First-Price Auctions with Speculative Resale  
Part I: Optimal Revenue  
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Abstract  
In this paper, we investigate the model of speculative resale in auctions. Resale (or secondary trade) is allowed for each bidder, but in equilibrium we show that there is only speculative resale. We consider a standard first-price auction in the first stage with symmetric independent private values (IPV) among N bidders and several speculators. In the second stage there is resale among the bidders. The winner in the first stage auction uses an optimal mechanism to sell the object to the losing bidders. We establish a supermodularity property without assuming monotonicity or symmetry in bidding. We show the equilibrium bidding function of the regular bidders to be symmetric and increasing. We give a simple computable equilibrium solution and prove it to be unique. The Myerson (1981) revenue formula is extended to our model with speculative resale. Revenue monotonicity holds, and the Bulow-Klemperer (1996) argument favoring participation rather than the choice of optimal reservation price is also valid here with even greater force. Speculators’ active participation enhances the seller revenue, but for optimal revenue, the seller should prevent speculative resale by setting a sufficiently high reservation price. Thus first-price auction provides an implementation of optimal auctions with a unique equilibrium, while allowing speculative resale.

1 Introduction  
Whenever there is asymmetry between bidders, a winner of an auction will be interested in resale as the allocation is often inefficient. When we allow resale, many interesting phenomena occur. For example, if you start with private value auctions, but allow resale, then it introduces value dependence between the bidders as the resale revenue is interdependent. This means that the standard benchmark model with private values no longer serves as a good guidance in the analysis of auctions. In auctions with resale, the typical case involves a mixture of private-value and common-value elements. The dependence between valuation among the bidders introduced by resale is probably the most interesting and important type of value dependence in the real world. However it has not received sufficient attention in the literature. Hazlett and Oh (forthcoming) have argued that secondary trade or resale is more important than exact regulation on spectrum rights to avoid the inefficiency from harmful interference. Some of the recent discussions of incentive auctions (Hazlett, Porter and Smith (2012)) have information revelation issues that are also relevant in equilibrium with secondary trade. Resale also is important in explaining many problems of information aggregation or herding behavior when valuation becomes dependent through resale.

Consideration of resale impact on auctions has its practical importance and relevance in policy making. Many spectrum auctions held across the world prohibit resale after a bidder wins the auction. Such restriction

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on bidder behavior has poor theoretical basis or grounding. Furthermore such restrictions can be easily circumvented by selling the shares or the whole company owning the license rather than license itself. This has indeed happened. In the 2000 UK 3G auction, resale was prohibited. Orange won the license E, and was quickly sold to France Telecom. The most valuable license A was won by TIW. Within three months, Hutchison then sold 35% of its share in TIW to KPN and NTT DoCoMo, making billions of pounds in the process. Similarly, after the European 3G auctions, Telia, the biggest telecom company in northern Europe, took over Sonera, a smaller and debt-burdened telecom company, and obtained the licenses that Sonera had won in Germany, Italy, Spain, and Norway. In the 2003 UK 3.4G spectrum auction, Pacific Century Cyberworks (PCCW), a Hong-Kong telecom company participated in the auction, lost two licenses to Red Spectrum and Public Hub respectively. After six months it obtained the two licenses back through purchase of the two companies (for more details, see Pagnozzi (2010)). So far we don’t seem to have a clear theoretical guidance as to whether speculators will enhance the auctioneer revenue or over all efficiency.

Most of the recent studies on the optimal design of auctions with resale were based on second-price or English auctions with resale (for instance on the theoretical side: Zheng (2002), Garratt et al. (2009), Mylovanov and Tröger (2009), Lebrun (2012); and on the experimental side: Georganas (2011), Georganas and Kagel (2011), Pagnozzi and Saral (2013), Filiz-Ozbay Lopez-Vargas and Ozbay (2013)). This is understandable because the actual spectrum auctions were ascending auctions more similar to second-price or English auctions. However in second-price auctions or English auctions with resale, there are typically indeterminacy in equilibria, even infinitely many equilibria. The revenue with resale can be lower or higher than auction revenue without resale depending on which equilibrium prevails (see GT(supp)). Therefore the second-price or English auctions are not the ideal model to evaluate whether resale enhances auction revenue or not. In fact, Lebrun (2012) has argued that optimal auction with resale constructed in his paper and in Zheng (2002), Mylovanov and Troger (2009) and Garratt et al. (2009) all suffer from the indeterminacy of equilibrium problem aside from the strong assumptions needed for such design. In principle, the lack of determinacy makes it harder to have clear predictions for testing. It is also harder for bidders to have stable equilibrium bidding behavior so that clear patterns of outcomes can be observed and tested. Filiz-Ozbay, Lopez-Vargas and Ozbay (2013) perform experiments showing that there is no significant difference between revenues with or without resale in a multi-unit English auction setting. The first-price or Dutch auctions with resale present itself as an alternative design choice because it has a unique equilibrium.

We will study first-price auctions with resale in this paper. In particular, we will focus on speculative resale involving speculators who have no value for the object on sale except for the purpose of winning and selling to other losing bidders. Lebrun (2012) also expressed the need to find a unique implementation of optimal auctions with resale. In our model, we will provide a unique implementation of auctions with resale achieving optimal revenue the same as in auctions without resale.

Our model is very general in terms of the specification of the valuation distribution $F(\cdot)$ over $[0, \beta]$ of the $N$ regular buyers. In fact, we allow $F(\cdot)$ to have the possibility of an atom at 0, meaning that a regular buyer may have a positive probability of having 0 value. Take the benchmark model of symmetric private-value auctions for a single object without resale, add speculators who have no value for the object, and allow the auction winner in the first-stage to sell the object to other bidders in the second stage. That is the framework of our analysis. There are of course many ways of specifying the resale market. We will assume that the winner will sell the object using an optimal mechanism in the second stage. We call our model "auctions with speculative resale", because in equilibrium we can show that only speculators will be a seller in the second stage. By allowing the speculators to maximize revenue using the optimal auctions of Myerson (1981) in the second stage, we are giving the speculators the maximum bargaining power subject to the uncertainty remaining in the resale market. This endogenous price formation in the resale market is probably least controversial, and gives the speculators the highest capacity to be active in the first-stage auction. The auctioneer will also set a reservation price $\rho$ in the first-stage first-price auction. In specifying this resale market, we do not have the certainty resale game often occurring in the second-price auctions with resale. The share of resale surplus is dependent on the winning bid in the first-stage auction, and the resale outcome is often efficient as resale may fail due to uncertainty.

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1TIW (Telesystem International Wireless) was a Canadian company based in Montreal and largely owned by Hutchison Whampoa, the Hong Kong conglomerate.

2Our resale occurs between bidders in the first-stage auction, sometimes called secondary trade. Bikchandani and Huang (1989), Bose and Deltas (2002) deal with resales to consumers who are not participants in the original auction.
In between the two stages, there is also a need to specify what information is revealed after the auction and before the resale. We will adopt the natural minimal approach regarding bid revelation: only the winning bid is revealed. From this information, the winner of the first-stage auction can only infer about the maximum value of the losers. This information uncertainty means that the object may not be sold, and resale cannot correct all the inefficiency of the first-stage auction. Thus we have an interesting model to study the trade-off of efficiency between the two stages: while resale can correct some inefficiency in the first-stage allocation, the resale itself has its own inefficiency problem. Moreover, resale opportunity may cause more inefficiency in the first-stage outcome, an issue that was widely recognized.

We will first provide a simple equilibrium solution for the model. The equilibrium solution is easily computable as long as \( F(\cdot) \) is analytic (or computable), and equilibrium strategies are given by explicit formulas. The revenue is also easily computable. The equilibrium property is almost as easy to analyze as a private-value model, but has the great advantage of allowing resale considerations and value-dependence in bidding behavior. The revenue equivalence result no longer holds in this model, and there are many issues about auction design that can now be evaluated in the model with speculative resale.

We will then extend the Myerson revenue formula for the private value model without resale to our model of speculative resale. Let \( F(\cdot) \) defined on \([0, \beta]\) be the use value distribution of a regular buyer (non speculator), and assume there are \( N \) regular buyers. Let the reservation price set by the auctioneer be \( \rho \), and \( r \) is the optimal reservation price according to Myerson (1981). Let \( J(v) \) denote the virtual value according to Myerson (1981). The new formula takes a very simple form. With speculators competing in the first-stage auction, the winning probability of a regular buyer with value \( v \) is \( t(v)F^{N-1}(v) \), where \( t(v) \) is either 1 or some number between 0 and 1. When a regular buyer has value above \( r \), we have \( t(v) = 1 \), otherwise it is less than one. This is because a regular buyer can buy back from the speculator even if he fails to win in the first-stage, and the buyback occurs with probability 1 when \( v \geq r \), and less than 1 when \( v < r \). We have the following formula for the revenue of an auction with speculative resale

\[
\int_{\rho}^{\beta} t(v)J(v)dF^N(v).
\]

The virtual value, \( J(v) \), is now discounted (keeping the same sign) by \( t(v) \), otherwise the revenue remains the same as before. The discount function \( t(v) \) is also given by an explicit formula related to the bid distribution of the speculators. From the revenue formula, we then provide simple answers to the following questions: (1) Are speculative resale beneficial to the auctioneer revenue? (2) What is the optimal mechanism for revenue when resale is allowed? (3) Do we have revenue monotonicity when there are more buyer competition? The answers to these questions are: (1) Speculative resale increases revenue when speculators are active in equilibrium, (2) The same reservation price \( r \) is also optimal in this model of speculative resale, and the optimal revenue is the same as well, (3) We have revenue monotonicity. It should be noted that revenue monotonicity is not obvious at all, as the monotonicity of revenue from more participation cannot be taken for granted once we go beyond the benchmark private value symmetric model. It is particularly serious in combinatorial auctions as shown in Ausubel and Milgrom (2006). Even in single object auctions with symmetric buyers, such as common value models or affiliated signal models, the revenue monotonicity may be false (see also Pinkse and Guofu (2005)). Revenue monotonicity is a general result in our model. Hafalir and Krishna (2009) show that resale enhances revenue in a two-bidder model of weak-strong buyers allowing resale, if the value distribution belongs to three families of functions. Our result is quite general in function specifications, but is focused on speculative resale.

We also show that speculators are inactive when the auctioneer sets the optimal reservation price \( r \). When speculators are active, the auctioneer revenue will be higher than the auction without the speculators, given the reservation price \( \rho \). However, somewhat paradoxically, in order to achieve the optimal revenue in the auction with speculative resale, it is actually better for the auctioneer to set a sufficiently high reservation price so that the speculators are inactive in equilibrium, and have no impact on the auction revenue.

Determining an optimal reservation price is often difficult and requires plenty of information in practice. Bulow and Klemperer (1996) argued that the revenue is higher with one more bidder without any reservation.

\footnote{This is also an issue that causes awkwardness in second-price or English auctions with resale. If the winning bid is announced, as usually is the case in practice, the winner becomes aware of the private information of the highest losing bidder. This information leakage leads to a less interesting resale market for analysis. Lebrun (2010) analyses the problem of bid disclosure.}
price than the highest possible revenue from choosing the right reservation price. Thus the auctioneer does better by focusing on expanding participation, rather on choosing the reservation price optimally. We show that in our model of speculative resale, the Bulow-Klemperer result holds with even more persuasion. The difference in revenue is greater when speculators are active in equilibrium without any reservation price. The question of whether in real world auctions, the reservation price has been optimally chosen has been studied in the literature for a variety of auctions (McAfee and Vincent, 1992; Paarsch, J. (1997); McAfee, Quan, and Vincent, 2002; Athey, Cramton, and Ingraham, 2002; Haile and Tamer, 2003; Tang, 2009). Taken together, the results of these papers found that reserve prices actually observed in real-world auctions are substantially lower than the theoretically optimal ones.

We allow strategies to be non-monotonic, and non-symmetric. An equilibrium strategy is shown to be increasing and symmetric. We say that an equilibrium is symmetric if all regular buyers use the same equilibrium strategy. GT (supp) restrict their analysis to the symmetric equilibrium (quasi symmetric equilibrium in their terminology). It is an open question whether there exists a non symmetric equilibrium in the model. For second-price auctions with resale, it is well-known that there are many non-symmetric equilibria in the model. We show in this paper that the equilibrium must be symmetric for the first-price auction with resale. This is another contribution of the paper. We show that the equilibrium must be unique among all possible non-monotonic, non-symmetric strategies. The uniqueness of equilibrium we obtain in our result is important because it allows us to have a unique implementation of optimal auctions allowing resale.

We show a supermodularity property which has a strong form and a weak form. The strong form is stronger than the single-crossing property discussed in Chapter 4 of Milgrom (2004). The weak form can occur in the model of speculative resale. The combination of strong and weak forms of supermodularity shown in the paper are sufficient for our analysis of equilibrium behavior. The supermodularity (consequently the single crossing) property for the auction with resale model is an important property for the analysis of auctions. The supermodularity property is proved under very general conditions without requiring any monotonicity or symmetry of the bidding strategies and the value distributions can be different for different regular buyers. A discrete version of the property has been given by Zheng (2012), Proposition 3, in a general model of auctions with resale without speculators.

In providing a proof of the symmetry property, we establish first a special case of the symmetry result with no speculators. When there are no speculators in the auction with resale model, one consequence of our result is that the equilibrium is unique and must be the same as the well-known equilibrium in the symmetric auction without resale. This result, although quite intuitive, and implicitly used by many, has not been shown before, and must be dealt with in our analysis. When you allow resale, it is by no means obvious that the only equilibrium possible is the one without resale. Even though there is no resale in equilibrium, resale may occur out-of-equilibrium. It may also be possible that there is a non-symmetric equilibrium with resale between the bidders as in the case of second-price auctions with resale. We need to rule out the possibility that some bidders may be able to bid lower and buy back later during resale in equilibrium. In fact its proof is almost as involved as the more general case when we allow speculators.

After we establish the increasing symmetry property of the equilibrium, we prove the existence and uniqueness of equilibrium by analyzing the equilibrium properties in active and inactive intervals. Active bid intervals are intervals inside the support of the (cumulative) bid distribution of the speculators, and inactive bid intervals are intervals in which the (cumulative) bid distribution is constant. Active value

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4Bulow and Klemperer (1996) show the result using English auctions without resale. There is no need for resale in their framework, as the auction is efficient. They allow affiliated signals, and assume that equilibria are symmetric. In our model, the auction with resale need not be efficient. We use the first-price auction with resale which has the symmetry property, and achieves the optimal revenue. The bidding in our model is mutually dependent, but the value dependence is endogenously determined, and the associated common-value model need not have the affiliated signal property. Adding more speculators has no impact on revenue in our model.

5However, Ostrovsky and Schwarz (2009) argued that the revenue could be increased if the reservation price were raised upward.

6The uniqueness of equilibrium is also shown in GT (supp). However, they do this under the assumption that the equilibrium is symmetric and increasing. The uniqueness result we have here is among all possible non-monotonic and non-symmetric strategies.

7His result should be easily extendable to include speculators or perhaps even continuous distributions. However, it seems that his property is a weak form rather than a strong form.
intervals are intervals in which regular buyers bid in active bid intervals, and inactive intervals are intervals in which they bid in inactive intervals. Strategies on active or inactive intervals can be determined relatively easily. This allows us to establish the existence and uniqueness of equilibrium over all. More importantly, this is how we obtain a simple solution to the equilibrium strategies, as it can be shown that the equilibrium solution is simple in either active or inactive intervals. We believe this is the best way to develop and present the theory of auction with speculative resale.

When there are at least two regular buyers, equilibrium with inactive speculators can occur quite often with no reservation price. This occurs when we take many commonly-used value distributions. When a reservation price is set, it is even more likely to lead to inactive speculators. In Part II of the paper we provide more exhaustive analysis of this issue. An interesting question then arises: What are economic factors that make the speculators active in equilibrium? We provide a necessary and sufficient condition for the speculators to be active in equilibrium. The condition basically says that there is a cost function and a revenue function that can be defined from \( F(\cdot) \). The speculators become active in equilibrium if and only if the revenue exceeds the cost at some point. The next natural question then arises: If speculators are active in equilibrium, what are the bid intervals in which they are active? In equilibrium, typically, speculators are active in disjoint open intervals (an intuitive explanation of this is offered right before section 3.5). We also provide a necessary and sufficient condition to characterize such intervals. There is a simple algorithm that will determine and compute all the active and inactive intervals. In general there is a collection of countably many disjoint active intervals separated by inactive intervals. There is a generic condition that will imply a finite number of such intervals. It is also possible to construct examples with as many active intervals as you want. These results will be provided in Part II of the paper which also answers the question of optimal efficient outcome of the model.

An example of equilibrium with an active speculator has been given by GT (2006). We offer a different but more transparent example at the end of section 6. The easiest way to find an example with active speculators is to allow an atom. GT (2006) has such an example with an atom at the end. Our example has an atom\(^8\) at the beginning point 0. This is one reason we allow \( F(0) > 0 \) in our formulation. In our example we show how the revenue function is lifted while the cost function is lowered with an atom at 0. The revenue exceeds the cost everywhere, and speculators become active in equilibrium. However they are not fully active in the sense that they do not bid in all bids of the support of the regular buyer’s bid distribution. We also use this example to illustrate how the active interval can be determined by simple functions that are derived from \( F(\cdot) \).

We only allow two stages in the auctions with resale model, and there is only one resale stage in our model. Although this is a restriction, it is rather harmless for the equilibrium strategies. Our symmetry property implies that when a speculator sells in the resale market, he is selling in a symmetric auction which allocates the item efficiently to the buyer with the highest value. Therefore the winner of the resale auction has no more incentive to sell it to other buyers. Therefore, in equilibrium, having only one resale stage is not a limitation. However it could be a limitation when we analyze what could happen out of the equilibrium path in which allocation in the resale stage need not go to the bidder with the highest value. There is also another incentive issue when a seller fails to sell in a stage. The seller may try to sell the same object again. We assume that in this case, the seller keeps the object for himself. The temptation to sell it again at a possibly lower price is an intertemporal issue. This is an issue of commitment rather unrelated to the resale incentive issue. We will assume in this paper there is no commitment problem in the model. Under this assumption, the limitation of one resale opportunity is not a serious one. We should also point out that the optimal reservation price we obtain is the same for all bidders in the first-stage. This is an attractive feature as in equilibrium it could be difficult to identify who is a speculator and who is a regular buyer. The same reservation price applies to all bidders, and an optimal revenue outcome is achieved.

We assume no discounting in our analysis, although the analysis can be easily adapted for this case. Virag (2011) considered auctions with resale with two types of regular buyers. The bid distribution of the two types of buyers are not the same. Ours can be considered a special case of theirs and similar results hold. However, they did not show the symmetry property, and the model of speculative resale here makes it easier to obtain a simple formula of the equilibrium strategy.

\(^8\)Atoms are neither sufficient nor necessary for the speculators to be active in equilibrium. They just provide simple examples in which we can demonstrate the possibility of active speculation. Examples of active speculation without any atoms are given in Part II of the paper.
In many important auctions, for instance the government sponsored spectrum auctions, revenue is not the main objective. In Part II, we will consider the problem of achieving the most efficient outcome with speculative resale. With this different objective function, we will show that speculator participation may provide the most efficient outcome, an interesting contrast to the optimal revenue case. However other optimal efficient outcome may also arise, depending on the properties of the valuation distribution \( F(.) \). A solution to the efficiency problem with speculative resale is given in Part II.

In section 2, we describe the model of speculative resale. In section 3, we focus on symmetric increasing strategies, and characterize active and inactive intervals and equilibrium properties in such intervals. In section 4, we give a simple computable solution to the equilibrium under the assumption that they are increasing and symmetric. In section 5, we offer the general version of the supermodularity property of the model. In section 6, we show that the equilibrium must be increasing and symmetric. Section 7 gives the revenue formula using the idea of virtual value in Myerson (1981). It also shows the revenue monotonicity with higher number of regular buyers, and the Bulow-Klemperer result in our model with speculative resale. In section 8, we provide the proofs for the supermodularity property, and the monotonicity and symmetry properties of equilibrium, as well as the lemmata in section 3.

2 A Model of First-price Auction with Speculative Resale

There are two types of buyers: regular buyers and speculators. Regular buyers have use value for the object, while speculators have no use value for the object and participate in the auction only for resale. There are \( N \) regular symmetric buyers and \( S \) speculators bidding for one object sold by the auctioneer (sometimes called the original seller to distinguish from the seller in the resale market). A regular buyer has a use value distribution \( F(v) \) over \([0, \beta]\), which is assumed to be \( C^2 \) smooth. Let \( f(v) \) denote the density function of \( F(.) \). We allow \( F(0) > 0 \), so that \( F \) may have an atom at 0 as in the case of convex exponential distributions. It is convenient to use the notation \( F(\cdot|v) = \frac{F(\cdot)}{F(v)} \) for the conditional distribution of \( F(.) \) if the use value has an upper bound \( v \). The speculators will be indexed by \( s \), while the regular buyers are indexed by \( i = 1, 2, ..., N \). Although it may seem more general to assume more than one speculator, we will show that indeed a model with one speculator will be sufficient for the study of the effect of speculation on bidding behavior in this paper. The multi-speculator model is equivalent to a single speculator model. So readers may want to limit their attention to this case.

We study the first-price auctions with resale\(^9\). The first-price auction with resale is a two-stage game. The first-stage auction is a first-price auction with a reservation price \( \rho \) for the speculators and regular buyers. A winner of the auction in the first stage may sell to the losers in the second stage when it is profitable. The seller in the resale stage chooses an optimal mechanism. By the revelation principle, we can assume that the resale mechanism is a modified second-price auction with optimally chosen reservation prices. At the end of the first-stage auction and before the beginning of the resale stage, the winning bid (the highest bid) is announced. We will assume no discounting between the first stage and the second stage.

Let \( b_i(v), i = 1, 2, ..., N \) be the bidding strategy of regular buyer \( i \). We allow \( b_i(.) \) to be non-monotonic, but exclude the possibility of having an atom in the bid distribution of \( \frac{b_i(v)}{F(v)} \), except possibly at \( \rho \). This is because atoms above \( \rho \) are not compatible with equilibirum requirements. A speculator uses a mixed strategy of bidding, represented by a cumulative bid distribution function \( H_s(b) \). We allow \( H_s(.) \) to be degenerate at 0, or has an atom at or below \( \rho \), but is atomless at any \( b > \rho \). We also assume that \( H_s(b) \)

\(^9\)We can allow \( F(.) \) to be piecewise \( C^2 \) smooth. This is a more natural framework for the analysis, as the equilibrium strategy in general is a maximum of two smooth functions. A typical example of a piecewise \( C^2 \) smooth function is the maximum or minimum of two smooth \( C^2 \) functions. Formally a function is called piecewise \( C^2 \) smooth if (i) it is \( C^2 \) except at a countable closed set \( D \), (ii) At a point in \( D \), the left and right derivatives (up to the second order) exist and are continuous from the right and left respectively.

\(^10\)Haile (2000,2001,2003) did the pioneering works on auctions with resale. However, in his models, resale arises out of new information. Our models are fashioned along the lines of Hafalir and Krishna (2008), Garratt and Troger (2006), Cheng and Tan (2010) and Lebrun (2010), in which there is an incentive for resale due to asymmetry between buyers in the first stage auction. Second-price auctions with resale are studied in Pagnozzi (2007,2010), Garratt, Tröger, and Zheng (2009), Lebrun (2012).
is weakly increasing. Let the joint bid distribution of the rival speculators be $H_{-s}(b) = \prod_{k \neq s} H_k(b)$. For convenience, we adopt the same bid distribution $H(.)$ for each speculator, even though this need not be the case in equilibrium. The joint bid distribution of the rival speculators is then $H^{S-1}(b)$. A speculator may become inactive over some interval, and $H(b)$ is a constant over such an interval. We will show in Part II that a speculator typically is active over disjoint intervals. In other words, the support of $H(b)$ may not be connected. Let $\bar{b}_j, \underline{b}_j, \bar{b}_s, \underline{b}_s$ be the maximum and minimum of the support of the bid distribution of regular buyer $j$ or a speculator, respectively. In equilibrium, only the joint bid distribution of the speculators is uniquely determined, and it will be denoted by $\bar{H}(\cdot) = H^S(.)$.

The winner of the first-stage auction, i.e. the seller in the resale auction, will revise his or her belief about the losers’ value distributions in the resale stage. When a speculator or buyer $i$ wins the first-stage auction with a bid $b$, the only information known about a losing regular bidder $j$ is that $b_j(\cdot) \leq b$. Let $G_j(\cdot,b)$ be the induced probability distribution of the losing buyer $j$ conditional on this information after winning, and $Q_j(b)$ be the probability of buyer $j$ bidding below $b$. Let $w_j(b)$ be the maximum of the support of $G_j(\cdot,b)$. The function $w_j(b)$ may not be continuous, but $Q_j(b)$ is a continuous function. If $b_j(\cdot)$ is strictly increasing, let $\phi_j(\cdot)$ be the inverse function of $b_j(\cdot)$. This updated belief about buyer $j$ is simply the conditional distribution $F(\cdot|\phi_j(b))$.

Since the bidding strategies $b_j(\cdot), j \neq i$, may not be the same for different losing buyers, the seller in the resale auction faces an asymmetric auction environment. Such revision of beliefs is common knowledge as the winning bid is public information. Since the game in the resale auction is well-understood, we can summarize the result of the resale auction by a profit function without specifying the second period strategies, so that we can focus on the first-stage bidding behavior anticipating the optimal resale outcome in the second stage.

To define an equilibrium strategy of the auction with resale, let $\sigma$ denote the strategy profile $H(\cdot), b_i(\cdot), i = 1, 2, ..., N$. Let $\pi_s(b,\sigma)$ denote the optimal resale profit (or payoff) during the resale stage. The speculator $s$ chooses $b$ to maximize the overall profit

$$u_s(b,\sigma) = \pi_s(b,\sigma) - bH^{S-1}(b) \prod_{i=1}^{N} Q_i(b),$$

where $\pi_s(b,\sigma)$ is simply the speculator’s expected revenue from resale. When $b_i(\cdot)$ is strictly increasing for each $i$, we can write

$$u_s(b,\sigma) = \pi_s(b,\sigma) - bH^{S-1}(b) \prod_{i=1}^{N} F(\phi_i(b)).$$

Given $\sigma$, we say that $b$ is an optimal bid for the speculator $s$ if $b$ is an optimal solution of the above maximization problem. Since a speculator uses a mixed strategy, all bids in the support of the equilibrium bid distribution $H(b)$ must yield the same payoff to the speculator $s$.

After a regular buyer $i$ with use value $v_i$ submits a bid $b$, and wins the auction, he or she updates the belief regarding the other regular buyers the same way as the speculators, and believes that buyer $j$ has the value distribution $G_j(\cdot,b)$. If $b \geq \bar{b}_j$, we assume that there is no change in belief regarding buyer $j$. After winning the first-stage auction, buyer $i$ may sell the object to buyer $j$ during the resale stage if $w_j(b) > v_i$. If buyer $i$ loses the auction, and the winner is a speculator or some regular buyer $j$ with $w_j(b) < v_i$, buyer $i$ may bid for the object and buy it from the winner during resale. Since the winner will use the winning bid and the bidding strategy of the losing buyer $i$ in this case to update belief and determine the reservation price, the payoff of buyer $i$ after losing the auction will depend not only on the strategy profile of the other buyers, but also on his own choice of the bidding strategy. When resale fails to materialize after a winner wins the object in the first stage, the winner keeps the object.

Let

$$\pi_{1i}(v_i, b, \sigma_{-i}) = (v_i - b)\bar{H}(b) \prod_{j \neq i} Q_j(b)$$

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11 However there are complications that arise when the bidding strategy is non-monotone. In this case the updated belief may violate the assumptions used for the value distribution in Myerson (1981). This technical issue can be resolved. See the proof for Theorem 6 in section 7.1.
be the payoff of the first-stage auction. Let \( \pi_{wi}(v_i, b, \sigma_{-i}) \) be the expected payoff of the bidder \( i \) in the resale market after winning, and \( \pi_{ii}(v_i, b, \sigma) \) the corresponding amount after losing. Then the overall payoff from bidding \( b \) is
\[
    u_i(v_i, b, \sigma) = \pi_{ii}(v_i, b, \sigma_{-i}) + \pi_{wi}(v_i, b, \sigma_{-i}) + \pi_{ii}(v_i, b, \sigma).
\]  
(1)

When all regular buyers use the same increasing strategy, more explicit formulas for the payoffs are given below.

We say that \( b \) is an optimal or equilibrium bid for the regular buyer \( i \) with use value \( v_i \) if it maximizes \( u_i(v_i, b, \sigma) \). We say that \( \sigma \) is a perfect Bayesian equilibrium of the auction with resale if (i) for each speculator \( s \), any bid \( b \) in the support of \( H(b) \) is an optimal bid for the speculator, (ii) for each regular buyer \( i \), \( b_i(v_i) \) is an optimal bid maximizing (1).

A perfect Bayesian equilibrium of the auction with resale should describe strategies in both stages of the game. For convenience, we shall abuse the language somewhat and refer to \( \sigma \) as a perfect Bayesian equilibrium. This is due to the fact that the resale game is not explicitly spelled out, and only summarized. See section 8.1 for more details about the analysis of the resale auction revenue. We say that the equilibrium has monotonicity and symmetry property if \( b_i(.) = b_j(.) = b(.) \) for any two regular buyers, and \( b(.) \) is strictly increasing. We will show that monotonicity and symmetry must hold for any equilibrium. The equilibrium strategy of a regular buyer will be denoted by one single bidding function \( B(.) \). We say that the equilibrium is unique if the profile \( \sigma = (B(.), \hat{H}(.)) \) is uniquely determined. We say that speculators are inactive in equilibrium if they all bid strictly below \( \rho \) with probability 1. We take the convention that when speculators are inactive in equilibrium they bid 0 for sure (although bidding below \( \rho \) yields the same outcome). We also take the convention that regular buyers with \( v < \rho \) all bid 0. If one speculator is not inactive in equilibrium, we say that the speculator is active in an equilibrium, and in this case all speculators are active in equilibrium.

### 2.1 Payoff Expressions

When the symmetry property holds, the resale auction by the speculator is a symmetric auction with a single reservation price. We will write down the detailed payoff expression in this case. It is convenient for the presentation to assume that the virtual value is increasing, so that there is a unique optimal reservation price in the resale stage. Let
\[
    J(x, w) = x - \frac{F(w) - F(x)}{f(x)}
\]
denote the conditional virtual value of \( x \) when the buyer value upper bound is \( w \). If \( J(x, \beta) \) is strictly increasing in \( x \), then \( J(x, w) \) is also strictly increasing for all \( w \). If the seller has use value \( v_0 \), the optimal reservation price \( r(v_0, w) \) conditioned on the upper bound \( w \) is determined by the solution of the following equation in \( x \)
\[
    J(x, w) = v_0,
\]

We let \( v_0 = 0 \) when the seller is a speculator. The increasing virtual value property of \( J(x, \beta) \) is made to insure the uniqueness of the optimal reservation price. When the seller is the speculator, we may also use the simpler notation \( r(w) \), or \( r(v) \) when \( w = v \). This optimal reservation price \( r(v) \) is independent of the number of buyers, and satisfies the following equation
\[
    r(v)f(r(v)) + F(r(v)) = F(v).
\]  
(2)

For the increasing symmetric strategy \( b(.) = \phi^{-1}(.) \), and \( \hat{H}(.) \), the payoff in (1) of a regular buyer can now be written more explicitly. Without the speculators and without resale, \( N > 1 \), the payoff function can be written as
\[
    u_0(v_i, b, \phi) = F^{N-1}(\phi(b))(v_i - b).
\]

When there are speculators and resale is allowed, consider a regular buyer \( i \) bidding \( b \) with \( \phi(b) < v_i \), there is no payoff from resale after winning the first-stage auction \( (\pi_{wi} = 0) \), but there is payoff after losing. If \( N > 1 \), the winner of the first-stage auction may be a regular buyer. The winner believes that he or she has
the highest value, and is indifferent between no resale or offering it for resale at the price equal to his own value. Hence either there is no payoff buying from the winner, or the payoff is equal to

$$\int_b^{\phi(y)} (v_i - \phi(y)) \tilde{H}(y) dF^{N-1}(\phi(y)).$$  \hspace{1cm} (3)$$

Let $\pi_{1i0}$ be either 0 or (3). When a speculator wins the first-stage auction with a bid $y$, the optimal reservation price set by the speculator during resale is $r(\phi(y))$. Let $\tilde{y}$ be the unique solution of the equation $r(\phi(y)) = v_i$. The payoff of buying from a speculator during resale is

$$\pi_{1i}(v_i, b, \sigma) = \int_b^{\tilde{y}} \left( \int_{r(\phi(y))}^{v_i} F^{N-1}(x) dx \right) d\tilde{H}(y) + \pi_{1i0}.  \hspace{1cm} (4)$$

The first-stage payoff is given by

$$\pi_{1i}(v_i, b, \sigma_{-i}) = \tilde{H}(b) u_0(v_i, b, \phi).$$

Hence the overall payoff is given by

$$u(v_i, b, \sigma) = \tilde{H}(b) u_0(v_i, b, \phi) + \pi_{1i}(v_i, b, \sigma), v_i > \phi(b).$$  \hspace{1cm} (5)$$

For the bid $b$ with $\phi(b) \geq v_i$. There may be payoff from resale after winning or losing the first-stage auction. If he or she loses the first-stage auction, and the winner is a speculator, then the resale payoff is given by (4) with $\pi_{1i0} = 0$. If the winner is a regular buyer, there is no payoff from resale. If he or she wins the auction, there is payoff from selling to the losing regular buyers. Let the optimal reservation price in resale be $r(v_i, \phi(b))$. The payoff after winning the auction is

$$\pi_{ui}(v_i, b, \sigma) = \tilde{H}(b) \int_{r(v_i, \phi(b))}^{\phi(b)} (J(x, \phi(b)) - v_i) dF^{N-1}(x).$$  \hspace{1cm} (6)$$

Hence the total payoff in this case is given by

$$u(v_i, b, \sigma) = \tilde{H}(b) u_0(v_i, b, \phi) + \int_b^{\tilde{y}} \left( \int_{r(\phi(y))}^{v_i} F^{N-1}(x) dx \right) d\tilde{H}(y) + \pi_{ui}(v_i, b, \sigma), v_i \leq \phi(b).$$  \hspace{1cm} (7)$$

### 2.2 Equilibrium without Speculators

If there are no speculators, $N > 1$, and resale is not allowed, the equilibrium strategy is known to be strictly increasing, symmetric and unique. Let $\phi(.)$ be the inverse of the equilibrium bidding strategy. The first-order condition of the equilibrium satisfies

$$\frac{\partial}{\partial b} u_0(v, b, \phi)|_{b=b(v)} = 0.$$  \hspace{1cm} (8)$$

Let $b_\rho(.)$. $\phi_\rho(.) = b_\rho^{-1}(.)$ denote the equilibrium bidding strategy and its inverse when the reservation price is $\rho$. We have the formula

$$b_\rho(v) = v - \int_\rho^{\phi(v)} F^{N-1}(x|v) dx = \rho F^{N-1}(\rho|v) + \int_\rho^{\phi(v)} x dF^{N-1}(x|v), \text{for } v \geq \rho.$$  \hspace{1cm} (9)$$

Note that for $N = 1$, we have $b_\rho(v) = \rho$ for $v \geq \rho$.

When resale is allowed, but there are no speculators (or $H(0) = 1$), we can write the payoff of a regular buyer with value $v$ bidding $b, \phi(b) < v$, as

$$u(v, b, \phi) = u_0(v, b, \phi) + \int_{b}^{\phi(v)} (v - \phi(y)) dF^{N-1}(\phi(y)).$$
and we have the first-order condition
\[
\frac{\partial}{\partial b} u(v,b,\phi) = \frac{\partial}{\partial b} u_0(v,b,\phi) = 0. \tag{10}
\]

When \( \phi(b) \geq v \), we have
\[
u(v,b,\phi) = u_0(v,b,\phi) + \int_{r(v,\phi(b))}^{\phi(b)} (J(x,\phi(b)) - v) dF^{N-1}(x),
\]
and we have the same first-order condition (10). Therefore if we allow resale without speculators, the first-order condition of an equilibrium is the same as that of a model with no speculators and no resale. Combine this with the supermodularity property, and we know that equilibrium must be unique in a model with speculators but allowing resale, and is the same one well-known in the literature. Once we establish the increasing symmetry property of the equilibrium, the uniqueness follows from the well-known result of the auction without resale model. Hence we immediately have the following result.

**Theorem 1** If there are no speculators, the equilibrium with resale is unique and is the same as the one without resale given by (9).

### 3 Equilibrium Properties

Once the equilibrium has been shown to be symmetric and increasing, GT (supp) have proved that the equilibrium exists and is unique. They use the uniqueness result of the solution of a system of differential equations to prove uniqueness. We shall adopt a different mode of analysis that is more transparent and eventually leads to an explicit simple solution of the equilibrium strategy.

The main ideas for our analysis will be implemented in several steps. We give an informal description before presenting the formal analysis. First we discuss the supermodular property for symmetric increasing strategies needed for our analysis. This will be generalized to non-monotone non-symmetric strategies in section 5. Then we introduce some useful functions which are important for our analysis and provide some preliminary lemmata. Thirdly we prove the first-order conditions of equilibrium. Fourthly, we give a simple characterization of active and inactive intervals based on revenue and cost function comparisons. In the fifth step, we show that active and inactive intervals do not depend on which equilibrium we take, and are uniquely determined. In the sixth step, we show that the cost function dominates the revenue function in inactive intervals, while the reverse occurs in active intervals. In the final step, done in section 4, we define the equilibrium strategy to be the maximum of the cost and revenue functions. The joint bid distribution of the speculators is also easily defined from the first-order conditions. This construction gives us an equilibrium strategy profile. The uniqueness of the equilibrium holds because it holds on each active and inactive intervals.

#### 3.1 Supermodularity

For increasing symmetric strategies, supermodularity can be easily shown. We say that the payoff function has the supermodularity property in strong form if
\[
\frac{\partial}{\partial b} \frac{\partial}{\partial v_i} u(v_i,b,\sigma) > 0. \tag{11}
\]

It has the supermodularity in weak form if
\[
\frac{\partial}{\partial b} \frac{\partial}{\partial v_i} u(v_i,b,\sigma) = 0. \tag{12}
\]
Consider the case \( \phi(b) \geq v_i > \rho \). Taking the partial derivatives of (7), we get
\[
\frac{\partial}{\partial b} \frac{\partial}{\partial v_i} u(v_i, b, \sigma) = \frac{\partial}{\partial b} [\tilde{H}(b) F^{N-1}(\phi(b))] > 0.
\]
Hence we have supermodularity in strong form when \( v_i \leq \phi(b) \).

Consider the other case \( v > \phi(b) > \rho \). When \( \pi_i(b) = 0 \), from the expression in (5), we have
\[
\frac{\partial}{\partial v_i} \pi_i(v_i, b, \sigma) = \int_b^\phi F^{N-1}(v_i) d\tilde{H}(y).
\]
We also have
\[
\frac{\partial}{\partial b} \tilde{H}(b) u_0(v_i, b, \phi) = \tilde{H}(b) F^{N-1}(\phi(b))
\]
Hence
\[
\frac{\partial}{\partial b} \frac{\partial}{\partial v_i} u(v_i, b, \sigma) = \tilde{H}(b) \frac{d}{db} F^{N-1}(\phi(b)) > 0.
\]
When \( \pi_i(b) \) is given by (3), we have
\[
\frac{\partial}{\partial v_i} \pi_i(v_i, b, \sigma) = \int_b^\phi F^{N-1}(v_i) d\tilde{H}(y) + \int_{b(v_i)}^{b(v_i)} \tilde{H}(y) F^{N-1}(\phi(y))
\]
the right-hand side above is simply the probability of winning the object during resale (either from a speculator or a regular buyer). While the right-hand side of (13) is the probability of winning the object in the first-stage auction. We have
\[
\frac{\partial}{\partial b} \frac{\partial}{\partial v_i} u(v_i, b, \sigma) = \frac{\partial}{\partial v_i} \tilde{H}(b) u_0(v_i, b, \phi) + \frac{\partial}{\partial v_i} \pi_i(v_i, b, \sigma) = 0.
\]
Thus we only get the weak form in this case.

The intuition in the equation (14) is that the sum of the two terms inside the bracket above does not depend on \( b \). When the regular buyer bids higher, the probability of winning in the first-stage auction becomes larger, but is cancelled by the lower probability of winning the object during resale. As long as \( b \) is inside the region in which there is no payoff from selling the object after winning, and there is resale (for sure) from buying after losing the auction in the first-stage, then the two probabilities must compensate each other and the sum stays the same. This intuition remains the same when we consider asymmetric and non-monotone strategies, and will be generalized later.

The above supermodularity property is sufficient for the first-order condition \( \frac{\partial}{\partial b} u(v_i, b, \sigma) |_{b=b(v_i)} = 0 \) to yield the optimality property for the strategy \( b(v_i) \). Take \( b' > b(v_i) \), \( \frac{\partial}{\partial v_i} u(v_i^', b, \sigma) |_{b=b(v_i)} = 0 \), (11) implies \( \frac{\partial}{\partial v_i} u(v_i, b, \sigma) |_{b=b(v_i)} < 0 \). For \( b' < b(v_i) \), we have \( \frac{\partial}{\partial v_i} u(v_i, b, \sigma) |_{b=b(v_i)} \geq 0 \). Hence \( b(v_i) \) is an optimal bid. We will see later that the regular buyer’s strategy is also uniquely determined. It is also clear that the supermodularity property above is also sufficient to insure that the equilibrium strategy \( b(v) \) of a regular buyer must be strictly increasing.

The supermodularity property, when generalized to asymmetric and non-monotone strategies, will allow us to conclude that an equilibrium strategy must be increasing and the first-order condition will be a sufficient condition for an optimal strategy. We will first use the supermodularity property for the symmetric increasing strategies established here to find a solution of the symmetric equilibrium.

### 3.2 Speculator Equilibrium Condition

Let \( \phi(\cdot) \) be the inverse equilibrium strategy of the regular buyers. After winning the auction bidding \( b \), a speculator sells to \( N \) regular buyers with the value distribution \( F(\cdot|\phi(b)) \) in the resale market. Let \( v = \phi(b) \), and \( B(v) \) be the expected total revenue during the resale. The function \( B(v) \) can be defined for any \( v \), and according to the Myerson (1981) revenue formula, it is given by
\[
B(v) = \int_{r(v)}^v J(x, v) dF^N(x|v) = \frac{N}{F^N(v)} \int_{r(v)}^v [xf(x) + F(x) - F(v)] F^{N-1}(x) dx.
\]
We shall call this the revenue function of the speculators. When \( N = 1 \), we have \( B(v) = r(v)(1 - \frac{F(r(v))}{F(v)}) \). The function \( B(.) \) is not defined at \( v = 0 \), but we can let \( B(0) = 0 \), which will make \( B(v) \) a continuous function over \([0, \beta]\). The inverse function of \( B(.) \) will be denoted by \( \eta(.) \). The speculator profit can be written as

\[
u_s(b, \phi) = F^N(\phi(b))(B(\phi(b)) - b).
\]

Equilibrium profit for a speculator bidding \( b \) in the support of the \( H(.) \) must be a constant independent of \( b \). It is known that a speculator makes zero profit in equilibrium. This has been proved in Lemma 7 of \( GT(supp) \). It is useful to provide the reasons here as it is a central part of our analysis. Let \( b_s \geq \rho > 0 \) be the minimum bid of a speculator. The minimum bid of a regular buyer must be \( b_s \) as well. Let a regular buyer with value \( v \) bid \( b_s \) in equilibrium. Since non-negative profit condition implies \( B(v) \geq b_s \), we must have \( v > b_s \). If \( H(b_s) = 0 \), the regular buyer with value \( v \) gets 0 payoff bidding \( b_s \). The deviation to a slightly higher bid yields a positive profit, hence we have a contradiction. If \( H(b_s) > 0 \), again a regular buyer can increase the probability of winning by bidding slightly higher, and increase the profit. Hence we have another contradiction. This implies that we must have \( b_s < \rho \), or \( \rho = 0 \). In either case, the speculator profit is zero. Another useful intuition we want to mention here is that in equilibrium the speculators as a whole bids less aggressively than a regular buyer (see Theorem 3). This means they bid below \( \rho \) with positive probability, and the profit is zero when they do so. For convenience, we state it as a lemma here.

**Lemma 1** In any equilibrium all speculators make zero profit.

From the zero profit condition, we know that if \( b \) is in the support of the \( H(.) \), we must have \( B(\phi(b)) - b = 0 \), or \( B(v) = \phi^{-1}(v) \). In other words, \( B(.) \) is the equilibrium bidding strategy of the regular buyers. This must be true for a regular buyer bidding \( b \) in the support of \( H(.) \). A speculator may become inactive, so that the support of \( H(.) \) is 0. We need to determine whether a speculator is active or not in equilibrium. The revenue function \( B(.) \) plays essential roles in understanding whether speculators become active or not in equilibrium and, if active, in determining the support of the bid distribution (or intervals in which they are active).

The equilibrium condition for a speculator is that \( \pi_s(b, \phi) = 0 \) for \( b \) in the support of \( H(.) \), and \( \pi_s(b, \phi) \leq 0 \) outside the support of \( H(.) \). An important part of our analysis is to separate the equilibrium conditions depending on whether speculators are active or not. We shall give a formal definition of this idea.

A countable closed subset of an interval \((c, d)\) is the intersection of \((c, d)\) with a countable closed set in the set of real numbers. If two continuous functions \( g_1(.) \), \( g_2(.) \) defined over \((c, d)\) are not the same over any open interval in \((c, d)\), then \( g_1(.) = g_2(.) \) can only occur on a countable closed subset.

We say that the speculators are inactive in equilibrium if \( H(0) = 1 \). They bid 0 for sure in equilibrium in this case, and have no impact on the outcome of the auction. We say that they are fully active in equilibrium if the support of \( H(.) \) is the same as that of the bid distribution of \( B(.) \). In equilibrium, a speculator typically bids only in certain intervals with positive probability. A useful concept in our equilibrium analysis is the active or inactive bid intervals of the speculators. To define an interval in which the speculators are active, we say that an open bid interval \((a_1, a_2)\) is an active interval if \( H'(.) > 0 \) in the interval except a countable closed subset. A maximal active interval is an active open interval which is not a proper subset of another active interval. When \((a_1, a_2)\) is a maximal active interval, we say that \( a_1 \) is the beginning of an active interval, and \( a_2 \) is the end of an active interval. An interval \([a_1, a_2]\) is inactive if it contains no active intervals. This is the same as saying \( H(.) \) is constant over the interval. It is convenient to call an interval \([z_1, z_2]\) of use values an active interval if \( (B(z_1), B(z_1)) \) is an active interval of bids. Therefore an active value interval is simply the value-types of a regular buyer who bids inside an active bid interval in equilibrium. A maximal value interval can be similarly defined, so are the beginning and end of an active value interval, and the inactive intervals. The concept of active and inactive intervals plays an important part of our analysis.

### 3.3 Equilibrium in Inactive Intervals

The equilibrium condition for a regular buyer in inactive intervals is very similar to the condition without speculators. It can be easily seen that in inactive intervals, (8) is also the first-order condition of equilibrium.
Therefore if the boundary condition is determined, then (8) uniquely determines the bidding behavior of the regular buyers in inactive intervals. We can rewrite the first-order condition (8) in the variable \( v = \phi(b) \) as follows

\[
    b'(v) = (N - 1)(v - b(v)) \frac{f(v)}{F(v)}.
\]

When we solve (16) with the initial condition \( b(z) = b_0 \) at \( z > \rho \), the solution is given by

\[
    b(v) = b_0 F^{N-1}(z|v) + \int_z^v x dF^{N-1}(x|v), \quad v \in [z, \beta].
\]

We often take \( z \) to be either \( \rho \), the beginning or the end of an active interval, and \( b_0 = \bar{B}(z) \). In this case, we use the notation \( B^*_z(.) \) to denote the solution (17). When \( b_0 = \bar{B}(z) = B(z) \), we have

\[
    B^*_z(.) = B(z) F^{N-1}(z|v) + \int_z^v x dF^{N-1}(x|v).
\]

In a symmetric increasing equilibrium only speculators will sell the object after winning it in the first-stage auction. In determining whether the speculators become active, the two functions \( b_\rho(.) \), \( B^*_\rho(.) \) serve as "cost functions" for a speculator because it is the cost needed to win the object to be able to sell to \( N \) regular buyers with value distribution \( F(.|v) \). We also call \( B^*_\rho(.) \) an endogenous cost function as it depends on the equilibrium bid \( \bar{B}(z) \) at \( z \), while \( b_\rho(.) \) does not. We can regard \( b_\rho(.) \) as a special case of \( B^*_\rho(.) \) when \( z = \rho = b_0 \). Both functions are strictly increasing (when \( N > 1 \)) and continuous. From (16), we immediately have the following.

**Lemma 2** When \( N > 1 \), the function \( B^*_z(.) \) has the following properties: (i) \( B^*_z(z) = \bar{B}(z) \), (ii) \( B^*_z(.) \) is strictly increasing in \([z, \beta]\), and (iii) The derivative is given by

\[
    B^*_z'(v) = \frac{(N - 1)f(v)}{F(v)}(v - B^*_z(v)).
\]

### 3.4 Equilibrium in Active Intervals

To state the first-order condition of equilibrium in active intervals, we will introduce another useful function. For \( v \leq w \), let

\[
    B^t(v, w) = v - \int_{r(w)}^v F^{N-1}(x|w)dx.
\]

When \( N = 1 \), \( B^t(v, w) = r(w) \). We use a simpler notation when \( v = w \) :

\[
    B^t(v) = v - \int_{r(v)}^v F^{N-1}(x|v)dx.
\]

The function \( B^t(v) \) is the amount of payment to a speculator by the "top" type when a speculator sells to \( N \) symmetric buyers with the use value distribution \( F(.|v) \). We have the following useful alternative formula for \( B(v) \) :

\[
    B(v) = \int_{r(v)}^v B^t(x, v) dF^N(x|v).
\]

Since \( B^t(x, v) \) is strictly increasing in \( x \), we have \( B^t(x, v) < B^t(v) \) when \( x < v \). Hence we have

\[
    B(v) < B^t(v) \int_{r(v)}^v dF^N(x|v) < B^t(v).
\]

For convenience, we state this as a lemma.
Lemma 3 We have 
\[ B^t(v) > B(v) \text{ for all } v \in (0, \beta). \]

In active intervals, the first-order condition for a regular buyer will be shown (in Lemma 9) to be
\[ \frac{\dot{H}'(b)}{H(b)} = -\frac{1}{F^{N-1}(\eta(b))[B^t(\eta(b)) - b]} \frac{\partial u_0(v, b, \eta)}{\partial b}\bigg|_{b = B(v)}. \] (19)

To simplify the expressions further, we introduce other useful functions. Let
\[ B^c(v) = v - N \int_{r(v)}^v F^{N-1}(x|v)dx. \]
Note that, by definition, we have
\[ v - B^c(v) = N(v - B^t(v)). \] (20)

When \( N = 1, B^c(v) = B^t(v) = r(v). \) We will show (in Lemma 10) that
\[ \frac{\partial u_0(v, b, \eta)}{\partial b}\bigg|_{b = B(v)} = -\frac{1}{N} F^{N-1}(v) \frac{B^c(v) - B(v)}{B^t(v) - B(v)} \] (21)

Let
\[ L(v) = \frac{B^c(v) - B(v)}{B^t(v) - B(v)}. \]
\[ L(b) = L(\eta(b)) = L(v). \]

Then the first-order condition of a regular buyer in active intervals takes the following form
\[ \frac{\dot{H}'(b)}{H(b)} = \frac{L(b)}{N[B^t(\eta(b)) - b]}, \] (22)
which allows us to solve the joint bid distribution \( \dot{H}(\cdot). \)

From this simple first-order condition, we can see the roles of the three functions \( B(\cdot), B^t(\cdot), B^c(\cdot). \) These results and their implications will be shown below. Now we need to prove some preliminaries on the derivatives of the functions, and crossing properties of the three functions \( B^c(\cdot), B(\cdot), B^z(\cdot). \)

Lemma 4 The derivative of the function \( B(v) \) in (15) is given by
\[ B'(v) = \frac{Nf(v)}{F(v)} \left[ B^t(v) - B(v) \right] > 0, \text{ for } v > 0. \] (23)

Lemma 5 We have
\[ B'(v) - B^z'(v) = \frac{Nf(v)}{F(v)} \left[ \frac{1}{N} B^c(v) + \frac{N - 1}{N} B^z(v) - B(v) \right], \text{ for } v > 0. \] (24)

Lemma 6 If any two of the functions \( B(\cdot), B^z(\cdot), B^c(\cdot) \) are the same over some interval \([v_1, v_2]\), then all three functions are equal to each other over \([v_1, v_2]\).
The following says that at the point where \( B^c(\cdot) \) crosses \( B(\cdot) \), we also have the crossing of \( B_1(\cdot) \) and \( B(\cdot) \) in the other direction.

**Lemma 7** Let \( N > 1 \), and \( B_1(\cdot) \) be defined in \((z,v)\) or \((v', z)\) with the boundary condition \( B_1'(z) = B(z) \), or \( B_1'(\cdot) = b_0(\cdot) \) at \( z = 0 \). Then the following three statements hold (with the understanding that the inequalities hold in \((z,v)\) or \((v', z)\) except a countable closed subset):

\[
\begin{align*}
(a) & \quad B^c(\cdot) > B(\cdot) \text{ if and only if } B(\cdot) > B_1^c(\cdot), \\
(b) & \quad B^c(\cdot) < B(\cdot) \text{ if and only if } B(\cdot) < B_1^c(\cdot), \\
(c) & \quad B^c(\cdot) = B(\cdot) \text{ if and only if } B(\cdot) = B_1^c(\cdot).
\end{align*}
\]

The following is a very useful sufficient condition for the equilibrium with inactive speculators to be shown later.

**Lemma 8** Let \( N > 1 \). Given an equilibrium strategy \( \bar{B}(\cdot) \), we can define \( B_1^c(\cdot) \) at \( z \) with the initial condition \( B_1^c(z) = B(z) \). Then over the interval \([z, z']\), we have

\[
B^c(\cdot) \leq B_1^c(\cdot) \text{ implies } B(\cdot) \leq B_1^c(\cdot)
\]

In particular when \( B^c(\cdot) \leq b_\beta(\cdot) \) over \([0, \beta]\), then \( B(\cdot) \leq b_\rho(\cdot) \) over \([0, \beta]\).

In equilibrium, typically speculators are active in disjoint open intervals. We now offer an intuitive explanation of this equilibrium behavior. Assume that initially there is no speculator. According to the standard first-price symmetric auction, bidders are bidding \( b_\rho(\cdot) \) in equilibrium. A speculator does not enter bidding if there is no bid such that the revenue from resale \( B(\cdot) \) exceeds the cost of winning \( b_\rho(\cdot) \). If there is a bid yielding positive profit, a speculator will enter. Once a speculator becomes active in bidding, the regular buyers will respond to the entry, and bid \( B(\cdot) \) which reduces a speculator profit to 0. Thus in active intervals, regular buyers bid the revenue function. At a point where the function \( B^c(\cdot) \) crosses \( B(\cdot) \) from above, the cost function crosses the revenue function from below. When this happens, the active interval ends. A regular buyer raises the bid to \( B^c(\cdot) \) which makes the speculator inactive as now the new cost exceeds the revenue. Given the new endogenous cost function, a speculator may find another higher profitable bid. This will end the inactive interval, and brings in a new active interval. So it may continue. During the inactive intervals, the bidding function is the cost function. An example in Part II shows that speculators in equilibrium bid (with positive probability) on two disjoint intervals, separated by inactive intervals. This pattern of behavior should be kept in mind as we analyze the equilibrium in the following two sections.

### 3.5 First-order condition of a regular buyer

The following gives the derivative of a regular buyer’s payoff with respect to \( b \).

**Lemma 9** We have the following derivative

\[
\frac{\partial u(v, b, \sigma)}{\partial b}|_{b=b(v)} = \tilde{H}(b)F^{N-1}(\phi(b))[B'(\phi(b)) - b] + \tilde{H}(b)\frac{\partial u_0(v, b, \phi)}{\partial b}|_{b=b(v)} \text{ for } b > \rho. \tag{25}
\]

**Proof.** For the term \( v_{t_{b_0}} \), \( v_t \) is now denoted by \( v \). It is either zero, or given by (13). Taking the derivative with respect to \( b \), we have

\[
\frac{\partial}{\partial b}\int_b^{b(v)} (v - \phi(y))\tilde{H}(y)dF^{N-1}(\phi(y))|_{b=b(v)} = 0.
\]
For the case \( \phi(b) < v \), taking the derivative of (5) with respect to \( b \), we have
\[
\frac{\partial u(v, b, \sigma)}{\partial b}|_{b=b(v)} = \hat{H}'(b)F^{N-1}(\phi(b))[v - b - \int_{v} F^{N-1}(x|\phi(b))dx] + \hat{H}(b)\frac{\partial u_0(v, b, \phi)}{\partial b}|_{b=b(v)},
\]
which is the same as (25). The formula (25) holds in the other case as well if
\[
\frac{\partial u_0(v, b, \phi)}{\partial b}|_{b=b(v)} = 0.
\]
Since \( J(\phi(b), \phi(b)) = v \), \( J(r(v, \phi(b)), \phi(b)) = v \), \( r(\phi(b), \phi(b)) = \phi(b) \), we have
\[
\frac{\partial u_0(v, b, \phi)}{\partial b}|_{b=b(v)} = \frac{\partial u_0(v, b, \phi)}{\partial b}|_{b=b(v)} = \hat{H}'(b)\int_{v}^{\phi(b)} (J(x, \phi(b)) - v)dF^{N-1}(x)|_{b=b(v)} = 0.
\]
Hence the derivative in the case \( \phi(b) \geq v \) leads to the same formula (25).

**Lemma 10** We have
\[
\frac{\partial u_0(v, b, \eta)}{\partial b}|_{b=B(v)} = -\frac{1}{N}F^{N-1}(v)L(v).
\]

**Proof.** We have
\[
\frac{\partial u_0(v, b, \eta)}{\partial b}|_{b=B(v)} = -F^{N-1}(\eta(b)) \left[ 1 - (N-1)(\eta(b) - b) \frac{f(\eta(b))}{F(\eta(b))} \right],
\]
where the expression inside the bracket is
\[
1 - (N-1)(v - B(v)) = \frac{f(v)}{F(v)B'(v)} = \frac{B'(v) - (N-1)(v - B(v))}{B'(v)}. \tag{27}
\]
Using Lemma 4, and
\[
v - B^c(v) = N(v - B^t(v)),
\]
we can rewrite (27) as
\[
\frac{1}{B'(v)F(v)} (N(B^t(v) - B(v)) - (N-1)(v - B(v)))
\]
\[
= \frac{1}{N(B^t(v) - B(v))} (N(B^t(v) - B(v)) - (N-1)(v - B(v)))
\]
\[
= \frac{1}{N(B^t(v) - B(v))} (N(v - B(v)) - N(v - B^t(v)) - (N-1)(v - B(v))
\]
\[
= \frac{(v - B(v)) - (v - B^c(v))}{N(B^t(v) - B(v))} = \frac{B^t(v) - B(v)}{N(B^t(v) - B(v))} = \frac{1}{N}L(v).
\]

The following summarizes the first-order conditions of a regular buyer in active and inactive intervals mentioned in the last section.

**Lemma 11** Let \( H(., b(.)) \) be an equilibrium strategy profile, \( \phi(.) = b^{-1}(.) \). In inactive intervals, we have
\[
\frac{\partial u_0(v, b, \phi)}{\partial b}|_{b=b(v)} = 0, b > \rho. \tag{28}
\]
In active intervals, we have \( \phi(b) = \eta(b) \), and

\[16\]
\[ \frac{\ddot{H}'(b)}{H(b)} = \frac{L(b)}{N(B'(\eta(b) - b))}, \quad b > \rho. \] (29)

**Proof.** In inactive intervals, \( H'(\cdot) = 0 < H(\cdot) \) implies (28). Let the speculators be active in any open interval \( I \). By Lemma 1, we must have \( \phi(b) = \eta(b) \) in the interval. Setting the derivative (25) equal to zero in equilibrium, and apply Lemma 10, we have

\[ \ddot{H}'(b)F^{N-1}(\eta(b))[B'(\eta(b)) - \bar{b}] - \ddot{H}(b)\frac{1}{N}F^{N-1}(v)\mathcal{L}(v) = 0. \]

Since \( b = B(v) \), we have

\[ \frac{\ddot{H}'(b)}{H(b)} = \frac{\mathcal{L}(v)}{N(B'(v) - B(v))} = \frac{L(b)}{N(B'(\eta(b) - b))}. \] (30)

We have \( B^c(v) \leq B^r(v) \) by definition, and the equality holds only if \( N = 1 \) or \( v = 0 \). In general we have \( \mathcal{L}(v) \leq 1 \). When \( N = 1 \), we have \( B'(v) = B^r(v) = \tau(v) \), and \( \mathcal{L}(v) = 1 > 0 \). When \( N > 1 \), we may have \( \mathcal{L}(v) < 0 \), and (29) implies a restriction on where a speculator can be active. The following is an immediate implication of Lemma 11).

**Lemma 12** In an active open interval \( I \) above \( \rho \) of the speculators, we must have \( L(b) > 0 \), except a countable closed subset, or equivalently \( B^c(v) > B(v) \) in the active interval \( \eta(I) \), except a countable closed subset.

**Lemma 13** If \((z_1, z_2)\) is a maximal active interval, then we have \( B^c(z_2) = B(z_2) \).

Proof of Lemma 13: If \( B^c(z_2) > B(z_2) \), we will obtain a contradiction. Since \((z_1, z_2)\) is an active interval, there exists an equilibrium strategy \( H(\cdot), \dot{B}(\cdot) \) such that \( H'(\cdot) > 0 \) in \((z_1, z_2)\). By Lemma 11, we have \( H'(b) > 0 \) in a neighborhood of \( z_2 \) if \( B^c(z_2) > B(z_2) \). By definition, then \((z_1, z_2 + \varepsilon)\) is also an active interval, violating the maximal property.

### 3.6 Characterization of active and inactive intervals

It can occur quite often that speculators are not active in equilibrium. When speculators are not active in equilibrium, they have no influence on the outcome of the auction, and regular bidders bid as if there are no speculators. We now give a condition that tells us precisely when speculators are active in equilibrium. Assume that initially there are no speculators, and according to the standard first-price symmetric auction, bidders are bidding \( b_p(\cdot) \) in equilibrium. If the revenue \( B(\cdot) \) exceeds the cost \( b_p(\cdot) \) at some \( v = \phi(b) \), then there is profit to be made, and a speculator will probably enter and bid actively. Otherwise, there will be no entry, and we should have an equilibrium with inactive speculators in this case. Thus, intuitively, for any possible entry of a speculator in the bidding, there should be a bid \( b > \rho \) at which a speculator can make a profit. This intuition can be formally verified in the following theorem.

Let \( \pi_p(b) = B(\phi_p(b)) - b \), \( b \in [\rho, b_p(\cdot)] \) be the profit function for the speculator when the regular bidders use the bidding function \( b_p(\cdot) \). When \( N = 1 \), the function \( \pi_p(b) \) is defined only for \( b = \rho \).

**Theorem 2** If \( \pi_p(b) \leq 0 \) for all \( b \in [\rho, b_p(\cdot)] \), then in equilibrium speculators must be inactive, and regular bidders bid \( b_p(v) \). The equilibrium is unique, and is identical to the one without speculators and without resale. If \( \pi_p(b) > 0 \) for some \( b \in [\rho, b_p(\cdot)] \), then speculators must be active in an equilibrium.
Proof. From the assumption, \( \pi_\rho(b) = B(\phi_\rho(b)) - b \leq 0 \), we have \( B(v) - b_\rho(v) \leq 0 \) for all \( v \). Let \( \phi(.) \) be the inverse equilibrium bidding strategy of the regular bidders. If a speculator is active in equilibrium, by Lemma 1, there exists an active interval \([b_1, b_2]\) such that \( \eta(b) = \phi(b) \) on the interval. We claim that on the interval \([\eta(b_1), \eta(b_2)]\) \( [x_1, x_2] \) we have \( B(.) = b_\rho(.) \). Otherwise, there exists an open interval \( J \) in \([b_1, b_2]\) such that \( \pi_\rho(b) = B(\phi_\rho(b)) - b > 0 \) contradicting Lemma 1. However, Lemma 6 implies that \( B^c(.) = B(.) \) over \([x_1, x_2]\) as well, hence we have \( B^c(.) = B(.) \) over \([x_1, x_2]\) and this contradicts Lemma 12. Thus we have shown that no speculator can be active in equilibrium. We obtain the same equilibrium \( b_\rho(.) \) in the model without resale as shown in Theorem 1. If \( \pi_\rho(b) > 0 \) for some \( b \in [\rho, b_\rho(\beta)] \), and speculators are inactive in an equilibrium, then regular bidders will bid \( b_\rho(.) \) in equilibrium, but in this case speculators can make a strictly positive profit by bidding \( b \), contradicting the zero profit condition. Hence the theorem is proved.

Theorem 2 tells us precisely when speculators are inactive in equilibrium. It can be rephrased as follows: The bid interval \([\rho, b(\beta)]\) is an inactive interval (equivalently, \([\rho, \beta]\) is an inactive value interval) if and only if \( B(.) \leq b_\rho(.) \) in \([\rho, \beta]\). We now generalize this result to all inactive intervals and give a simple characterization of inactive intervals by endogenous cost functions. Given any equilibrium strategy \( B(.) \) and interval \([z, z']\), we can define the endogenous cost function \( B^z(.) \) with the initial condition \( B^z(z) = B(z) \). Let \( \phi^z(.) \) be its inverse. Define \( \pi^z(b) = B(\phi^z(b)) - b \) for \( b \geq B(z) \). If speculators are inactive in the interval, we know that the first-order condition (8) must be satisfied, and both \( B^z(.) \) and \( B(.) \) have the same initial condition, therefore we have \( B^z(.) = B(.) \) in \([z, z']\). The speculators make maximum profit 0 in equilibrium, and we must have \( \pi^z(b) \leq 0 \). This condition also is sufficient for being an inactive interval as stated in the following. The proof is the same as that of Theorem 2.

Lemma 14 The interval \([z, z']\) is an inactive interval if and only if \( \pi^z(b) \leq 0 \) for all \( b \in [B(z), B^z(z')] \), or equivalently, \( B(.) \leq B^z(.) \) in \([z, z']\).

Note that a corollary of the characterization of inactive equilibrium is that when the speculator is inactive in one equilibrium, he must be inactive in another equilibrium, as the condition \( \pi_\rho(b) \leq 0 \) for all \( b \in [\rho, b_\rho(\beta)] \) is independent of the equilibrium profile. Hence the equilibrium is unique when \( H(\rho) = 1 \) in an equilibrium. We define inactive intervals through \( H(.) \). Now we want to show that if an interval is inactive in one equilibrium, it must be inactive in another equilibrium, if there is another one.

Lemma 15 If \([z, z'], z \geq \rho \), is inactive in one equilibrium, it must be inactive in any other possible equilibrium.

Proof. It is sufficient to prove this for a maximal inactive interval. Let \([z, z']\) be a maximal inactive interval in the equilibrium \( H(.) \), \( B(.) \). Either we have \( z = \rho \), or \( \bar{B}(z) = B(z) \). If \( z = \rho \), then the non-positive profit condition implies \( B(.) \leq \bar{B}(.) \) in \([z, v]\), an Lemma 7 implies that \( B^c(.) \leq B(.) \) in \([z, v]\). If \( \bar{B}(z) = B(z) \), define \( B^z(.) \) with the initial condition \( B^z(z) = B(z) \), then the same arguments in the proof of Lemma 7 shows that we must have \( B^c(.) \leq B(.) \) in \([z, v]\) as well. Assume that there is another equilibrium \( \bar{H}(.), \bar{B}(.) \) such that \([z, z']\) is not an inactive interval. There is a maximal active interval \((c, d)\) of \( H(.) \) that overlaps with \([z, z']\) in some open interval \((x, x')\). By continuity, the speculator profit must be 0 at \( c \), hence \( \bar{B}(c) = \bar{B}(c) \). Define \( B^z(.) \) with the initial condition \( B^z(z) = \bar{B}(c) \). By Lemma 12, we have \( B^c(.) > B(.) \). This is a contradiction. The contradiction proves that \([z, v]\) must be inactive in another equilibrium.

From Lemma 14 and 8, we immediately have the following simple necessary condition based on \( F(.) \) that tells us where inactive intervals are located.

Lemma 16 Let \( B^z(.) \) be defined over \([z, z']\) by the initial condition \( B^z(z) = B(z) \). The value interval \([z, z']\) is an inactive interval if \( B^c(.) \leq B^z(.) \) in the interval. In particular if \( B^c(.) \leq b_\rho(.) \) in \([\rho, \beta]\), then speculators must be inactive in equilibrium.

Lemma 16 is actually a very powerful tool for showing an inactive interval. We provide more results of this nature in Part II. Here we will just illustrate with examples the usefulness of this lemma.
Example 1 Suppose there are two regular buyers and one speculator and $\rho = 0$. If $F(v) = v$ is the uniform distribution, we have $b_0(v) = \frac{1}{2}v$ and

$$B^c(v) = v - \frac{2}{v} \int_0^v x \, dx = 0 < b_0(v).$$

Hence Lemma 16 implies that the speculator is inactive in equilibrium when $F(v)$ is a uniform distribution. This is true for any number of regular buyers. If $F(v) = v^2$, we have $b_0(v) = \frac{2}{3}v$, and

$$B^c(v) = v - \frac{2}{v^2} \int_0^v x^2 \, dx = \frac{1}{3}v < b_0(v).$$

Hence the speculator is inactive in equilibrium, when $F(v) = v^2$, and this is also true for any number of regular buyers. In either case, a direct proof is not a simple matter, but Lemma 16 provides an easy answer\(^{12}\).

Since the active intervals are complementary to inactive intervals, Lemma 16 also implies that the active intervals are uniquely determined regardless of the $H(.)$ used to define it.

Lemma 17 If a value interval $(x_1, x_2)$ is active in one equilibrium, it must also be active in another equilibrium.

From Lemma 12 and Lemma 7, we immediately have the following necessary condition of an active interval.

Lemma 18 If $(v_1, v_2), v_1 \geq \rho$, is an active interval, then for any $B^*_z(.)$ defined at $z < v_1$ with the initial condition $B^*_z(z) = \bar{B}(z)$, we must have $B^c(.) > B(.) > B^*_z(.)$ over $(v_1, v_2)$ except a countable closed subset.

We will show the conditions to be sufficient for an active interval as well after we construct an equilibrium. In any equilibrium, Lemma 11 implies that $H^\prime(.) > 0$ in the interval when $B^c(.) > B(.)$. By definition, this means that the interval must be an active interval. Thus when we have the existence of equilibrium, the necessary conditions are also sufficient for an active interval.

4 Equilibrium Solution

We have the existence and uniqueness of equilibrium in Theorem 2 when speculators are inactive. The equilibrium bidding strategy of a regular buyer is simply given by $b_\rho(.)$. We will provide the equilibrium solution in this section under the assumption that the equilibrium strategy is increasing and symmetric.

Now we show the existence and uniqueness of equilibrium and give a simple solution when speculators are active in equilibrium. This occurs if and only if $\pi_\rho(b) > 0$ for some $b \in [\rho, b_\rho(\beta)]$. We know that the active intervals are the same for all equilibria. In equilibrium, we must have $\bar{B}(.) = \bar{B}(.)$ over an active interval. For an inactive interval, the equilibrium $\bar{B}(.)$ is the endogenous cost function defined at the end point $z$ of each active interval, with the initial condition $B^*_z(z) = \bar{B}(z)$. The job is easily done if we can construct all the active intervals. When a speculator is active in equilibrium, in general he or she is active over several disjoint bidding intervals. We start with the case when there is only one maximal active interval.

\(^{12}\)In fact the same conclusion holds for any power function, concave function, or many often-used value distributions. This is treated more extensively in Part II of the paper.
4.1 Finite-Interval Case

The following gives the necessary and sufficient condition for the existence of an equilibrium which is unique and has exactly one maximal active interval.

**Lemma 19** Let \( N > 1 \). The interval \((z_1, v_1)\), \( z_1 \geq \rho \), is a unique maximal active interval if and only if all the following three conditions hold: (i) \( B(.) \leq b_p(.) \) in \([\rho, z_1]\); (ii) \( B^*(.) > B(.) \) on \((z_1, v_1)\), except a countable closed subset; (iii) \( B(.) \leq B^*_{v_1}(.) \) in \([v_1, \beta]\). The equilibrium strategy profile in this case is unique and is given by

\[
\bar{B}(v) = \begin{cases} 
  b_p(v), & v \in [\rho, z_1], \\
  B(v), & v \in [z_1, v_1], \\
  B^*_{v_1}(v), & v \in [v_1, \beta],
\end{cases}
\]

(31)

and

\[
\bar{H}(b) = \exp\left(-\int_b^{B(v_1)} \frac{L(y)}{N(B^*(\eta(y)) - B(\eta(y)))} dy, b \in [\rho, B(v_1)].
\]

(32)

The speculator is active in the bid interval \((B(z_1), B(v_1)) = (a_1, b_1)\), and inactive in \([\rho, a_1]\) and above \( b_1 \).

**Proof.** The three conditions are clearly necessary for the maximal active interval. To prove the sufficiency, Note that condition (i) and Lemma 14 together implies that the speculator is inactive in \([\rho, b_p(z_1)]\). Similarly, condition (iii) also implies that \([v_1, \beta]\) in an inactive interval. To show that \((z_1, v_1)\) is an active interval, we will prove that the equilibrium strategy profile specified in the statement is an equilibrium strategy. Since the speculators are not active in \([\rho, a_1]\), \( b_p(.) \) is an equilibrium strategy for a regular buyer in \([\rho, z_1]\). The speculators make zero profit bidding in the interval \((a_1, b_1)\), and non-positive profit elsewhere, hence this is optimal for them. For the regular buyers, the first-order condition of equilibrium is satisfied because of the way \( \bar{H}(.) \) is defined. The supermodularity property insures that \( B(.) \) is an optimal strategy for a regular buyer in \((z_1, b_1)\). The optimality of the strategy \( B^*_{v_1}(.) \) in \([v_1, \beta]\) is also clear as the speculators are not active above \( b_1 \). The uniqueness is also clear as the active and inactive intervals are uniquely determined, and on active intervals, regular buyers must bid \( B(.) \), while on inactive intervals, the boundary conditions are clearly uniquely determined, and hence the strategy of a regular buyer must be the same as the one given in (31). Once \( B(.) \) is uniquely determined, \( \bar{H}(.) \) is also clearly determined, and must be given by (32).

If there are finitely many active intervals, at the end of one active interval, the following gives the necessary and sufficient condition for the next maximal active interval. This gives us the recursive construction of all the active intervals. Given the initial maximal active interval, and the end point \( v_1 \) of the interval, we can define the endogenous cost function \( B^*_1(.) = B^*_{v_1}(.) \) with the initial condition \( B^*_1(v_1) = B(v_1) \). We can obtain the next maximal active interval \((z_2, v_2)\) in a similar way. More generally, let \( v_k \) be the end point of an active interval. Let \( B^*_{v_k}(.) = B^*_{v_k}(.) \) be the endogenous cost function with the initial condition \( B^*_{v_k}(v_k) = B(v_k) \). We have the following recursive characterization result for the next maximal active interval. The proof is the same.

**Lemma 20** Let \( N > 1 \), and \( v_k \) be the end point of an active interval. Let \( z_{k+1} > v_k \), then \((z_{k+1}, v_{k+1})\) is the next maximal active interval if and only if all the following three conditions hold: (i) \( B(.) \leq B^*_{v_k}(.) \) in \([v_k, z_{k+1}]\), (ii) \( B^*(.) > B(.) \) on \((z_{k+1}, v_{k+1})\) except a countable closed subset; (iii) \( B(.) \leq B^*_{v_{k+1}}(.) \) in some interval \([v_{k+1}, v_{k+1} + \epsilon]\). The equilibrium strategy profile is uniquely determined for the range stated below, if \( b_{k+1} = \bar{H}(v_{k+1}) \) is uniquely determined, and must be given by

\[
\begin{align*}
\bar{B}(.) &= B^*_{v_{k+1}}(.) \text{ in the inactive interval } [v_k, z_{k+1}], \\
B(.) &= B(.) \text{ in the active interval } [z_{k+1}, v_{k+1}],
\end{align*}
\]

and

\[
\bar{H}(b) = \exp\left(-\int_b^{b_{k+1}} \frac{L(y)}{N(B^*(\eta(y)) - B(\eta(y)))} dy, b \in [\rho, b_{k+1}].
\]
Assume that the following assumption is satisfied.

(F) There is a finite number of solutions in the equation $B^c(\cdot) = B(\cdot)$.

The generic property of this assumption will be investigated in Part I of the paper. When $F(\cdot)$ is analytic, assumption (F) always holds. Hence (F) is essentially satisfied by all computable function $F(\cdot)$. Let $(x_k), k = 1, 2, \ldots, m'$ be the list of all intersection points above but except $\rho$, arranged according to its size. Drop the point $x_0$ from the list if $B^c(\cdot) - B(\cdot)$ has the same sign to its left and right. Use the same notation $(x_k), k = 1, 2, \ldots, m''$ for the new list. Let $x_0 = \rho$. If $B^c(\cdot) - B(\cdot) > 0$ in $(x_{k-1}, x_k)$, it is called a positive range, otherwise, it is called a negative range. Because of our choice, positive and negative ranges alternate.

Under the assumption (F), we can show that there are only finitely many maximal active intervals. Now we describe the recursive process of constructing all the maximal active intervals. Starting with the cost function $b_\rho(\cdot)$.

Step one: If $B(x_k) < b_\rho(x_k)$ for all end points of positive ranges $x_k$, stop, and we will show that the speculator is inactive in equilibrium, and there is no active interval.

Step two: Assume there exists some $x_k$ such that $B(x_k) \geq b_\rho(x_k), (x_{k-1}, x_k)$ is a positive range, and the following properties are satisfied

Either $b_\rho(\cdot) < B(\cdot)$ in $(\rho, x_k)$, or $b_\rho(\cdot)$ crosses $B(\cdot)$ from above at a point in $[x_{k-1}, x_k]$.  \hspace{1cm} (33)

If no $x_k$ has the above properties, then there is no active interval, and speculators are inactive in equilibrium. Let $x_{k'}$ be the first $x_k$ with the above properties, and rename it $v_1$. We will show that $v_1$ is the end point of the first active interval. Note that the first possibility in (33) only occurs when $\rho = 0$. When $\rho > 0$, we have $b_\rho(\rho) > B(\rho)$ for $N > 1$. Thus only the second possibility is applicable when $\rho > 0$.

Step three: If $B(x_k) < B^*_v(x_k)$ for all end points of positive ranges $x_k > v_1$, stop, and there is no more active intervals. Check whether there exists some $x_k > v_1$ such that $B(x_k) \geq b_\rho(x_k), (x_{k-1}, x_k)$ is a positive range and

$B^*_v(\cdot)$ crosses $B(\cdot)$ from above at a point in $[x_{k-1}, x_k]$. \hspace{1cm} (34)

If there is no such $x_k$, there is no more active intervals, the process stops.

Step four: If there exists some $x_k > v_1$ such that $B(x_k) > B^*_v(x_k), (x_{k-1}, x_k)$ is a positive range, and (34) holds, choose the first one and rename it $v_2$. We will show $v_2$ to be the end point of the second active interval.

This process continues using the property (34) in each step. The process stops when we arrive at the last end point $v_m$ of an active interval, if not earlier. At the end, we have a list $(v_1, v_2, \ldots, v_m)$ if we go beyond the first step. For convenience, let $v_{m+1} = \beta$, if $v_m \neq \beta$. Define $B^*_k(\cdot) = B^*_v(\cdot)$ with the initial condition $B^*_k(v_k) = B(v_k)$. Define

\[
B^*(\cdot) = \begin{cases} 
    b_\rho(\cdot) & \text{in } [\rho, v_1), \\
    B^*_k(\cdot) & \text{in } [v_k, v_{k+1}), k < m, \\
    B_m(\cdot) & \text{in } [v_m, \beta], \text{ if needed}.
\end{cases}
\]

This uniquely defines $B^*(\cdot)$, which is continuous everywhere, except possibly at $v_k$, where we may have an up-jump, and we only have right-continuity at such points. However, the function $\max\{B(\cdot), B^*(\cdot)\}$ is a continuous function everywhere.

From Lemma 14, we know that in an inactive interval, we have $B(\cdot) \leq B^*_v(\cdot)$, but in an active interval, Lemma 18 says that the ranking is reversed. In either the active or inactive intervals, $B(\cdot)$ is the larger of the two. We have the following equilibrium solution which contains Theorem 2 as a special case. We used $b_\rho, b$ for the maximum bids of the speculators and the regular buyers respectively. Note that when $b_\rho = 0$ if and only if the speculators are inactive in equilibrium.

**Theorem 3** We make the assumption (F). In the first-price auction with speculative resale model, either (i) we have an equilibrium with inactive speculators, or (ii) there exists a uniquely determined sequence
of intervals \((z_k, v_k), k = 1, 2, \ldots, m\), and a unique equilibrium with speculation active in each \((z_k, v_k)\), and inactive elsewhere. The equilibrium is given by

\[
\hat{B}(\cdot) = \max\{B(\cdot), B^*(\cdot)\} \text{ in } [\rho, \beta],
\]

\[
\hat{H}(b) = \exp \left( - \int_b^{\hat{b}_s} \frac{\hat{L}(y)}{N(B^*(\eta(y) - y))} \, db \right) \text{ in } [\rho, \hat{b}_s],
\]

where \(\hat{b}_s = B(v_m)\) is the maximum equilibrium bid of the speculators. Moreover the equilibrium has the following first-order stochastic dominance property

\[
\hat{H}(b) > F(\phi(b)) \text{ for all } b < \hat{b}.
\]

**Proof.** In step one, assume that there is no \(x_k\) such that \(b_p(x_k) \leq B(x_k)\). We will prove that \(b_p(\cdot) \geq B(\cdot)\)

in \([\rho, \beta]\). Assume it is false, then \(b_p(\cdot)\) must cross \(B(\cdot)\) somewhere from above, and we will end up with a contradiction. If the first interval \((\rho, x_1)\) is a positive range, by Lemma 7, we have \(b_p(x_1) \leq B(x_1)\), a contradiction. If \((\rho, x_1)\) is a negative range, we have \(b_p(\cdot) > B(\cdot)\) by the same Lemma. If \(b_p(\cdot)\) crosses \(B(\cdot)\) at \(x_1\), by the same Lemma, we have \(b_p(x_2) \leq B(x_2)\) which is a contradiction. Hence \(b_p(\cdot)\) must cross \(B(\cdot)\) from above at some \(v > x_1\). Assume it crosses in \((x_k, x_{k+1})\) where \(B^c(\cdot) - B(\cdot)\) is positive. Let \(z\) be the crossing point. Then apply Lemma 7 again for the range \((z, x_{k+1})\), we get another contradiction \(b_p(x_{k+1}) \leq B(x_{k+1})\). However \(b_p(\cdot)\) cannot cross \(B(\cdot)\) at a point \(z \in [x_k, x_{k+1})\) which is a negative range, by Lemma 5. The conclusion is that \(b_p(\cdot)\) cannot cross \(B(\cdot)\) from above, and we must have \(b_p(\cdot) \geq B(\cdot)\) in \([\rho, \beta]\). Speculators will be inactive in equilibrium by Lemma 14. In step two, if there exists no \(x_k\) with the specified properties, clearly, there cannot be any active interval from the necessary conditions we have established. Now assume we have \(v_1\) and consider the first case in (33). By Lemma 19, \((\rho, v_1)\) is an active interval. In the second case, we have \(v_1\) with the property that \(b_p(v_1) \leq B(v_1)\), but \(b_p(x_k) > B(x_k)\) for all \(x_k < v_1\). Using the same arguments in the step one, we can prove that \(b_p(\cdot) \geq B(\cdot)\) in \([\rho, v_1]\), where \(x_k\) is the neighboring point to the left of \(v_1\). Since \(b_p(\cdot)\) crosses \(B(\cdot)\) from above at some \(z_1 \in [x_{k'}, v_1)\). By Lemma 7 again, we have \(B(\cdot) > b_p(\cdot)\) in \((z_1, v_1)\) except a countable closed subset. We also know that \(B^c(\cdot) > B(\cdot)\) in \((z_1, v_1)\), and \(b_p(\cdot) \geq B(\cdot)\) in \([\rho, z]\). Therefore by Lemma 19, \((z_1, v_1)\) is the first active interval. This proves the statements in step two. Now we are in step three. By the same arguments in step one, using the cost function \(B^*_1(\cdot)\) instead of \(b_p(\cdot)\), we know there is no more active intervals after \(v_1\). In step four, we repeat the arguments in step two to prove the assertions made in step four. This continues until we stop at the last active interval \((z_m, v_m)\). We have also constructed the equilibrium strategy profile through Lemma 20, and the bidding strategy of a regular buyer is always the maximum of the two functions \(B(\cdot), B^*_1(\cdot)\) in each interval \((z_k, v_k)\). Furthermore the equilibrium is uniquely determined as long as each \(b_k = B(v_k)\) is uniquely determined. Since we must have \(\hat{H}(v_m) = 1\), and we have a unique equilibrium given by (35),(36). To show (37), consider a maximal active interval \((a_k, b_k) = (B(z_k), B(v_k))\), we have

\[
\ln \hat{H}(b_k) - \ln \hat{H}(a_k) = \int_{a_k}^{b_k} \frac{L(b)}{N(B^*(\eta(b) - b))} \, db = \int_{z_k}^{v_k} \frac{B^c(v) - B(v)}{N[B^*(v) - B(v)]^2} B'(v) \, dv
\]

\[
= \int_{z_k}^{v_k} \frac{L(v)}{F(v)} \, dv = \int_{z_k}^{v_k} L(v) \, d\ln(F(v))
\]

\[
< \int_{z_k}^{v_k} d\ln(F(v)) = \ln F(\eta(b_k)) - \ln F(\eta(a_k)).
\]

Over a maximal inactive interval \([v_k, z_{k+1}]\), we also have

\[
\ln \hat{H}(a_{k+1}) - \ln \hat{H}(b_k) < \ln F(\eta(a_{k+1})) - \ln F(\eta(b_k)),
\]

as the left-hand side is always 0. Hence for all \(b < \hat{b}_s\), we must have

\[
\ln \hat{H}(b_s) - \ln \hat{H}(b) < \ln F(\eta(b_s)) - \ln F(\eta(b)) < -\ln F(\eta(b)),
\]

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and we have
\[ \ln \hat{H}(b) > \ln F(\eta(b)), \]
or
\[ \hat{H}(b) > F(\eta(b)). \]
Rewrite in the variable \( v \),
\[ \hat{H}(B(v)) > F(v) \text{ for all } v < \eta(b). \]
Note that we have \( B(v) = \tilde{B}(v) \) over active intervals, and \( \hat{H}(.) \) is constant over inactive intervals. We have
\[ \hat{H}(B(v)) = \hat{H}(\tilde{B}(v)), \]
and this translates into
\[ \hat{H}(\tilde{B}(v)) > F(v) \text{ for all } v < \eta(b), \]
which is equivalent to
\[ \hat{H}(b) > F(\phi(b)) \text{ for } b < \tilde{b}, \]
which is also valid for \( b < b \), as \( \hat{H}(b) = 1 \) when \( b \geq \tilde{b} \). The proof is complete.

We now give an easily computable revenue formula. Let \( (z_k, v_k), k = 1, 2, ..., m \) be the active intervals, then \([v_{k-1}, z_k], k = 1, 2, ..., m, \) and \([v_m, \beta]\) are the inactive intervals. Let \( \beta = z_{m+1} \).

Let \( R(N, \rho) \) denote the revenue when there are \( N \) regular buyers, and the reservation price is \( \rho \).

**Theorem 4** When \( N > 1 \), the revenue is given by
\[ R(N, \rho) = \sum_{k=1}^{m} \int_{v_k}^{v_{k-1}} B(v)H_v(\rho) \left( 1 + \frac{1}{N} \mathcal{L}(v) \right) df^N(v) dv + \sum_{k=1}^{m+1} \int_{v_{k-1}}^{v_k} B^*(v) df^N(v), \]
where
\[ H_v(\rho) = \exp(-\int_{v}^{\beta} \mathcal{L}(x) \frac{f(x)}{F(x)} dx) \]
When \( N = 1 \), the revenue for \( \rho \leq \tilde{\rho} \) is given by
\[ \rho(F(\eta(\rho)) - F(\rho)) + \int_{\eta(\rho)}^{1} B(b) df^2(v). \]

**Proof.** We have, for each active interval \((z_k, v_k)\), the revenue
\[ \int_{a_k}^{b_k} db(F^N(\eta(b))\hat{H}(b)) = \int_{a_k}^{b_k} bF^N(\eta(b))\hat{H}'(b) db + \int_{a_k}^{b_k} b\hat{H}(b)db(F^N(\eta(b))) \]
\[ = \int_{a_k}^{b_k} bF^N(\eta(b))\hat{H}(b) \frac{L(b)}{N(\eta(b))} db + \int_{z_k}^{v_k} B(v)\hat{H}(v)df^N(v) \]
\[ = \int_{z_k}^{v_k} B(v)F^N(v)\hat{H}(v) \frac{\mathcal{L}(v)}{N(B'(v) - \eta(v))} (\hat{H}(v) - \hat{H}(v)) dv + N \int_{z_k}^{v_k} F^{N-1}(v)f(v)B(v)\hat{H}(v)dv \]
\[ = \int_{z_k}^{v_k} B(v)\hat{H}(v)F^{N-1}(v)f(v)\mathcal{L}(v)dv + N \int_{z_k}^{v_k} B(v)\hat{H}(v)F^{N-1}(v)f(v)dv, \]
\[ = \int_{z_k}^{v_k} B(v)\hat{H}(v)(1 + \frac{1}{N} \mathcal{L}(v)) df^N(v). \]
For each inactive interval, we have the revenue
\[ \hat{H}(b_{k-1}) \int_{b_{k-1}}^{a_k} dbF^N(\phi(b)) = \hat{H}(v_{k-1}) \int_{v_{k-1}}^{z_k} B^*(v) df^N(v). \]
Hence we just add up all the revenues. The case of \( N = 1 \) is quite simple. The proof is complete.
4.2 An illustrative Example of Active Speculation

Distribution functions that yield an active speculator in equilibrium are the ones that allow the optimal revenue \( B(v) \) of selling to \( N \) regular buyers with value distribution \( F(\cdot|v) \) to be higher at some point \( v \) than the expected revenue \( b_0(v) \) of the joint distribution of \( N - 1 \) regular buyers with the same value distribution.

A simple example of such a case can be given in discrete distributions. Suppose that the use value is either \( v = 0 \) with probability 0.5, and 1 with probability 0.5. The expected value of \( F(\cdot) \) is 0.5. However, a speculator can set the optimal reservation price \( r = 1 \), with the optimal revenue of selling to two regular buyers given by \( 1 - 0.5^2 = 0.75 > 0.5 \). Therefore when there are two regular buyers with this discrete distribution, a speculator must be active in equilibrium.

This idea can be applied to a continuous distribution through approximation as follows. Let

\[
F(v) = 0.5 + 0.5v^n \text{ over } [0, 1].
\]

With two regular buyers, we have

\[
b_0(1) = 1 - \int_0^1 0.5(1 + x^n)dx = 0.5 - \frac{0.5}{n+1}.
\]

We have \( r(1) = (n + 1)^{-\frac{1}{n}} \to 1 \) as \( n \to \infty \), and

\[
B(1) = 0.5 \int_{(n+1)^{-\frac{1}{n}}}^1 ((n + 1)x^n - 1)(1 + x^n)dx.
\]

Since

\[
\lim_{n \to \infty} B(1) = \lim_{n \to \infty} 0.5(1 - (n + 1)^{-\frac{n+1}{n}} + \frac{n + 1}{2n + 1} (1 - r^{2n})) = 0.75.
\]

Hence for large \( n \), a speculator must be active in equilibrium. In the limit there is an atom at the end point \( v = 1 \), hence a speculator becomes active for large \( n \). The idea of an atom at the end was used in the example of Garrett and Troger (2006).

Here we will give another more interesting example of active speculation based on the existence of an atom at 0. The example will also be used to illustrate many results in this section. Suppose \( F_0(v) = v^2 \). There are two regular buyers, and one speculator. It is shown earlier that the speculator is inactive in equilibrium in this case. Consider the change from \( F_0(v) = v^2 \) to the value distribution \( F_1(v) = 0.5(v^2 + 1) \) over \([0, 1]\), with an atom at 0 of size 0.5. This change is a down the rank in the first-order stochastic dominance. The atom means that with probability 0.5, a regular buyer has 0 value. Let \( \rho = 0 \). The cost function \( b_0(\cdot) \) is now lower than before as

\[
b_0(\cdot) = \frac{1}{F_1(v)} \int_0^v xdF_1(x) < \frac{1}{F_0(v)} \int_0^v xdF_0(x).
\]

However the revenue function \( B(\cdot) \) is higher as the upper range of the value has a higher density than before. This will be shown below. The change will allow the revenue function to overtake the cost function, leading to active speculation.

**Example 2** There are two regular buyers, and one speculator. Let the regular buyer’s use value distribution be

\[
F(v) = 0.5(1 + v^2) \text{ over } [0, 1].
\]

We have

\[
b_0(v) = \frac{2v^3}{3(1 + v^2)}.
\]

The revenue without the speculator or without resale is

\[
\int_0^1 b_0(x)dF^2(x) = 2 \cdot 0.5^2 \int_0^1 \frac{2x^3}{3(1 + x^2)}(1 + x^2)2xdx
\]
The virtual value is
\[ J(x, \beta) = x - \frac{1 - 0.8(1 + x^2)}{1.6x}. \]

The optimal reservation price for revenue is given by \( r(v) = \frac{v}{\sqrt{3}} \). We have
\[
B(v) = \frac{4v^3}{(1 + v^2)^2} \left[ \frac{1}{3\sqrt{3}} + \frac{1}{15}(2 + \frac{1}{3\sqrt{3}})v^2 \right],
\]
\[
B^*(v) = \frac{v}{1 + v^2} \left[ \frac{1}{\sqrt{3}} + \frac{1}{3}(2 + \frac{1}{3\sqrt{3}})v^2 \right],
\]
\[
B^c(v) = \frac{v}{1 + v^2} \left[ \frac{2}{\sqrt{3}} - 1 + \frac{1}{3}(1 + \frac{2}{3\sqrt{3}})v^2 \right].
\]

We have
\[
B(v) - b_0(v) = \frac{4v^3}{1 + v^2} \left[ \frac{1}{135\sqrt{3}} - \frac{1}{30}v^2 + \frac{1}{9\sqrt{3}} - \frac{1}{6} \right] > 0
\]
when \( v \leq 1 \). Hence \( B(\cdot) > b_0(\cdot) \) in \([0, 1]\). This however does not mean that the speculator is fully active. To determine active or inactive intervals, we first solve
\[
B^c(v) - B(v) = \frac{v}{(1 + v^2)^2} \left[ \left( \frac{8}{9\sqrt{3}} - \frac{2}{3} \right) v^2 + \left( \frac{2}{15\sqrt{3}} - \frac{1}{5} \right) v^4 + \frac{2}{\sqrt{3}} - 1 \right] = 0.
\]

We have a unique positive solution \( v_0 = 0.8120631 \), and \( B(v_0) = 0.2216755 \). Since \( B^c(\cdot) > B(\cdot) \) in \((0, v_0)\),
we know \((0, v_0)\) is an active value interval, and \((v_0, 1)\) is an inactive interval as \( B^c(\cdot) < B(\cdot) \) in this range.
The speculator is active in the bid interval \((0, B(v_0))\), and \( B(v_0) = b^*_v \) is the maximum equilibrium bid of the speculator. The endogenous cost function at \( v_0 \) is
\[
B^*_v(v) = \frac{0.01582887 + \frac{5}{3}v^3}{1 + v^2}.
\]

In equilibrium, a regular buyer bids \( B(\cdot) \) in \([0, v_0]\), and bids \( B^*_v(\cdot) \) in \([v_0, 1]\). The bid distribution of the speculator, expressed as a function of \( v \), is
\[
H(v) = \exp \left( -\int_v^{v_0} \frac{B^c(x) - B(x)f(x)}{B^*(x) - B(x)} F(x) dx \right)
\]
\[
= \exp \left( -\int_v^{0.8120631} \frac{\left( \frac{8}{9\sqrt{3}} - \frac{2}{3} \right) v^2 + \left( \frac{2}{15\sqrt{3}} - \frac{1}{5} \right) v^4 + \frac{2}{\sqrt{3}} - 1}{\left( \frac{2}{3} - \frac{2}{9\sqrt{3}} \right) x^2 + \left( \frac{1}{45\sqrt{3}} + \frac{2}{15} \right) x^4 + \frac{1}{\sqrt{3}} + \frac{2x}{1 + x^2} dx} \right).
\]

The revenue can be computed as follows
\[
R = 0.5^2 \int_0^{0.8120631} B(x)F(x)f(x)H(x)(2 + \frac{B^c(x) - B(x)}{B^*(x) - B(x)} dx + 0.5^2 \int_{0.8120631}^1 2B^*_v(x)(1 + x^2) dx.
\]
\[
= 0.1405397.
\]

The revenue is higher than the revenue without resale.
In the following graph, $B(\cdot), B_{v_0}^*(\cdot)$ in $(v_0, 1)$ are very close, but the difference can be seen when magnified.
4.3 General Treatment

We now drop the assumption (F). The open set \( \{ v : B^c(v) > B(v) \} \) is a union of countably many open intervals \((c_k, d_k), k \in K\). Note that it is possible \( d_k = c_i \) for some pair \( k, j \) and in this case we combine the two intervals \((c_k, d_k), (c_j, d_j)\) into one \((c_k, d_j)\). This is to be done for all such intervals, countably many if needed, so that we don’t have \( d_k = c_j \), for any pair, and each \((c_k, d_k)\) is separated from another \((c_j, d_j)\) by intervals in which \( B^c(.) \leq B(.)\). At each \( d_k \), define the cost function \( B^*_d(.) \) with the initial condition \( B^*_d(d_k) = B(d_k) \) as before. Let

\[
g(d_k) = \sup_{d_j < d_k} B^*_d(d_k) \tag{38}
\]

Eliminate any interval \((c_k, d_k)\) unless the function \( g(d_k) \) crosses \( B(.) \) from above at some point in \([c_k, d_k)\) . Denote the remaining intervals by the same notation \((c_k, d_k), k \in K\), but rename \( d_k \) as \( v_k \). Let

\[
v_k = \sup \{ v_j : v_j < v_k \},
\]

and define \( B^*_k(.) \) to be the endogenous cost function \( B^*_v(.) \) with the initial condition \( B^*_v(v_k) = B(v_k) \). Note that \( v_k \leq c_k < v_k \). We now have a countable collection of intervals \((c_k, v_k), k \in K\), such that \( B^*_k(.) \) crosses \( B(.) \) from above at some point in \([c_k, v_k)\). If \((c_j, v_j)\) is the interval to the immediate left of \((c_k, v_k)\), then we have \( v_k = v_j \). Let \( z_k \in [c_k, v_k) \) be the first point at which \( B^*_k(.) \) crosses \( B(.) \) from above. From our characterization of active intervals, we can show as before \((z_k, v_k)\) is a maximal active interval, and the countable collection is the set of all maximal active intervals.

Define \( B^*(.) = B^*_k(.) \) in each \([v_k, v_k)\). Let \( \bar{v} = \sup \{ v_k : k \in K \} \), and \( B^*(.) = B^*_\bar{v}(.) \) in \([v_m, \beta]\). This uniquely defines the function \( B^*(.) \). Let \( a_k = B(z_k), b_k = B(v_k) \). Define \( L(b) = \bar{L}(b) \) in each \((a_k, b_k)\), \( \bar{L}(b) = 0 \) elsewhere.

We can now state the general solution of the equilibrium as follows.

**Theorem 5** In the first-price auction with speculative resale model, either (i) we have an equilibrium with inactive speculators, or (ii) there exists a uniquely determined countable collection of intervals \((z_k, v_k), k \in K\), and a unique equilibrium with speculators active in each \((z_k, v_k)\), and inactive elsewhere. The equilibrium is given by

\[
\bar{B}(v) = \max \{ B(v), B^*(v) \}, v \in [\rho, \beta].
\]

\[
\bar{H}(b) = \exp(- \int_b^{b^*} \bar{L}(y)dy), b \in [0, b^*]
\]

where \( b^* = B(\bar{v}) \) is the maximum bid of the speculators. Furthermore, we have \( H(b) > F(\phi(b)) \) for all \( b < \bar{b} \).

5 Supermodularity Property

In this section, we examine the supermodularity property of the auctions with speculative resale model when strategies are not monotonic and different among the buyers. The results will apply to a model without speculators (set \( \bar{H}(0) = 1 \)) and with different value distributions \( F_i(.) \) for different bidders. As explained by Milgrom (2004, Chapter 4, section 4.1), the proper version of this property insures that the first-order condition of a bid in the maximization of payoffs is sufficient to guarantee the optimal property. It is also important for the increasing property of the equilibrium strategies. We will prove the symmetry property and increasing property for the equilibrium strategy profile. Symmetry property for the speculators in equilibrium does not hold in general, and will not be needed for our analysis. The proofs are offered in section 8. The version for the symmetric case has been discussed in section 3.1.
Given any profile of possibly non-symmetric and non-monotone bidding strategies \( \sigma \), we can define the payoff function \( u_i(v_i, b, \sigma) \) as in (1). We assume that \( b_j(\cdot), H(\cdot) \) are continuously differentiable\(^\text{13}\). This will insure that the payoff function has left and right derivatives with respect to \( b \). Even though we assume symmetric regular bidders in the model, Theorem 6 holds more generally with asymmetric bidders. The proof we provide does not require \( F(\cdot) \) to be the same for all buyers. Let \( \frac{\partial}{\partial b_+} \) denote the partial derivative from the right.

**Theorem 6** Let \( N > 1 \), and \( \sigma \) be a profile of possibly non-symmetric and non-monotone bidding strategies. Assume that regular buyer \( i \) with value \( v_i \) bids \( b \in (\max_{j \neq i} b_j, \max_{j \neq i} b_j), H(b) > 0 \). If there exists at least one regular buyer \( j \neq i \) such that
\[
v_i < w_j(b) < \beta.
\]
then we have
\[
\frac{\partial}{\partial b_+} \frac{\partial}{\partial v_i} u(v_i, b, \sigma) > 0.
\]
If condition (39) fails, then we have
\[
\frac{\partial}{\partial b} \frac{\partial}{\partial v_i} u(v_i, b, \sigma) = 0.
\]

Note that in section 3.1, we have seen why the condition (39) is needed for the strong form of supermodularity. The condition simply says that if there is payoff from selling the object after winning it in the first-stage auction, then we have the strong form of supermodularity. Otherwise, if there is only payoff from buying in the resale market after losing the first-stage auction, and no payoff after winning the object, then we only get the weak form.

The supermodularity property we have in Theorem 6 is sufficient for our equilibrium analysis. We have explained in section 3.1 that it is strong enough to guarantee that the first-order condition is sufficient for optimality. In the next section, we will show that it is also strong enough to imply that equilibrium strategies must be increasing.

Let \( Q_{1i}(v_i, b) \) be the probability of the event \( E_{1i}(v_i, b) \) that regular buyer \( i \) wins the first-stage auction bidding \( b \), but fails to sell in the resale stage. We prove the following lemma.

**Lemma 21** The functions \( \pi_{1i}(v_i, b, \sigma), \pi_{wi}(v_i, b, \sigma) \) are differentiable with respect to \( v_i \), and we have
\[
\frac{\partial}{\partial v_i} [\pi_{1i}(v_i, b, \sigma_{-i}) + \pi_{wi}(v_i, b, \sigma_{-i})] = Q_{1i}(v_i, b).
\]

The proof of Theorem 6 is simpler when the strategies are monotone, and \( w_j(b) > v_i \) for all \( j \neq i \). In this case there is no payoff for a regular buyer \( i \) after losing the first-stage auction. It is then a simple matter to show that \( \frac{\partial}{\partial b_+} Q_{1i}(v_i, b) > 0 \). This proves (40) for the case when there is no payoff from losing the first-stage auction.

When there is payoff after losing the first-stage auction and buying from the winner during the resale, additional arguments are needed. Let \( Q_{2i}(v_i, b) \) be the probability of the event \( E_{2i}(v_i, b) \) that regular buyer \( i \) loses the first-stage auction, but then wins it back during resale. We prove the the following lemma.

**Lemma 22** The function \( \pi_{1i}(v_i, b, \sigma) \) is differentiable with respect to \( v_i \), and we have
\(^\text{13}\)Only piecewise continuous differentiability assumption is needed here. Even if they are continuously differentiable, the revenue function need not be and are only piecewise continuously differentiable. See also footnote 9.
\[ \frac{\partial}{\partial v_i} \pi_{ii}(v_i, b, \sigma) = Q_{2i}(v_i, b). \]

Now consider the other extreme case when \( w_j(b) < v_i \) for all \( j \neq i \). Then \( \pi_{wi}(v_i, b, \sigma) = 0 \). Hence we have
\[
\frac{\partial}{\partial b} \frac{\partial}{\partial v_i} u(v_i, b, \sigma) = \frac{\partial}{\partial b} [Q_{1i}(v_i, b) + Q_{2i}(v_i, b)].
\]
But the sum of probabilities in the bracket above is independent of \( b \). When \( b \) is higher, there is higher probability from winning, but lower probability from possessing it during resale. We just shift the probabilities between the two stages without changing the probability of owning the object at the end. Hence we immediately have the following.

**Lemma 23** If condition (39) fails, then we have
\[
\frac{\partial}{\partial b} \frac{\partial}{\partial v_i} u(v_i, b, \sigma) = 0. \tag{42}
\]

Now assume that there is some \( j \neq i \) such that \( v_i \leq w_j(b) < \beta \). First assume that strategies are monotone. When \( b \) is higher, there is a higher probability of winning against such a buyer, but no loss of the probability of buying from him or her after losing the first-stage auction, as there was no probability of such buying anyway at the bid \( b \). For other buyers with \( w_j(b) < v_i \), the higher probability of winning against such buyers is cancelled by the lower probability of buying from them after losing the first-stage auction. Therefore we must have the strong form of supermodularity. When strategies are not monotone, this result still holds. If a regular buyer \( i \) loses to a buyer with \( w_j(b) > v_i \), buyer value \( v_j \) may be in a region below \( v_i \), and in this case there may be buying from buyer \( j \) just like the case when \( w_j(b) < v_i \). In this case, probability of winning the first-stage auction substitutes that of winning the second stage auction after losing when \( b \) is higher. However, there is a positive probability that \( v_j \) lies in a region above \( v_i \). Therefore we have the strong form of supermodularity as long as there is one buyer \( j \) with \( v_i \leq w_j(b) < \beta \).

Intuitively, we can think of \( E_{1i}(v_i, b) \) as a circular region with a hole (a donut). When \( b \) is raised to \( b' \), and condition (39) holds, the outer boundary expands, while the inner boundary shrinks. The shrinking in the inner hole compensates the loss of probability in \( E_{2i}(v_i, b) \). But the expansion of the outer boundary leads to an additional increase in probability. When (39) fails, the outer boundary does not expand, and we have (39). This is the intuition behind the two forms of supermodularity property. See the details of the proof in section 8.1.

## 6 Monotonicity and Symmetry

We will prove both properties together in our treatment. We will establish the result in several steps. First we shall deal with the symmetry property of the symmetric auction allowing resale, but without speculators. This is not a trivial issue, as we know that in a symmetric second-price auction with resale without speculators, there are many non-symmetric equilibria. We need to show that this would not happen in a symmetric first-price auction with resale without speculators. When we add the speculators, the symmetry property also holds, and the proof is very similar to the case without speculators.

**Lemma 24** The equilibrium in the auction with resale model without speculators must have the symmetry and increasing property for all regular bidders.
Now we state the following major result on the increasing symmetry property for the model with speculators.

**Theorem 7** The equilibrium strategy of the regular buyers in our model must be symmetric and increasing.

The arguments are presented in section 8. Intuitively, the proof goes as follows. We show that if all regular buyers have the same maximum bid, then a regular buyer with value $\beta$ must bid $b_1$ and we have increasing property for the equilibrium strategies near $\beta$. Then we show that the differential equations satisfied by the inverse equilibrium strategies force all of them to have the symmetry property near the maximum bid. The symmetry and increasing property can be further extended downward to $\rho$. To show that the maximum bid must be the same for all regular buyers, we note that if there are two different maximum equilibrium bids, the buyer with the highest maximum bid faces less or the same degree of competition than the buyer with the lower maximum bid, leading to a contradiction. This essentially means that the former can lower the bid to increase his or her payoff, violating the equilibrium condition.

### 7 Optimal Revenue with Speculative Resale

In this section, we will extend the optimal auction analysis of Myerson (1981) to our model of speculative resale. For this purpose, we shall first extend the revenue formula of Myerson (1981) to our model. The optimal auction result will follow easily from the formula as in the case of Myerson (1981). Bulow and Klemperer (1996) argue that focusing on more participation is more effective in raising revenue than the choice of an optimal reservation price. We will show that their argument also applies in auctions with speculative resale. For this purpose, we need to show first that adding one more regular buyer will indeed raise the original seller’s revenue, and we show that the Bulow-Klemperer argument is even more persuasive as the revenue increase from one more regular buyer can be even greater, compared to the revenue increase from setting the optimal reservation price.

#### 7.1 Myerson Revenue Formula

In this model of speculative resale, we will establish the revenue formula similar to that of Myerson (1981) with some natural adjustments needed. Although regular buyers make payments to both the auctioneer and the speculators, all revenue ends up in the hands of the auctioneer as speculators make zero profit in equilibrium. The winning probability is the sum of the winning probability in the first-stage auction and the winning probability in the resale auction. Let $r$ be the optimal reservation price set by a speculator facing a buyer with value distribution $F(.)$ according to the Myerson (1981) theory. When a regular buyer has value $v$ higher than $r$, then $v$ is always higher than the reservation price set by the winning speculator during resale. There will be resale to the buyer if he has the highest value among the losers. This means the regular buyer will get it back even if he loses in the first-stage auction. Hence the regular buyer has the winning probability $F^{N-1}(v)$ the same as that of a regular buyer in the auction with resale, if the equilibrium is symmetric (all regular bidders bid the same way). If the regular buyer has value below $r$, then speculators may set a reservation price above his value in the resale market, and as a result fail to sell to the buyer even if he has the highest value among all the losing bidders. In this case the winning probability will be of the form $t(v)F(v)^{N-1}$ for a regular bidder with value $v$. The term $t(v)$ depends only on $F(.)$ and the joint bid distribution $\hat{H}(.)$ of the speculators. We have $t(v) = 1$ when $v \geq r$, and $t(v) < 1$ when $v < r$. Computing the revenue using this information about the winning probabilities gives us a formula similar to that of Myerson (1981). The virtual value, $J(v)$ is now discounted (keeping the same sign) by $t(v)$, which is equal to one when $v \geq r$. We have the following formula for the revenue of an auction with speculative resale

$$\int_{\rho}^{\beta} t(v)J(v) dF^N(v).$$
Since the virtual value is positive above \( r \) and negative below \( r \), we immediately have three implications from this revenue formula. Firstly speculative resale would not change the optimal reservation price \( r \). Secondly, resale would not affect the optimal revenue as \( t(v) = 1 \) for \( v \geq r \). Thirdly, since \( t(v) < 1 \), \( J(v) < 0 \) when \( v < r \), we have \( t(v)J(v) = J(v) \) for all \( v \geq r \), and \( t(v)J(v) > J(v) \) for \( v < r \). Thus we conclude that for all reservation price set by the auctioneer, the revenue is strictly higher in auctions with resale than without resale, as long as the speculators are active in equilibrium.\(^{14}\)

The winning probability of a regular buyer is now the sum of the probability of winning from the original seller and that of winning from the speculators. Since speculators make zero profit, the total contribution to revenue by a regular buyer to the speculators and the original seller can be added up to get the total revenue of the original seller. The probability of winning is now a fraction of the winning probability in the auction without resale model. To express this discount, let \( \hat{H}(s) \), \( \hat{B}(s) \) be the equilibrium strategy profile. Since we have \( \hat{B}(s) = B(s) \) in active intervals, and \( \hat{H}(s) \) is constant in inactive intervals, we have

\[
\hat{H}(u) = \hat{H}(B(v)) = \hat{H}(B(v)).
\]

We define

\[
t(x) = \begin{cases} 
\hat{H}(v(\rho)) = \hat{H}(a(\rho)) & \text{if } \rho \leq x \leq r(v(\rho)) \\
\hat{H}(r^{-1}(x)), & \text{if } r(v(\rho)) \leq x < r(\beta), \\
1, & \text{if } x \geq r(\beta).
\end{cases}
\]

When \( N = 1 \), we have the simpler expression

\[
t(x) = \begin{cases} 
F(\eta(\rho)) & \text{if } \rho \leq x \leq r(\eta(\rho)) \\
F(r^{-1}(x)), & \text{if } r(\eta(\rho)) \leq x < r(\beta), \\
1, & \text{if } x \geq r(\beta).
\end{cases}
\]

We will show that the winning probability of a regular buyer with value \( v \) is given by \( t(v)F^{N-1}(v) \).

**Theorem 8** Let \( \rho \geq 0 \) be the reservation price. We have the following equilibrium revenue formula

\[
\int_{\rho}^{\beta} t(x)J(x, \beta)dF^{N}(x).
\]

**Proof.** First let \( N > 1 \). We want to show that the equilibrium winning probability of a regular buyer with value \( x \) is \( t(x)F^{N-1}(x) \). In the equilibrium, we first show that a regular buyer with value \( x \geq r(\beta) \) has winning probability \( F^{N-1}(x) \). When a speculator wins, \( t(x) \) is the resale auction, the optimal reservation price is set below \( r(\beta) \), and the regular buyer will win as long as he has the highest value among the regular buyers. Therefore his winning probability is \( F^{N-1}(x) \). For \( x \in [r(v(\rho)), r(\beta)) \), the winning probability is \( \hat{H}(r^{-1}(x))F^{N-1}(x) \) because he can win in the resale market only if speculators win by bidding \( b \leq \hat{B}(r^{-1}(x)) \). The probability of speculators bidding below \( B(v(\rho)) \) is \( \hat{H}(B(v(\rho))) = \hat{H}(r^{-1}(x)) = t(x) \). Hence the winning probability is given by \( t(x)F^{N-1}(x) \). For \( x \in [\rho, r(\beta)) \), the winning probability is the same as the winning probability in the first-stage auction, because when a regular loses the first-stage auction, a winning speculator will set a reservation price for him to accept. Hence the winning probability is \( \hat{H}(a(\rho))F^{N-1}(x) = t(x)F^{N-1}(x) \). By the Myerson (1981) argument, we must have the following expected contribution of revenue from a regular buyer with value \( v \)

\[
F^{N-1}(v)t(v)v - \int_{\rho}^{v} F^{N-1}(x)t(x)dx.
\]

Hence the expected revenue of the auction with resale is given by

\[
N \int_{\rho}^{\beta} \left( F^{N-1}(v)t(v)v - \int_{\rho}^{v} F^{N-1}(x)t(x)dx \right) dF(v)
\]

\(^{14}\)GT(supp) have shown with a different proof that with zero reservation price, the revenue is higher when speculators are active in equilibrium. This is a special case of our result which holds for all reservation price. Our proof is much simpler and more intuitive. Zhang and Wang (2013) also looked at the optimal auction with speculative resale with one speculator and one regular buyer, and showed that Myerson’s optimal revenue is achieved. They also considered more general resale markets than ours.
The optimal revenue is obtained by setting the reservation price \( \rho = r(\beta) \) and the revenue is the same as the optimal revenue without resale.

In fact, we can also show directly that at the reservation price \( \rho = r(\beta) \), speculators are inactive in equilibrium, so that the revenue is the same as the the one without resale when \( \rho = r(\beta) \). This is another confirmation that the optimal revenue without resale is the same as that with resale.

**Corollary 9** The optimal revenue is obtained by setting the reservation price \( \rho = r(\beta) \) and the revenue is the same as the optimal revenue without resale.

From the revenue formula, we can derive many important properties. Since \( J(x, \beta) \) has the same sign as \( t(x)J(x, \beta) \), the same argument in Myerson (1981) implies that the revenue is maximized in the same way. For the case of \( N = 1 \), both terms in (43) are increasing in \( \rho \leq r(\rho) \), hence the revenue is optimal at \( \rho = r(\beta) \) as well.

**Corollary 10** The speculators are inactive at the reservation price \( \rho = r(\beta) \).

**Proof.** First we look at the case \( N = 1 \). In this case, the regular buyer bids \( r(\beta) \), and the speculators do not participate because the cost of winning is at least \( r(\beta) \), but the revenue is \( B(\beta) < r(\beta) \). When \( N > 2 \), \( \rho = r(\beta) \), we have

\[
\frac{b_\rho(v) - B^*(v)}{N} = \int_{\rho(v)}^{v} F^{N-1}(x)dx - \int_{r(\beta)}^{v} F^{N-1}(x)dx 
\geq N \int_{r(\beta)}^{v} F^{N-1}(x)dx - \int_{r(\beta)}^{v} F^{N-1}(x)dx = (N - 1) \int_{r(\beta)}^{v} F^{N-1}(x)dx > 0.
\]

for all \( v \in [\rho, \beta] \). Hence by Lemma 8, the speculators are inactive in equilibrium. 

Note that \( J(x, \beta) \geq 0 \) for \( x \geq r(\beta) \), and in this case \( t(x) = 1 \). When \( x < r(\beta) \), and the speculators are active in equilibrium, then \( B(b_\beta^*) > \rho \), and \( t(x) < 1 \) for all \( x < B(b_\beta^*) \). Since \( J(x, \beta) < 0 \) for such \( x \), the multiplicative factor increases the value of the revenue integral formula. Hence we know that the revenue is strictly higher when speculators are active in equilibrium.

Earlier, we define \( \tilde{\rho} \) be the maximum of all \( \rho \) such that speculators are active in equilibrium at the reservation price \( \rho \). This can be defined more generally. If there exists no such \( \rho \), we let \( \tilde{\rho} = 0 \). When \( N = 1 \), \( \tilde{\rho} = B(\beta) \). When \( N > 1 \), \( \tilde{\rho} \) is defined to be the smallest \( \rho \) satisfying the following equation

\[
B(v) \leq b_\rho(v) \text{ for all } v \geq \rho.
\]

From Lemma 10, we know that \( \tilde{\rho} \leq r(\beta) \). In fact, we must have \( \tilde{\rho} < r(\beta) \) from (??). We will show in Part II that given \( F(\cdot) \), \( \tilde{\rho} \) is essentially software-computable. The following is another important corollary of Theorem 8. It tells us precisely when the speculators are active in equilibrium, and that the revenue is higher with resale than without resale when they are active.

**Corollary 11** The speculators are active in equilibrium if and only if \( \rho < \tilde{\rho} \). When the speculators are active in equilibrium, the revenue with resale is strictly higher than the auction without resale. When they are inactive in equilibrium, there is no difference in the revenue with or without resale.
7.2 Revenue Monotonicity

When we add more speculators, there is no effect on the revenue. It only changes the bid distribution of each speculator, and the joint bid distribution is unchanged. The regular buyers bid the same way, and the revenue remains unchanged. Now we show that when we add one more regular buyer, the revenue will increase.

**Theorem 12** Let $b_N(.)$ be the equilibrium bidding strategy of a regular buyer when there are $N$ of them, then we have

\[ b_{N+1}(v) > b_N(v) \text{ for all } v > 0. \]

**Proof.** We call the equilibrium with $N+1$, $N$ regular buyers $N+1$-equilibrium, $N$-equilibrium respectively. Assume that the theorem is false, let $v_0 = \max\{v : b_{N+1}(x) \geq b_N(x) \text{ for all } x \leq v\}$. There are three cases to consider: (1) for some $\varepsilon > 0$, the interval $(v_0, v_0 + \varepsilon)$ is an inactive interval for both equilibria, (2) for some $\varepsilon > 0$, the interval $(v_0, v_0 + \varepsilon)$ is an active interval in the $N+1$-equilibrium, (3) for each $\varepsilon > 0$, there is an interval $(v_0 + \varepsilon - \epsilon, v_0 + 2\epsilon)$ which is an active interval in the $N+1$-equilibrium. We will arrive at a contradiction in each case, and our proof will be complete after that. In case 1, we have, for $v \in (v_0, v_0 + \varepsilon)$.

\[ b_{N+1}(v) = b_{N+1}(v_0) + \int_{v_0}^{v} F^N(x|v)dx, \]
\[ b_N(v) = b_N(v_0) + \int_{v_0}^{v} F^{N-1}(x|v)dx < b_{N+1}(v) \]

In case 2, we can show that $(v_0, v_0 + \varepsilon)$ is also an active interval of the $N$-equilibrium. By Theorem 7, we know (for the case $N+1$), $B^e(.) > b_\rho(.)$, except a countable closed subset, or

\[ 0 < \int_{\rho}^{v} F^N(x)dx - (N + 1) \int_{\rho}^{v} F^N(x)dx = \int_{\rho}^{v} F^N(x)dx - N \int_{\rho}^{v} F^N(x)dx \]
\[ = F(\rho) \left[ \int_{\rho}^{v} F^{N-1}(x)dx - N \int_{\rho}^{v} F^{N-1}(x)dx \right] \]
\[ = F(\rho) \left[ (N + 1) \int_{\rho}^{v} F^{N-1}(x)dx - \int_{\rho}^{v} F^{N-1}(x)dx \right] \]
\[ < F(\rho) \left[ N \int_{\rho}^{v} F^{N-1}(x)dx - \int_{\rho}^{v} F^{N-1}(x)dx \right] \]

which means $(v_0, v_0 + \varepsilon)$ is also active in the $N$-equilibrium. However the zero profit condition then implies that $b_{N+1}(.) > b_N(.)$, a contradiction. In case 3, we reach the same contradiction as in case 2. The proof is now complete.

The increasing property of the equilibrium bidding strategy with respect to $N$ is not quite sufficient on its own to insure the revenue monotonicity. The effect of $N$ on $\tilde{H}(.)$ is still a factor to take into account. However, we can still prove the revenue monotonicity of our model. Note that adding more speculators has no impact on the revenue.

**Theorem 13** Let $R(N+1, \rho)$, $R(N, \rho)$ be the revenues with $N+1$, $N$ regular buyers respectively. Then we have

\[ R(N, \rho) < R(N+1, \rho), \rho < \beta. \]

We also have monotonicity in $\rho \in [0, r(\beta)]$, $R(N, \rho) < R(N, \rho')$ when $\rho < \rho' \leq r(\beta)$. 

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Proof. We know from Theorem 12 that, letting \( \phi_{N+1}(.) = b_{N+1}^{-1}(.) \), \( \phi_N(.) = b_N^{-1}(.) \), we have

\[
\phi_{N+1}(b) < \phi_N(b), \quad b < b^*
\]

hence, from Theorem 3, we have

\[
F^{N+1}(\phi_{N+1}(b)) < F^N(\phi_N(b)) F(\phi_N(b)).
\] (44)

The inequality (44) is precisely the first-order stochastic dominance of the joint bid distribution of the \( N + 1 \) regular buyers in the \( N + 1 \)-equilibrium over the joint bid distribution of the \( N \)-equilibrium of all buyers. This is sufficient to insure \( R(N + 1), \rho) > R(N, \rho) \). ■

7.3 Bulow-Klemperer Result: competition vs. optimal reservation price

Now we show the Bulow-Klemperer result in the auctions with speculative resale. To our knowledge, this is the first extension of their result beyond the benchmark symmetric private value model without resale. The outcome of our model is in fact equivalent to the outcome of a common value auction with the common value defined by the resale value.

The following is the Bulow-Klemperer result for the auctions with speculative resale. We make use of their result, rather than providing a more direct proof.

**Theorem 14** In our model of auctions with speculative resale, adding just one more bidder (and setting a zero reserve price) is always preferable to setting the optimal reserve price.

**Proof.** Let \( R^*(N, \rho) \) be the revenue of the model with \( N \) regular buyers and reservation price \( \rho \), without resale. From Corollary 9, \( R^*(N, r) = R(N, r) \). By the Bulow Klemperer result, we have \( R^*(N + 1, 0) > R^*(N, r) \). By Corollary 11, we have \( R(N + 1, 0) > R^*(N + 1, 0) \). Hence we have

\[
R(N + 1, 0) > R^*(N + 1, 0) > R^*(N, r) = R(N, r),
\]

and the Theorem is proved. ■

In the symmetric auction without speculators, let \( R^*(N + 1, 0) - R^*(N, 0) \) be the revenue increase from one additional buyer, and \( R^*(N, r) - R^*(N, 0) \) be the revenue increase from setting the right reservation price. The Bulow-Klemperer result says that

\[
D^* = R^*(N + 1, 0) - R^*(N, r) = (R^*(N + 1, 0) - R^*(N, 0)) - (R^*(N, r) - R^*(N, 0)) > 0.
\]

When there are speculators added to the \( N \) regular buyers, we use the corresponding notations \( R(N + 1, 0) - R(N, 0), R(N, r) - R(N, 0) \) to denote the two revenue increases. Theorem 14 says that

\[
D = R(N + 1, 0) - R(N, r) = ((N + 1, 0) - R(N, 0)) - (R(N, r) - R(N, 0)) > 0.
\]

The following result says that the difference \( D \) is greater than \( D^* \). Therefore the Bulow-Klemperer argument is even more persuasive, as the increase in revenue from adding a regular buyer is even greater if the speculators are active in equilibrium

**Corollary 15** We have \( D \geq D^* \), and \( D > D^* \) holds when the speculators remain active after adding one more regular buyer.

**Proof.** From Corollary 9, we know that \( R(N, r) = R^*(N, r) \). Hence we have

\[
D - D^* = R(N + 1, 0) - R^*(N + 1, 0) \geq 0,
\]

and strict inequality holds when the speculators are active with \( N + 1 \) regular buyers and \( \rho = 0 \). ■
Using the example in the last section, we can compute the revenue for $\rho = 0$ as follows:

$$t(v) = H(\sqrt{3}v) = \exp\left(- \int_{\sqrt{3}v}^{0.5} B'(x) - B(x) \frac{2x}{B'(x) - B(x) 1 + x^2} dx\right)$$

The revenue using the Myerson formula is

$$2 * \frac{1}{4} \left( \int_0^{\frac{0.8129031}{\sqrt{3}}} (3x^2 - 1) H(\sqrt{3}x)(1 + x^2) dx + \int_{\frac{0.8129031}{\sqrt{3}}}^1 (3x^2 - 1)(1 + x^2) dx \right)$$

$$= 0.1405397.$$

which confirms the validity of the formulas. This revenue is higher than the one without the speculator, which confirms the validity of the formulas. This revenue is higher than the one without the speculator.

With speculative resale, it is still possible to obtain the optimal revenue without resale as in Myerson (1981). The original seller can set the same optimal reservation price $\rho = r(\beta)$ without resale. With this reservation price, speculators are not active in equilibrium, and hence the equilibrium is the same as if there is no speculators and no resale. The revenue is the same as that of the optimal revenue without resale. Garratt, Tröger and Zheng (2009) have shown that optimal revenue can be achieved through a second price auction with resale. In the first-price auction with resale, the equilibrium is unique, while in second-price auctions, there are many different equilibria. Our result has the additional benefit that the equilibrium is uniquely determined, and the same reservation price applies to all buyers.

Although the active participation of speculators increases the original seller’s revenue with the same reservation price, the optimal revenue possible is actually to discourage the active participation of speculators by making the reservation price sufficiently high. To illustrate this somewhat paradoxical point, we will use a simple uniform distribution example with one speculator, one regular buyer, hence $N = 1, F(v) = v$. If the reservation price is 0, the speculator is active at all bids. Without the participation of the speculator, the revenue would be zero. With the speculator participation, the equilibrium is given by Theorem 27, and the original seller’s revenue is

$$\int_0^{\frac{1}{4}} b d(4b)^2 = 32 \int_0^{\frac{1}{4}} b^2 db = \frac{1}{6}.$$

Hence the speculator participation increases the revenue. To compute the revenue for a general reservation $\rho \in [0, 0.25]$, by Lemma 19, the revenue is

$$12\rho^3 + 32 \int_\rho^{\frac{1}{4}} b^2 db = \frac{4}{3}\rho^3 + \frac{1}{6}.$$

This revenue is increasing in $[0, \frac{1}{4}]$. When $\rho > 0.25$, the speculator is inactive in equilibrium. The regular buyer bids $\rho$ when his value is above $\rho$. The revenue is $\rho(1 - \rho)$ which is increasing in $[0.25, 0.5]$. The maximum is obtained at $\rho = 0.5$, with the optimal revenue $\frac{1}{4}$. This is also the optimal revenue without the speculator. Alternatively, we can compute the revenue according to (43),

$$t(x) = \begin{cases} 
4\rho & \text{for } \rho \leq x \leq 2\rho, \\
2x & \text{for } 2\rho \leq x \leq \frac{1}{2}, \\
1 & \text{for } x \geq \frac{1}{2}.
\end{cases}$$

We obtain the same revenue

$$4\rho \int_\rho^{2\rho} (2x - 1) dx + \int_{2\rho}^{\frac{1}{2}} 2x(2x - 1) dx + \int_{\frac{1}{2}}^{1} (2x - 1) dx = \frac{4}{3}\rho^3 + \frac{1}{6}$$

for $\rho \leq 0.25$. 

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8 Proofs

8.1 Proof for the supermodularity Property

Proof of Theorem 6:

Note that when $b_j(\cdot)$ is not monotonic, the probability distribution $G_j(b)$ of the updated belief about a losing bidder $j$ need not be regular even if $F(\cdot)$ is regular. Moreover, the support of the distribution is not a connected interval. The density of $G_j(b)$ now is $> 0$, but may have kinks or infinite derivative when $b'_j(\cdot) = 0$. Myerson (1981) has shown that the optimal auction mechanism works with non-regular distributions. Furthermore, the analysis can be adapted to deal with non-connected intervals and kinks as well. For the case of infinite derivative, it just means that the virtual value is the same as the value itself.

The virtual value $J(x, w)$ needs to be modified so that it is weakly monotone and the allocation is based on this monotone virtual value which we should denote by the same notation in this proof. The same proof that Myerson (1981) gives for the case of non-regular distributions applies in our case. The gaps between intervals can be included as regions with $f(\cdot) = 0$, $J(x, w) = -\infty$.

When regular buyer $i$ with the value $v_i$ bids $b$, there are payoffs from (i) winning the first-stage auction; (ii) resale to some regular buyer $k$ after winning the auction; (iii) losing the auction to some speculator or regular buyer $j \neq i$, then buys the object from winner. Let $\pi_{1i}(v_i, b), \pi_{wi}(v_i, b), \pi_{1i}(v_i, b)$ be the payoffs from each part respectively. We have

$$u_i(v_i, b, \sigma) = \pi_{1i}(v_i, b) + \pi_{wi}(v_i, b) + \pi_{1i}(v_i, b).$$

Let $K_1 = \{ j \neq i : v_i \leq w_j(b) < \beta \}$. When condition (39) holds, $K_1$ is not empty.

For $j \in K_1$, let the optimal reservation price in the resale market be denoted by $\gamma_j(v_i, b) = \inf\{ v_j : b_j(v_j) \leq b, J(v_j, w_j(b)) \geq v_i \}$. When $b_j(\cdot)$ is monotonic, $\gamma_j(v_i, b)$ is simply the optimal reservation price for buyer $j$ set by buyer $i$ during the resale. For convenience, we also define $\gamma_i(v_i, b) = v_i$ when $j \not\in K_1$. We have $\frac{\partial}{\partial b_n} \gamma_j(v_i, b) > 0$ when $j \in K_1$. Let $v_{-i}$ be the vector of realized values $v_j, j \neq i$. Let $b_n$ denote the realized maximum bid of the speculators. Let $E_{1i}(v_i, b)$ be the event that regular buyer $i$ wins the first-stage auction, but fails to sell to the losing bidders. The event $E_{1i}(v_i, b)$ is characterized (allowing exception for a set of measure 0) by the property that

$$b_n < b, b_j(v_j) < b \text{ and } v_j < \gamma_j(v_i, b), \text{ for all } j \neq i.$$ 

Hence we can write

$$E_{1i}(v_i, b) = \{ (b_n, v_{-i}) : \max(b_n, v_{-i}) < b, v_j < \gamma_j(v_i, b) \text{ for all } j \neq i \}.$$ 

Let $Q_{1i}(v_i, b)$ be the probability of the event $E_{1i}(v_i, b)$. Let $P_j(v_i, b)$ be the probability of $\{ v_j : v_j < \gamma_j(v_i, b), b_j(v_j) \leq b \}$ for $j \in K_1$. Note that although $\gamma_j(v_i, b)$ may be discontinuous, $P_j(v_i, b)$ is a continuous function. In the resale, buyer $i$ only sells to buyer $j \in K_1$ with $v_j > \gamma_j(v_i, b)$. We have

$$Q_{1i}(v_i, b) = \bar{H}(b) \prod_{j \not\in \{i, K_1\}} Q_j(b) \prod_{j \in K_1} P_j(v_i, b).$$

(45)

Proof of Lemma 21: The first part of the payoff is

$$\pi_{1i}(v_i, b) = (v_i - b)\bar{H}(b) \prod_{j \neq i} Q_j(b),$$

and we have

$$\frac{\partial}{\partial v_i} \pi_{1i}(v_i, b) = \bar{H}(b) \prod_{j \neq i} Q_j(b).$$
Let $\pi_{wij}$ be the payoff from selling to buyers $j$ in $K_1$. When buyer $j$ with value $x > \gamma_j(v_i, b)$ wins the resale auction, the expected price it pays conditional on winning the resale auction depends only on $x$, not on $v_i$, and is denoted by $p(x)$. Let $P_{ij}(x)$ be the cumulative (unconditional) probability of buyer $i$ winning the first-stage auction, with the resale winner being buyer $j \in K_1$ with value $v_j \leq x$. Then

$$
\pi_{wij} = \int_{\gamma_j(v_i, b)}^{w_{ij}(b)} (v_i - p(x)) dP_{ij}(x).
$$

Since $p(x) = v_i$ when $x = \gamma_j(v_i, b)$, we have

$$
\frac{\partial}{\partial v_i} \pi_{wij} = P_{ij}(w_{ij}(b)) - P_{ij}(\gamma_j(v_i, b)),
$$

and

$$
\frac{\partial}{\partial v_i} \pi_{w_i}(v_i, b) = \sum_{j \in K_1} \left( P_{ij}(w_{ij}(b)) - P_{ij}(\gamma_j(v_i, b)) \right)
= H(b) \left( \prod_{j \notin \{i, K_1\}} Q_j(b) \prod_{j \neq i} P_j(b) - \prod_{j \neq i} Q_j(b) \right)
= Q_{1i}(v_i, b) - \frac{\partial}{\partial v_i} \pi_{1i}(v_i, b),
$$

and the proof is complete.

There is a simpler proof for the case when there is no payoff after losing the auction, i.e. no payoff from (iii). This is the case, for example, when $b_j(\cdot)$’s are monotonic and $K_1$ contains every buyer except $i$. We have $H(b) > 0, Q_j(b) > 0$ by assumption. Since $\frac{\partial}{\partial v_i} \gamma_j(v_i, b) > 0$, for $j \in K_1$, we have $\frac{\partial}{\partial v_i} P_j(v_i, b) > 0$ when $j \in K_1$. We conclude from Lemma 21 and (45) that

$$
\frac{\partial^2}{\partial b_i \partial v_i} u(v_i, b) = \frac{\partial^2}{\partial b_i \partial v_i} \left[ \pi_{1i}(v_i, b) + \pi_{wi}(v_i, b) \right]
= \frac{\partial}{\partial b_i} \left[ \tilde{H}(b) \prod_{j \notin \{i, K_1\}} Q_j(b) \prod_{j \in K_1} P_j(v_i, b) \right] > 0
$$

This gives the proof when there is no payoff from (iii).

To deal with the case when there is payoff from (iii), we first prove Lemma 22. Let $K_2 = \{ j \neq i : w_{ij}(b) < v_i \}$. If $K_1$ is empty, then $K_2$ is not empty from our assumptions. Let $E_{2i}(v_i, b)$ be the event that buyer $i$ loses the first-stage auction, but then wins it back during the resale. Let $Q_{2i}(v_i, b)$ be the probability of $E_{2i}(v_i, b)$. When regular buyer $j \neq i$ with $v_j$ wins the first-stage auction bidding $b^* > b$, and sells to buyer $i$ during resale, we can also define $\gamma_j(v_j, b^*)$ in a similar way. Let $\gamma_i(s, b^*)$ be the corresponding function for a speculator. We have the characterization (except a set of measure 0) of $E_{2i}(v_i, b)$ as a set of realized $(b_s, v_{-i})$ satisfying the following two properties:

If $b_s > \max_{j \neq i} b_j(v_j)$, then we have $J(v_j, w_j(b_s)) \leq J(v_i, w_i(b_s))$ for all $j \neq i$,

If $b_j(v_j) = b > \max\{ b_s, \max_{k \neq i, j} b_k(v_k) \}$, $j \in K_2$, then $J(v_k, w_k(b')) \leq J(v_i, w_i(b'))$ for $k \neq i, j$, and $\gamma_i(v_j, b') \leq v_i$.

Proof of Lemma 22: Let $y_j(v_i) = \sup\{ b' : b' \geq \tilde{b}_i \text{ such that } \gamma_j(v_j, b') \leq v_i \}$. Similarly, let $y_s(v_k) = \sup\{ b' : b' \leq \tilde{b}_i \text{ such that } \gamma_j(s, b') \leq v_i \}$. Let $\pi_{1is}$ be the payoff of buying from regular bidders $j$ in $K_2$ during resale after buyer $i$ loses the first-stage auction. Let $\pi_{1is}$ be the similar payoff of buying from the speculators during resale. When a buyer $j$ with value $v_j$ wins the first-stage auction bidding $y = b_j(v_j)$, the expected price
he gets conditional on selling to buyer $i$ during resale depends only on $y, v_j$ not on $v_i$, and is denoted by $q(y, v_j)$. Let $Q_{ij}(x)$ be the cumulative (unconditional) probability of buyer $i$ losing the first-stage auction, but buying from buyer $j \in K_2$ who has a winning bid $b_j(v_j) \leq x$. Then

$$\pi_{ij} = \int_b^{y_j(v_i)} (v_i - q(x, v_j))dQ_{ij}(x).$$

Since either we have $y_j(v_i) = b_i$ or we have $q(x, v_j) = v_i$ when $x = y_j(v_i)$, we must have

$$\frac{\partial}{\partial v_i} \pi_{ij} = Q_{ij}(y_j(v_i)) - Q_{ij}(b).$$

Similarly, let $Q_{is}(x)$ be the corresponding cumulative (unconditional) probability of buying from speculators whose winning bid is $\leq x$, we have

$$\frac{\partial}{\partial v_i} \pi_{is} = Q_{is}(y_s(v_i)) - Q_{is}(b).$$

Hence

$$\frac{\partial}{\partial v_i} \pi_{is}(v_i, s) = \frac{\partial}{\partial v_i} \pi_{is} + \sum_{j \in K_2} \frac{\partial}{\partial v_i} \pi_{ij} = Q_{is}(v_i, b).$$

This completes the proof of the lemma.

### 8.2 Proof for the Increasing Symmetry Property

We apply the following standard result on the existence and uniqueness of the solution to a system of differential equations with the initial boundary conditions.

Picard–Lindelöf Theorem: Let $b$ be a real variable, $z$ be a vector, and $h(b, z)$ be vector of functions continuous in $b$, and Lipschitz continuous in $z$. Then the system of differential equations $y'(b) = h(b, y(b))$ with the initial boundary condition $y(b_0) = z_0$ has a unique solution over $[b_0, b_1]$.

Proof of Lemma 24:

First assume that all regular bidders have the same maximum bid $\bar{b}$ in equilibrium. Lemma 25 says that the inverse strategies must be strictly increasing near $\bar{b}$. Lemma 26 says that the first order condition at $\bar{b}$ is also the same as that of the auction without resale model for non-symmetric strategies. Lemma 27 proves the symmetry and increasing property under the assumption that all bidders have the same maximum bid in the support of the bid distributions. Lemma 28 then shows that in equilibrium, all bidders must have the same maximum bid in the support of the bid distributions, making use of Theorem 6 again.

**Lemma 25** Assume that all buyers have the same maximum bid $\bar{b}$ in their support of the equilibrium bid distributions. In equilibrium strategies $b_i(\cdot)$, we must have $b_i(\beta) = \bar{b}$, and $b_i(\cdot)$ is strictly increasing in a neighborhood of $\beta$.

**Proof.** Suppose it is false. There is some $v_0 < \beta$ such that $b_i(v_0) = \bar{b}$. There is an interval $(v_i', v_i'')$ to the right of $v_0$, but nearby, such that $b_i(\cdot)$ is strictly decreasing and we have $v_i < v_j(b), b < b$ for all $v_i \in (v_i', v_i'')$, $b$ near $b_i(v_i')$. Condition (39) holds for such $v_i, b$, hence the strong form of supermodularity in Theorem 6 implies that $b_i(\cdot)$ must be increasing which is a contradiction. Hence we have shown that $b_i(\beta) = \bar{b}$ for all $i$. The same argument also shows that $b_i(\cdot)$ must be strictly increasing in a neighborhood of $\beta$. ■

The same argument in the above lemma also shows that $\max_{v \leq v_0} b_i(v) < \bar{b}$ if $v_0 < \beta$. Hence $b_i(\cdot)$ has a uniquely defined inverse function $\phi_i(\cdot)$ in a neighborhood of $\beta$, and we have $Q_i(b) = F(\phi_i(b))$ in the neighborhood.
Lemma 26 Assume that all buyers have the same maximum bid $\bar{b}$ in their support of the equilibrium bid distributions, then the first-order condition for equilibrium at $\bar{b}$ is

$$\frac{d}{db} \left( (\beta - b) \prod_{k \neq i} F(\phi_k(b)) \right) |_{b = \bar{b}} = 0,$$

which is the same as if there is no resale. Furthermore $\phi'_i(\bar{b}) = \phi'_j(\bar{b})$ for all $i, j$.

Proof. Let buyer $i$ with use value $b$ bids $b < \bar{b}$. There is no resale if the buyer wins the auction, but there is resale when he loses the auction. After losing the auction to some buyer $k$, he may buy it from buyer $k$.

Let $\gamma_k(y)$ be the optimal reservation price set by the winning buyer $k$ with value $\phi_k^{-1}(y)$ after winning with the bid $y$. Let $P_k(y)$ be the probability of winning bid being below $y$. The expected payoff of buyer $i$ bidding $b < \bar{b}$, losing to buyer $k$, and buying it back, is given by

$$\int_{b}^{\bar{b}} (v_i - \gamma_k(y)) dP_k(y).$$

(46)

At $y = \bar{b}$, we have $\gamma_k(y) = v_i$. The derivative from the left of (46) with respect to $b$, evaluated at $\bar{b}$, is 0. Since this is true for all winning buyer $k$, the first-order condition for the auction with resale at $\bar{b}$ is

$$\frac{d}{db} \left( (\beta - b) \prod_{j \neq i} F(\phi_j(b)) \right) |_{b = \bar{b}} = 0.$$

From the first-order condition, we get

$$\sum_{j \neq i} \phi'_j(\bar{b}) = \frac{1}{f(\beta)(\beta - b)}$$

for all $i$, which implies that

$$\phi'_i(\bar{b}) = \frac{1}{(N - 1)f(\beta)(\beta - b)}$$

for all $i$, and the proof is complete. \(\blacksquare\)

The following is a more restrictive version of the symmetry result without speculators. It will be used for the proof of the full version of the symmetry property. The idea of the proof will also be used in the symmetry result when there are speculators.

Lemma 27 Assume that there are no speculators. In the auction with resale, assume that all buyers have the same maximum bid $\bar{b}$ in their support of the equilibrium bid distributions. Then the equilibrium bidding strategy $b_i$ must be symmetric and strictly increasing, and $b_i(v_i) < v_i$ for all $v_i > \rho$.

Proof. First we show that the maximum equilibrium bid $\bar{b}$ is strictly less $\beta$. If not, since it is a common bid by all buyers with the use value $\beta$, the equilibrium payoff of a buyer with the use value $\beta$ must be 0. By bidding slightly lower, such a buyer has a positive probability of winning at a price lower than his use value, thus the payoff will be higher than the equilibrium payoff, a contradiction. Hence we must have $\bar{b} < \beta$ in equilibrium. Now we will show symmetry. We consider a regular bidder $i$ with use value $v_i$ bidding $b$ near $\bar{b}$. Let $\vec{\phi}(b)$ be the vector $(\phi_1(b), \phi_2(b), ..., \phi_N(b))$. The first-order conditions of the equilibrium of the model can be written as

$$\sum_{k \neq i} a_{ik}(b, \vec{\phi}(b)) \phi'_k(b) = c_i(b, \vec{\phi}(b)).$$
These properties are insured by the piecewise $C^2$ smooth assumptions of $b_i(.), F(.)$. Thus we can write the system of linear equations in $\phi'_j(b)$ as

$$A\phi''(b) = \mathbf{c}.$$  

(47)

When $v_i = \beta, b = \bar{b}$, from Lemma (26), we have

$$\phi'_k(\bar{b}) = \frac{1}{(N-1)(\beta - b)f(\beta)}$$ for all $i$.

In other words, the matrix $A$ at $b = \bar{b}$ can be written as an identity matrix and $c_i(\bar{b}, \phi(\bar{b})) = \frac{1}{(N-1)(\beta - b)f(\beta)}$. Hence for $b$ near $\bar{b}$, by continuity, $A$ must be a singular matrix, and (47) can be solved uniquely for $\phi(\bar{b})$. Therefore we have a system of differential equations satisfying the conditions of the Picard–Lindelöf Theorem. The solution is unique with the initial condition $\phi_i(\bar{b}) = \beta$ for all $i$. Since it is obvious that a symmetric increasing solution $b_p(.)$ satisfying the boundary conditions exists, the uniqueness of the solution implies we have the symmetric and increasing property in a neighborhood of $\bar{b}$. In the next step, we want to show that the symmetric increasing property holds for all $b > \rho$. Assume this is not true, then we must have some $b_0 = b_i(v_0) > \rho$ such that the symmetric increasing property holds for all $b \geq b_0$, but fails in any neighborhood to the left of $b_0$. We claim that $v_0 > b_0$, otherwise bidder $i$ can bid slightly lower to get a positive payoff while in the solution the payoff is 0, violating the optimal property of the bid $b_0$. Therefore we have $\phi_i(b_0) > b_0$. Lemma 26 is valid at $b_0$ instead of $\bar{b}$ with the same proof. Furthermore, Lemma 25 with $\bar{b}$ replaced by $b_0$ also holds, meaning $b_i(.)$ must be increasing in a neighborhood $(v_0 - \varepsilon, v_0)$, as the stronger supermodularity holds for each regular buyer. For the same reason, we also know that $\max_{v' \leq \beta} b_i(.) < b_0$ if $v' < v_0$. Hence $b_i(.)$ has a uniquely defined inverse in the neighborhood of $v_0$. Repeating the arguments above, we can extend the definition below $b_0$, so that the symmetric increasing property must hold in a neighborhood of $b_0$. This is a contradiction, and the contradiction implies that we must have the symmetry for all $b > \rho$. The rest is simple and our proof for the Lemma is complete.  

We now show that the maximum bid of all the buyers must be the same. This gives us the full symmetry result.

**Lemma 28** Assume that there are no speculators. In the auction with resale, the maximum equilibrium bid of all buyers must be the same, and hence the equilibrium bidding strategies must be symmetric and increasing.

**Proof.** Assume that, by relabeling, buyer one has the largest maximum equilibrium bid $\bar{b}$. There is at least another buyer, say buyer two, who also has the same maximum equilibrium bid. Let there be $m \geq 2$ buyers with the maximum equilibrium bid $\bar{b}$, and let buyer $m + 1$ be the buyer with the next highest maximum equilibrium bid $b_* \leq \bar{b}$. Using the same argument in Lemma 27, we can show that for each all $b$, $\phi_i(b), i = 1, 2, .., m$ are defined over $[b_*, \bar{b}]$ and are all equal and $\phi'_i(b) > 0$ over the interval $[b_*, \bar{b}]$. Let $\phi(b)$ denote $\phi_i(b)$ for all $i \leq m$. For $b \leq \bar{b}$, when bidder $i = 1$ with use value $\beta$ bids $b \leq \bar{b}$, Lemma 26 gives us the first-order condition for the equilibrium bid at $b = \bar{b}$

$$(m - 1)f(\beta)\phi'(\bar{b})(\beta - \bar{b}) - 1 = 0.$$  

(48)

For bidder $j = m + 1$ with use value $\beta$ bidding $b \leq \bar{b}$, there is no resale after winning the auction. When he loses the auction, there may be resale. In the resale, the winner bidder $k, m \geq k > 1$ resells to other bidders, believing that buyers $j \leq m, j \neq k$ has use value upper bound $\phi(b)$, while buyers $j \geq m + 1$ are identical with the use value upper bound $\beta$. The first-order derivative at $\bar{b}$ is

$$mf(\beta)\phi'(\bar{b})(\beta - \bar{b}) - 1.$$  

(49)

Since the optimal bid of bidder $j = m + 1$ is below $\bar{b}$, Theorem 6 (39) applies, and the derivative of the payoff (from the left) at $\bar{b}$ is

$$mf(\beta)\phi'(\bar{b})(\beta - \bar{b}) - 1 < 0.$$  

(50)
Obviously, (48) and (50) are contradictory. This contradiction proves the Lemma.

Since the differential equations determining the equilibrium strategies of the auction with resale model are the same as those without resale for symmetric strategies, and the boundary conditions are the same, we immediately have \( b_\rho(.) \) as the unique equilibrium bidding strategy of the model.

Proof of Theorem 7:
The proof requires similar steps above. Let \( b_s = \inf \{ b : H(b) > 0 \} \). Note that regular buyers with value above \( b_s \) would not bid below \( b_s \) because the winning probability is zero in this case. Hence we only need to prove symmetry for all \( v \geq b_s \). Since we know that \( H(\rho) > 0 \), then we have symmetry increasing property for all \( v \geq \rho \).

Note that Lemma 25 holds without any change of the proof. We have a well-defined inverse bidding strategy for all \( b_i(b) \) near \( \beta \), when all regular buyers have the same maximum bid. We will be rather brief as the idea of the proof is the same, and the arguments only require minor modifications.

**Lemma 29** In the auction with resale with speculators, assume that all regular buyers have the same maximum bid \( b \) in their support of the equilibrium bid distributions, then the first-order condition for the equilibrium of a regular buyer at \( \bar{b} \) is

\[
\phi_i'(\bar{b}) = \frac{1}{(N-1)f(\beta)} \left( \frac{1}{\beta - b} - \bar{H}'(b) \right) \quad \text{for all } i.
\]

(51)

Hence \( \phi_i'(\bar{b}) = \phi_j'(\bar{b}) \) for all \( i, j \).

**Proof.** The proof is the same as in the proof of Lemma 26, except that we now have an additional possibility that speculators may be the winner in the first-stage auction. When speculators win, the proof is also similar. From the first order condition

\[
\frac{d}{db} \left[ \bar{H}(b) (\beta - b) \prod_{j \neq i} F(\phi_j(b)) \right] \bigg|_{b=\bar{b}} = 0,
\]

we have

\[
f(\beta) \sum_{j \neq i} \phi_j'(\bar{b}) = \frac{1}{\beta - b} - \bar{H}'(\bar{b}),
\]

hence

\[
\phi_i'(\bar{b}) = \frac{1}{(N-1)f(\beta)} \left( \frac{1}{\beta - b} - \bar{H}'(b) \right) \quad \text{for all } i.
\]

Hence the lemma is proved.

Now we can establish the following symmetry property when all regular bidders bid the same maximum amount.

**Lemma 30** Assume that there are speculators in the auctions with resale, and assume that all regular buyers have the same maximum bid \( \bar{b} \) in their support of the equilibrium bid distributions. Then the equilibrium strategy must be symmetric and increasing for all \( b > \bar{b}_s \), and \( \phi(b) > b \) when \( b > \bar{b}_s \).

**Proof.** The proof is the same as in Lemma 27 except that we can only obtain symmetry for all \( b > \bar{b}_s \). The first-order condition takes a slightly different form having the same properties needed for the proof.

Next we show that the maximum bid must be the same among the regular bidders.
Lemma 31 In equilibrium, the support of all regular buyers must be the same, and hence the equilibrium bidding strategies must be symmetric and increasing.

Proof. The idea is similar to the proof of Lemma 28. Let \( m \) be the number of regular buyers or speculators with the maximum bid \( \hat{b} \). There is at least one regular buyer with the maximum bid \( \hat{b} \). When there are no speculators active at the bid \( \hat{b} \), the proof is identical. If there are \( k \) speculators active at the bid \( \hat{b} \), we have \( k < m \), and the first-order condition of the equilibrium bid at \( \hat{b} \) for such a regular buyer is

\[
(m - k - 1)f(\beta)\phi'(\hat{b}) = \frac{1}{\beta - b} - k \frac{\hat{H}'(\hat{b})}{\hat{H}(\hat{b})}.
\] (52)

When there is only one regular buyer with the maximum bid \( \hat{b} \), the first-order condition still holds with 0 for the left-hand side. For a regular bidder having a lower maximum bid, but with use value bidding \( b < \hat{b} \), the derivative of the logarithm of payoff with respect to \( b \) at \( \hat{b} \) is

\[
(m - k)f(\beta)\phi'(\hat{b}) + k \frac{\hat{H}'(\hat{b})}{\hat{H}(\hat{b})} - \frac{1}{\beta - b}.
\]

The equilibrium bid is less than \( \hat{b} \). Again from Theorem 6, we must have

\[
(m - k)f(\beta)\phi'(\hat{b}) < \frac{1}{\beta - b} - k \frac{\hat{H}'(\hat{b})}{\hat{H}(\hat{b})}.
\] (53)

The two statements (52) and (53) are clearly contradictory, and this proves the lemma.

8.3 Proofs for the lemmata in Section 3

Lemma 2 is obvious. We start with the proof of the next one.

Proof of Lemma 3:

Proof of Lemma 4: Rewrite (15) as

\[
B(v) = \int_{r(v)}^{v} [x - \frac{1 - F(x|v)}{f(x|v)}]dF^N(x|v).
\]

Since the integrand converges to 0 as \( v \to 0 \), we must have \( B(v) \to 0 \). We use the following formula for \( B(v) \)

\[
B(v) = \frac{N}{F^N(v)} \int_{r(v)}^{v} [xF^N-1(x) - \int_{r(v)}^{x} F^N-1(y)dy]dF(x).
\]

Since \( r(v) \) is optimally chosen in a maximization problem, by the envelop theorem, we have

\[
B'(v) = \frac{N f(v)}{F(v)} \left( vF^N-1(v) - \int_{r(v)}^{v} F^N-1(x)dx \right) - \frac{N f(v)}{F(v)} B(v)
\]

\[
= \frac{N f(v)}{F(v)} \left[ v - \frac{1}{F^N-1(v)} \int_{r(v)}^{v} F^N-1(x)dx - B(v) \right] = \frac{N f(v)}{F(v)} [B'(v) - B(v)].
\] (54)

Lemma 3 implies that \( B'(v) > 0 \).
Proof of Lemma 5: For $v \geq \rho$, we have

$$B'(v) - B_z^+(v) = \frac{f(v)}{F(v)} \left[ N(B^z(v) - B(v)) - (N - 1)(v - B_z^+(v)) \right]$$

$$= \frac{f(v)}{F(v)} \left[ Nv - \frac{N}{F_{N-1}(v)} \int_{r(v)}^{v} F^{N-1}(x)dx - (N - 1)(v - B_z^+(v)) \right]$$

$$= \frac{f(v)}{F(v)} \left[ B^c(v) - NB(v) + (N - 1)B_z^+(v) \right] = \frac{Nf(v)}{F(v)} \left[ \frac{1}{N} B^c(v) + \frac{N - 1}{N} B_z^+(v) - B(v) \right].$$

Proof of Lemma 6: Suppose $B(.) = B_z^+(.)$ in some interval $[v_1, v_2]$. From Lemma 5, we get $B^c(.) = B_z^+(.)$ in the same interval. If $B^c(.) = B_z^+(.)$ in the interval, from Lemma 5, we get

$$B'(v) - B_z^+(v) = \frac{Nf(v)}{F(v)} [B_z^+(v) - B(v)],$$

which means that whenever $B'(v) > B_z^+(v)$, we must have $B(.) < B_z^+(.)$ in a neighborhood $(v, v + \varepsilon)$, a contradiction. Similarly, we cannot have $B'(v) < B_z^+(v)$. We conclude that $B'(v) = B_z^+(v)$ in the interval, and we have $B(.) = B_z^+(.)$. In the third case, if $B(.) = B_z^+(.)$ in the interval, from Lemma 5, we get

$$B'(v) - B_z^+(v) = (N - 1)f(v) \frac{B_z^+(v) - B(v)}{F(v)},$$

and the same contradiction is obtained.

Proof of Lemma 7: To prove (a), assume that $B^c(.) > B(.)$ except a countable closed subset. We want to show that $B(.) > B_z^+(.)$ except a countable closed subset. First we show that $B(.) \leq B_z^+(.)$ in $(z, z + \varepsilon)$ can not be true. Otherwise, Lemma 5 implies that $B'(.) > B_z^+(.)$ in $(z, z + \varepsilon)$ except a countable closed subset. This further implies that $B(.) > B_z^+(.)$ in $(z, z + \varepsilon)$, which is a contradiction. Hence if it is not true that $B(.) > B_z^+(.)$ except a countable closed subset, we can find an interval $(x, x + \varepsilon)$ such that $z < x$, $B^c(x) > B(x)$, $B(x) = B_z^+(x)$, $B(.) \leq B_z^+(.)$ in $(x, x + \varepsilon)$. By Lemma 5 again, we must have $B'(x) > B_z^+(x)$, which implies $B(.) > B_z^+(.)$ in some $(x, x + \varepsilon')$, a contradiction. Now we want to prove the converse. The arguments are quite similar. Assume that $B(.) > B_z^+(.)$ except a countable closed subset. We cannot have $B^c(.) \leq B(.)$ in some $(z, z + \varepsilon)$, otherwise we would have $B'(.) < B_z^+(.)$ in $(z, z + \varepsilon)$ except a countable closed subset, which leads to the contradiction $B(.) < B_z^+(.)$. Now we can find an interval $(x, x + \varepsilon)$ such that $z < x$, $B(x) > B_z^+(x)$, $B^c(x) = B(x)$, $B(.) \leq B(.)$ in $(x, x + \varepsilon)$. By Lemma 5, we have $B'(x) - B_z^+(x) < 0$ leading to the contradiction $B(.) - B_z^+(.) < 0$ in some $(x, x + \varepsilon')$. All the arguments and Lemmata we use apply to $B_z^+(.) = b_n(.)$ when $z = \rho = 0$.

The proof of (b) is similar, and (c) is a consequence of Lemma 6.

Proof of Lemma 8:

Assume that $B^c(v) \leq B_z^+(v)$ for all $v \in [z, z']$. If the lemma is false, let $(x, x')$ be a maximal open subinterval of $[z, z']$ such that $B(.) > B_z^+(.)$. At $x$, we must have $B(x) = B_z^+(x)$. If $B^c(x) < B(x)$, by Lemma 5, we have $B'(x) < B_z^+(x)$ which implies $B(.) < B_z^+(.)$ in some neighborhood $(x, x + \varepsilon)$, a contradiction. If $B^c(x) = B_z^+(x)$, define $B_z^+(.)$ by the initial condition $B_z^+(x) = B(x)$, then $B(.) > B_z^+(.) = B_z^+(.)$ over $(x, x')$. By Lemma 7, we must have $B^c(.) > B(.)$ over $(x, x')$, which again is a contradiction. Hence the lemma must be true.
References


Mimeo, University of Mannheim.


