi_k(1 + \kappa - \delta)k,$$

while $z^{ma} = z^m + z^a$, $z^{mk} = z^m + z^k$, $z^{ak} = z^a + z^k$ and $z^e = z^m + z^a + z^k$.

Often χ_j is called the pledgeability of asset j. To see why, consider two interpretations of payments. First, imagine DM buyers turn over assets for immediate settlement, in which case $\chi_j < 1$ can be microfounded using information theory (see Li et al. 2012 and references therein). Second, imagine buyers promise payment in the next CM, in which case $\chi_j < 1$ can be motivated by saying buyers might renege, and the only punishment is to seize a fraction χ_j of each asset j while defaulters abscond with the rest (as in work following Kiyotaki and Moore 1997). So sellers only accept promises up to χ_j times holdings of asset j, explaining the pledgeability label. Here it does not matter which story one adopts – assets as media of exchange or collateral – since the equations apply to both.

Following Gu and Wright (2016), the DM terms of trade (p, q) are set by a generic mechanism $v(\cdot)$, which means that to get q buyers must pay p where $p/\omega = v(q)$ (here $1/\omega$ simply converts p into utility). As usual, assume $v(0) = 0$, $v'(q) > 0$ and this: let $p^*/\omega = v(q^*)$ where q^* is efficient, defined by $u'(q^*) = c'(q^*)$; then

$$p^* \leq z \Rightarrow p = p^* \text{ and } q = q^*, \text{ while } p^* > z \Rightarrow p = z \text{ and } q = v^{-1}(z/\omega). \tag{8}$$

This says that buyers, if they can, pay p^* and get q^*, while otherwise they pay $p = z$ and q is determined by $v(\cdot)$. A special case is the Kalai bargaining solution, which has been popular in monetary economics since Aruoba et al. (2007), and implies $v(q) = \theta c(q) + (1 - \theta)u(q)$ where θ is buyers' share. Generalized Nash bargaining gives the same outcome when $p^* \leq z$ is slack, but otherwise implies

$$v(q) = \frac{\theta u'(q)\, c(q) + (1 - \theta)\, c'(q)\, u(q)}{\theta u'(q) + (1 - \theta)\, c'(q)}.$$

Other special cases include some strategic bargaining solutions (Zhu 2019), mechanisms designed for efficiency (Hu et al. 2009), and Walrasian pricing, which can be justified by having multilateral instead of bilateral DM meetings (Rocheteau and Wright 2005). We do not rely on exotic mechanisms below, and indeed many of the examples use BTA (buyer-take-all) bargaining, $v(q) = c(q)$, which is actually the same as Walrasian pricing when $c(q) = q$. Still, a generic $v(q)$ is desirable for generality in the analytic results of Sections 4, 8 and 9, and it is nice to be able to check how the mechanism might matter for numerical results.

The above discussion concerns buyers. Sellers' problem is similar but they take no cash out of the CM. To clarify, consider for ease of exposition a stationary outcome where z_t^m is constant, so the nominal price level $1/\phi_t^m$ rises at the same rate as the money supply, and the inflation rate π is given by $1+\pi = \phi_t^m/\phi_{t+1}^m = 1+\mu$. Then define the illiquid nominal interest rate ι by asking agents how many dollars they would need in the next CM to give up a dollar in this CM; the answer will be $1 + \iota = (1 + \pi)/\beta$, which means $1 + \iota = (1 + \mu)/\beta$ when z_t^m is constant. In any case, as is standard, we assume $\iota > 0$, which guarantees sellers hold no cash, although we can consider the limit $\iota \to 0$, which is the Friedman rule. Of course, the reason sellers hold no cash when $\iota > 0$ is that ι is the opportunity cost of the liquidity services cash provides, and liquidity is of no benefit to them.

Given this, the money market clearing condition is $\hat{m} = (1 + \mu) M$. For a, while buyers hold $\hat{a}$, sellers may also hold some, say $\tilde{a}$, so market clearing is $\hat{a} + n\tilde{a} = 1$ where 1 is the normalized supply and we recall that n is the measure of sellers while 1 is the measure of buyers. The reason sellers may demand $\tilde{a} > 0$ is that real assets may be a good saving vehicle even if their liquidity services are not valued, which is not true for cash. Similarly, $\hat{k} + n\tilde{k} = K$, where $\tilde{k}$ is capital held by sellers and K is the endogenous supply. Market clearing for x is $F(h, k) + (1 - \delta)k + \rho = (1 + n)x + G + k_{+1}$ where we recall that ρ is the output (dividends) provided by A and G is government spending. Then, if the markets for m, a and x clear, the labor market clears automatically (Walras' law).

Now *equilibrium* is defined as a list of time paths $\{x, h, \hat{\zeta}, \phi^a, \phi^m, \omega, \kappa, q^j, p^j\}$ satisfying $\forall t$: (i) firm CM optimization (1); (ii) household CM optimization (3); (iii) DM trade as described by (8); (iv) CM clearing as discussed above; and (v) an initial condition ζ_0. Note that equilibrium implies $\lim_{t\to\infty}\beta^t z_t^j = 0$, a standard transversality condition required for household optimization (see, e.g., Rocheteau and Wright 2013). A *stationary equilibrium* is one where the variables at t might depend on k_t but do not depend directly on t. A *monetary equilibrium* has $\phi_t^m > 0$ $\forall t$. A *monetary steady state*, or MSS, satisfies conditions (i)-(iv) with all variables constant over time, except m_t and ϕ_t^m, although the product $\phi_t^m m_t > 0$ is constant.

Without the DM this is a textbook growth model, where the unique equilibrium has $k_t \to \bar{k}$ monotonically $\forall k_0 > 0$, with $\bar{k}$ denoting steady state. With a DM, we show below there are equilibria where k fluctuates forever – a limit cycle – which is somewhat novel because cycles in k are not easy to get in models without liquidity considerations (Boldrin and Rustichini 1994). To pursue this, let $\lambda(q)$ be the Lagrange multiplier on the constraint $p \leq z$, which as usual in these models satisfies $\lambda(q) = u'(q)/v'(q) - 1$. This is often called the liquidity premium, the benefit of holding an asset over and above its return, coming from relaxing $p \leq z$.[10]

To proceed, it is useful to let $L(z/\omega) = \lambda \circ v^{-1}(z/\omega)$, where $\circ$ denotes the composite of two functions, so that L is the liquidity premium with z as its argument, rather than q. Also, we write liquidity in terms of utility as $\tilde{z} = zU'(x) = z/\omega$, since $U'(x) = 1/\omega$ from the FOC for x in the CM. Then it is routine (see Appendix A2 for details) to derive the Euler equations for $\tilde{z}^m$, $\tilde{z}^a$ and $\tilde{z}^k$,

$$\tilde{z}_t^m = \frac{\beta \tilde{z}_{t+1}^m}{1+\mu}\left[1 + \chi_m \sum_{j\in\mathcal{S}_m} \alpha_j L\left(\tilde{z}_{t+1}^j\right)\right] \tag{9}$$

$$\tilde{z}_t^a = \chi_a \rho U'(x_t) + \beta \tilde{z}_{t+1}^a \left[1 + \chi_a \sum_{j\in\mathcal{S}_a} \alpha_j L\left(\tilde{z}_{t+1}^j\right)\right] \tag{10}$$

$$U'(x_t) = \beta(1+\kappa_{t+1}-\delta)U'(x_{t+1})\left[1 + \chi_k \sum_{j\in\mathcal{S}_k} \alpha_j L\left(\tilde{z}_{t+1}^j\right)\right], \tag{11}$$

where $\mathcal{S}_j$ is the set of meetings where asset j is accepted, e.g. $\mathcal{S}_m = \{m, ma, mk, e\}$. Also, it is useful below to define real returns on the assets as

$$1 + r_t^m = \phi_{t+1}^m/\phi_t^m,\ 1 + r_t^a = \left(\phi_{t+1}^a + \rho\right)/\phi_t^a, \text{ and } 1 + r_t^k = 1 + \kappa_t - \delta.$$

[10] It can also be interpreted as the "convenience yield" in Krishnamurthy and Vissing-Jorgensen (2012), but here "convenience" is modeled explicitly by assets' value as payment instruments.

4 Steady State

A MSS is a $\tilde{\mathbf{z}} = \left(\tilde{z}^m, \tilde{z}^a, \tilde{z}^k\right)$ solving (9)-(11), without t subscripts, since from $\tilde{\mathbf{z}}$ all other variables easily follow. Existence and uniqueness of MSS in related models is discussed elsewhere (e.g., Gu and Wright 2016); here the interest is in comparative statics. For this, let $F(h,k) = h + \varepsilon f(k)$, define $g(z^k) = k$ implicitly by $z^k = \chi_k \left[1 + \varepsilon f'(k) - \delta\right] k$, and impose the fairly weak restriction $g'(z^k) > 0$. Then Table 1 lists the impact on the z's and r's of ι, ρ, ε and the χ's.[11]

Table 1: Comparative Statics in the Baseline Model.

	z^m	z^a	z^k	z^{ma}	z^{mk}	z^{ak}	z^e	r^m	r^a	r^k
ι	$-$	$+^3$	$+^2$	$-$	$-$	$+$	$-^1$	$-$	$-^3$	$-^2$
ρ	$-^3$	$+$	$-^1$	$+$	$-$	$+$	$+^2$	0	$+$	$+^1$
ε	$-^2$	$-^1$	$+$	$-$	$+$	$+$	$+^3$	0	$+^1$	$+$
χ_m	$+$	$-^3$	$-^2$	$+$	$+$	$-$	$+^1$	0	$+^3$	$+^2$
χ_a	$-^3$	$+$	$-^1$	$+$	$-$	$+$	$+^2$	0	?	$+^1$
χ_k	$-^2$	$-^1$	$+$	$-$	$+$	$+$	$+^3$	0	$+^1$	?

Notes: 1 $\Rightarrow$ holds if $\ell_{ma}\ell_{mk}= 0$; 2 $\Rightarrow$ holds if $\ell_{ma}\ell_{ak}= 0$; 3 $\Rightarrow$ holds if $\ell_{mk}\ell_{ak}= 0$

Many effects are unambiguous, including all direct effects: making M better by lowering ι increases z^m; making A better by raising ρ increases z^a; and making K better by raising ε increases z^k. Some others depend on the ℓ's, where $\ell_j \equiv \alpha_j L'(z^j)$ is the expected marginal value of liquidity in type j meetings. This can be understood via the Mundell-Tobin effect, $\partial k/\partial\pi > 0$. In MSS $\partial k/\partial\pi$ has the same sign as $\partial z^k/\partial\iota$ and $\partial z^k/\partial\mu$, so Mundell-Tobin holds if $\ell_{ma}\ell_{ak} = 0$ but *not* in general. Now $\ell_{ma}\ell_{ak} = 0$ means either: (i) meetings where m and a but not k are accepted never occur, or if they do liquidity is not scarce in those meetings; or (ii) meetings where A and K but not M are accepted never occur, or if they do liquidity is not scarce in those meetings.

Now notice that with $\ell_{ma}\ell_{ak} \neq 0$ the following can happen: higher ι lowers z^m, which tightens liquidity in ma meetings; so agents demand more z^a, which relaxes

[11] Other comparative static results, including the impact of the α's, are contained in Appendix A3. Additionally, there we take up the question (suggested by the Editor), what makes assets react more strongly to changes in inflation? It is proved that the elasticity of an asset's return with respect to ι is bigger in absolute value when the asset is a better payment instrument.

liquidity in ak meetings; so they decrease z^k. However, as a referee pointed out, independent of the ℓ's a *generalized Mundell-Tobin effect* always holds: $\partial z^{ak}/\partial \iota > 0$ is guaranteed even if $\partial z^k/\partial \iota > 0$ and $\partial z^a/\partial \iota > 0$ are not.

*** Figure 1 about here ***

That K can go up or down is shown by the two panels in Figure 1, indicating that $\partial z^k/\partial \iota < 0$ is possible, and it is not merely that we failed to prove $\partial z^k/\partial \iota > 0$. As in all of our numerical work, unless specified otherwise, as is the case in the policy analysis where we consider alternatives, Figure 1 is drawn assuming

$$u(q) = \frac{(q+\varsigma)^{1-\gamma} - \varsigma^{1-\gamma}}{1-\gamma},\ c(q) = v(q) = q \text{ and } F(h,k) = Ch + \varepsilon k^{\sigma}. \tag{12}$$

The parameters for these graphs, listed in Table 2, were selected as follows. For the CM, β, δ and ρ are standard in macro calibrations.[12] Then $C = 1$ and $\varepsilon = 1$ are normalizations. Our σ is in the standard range, even if F is somewhat nonstandard (with $\sigma < 1$ it is not CRS, but that is not a problem). While it does not matter for Figure 1, $U(x)$ matters in Section 8.3 and is discussed there. Similarly, utility over h does not matter here, but can be set as usual to match hours worked.

Table 2: Parameter Values for Figure 1.

β	ς	γ	ρ	σ	δ	ε	C
0.95	0.001	0.67	0.01	0.25	0.1	1	1

α_m	α_a	α_k	α_{ma}	α_{mk}	α_{ak}	α_e	χ_m	χ_a	χ_k
0.05	0	0	0.1	0.1	0	0.05	1	0.45	0.17

Note: This is case (i); case (ii) has α_{mk}= 0.01 and α_{ak}= 0.09.

The DM is less straightforward. For preferences, ς is set small enough that $u(q)$ is close to CRRA, while $\gamma = 0.67$ is taken from a recent calibration by Bethune et al. (2020) of a related model. Similarly, χ's are taken from Venkateswaran and

[12] At least, these are standard in annual models, which is our preferred specification for numerical work, mainly because it facilitates comparison with related papers. In case one worries that this means each household makes at most one DM purchase per year, we can interpret agents in the model as members of large households as in Shi (1997), and then a household can consume many DM goods each period even if each member buys no more than one.

Wright (2014). For α's, appropriate values depend on the type of DM one wants to consider, which differs across applications; we simply set them so that buyers' DM meeting probability is 0.3, and in any meeting the probability they can use M is 1, while the probability they can use A or K is 0.5. This is case (i) in Figure 1; case (ii) adjusts α_{mk} and α_{ak} slightly to show that $\partial k/\partial \mu < 0$ is possible.

The graph shows some of the z' (others can be inferred since, e.g., $z^{ma} = z^m + z^a$). It also shows asset returns, $1 + r^j$, as well as the illiquid real return $1 + r = 1/\beta$, defined parallel to the way ι is defined as the amount of x agents require in the next CM to give up 1 unit of x in this CM. In terms of general observations, first, now that it is clear how the impact of μ on K depends on assets' DM interactions, we can tell a similar tale about A: the impact of μ on its price and return is unambiguous if $\ell_{mk}\ell_{ak} = 0$, but not in general. Also note that anything affecting the z's affects welfare, since z's are instrumental in DM trade, as we discuss further below. As regards DM volume Q, for most parameter changes some z's go up and others down, so there are no general results, but numerically we get $\partial Q/\partial \mu < 0$ in Figure 1. As mentioned, low Q is often used as a measure of high frictions in decentralized markets. Here an increase in μ shows up as more severe frictions, the point being that this empirical measure is not invariant to policy.

Notice that higher ι always reduces r^m, and reduces r^a and r^k at least if $\ell_{mk}\ell_{ak}$ and $\ell_{ma}\ell_{ak}$ are not too big. This is related to Wallace (1980):

> In general, fiat money issue [equivalent here to higher ι] is not a tax on all saving. It is a tax on saving in the form of money. But it is important to emphasize that the equilibrium rate-of-return distribution on the equilibrium portfolio does depend on the magnitude of the fiat money financed deficit... the real rate-of-return distribution faced by individuals in equilibrium is less favorable the greater the fiat money-financed deficit. Many economists seem to ignore this aspect of inflation because of their unfounded attachment to Irving Fisher's theory of nominal interest rates... [and that] accounts for why economists seem to have a hard time describing the distortions created by anticipated inflation.

This sounds right, although in the OLG economies Wallace was using back then, the return distribution is degenerate (by the law of one price, which applies since those models are basically Walrasian) unless assets differ in risk. While risk was at one time suggested as a foundation for money demand (Tobin 1958), our approach provides an alternative. The nondegenerate return distribution here is based on liquidity, and its dependence on inflation is exactly what Figure 1 shows.

We mention one more comparative static as a segue to Section 5: it is ambiguous how r^a moves with χ_a (and similarly for r^k and χ_k). To see why, suppose $\chi_a = 0$, so a provides no liquidity. Then $\phi^a = \rho/r$. As χ_a increases, liquidity constraints get relaxed, implying a premium that raises ϕ^a and lowers r^a. As χ_a increases further, however, constraints relax further and eventually become slack, so the premium falls and the changes in ϕ^a and r^a are reversed. This is one way liquidity considerations lead to nonmonotonicities crucial for equilibrium dynamics and it distinguishes these models from those in, e.g., Azariadis (1993), where backward-bending savings or labor supply curves can be said to drive the results.

5 One-Asset Dynamics

For simplicity, set $F(h,k) = h + f(k)$, and first consider $\alpha_{ma} = \alpha_{mk} = \alpha_{ak} = \alpha_e = 0$, so in every meeting only one asset is accepted. Then (9)-(11) reduce to:

$$z_t^m = \frac{\beta z_{t+1}^m}{1+\mu}\left[1 + \alpha_m \chi_m L\left(z_{t+1}^m\right)\right] \tag{13}$$

$$z_t^a = \chi_a \rho + \beta z_{t+1}^a \left[1 + \alpha_a \chi_a L\left(z_{t+1}^a\right)\right] \tag{14}$$

$$1 = \frac{\beta z_{t+1}^k}{\chi_k g(z_{t+1}^k)}\left[1 + \alpha_k \chi_k L\left(z_{t+1}^k\right)\right] \tag{15}$$

These are three independent equations, which is not our preferred specification, but is useful because each asset behaves *as if* it were the only liquid asset. Note that elsewhere below we also set some α's to 0, but only for tractability – by continuity the outcomes are similar if they are positive but not too big. In contrast, for asset independence, here we set α_{ma}, α_{mk}, α_{ak} and α_e to exactly 0.

5.1 Money

Write $z_t^m = \Phi_m(z_{t+1}^m)$ where Φ_m is defined by the RHS of (13). As Figure 2 shows, there are two steady states, 0 and $z_s^m > 0$, and the former is stable. So there are solutions to $z_t^m = \Phi_m(z_{t+1}^m)$ starting at any z_0^m in some range, with $z_t^m \to 0$, and these constitute equilibria (and to be clear, as always, solutions with $z_t^m \to \infty$ do not constitute equilibria as they violate transversality). Standard results (Azariadis 1993) tell us $\Phi_m'(z_s^m) < -1$ implies there are cycles of period 2, i.e., solutions to $z_2^m = \Phi_m(z_1^m)$ and $z_1^m = \Phi_m(z_2^m)$, or fixed points of $\Phi_m^2 = \Phi_m \circ \Phi_m$, with $z_1^m \neq z_2^m$.

*** Figure 2 about here ***

This is shown in Figure 2(b) where (z_1^m, z_2^m) occurs at intersections of Φ_m with its inverse Φ_m^{-1} off the 45^o line. When z^m cycles, so do asset returns: given $z_1^m > z_s^m > z_2^m$, one can check $r_1^m < r_s^m < r_2^m$. Also, when 2-cycles exist, there are stochastic sunspot equilibria where z^m fluctuates randomly (see below).

Table 3: Parameter Values for Figure 2.

β	ς	γ	α_m	χ_m	μ
0.9	0.5	1.5	0.9	1	0.01

Note: The right panel is the same except $\gamma = 6$.

For the $u(q)$ in (12), as γ increases we eventually get $\Phi_m'(z_s^m) < -1$ and hence 2-cycles. As γ increases further, higher-order cycles emerge, including 3-cycles, fixed points of Φ_m^3. When 3-cycles exist, the Sarkovski and Li-Yorke Theorems say there are N-cycles for any integer N plus chaos. Thus monetary economies admit a variety of dynamic equilibria due to money's self-referential nature: you value it more when others value it more.[13]

[13]These results are well known, although one might also say that they are sometimes under-appreciated, since it is commonly believed that some of them rely on discrete time. It is true that usually deterministic cycles cannot occur in continuous-time univariate systems (see Oberfield and Trachter 2012 or Choi and Rocheteau 2020 for discussions in monetary models). We do not find this too problematic, for the following reasons: (i) It is not obvious continuous time is a better modeling choice. (ii) As mentioned in fn. 1, seeing how belief-based dynamics arise in simple

5.2 Real Assets

Now write $z_t^a = \Phi_a(z_{t+1}^a)$ where Φ_a is the RHS of (14). Different from M, $\rho > 0$ rules out equilibria where $\phi_t^a = 0$ or $\phi_t^a \to 0$. One can show there is a unique steady state $z_s^a > 0$, but if $\Phi_a'(z_s^a) < -1$, there are cyclic and stochastic equilibria. Therefore, having real assets used as payment instruments may eliminate the degenerate steady state 0, as well as paths leading to 0, that appear with fiat money, but this does not rule out belief-based dynamics. That is perhaps not as well known as the results for fiat money, although one might argue it is more relevant for financial economics.

5.3 Capital

In the current simple specification, (15) is not a difference equation. Hence, there are no belief-based dynamics in K, which is nice for our purposes, since it makes it all the more interesting below when we do get K fluctuations.

6 Two-Asset Dynamics

6.1 Money and Real Assets

Suppose $\alpha_{ma} > 0 = \alpha_{mk} = \alpha_{ak} = \alpha_e$ and $F(h,k) = h + f(k)$. Then the K equation is as above but the M and A equations are not independent:

$$z_t^m = \frac{\beta z_{t+1}^m}{1+\mu}\left[1 + \alpha_m \chi_m L\left(z_{t+1}^m\right) + \alpha_{ma}\chi_m L\left(z_{t+1}^{ma}\right)\right] \tag{16}$$

$$z_t^a = \chi_a \rho + \beta z_{t+1}^a\left[1 + \alpha_a \chi_a L\left(z_{t+1}^a\right) + \alpha_{ma}\chi_a L\left(z_{t+1}^{ma}\right)\right]. \tag{17}$$

Write this as $\hat{\mathbf{z}}_\mathbf{t} = \mathbf{\Phi}_{ma}(\hat{\mathbf{z}}_{\mathbf{t+1}})$ where

$$\hat{\mathbf{z}}_\mathbf{t} = \begin{bmatrix} z_t^m \\ z_t^a \end{bmatrix} \text{ and } \mathbf{\Phi}_{ma}(\hat{\mathbf{z}}_\mathbf{t}) = \begin{bmatrix} \Phi_m\left(\hat{\mathbf{z}}_\mathbf{t}\right) \\ \Phi_a\left(\hat{\mathbf{z}}_\mathbf{t}\right) \end{bmatrix},$$

models, even if they rely on discrete time, may affect one's priors about whether they can arise in actual economies. (iii) Even if continuous-time deterministic cycles do not arise in univariate models, they can in multivariate models (e.g., Coles and Wright 1998). (iv) Sunspot cycles exist in univariate continuous-time models. (v) Rocheteau and Wang (2022) show deterministic cycles actually *can* exist in univariate continuous-time models with liquidity modeled carefully.

and Φ_m and Φ_a are given by the RHS of (16)-(17). For a 2-cycle, we seek a fixed point of $\mathbf{\Phi}_{ma}^2$ with $\hat{\mathbf{z}}_\mathbf{1} \neq \hat{\mathbf{z}}_\mathbf{2}$.

Table 4: Types of 2-Cycles

cycle	z^m	z^a	description
i	$z_1^m > z_2^m$	$z_1^a \approx z_2^a$	m fluctuates a lot, a fluctuates a little
ii	$z_1^m \approx z_2^m$	$z_1^a > z_2^a$	a fluctuates a lot, m fluctuates a little
iii	$z_1^m > z_2^m$	$z_1^a > z_2^a$	both fluctuate; positive correlation
iv	$z_1^m < z_2^m$	$z_1^a > z_2^a$	both fluctuate; negative correlation

We find numerically that four types of 2-cycles exist *for the same parameters*, as summarized in Table 4. Intuitively, Section 5 shows 2-cycles and steady states exist for M and for A when they are independent. The outcomes in Table 4 are basically combinations of those possibilities: (i) z^m fluctuates but z^a is approximately constant; (ii) z^a fluctuates but z^m is approximately constant; (iii) z^m and z^a fluctuate together; and (iv) z^m and z^a fluctuate in opposition. The first two are approximations to one variable being in steady state, because with $\alpha_{ma}\chi_m L\left(z_{t+1}^{ma}\right) > 0$, one asset cannot fluctuate without the other doing so. Depending on the correlation between z^m and z^a, the cycles have different implications for liquidity and asset returns: (iii) implies positive and (iv) implies negative correlation between r^m and r^a. Hence various patterns of returns and inflation can emerge, consistent with Table 1, but the results here entail genuine dynamics, not comparative statics.

*** Figure 3 about here ***

Figure 3 provides examples with parameters in Table 5. Given the system $\hat{\mathbf{z}}_\mathbf{t} = \mathbf{\Phi}_{ma}(\hat{\mathbf{z}}_\mathbf{t+1})$, vertical curves are solutions to $z^m = \Phi_m[\Phi_m(z^m, z^a), \Phi_a(z^m, z^a)]$ for a fixed z^a and horizontal curves are solutions to $z^a = \Phi_a[\Phi_m(z^m, z^a), \Phi_a(z^m, z^a)]$ for a fixed z^m. Then each crossing is a fixed point of $\mathbf{\Phi}_{ma}^2$. There are four proper 2-cycles consisting of two fixed points $\hat{\mathbf{z}}_\mathbf{1} = \mathbf{\Phi}_{ma}(\hat{\mathbf{z}}_\mathbf{2})$ and $\hat{\mathbf{z}}_\mathbf{2} = \mathbf{\Phi}_{ma}(\hat{\mathbf{z}}_\mathbf{1})$, plus a steady state, for a total of nine fixed points. The letters indicate which points form cycles: E in the middle is the steady state; D and D' represent a 2-cycle of type (i); B and B' form a 2-cycle of type (ii); C and C' represent a 2-cycle of type (iii); and A, A' form a 2-cycle of type (iv).

Table 5: Parameter for the Example with m and a.

β	ς	γ	α_m	α_a	α_{ma}	χ_m	χ_a	μ	ρ
0.6	0.01	5.5	0.1	0.1	0.1	1	0.01	0.2	0.01

6.2 Money and Capital

Next consider $\alpha_{mk} > 0 = \alpha_{ma} = \alpha_{ak} = \alpha_e$, assuming $F(h,k) = h + \varepsilon k$ with $0 \leq \varepsilon - \delta \leq (1-\beta)/\beta$, so $\omega = 1$ and $\kappa = \varepsilon$. Having F linear in h and k makes things easier but is not necessary: Appendix A4 shows similar results when F is a CES function. Now the A equation is as in Section 5, but the M and K equations are not independent:

$$z_t^m = \frac{\beta z_{t+1}^m}{1+\mu}\Big[1 + \alpha_m \chi_m L\left(z_{t+1}^m\right) + \alpha_{mk}\chi_m L\left(z_{t+1}^{mk}\right)\Big] \tag{18}$$

$$1 = \beta(1+\varepsilon-\delta)\Big[1 + \alpha_k \chi_k L\left(z_{t+1}^k\right) + \alpha_{mk}\chi_k L\left(z_{t+1}^{mk}\right)\Big]. \tag{19}$$

For illustration, set $\alpha_k = 0$. Then, assuming $L\left(z_{t+1}^{mk}\right) > 0$, we solve for it from (19) and insert it into (18) to get $z_t^m = \Psi\left(z_{t+1}^m\right)$ where

$$\Psi\left(z^m\right) = \frac{\beta z^m}{1+\mu}\left\{1 + \alpha_m \chi_m L\left(z^m\right) + \frac{\chi_m}{\chi_k}\left[\frac{1}{\beta\left(1+\varepsilon-\delta\right)} - 1\right]\right\}.$$

This reduces equilibrium to a univariate system in z^m, with steady state z_s^m. If $\Psi'\left(z_s^m\right) < -1$ there is a 2-cycle with $z_1^m < z_s^m < z_2^m$, then (19) determines $\left(z_1^k, z_2^k\right)$ with $z_1^k > z_2^k$. Appendix A5 shows $\Psi'\left(z_s^m\right) < -1$ is possible. Hence we get cycles in K, which were not possible in Section 5.3. The reason is simple: first, it is easy to get z^m fluctuating; then, since M and K are substitutes in DM payments, z^k fluctuates. An example is provided by the parameters in Table 6. The steady state is $\left(z_s^m, z_s^k\right) = (0.9385, 0.0508)$, and there is a 2-cycle with

$$\left(z_1^m, z_1^k\right) = (0.9144{,}0.0750) \text{ and } \left(z_2^m, z_2^k\right) = (0.9661{,}0.0233).$$

Table 6: Parameter Values for the Example with m and k.

β	ς	γ	α_m	α_{mk}	χ_m	χ_k	μ	δ	ε
0.95	0.01	5.57	0.3	0.7	1	1	0.05	0.1	0.15

6.3 Real Assets and Capital

That there are cycles in z^a and z^k is an obvious extension of Section 6.2. Still, it shows K fluctuations do not require money per se, just another liquid asset. Moreover, Appendix A1 has related results with 1-period assets or bonds instead of K: there are cycles in those assets iff they interact with A or M the way K interacts with A or M here. The general point is that modeling liquidity explicitly changes the nature of the equilibrium set in asset-pricing theory.

7 Three-Asset Dynamics

Suppose $\alpha_e > 0 = \alpha_{ma} = \alpha_{mk} = \alpha_{ak}$ and $F(h,k) = h + \varepsilon k$. Given $\alpha_e > 0$ all three assets are substitutes in some meetings as can be seen in:

$$z_t^m = \frac{\beta z_{t+1}^m}{1+\mu}\Big[1 + \alpha_m \chi_m L\left(z_{t+1}^m\right) + \alpha_e \chi_m L\left(z_{t+1}^e\right)\Big] \tag{20}$$

$$z_t^a = \chi_a \rho + \beta z_{t+1}^a \Big[1 + \alpha_a \chi_a L\left(z_{t+1}^a\right) + \alpha_e \chi_a L\left(z_{t+1}^e\right)\Big] \tag{21}$$

$$1 = \beta(1+\varepsilon-\delta)\Big[1 + \alpha_k \chi_k L\left(z_{t+1}^k\right) + \alpha_e \chi_k L\left(z_{t+1}^e\right)\Big]. \tag{22}$$

7.1 Cycles

Here we use $\alpha_k = 0$, but that is just a trick to get sharp results, since the outcome is similar if $\alpha_k > 0$ is not too big (recall the discussion at the start of Section 5). Then (22) can be rearranged as

$$\alpha_e L\left(z_{t+1}^e\right) = \frac{1}{\chi_k}\left[\frac{1}{\beta(1+\varepsilon-\delta)} - 1\right]. \tag{23}$$

Using this, we write the M and A equations as

$$z_t^m = \frac{\beta z_{t+1}^m}{1+\mu}\left\{1 + \alpha_m \chi_m L\left(z_{t+1}^m\right) + \frac{\chi_m}{\chi_k}\left[\frac{1}{\beta(1+\varepsilon-\delta)} - 1\right]\right\} \tag{24}$$

$$z_t^a = \chi_a \rho + \beta z_{t+1}^a \left\{1 + \alpha_a \chi_a L\left(z_{t+1}^a\right) + \frac{\chi_a}{\chi_k}\left[\frac{1}{\beta(1+\varepsilon-\delta)} - 1\right]\right\}. \tag{25}$$

Now z^e is determined by (23), while z^m and z^a are determined by (24) and (25). As in Section 6.1, either z^m or z^a can move a little or a lot, and their correlation can be positive or negative. Further, if the sum $z^m + z^a$ cycles, z^k does too, because (23) implies total liquidity z^e is constant.

With the parameters in Table 7, we find the following 2-cycles:

$$\mathbf{z_1} = (0.6153, 0.3253, 0.0465) \text{ and } \mathbf{z_2} = (0.6153, 0.3422, 0.0297)$$
$$\mathbf{z_1} = (0.6021, 0.3325, 0.0525) \text{ and } \mathbf{z_2} = (0.6312, 0.3325, 0.0235)$$
$$\mathbf{z_1} = (0.6021, 0.3253, 0.0597) \text{ and } \mathbf{z_2} = (0.6312, 0.3422, 0.0138)$$
$$\mathbf{z_1} = (0.6021, 0.3422, 0.0429) \text{ and } \mathbf{z_2} = (0.6312, 0.3253, 0.0307).$$

In the first case, z^m is constant while z^a and z^k are negatively correlated. In the second, z^a is constant while z^m and z^k are negatively correlated. In the third, all three fluctuate and z^m is positively correlated with z^a. In the fourth, all three fluctuate but now z^a and z^k are positively correlated.

Table 7: Parameter Values for Figure 4.

β	ς	γ	α_m	α_a	α_e	χ_m	χ_a	χ_k	ρ	μ	δ	ε
0.725	0.01	7.68	0.01	0.01	0.98	1	0.01	1	0.17	0	0.1	0.45

Note: The Right Panel is the Same except $\gamma = 8$, χ_m= 0.01, χ_a= 0.2, $\rho = 0.01$.

*** Figure 4 about here ***

Figure 4 highlights one reason it is interesting to have three assets: while Section 6.2 or 6.3 had K cycles, there was a tight relationship between z^k and z^m or between z^k and z^a, while more varied patterns emerge here. Also, these results provide a simple example of the critique of econometric policy evaluation in Lucas (1976). First note that time series from cyclic equilibria provide correlations between, e.g., z^m and z^k. One might think a policy that permanently affects z^m would affect z^k as predicted by that correlation, but it may not. Suppose we are in an equilibrium with z^m and z^k positively correlated, and that a policy maker desires a higher z^k. An observer/advisor might then recommend lowering ι to raise z^m, because based on the time series it is predicted that z^k will rise. But when ι is lowered, while z^m indeed rises, for the parameters in Table 7 it turns out that z^k falls.[14]

7.2 Sunspots

Next consider stochastic, or sunspot, equilibria. Assume there is a commonly-observed variable $s \in \{A, B\}$ that does not affect fundamentals but may affect behavior. Given s at t, with probability τ_s it changes to $s' \neq s$ at $t+1$. A proper sunspot equilibrium has the z's fluctuating with s. We seek a solution $\mathbf{z}^A \neq \mathbf{z}^B$ to

$$\mathbf{z^A} = (1-\tau_A)\mathbf{\Phi}_e(\mathbf{z^A}) + \tau_A \mathbf{\Phi}_e(\mathbf{z^B}) \tag{26}$$

$$\mathbf{z^B} = (1-\tau_B)\mathbf{\Phi}_e(\mathbf{z^B}) + \tau_B \mathbf{\Phi}_e(\mathbf{z^A}). \tag{27}$$

where $\mathbf{z_t} = \mathbf{\Phi}_e(\mathbf{z_{t+1}})$ and

$$\mathbf{\Phi}_e(\mathbf{z_t}) = \begin{bmatrix} \Phi_m(\mathbf{z_t}) \\ \Phi_a(\mathbf{z_t}) \\ \Phi_k(\mathbf{z_t}) \end{bmatrix},$$

while $\Phi_m(\mathbf{z_t})$, $\Phi_a(\mathbf{z_t})$ and $\Phi_k(\mathbf{z_t})$ are given by the RHS of (20)-(22).

If $\tau_A = \tau_B = 1$ this is a 2-cycle, which we already know exists. By continuity, sunspot equilibria exist for τ's not too far from 1. For the parameters in Table 7 plus $\tau_A = \tau_B = 0.9$, we find four types of sunspot equilibria generating patterns like Figure 4, except the transitions are random:

$$\begin{aligned}
\mathbf{z^A} &= (0.6104, 0.3292, 0.0475) \text{ and } \mathbf{z^B} = (0.6202, 0.3367, 0.0303) \\
\mathbf{z^A} &= (0.6070, 0.3286, 0.0516) \text{ and } \mathbf{z^B} = (0.6261, 0.3379, 0.0232) \\
\mathbf{z^A} &= (0.5997, 0.3279, 0.0595) \text{ and } \mathbf{z^B} = (0.6337, 0.3394, 0.0141) \\
\mathbf{z^A} &= (0.6070, 0.3367, 0.0435) \text{ and } \mathbf{z^B} = (0.6261, 0.3305, 0.0306).
\end{aligned}$$

We can construct sunspot equilibria in Sections 4 and 5, too, but omit that exercise in the interest of space. A reason these are interesting is that one might think stochastic fluctuations look more like data than low-order deterministic cycles.

[14] Another feature of these numerical examples is that welfare comparisons are generally ambiguous across cyclic equilibria. Also, cyclic equilibria can dominate steady states, or vice-versa, depending on parameters and on where in the cycle we start. This is not especially novel (see, e.g., Gu et al. 2013), but it highlights that stability is not the same as efficiency.

8 Monetary Policy

The next step is to analyze short- and long-run implications for policy.

8.1 Steady State Implications

In MSS $\pi = \mu$, so it is equivalent to peg inflation or the growth rate of the money supply. Also, $1 + \iota = (1 + \pi)/\beta$, so it is equivalent to peg ι. If that is well known, it is less standard to note that we can also target a measure of liquidity captured by one (or a combination) of the z's by letting μ be whatever it takes: simply use the relevant curve in Figure 1 to read μ off the horizontal axis. There are of course bounds on any target, corresponding to a lower bound of 0 on ι, and an upper bound that is the maximum consistent with the existence of MSS.

In many policy discussions and papers on the topic, it seems to be taken for granted that the central bank controls real interest rates, but the transmission mechanism is nebulous. Presumably policy can determine μ, the growth rate of M (at least if ignoring fractional-reserve banking and the money multiplier, but even with those features Altermatt (2022) shows how the path of M can be controlled). In MSS $1 + r^m = 1/(1 + \mu)$, and as Figure 1 shows, changes in r^m get passed through to other returns, but it is important to know how and how much. The illiquid real rate solves $1 + r = 1/\beta$ in MSS, independent of μ as in Fisher (1930), so policy cannot control r, but it can control r^m and to some extent r^a and r^k when A and K convey liquidity.

However, we know $r^j = r$ if $\alpha_j = 0$ or $\chi_j = 0$, and $r^j = r$ is likely if $\alpha_j > 0$ and $\chi_j > 0$ are very big because then liquidity constraints tend to be slack. So μ affects r^j iff α_j and χ_j are positive but not big. Further, even if r^j can be controlled its range is limited: r^j must be in the image of $r^j = r^j(\mu)$ for μ consistent with the existence of MSS. Further still, when policy can hit a target r^j, we must be willing to bear the consequences. In case (i) of Figure 1, r^a can be reduced from over 5% to close to 2%, but only by pushing μ above 10%. That makes Q fall from 0.3 to about

0.15, which has welfare implications, as discussed below.[15]

8.2 Dynamic Implications

Does it matter if we use a money supply rule that pegs μ, or an interest rate rule that pegs ι, when we think beyond steady state? It is easy to show that when M is the only liquid asset fixing ι leads to a unique equilibrium, ruling out all belief-based dynamics. To verify this, set $\alpha_j = 0\,\forall j \neq m$, so the Euler equation for z^m becomes

$$\phi_t^m U'(x_t) = \beta U'(x_{t+1})\phi_{t+1}^m \left[1 + \alpha_m \chi_m L\left(z_{t+1}^m/\omega_{t+1}\right)\right]. \tag{28}$$

Now, without imposing steady state, the inflation rate and illiquid real interest rate are $1+\pi_t = \phi_t^m/\phi_{t+1}^m$ and $1+r_t = U'(x_t)/\beta U'(x_{t+1})$. Then the illiquid nominal rate is $1+\iota_t = (1+\pi_t)(1+r_t)$, and (28) reduces to

$$\iota_t = \alpha_m \chi_m \lambda\left(q_{t+1}^m\right), \tag{29}$$

where we write the liquidity premium as $\lambda\left(q_{t+1}^m\right)$ to emphasize a point. The point is that $\iota_t = \iota\,\forall t$ implies $q_t^m = q\,\forall t$. Although it is predicated on M being the only liquid asset, this conclusion is otherwise quite robust – e.g., it holds for a generic mechanism $v(q)$ determining the terms of trade and general production function $F(h,k)$.

While pegging $\iota_t = \iota\,\forall t$ implies $q_t = q\,\forall t$, it makes the path of M endogenous. To solve for it, use $v\left(q_t^m\right) = \chi_m \phi_t^m M_t/\omega_t$ to eliminate the ϕ's in (28),

$$\frac{v\left(q_t^m\right)}{M_t} = \frac{v\left(q_{t+1}^m\right)\beta}{M_{t+1}}\left[1 + \alpha_m \chi_m \lambda\left(q_{t+1}^m\right)\right]. \tag{30}$$

Since constant ι entails constant q,

$$\frac{M_{t+1}}{M_t} = \beta\left[1 + \alpha_m \chi_m \lambda(q)\right]. \tag{31}$$

[15]Another point is there is no general ordering of r^m, r^a and r^k. As an implication, consider nominal returns on real assets – e.g., invest a dollar in A at t, take the payoff at $t+1$ and turn it into $1+\iota^a = (1+r^a)(1+\mu)$ dollars. These nominal yields can be negative, as is sometimes observed in data, but impossible in standard theory. It is easy to see $\iota^j < 0 \Leftrightarrow r^j < r^m$, so in Figure 1(ii), low μ implies $\iota^a, \iota^k < 0$. With payment frictions, $\iota^j < 0$ is not so anomalous.

Hence, μ is time invariant. Pegging $\iota_t = \iota$ and letting M follow (31) eliminates all endogenous dynamics when M is the only liquid asset, but that does not generalize to multiple liquid assets.

The economic intuition, discussed in the Introduction, on this can now be made precise. Suppose K is not liquid, while M and A are, with $\alpha_m, \alpha_a, \alpha_{ma} > 0$. Then set $\chi_m = \chi_a = 1$, $F(h,k) = h + f(k)$, mainly to reduce notation, and consider fixed ι. Then the dynamic system becomes

$$\iota = \alpha_m L(z^m_{t+1}) + \alpha_{ma} L(z^m_{t+1} + z^a_{t+1}) \tag{32}$$

$$z^a_t = \rho + \beta z^a_{t+1}[1 + \alpha_a L(z^a_{t+1}) + \alpha_{ma} L(z^m_{t+1} + z^a_{t+1})]. \tag{33}$$

Now (32) defines $z^m = P(z^a)$, leading to $z^a_t = \Phi_P(z^a_{t+1})$ where

$$\Phi_P(z^a) = \rho + \beta z^a\{1 + \alpha_a L(z^a) + \alpha_{ma} L[P(z^a) + z^a]\}.$$

Under the usual conditions there is a cycle where both z^m and z^a fluctuate. One can get the endogenous path of M over the cycle, and show it is not constant: a 2-cycle with $z^m_1 > z^m_2$ implies $\mu_2 > \mu_1$.

Table 8: Parameter Values for Two Assets with Fixed ι.

β	ς	γ	α_m	α_a	α_{ma}	ι	ρ
0.9	0.01	5.65	0	0.3	0.7	0.0001	0.01

Table 8 has a parametric example. The steady state is $(z^m_s, z^a_s) = (0.0493, 0.9407)$. There is also a 2-cycle with $(z^m_1, z^a_1) = $ (0.0056,0.9844) and $(z^m_2, z^a_2) = $ (0.0853,0.9046). Appendix A6 also constructs cycles with fixed ι when all three assets are liquid, so that K fluctuates. The bottom line is that with multiple liquid assets fixing ι does not rule out belief-based fluctuations. Is there a policy that does? We do not know, in general, but in the special case where M is accepted by all sellers the answer is easy: the Friedman rule $\iota = 0$.[16]

[16] While we do not know of a policy that eliminates multiplicity and endogenous dynamics with multiple liquid assets and away from Friedman rule, the way pegging ι does with only M, the Editor offered an interesting conjecture: perhaps with N liquid assets policy needs to somehow peg N different returns. This seems worth investigating in future work.

8.3 Welfare

Next the quantitative specification in Section 4 is used to measure the impact of inflation on welfare. Here $U(x) = D \log(x)$ with D and bargaining power θ calibrated. To explain, first, we consider several specifications differentiated on two dimensions: we vary the bargaining solution; and compare the baseline model to an alternative with $\alpha_m = 0.3$ and $\alpha_j = 0 \, \forall j \neq m$, which keeps the DM meeting probability the same but makes M the only liquid asset.

For each specification D is set to match the money demand curve – i.e., the relationship between real balances (scaled by output to render the series stationary) and nominal interest rates – in the data and model. Our sample is based on US data from 1955-2008. For money, while an argument can be made for M2 (Alvarez et al. 2009), we use M1J from Lucas and Nicolini (2015), which they argue is stable over the period after scaling by GDP. We do not use M0, since money in your pocket and money in your checking account are near-perfect substitutes for many purposes. For ι we use AAA corporate bond rates, although results are similar with T-Bill rates. Figure 5 and Table 9 display the results.[17]

Table 9: Specifications for Welfare Analysis

Liquidity	Bargaining	D Estimate	DM size	WC: $\mu = 0.1$	WC: $\chi_m = 0$
m, a, k	BTA $\theta = 1$	0.885	0.111	1.458%	6.410%
m, a, k	Kalai $\hat{\theta} = 0.842$	1.283	0.103	1.516%	5.799%
m, a, k	Nash $\hat{\theta} = 0.815$	0.793	0.114	3.378%	7.766%
m, a, k	Kalai $\theta = 0.5$	1.901	0.097	3.764%	6.536%
m, a, k	Nash $\theta = 0.5$	0.585	0.122	8.021%	9.612%
only m	BTA $\theta = 1$	1.314	0.071	1.156%	17.548%
only m	Kalai $\hat{\theta} = 0.840$	1.740	0.071	1.277%	13.458%
only m	Nash $\hat{\theta} = 0.817$	1.225	0.071	2.697%	17.695%
only m	Kalai $\theta = 0.5$	2.388	0.071	3.107%	9.483%
only m	Nash $\theta = 0.5$	0.996	0.071	5.688%	17.743%

[17] One can in principle calibrate two parameters to match two moments of money demand: D can be set to hit $\mathbb{E}(M/PY)$ at $\mathbb{E}(\iota)$, while something else, say the γ parameter in $u(q)$, can be set to hit the elasticity. We chose not to do so because it is not clear how precisely γ can be identified. Hence, we keep $\gamma = 0.67$, from Table 2, and estimate only D.

*** Figure 5 about here ***

For each case, the welfare cost, WC, of inflation rate $\pi = \mu$ is the fraction of consumption of x agents would give up to move from MSS at μ to the Friedman rule. In Figure 5, for various bargaining solutions, the top row is fitted money demand and the bottom is WC, while the left column is the baseline model and the right is the alternative where M is the only liquid asset. The most obvious feature is that in all panels money demand fits well, yet WC varies a lot.

To explain this, let us begin with bargaining. The cases considered are: BTA bargaining, i.e. Nash or Kalai with $\theta = 1$; and Nash or Kalai with $\theta = \hat{\theta}$ calibrated as explained below; and Nash or Kalai with $\theta = 0.5$, which has the virtue of symmetry, but is included mostly to give a feel for how θ matters. With $\theta = 1$, WC is 1.458% of x consumption, higher than the number typically found in CIA/MIU models, which is closer to 0.5%, but less than some search-based models (e.g., see Bethune et al. 2020). More relevant is the case where $\hat{\theta}$ is calibrated to deliver an average markup, defined as price over marginal cost, set to 1.4 consistent with retail trade survey data.[18]

At $\theta = \hat{\theta}$, WC is a little higher with Kalai bargaining, 1.516%, and a lot higher with Nash bargaining, 3.378%. The difference between Nash and Kalai, with θ in each case set to match the same markup, reflects the relative inefficiency of the Nash solution – e.g., at $\iota = 0$ it implies $q < q^*$ while Kalai implies $q = q^*$ in all meetings. A conclusion is that microfoundations matter: the two specifications fit the money demand and markup data equally well, but the implications for WC are very different. Also included, for comparison, is the much bigger WC at $\theta = 0.5$, even if it means the markup is too high.

Now compare the models with multiple assets and with one liquid asset M. As Table 9 and Figure 5 show, WC is *higher* with multiple assets. To explain this,

[18] The markup data is available at see https://www.census.gov/retail. It is considered reliable, in part because marginal cost is relatively easy to measure in retail by the wholesale price, and has been used in the related literature since Faig and Jerez (2005).

first, one has to be careful not to be misled by partial equilibrium reasoning: an individual may be better off with access to alternative liquid assets, since when the inflation tax is high one agent can substitute from M to A or K; but in equilibrium, if all agents do so, it leads to bigger distortions based on a basic principle of public finance, that it is more inefficient to tax things that are more elastic. Second, when M is the only payment instrument D gets recalibrated, upwards, making the CM more and DM less important, so inflation is less painful. This illustrates well the importance of modeling multiple liquid assets explicitly.

The size of DM trade relative to total output is not calibrated, but can be calculated by the formula

$$\frac{\sum_j \alpha_j p_j}{(1+n)x + \delta k + \sum_j \alpha_j p_j},$$

where p_j is the payment in a meeting of type j in terms of numeraire. When M is the only liquid asset, matching money demand pins down the relative size of the DM fairly precisely, so the variation is small across specifications. With A and K also liquid, matching money demand does not pin down the relative size of the DM, but the variation across specifications is still not too large. The DM is larger with more liquid assets, since there is more DM trade given the same money demand.

The take-away is that the welfare cost of inflation is higher with multiple liquid assets, and this is not mainly due to having a bigger DM. Our estimates of DM trade relative to total output, between 7% and 12%, are roughly in line with previous studies (e.g., 10% in Lagos-Wright 2005 or 5% in Aruoba et al. 2011), and hence differences from those studies are not due to huge differences in DM size.

As another application, consider Rogoff's (2017) proposal to completely eliminate currency. Of course, a more tempered version is to eliminate only some forms – e.g., large bills that tend to be used in untoward activities – but let us try the more extreme experiment for the sake of illustration. Whatever the benefits from this experiment, they should be weighed against the cost, which can be computed easily in the model by comparing equilibrium with $\chi_m = 1$ and $\chi_m = 0$ at the average ι

from the data. As shown in the last column of Table 9, for all specifications the WC of imposing $\chi_m = 0$ is big, and in some cases extremely big – e.g., almost 18% when M is the only liquid asset and we use symmetric Nash bargaining. WC changes to 10% when there are multiple liquid assets and we use the same bargaining solution, and changes further if we use Kalai rather than Nash or $\hat{\theta}$ rather than 0.5, but it is never below 5.0%, which is very big by the standards of the literature.

To be clear, the objective is to illustrate the method and show how microfoundations matter, not to provide a "final answer" to the cost-of-inflation or cost-of-eliminating-currency question. One reason is that a "final answer" will depend on other factors, like whether search is random or directed, whether there is entry or a fixed number of agents, and whether there is symmetric or asymmetric information (again see Bethune et al. 2020). Also, one might think liquidity as captured by the α's or χ's should be endogenous, although the net impact is not obvious, and so we take that up explicitly in the next exercise.

8.4 Endogenous Acceptability

Lester et al. (2012) endogenize α by assuming sellers have to pay a cost to become informed – i.e., to recognize the quality of some assets, and thus avoid inferior versions, including as an extreme counterfeits. They also assume buyers can produce low-quality – in fact, worthless – assets on the spot for free. Hence, buyers can always give worthless assets to uninformed sellers, so those sellers never accept anything they cannot recognize. A reasonable case has M recognized and accepted by everyone, but A and K only by sellers that pay a cost. In general, there is a distribution of this cost across sellers determining the fraction that accept those assets. It can be a one-time cost, or one that must be paid each period, and here we adopt the latter formulation.[19]

[19]Models like this can have multiple steady states. Intuitively, if more sellers pay to become informed and an asset is more acceptable in the DM; that raises its price in the CM; and that makes sellers more willing to pay the cost to accept it. We focus here on cases where either steady state is unique, or if it is not the economy is in the better steady state.

An implication is that after a change in the environment, including a change in monetary policy, a different fraction of sellers may be willing to incur the cost and hence acceptability can change. That allows A and K to partially step in for M if inflation goes up, or if we eliminate cash altogether. The net impact of a policy change therefore depends on how many sellers end up paying the cost, but while an increase in acceptability is generally good, the cost of information has to be taken into account in welfare calculations. The net impact is unclear; e.g., close to $\iota = 0$ the cost is a dead-weight loss, since at the Friedman rule M can fully satisfy liquidity demand when everyone accepts cash.

To implement this idea, we let all sellers costlessly recognize M, so $\alpha_a = \alpha_k = \alpha_{ak} = 0$, as in our baseline calibration. Then to keep things relatively simple, we assume α_{ma} and α_{mk} do not change – say, some sellers are endowed with the technology to recognize either A or K, but it is prohibitively costly for them to recognize the other asset. This leaves α_m and α_e to adjust endogenously: sellers either pay no cost and recognize/accept only M; or pay a cost that allows them to recognize both A and K. The idea here is that a lower bound for the cost can be estimated by the difference in profit between sellers that accept everything and those that accept only M.

Table 10: WC of $\chi_m = 0$

Bargaining	Estimated D	Exogenous α	Endogenous α
Kalai $\hat{\theta} = 0.842$	1.283	5.799%	2.454%
Nash $\hat{\theta} = 0.815$	0.793	7.766%	3.011%
Kalai $\theta = 0.5$	1.901	6.536%	4.180%
Nash $\theta = 0.5$	0.585	9.612%	4.889%

We first reconsider the WC of eliminating currency. Table 10 reports numbers with exogenous α, the same as the results in Table 9, plus new numbers given α_m and α_e adjust.[20] The interpretation is that all sellers pay the cost after the policy

[20]Endogenous acceptability is mainly relevant if there are multiple liquid assets, so we do not report results for economies where M is the only one. Also, we do not report results for BTA bargaining, since then sellers make zero profit and hence are unwilling to pay to accept an asset.

change, which can be seen as a lower bound on WC, while the case with exogenous α's can be seen as an upper bound. As can be seen in Table 10, the bounds are tight. As could have been anticipated, allowing α's to adjust lowers the cost of eliminating cash, but not by very much, in part due to the cost of sellers becoming informed.

Table 11: WC of $\mu = 0.1$

Bargaining	Estimated D	Exogenous α	$\alpha_e\|_{\mu=0.1} = 0.05$	$\alpha_e\|_{\mu=0.1} = 0.1$
Kalai $\hat{\theta} = 0.842$	1.283	1.516%	1.556%	1.371%
Nash $\hat{\theta} = 0.815$	0.793	3.378%	3.765%	3.800%
Kalai $\theta = 0.5$	1.901	3.764%	3.910%	3.945%
Nash $\theta = 0.5$	0.585	8.021%	9.054%	11.188%

Table 11 reports the results of a similar exercise for the WC of inflation. Things are more complicated here, given we are comparing economies at the Friedman rule and at $\mu = 0.1$, since at these values the α's differ from the baseline calibration. However, it is clear how the α's adjust at the Friedman rule: since all sellers costlessly accept M, there is no extra profit from accepting A and K, so $\alpha_m = 0.1$ while $\alpha_e = 0$. Things are less clear at $\mu = 0.1$, so we report WC for two extreme cases. In the column labeled $\alpha_e|_{\mu=0.1} = 0.05$, we assume the measure of sellers accepting everything remains unchanged relative to the baseline. Thus, the difference in results relative to exogenous α comes solely from the fact that at the Friedman rule no sellers pay the cost. In the column labeled $\alpha_e|_{\mu=0.1} = 0.1$, we instead assume that at 10% inflation all sellers who accept only M at the baseline now pay the cost to accept A and K, implying $\alpha_m = 0$ and $\alpha_e = 0.1$.

Notice from Table 11 that endogenizing the measure of sellers that accept everything does not necessarily lower the WC of inflation. The reason is as follows: when α_e increases while α_m decreases, this is good for buyers who otherwise would have an M only meeting; but it is bad for buyers in other meetings. With $\alpha_m = 0.05$, buyers carry substantial amounts of real money balances even at 10% inflation, since they may end up in a meeting where only M is accepted. If $\alpha_m = 0$, money is less important to buyers, so they demand lower z^m, and that reduces trade in meetings where M is also accepted, because the reduction in z^m is not fully offset by increases in z^a and z^k. Hence, even though endogenous acceptability could mitigate the cost of inflation, it may actually increase it through these GE effects.

9 Indirect Liquidity and OTC Trade

So far, A and K provide direct liquidity: buyers can trade them for q. Suppose they provide indirect liquidity: only M buys q, while agents trade A or K for M in an OTC market, called the OM, which convenes between the CM and DM. Gains from trade arise in the OM because a fraction ψ of q buyers learn after exiting the CM they will have a trade opportunity in the DM. In general this can mean they learn they have a desire or need for q, but it is still random if they meet a seller; here for simplicity assume they meet a seller for sure. Thus, a measure ψ of q buyers enter the OM trying to trade A or K for M, to get DM liquidity, and are called asset sellers, while a measure $1-\psi$ are asset buyers.

For simplicity, let asset sellers have all the bargaining power in the OM. Also, assume $F(h,k) = h + \varepsilon f(k)$ and set $\chi_j = 1\,\forall j$, mainly to reduce notation. Now note that there can be three types of meetings in the OM: with probability η_a an asset seller meets an asset buyer that will give up M for A; with probability η_k he meets one that will give up M for K; and with probability η_e he meets one that will give up M for A and K. The DM value function is $V(\zeta) = W(\zeta) + u(q) - p$, with p constrained by m. Therefore,

$$W_t(\zeta) = \Omega + \max_x \{U(x) - x\} + \max_{\hat{\zeta}} \{-\phi_t^m \hat{m} - \phi_t^a \hat{a} - \hat{k} + \psi\beta O_{t+1}(\hat{\zeta}) + (1-\psi)\beta W_{t+1}(\hat{\zeta})\}$$

where with probability ψ the continuation value is $O(\zeta)$, which is the OM value function for asset sellers, while with probability $1-\psi$ it is $W(\zeta)$, because asset buyers have 0 bargaining power and hence get 0 surplus from OM trade. Then

$$\begin{aligned} O(\zeta) \;=\;& \eta_a V(m+m_a, a-a_a, k) + \eta_k V(m+m_k, a, k-k_k) \\ &+ \eta_e V(m+m_e, a-a_e, k-k_e) + (1-\eta_a-\eta_k-\eta_e)\, V(\zeta), \end{aligned} \tag{34}$$

where m_j, a_j and k_j denote the amounts of M, A and K exchanged in type j meetings.

In OM meetings between an asset seller with (m, a, k) and an asset buyer with (M, A, K) there are constraints $a_a, a_e \leq a$, $k_k, k_e \leq k$ and $m_a, m_k, m_e \leq M$. Different configurations arise depending which of these binds, as detailed in Appendix A7. In any case, since asset sellers have all the bargaining power in the OM, the terms of trade are given by $(\phi^a + \rho)a_a = \phi^m m_a$, $(1 + \kappa - \rho)k_k = \phi^m m_k$ and $(\phi^a + \rho)a_e + (1 + \kappa - \rho)k_e = \phi^m m_e$, which says asset sellers convert illiquid a or k into liquid m according to their CM values.

The general Euler equations are in Appendix A7; here we focus on cases where OM trade is not constrained by asset buyers' money holdings, which means z^m is not too small and holds as long as ι is not too big. Then the M equation is

$$\begin{aligned} z_t^m \;\; = \;\; & \frac{\beta z_{t+1}^m}{1+\mu} \left[1 + \psi\eta_a L(z_{t+1}^{ma}) + \psi\eta_k L(z_{t+1}^{mk}) + \psi\eta_e L(z_{t+1}^{e}) \right. \\ & \left. + \psi(1 - \eta_a - \eta_k - \eta_e) L(z_{t+1}^m)\right]. \end{aligned} \tag{35}$$

Notice the argument of $L\left(z_{t+1}^{ma}\right)$, which is liquidity in the baseline model in DM meetings where q sellers accept both M and A. This reflects the fact that, while DM sellers now only take cash, DM buyers have m from the CM plus m_a from the OM, and the total gets the same q they get in the baseline model in ma meetings. The story is similar for the other terms in (35).

The other Euler equations are

$$z_t^a = \rho + \beta z_{t+1}^a \left[1 + \psi\eta_a L(z_{t+1}^{ma}) + \psi\eta_e L(z_{t+1}^{e})\right] \tag{36}$$

$$1 = \beta(1 + \kappa_{t+1} - \delta) \left[1 + \psi\eta_k L(z_{t+1}^{mk}) + \psi\eta_e L(z_{t+1}^{e})\right]. \tag{37}$$

Notice that (35)-(37) are *exactly the same* as the conditions in the baseline model, where a and k directly trade for q, if we set $\alpha_{ma} = \psi\eta_a$, $\alpha_{mk} = \psi\eta_k$, $\alpha_m = \psi(1 - \eta_a - \eta_k - \eta_e)$ and $\alpha_a = \alpha_k = \alpha_{ak} = 0$. It is no coincidence that the α's not involving M are all zero, since to replicate the indirect liquidity model M must be accepted in all meetings, but that is not a drastic assumption.

Table 12: Comparative Statics with OTC Trade.

	z^m	z^a	z^k	z^{ma}	z^{mk}	z^{ak}	z^e	r^m	r^a	r^k
ι	$-$	$+$	$+$	$-$	$-$	$+$	$-^1$	$-$	$-$	$-$
ρ	$-$	$+$	$-^1$	$+$	$-$	$+$	$+$	0	$+$	$+^1$
ε	$-$	$-^1$	$+$	$-$	$+$	$+$	$+$	0	$+^1$	$+$
η_a	$-$	$+$	?	?	?	$+$	?	0	$-$	?
η_k	$-$	?	$+$	?	?	$+$	?	0	?	$-$
η_e	$-$	$+$	$+$	?	?	$+$	?	0	$-$	$-$

Notes: 1 $\Rightarrow$ holds if $\ell_{ma}\ell_{mk}$= 0.

Given these α's, suppose $m_j \leq M$ is slack. Then the results are the same as the baseline model with α's as indicated above. Table 12 reports the result of changes in ι, ρ, ε and η's, where to be clear, this is not a generalization of the results in Table 1, but a specialization to $\alpha_a = \alpha_k = \alpha_{ak} = 0$, which eliminates some ambiguities (e.g., higher ι now must raise both z^a and z^k). Further, going beyond comparative statics, we can obtain similar results on dynamics as above, and in particular there are cycles with various kinds of fluctuations. Further still, we can apply quantitative welfare results like those derived above directly to OTC markets.

For more intuition, consider A (similar points apply to K). In the baseline model, A is valued for its return augmented by a liquidity premium: with probability α_{ma} you are a DM buyer that meets a seller who takes M and A, and there liquidity is worth $L(z^{ma})$. In the model with indirect liquidity, A's return is also augmented by a liquidity premium, even if now you cannot spend a on q directly, since with probability $\psi\eta_a$ you are a q buyer in the next DM and can trade a for m_a in the OM. Since using $m + m_a$ in the DM conveys the same benefit as using m plus a in the baseline model, it is *as if* the q seller accepts a.

10 Conclusion

This paper studied economies with multiple assets that are more or less substitutes for payment purposes in decentralized exchange. This can be important even if consumers mostly use M, or claims on M in their banks, to buy goods, given that firms and financial institutions regularly use A, K and B (bonds as in Appendix A1) to facilitate trade. Also – and here the formulation with indirect liquidity is especially relevant – before buying q consumers often need to sell other assets to get M, and we showed that can make it *as if* they use other assets to get q.

We think the approach should be useful not only for monetary theorists, but for anyone interested in asset markets and liquidity. In particular, knowing how assets facilitate payments and how that impacts the macro economy seems especially topical with intriguing new instruments looming on the horizon, if not already in play, including e-monies like Bitcoin, and Central Bank Digital Currency. While we did not discuss e-money explicitly, it should be clear how our methods apply.

Instead of featuring one main finding or theorem, we characterized various properties of the framework analytically or numerically. Classic issues revisited include: Mundell-Tobin effects of inflation on investment and the stock market; Wallace's view on rate of return distributions; and Lucas' message on the perils of policy prescriptions based on correlations. Novel results on endogenous dynamics literature include the coexistence of dynamic equilibria with very different volatility and correlation patterns. An overall conclusion is that rich dynamics can emerge especially when there are multiple liquid assets.

Another implication is that the *composition* of liquid assets in portfolios matters, not just some aggregate level of liquidity. In terms of policy, we showed that a nominal interest rate peg eliminates endogenous fluctuations if M is the only liquid asset, but not with multiple liquid assets. We also showed the cost of inflation is higher with multiple liquid assets, and analyzed the suggestion to eliminate cash altogether. And we illustrated how the impact of policies can be affected by endogenizing acceptability. There is much more that can be done on models with multiple liquid assets by those working on monetary and financial economics, and as such the framework constitutes a promising area for future research.

Appendix

A1. Finite Assets and Bonds: Suppose A now yields ρ for N of periods, where the text has $N = \infty$. Here we assume $N = 1$, but any $N < \infty$ can be analyzed, with similar results, using backward induction. Moreover, A is now equivalent to a bond (and actually similar to storage in the benchmark model, i.e., to K with technology separable in k and h). Framing the discussion that way, assume B one-period bonds are issued by government in every CM, paying 1 unit of x in the next CM, with any fiscal implications offset by T, and for simplicity assume no capital. The CM problem now has budget equation $x = h - \phi^m(\hat{m} - m) - \phi^b \hat{b} + b + T$. Note the value of b maturing bonds is simply b, as there is no resale value the way there is with a in the text (which is what makes B similar to storage).

The usual methods lead to

$$\begin{aligned} z_t^m &= \frac{\beta z_{t+1}^m}{1+\mu}[1 + \alpha_m \chi_m L(z_{t+1}^m) + \alpha_{mb}\chi_m L(z_{t+1}^m + \chi_b B)] \\ \phi_t^b &= \beta[1 + \alpha_b \chi_b L(\chi_b B) + \alpha_{mb}\chi_b L(z_{t+1}^m + \chi_b B)], \end{aligned}$$

using bond market clearing, $\hat{b} = B$. If $\alpha_{mb} = 0$, $\phi_t^b = \beta[1 + \alpha_b \chi_b L(\chi_b B)]$, so while it can exceed its fundamental price, there are no dynamics in ϕ_t^b. But if $\alpha_{mb} > 0$ then $z_t^m = \Phi(z_{t+1}^m)$, where

$$\Phi(z^m) = \frac{\beta z^m}{1+\mu}[1 + \alpha_m \chi_m L(z^m) + \alpha_{mb}\chi_m L(z^m + \chi_b B)].$$

If $\Phi'(z_s^m) < -1$ there is a 2-cycle in z^m. If z^m cycles then so must ϕ_t^b.

A2. Deriving the Euler Equations: Since $z^j/\omega = v(q^j)$, we have $\partial q^j/\partial m = \chi_m \phi^m / \omega v'(q^j)$ for each $j \in \mathcal{S}_m$. Then differentiate $V(\zeta)$ wrt m to get

$$\begin{aligned} \frac{\partial V(\zeta)}{\partial m} &= \frac{\phi^m}{\omega} + \sum_{j \in \mathcal{S}_m} \alpha_j \left\{ u'[q^j(z^j/\omega)] \frac{\partial q^j(z^j/\omega)}{\partial m} - \frac{\chi_m \phi^m}{\omega} \right\} \\ &= \frac{\phi^m}{\omega} + \frac{\chi_m \phi^m}{\omega} \sum_{j \in \mathcal{S}_m} \alpha_j \left\{ \frac{u'[q^j(z^j/\omega)]}{v'[q^j(z^j/\omega)]} - 1 \right\}. \end{aligned}$$

Insert this into (4), being careful with the t subscripts, to get

$$\frac{\phi_t^m}{\omega_t} = \beta \frac{\phi_{t+1}^m}{\omega_{t+1}} \left[1 + \chi_m \sum_{j \in \mathcal{S}_m} \alpha_j L\left(\frac{z_{t+1}^j}{\omega_{t+1}} \right) \right].$$

Finally use $U'(x_t) = 1/\omega_t$ to rewrite this in terms of $\tilde{z}^m = \chi_m \phi^m m/\omega$, which yields (9). The procedures for z^a and z^k are similar and hence omitted.

A3. More on Comparative Statics: Here are the results for the α's.

Table 13: Effects of Changes in α's.

	z^m	z^a	z^k	z^{ma}	z^{mk}	z^{ak}	z^e	r^m	r^a	r^k
α_m	+	−[3]	−[2]	+	+	−	+[1]	0	+[3]	+[2]
α_a	−[3]	+	−[1]	+	−	+	+[2]	0	−	+[1]
α_k	−[2]	−[1]	+	−	+	+	+[3]	0	+[1]	−
α_{ma}	+	+	−	+	?	?	+	0	−	+
α_{mk}	+	−	+	?	+	?	+	0	+	−
α_{ak}	−	+	+	?	?	+	+	0	−	−
α_e	+[1]	+[2]	+[3]	+	+	+	+	0	−[2]	−[3]

Notes: 1 $\Rightarrow$ holds if $\ell_{ma}\ell_{mk} = 0$; 2 $\Rightarrow$ holds if $\ell_{ma}\ell_{ak} = 0$; 3 $\Rightarrow$ holds if $\ell_{mk}\ell_{ak} = 0$.

Other results can be investigated, including the effects of the wage $\omega = C$ when $F(k,h) = Ch + \varepsilon f(k)$. This turns out to be quite complicated, which is why it is useful to eliminate GE wage effects for the other comparative statics. For details of all these derivations, go to the online Appendix at:

`https://drive.google.com/file/d/18rnSFSIMcI8xLpo_7Y7S3tGX9PALyYVD/view`

Additionally, we want to investigate the elasticity of returns with respect to inflation, as mentioned in fn. 11. Consider the case where $\alpha_j = 0$ for each $j \neq e$ and $\alpha_e > 0$. Then,

$$\frac{\partial r^a}{\partial \iota} = -\frac{\chi_a(1+r^a)^2}{2\chi_m(1+r)} < 0.$$

After some algebra, the elasticity is

$$E_{r^a,\iota} \equiv \frac{\partial r^a}{\partial \iota}\frac{1+\iota}{1+r^a} = -\frac{(1+\iota)\chi_a(1+r^a)}{2\chi_m(1+r)}.$$

Hence,

$$\frac{\partial E_{r^a,\iota}}{\partial \chi_a} = -\frac{(1+\iota)(1+r^a)^2(2+r)}{4\chi_m(1+r)^2} < 0,$$

which says that when a real asset is a better substitute for money – i.e., a better payment instrument in the sense that χ_a is larger – the elasticity of its return with respect to inflation is greater in absolute value (i.e., more negative).

A4. CES Technology: To show that for cycles we do not need F separable and linear in h, consider

$$F(h,k) = (1+\varepsilon)\left(\frac{1}{1+\varepsilon}h^{\sigma} + \frac{\varepsilon}{1+\varepsilon}k^{\sigma}\right)^{\frac{1}{\sigma}}.$$

Here we need CM utility, and use $U(x) = D[\ln(x+\varsigma) - \ln(\varsigma)]$. Parameters are in Table 7 plus $\sigma = 0.999$ and $D = 40.5$. The following is a 2-cycle:

$$\begin{aligned}
\mathbf{z_1} &= (0.6239, 0.3185, 0.0447) \text{ and } \mathbf{z_2} = (0.6067, 0.3517, 0.0287) \\
\mathbf{z_1} &= (0.6082, 0.3242, 0.0547) \text{ and } \mathbf{z_2} = (0.6217, 0.3425, 0.0229) \\
\mathbf{z_1} &= (0.5961, 0.3302, 0.0607) \text{ and } \mathbf{z_2} = (0.6375, 0.3353, 0.0144) \\
\mathbf{z_1} &= (0.5985, 0.3442, 0.0443) \text{ and } \mathbf{z_2} = (0.6349, 0.3234, 0.0287).
\end{aligned}$$

While $\sigma = 0.999$ makes this example close linear, it still verifies the point.

A5. Verifying $\Psi'(z_s^m) < -1$: With the functional forms in (12), we have

$$\Psi(z^m) = \frac{\beta z^m}{1+\mu}\left\{1 + \alpha_m \chi_m\left[(z^m+\varsigma)^{-\gamma} - 1\right] + \frac{\chi_m}{\chi_k}\left[\frac{1}{\beta(1+\varepsilon-\delta)} - 1\right]\right\}.$$

Hence,

$$\Psi'(z^m) = \frac{\beta}{1+\mu}\left\{1 + \alpha_m \chi_m\left[\frac{(1-\gamma)z^m+\varsigma}{(z^m+\varsigma)^{1+\gamma}} - 1\right] + \frac{\chi_m}{\chi_k}\left[\frac{1}{\beta(1+\varepsilon-\delta)} - 1\right]\right\}.$$

In MSS

$$1 = \frac{\beta}{1+\mu}\left\{1 + \alpha_m \chi_m\left[(z_s^m+\varsigma)^{-\gamma} - 1\right] + \frac{\chi_m}{\chi_k}\left[\frac{1}{\beta(1+\varepsilon-\delta)} - 1\right]\right\}.$$

The existence of MSS requires

$$\frac{\beta}{1+\mu}\left\{1 + \frac{\chi_m}{\chi_k}\left[\frac{1}{\beta(1+\varepsilon-\delta)} - 1\right]\right\} < 1.$$

Then

$$(z_s^m+\varsigma)^{-\gamma} = 1 + \frac{1}{\alpha_m\chi_m}\left\{\frac{1+\mu}{\beta} - 1 - \frac{\chi_m}{\chi_k}\left[\frac{1}{\beta(1+\varepsilon-\delta)} - 1\right]\right\} > 1.$$

Hence, $z_s^m + \varsigma < 1$ and $\Psi'(z_s^m) < -1$ for large γ.

A6. Three Liquid Assets with Fixed ι: For simplicity, set $\alpha_{ak} = \alpha_e = 0$, $\chi_m = \chi_a = \chi_k = 1$ and $F(h,k) = h + \varepsilon k$. Then the system is

$$\begin{aligned}
\iota &= \alpha_m L(z^m_{t+1}) + \alpha_{ma} L(z^m_{t+1} + z^a_{t+1}) + \alpha_{mk} L(z^m_{t+1} + z^k_{t+1}) \\
z^a_t &= \rho + \beta z^a_{t+1}\left[1 + \alpha_a L\left(z^a_{t+1}\right) + \alpha_{ma} L(z^m_{t+1} + z^a_{t+1})\right] \\
1 &= \beta(1+\varepsilon-\delta)\left[1 + \alpha_k L(z^k_{t+1}) + \alpha_{mk} L\left(z^m_{t+1} + z^k_{t+1}\right)\right].
\end{aligned}$$

Set $\alpha_k = 0$ and rearrange the k equation as

$$\alpha_{mk} L(z^m_{t+1} + z^k_{t+1}) = \frac{1}{\beta(1+\varepsilon-\delta)} - 1.$$

Use this to write the m equation as

$$\iota = \alpha_m L(z^m_{t+1}) + \alpha_{ma} L(z^m_{t+1} + z^a_{t+1}) + \frac{1}{\beta(1+\varepsilon-\delta)} - 1.$$

Suppose $\alpha_{ma} > 0$. Then, as in the model with only m and a liquid, if $\alpha_m > 0$ or $\alpha_a > 0$ there is a cycle where z^m and z^a fluctuate, and now z^k also fluctuates. An example is in Table 14. The steady state is $\left(z^m_s, z^a_s, z^k_s\right) = (0.0487, 0.9410, 0.9244)$, and there is a 2-cycle with $\left(z^m_1, z^a_1, z^k_1\right) =$ (0.0103,0.9795,0.9628) and $\left(z^m_2, z^a_2, z^k_2\right) =$ (0.0811,0.9087,0.8921).

Table 14: Parameter Values for Three Assets with ι Fixed.

β	ς	γ	α_m	α_a	α_{ma}	α_{mk}	ι	ρ	ε	δ
0.9	0.01	5.65	0	0.3	0.6	0.1	0.011	0.01	0.2	0.1

A7. More on Indirect Liquidity: Deriving the Euler equations is similar to but not the same as in the baseline model. First, from (34) we derive

$$\begin{aligned}
\frac{\partial O}{\partial m} &= \eta_a \left[\frac{\partial V^a}{\partial m}\left(1 + \frac{\partial m_a}{\partial m}\right) - \frac{\partial V^a}{\partial a}\frac{\partial a_a}{\partial m}\right] + \eta_k \left[\frac{\partial V^k}{\partial m}\left(1 + \frac{\partial m_k}{\partial m}\right) - \frac{\partial V^k}{\partial k}\frac{\partial k_k}{\partial m}\right] \\
&\quad + \eta_e \left[\frac{\partial V^e}{\partial m}\left(1 + \frac{\partial m_e}{\partial m}\right) - \frac{\partial V^e}{\partial a}\frac{\partial a_e}{\partial m} - \frac{\partial V^e}{\partial k}\frac{\partial k_e}{\partial m}\right] + (1-\eta_a-\eta_k-\eta_e)\frac{\partial V^0}{\partial m}
\end{aligned}$$

where the superscript on V^a indicates V is evaluated in the DM at the portfolio after the agent meets an asset buyer in the OM who accepts a, and similarly for V^k and V^e, while V^0 is evaluated at the portfolio after having no OM meeting. The next step is to substitute into this expression the derivatives of V^j as well as the derivatives of $m_a, a_a, ...$ wrt m.

As noted in the text, since the asset seller has all the bargaining power in the OM, $(\phi^a+\rho)a_a = \phi^m m_a$, $(1+\kappa-\rho)k_k = \phi^m m_k$ and $(\phi^a+\rho)a_e+(1+\kappa-\rho)k_e = \phi^m m_e$. Also $V(\zeta) = W(\zeta) + u(q) - p$, and DM bargaining implies $v(q^{ma}) = \phi^m(m + m_a)$, $v(q^{mk}) = \phi^m(m + m_k)$, $v(q^e) = \phi^m(m + m_e)$, and $v(q^m) = \phi^m m$. Together these imply the derivatives wrt a and k are

$$\frac{\partial V^j}{\partial a} = \frac{\partial W}{\partial a} = \phi^a + \rho \text{ and } \frac{\partial V^j}{\partial k} = \frac{\partial W}{\partial k} = 1 + \kappa - \delta, \text{ for } j = a, k, e, 0,$$

the derivatives wrt m are

$$\frac{\partial V^a}{\partial m} = \frac{\partial W}{\partial m} + u'(q^{ma})\frac{\partial q^{ma}}{\partial m} - \phi^m = \frac{\phi^m u'(q^{ma})}{v'(q^{ma})},$$

and similarly for the remaining $\partial V^j/\partial m$.

We can insert $\partial V^j/\partial m$ into $\partial O/\partial m$ and insert that into the FOC for $\hat{m}$ from the CM, $\phi_t^m = \beta\psi\partial O_{t+1}/\partial\hat{m} + (1-\psi)\beta\partial W_{t+1}/\partial\hat{m}$. The result is

$$\begin{aligned}
\phi_t^m = \beta\phi_{t+1}^m\Bigg\{ & 1 + \psi\eta_a\left[\frac{u'(q_{t+1}^{ma})}{v'(q_{t+1}^{ma})}\left(1 + \frac{\partial m_a}{\partial m}\right) - 1 - \frac{\phi_{t+1}^a + \rho}{\phi_{t+1}^m}\frac{\partial a_a}{\partial m}\right] \\
& + \psi\eta_k\left[\frac{u'(q_{t+1}^{mk})}{v'(q_{t+1}^{mk})}\left(1 + \frac{\partial m_k}{\partial m}\right) - 1 - \frac{1 + \kappa_{t+1} - \delta}{\phi_{t+1}^m}\frac{\partial k_k}{\partial m}\right] \\
& + \psi\eta_e\left[\frac{u'(q_{t+1}^{e})}{v'(q_{t+1}^{e})}\left(1 + \frac{\partial m_e}{\partial m}\right) - 1 - \frac{\phi_{t+1}^a + \rho}{\phi_{t+1}^m}\frac{\partial a_e}{\partial m} - \frac{1 + \kappa_{t+1} - \delta}{\phi_{t+1}^m}\frac{\partial k_e}{\partial m}\right] \\
& + \psi(1 - \eta_a - \eta_k - \eta_e)\left[\frac{u'(q_{t+1}^{m})}{v'(q_{t+1}^{m})} - 1\right]\Bigg\}.
\end{aligned}$$

Similarly, for a and k, the results are

$$\begin{aligned}
\phi_t^a = \beta(\phi_{t+1}^a + \rho)\Bigg\{ & 1 + \psi\eta_a\left[\frac{u'(q_{t+1}^{ma})}{v'(q_{t+1}^{ma})}\frac{\phi_{t+1}^m}{\phi_{t+1}^a + \rho}\frac{\partial m_a}{\partial a} - \frac{\partial a_a}{\partial a}\right] \\
& + \psi\eta_e\left[\frac{u'(q_{t+1}^{e})}{v'(q_{t+1}^{e})}\frac{\phi_{t+1}^m}{\phi_{t+1}^a + \rho}\frac{\partial m_e}{\partial a} - \frac{\partial a_e}{\partial a} - \frac{1 + \kappa_{t+1} - \delta}{\phi_{t+1}^a + \rho}\frac{\partial k_e}{\partial a}\right]\Bigg\} \\
1 = \beta(1 + \kappa_{t+1} - \delta)\Bigg\{ & 1 + \psi\eta_k\left[\frac{u'(q_{t+1}^{mk})}{v'(q_{t+1}^{mk})}\frac{\phi_{t+1}^m}{1 + \kappa_{t+1} - \delta}\frac{\partial m_k}{\partial k} - \frac{\partial k_k}{\partial k}\right] \\
& + \psi\eta_e\left[\frac{u'(q_{t+1}^{e})}{v'(q_{t+1}^{e})}\frac{\phi_{t+1}^m}{1 + \kappa_{t+1} - \delta}\frac{\partial m_e}{\partial k} - \frac{\phi_{t+1}^a}{1 + \kappa_{t+1} - \delta}\frac{\partial a_e}{\partial k} - \frac{\partial k_e}{\partial k}\right]\Bigg\}.
\end{aligned}$$

As we did in Appendix A2 for the baseline model, we can rewrite these in terms of $\tilde{z}^m$, $\tilde{z}^a$ and $\tilde{z}^k$.

This yields the general form of the Euler equations, although we still need to insert $\partial m_a/\partial m$, $\partial a_a/\partial m$, ... $\partial k_e/\partial k$. These will be different depending on which OM constraints bind. In each type j meeting in the OM, $j = a, k, e$, there are in principle several cases, since the asset seller's asset holdings may or may not bind, and the same for the asset buyer's money holdings. Not all combinations are possible – e.g., if type a and k meetings are unconstrained by a and k then type e meetings are too – but there are still many possibilities. In any case, we now discuss how some of the derivatives in the above equations vanish.

Consider the case where in all OM meetings asset buyers' and asset sellers' constraints are slack. To describe this, let $\hat{m}$ solve $v(\hat{q}) = \phi^m \hat{m}$, where $\hat{q}$ maximizes the buyers' surplus from DM trade (e.g., $\hat{q} = q^*$ for Kalai bargaining, but $\hat{q} < q^*$ for Nash with $\theta < 1$). Then in type a meetings $m_a = \hat{m} - m$ and $a_a = \phi^m(\hat{m} - m)/(\phi^a + \rho)$, so the only nonzero derivatives in the Euler equations are

$$\frac{\partial m_a}{\partial m} = -1; \; \frac{\partial a_a}{\partial m} = -\frac{\phi^m}{\phi^a + \rho}.$$

In type k meetings, $m_k = \hat{m} - m$ and $k_k = \phi^m(\hat{m} - m)/(1 + \kappa - \delta)$, so the nonzero derivatives are

$$\frac{\partial m_k}{\partial m} = -1; \; \frac{\partial k_k}{\partial m} = -\frac{\phi^m}{1 + \kappa - \delta}.$$

In type e meetings, $m_e = \hat{m} - m$, $a_e = \phi^m(\hat{m} - m)/(\phi^a + \rho) - (1 + \kappa - \delta)k_e/(\phi^a + \rho)$, and $k_e = \phi^m(\hat{m} - m)/(1 + \kappa - \delta) - (\phi^a + \rho)a_e/(1 + \kappa - \delta)$, so the nonzero derivatives are

$$\frac{\partial m_e}{\partial m} = -1; \; \frac{\partial a_e}{\partial m} = -\frac{\phi^m}{\phi^a + \rho}; \; \frac{\partial k_e}{\partial m} = -\frac{\phi^m}{1 + \kappa - \delta}.$$

Next consider cases where OM trade is constrained by M. If type a meetings are M constrained, $m_a = M$, so $a_a = \phi^m M/(\phi^a + \rho)$; if type k meetings are M constrained, $m_k = M$, $k_k = \phi^m M/(1 + \kappa - \delta)$; and if type e meetings are M constrained, $m_e = M$, $a_e = \phi^m M/(\phi^a + \rho) - (1 + \kappa - \delta)k_e/(\phi^a + \rho)$, and $k_e = \phi^m M/(1 + \kappa - \delta) - (\phi^a + \rho)a_e/(1 + \kappa - \delta)$. Thus, all derivatives in the Euler equations vanish.

Now consider cases where OM trade is constrained by a or k. If type a meetings are a constrained, $a_a = a$, $m_a = (\phi^a + \rho)a/\phi^m$, so the nonzero derivatives are

$$\frac{\partial m_a}{\partial a} = \frac{\phi^a + \rho}{\phi^m}; \ \frac{\partial a_a}{\partial a} = 1.$$

If type k meetings are k constrained, $k_k = k$, $m_k = (1+\kappa-\delta)k/\phi^m$, and the nonzero derivatives are

$$\frac{\partial m_k}{\partial k} = \frac{1+\kappa-\delta}{\phi^m}; \ \frac{\partial k_k}{\partial k} = 1.$$

Finally, if type e meetings are $a+k$ constrained, $a_e = a$, $k_e = k$, and $m_e = (\phi^a + \rho)a/\phi^m + (1+\kappa-\delta)k/\phi^m$, and the nonzero derivatives are

$$\frac{\partial m_e}{\partial a} = \frac{\phi^a + \rho}{\phi^m}; \ \frac{\partial m_e}{\partial k} = \frac{1+\kappa-\delta}{\phi^m}; \ \frac{\partial a_e}{\partial a} = \frac{\partial k_e}{\partial k} = 1.$$

Substituting these derivatives into the Euler equations yields the result for all possible cases.

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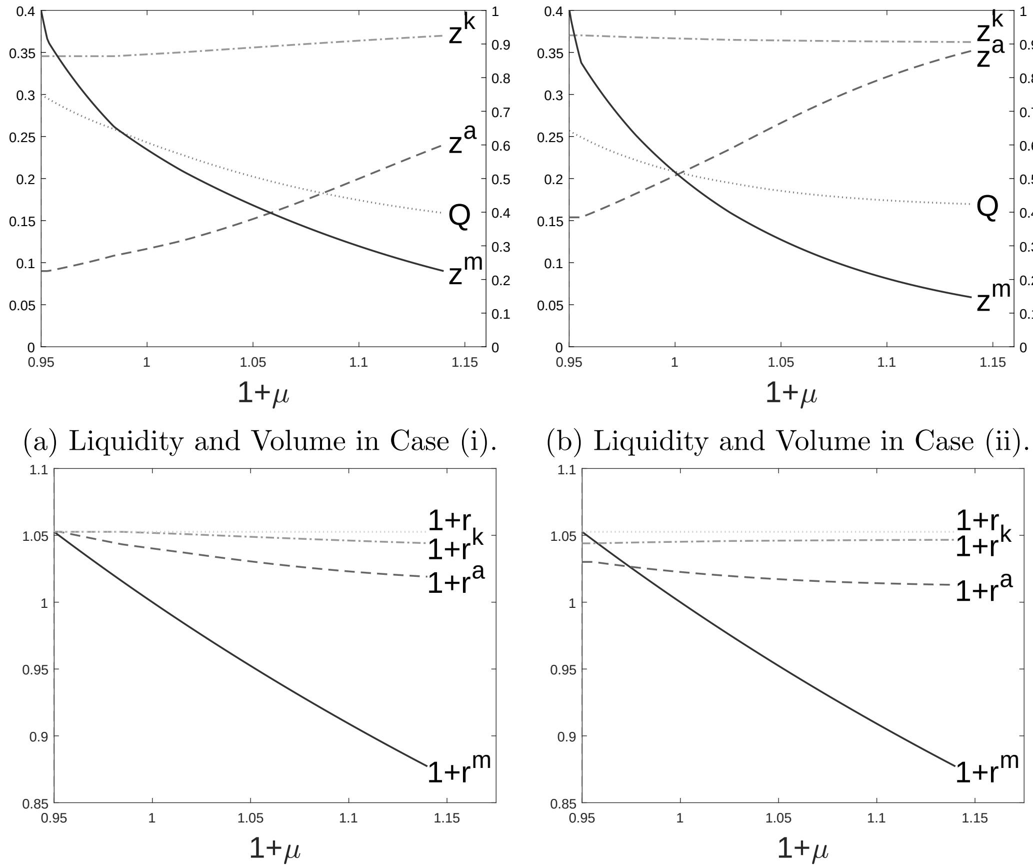


(a) Liquidity and Volume in Case (i).

(b) Liquidity and Volume in Case (ii).

(c) Returns in Case (i).

(d) Returns in Case (ii).

Figure 1: The Effects of Changing μ.

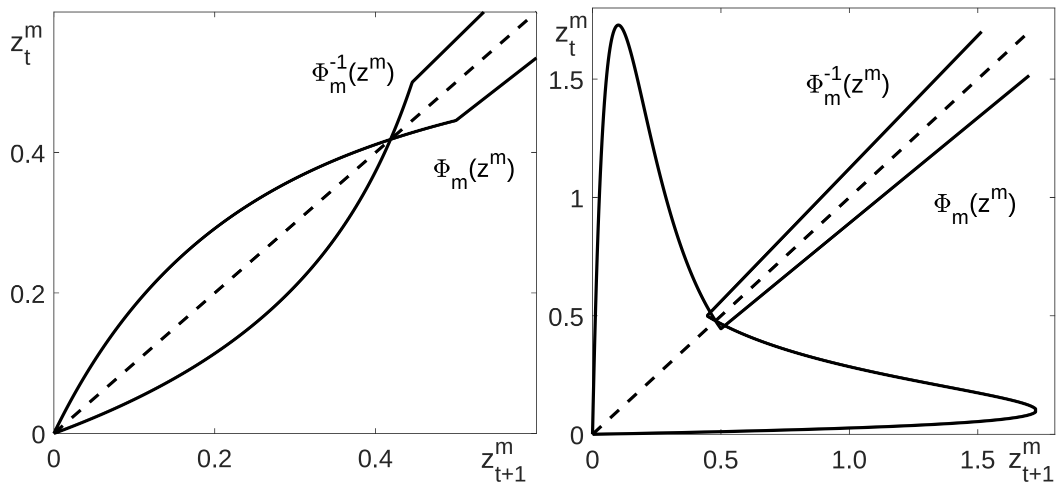


Figure 2: (a) Monotone Φ and (b) Nonmonotone Φ.

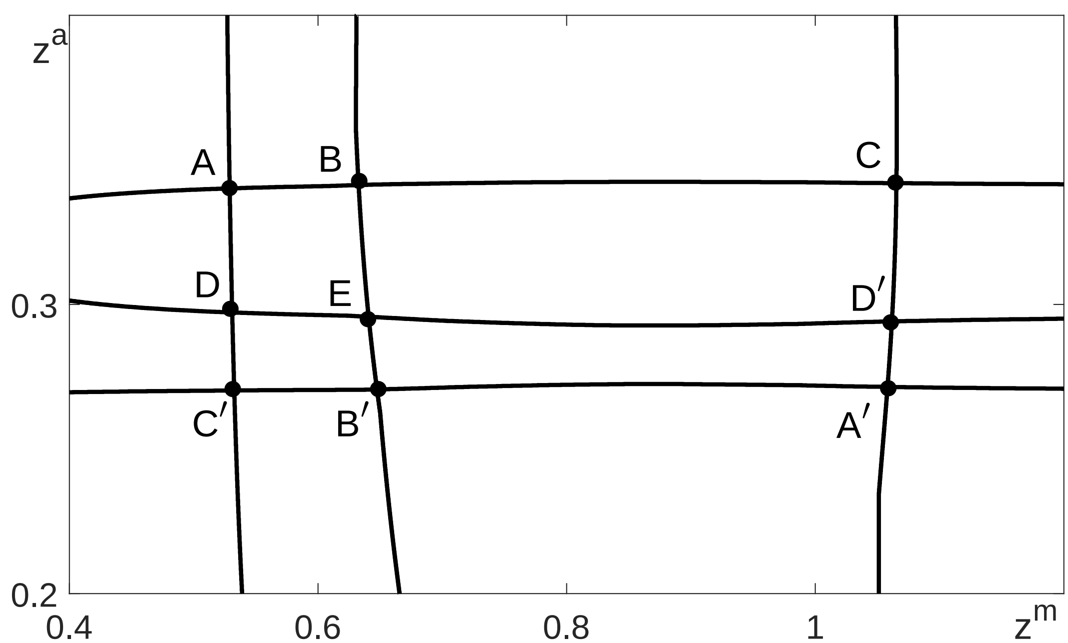


Figure 3: Different 2-Cycles when m and a Interact.

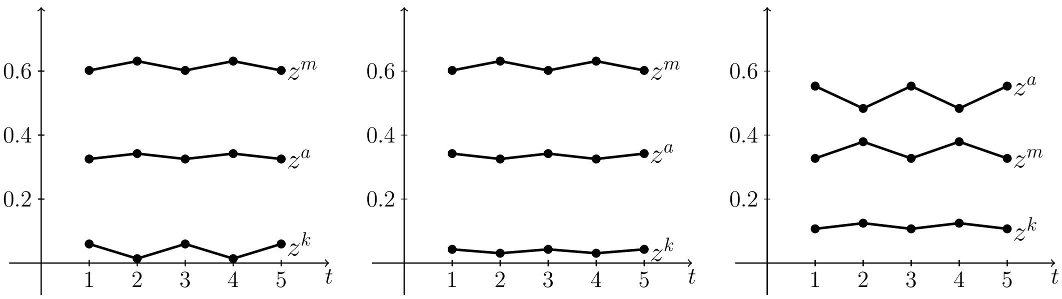
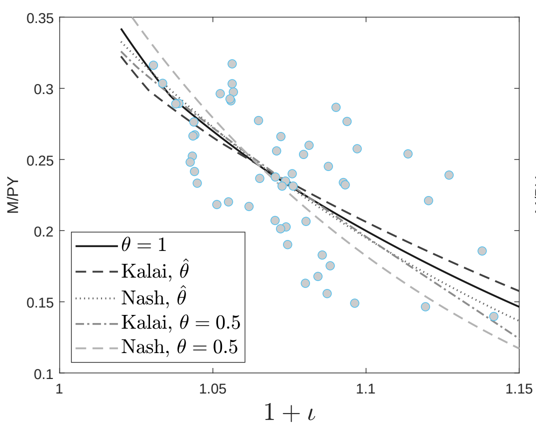
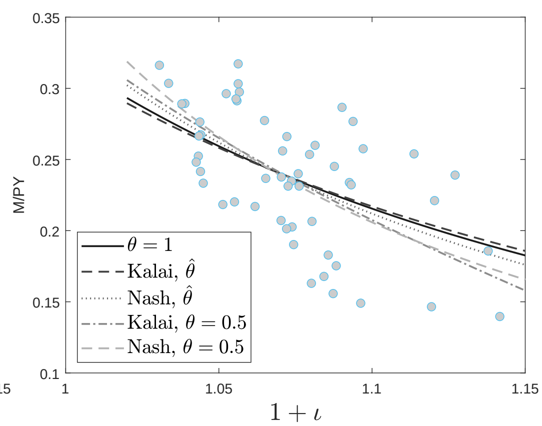


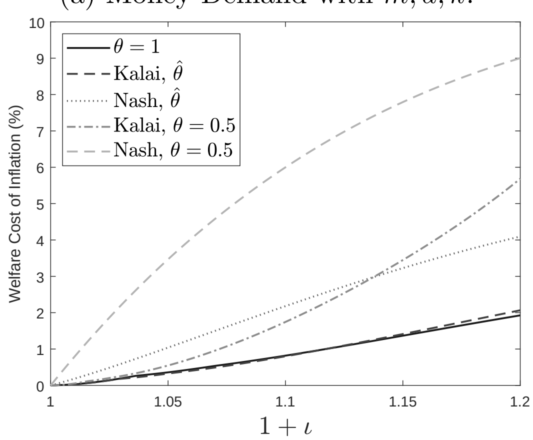
Figure 4: Cycles with Different Patterns.



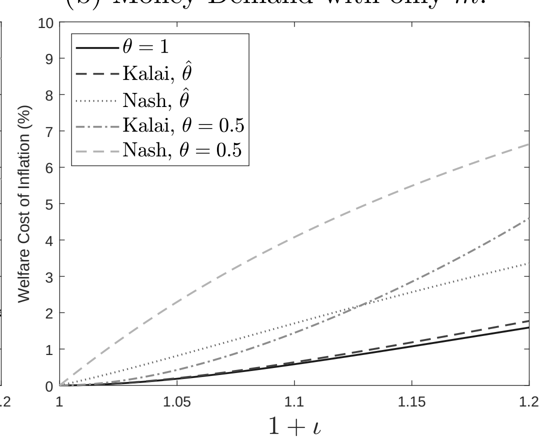
(a) Money Demand with m, a, k.



(b) Money Demand with only m.



(c) Welfare Cost with m, a, k.



(d) Welfare Cost with only m.

Figure 5: Money Demand and Welfare Cost of Inflation.