Compellingness in Nash Implementation*

Shurojit Chatterji, Takashi Kunimoto, and Paulo Ramos†

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Abstract

A social choice function (SCF) is said to be Nash implementable if there exists a mechanism in which every Nash equilibrium outcome coincides with that specified by the SCF. The main objective of this paper is to assess the impact of considering mixed strategy equilibria in Nash implementation. To do this, we focus on environments with two agents and restrict attention to finite mechanisms. We call a mixed strategy equilibrium "compelling" if its outcome Pareto dominates any pure strategy equilibrium outcome. We show that if the finite environment and the SCF to be implemented jointly satisfy what we call *Condition P+M*, we construct a finite mechanism which Nash implements the SCF in pure strategies and possesses no compelling mixed strategy equilibria. This means that the mechanism might possess mixed strategy equilibria which are "not" compelling. Our mechanism has several desirable features: transfers can be completely dispensable; only finite mechanisms are considered; integer games are not invoked; and agents' attitudes toward risk do not matter. These features make our result quite distinct from many other prior attempts to handle mixed strategy equilibria in the theory of implementation. We also illustrate the difficulty of extending our result to the case of more than two agents.

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Keywords: implementation, compelling equilibria, ordinality, mixed strategies, Nash equilibrium.

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†School of Economics, Singapore Management University, Singapore; shurojitc@smu.edu.sg, tkunimoto@smu.edu.sg, pauloramos.2018@phdecons.smu.edu.sg

1 Introduction

The theory of implementation attempts to answer two questions. First, can one design a mechanism that successfully structures the interactions of agents in such a way that, in each state of the world, they always choose actions which result in the socially desirable outcomes for that state? Second, if agents possess information about the state and interact through a given mechanism, what properties do the resulting outcomes, viewed as a map from states to outcomes (and called social choice functions - henceforth, SCFs), possess? In answering these, the consequences of a given mechanism are predicted through the application of game theoretic solution concepts.[1]

In this paper we adopt Nash equilibrium as a solution concept, consider complete information environments, and ask if a given SCF is implementable, i.e., when we can design a mechanism in which "every" Nash equilibrium induces outcomes consistent with the SCF. Although the literature claims to care about all equilibria, it often ignores mixed strategy equilibria and only focuses on pure strategy equilibria. Jackson (1992) provides the most forceful argument for why the omission of mixed strategy equilibria brings about a serious consequence. In his Example 4, Jackson (1992) constructs a two-person environment and an SCF such that (i) there is a finite mechanism that pure Nash implements the SCF; and (ii) every finite pure Nash implementing mechanism always has a mixed strategy equilibrium that gives a lottery that is preferred by both agents to the outcome of the SCF. Thus, if we insist on using finite mechanisms, which is to be anticipated in an environment with finite number of alternatives and agents, we must question why agents would limit themselves to playing only pure strategies, particularly when there is a mixed strategy equilibrium that would be strictly preferred by both of them than any pure strategy equilibrium. We call such a mixed strategy equilibrium *compelling*. In this paper, we revisit Jackson's example in Subsection 3.1 and show in Subsection 3.2 that the issue of compelling mixed strategy equilibria identified by Jackson's example is not specific to the two-person environment but generalized into a three-person environment where each agent plays an indispensable role in at least one state.

To obtain the main result of the paper, we consider a two-person finite environment with respect to an SCF on which we impose Condition P+M, which delineates a set of conditions where it is always possible to construct a finite mechanism which pure Nash implements the SCF without compelling mixed strategy equilibria.[2] We call such a notion of implementation *compelling implementation*.

[1] See Jackson (2001), Maskin and Sjöström (2002), and Serrano (2004) for the survey of implementation theory.

[2] Note that Dutta and Sen (1991) and Moore and Repullo (1990) independently identify a necessary and sufficient condition (called Condition β and Condition $\mu 2$, respectively) for two-

Importantly, compelling implementation allows the implementing mechanism to admit mixed strategy equilibria that result in outcomes not consistent with the ones induced by the SCF, provided these mixed equilibria are not compelling. Hence, compelling implementation is considered a compromise between pure Nash implementation where only pure strategy equilibria considered and mixed Nash implementation where all mixed strategy equilibria are fully considered.

To locate our contribution in a broader context, we first acknowledge that every prior work cited in the table below exploits some combination of the following five ingredients to handle mixed strategy equilibria in complete information environments: (i) infinite mechanisms; (ii) rationalizability as a stronger requirement than Nash equilibrium;[3] (iii) refinements of Nash equilibrium, such as subgame perfect equilibrium and undominated Nash equilibrium; (iv) environments with transfers or ones similar to separable environments of Jackson, Palfrey, and Srivstava (1994); and (v) cardinal utilities.[4]

Combination of ingredients used	Previous works which handle mixed strategy equilibria in complete information environments
(i)	Kartik and Tercieux (2012), Maskin (1999), Maskin and Sjöström (2002), Mezzetti and Renou (2012a)
(i) × (v)	Kunimoto (2019), Serrano and Vohra (2010)
(i) × (ii) × (v)	Bergemann, Morris, and Tercieux (2011), Jain (2021), Kunimoto and Serrano (2019), Xiong (2022)
(ii) × (iv) × (v)	Abreu and Matsushima (1992), Chen, Kunimoto, Sun, and Xiong (2021)
(iii) × (iv)	Goltsman (2011), Jackson, Palfrey, and Srivastava (1994), Moore and Repullo (1988), Sjöström (1994)
(iii) × (iv) × (v)	Abreu and Matsushima (1994)
(iv)	Mezzetti and Renou (2012b)
(iv) × (v)	Chen, Kunimoto, Sun, and Xiong (2022)

Table 1: The list of prior works handling mixed strategy equilibria in complete information environments.

We next emphasize that we obtain the main result of the paper without using any of the five ingredients used in the previous works. Of course, there is the cost associated with this result, as our implementing mechanism might admit mixed strategy equilibria which are not compelling. In addition to the information about the agents' ordinal strict preferences, what is required is the information regarding the smallest difference in cardinal utilities between any two distinct alternatives. We can think of such information as the smallest unit in which the agents' utilities

person Nash implementation where only pure strategies are considered.

[3]Rationalizability is a more demanding requirement than Nash equilibrium because every action played with positive probability in a mixed strategy Nash equilibrium is rationalizable.

[4]This table, by no means, exhausts all related papers.

are measured. As long as that unit of measure is positive, we can construct a mechanism that compellingly implements the SCF. In this sense, while compelling implementation is not completely ordinal, it can be made as ordinal as it can possibly be. We consider Nash implementation as the right notion of implementation if we insist on the robustness to information perturbations. This is so because Aghion, Fudenberg, Holden, Kunimoto, and Tercieux (2012) and Chung and Ely (2003) both show that Maskin monotonicity, a necessary condition for Nash implementation, is also necessary if we want implementation using refinements of Nash equilibria to be robust to information perturbations.[5] Our mechanism is finite so that it does not use the *integer games* which are often considered a questionable devise in the literature.[6] The use of transfers can be dispensed with completely, which allows us to apply our result to an important class of environments including the models of voting and matching in which monetary transfers are simply unavailable.

We finally take up Korpela (2016) which is perhaps the closest to our paper.[7] Korpela (2016) uses a weaker notion of implementation than our compelling implementation in the sense that his notion of implementation ignores all Nash equilibria which are not compelling, "regardless of whether they are pure or mixed." We mention three main differences: first, Korpela's (2016) notion of implementation does not necessarily imply pure strategy Nash implementation, whereas our compelling implementation does. Second, our Condition P+M is significantly weaker than Korpela's (2016) "essentially finite economic environments" needed for his result. Third, we focus on the case of two agents, while Korpela (2016) handles any number of agents.[8]

We organize the rest of the paper as follows: Section 2 presents the environment, notation, mechanisms and solution concepts, as well as a small discussion on Maskin Monotonicity. Section 3 consists of two subsections: Subsection 3.1 revisits Example 4 of Jackson (1992), which motivates our inquiry and Subsection 3.2 generalizes the insights obtained by Jackson's example into a three-person environment in which each agent plays an indispensable role in at least one state. From Section 4 till the end of the paper, we focus entirely on two-person environments. Section 4 slightly modifies Example 4 of Jackson (1992) and presents an illustra-

[5]Aghion, Fudenberg, Holden, Kunimoto, and Tercieux (2012) and Chung and Ely (2003) adopt subgame perfect equilibrium and undominated Nash equilibrium as a solution concept, respectively.

[6]In the integer game, each agent announces some integer and the person who announces the highest integer gets to name his favorite outcome.

[7]This paper has been developed independently of Korpela (2016) and we only became aware of it after we completed the first draft of the paper.

[8]In the rest of the paper, we will further make the connection to Korpela (2016) wherever necessary.

tion of this paper's main result. Section 5 contains the main result of the paper together with a specific family of mechanisms that can achieve compelling implementation under Condition P+M. Section 6 shows that part of Condition P+M is necessary for pure Nash Implementation and the other part of Condition P+M is indispensable for compelling implementation in the sense that our mechanism fails to achieve compelling implementation when the other part of Condition P+M is not satisfied. Section 7 compares our mechanism with the canonical mechanism of Moore and Repullo (1990), showing that there is a class of environments satisfying Condition P+M in which the canonical mechanism of Moore and Repullo (1990) admits a compelling mixed strategy equilibrium. In Section 8, we illustrate the difficulty of extending our result to the case of more than two agents. Section 9 concludes the paper and the Appendix contains the proofs omitted from the main body of the paper.

2 Preliminaries

Throughout the paper, we consider an environment in which there is a finite set of agents, denoted by $I = \{1, 2, \ldots, n\}$. Let Θ be the finite set of states. It is assumed that the underlying state $\theta \in \Theta$ is commonly certain among the agents. This is the complete information assumption. Let A denote the set of social alternatives, which are assumed to be independent of the information state. We shall assume that A is finite, and denote by $\Delta(A)$ the set of probability distributions over A. Associated with each state θ is a preference profile $\succeq^\theta = (\succeq_i^\theta)_{i\in I}$ where $\succeq_i^\theta$ is agent i's preference relation over A at θ. We write $a \succeq_i^\theta a^{'}$ when agent i weakly prefers a to $a^{'}$ in state θ. We also write $a \succ_i^\theta a^{'}$ if agent i strictly prefers a to $a^{'}$ in state θ and $a \sim_i^\theta a^{'}$ if agent i is indifferent between a and $a^{'}$ in state θ. We can now define an *environment* as $\mathcal{E} = \left(I, A, \Theta, (\succeq_i^\theta)_{i\in I, \theta\in\Theta}\right)$, which is implicitly understood to be commonly certain among the agents. Throughout the paper, we assume that the environment $\mathcal{E}$ admits strictly preferences only, that is, for any $i \in I,\ \theta \in \Theta$, and $a, a^{'} \in A$, it follows that either $a \succ_i^\theta a^{'}$ or $a^{'} \succ_i^\theta a$.

We assume that any preference relation $\succeq_i^\theta$ is representable by a von Neumann-Morgenstern utility function $u_i(\cdot,\theta) : \Delta(A) \to \mathbb{R}$. We say that $u_i(\cdot,\theta)$ is consistent with $\succeq_i^\theta$ if, for any $a, a^{'} \in A$, $u_i(a,\theta) \geq u_i(a^{'},\theta) \Leftrightarrow a \succeq_i^\theta a^{'}$. We denote by $\mathcal{U}_i^\theta$ the set of all possible cardinal representations $u_i(\cdot,\theta)$ that are consistent with $\succeq_i^\theta$. We formally define $\mathcal{U}_i^\theta$ as follows:

$$\mathcal{U}_i^\theta = \left\{ u_i(\cdot,\theta) \in [0,1]^{|A|} \;\middle|\; \begin{array}{l} u_i(\cdot,\theta) \text{ is consistent with } \succeq_i^\theta;\ \min_{a\in A} u_i(a,\theta) = 0; \\ \text{and } \max_{a\in A} u_i(a,\theta) = 1 \end{array} \right\},$$

where $|A|$ denotes the cardinality of A. Let $\mathcal{U}^\theta \equiv \times_{i\in I}\mathcal{U}_i^\theta$ and $\mathcal{U} \equiv \times_{\theta\in\Theta}\mathcal{U}^\theta$. We denote any subset of $\mathcal{U}_i^\theta$ by $\hat{\mathcal{U}}_i^\theta$ and any subset of $\mathcal{U}^\theta$ by $\hat{\mathcal{U}}^\theta$, respectively

The planner's objective is specified by a *social choice function (henceforth, SCF)* $f : \Theta \to \Delta(A)$. Although many papers deal with multi-valued social choice correspondences in the literature of Nash implementation, we focus only on single-valued SCFs.

2.1 Mechanisms and Solution Concepts

Let $\Gamma = ((M_i)_{i\in I}, g)$ be a finite mechanism where M_i is a nonempty *finite* set of messages available to agent i; $g : M \to A$ (where $M \equiv \times_{i\in I} M_i$) is the *outcome function*. At each state $\theta \in \Theta$ and profile of representations $u \in \mathcal{U}$, the environment and the mechanism together constitute a *game with complete information* which we denote by $\Gamma(\theta, u)$. By $\Gamma(\theta)$ we mean the game in which the preference profile $(\succeq_i^\theta)_{i\in N}$ is commonly certain among the agents so that any representation $u \in \mathcal{U}$ is admissible. Note that the restriction of M_i to a finite set rules out the use of integer games (See, for example, Maskin (1999)).

Let $\sigma_i \in \Delta(M_i)$ be a mixed *strategy* of agent i in the game $\Gamma(\theta, u)$. A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_N) \in \times_{i\in I}\Delta(M_i)$ is said to be a mixed-strategy *Nash equilibrium* of the game $\Gamma(\theta, u)$ if, for all agents $i \in I$ and all messages $m_i \in \text{supp}\,(\sigma_i)$ and $m_i' \in M_i$, we have

$$\sum_{m_{-i}\in M_{-i}} \prod_{j\neq i} \sigma_j(m_j) u_i(g(m_i, m_{-i}), \theta) \geq \sum_{m_{-i}\in M_{-i}} \prod_{j\neq i} \sigma_j(m_j) u_i(g(m_i', m_{-i}), \theta).$$

A *pure-strategy* Nash equilibrium is a mixed-strategy Nash equilibrium σ such that each agent i's mixed-strategy σ_i assigns probability one to some $m_i \in M_i$. Let $NE(\Gamma(\theta, u))$ denote the set of mixed-strategy Nash equilibria of the game $\Gamma(\theta, u)$ and $pureNE(\Gamma(\theta))$ denote the set of pure strategy Nash equilibria of the game $\Gamma(\theta)$. As far as we are only concerned with pure strategy equilibria, we only need ordinal preferences so that we can write $pureNE(\Gamma(\theta))$. We also define

$$NE(\Gamma(\theta)) = \bigcup_{u\in\mathcal{U}^\theta} NE(\Gamma(\theta, u))$$

as the set of all Nash equilibria of the class of games $\Gamma(\theta, u)$ across all possible representation $u \in \mathcal{U}^\theta$. Since it does not depend upon cardinal utilities, $NE(\Gamma(\theta))$ is defined only in terms of ordinal preferences. We introduce the notion of pure strategy Nash implementation.

Definition 1 *An SCF f is* ***pure Nash implementable*** *if there exists a finite mechanism $\Gamma = (M, g)$ such that for every state $\theta \in \Theta$, (i) $pureNE(\Gamma(\theta)) \neq \emptyset$; and (ii) $m \in pureNE(\Gamma(\theta)) \Rightarrow g\,(m) = f\,(\theta)$.*

For each $\theta \in \Theta$, define

$$\text{supp}(NE(\Gamma(\theta))) = \left\{m \in M | \ \exists \ u \in \mathcal{U}^{\theta} \ \text{ such that } \ \exists \ \sigma \in NE(\Gamma(\theta, u)) \ \text{ with } \sigma(m) > 0\right\}$$

as the set of message profiles which can be played with positive probability in a Nash equilibrium of the game $\Gamma(\theta, u)$ associated with some $u \in \mathcal{U}^{\theta}$. We next introduce a notion of mixed strategy Nash implementation.

Definition 2 *An SCF f is* ***mixed Nash implementable*** *if there exists a finite mechanism $\Gamma = (M, g)$ such that for every state $\theta \in \Theta$, (i) pure$NE(\Gamma(\theta)) \neq \emptyset$; and (ii) $m \in$ supp$(NE(\Gamma(\theta))) \Rightarrow g(m) = f(\theta)$.*

This definition is proposed by Maskin (1999) but we differ from Maskin (1999) because he allows for infinite mechanisms. The notion of mixed Nash implementation is stronger than that of pure Nash implementation because the former guarantees that every message profile that can be played with positive probability in a Nash equilibrium results in the outcome specified by the SCF. Since it is extremely demanding to take care of all mixed strategy equilibria, we propose a notion of compellingness, which singles out the class of mixed strategy equilibria on which we give a serious consideration.

Definition 3 *Fix $\theta \in \Theta$ and $u \in \mathcal{U}^{\theta}$. We say that σ is a* ***compelling*** *mixed strategy equilibrium of the game $\Gamma(\theta, u)$ if, for any $m \in$ pure$NE(\Gamma(\theta))$ and $i \in I$,*

$$\sum_{\tilde{m} \in M} \sigma(\tilde{m}) u_i(g(\tilde{m}), \theta) \geq u_i(g(m), \theta),$$

with at least one strict inequality for some $i \in I$.

For each $\theta \in \Theta$, we denote by $\hat{\mathcal{U}}^{\theta}$ an arbitrary subset of $\mathcal{U}^{\theta}$. We write $\hat{\mathcal{U}} \equiv \times_{\theta \in \Theta} \hat{\mathcal{U}}^{\theta}$. We now introduce what we call *compelling implementation* which takes $\hat{\mathcal{U}}$ as the set of admissible cardinal utilities explicitly. The basic tenet underlying our notion of Nash implementation is that we ignore mixed strategy equilibria which are "not" compelling, while we take compelling mixed strategy equilibria seriously.

Definition 4 *Let $\hat{\mathcal{U}} \subseteq \mathcal{U}$. An SCF f is* ***compellingly implementable (C-implementable) with respect to*** *$\hat{\mathcal{U}}$ if there exists a finite mechanism $\Gamma = (M, g)$ such that for every state $\theta \in \Theta$, (i) pure$NE(\Gamma(\theta)) \neq \emptyset$; (ii) $m \in$ pure$NE(\Gamma(\theta)) \Rightarrow g(m) = f(\theta)$; and (iii) for any $u \in \hat{\mathcal{U}}^{\theta}$, the game $\Gamma(\theta, u)$ has no compelling mixed strategy equilibria.*

Our notion of compelling implementation strengthens the definition of pure Nash implementation with the following additional requirement: there be no compelling mixed strategy equilibria within the class of games $\Gamma(\theta, u)$ across all representation $u \in \hat{\mathcal{U}}^{\theta}$. On the other hand, our notion of compelling implementation weakens the definition of mixed Nash implementation by allowing the following possibilities: (i) there might exist a state $\theta \in \Theta$ and a message profile $m \in \text{supp}(NE(\Gamma(\theta))$ such that $g(m) \neq f(\theta)$ and (ii) there might exist $\theta \in \Theta$, $u \in \mathcal{U}^{\theta} \backslash \hat{\mathcal{U}}^{\theta}$, and $\sigma \in NE(\Gamma(\theta, u))$ such that σ is compelling. The first possibility means that our implementing mechanism might admit a bad mixed strategy Nash equilibrium that is not compelling. The second possibility means that our implementing mechanism might admit a compelling equilibrium if we allow a possible representation u to be outside of $\hat{\mathcal{U}}$.

2.2 Maskin Monotonicity

We now restate the definition of Maskin monotonicity that Maskin (1999) proposes for Nash implementation.

Definition 5 *An SCF f satisfies* ***Maskin monotonicity*** *if, for every pair of states $\tilde{\theta}$ and θ with $f(\tilde{\theta}) \neq f(\theta)$, some agent $i \in I$ and some allocation $a \in A$ exist such that*

$$f(\tilde{\theta}) \succeq_i^{\tilde{\theta}} a \text{ and } a \succ_i^{\theta} f(\tilde{\theta}). \tag{1}$$

To show that Maskin monotonicity a necessary condition for compelling implementation, suppose that the SCF f is C-implementable by a mechanism $\Gamma = (M, g)$. When $\tilde{\theta}$ is the true state, there exists a pure-strategy Nash equilibrium $m \in M$ in $\Gamma(\tilde{\theta})$ which induces $f(\tilde{\theta})$. If $f(\tilde{\theta}) \neq f(\theta)$ and θ is the true state, then m cannot be a Nash equilibrium, i.e., there exists some agent i who has a profitable deviation. Suppose that the deviation induces outcome a, i.e., agent i strictly prefers a to $f(\tilde{\theta})$ at state θ. Since m is a Nash equilibrium at state $\tilde{\theta}$, such a deviation cannot be profitable in state $\tilde{\theta}$; that is, agent i weakly prefers $f(\tilde{\theta})$ to a at state $\tilde{\theta}$. In other words, a belongs to agent i's lower contour set at $f(\tilde{\theta})$ of state $\tilde{\theta}$, whereas it belongs to the strict upper-contour set at $f(\tilde{\theta})$ of state θ. Therefore, Maskin monotonicity is a necessary condition for compelling implementation; in fact, it is a necessary condition even for pure Nash implementation.

3 The Relevance of Mixed Strategy Equilibria in Nash Implementation

In this section, we articulate a compelling reason why we need to be worried about mixed strategy equilibria in Nash implementation. To do so, we decompose our argument into two subsections. In Subsection 3.1, we revisit Example 4 of Jackson (1992), which focuses on an environment with two agents and shows that the omission of mixed strategy equilibria brings about a serious blow to Nash implementation. In Subsection 3.2, we show that all the insights exhibited by Example 4 of Jackson (1992) can be extended to an environment with three agents and moreover, this extension can be made in a nontrivial manner, i.e., each agent plays an indispensable role in at least one state. Hence, the extension achieved in Subsection 3.2 emphasizes that the relevance of mixed strategy equilibria in Nash implementation is rather a general phenomenon, not specific to the two-person environment.

3.1 Example 4 of Jackson (1992)

Suppose that there are two agents $I = \{1, 2\}$; four alternatives $A = \{a, b, c, d\}$; and two states $\Theta = \{\theta, \theta'\}$. Suppose that agent 1 has the state-independent preference $a \succ_1 b \succ_1 c \sim_1 d$ and agent 2 has the preference $a \succ_2^{\theta} b \succ_2^{\theta} d \succ_2^{\theta} c$ at state θ and preference $b \succ_2^{\theta'} a \succ_2^{\theta'} c \sim_2^{\theta'} d$ at state θ'. Consider the SCF f such that $f(\theta) = a$ and $f(\theta') = c$.

First, Jackson (1992) constructs a finite mechanism $\Gamma = (M, g)$ (described in Table 2) that implements the SCF f in pure-strategy Nash equilibria:

$g(m)$		Agent 2		
		m_2^1	m_2^2	m_2^3
Agent 1	m_1^1	c	d	d
	m_1^2	d	a	b
	m_1^3	d	b	a

Table 2: The mechanism introduced in Example 4 of Jackson (1992).

There are two pure strategy Nash equilibria, (m_1^2, m_2^2) and (m_1^3, m_2^3), in the game $\Gamma(\theta)$, both of which result in outcome a. In the game $\Gamma(\theta')$, the unique pure-strategy Nash equilibrium is (m_1^1, m_2^1), which results in outcome c. Thus, the SCF f is implementable by the above finite mechanism in pure-strategy Nash equilibria. Due to the necessity of Maskin monotonicity for Nash implementation, we know that the SCF f satisfies Maskin monotonicity. However, in the game $\Gamma(\theta')$, there is a mixed-strategy Nash equilibrium, where each agent i plays m_i^2 and m_i^3 with equal probability, which results in outcomes a and b, each with probability 1/2. Both agents strictly prefer any outcome of the mixed-strategy equilibrium to the outcome of the pure-strategy equilibrium. Thus, according to our terminology, this mixed strategy Nash equilibrium is compelling. Note that there is a conflict of interests between the two agents over a and b in state θ', i.e., while agent 1 prefers a to b, agent 2 prefers b to a. This conflict of interests allows us to have the unique pure strategy Nash equilibrium in the game $\Gamma(\theta')$, which results in outcome c. At the same time, this logic for the uniqueness of the pure-strategy equilibrium is extremely dubious because outcomes a and b are strictly better for both agents than outcome c.

Jackson (1992) further shows that his argument applies to *any* finite implementing mechanism. That is, for any finite mechanism which implements the SCF f in pure-strategy Nash equilibria, there must also exist a compelling mixed-strategy Nash equilibrium at state θ' inducing a lottery different from c, which is the socially desirable outcome by the SCF f at state θ'. Therefore, the SCF f is "not" C-implementable with respect to $\mathcal{U}$, which is the set of "all" cardinal utility representations, or any of its subsets. It thus follows that the identified compelling mixed strategy equilibrium persists independently of any cardinal representation.

3.2 The Case of More Than Two Agents

The previous subsection presented a scenario where any finite pure Nash implementing mechanism suffers from the existence of compelling mixed equilibria, undermining the competence of pure Nash implementation. We now argue that the problem exemplified in Subsection 3.1 is not specific to the two-person environment. More specifically, we consider an environment with three agents in which each player plays an indispensable role in at least some state and nevertheless, we are able to show that the same insights obtained in Subsection 3.1 can be extended to this three-person environment.

Consider an environment $\mathcal{E}^*$ in which there are three agents, $I = \{1, 2, 3\}$; four alternatives, $A = \{a, b, c, d\}$; and four states, $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$, as well as the SCF f to be implemented, which all are depicted by the table below:

State	Agent 1	Agent 2	Agent 3	$f(\cdot)$
θ_1	$a \succ_1^{\theta_1} b \succ_1^{\theta_1} c \sim_1^{\theta_1} d$	$a \succ_2^{\theta_1} b \succ_2^{\theta_1} d \succ_2^{\theta_1} c$	$a \succ_3^{\theta_1} b \succ_3^{\theta_1} c \sim_3^{\theta_1} d$	$f(\theta_1) = a$
θ_2	$a \succ_1^{\theta_2} b \succ_1^{\theta_2} c \sim_1^{\theta_2} d$	$b \succ_2^{\theta_2} a \succ_2^{\theta_2} c \sim_2^{\theta_2} d$	$a \succ_3^{\theta_2} b \succ_3^{\theta_2} c \sim_3^{\theta_2} d$	$f(\theta_2) = c$
θ_3	$a \succ_1^{\theta_3} b \succ_1^{\theta_3} c \sim_1^{\theta_3} d$	$b \succ_2^{\theta_3} a \succ_2^{\theta_3} c \sim_2^{\theta_3} d$	$b \succ_3^{\theta_3} a \succ_3^{\theta_3} d \succ_3^{\theta_3} c$	$f(\theta_3) = b$
θ_4	$d \succ_1^{\theta_4} a \succ_1^{\theta_4} b \succ_1^{\theta_4} c$	$b \succ_2^{\theta_4} a \succ_2^{\theta_4} c \sim_2^{\theta_4} d$	$a \succ_3^{\theta_4} b \succ_3^{\theta_4} d \succ_3^{\theta_4} c$	$f(\theta_4) = d$

Table 3: The environment $\mathcal{E}^*$ and the SCF f

The environment $\mathcal{E}^*$ and the SCF f differ from those presented by Example 4 of Jackson (1992) in a non-trivial way, as there are more states and none of the agents has preferences that are a replication of another agent. Moreover, every outcome in the set of alternatives can be chosen by the SCF in some state. As such, it is impossible to devise a mechanism that simply ignores one agent by giving him a trivial message space, which essentially reduces the problem of implementation to the two-person problem. In this sense, we deal with a genuinely three-person problem. Let us consider the following mechanism (M, g) in which each agent chooses an integer from $\{0, 1, 2\}$; agent 1 chooses a row; agent 2 chooses a column; and agent 3 chooses a matrix:

$g(m_1, m_2, 0)$		Agent 2		
		2	1	0
Agent 1	2	c	d	d
	1	d	a	b
	0	d	b	a

$g(m_1, m_2, 1)$		Agent 2		
		2	1	0
Agent 1	2	d	c	c
	1	a	a	b
	0	b	a	b

$g(m_1, m_2, 2)$		Agent 2		
		2	1	0
Agent 1	2	d	c	c
	1	b	b	a
	0	a	b	a

Table 4: The mechanism that pure Nash implements f in the environment $\mathcal{E}^*$

The mechanism above pure Nash implements the SCF f. However, in state θ_2, there is a compelling mixed equilibrium, with agents 1 and 2 uniformly randomizing between messages 1 and 0 and agent 3 uniformly randomizing between messages 1 and 2. This is so regardless of any cardinal representation. As in the same as Example 4 of Jackson (1992), this is a persistent feature possessed by *any* finite mechanism that pure Nash implements f:

Proposition 1 *We consider the environment $\mathcal{E}^*$. Then, the SCF f is pure Nash implemented by finite mechanisms. Moreover, if the SCF f is pure Nash implemented by a finite mechanism Γ, for any $u \in \mathcal{U}$, the game $\Gamma(\theta_2, u)$ has a compelling mixed strategy equilibrium.*[9]

This proposition (whose proof can be found in the Appendix) highlights that the issue of compelling mixed strategy equilibria in pure Nash implementation is not specific to two-person environments but can be extended to much more general environments.

[9]The proof of this proposition essentially follows the same logic of the remark made by Jackson (1992, p.770). We adapt Jackson's argument to an environment where there are three agents; there are four states; and each agent plays an indispensable role defining the outcome induced by the SCF in some state.

4 Illustration of the Main Result

In the rest of the paper, due to our technical difficulty, we will focus entirely on the two-person environments, i.e., $I = \{1, 2\}$. We defer our discussion on this difficulty to the conclusion section. The main objective of this paper is to identify a class of environments where the issue of compelling mixed strategy equilibria can be avoided by carefully designing an implementing mechanism. In this section, we illustrate how we resolve this issue in the slightly modified version of Example 4 of Jackson (1992).

One crucial feature Jackson's Example 4 has is that its argument seems to rely heavily on the extreme inefficiency of the SCF, i.e., the SCF f assigns the common worst outcome in state θ'.[10] To investigate how robust Jackson's argument is, we only make the following modification: both agents now strictly prefer c to d in state θ', i.e., $c \succ_i^{\theta'} d$ for each $i = 1, 2$. Recall that this modification is consistent with our setup, as the environment this paper considers only admits strict preferences.

We summarize the basic setup. Agent 1 has the state-independent preference $a \succ_1 b \succ_1 c \succ_1 d$ and agent 2 has the preference $a \succ_2^\theta b \succ_2^\theta d \succ_2^\theta c$ at state θ and preference $b \succ_2^{\theta'} a \succ_2^{\theta'} c \succ_2^{\theta'} d$ at state θ'. Consider the same SCF f such that $f(\theta) = a$ and $f(\theta') = c$. This way the SCF never assigns the worst outcome for any agent in either state (a feature that will also be implied by our sufficient condition).

With this modification, we are able to construct a mechanism that not only implements the SCF in pure-strategy Nash equilibrium but also guarantees that all mixed-strategy equilibria of the constructed mechanism give each agent the expected payoff arbitrarily close to that of d, which is worse than that of c, the outcome induced by the SCF f at state θ'. Hence, we essentially overturn the implication of Jackson's Example 4 by assuming that there is a uniform bound for the utility difference.

For each integer $k \geq 2$, we define $\Gamma^k = (M^k, g^k)$ as a mechanism with the following properties: (i) for each $i \in N$, $M_i^k = \{0, 1, \ldots, k\}$ and (ii) the outcome function $g^k : M^k \to A$ is given by the following rules: for each $m \in M^k$,

- If $m = (k, k)$, then $g^k(m) = c$;
- If there exists an integer h with $0 \leq h \leq k-1$ such that $m = (h, h)$, then $g^k(m) = a$;
- If there exists an integer h with $0 \leq h \leq k-1$ such that $m = (h, (h + 1 \bmod k))$, then $g^k(m) = b$; and
- Otherwise, $g^k(m) = d$.

[10] Jackson (1992, p.770) is well aware of this point.

We illustrate this mechanism as follows:

$g^k(m)$		Agent 2								
		k	$k-1$	$k-2$	$k-3$	$\cdots$	3	2	1	0
Agent 1	k	c	d	d	d	$\cdots$	d	d	d	d
	$k-1$	d	a	d	d	$\cdots$	d	d	d	b
	$k-2$	d	b	a	d	$\cdots$	d	d	d	d
	$k-3$	d	d	b	a	$\cdots$	d	d	d	d
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	3	d	d	d	d	$\cdots$	a	d	d	d
	2	d	d	d	d	$\cdots$	b	a	d	d
	1	d	d	d	d	$\cdots$	d	b	a	d
	0	d	d	d	d	$\cdots$	d	d	b	a

Table 5: $\Gamma^k = (M^k, g^k)$ where $k \geq 3$.

When $k = 2$, our mechanism is reduced to the one introduced by Jackson (1992) where we set $m_i^1 = 2; m_i^2 = 1$; and $m_i^3 = 0$ for each $i \in \{1, 2\}$.

$g(m)$		Agent 2		
		2	1	0
Agent 1	2	c	d	d
	1	d	a	b
	0	d	b	a

Table 6: $\Gamma^k = (M^k, g^k)$ where $k = 2$.

For each $\theta \in \Theta$, $i \in \{1, 2\}$, and $\varepsilon > 0$, we define $\mathcal{U}_i^{\theta,\varepsilon}$ as a subset of $\mathcal{U}_i^\theta$ as follows:

$$\mathcal{U}_i^{\theta,\varepsilon} = \left\{ u_i \in \mathcal{U}_i^\theta \middle| \; |u_i(a, \theta) - u_i(a', \theta)| \geq \varepsilon, \; \forall a \in A, \forall a' \in A \backslash \{a\}, \; \forall \theta \in \Theta \right\}.$$

Let $\mathcal{U}^{\theta,\varepsilon} \equiv \times_{i \in N} \mathcal{U}_i^{\theta,\varepsilon}$ and $\mathcal{U}^\varepsilon \equiv \times_{\theta \in \Theta} \mathcal{U}^{\theta,\varepsilon}$. We observe that $\mathcal{U}^\varepsilon$ possesses the following monotonicity:

$$\varepsilon > \varepsilon' > 0 \Rightarrow \mathcal{U}^\varepsilon \subsetneq \mathcal{U}^{\varepsilon'} \subseteq \mathcal{U} \subseteq \mathcal{U}^0.$$

Loosely speaking, if we choose $\varepsilon > 0$ small enough, we can approximate $\mathcal{U}$ by $\mathcal{U}^\varepsilon$ to an arbitrary degree. We are now ready state the main result of this section.

Proposition 2 *For any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ large enough such that the SCF f is C-implementable with respect to $\mathcal{U}^{\varepsilon}$ by the mechanism Γ^K.*

Proof: The proof is completed by a series of lemmas. For the moment, we fix k in the proof and we ignore the dependence of the mechanism on k. We first show pure Nash implementation by the mechanism Γ^k.

Lemma 1 *The mechanism Γ^k implements the SCF in pure-strategy Nash Equilibrium.*

Proof: The message profile $(1,1)$ is a Nash equilibrium of the game $\Gamma^k(\theta)$, as it yields a which is their most preferred outcome for both agents so that no agent can find a profitable deviation. We claim that a is the unique Nash equilibrium outcome of the game $\Gamma^k(\theta)$. Let m be a message profile such that $g(m) \neq a$. We will show that m is "not" a Nash equilibrium in the game $\Gamma^k(\theta)$:

- If $g(m) = b$, there exists an integer h with $0 \leq h \leq k-1$ such that $m = (h, (h+1 \bmod k))$. Then agent 1 has an incentive to send a message $h + 1 \bmod k$ so that outcome a is induced.
- If $g(m) = c$, then $m = (k,k)$. Then, agent 2 has an incentive to send any message other than k so that outcome d is induced, as he strictly prefers outcome d to outcome c at state θ.
- If $g(m) = d$, then we have $m = (m_1, m_2)$ where $m_1 \neq m_2$. If $m_1 > m_2$ then, agent 1 has an incentive to deviate from m_1 to m_2 so that outcome a is induced. Conversely, if $m_2 > m_1$, then agent 2 has an incentive to deviate from m_2 to m_1, so that outcome a is induced.

We next claim that (k,k) is a Nash equilibrium of the game $\Gamma^k(\theta')$ because any unilateral deviation from (k,k) yields d, which is inferior to c induced by (k,k) for both agents. Moreover, no other outcome can be induced by a Nash equilibrium in this game: every message profile $m = (m_1, m_2)$ where $m_2 < k$ and $g(m) \neq a$ has a profitable deviation for player 1 at $m_1' = m_2$, while every message profile $m = (m_1, m_2)$ where $m_1 < k$ and $g(m) \neq b$ has a profitable deviation for player 2 at $m_2' = m_1 + 1 \bmod k$. Since $g(m) = a$ implies $m_1 < k$ and $g(m) = b$ implies $m_2 < k$, we have that there are no possible Nash equilibria with either $m_1 < k$ or $m_2 < k$. Thus, the only possible Nash equilibrium in pure strategies for this game is (k,k). ■

The following lemma is our key result, characterizing the set of Nash equilibria of the mechanism Γ^k in state θ'.

Lemma 2 *For each $i \in \{1,2\}$, let $\sigma_i = (\sigma_i(0), \sigma_i(1), ..., \sigma_i(k))$ denote agent i's strategy and for each $x \in \{0, 1, \ldots, k\}$, let $\sigma_i(x)$ denote the probability that agent i chooses x. If $\sigma = (\sigma_1, \sigma_2)$ is a Nash equilibrium in the game $\Gamma^k(\theta^{'})$, then, for each $i \in \{1,2\}$, there is a number $p^i \in [0,1]$ such that $\sigma_i(x) = p^i/k$ for each $x \in \{0, \ldots, k-1\}$. Moreover, $p^1 = 0$ if and only if $p^2 = 0$.*

Proof: Recall that we set $u_i(d; \theta^{'}) = 0$ for each $u_i \in \mathcal{U}_i^{\theta^{'}}$ and $i \in \{1,2\}$. Let σ be a Nash equilibrium of the game $\Gamma^k(\theta^{'})$. If $\sigma_i(k) = 1$ for each $i \in \{1,2\}$, such p^i in the lemma is guaranteed to exist by setting $p^i = 0$. Thus, we assume that there exists $i \in \{1,2\}$ for whom $\sigma_i(k) < 1$. We divide the proof into a series of steps, whose proofs will be found in the Appendix:

Step 1a: If there exists $x \in \{0, \ldots, k-1\}$ such that $\sigma_1(x) > 0$, then $\sigma_2(x) > 0$.

Step 1b: If there exists $x \in \{1, \ldots, k-1\}$ such that $\sigma_2(x) > 0$, then $\sigma_1(x-1) > 0$. Moreover, if $\sigma_2(0) > 0$, then $\sigma_1(k-1) > 0$.

Step 1c: If there exist $i \in \{1,2\}$ and $x^{'} \in \{0, \ldots, k-1\}$ for whom $\sigma_i(x^{'}) > 0$, then $\sigma_1(x) > 0$ and $\sigma_2(x) > 0$ for all $x \in \{0, \ldots, k-1\}$.

Step 2: If there exist $i \in \{1,2\}$ and $x, x^{'} \in \{0, \ldots, k-1\}$ such that $\sigma_i(x) > 0$ and $\sigma_i(x^{'}) > 0$, then $\sigma_i(x) = \sigma_i(x^{'})$.

It follows from both Steps 2 and 1c that $\sigma_i(x) = \sigma_i(x^{'})$ for every $x, x^{'} \in \{0, \ldots, k-1\}$ and $i \in \{1,2\}$. Thus, we can set $p^i = \sum_{x=0}^{k-1} \sigma_i(x)$. Since we assume $\sigma_i(k) < 1$ for each $i \in \{1,2\}$, we have $p^i > 0$. This completes the proof of Lemma 2. ■

As we can easily see in the proof of Lemma 1, there are no (compelling) mixed strategy Nash equilibria of the game $\Gamma^k(\theta)$ because, in state θ, the unique Nash equilibrium outcome is a, which is the best outcome for both agents. It thus remains to prove that there are no compelling mixed strategy equilibria in the game $\Gamma^k(\theta^{'})$.

If $k \geq 3$, we let σ^k be a nontrivial mixed-strategy Nash equilibrium in the game $\Gamma^k(\theta^{'})$. Then, the resulting outcome distribution induced by σ^k is given by

$$g \circ \sigma^k = \begin{cases} c & \text{w.p. } (1-p^1)(1-p^2) \\ a & \text{w.p. } (p^1 p^2)/k \\ b & \text{w.p. } (p^1 p^2)/k \\ d & \text{w.p. } ((k - 2p^1 p^2)/k) - ((1-p^1)(1-p^2)), \end{cases}$$

where $p^1, p^2 \in (0,1]$ and $p^i = \sum_{x=0}^{k-1} \sigma_i(x)$ for each $i \in \{1,2\}$. Recall the following pieces of notation:

$$\mathcal{U}_1^{\theta'} = \left\{u_1(\cdot;\theta') \in [0,1]^A \middle| \, 1 = u_1(a;\theta') > u_1(b;\theta') > u_1(c;\theta') > u_1(d;\theta') = 0\right\};$$
$$\mathcal{U}_2^{\theta'} = \left\{u_2(\cdot;\theta') \in [0,1]^A \middle| \, 1 = u_2(b;\theta') > u_2(a;\theta') > u_2(c;\theta') > u_2(d;\theta') = 0\right\}.$$

Let $\mathcal{U}^{\theta'} \equiv \mathcal{U}_1^{\theta'} \times \mathcal{U}_2^{\theta'}$. For each $\varepsilon \in (0,1)$, we have

$$\mathcal{U}_1^{\theta',\varepsilon} = \left\{u_1(\cdot;\theta') \in \mathcal{U}_1^{\theta'} \middle| \, u_1(c;\theta') \geq \varepsilon\right\};$$
$$\mathcal{U}_2^{\theta',\varepsilon} = \left\{u_2(\cdot;\theta') \in U_2^{\theta'} \middle| \, u_2(c;\theta') \geq \varepsilon\right\}.$$

Similarly, let $\mathcal{U}^{\theta',\varepsilon} \equiv \mathcal{U}_1^{\theta',\varepsilon} \times \mathcal{U}_2^{\theta',\varepsilon}$.

By the lemma below, we show that for each $\varepsilon > 0$, there exists $K \in \mathbb{N}$ large enough so that, for any $u \in \mathcal{U}^{\theta',\varepsilon}$, the game $\Gamma^K(\theta', u)$ has no compelling mixed strategy equilibria.

Lemma 3 *For each $\varepsilon > 0$, there exists an integer $K \in \mathbb{N}$ large enough so that for any $k \geq K$, $i \in \{1,2\}$, and $(u_1(\cdot,\theta'), u_2(\cdot;\theta')) \in \mathcal{U}^{\theta',\varepsilon}$,*

$$U_i(\sigma^k;\theta') \leq u_i(c;\theta'),$$

where $U_i(\sigma^k;\theta') = \sum_{x=0}^k \sigma_1^k(x) \sum_{x'=0}^k \sigma_2^k(x') u_i(g(x,x');\theta')$.

Proof: Fix $\varepsilon > 0$ and $i \in \{1,2\}$. We compute

$$U_i(\sigma^k;\theta') = \frac{p^1 p^2}{k}[u_i(a;\theta') + u_i(b;\theta')] + (1-p^1)(1-p^2)u_i(c;\theta').$$

For each $(p^1,p^2) \in [0,1]^2$, we define

$$k(p^1,p^2) = \frac{u_i(a;\theta') + u_i(b;\theta')}{u_i(c;\theta')} \left[\frac{1}{p^1} + \frac{1}{p^2} - 1\right]^{-1}.$$

In the rest of the proof, we make use of the following properties of $k(p^1,p^2)$:

- $k(\cdot,\cdot)$ is strictly increasing in both arguments over $[0,1]^2$.
- $k(p_h^1,p_h^2)$ converges to zero no matter how the sequence $\{(p_h^1,p_h^2)\}_{h=1}^\infty$ approaches $(0,0)$. Thus, $k(0,0) \equiv \lim_{(p^1,p^2)\to(0,0)} k(p^1,p^2) = 0$.
- $k(1,1) = [u_i(a;\theta') + u_i(b;\theta')]/u_i(c;\theta') = \max_{(p^1,p^2)\in[0,1]^2} k(p^1,p^2)$.

- We can conveniently rewrite $k(p^1, p^2)$ as

$$k(p^1, p^2) = \frac{u_i(a;\theta') + u_i(b;\theta')}{u_i(c;\theta')} \frac{p^1 p^2}{[1-(1-p^1)(1-p^2)]}.$$

We set $K = \min\{k \in \mathbb{N} | k \geq 2/\varepsilon\}$. As $2/\varepsilon \geq [u_i(a;\theta') + u_i(b;\theta')]/u_i(c;\theta')$ for any $u_i(\cdot;\theta') \in \mathcal{U}_i^{\theta'}[\varepsilon]$, we have that $K \geq k(1,1)$. Due to the strict monotonicity of $k(p^1, p^2)$ with respect to p^1 and p^2, we have that $K \geq k(p^1, p^2)$ for any $(p^1, p^2) \in [0,1]^2$. Hence, for any $k \geq K$:

$$\begin{aligned}
U_i(\sigma^k;\theta') &= \frac{p^1 p^2}{k}[u_i(a;\theta') + u_i(b;\theta')] + (1-p^1)(1-p^2)u_i(c;\theta') \\
&\leq \frac{p^1 p^2}{k(p^1,p^2)}[u_i(a;\theta') + u_i(b;\theta')] + (1-p^1)(1-p^2)u_i(c;\theta') \\
&\quad (\because\ k \geq K \geq k(p^1,p^2)\ \forall (p^1,p^2) \in [0,1]^2) \\
&= u_i(c;\theta')[1-(1-p^1)(1-p^2)] + (1-p^1)(1-p^2)u_i(c;\theta') \\
&= u_i(c;\theta').
\end{aligned}$$

This completes the proof of Lemma 3. ■

Combining Lemmas 1, 2, and 3 together, we complete the proof of Proposition 2. ■

5 The Main Result

The objective of this section is to generalize Proposition 2 which we obtained in the previous section. First, we introduce a notion of *acceptability*.

Definition 6 *Given two subsets of alternatives $\mathcal{A}, \mathcal{B} \subseteq A$, we say that alternative x is $(\mathcal{A}, \mathcal{B})$-**acceptable** at state θ if $x \in \mathcal{A} \cup \mathcal{B}$ and the following two conditions hold:*

- *There is no alternative $a \in \mathcal{A}$ such that $a \succ_1^\theta x$.*
- *There is no alternative $b \in \mathcal{B}$ such that $b \succ_2^\theta x$.*

The property that x is $(\mathcal{A}, \mathcal{B})$-acceptable at state θ guarantees that $\mathcal{A}$ is contained in agent 1's lower contour set at x in state θ and $\mathcal{B}$ is contained agent 2's lower contour set at x in state θ. In the rest of the argument below, we write $\Theta = \{\theta_0, \theta_1, \ldots, \theta_J\}$ where $J = |\Theta| - 1$. So, we are now ready to introduce the key condition for our characterization.

Definition 7 *The environment $\mathcal{E} = \left(\{1,2\}, A, \Theta, (\succeq_i^\theta)_{i\in\{1,2\},\theta\in\Theta}\right)$ satisfies **Condition P+M** with respect to the SCF f if there exist a function $z : \{0,\ldots,J\} \times \{0,\ldots,J\} \to A$ with Z as $z(\cdot)$'s image, and two collections of subsets $\{\mathcal{A}_j\}_{j=0}^J, \{\mathcal{B}_j\}_{j=0}^J \subseteq A$ such that*

1. *$z(j_1, j_2) \in \mathcal{A}_{j_2} \cap \mathcal{B}_{j_1}$ for all $(j_1, j_2) \in \{0,\ldots,J\} \times \{0,\ldots,J\}$;*
2. *For each state $\theta \in \Theta$ and each pair $(j_1, j_2) \in \{0,\ldots,J\} \times \{0,\ldots,J\}$, if $f(\theta) \neq z(j_1, j_2)$, there exists either $a_{(j_1,j_2)} \in \mathcal{A}_{j_2}$ such that $a_{(j_1,j_2)} \succ_1^\theta z(j_1, j_2)$ or $b_{(j_1,j_2)} \in \mathcal{B}_{j_1}$ such that $b_{(j_1,j_2)} \succ_2^\theta z(j_1, j_2)$;*
3. *For every $j \in \{0,\ldots,J\}$, $f(\theta_j)$ is $(\mathcal{A}_j, \mathcal{B}_j)$-acceptable at state θ_j;*
4. *For every $\theta \in \Theta$ and every $j \in \{0,\ldots,J\}$, if there exists $x \in A$ such that x is $(\mathcal{A}_j, \mathcal{B}_j)$-acceptable at θ, then $x = f(\theta)$,*
5. *For each $\theta \in \Theta$, if $f(\theta) \in Z$, there exists no $x \in \bigcup_{j=0}^J \mathcal{A}_j \cup \mathcal{B}_j$ such that $x \succ_i^\theta f(\theta)$ for all $i \in \{1,2\}$.*
6. *For each $\theta \in \Theta$, if $f(\theta) \notin Z$, then $f(\theta) \succ_i^\theta z$ for all $i \in \{1,2\}$ and $z \in Z$.*

In what follows, Properties 1, 2, 3, and 4 in Condition P+M are collectively called *Condition P* and Properties 5 and 6 in Condition P+M are collectively called *Condition M*, respectively. By "Condition P," we mean the property concerning "pure" Nash implementation and by "Condition M," we mean the property concerning "mixed" Nash implementation.

We next show that Condition P+M is satisfied in the modified version of Jackson's (1992) example discussed in Section 4. We set $J = 1$, $z(j_1, j_2) = d\ \forall (j_1, j_2) \in \{0,1\}^2$, $\mathcal{A}_0 = \mathcal{B}_0 = \{c, d\}$, $\mathcal{A}_1 = \mathcal{B}_1 = \{a, b, d\}$. Property 1 of Condition P can easily be verified, as $\mathcal{A}_0, \mathcal{A}_1, \mathcal{B}_0, \mathcal{B}_1$ all contain d. Property 5 of Condition M is vacuously satisfied, as there exists no $\tilde{\theta} \in \Theta$ such that $f(\tilde{\theta}) = d$. Property 2 of Condition P is satisfied because $z(j_1, j_2) = d$ for each (j_1, j_2); $a \in \mathcal{A}_0$; $c \in \mathcal{A}_1$; $a \succ_1^{\tilde{\theta}} d$ and $c \succ_1^{\tilde{\theta}} d$ for each $\tilde{\theta} \in \Theta$. Property 3 of Condition P is satisfied because $f(\theta) = a$, which is $(\mathcal{A}_1, \mathcal{B}_1)$-acceptable in state θ and $f(\theta') = c$, which is $(\mathcal{A}_0, \mathcal{B}_0)$-acceptable in state θ'. Since outcomes a and c are the only alternatives that are $(\mathcal{A}_j, \mathcal{B}_j)$-acceptable for some $j = 0, 1$ in some state, Property 4 of Condition P holds. Lastly, Property 6 is satisfied because $z(j_1, j_2) = d$ for each (j_1, j_2); $f(\theta) = a$; $f(\theta') = c$; both agents prefer a to d in state θ and c to d in state θ'. Thus, all properties in Condition P+M hold.

We recall the following notation. For each $\varepsilon > 0$, $\theta \in \Theta$, and $i \in \{1,2\}$, we define

$$\mathcal{U}_i^{\theta,\varepsilon} = \left\{ u_i \in \mathcal{U}_i^{\theta} \middle| \ |u_i(a,\theta) - u_i(a',\theta)| \geq \varepsilon, \ \forall a \in A, \forall a' \in A \backslash \{a\}, \ \forall \theta \in \Theta \right\}$$

as the set of agent i's cardinal utility representations in state θ such that the utility difference is bounded from below by ε. We let $\mathcal{U}^{\theta,\varepsilon} = \mathcal{U}_1^{\theta,\varepsilon} \times \mathcal{U}_2^{\theta,\varepsilon}$ and $\mathcal{U}^{\varepsilon} = \times_{\theta \in \Theta} \mathcal{U}^{\theta,\varepsilon}$.

Theorem 1 *Let f be an SCF. Suppose that the finite two-person environment $\mathcal{E} = \left(\{1,2\}, A, \Theta, (\succeq_i^{\theta})_{i \in \{1,2\}, \theta \in \Theta}\right)$ satisfies Condition P+M with respect to the SCF f. Then, for any $\varepsilon > 0$, the SCF f is C-implementable with respect to $\mathcal{U}^{\varepsilon}$.*

Proof: Suppose that the finite two-person environment $\mathcal{E} = \left(\{1,2\}, A, \Theta, (\succeq_i^{\theta})_{i \in \{1,2\}, \theta \in \Theta}\right)$ satisfies Condition P+M with respect to the SCF f. For each integer $k \geq 2$, we construct a mechanism $\Gamma^k = (M^k, g^k)$ as follows: For each $i \in \{1,2\}$, we set $M_i^k = \{0, 1, \ldots, (J+1)k - 1\} \times A$, i.e., each message $m_i = (o_i, x_i)$ agent i sends to the mechanism is composed of a pair of an integer which lies between 0 and $(J+1)k - 1$ and an alternative in A.

Define the function $n^k : \{0, \ldots, (J+1)k-1\} \to \{0, \ldots, J\}$ as follows: for each $o_i \in \{0, \ldots, (J+1)k - 1\}$,

$$n^k(o_i) = \max\{n \in \mathbb{N} | \ n \times k \leq o_i\}.$$

In words, we first compute o_i/k, then round the computed number down to the nearest integer, and finally set the obtained integer as $n^k(o_i)$. For example, if $o_i = 13$ and $k = 5$, we have $n^5(13) = 2$.

To define the outcome function below, we introduce the following permutation $\pi^k : \{0, \ldots, (J+1)k-1\} \to \{0, \ldots, (J+1)k-1\}$: for each $\tilde{o} \in \{0, \ldots, (J+1)k-1\}$,

$$\pi^k(\tilde{o}) = \begin{cases} n^k(\tilde{o})k & \text{if } \tilde{o} = (n^k(\tilde{o}) + 1)k - 1, \\ \tilde{o} + 1 & \text{otherwise.} \end{cases}$$

We can interpret π^k as a series of $J+1$ cycles which moves $\tilde{o}$ to $\tilde{o}+1$ for all $\tilde{o} \in \{n^k(\tilde{o})k, \ldots, (n^k(\tilde{o})+1)k - 2\}$, while moves $(n^k(\tilde{o})+1)k - 1$ to $n^k(\tilde{o})k$. The outcome function $g^k : M^k \to A$ is dictated by the following three rules: for each $m \in M^k$ where $m = (m_1, m_2) = ((o_1, x_1), (o_2, x_2))$,

Rule 1: If $o_1 = o_2$ and $x_1 \in \mathcal{A}_{n^k(o_1)}$, then $g^k(m) = x_1$.

Rule 2: If $o_2 = \pi^k(o_1)$ and $x_2 \in \mathcal{B}_{n^k(o_2)}$, then $g^k(m) = x_2$.[11]

[11] We note that $o_2 = \pi^k(o_1)$ implies both $o_1 \neq o_2$ and $n^k(o_1) = n^k(o_2)$, as these properties will be exploited later.

Rule 3: For all other cases, $g^k(m) = z(n^k(o_1), n^k(o_2))$.

We describe how the mechanism Γ^k can be played as follows: first, each agent selects a number between 0 and J (represented in the mechanism by the values of $n^k(o_1)$ for agent 1 and $n^k(o_2)$ for agent 2). If $n^k(o_1) \neq n^k(o_2)$, the outcome is given by $z(n^k(o_1), n^k(o_2))$. If $n^k(o_1) = n^k(o_2)$, then they proceed to play a particular form of modulo game: each announces a second number between 0 and $k-1$. If they both select the same number, which implies $o_1 = o_2$, agent 1 wins and he can select any outcome from the set $\mathcal{A}_{n^k(o_1)}$. If agent 2 picks a number o_2 that is higher than the number o_1 picked by agent 1 exactly by one unit (or picks 0, in case agent 1 picks $k-1$) but still $n^k(o_1) = n^k(o_2)$, then agent 2 wins and can pick any outcome from the set $\mathcal{B}_{n^k(o_2)}$. In any other case, the outcome is, once again, given by $z(n^k(o_1), n^k(o_1))$.[12] This is illustrated in Table 7, which uses the following notation: for each message $(o_1, x_1) \in M_1^k$, $(o_2, x_2) \in M_2^k$, and $j \in \{0, \ldots, J-1\}$, we write

$$a_j = \begin{cases} x_1 & \text{if } x_1 \in \mathcal{A}_j, \\ z(j,j) & \text{if } x_1 \notin \mathcal{A}_j, \end{cases} \quad \text{and} \quad b_j = \begin{cases} x_2 & \text{if } x_2 \in \mathcal{B}_j, \\ z(j,j) & \text{if } x_2 \notin \mathcal{B}_j. \end{cases}$$

We complete the rest of the proof in a series of lemmas.

Lemma 4 *If Condition P is satisfied,* Γ^k *pure Nash implements* f.

Proof of Lemma 4: The proof is in the Appendix. ■

Next, we show that if $f(\theta) \in Z$, there are no compelling mixed strategy equilibria in the game.

Lemma 5 *Assume that Condition P+M holds. If* $f(\theta) \in Z$*, then, for any* $u \in \mathcal{U}^\theta$*, the game* $\Gamma^k(\theta, u)$ *has no compelling mixed strategy equilibria.*

Proof of Lemma 5: The proof is in the Appendix. ■

It remains now for us to show that when $f(\theta) \notin Z$, for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that, for any $u \in \mathcal{U}^\varepsilon$, the game $\Gamma^K(\theta, u)$ has no compelling mixed strategy equilibria. The proof of this case requires us to take a series of steps.

For each $j \in \{0, \ldots, J\}$ and $\theta \in \Theta$, let a_j^θ be the best outcome for player 1 within $\mathcal{A}_j$ at state θ, and b_j^θ the best outcome for player 2 within $\mathcal{B}_j$ at state θ, respectively. In the rest of the proof, we fix $\theta \in \Theta$ throughout.

Lemma 6 *Consider the mechanism* $\Gamma^k = (M^k, g^k)$. *For each message* $m_1 = (o_1, x_1) \in M_1^k$*, we can define the following message* $m_1^*(m_1) = (o_1, a^\theta_{n^k(o_1)}) \in$

[12]This construction is reminiscent of what Korpela (2016) calls the flow game.

<table>
<tr><td colspan="3" rowspan="3">$g(m_1, m_2)$ where
$m_i = (o_i, x_i)$
for $i \in \{1, 2\}$</td><td colspan="17">Agent 2</td></tr>
<tr><td colspan="4">$n^k(o_2) = 0$</td><td colspan="4">$n^k(o_2) = 1$</td><td colspan="4">$n^k(o_2) = 2$</td><td>$\dots$</td><td colspan="4">$n^k(o_2) = J$</td></tr>
<tr><td>0</td><td>1</td><td>$\dots$</td><td>$k-1$</td><td>k</td><td>$k+1$</td><td>$\dots$</td><td>$2k-1$</td><td>$2k$</td><td>$2k+1$</td><td>$\dots$</td><td>$3k-1$</td><td>$\dots$</td><td>Jk</td><td>$Jk+1$</td><td>$\dots$</td><td>$(J+1)k-1$</td></tr>
<tr><td rowspan="17">Agent 1</td><td rowspan="4">$n^k(o_1) = 0$</td><td>0</td><td>a_0</td><td>b_0</td><td>$\dots$</td><td>$z_{0,0}$</td><td colspan="4" rowspan="4">$z_{0,1}$</td><td colspan="4" rowspan="4">$z_{0,2}$</td><td rowspan="4">$\dots$</td><td colspan="4" rowspan="4">$z_{0,J}$</td></tr>
<tr><td>1</td><td>$z_{0,0}$</td><td>a_1</td><td>$\dots$</td><td>$z_{0,0}$</td></tr>
<tr><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td><td>$\ddots$</td><td>$\vdots$</td></tr>
<tr><td>$k-1$</td><td>b_0</td><td>$z_{0,0}$</td><td>$\vdots$</td><td>a_0</td></tr>
<tr><td rowspan="4">$n^k(o_1) = 1$</td><td>k</td><td colspan="4" rowspan="4">$z_{1,0}$</td><td>a_1</td><td>b_1</td><td>$\dots$</td><td>$z_{1,1}$</td><td colspan="4" rowspan="4">$z_{1,2}$</td><td rowspan="4">$\dots$</td><td colspan="4" rowspan="4">$z_{1,J}$</td></tr>
<tr><td>$k+1$</td><td>$z_{1,1}$</td><td>a_1</td><td>$\dots$</td><td>$z_{1,1}$</td></tr>
<tr><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td><td>$\ddots$</td><td>$\vdots$</td></tr>
<tr><td>$2k-1$</td><td>b_1</td><td>$z_{1,1}$</td><td>$\dots$</td><td>a_1</td></tr>
<tr><td rowspan="4">$n^k(o_1) = 2$</td><td>$2k$</td><td colspan="4" rowspan="4">$z_{2,0}$</td><td colspan="4" rowspan="4">$z_{2,1}$</td><td>a_2</td><td>b_2</td><td>$\dots$</td><td>$z_{2,2}$</td><td rowspan="4">$\dots$</td><td colspan="4" rowspan="4">$z_{2,J}$</td></tr>
<tr><td>$2k+1$</td><td>$z_{2,2}$</td><td>a_2</td><td>$\dots$</td><td>$z_{2,2}$</td></tr>
<tr><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td><td>$\ddots$</td><td>$\vdots$</td></tr>
<tr><td>$3k-1$</td><td>b_2</td><td>$z_{2,2}$</td><td>$\dots$</td><td>a_2</td></tr>
<tr><td>$\vdots$</td><td>$\vdots$</td><td colspan="4">$\vdots$</td><td colspan="4">$\vdots$</td><td colspan="4">$\vdots$</td><td>$\ddots$</td><td colspan="4">$\vdots$</td></tr>
<tr><td rowspan="4">$n^k(o_1) = J$</td><td>Jk</td><td colspan="4" rowspan="4">$z_{J,0}$</td><td colspan="4" rowspan="4">$z_{J,1}$</td><td colspan="4" rowspan="4">$z_{J,2}$</td><td rowspan="4">$\dots$</td><td>a_J</td><td>b_J</td><td>$\dots$</td><td>$z_{J,J}$</td></tr>
<tr><td>$Jk+1$</td><td>$z_{J,J}$</td><td>a_J</td><td>$\dots$</td><td>$z_{J,J}$</td></tr>
<tr><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td><td>$\ddots$</td><td>$\vdots$</td></tr>
<tr><td>$J(k+1)-1$</td><td>b_J</td><td>$z_{J,J}$</td><td>$\dots$</td><td>a_J</td></tr>
</table>

Table 7: The mechanism Γ^k where we write $z(j_1, j_2)$ as z_{j_1,j_2}

M_1^k (possibly $m_1^*(m_1) = m_1$) such that $g^k(m_1^*(m_1), m_2) \succeq_1^\theta g^k(m_1, m_2)$ for each $m_2 \in M_2^k$. Moreover, if $g^k(m_1^*(m_1), m_2) \neq g^k(m_1, m_2)$ for some $m_2 \in M_2^k$, then $g^k(m_1^*(m_1), m_2) \succ_1^\theta g^k(m_1, m_2)$. Similarly, for each message $m_2 = (o_2, x_2) \in M_2^k$, we can define the following message $m_2^*(m_2) = (o_2, b^\theta_{n^k(o_2)}) \in M_2^k$ (possibly $m_2^*(m_2) = m_2$) such that $g^k(m_1, m_2^*(m_2)) \succeq_2^\theta g^k(m_1, m_2)$ for each $m_1 \in M_1^k$. Moreover, if $g^k(m_1, m_2^*(m_2)) \neq g^k(m_1, m_2)$ for some $m_1 \in M_1^k$, then $g^k(m_1, m_2^*(m_2)) \succ_2^\theta g^k(m_1, m_2)$.

Proof of Lemma 6: The proof is in the Appendix. ■

We introduce the following notation in the mechanism $\Gamma = (M, g)$: for any agent $i \in \{1, 2\}$ and mixed strategy $\sigma_i \in \Delta(M_i)$, we can define another mixed strategy $\sigma_i^*[\sigma_i]$ as follows: for each $m_i \in M_i$,

$$\sigma_i^*[\sigma_i](m_i) = \sum_{\tilde{m}_i : m_i = m_i^*(\tilde{m}_i)} \sigma_i(\tilde{m}_i).$$

Then, we establish the following lemma.

Lemma 7 *Fix $u \in \mathcal{U}^\theta$ and $\sigma \in NE(\Gamma(\theta, u))$. Then, $\sigma^*[\sigma] \in NE(\Gamma(\theta, u))$. Moreover, if σ is compelling in the game $\Gamma(\theta, u)$, $\sigma^*[\sigma]$ is also compelling in the same game $\Gamma(\theta, u)$.*

Proof of Lemma 7: This follows directly from Lemma 6. ■

Define

$$NE^*(\Gamma(\theta, u)) \equiv \bigcup_{\sigma \in NE(\Gamma(\theta,u))} \{\sigma^*[\sigma]\}.$$

By Lemma 7, we have $NE^*(\Gamma(\theta, u)) \subseteq NE(\Gamma(\theta, u))$. The contrapositive form of Lemma 7 says that if σ^* is not a compelling mixed strategy equilibrium, σ is also not a compelling mixed strategy equilibrium. This implies that there is no loss of generality to focus on $NE^*(\Gamma(\theta, u))$, as far as we are concerned with the nonexistence of compelling mixed strategy equilibria in the game $\Gamma(\theta, u)$. If we only focus on $NE^*(\Gamma(\theta, u))$, we can only focus on mixed strategies where the players randomize only on the integers they choose (the first component of the message), with the alternative (second component) being always the most preferred alternative from their choice set associated.

With this specific structure of mixed strategies, we introduce the following notation. Let $m_1(o_1) = (o_1, a^\theta_{n^k(o_1)})$ for each $o_1 \in \{0, \dots, (J+1)k - 1\}$ and $m_2(o_2) = (o_2, b^\theta_{n^k(o_2)})$ for each $o_2 \in \{0, \dots, (J+1)k-1\}$. Then, for each $i \in \{1, 2\}$, denote by σ_i the strategy that assigns probability $\sigma_i(o_i)$ to message $m_i(o_i)$, with $\sum_{o_i=0}^{(J+1)k-1} \sigma_i(o_i) = 1$.

Lemma 8 *Suppose that Condition P+M holds and $f(\theta) \notin Z$. Let $u \in \mathcal{U}^\theta$ and $\sigma = (\sigma_1, \sigma_2) \in NE^*(\Gamma^k(\theta, u))$ be a compelling mixed strategy equilibrium. Then, for each $i \in \{1, 2\}$ and $j \in \{0, \dots, J-1\}$ such that $a^\theta_j \succ^\theta_1 f(\theta)$ or $b^\theta_j \succ^\theta_2 f(\theta)$, there is a number $p^i_j \in [0, 1]$ such that $\sigma_i(x) = p^i_j/k$ for each x such that $n^k(x) = j$.*

Proof of Lemma 8: The proof is in the Appendix. ■

Lemma 8 needs to assume that there exists $j \in \{0, \dots, J-1\}$ such that $a^\theta_j \succ^\theta_1 f(\theta)$ or $b^\theta_j \succ^\theta_2 f(\theta)$ to characterize the structure of compelling mixed strategy equilibria. If such a condition is not satisfied, we do not know the structure of compelling equilibria by Lemma 8. Therefore, the next lemma guarantees that the premise for Lemma 8 is nonvacuous.

Lemma 9 *Suppose that Condition P+M holds and $f(\theta) \notin Z$. Let $u \in \mathcal{U}^\theta$ and $\sigma = (\sigma_1, \sigma_2) \in NE^*(\Gamma^k(\theta, u))$ be a compelling mixed strategy equilibrium. Then, there exists a $j \in \{0, \dots, J-1\}$ such that $a^\theta_j \succ^\theta_1 f(\theta)$ or $b^\theta_j \succ^\theta_2 f(\theta)$.*

Proof of Lemma 9: The proof is in the Appendix. ■

Using Lemmas 7, 8, and 9, we are able to establish the lemma below.[13]

Lemma 10 *Suppose that Condition P+M holds and $f(\theta) \notin Z$. Then, for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that there are no compelling mixed strategy equilibria of the game $\Gamma^K(\theta, u)$ for all $u \in \mathcal{U}^\varepsilon$.*

Proof of Lemma 10: The proof is in the Appendix. ■

The proof of Theorem 1 is completed as follows. By Lemma 4, the mechanism Γ^k pure Nash implements the SCF f. When $f(\theta) \in Z$, by Lemma 5, for any $u \in \mathcal{U}$, the game $\Gamma^k(\theta, u)$ has no compelling mixed strategy equilibria. When $f(\theta) \notin Z$, by Lemma 10, for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ large enough so that for any $u \in \mathcal{U}^\varepsilon$, the game $\Gamma^K(\theta, u)$ has no compelling mixed strategy equilibria. Thus, f is C-implementable with respect to $\mathcal{U}^\varepsilon$ by the mechanism Γ^k. ■

6 Indispensability of Condition P+M

In the previous section, we have shown that our Condition P+M is a *sufficient* condition for C-implementation. This section next investigates how necessary Condition P+M is for C-implementation. As mentioned earlier, Condition P+M can be decomposed into the following two conditions: Condition P regarding pure Nash implementation and Condition M regarding C-implementation, respectively. In this section, we argue that Condition P is a necessary and sufficient condition for pure Nash implementation by any finite mechanism, while Condition M is a necessary condition "if we require C-implementation by our proposed canonical mechanism." Hence, all of the properties in Condition P+M are indispensable for our Theorem 1.

6.1 Condition P is Necessary for pure Nash implementation

The lemma below establishes the necessity of Condition P for pure Nash implementation, which is implied by C-implementation.

Lemma 11 *Suppose that the SCF f is pure Nash implementable. Then, there exist two collections of subsets of alternatives $\{\mathcal{A}_j\}_{j=0}^J, \{\mathcal{B}_j\}_{j=0}^J \subseteq A$ and a function $z : \{0, \ldots, J\} \times \{0, \ldots, J\} \to A$ for which Condition P holds.*

[13] The reader is referred to the proof of Lemma 10 in the Appendix to see how these lemmas are combined.

Dutta and Sen (1991) and Moore and Repullo (1990) independently identify Condition β and Condition $\mu 2$, respectively as a necessary and sufficient condition for pure Nash implementation for the case of two agents. With the help of Lemma 4, the lemma above shows that Condition P is yet another necessary and sufficient condition for pure Nash implementation in finite mechanisms when there are two players. Since we can modify the canonical mechanisms proposed by Dutta and Sen (1991) and Moore and Repullo (1990) into a finite mechanism by replacing the integer game with the modulo game, our Condition P is equivalent to Condition β and Condition $\mu 2$.

6.2 Indispensability of Condition M

We shall show that if the environment and the SCF to be implemented violates either Properties 5 or 6 in Condition M, our canonical mechanism Γ^k possesses a compelling mixed strategy equilibrium, regardless of the size of k. More specifically, we conclude that Property 5 of Condition M is indispensable for C-implementation by "our canonical mechanism," while Property 6 of Condition M is indispensable for C-implementation by "any" finite mechanism.

We start with the indispensability of Property 6 in Condition M. To show this, we take up Jackson's Example 4 which features an environment and an SCF where all properties of Condition P+M are satisfied except Property 6. In this example, we have $\Theta = \{\theta, \theta'\}$ and $f(\theta) = a$ and $f(\theta') = c$. To see this, we set $J = 1$; $z(j_1, j_2) = d$; $\forall (j_1, j_2) \in \{0, 1\}^2$; $\mathcal{A}_0 = \mathcal{B}_0 = \{c, d\}$; and $\mathcal{A}_1 = \mathcal{B}_1 = \{a, b, d\}$. It is then easy to see that Property 5 is vacuously satisfied, as we have $f(\tilde{\theta}) \neq d$ for each $\tilde{\theta} \in \{\theta, \theta'\}$. Properties 1 through 4 are obviously satisfied, as they are necessary for pure Nash implementation. In state θ', we have $f(\theta') = c \neq d$ but $f(\theta') = c \sim_i^{\theta'} d$ for each $i \in \{1, 2\}$ for which Property 6 of Condition M is violated. Jackson (1992) shows that any finite mechanism which pure Nash implements the SCF f possesses a compelling mixed strategy equilibrium in the game $\Gamma(\theta')$. Therefore, Property 6 is indispensable for C-implementation by any finite mechanism.

To establish the indispensability of Property 5 in Condition M, we consider the same environment and the SCF f as the one discussed in Section 4. We set $J = 1$; $\mathcal{A}_0 = \mathcal{B}_0 = \{c, d\}$; $\mathcal{A}_1 = \mathcal{B}_1 = \{a, b, d\}$; $z(1, 1) = c$; and $z(1, 0) = z(0, 1) = z(0, 0) = d$. It is straightforward to see that Properties 1 through 4 of Condition P as well as Property 6 of Condition M are satisfied. To check Property 5 of Condition M, we consider state θ'. Since $Z = \{c, d\}$, we have $f(\theta') = c \in Z$. Since $\bigcup_{j=0,1} \mathcal{A}_j \cup \mathcal{B}_j = \{a, b, c, d\}$, we know that $a \in \bigcup_{j=0,1} \mathcal{A}_j \cup \mathcal{B}_j$ such that $a \succ_i^{\theta'} f(\theta') = c$ for each $i \in \{1, 2\}$. Hence, Property 5 of Condition M fails.

Fix $k \geq 3$ arbitrarily. We now use our canonical mechanism Γ^k introduced in Theorem 1 which is described in terms of $\{\mathcal{A}_j, \mathcal{B}_j\}_{j=0,1}$, and $\{z(0, 0), z(0, 1), z(1, 0), z(1, 1)\}$.

In what follows, we focus on state θ' and therefore consider the game $\Gamma^k(\theta')$. In the game $\Gamma^k(\theta')$, outcome a is the best outcome for agent 1, while outcome b is the best outcome. Since we are only concerned with compelling mixed strategy equilibria, there is no loss of generality to focus on $NE^*(\Gamma^k(\theta'))$ in which agent 1 always chooses a and agent 2 always chooses b in the second component of their message.[14] Then, the game $\Gamma^k(\theta')$ is illustrated as follows:

$g^k(m_1, m_2)$ where			Agent 2							
$m_1 = (o_1, a)$			$n^k(o_2) = 0$				$n^k(o_2) = 1$			
$m_2 = (o_2, b)$			0	1	$\ldots$	$k-1$	k	$k+1$	$\ldots$	$2k-1$
Agent 1	$n^k(o_1) = 0$	0	a	b	$\ldots$	d				
		1	d	a	$\ldots$	d				
		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$		d		
		$k-1$	b	d	$\vdots$	a				
	$n^k(o_1) = 1$	k					a	b	$\ldots$	c
		$k+1$					c	a	$\ldots$	c
		$\vdots$		d			$\vdots$	$\vdots$	$\ddots$	$\vdots$
		$2k-1$					b	c	$\ldots$	a

Table 8: The mechanism Γ^k where $m_1 = (o_1, a)$ and $m_2 = (o_2, b)$

We consider a mixed strategy profile σ in which each agent randomizes uniformly over $\{k, \ldots, 2k-1\}$ with equal probability $1/k$. Given agent $j \neq i$'s mixed strategy σ_j, any message from $\{k, \ldots, 2k-1\}$ induces outcome a with probability $1/k$, outcome b with probability $1/k$, and outcome c with probability $(1 - 2/k)$. Since any pure message in the support of σ_i generates the same expected utility given σ_j, σ is a mixed strategy Nash equilibrium of the game $\Gamma^k(\theta')$. In this mixed strategy equilibrium σ, both outcomes a and b are realized with positive probabilities and outcome c is realized with the rest of the probability. Thus, regardless of agent i's cardinal utility representation, each agent i's expected utility of playing σ is always strictly above i's utility which comes from c, which corresponds to $f(\theta')$, which is the unique pure strategy Nash equilibrium outcome of the game $\Gamma^k(\theta')$. Hence, σ is a compelling mixed strategy equilibrium of the game $\Gamma^k(\theta')$. Note how this argument does not depend on the size of k at all.

[14]See the discussion after Lemma 7 where we define $NE^*(\Gamma^k(\theta'))$.

7 Comparison with the Canonical Mechanism of Moore and Repullo (1990)

The objective of this section is to compare the mechanism introduced in our Theorem 1 with the natural finite version of the canonical mechanism developed by Moore and Repullo (1990). The natural finite adaptation of the canonical mechanism that pure Nash implements an SCF is obtained by replacing the integer game with a modulo game instead. The modulo game is regarded as a finite version of the integer game in which agents announce integers from a finite set. The agent who matches the modulo of the sum of the integers gets to name an allocation. The modulo game is used in Jackson (1991), McKelvey (1989), and Saijo (1988). We will show that, in the setting of our modified version of Example 4 of Jackson (1992), the finite version of Moore and Repullo's canonical mechanism still admits a compelling mixed strategy equilibrium. In contrast, such a compelling mixed strategy equilibrium ceases to exist if we use the mechanism proposed in Section 4.

We recap the setup in Section 4. Suppose that agent 1 has the state-independent preference $a \succ_1 b \succ_1 c \succ_1 d$ and agent 2 has the preference $a \succ_2^{\theta} b \succ_2^{\theta} d \succ_2^{\theta} c$ at state θ and preference $b \succ_2^{\theta'} a \succ_2^{\theta'} c \succ_2^{\theta'} d$ at state θ'. Let f be the SCF such that $f(\theta) = a$ and $f(\theta') = c$.

Let $\Gamma^{MR} = (M^{MR}, g^{MR})$ be the finite version of the Moore and Repullo's canonical mechanism in which $M_i^{MR} = \{(\theta_i, b_i, n_i) \in \{\theta, \theta'\} \times \{a, b, c, d\} \times \{0, 1, 2\}\}$ for each $i \in \{1, 2\}$, and an outcome function $g^{MR} : M^{MR} \to \{a, b, c, d\}$ possesses the following rules: for each $m = (m_1, m_2) = ((\theta_1, b_1, n_1), (\theta_2, b_2, n_2)) \in M^{MR}$,

Rule I: If there exists $\tilde{\theta} \in \{\theta, \theta'\}$ such that $\theta_1 = \theta_2 = \tilde{\theta}$, then $g^{MR}(m) = f(\tilde{\theta})$.
Rule II: If $\theta_1 \neq \theta_2$ and either $n_1 = 0$ or $n_2 = 0$, then $g^{MR}(m) = d$.
Rule III: If $\theta_1 \neq \theta_2$ and $n_1 = n_2 \neq 0$, then $g^{MR}(m) = b_2$.
Rule IV: If $\theta_1 \neq \theta_2$ and $n_1 \neq n_2$ with $n_1, n_2 > 0$, then $g^{MR}(m) = b_1$.

We consider the following mixed strategy profile σ: player 1 randomizes uniformly between messages $(\theta, a, 1)$ and $(\theta, a, 2)$, while player 2 randomizes uniformly between messages $(\theta', b, 1)$ and $(\theta', b, 2)$. We claim that this mixed strategy profile σ is an equilibrium of the game $\Gamma^{MR}(\theta')$. Given σ_2, both $(\theta, a, 1)$ and $(\theta, a, 2)$ generate the same lottery in which outcomes a and b are realized with equal probability. Similarly, given σ_1, both $(\theta', b, 1)$ and $(\theta', b, 2)$ generate the same lottery in which outcomes a and b are realized with equal probability. Thus, σ is a mixed strategy equilibrium of the game $\Gamma^{MR}(\theta')$. Moreover, since $f(\theta') = c$ and $a \succ_1^{\theta'} b \succ_1^{\theta'} \succ_1^{\theta'} c$ and $b \succ_2^{\theta'} a \succ_2^{\theta'} \succ_2^{\theta'} c$, we further conclude that σ is a compelling mixed strategy equilibrium of the game $\Gamma^{MR}(\theta')$. Therefore, the mechanism Γ^{MR} fails to C-implement the SCF f. As we have discussed in Section 4, this is not the end of the story. In fact, we have shown that for each $\varepsilon \in (0,1)$, there exists $K \in \mathbb{N}$ such that the SCF f is C-implementable with respect to $\mathcal{U}^\varepsilon$ by the mechanism Γ^K proposed in Section 4. This insight is further generalized in our Theorem 1.[15]

The preceding argument sheds light on the importance of how to select from among all the implementing mechanisms. This point is particularly relevant when we are concerned with the malfunction of the mechanism due to the existence of compelling mixed equilibria. While the issue of compelling mixed strategy equilibria is unavoidable in Jackson's (1992) original example, there are many contexts including the modified version of Jackson's (1992) example discussed in Section 4 in which this problem can be resolved via a more careful selection of the implementing mechanism.

8 On the Difficulty of Extending Our Result to the Case of More Than Two agents

Our main result (Theorem 1) only applies to two-person environments. This section therefore examines whether and how we can extend our Theorem 1 to environments with three or more agents. To do this, we start from the following natural extension of Condition P+M to environments with more than two agents:

Definition 8 *Let $I = \{1, \dots, n\}$ be the set of agents where $n \geq 3$. The environment $\mathcal{E} = \big(I, A, \Theta, (\succeq_i^\theta)_{i \in I, \theta \in \Theta}\big)$ satisfies **Condition** α with respect to the SCF f if there exists a pair of agents $\{i_1, i_2\} \subseteq I$ for which the truncated environment $\mathcal{E}_{i_1,i_2} = \big(\{i_1, i_2\}, A, \Theta, (\succeq_i^\theta)_{i \in \{i_1,i_2\}, \theta \in \Theta}\big)$ satisfies Condition P+M with respect to the SCF f.*

With Condition α, we can establish the following result:

Fact 1 *Let f be an SCF and $I = \{1, \dots, n\}$ be the set of agents where $n \geq 3$. Suppose that the environment $\mathcal{E} = \big(I, A, \Theta, (\succeq_i^\theta)_{i \in I, \theta \in \Theta}\big)$ satisfies Condition α with respect to the SCF f. Then, for any $\varepsilon > 0$, the SCF f is C-implementable with respect to $\mathcal{U}^\varepsilon$.*

This result says that if Condition α is satisfied, we can construct a mechanism in which all agents other than i_1 and i_2 have a trivial message space, while i_1 and i_2 play the two-person mechanism used for proving Theorem 1. Thus, compelling implementation is achieved for any number of agents in an environment satisfying Condition α.

[15] This can be verified by directly checking Condition P+M is satisfied in the modified version of Jackson's (1992) example in Section 4. This can be found in the discussion right after Condition P+M is introduced.

Condition α essentially allows the planner to achieve compelling implementation by restricting attention to a single pair of agents for which the two-person mechanism proposed in Section 5 is employed. Hence, Condition α strikes us of being an extremely stringent condition. What makes Condition α so restrictive are attributed to Properties 3 and 4 of Condition P+M, which apply to a single pair of agents $\{i_1, i_2\}$. We elaborate on this point. These two properties together require that, for any $\theta, \theta^{'} \in \Theta$, whenever $f(\theta) \neq f(\theta')$, there exist an agent $i \in \{i_1, i_2\}$ and an outcome $x \in A$ such that $x \succ_i^{\theta} f(\theta')$, while $f(\theta^{'}) \succ_i^{\theta^{'}} x$. When there are only two agents, this is equivalent to the SCF f being Maskin monotonic. When there are more than two agents, however, this same condition requires that, for any two states where the SCF f induces two distinct outcomes, there be the right kind of preference reversal between the states for agent either i_1 or i_2. We stress that the preferences of other agents are redundant for determining the outcome specified by the SCF f. Thus, this requirement is a lot more stringent than Maskin monotonicity so that Condition α excludes a large class of environments from our consideration.

To illustrate the restrictiveness of Condition α, we revisit the environment $\mathcal{E}^*$ presented in Section 3.2. To account for the need of strict preferences in our argument, we slightly modify that environment in such a way that whenever an agent is indifferent between c and d in the original environment, that agent strictly prefers c to d in the modified environment. This modified environment would still violate Condition α, since each agent plays an essential role in the determination of the outcome by the SCF in the modified environment. More specifically, agent 3's preferences are essential because agent 3 only has the preference reversal over $f(\theta_2)$ and $f(\theta_3)$ between states θ_2 and θ_3; agent 1's preferences are essential because agent 1 only has the preference reversal over $f(\theta_3)$ and $f(\theta_4)$ between states θ_3 and θ_4; and agent 2's preferences are essential because agent 2 only has the preference reversal over $f(\theta_1)$ and $f(\theta_2)$ between states θ_1 and θ_2. Thus, the problem of implementation in this environment cannot be reduced to a two-person problem so that Condition α fails here.

9 Conclusion

We present a concept of compelling implementation, which strengthens the requirement of pure-strategy Nash implementation by taking care of what we call compelling mixed strategy equilibria, but ignoring other mixed strategy equilibria. We call a mixed strategy equilibrium compelling if its outcome Pareto dominates any pure strategy equilibrium.

The main contribution of this paper is to provide Condition P+M under which compelling implementation is possible by finite mechanisms in environments with two agents. We also show that Condition P+M is indispensable for our result. Our implementing mechanism has desirable properties: transfers are not needed at all; only finite mechanisms are used; integer games are not invoked; and agents' risk attitudes do not matter.

We conclude this paper with possible extensions. First, we assume throughout the paper that agents have strict preferences over alternatives. A slight relaxation of this assumption can be made, as we only rely on the best alternative for each agent in each choice set $\mathcal{A}_j$ and $\mathcal{B}_j$ to be unique. However, since the SCF determines the choice of both collection of sets, this places domain restrictions which depend upon the chosen SCF. Thus, indifferences must be allowed on a case-by-case basis.

When we extend our results to environments with three or more agents in a straightforward manner, as we did in Section 8, the class of environments in which compelling implementation is possible becomes very small. This happens because our Condition α introduced in Section 8 excludes a large class of environments that can be of interest. Identifying a condition which is weaker than Condition α in which compelling implementation is still possible remains an important open question that we pursue in future research.

10 Appendix

In this appendix, we provide the proofs we omitted in the main body of the paper.

10.1 Proof of Proposition 1

We first show that the SCF f is pure Nash implemented by the mechanism described by Table 4 (p.10). At state θ_1, we claim that message profile $(m_1, m_2, m_3) = (1, 1, 0)$ is a Nash equilibrium which induces outcome $a = f(\theta_1)$. This is easy to see because a is the best outcome for all agents in state θ_1. We next claim that outcome a is the unique pure Nash equilibrium outcome in state θ_1. This is because: i) at every message profile where c is the induced outcome, agent 2 can unilaterally induces d, which is better for agent 2 than c in state θ_1 and ii) At every message profile where d or b is the induced outcome, either agent 1 or agent 2 can unilaterally induce outcome a, which is better for both agents 1 and 2 than d or b in state θ_1. Thus, there are no other pure Nash equilibria in state θ_1.

At state θ_2, we claim that message profile $(m_1, m_2, m_3) = (2, 2, 0)$ is a Nash equilibrium which induces outcome $c = f(\theta_2)$. This is easy to see because the only other outcome any player can unilaterally induce is d, which is indifferent to c for all agents in state θ_2. We next claim that outcome c is the unique pure Nash equilibrium outcome in state θ_2. This is because: i) at every message profile where d is the induced outcome, either agents 1 or 2 can unilaterally induce a, which is better for agents 1 and 2 than d in state θ_2; ii) at every message profile where the induced outcome is b, agent 3 can unilaterally induce a, which is better for agent 3 than b in state θ_2; and iii) at every message profile where the induced outcome is a, agent 2 can unilaterally induce b, which is better for agent 2 than a in state θ_2. Thus, there are no other pure Nash equilibria in state θ_2.

At state θ_3, we claim that message profile $(m_1, m_2, m_3) = (1, 1, 2)$ is a Nash equilibrium which induces outcome $b = f(\theta_3)$. Agents 2 and 3 have no profitable deviations because b is the best outcome for them in state θ_3. Moreover, since agent 1 can induce only outcomes b or c and b is better for agent 1 than c in state θ_3, agent 1 has no profitable deviation. So, $(1, 1, 2)$ is a Nash equilibrium in state θ_3. We next claim that outcome b is the unique pure Nash equilibrium outcome in state θ_3. This is because: i) at every message profile where d is the induced outcome, either agents 1 or 2 can unilaterally induce a, which is better for agents 1 and 2 than d in state θ_3; ii) at every message profile where the induced outcome is a, agent 3 can unilaterally induce b, which is better for agent 3 than a in state θ_3; and iii) at every message profile where the induced outcome is c, agent 3 can unilaterally induce d, which is better for agent 3 than c in state θ_3. Thus, there are no other pure Nash equilibria in state θ_3.

At state θ_4, we claim that message profile $(m_1, m_2, m_3) = (2, 2, 2)$ is a Nash equilibrium which induces outcome $d = f(\theta_4)$. Agent 1 has no profitable deviations because d is the best outcome for agent 1 in state θ_4. Since agent 2 can unilaterally induce c and c is indifferent to d for agent 2 in state θ_4, agent 2 has no profitable deviations. Finally, since agent 3 can unilaterally induce c or d and d is better for agent 3 than c in state θ_4, agent 3 has no profitable deviations. So, $(2, 2, 2)$ is a Nash equilibrium in state θ_4. We next claim that outcome d is the unique pure Nash equilibrium outcome in state θ_4. This is because: i) at every message profile where c is the outcome, agent 3 can unilaterally induce d, which is better for agent 3 than c in state θ_4; ii) at every message profile where b is the induced outcome, agent 3 can unilaterally induce a, which is better for agent 3 than b in state θ_4; and iii) at every message profile where a is the induced outcome, agent 2 can unilaterally induce b, which is better for agent 2 than a in state θ_4. Thus, there are no other pure Nash equilibria in state θ_4.

Let $\Gamma = (M, g)$ be a finite mechanism that pure Nash implements the SCF f. It thus remains to prove that, for any $u \in \mathcal{U}$, the game $\Gamma(\theta_2, u)$ has a compelling mixed strategy equilibrium.

Define
$$M' = M_1^{'} \times M_2^{'} = \{m \in M |\ g(m) = a \text{ or } b\}$$
as the subset of M such that every message induces either a or b. For instance, in the mechanism presented in Table 4, $M^{'}$ excludes message $m_1 = 2$ for agent 1, since whenever he sends this message, the only outcomes it can induce are c

or d. Since f is pure Nash implemented by the mechanism Γ and $f(\theta_1) = a$ and $f(\theta_3) = b$, $M^{'}$ must be non-empty. We denote by $\Gamma^{'} = (M^{'}, g)$ the truncated message space $M^{'}$ along with the original outcome function g. For any $u \in \mathcal{U}$, by Nash's theorem, the finite game $\Gamma^{'}(\theta_2, u)$ has at least one possibly mixed strategy Nash equilibrium. Fix $u \in \mathcal{U}$. Since agents 1 and 2 prefer a to b and agent 3 prefers b to a, which exhibits a conflict of interests between agents, this equilibrium in the game $\Gamma^{'}(\theta_2, u)$ is genuinely a mixed strategy equilibrium which assigns positive probability on both outcomes a and b. Finally, the mixed strategy equilibrium in the truncated game $\Gamma^{'}(\theta_2, u)$ will also be a mixed strategy equilibrium in the original game $\Gamma(\theta_2, u)$, since any unilateral deviation that can lead to c or d is not profitable for any of the agents. As outcome c is the unique pure Nash equilibrium outcome and c and d are indifferent for all agents in state θ_2, this mixed strategy equilibrium induces a lottery which is strictly preferred to c for all agents, which makes this mixed strategy equilibrium compelling. This completes the proof. ■

10.2 Proof of Lemma 2

Proof of Step 1a: Assume by way of contradiction that there exists an integer $x \in \{0, \dots, k-1\}$ such that $\sigma_1(x) > 0$ and $\sigma_2(x) = 0$. Then, there are two possibilities: either there exists $x^{'} \in \{0, \dots, k-1\} \backslash \{x\}$ such that $\sigma_2(x^{'}) > 0$ or $\sigma_2(k) = 1$.

In the first case, let $x^{'} \in \arg\max_{x^{''} \in \{0,\dots,k-1\} \backslash \{x\}} \sigma_2(x^{''})$. The expected payoff for agent 1 when sending message x is

$$U_1(x, \sigma_2; \theta^{'}) = \begin{cases} \sigma_2(x+1) u_1(b; \theta^{'}) & \text{if } x < k-1 \\ \sigma_2(0) u_1(b; \theta^{'}) & \text{if } x = k-1, \end{cases}$$

where we take into account that $u_i(d; \theta^{'}) = 0$. On the other hand, the expected payoff for agent 1 when sending message $x^{'}$ is given by

$$U_1(x^{'}, \sigma_2; \theta^{'}) = \begin{cases} \sigma_2(x^{'}) u_1(a; \theta^{'}) + \sigma_2(x^{'}+1) u_1(b; \theta^{'}) & \text{if } x^{'} < k-1 \\ \sigma_2(x^{'}) u_1(a; \theta^{'}) + \sigma_2(0) u_1(b; \theta^{'}) & \text{if } x^{'} = k-1 \end{cases}$$

As $u_1(a, \theta^{'}) > u_1(b, \theta^{'})$ and $\sigma_2(x^{'}) \geq \sigma_2(x+1)$, sending message $x^{'}$ is strictly better for agent 1 than sending x against σ_2, thus contradicting the hypothesis that message x is played with positive probability in the Nash equilibrium σ.

Consider the second possibility where agent 2 sends k with probability 1. Then, agent 1's expected payoff of sending message x is $U_1(x, \sigma_2; \theta^{'}) = 0$, while agent 1's expected payoff of sending message k is $U_1(\sigma_2; k; \theta^{'}) = u_1(c, \theta^{'}) > 0$, contradicting the hypothesis that message x is played with positive probability in the Nash equilibrium σ. ■

Proof of Step 1b: Assume by way of contradiction that there exists $x \in \{0,\ldots,k-1\}$ such that $\sigma_2(x) > 0$ and $\sigma_1(x-1) = 0$ if $x \geq 1$ and $\sigma_1(k-1) = 0$ if $x = 0$. Then we decompose our argument into the following two cases: (i) there exists $x^{'} \in \{0,\ldots,k-1\}$ such that $\sigma_1(x^{'}) > 0$ or (ii) $\sigma_1(k) = 1$.

We first consider Case (i). We assume without loss of generality that $x^{'} \in \arg\max_{x^{''} \in \{0,\ldots,k-1\}} \sigma_1(x^{''})$. Agent 2's expected payoff of sending message x against σ_1 in the game $\Gamma(\theta^{'})$ is given by

$$U_2(\sigma_1, x; \theta^{'}) = \sigma_1(x)u_2(a; \theta'),$$

while agent 2's expected payoff of sending message ($x^{'} + 1$ mod k) against σ_1 in the game $\Gamma(\theta^{'})$ is given by

$$U_2(\sigma_1, x^{'} + 1 \text{ mod } k; \theta^{'}) = \begin{cases} \sigma_1(x^{'})u_2(b;\theta^{'}) + \sigma_1(x^{'}+1)u_2(a;\theta^{'}) & \text{if } x^{'} < k-1 \\ \sigma_1(x^{'})u_2(b;\theta^{'}) + \sigma_1(0)u_2(a;\theta^{'}) & \text{if } x^{'} = k-1, \end{cases}$$

where we take into account that $u_2(d; \theta^{'}) = 0$. Since $u_2(b; \theta^{'}) > u_2(a; \theta^{'}) > 0$, due to the way $x^{'}$ is defined, we have $U_2(\sigma_1, x^{'} + 1 \text{ mod } k; \theta^{'}) > U_2(\sigma_1, x; \theta^{'})$, which contradicts the hypothesis that message x is sent with positive probability in the Nash equilibrium σ.

We next consider Case (ii). Agent 2's expected payoff of sending message x against σ_1 in the game $\Gamma(\theta^{'})$ is given by

$$U_2(\sigma_1, x; \theta^{'}) = 0,$$

where we take into account that $u_2(d; \theta^{'}) = 0$. On the contrary, agent 2's expected payoff of sending message k against σ_1 in the game $\Gamma(\theta^{'})$ is given by

$$U_2(\sigma_1, k; \theta^{'}) = u_2(c; \theta^{'}).$$

Since $u_2(c; \theta^{'}) > u_2(d; \theta^{'}) = 0$, we have $U_2(\sigma_1, k; \theta^{'}) > U_2(\sigma_1, x; \theta^{'})$, contradicting the hypothesis that message x is sent with positive probability in the Nash equilibrium σ in the game $\Gamma(\theta^{'})$. ■

Proof of Step 1c: Assume first that $i = 1$; that is, there exists $x^{'} \in \{0,\ldots,k-1\}$ such that $\sigma_1(x^{'}) > 0$. By Step 1a, we first have that $\sigma_2(x^{'}) > 0$. Second, by Step 1b, $\sigma_2(x^{'}) > 0$ implies $\sigma_1(x^{'} - 1) > 0$ if $x^{'} \geq 1$ and $\sigma_1(k) > 0$ if $x^{'} = 0$. Third, using Step 1a once again, we conclude that $\sigma_2(x^{'} - 1) > 0$ if $x^{'} \geq 1$ and $\sigma_2(k) > 0$ if $x^{'} = 0$. Finally, iterating this argument, we are able to conclude that $\sigma_1(x) > 0$ and $\sigma_2(x) > 0$ for all $x \in \{0,\ldots,k-1\}$.

The case where $i = 2$ is analogous to the previous one, only that we start the loop by applying Step 1b first, before Step 1a. This completes the proof of Step 1c. ■

Proof of Step 2: Assume by way of contradiction that there exist $i \in N$ and $x, x^{'} \in \{0, \ldots, k-1\}$ such that $\sigma_i(x) > \sigma_i(x^{'}) > 0$. By Step 1c, we know that $\sigma_i(\tilde{x}) > 0$ for all $\tilde{x} \in \{0, \ldots, k-1\}$. Then, we can choose x and $x^{'}$ satisfying the following property:

$$x \in \arg \max_{\tilde{x} \in \{0,\ldots,k-1\}} \sigma_i(\tilde{x}) \text{ and } x^{'} \in \arg \min_{\tilde{x} \in \{0,\ldots,k-1\}} \sigma_i(\tilde{x}).$$

By Step 1c, we also know that $\sigma_j(\tilde{x}) > 0$ for each $\tilde{x} \in \{0, \ldots, k-1\}$, where $j \in \{1,2\} \backslash \{i\}$.

Assume that $i = 2$. The expected payoff for agent 1 of sending message $x^{'}$ against σ_2 in the game $\Gamma(\theta^{'})$ is given by

$$U_1(x^{'}, \sigma_2; \theta^{'}) = \begin{cases} \sigma_2(x^{'})u_1(a;\theta^{'}) + \sigma_2(x^{'}+1)u_1(b;\theta^{'}) & \text{if } x^{'} < k-1 \\ \sigma_2(x^{'})u_1(a;\theta^{'}) + \sigma_2(0)u_1(b;\theta^{'}) & \text{if } x^{'} = k-1 \end{cases}$$

On the other hand, The expected payoff for agent 1 of sending message x against σ_2 in the game $\Gamma(\theta^{'})$ is given by

$$U_1(x, \sigma_2; \theta^{'}) = \begin{cases} \sigma_2(x)u_1(a;\theta^{'}) + \sigma_2(x+1)u_1(b;\theta^{'}) & \text{if } x < k-1 \\ \sigma_2(x)u_1(a;\theta^{'}) + \sigma_2(0)u_1(b;\theta^{'}) & \text{if } x = k-1. \end{cases}$$

We compute

$$\begin{aligned} & U_1(x, \sigma_2; \theta^{'}) - U_1(x^{'}, \sigma_2; \theta^{'}) \\ = \; & [\sigma_2(x) - \sigma_2(x^{'})]u_1(a;\theta^{'}) + [\sigma_2(x+1 \bmod k) - \sigma_2(x'+1 \bmod k)]u_1(b;\theta^{'}) \\ \geq \; & [\sigma_2(x) - \sigma_2(x^{'})]u_1(a;\theta^{'}) - [\sigma_2(x) - \sigma_2(x^{'})]u_1(b;\theta^{'}) \\ & (\because [\sigma_2(x+1 \bmod k) - \sigma_2(x'+1 \bmod k)] \geq -[\sigma_2(x) - \sigma_2(x^{'})], u_1(b;\theta^{'}) > 0) \\ = \; & [\sigma_2(x) - \sigma_2(x^{'})](u_1(a;\theta^{'}) - u_1(b;\theta^{'}) \\ > \; & 0. \end{aligned}$$

This implies that message x is a strictly better response for agent 1 against σ_2 than $x^{'}$ in the game $\Gamma(\theta^{'})$, contradicting the hypothesis that $\sigma_1(x^{'}) > 0$.

We next assume $i = 1$. The expected payoff for agent 2 of sending message $x^{'}+1$ against σ_1 in the game $\Gamma(\theta^{'})$ is given by

$$U_2(\sigma_1, x^{'}+1; \theta^{'}) = \begin{cases} \sigma_1(x^{'}+1)u_2(a;\theta^{'}) + \sigma_1(x^{'})u_1(b;\theta^{'}) & \text{if } x^{'} < k-1 \\ \sigma_1(0)u_2(a;\theta^{'}) + \sigma_1(x')u_1(b;\theta^{'}) & \text{if } x^{'} = k-1 \end{cases}$$

On the other hand, The expected payoff for agent 2 of sending message $x+1$ against σ_1 in the game $\Gamma(\theta^{'})$ is given by

$$U_2(\sigma_1, x+1; \theta^{'}) = \begin{cases} \sigma_1(x+1)u_1(a;\theta^{'}) + \sigma_1(x)u_2(b;\theta^{'}) & \text{if } x < k-1 \\ \sigma_1(0)u_1(a;\theta^{'}) + \sigma_1(x)u_2(b;\theta^{'}) & \text{if } x = k-1. \end{cases}$$

We compute

$$
\begin{aligned}
& U_2(\sigma_1, x+1; \theta^{'}) - U_2(\sigma_1, x^{'}+1; \theta^{'}) \\
= \;& [\sigma_1(x+1) - \sigma_1(x^{'}+1)]u_2(a;\theta^{'}) + [\sigma_1(x) - \sigma_1(x')]u_2(b;\theta^{'}) \\
\geq \;& [\sigma_1(x+1) - \sigma_1(x^{'}+1)]u_2(b;\theta^{'}) - [\sigma_1(x) - \sigma_1(x^{'})]u_2(a;\theta^{'}) \\
& (\because [\sigma_1(x+1 \bmod k) - \sigma_1(x'+1) \bmod k)] \geq -[\sigma_1(x) - \sigma_1(x^{'})], u_2(a;\theta^{'}) > 0) \\
= \;& [\sigma_1(x) - \sigma_1(x^{'})](u_2(b;\theta^{'}) - u_2(a;\theta^{'})) \\
> \;& 0.
\end{aligned}
$$

This implies that message $x+1$ is a strictly better response for agent 2 against σ_2 than $x^{'}+1$ in the game $\Gamma(\theta^{'})$, contradicting the hypothesis that $\sigma_2(x^{'}+1) > 0$. This completes the proof of Step 2. ■

10.3 Proof of Lemma 4

Fix $j \in \{0, \ldots, J\}$. By Property 3 in Condition P, $f(\theta_j)$ is $(\mathcal{A}_j, \mathcal{B}_j)$-acceptable at state θ_j. This implies that $f(\theta_j) \in \mathcal{A}_j \cup \mathcal{B}_j$. Assume first that $f(\theta_j) \in \mathcal{A}_j$. We then define $m = (m_1, m_2) = ((jk, f(\theta_j)), (jk, z(j,j)))$. By construction, m induces Rule 1 in the mechanism Γ^k so that we have $g(m) = f(\theta_j)$. We consider $m_1^{'}$ as an arbitrary deviation strategy of agent 1 and argue that $m_1^{'}$ never be a better reply than m_1 against m_2. If $(m_1^{'}, m_2)$ induces Rule 1, by Property 3 of Condition P, $m_1^{'}$ is not a profitable deviation. If $(m_1^{'}, m_2)$ induces Rule 2, then we have $g(m_1^{'}, m_2) = z(j,j)$, which, by Property 1, is also a part of $\mathcal{A}_j$ and thus, by Property 3, not a profitable deviation either. If $(m_1^{'}, m_2)$ induces Rule 3, there exists $j_1 \in \{0, \ldots, J\}$ such that $g(m_1^{'}, m_2) = z(j_1, j)$. By Property 1, $z(j_1, j) \in \mathcal{A}_j$ and thus by Property 3 in Condition P, $m_1^{'}$ is not a profitable deviation.

We next consider $m_2^{'}$ as an arbitrary deviation strategy of agent 2 and argue that $m_2^{'}$ never be a better reply than m_2 against m_1. If $(m_1, m_2^{'})$ induces Rule 1, we have $g(m_1, m_2^{'}) = g(m_1, m_2)$ so that $m_2^{'}$ is not a profitable deviation. If $(m_1, m_2^{'})$ induces Rule 2, then $g(m_1, m_2^{'}) \in \mathcal{B}_j$ and it follows from Property 3 of Condition P that $m_2^{'}$ is not a profitable deviation, since $f(\theta_j)$ being $(\mathcal{A}_j, \mathcal{B}_j)$-acceptable at that state means there can be no element in $\mathcal{B}_j$ that is preferred to $f(\theta_j)$ by agent 2. Finally, if $(m_1, m_2^{'})$ induces Rule 3, there exists $(j, j_2) \in \{0, \ldots, J\} \times \{0, \ldots, J\}$ such that $g(m_1, m_2^{'}) = z(j, j_2)$. It follows from Property 1 that $z(j, j_2) \in \mathcal{B}_j$ and from Property 3 of Condition P that $m_2^{'}$ is not a profitable deviation. Thus, $(m_1, m_2) = ((jk, f(\theta_j), (jk, z(j,j))$ is a Nash equilibrium of the game $\Gamma^k(\theta_j)$ in this case.

Finally, consider the scenario when $f(\theta_j) \notin \mathcal{A}_j$, which, by the definition of acceptability, must imply that $f(\theta_j) \in \mathcal{B}_j$. Then, we define $m = (m_1, m_2) = ((jk, z(j,j)), (jk+1, f(\theta_j))$. This message induces Rule 2 and results in $f(\theta_j)$ as the outcome, as desired. To check that there are no profitable deviations, consider first m_1' as an arbitrary deviation by agent 1. If (m_1', m_2) induces Rule 1, by Property 3 this is not a profitable deviation. If it induces Rule 2, the outcome is unchanged, so again, no benefit for the agent. Finally, if it induces Rule 3, then there is a (j_1, j) such that $g(m_1', m_2) = z(j_1, j)$, but as $z(j_1, j) \in \mathcal{A}_j$ according to Property 1, we can once more invoke Property 3 to conclude that this is not a profitable deviation either.

We move on to deviation strategies attempted by agent 2, denoting by m_2' an arbitrary deviation. As above, Property 3 ensures that any deviation that induces Rule 2 cannot be profitable. If (m_1, m_2') induces Rule 3 instead, then we can find a (j, j_2) such that $g(m_1', m_2) = z(j, j_2)$, but as $z(j, j_2) \in \mathcal{B}_j$ according to Property 1, we can once more invoke Property 3 to conclude that this is not a profitable deviation either. Lastly, if it induces Rule 1, the outcome must be $z(j, j)$ and then by Properties 1 and 3 this cannot be a profitable deviation either.

Fix $\theta \in \Theta$. We shall show that $m \in pureNE(\Gamma^k(\theta))$ implies $g(m) = f(\theta)$. We assume by way of contradiction that there exists $m \in pureNE(\Gamma^k(\theta))$ such that $g(m) \neq f(\theta)$. We write $m = (m_1, m_2) = ((o_1, x_1), (o_2, x_2))$. We complete the proof by considering the following separate cases.

Case 1: m induces Rule 1

Assume that m induces Rule 1. Then, we have $g(m) = x_1$. Since $x_1 \neq f(\theta)$ from our hypothesis, Property 4 of Condition P implies that for every $j = \{0, \dots, J\}$, x_1 is not $(\mathcal{A}_j, \mathcal{B}_j)$-acceptable. As $x_1 \in \mathcal{A}_{n^k(o_2)}$, this implies that there must exist either $a \in \mathcal{A}_{n^k(o_2)}$ such that $a \succ_1^\theta x_1$ or $b \in \mathcal{B}_{n^k(o_2)}$ such that $b \succ_2^\theta x_1$. Assume it is the former. Then $m_1' = (o_1, a)$ is a profitable deviation for agent 1. If it is the latter, then $m_2' = (\pi^k(o_1), b)$ is a profitable deviation for agent 2.

Case 2: m induces Rule 2

Assume that m induces Rule 2. Then, we have $g(m) = x_2$. Since $x_2 \neq f(\theta)$ from our hypothesis, Property 4 of Condition P implies that for every $j = \{0, \dots, J\}$, x_1 is not $(\mathcal{A}_j, \mathcal{B}_j)$-acceptable. As $x_2 \in \mathcal{B}_{n^k(o_1)}$, this implies that there must exist either $a \in \mathcal{A}_{n^k(o_2)}$ such that $a \succ_1^\theta x_1$ or $b \in \mathcal{B}_{n^k(o_2)}$ such that $b \succ_2^\theta x_1$. Assume it is the former. Then $m_1' = (o_2, a)$ is a profitable deviation for agent 1. If it is the latter, then $m_2' = (o_2, b)$ is a profitable deviation for agent 2.

Case 3: m induces Rule 3

If m induces Rule 3, we have $g(m) = z(j_1, j_2)$ where $(j_1, j_2) = (n^k(o_1), n^k(o_2))$. By assumption, $z(j_1, j_2) \neq f(\theta)$, then we can invoke Property 2 and find either $a_{(j_1,j_2)} \in \mathcal{A}_{j_2}$ such that $a_{(j_1,j_2)} \succ_1^\theta z(j_1, j_2)$ or $b_{(j_1,j_2)} \in \mathcal{B}_{j_1}$ such that $b_{(j_1,j_2)} \succ_2^\theta z(j_1, j_2)$. Assume it is the former. Then agent 1 has $m_1' = (o_2, a_{(j_1,j_2)})$ as a profitable deviation strategy. If it is the latter, than agent 2 has $m_2' = (\pi^k(o_1), b_{(j_1,j_2)})$ as a profitable deviation strategy. ■

10.4 Proof of Lemma 5

Fix $u \in \mathcal{U}^{\theta}$. Suppose by way of contradiction that there is a compelling mixed strategy Nash equilibrium σ of the game $\Gamma^k(\theta, u)$. As $\Gamma^k = (M^k, g^k)$ pure Nash implements the SCF f under Condition P by Lemma 4, we have that for each $i \in \{1, 2\}$,

$$\sum_{\tilde{m} \in M} \sigma(\tilde{m}) u_i(g^k(\tilde{m}), \theta) \geq u_i(f(\theta), \theta),$$

with at least one strict inequality for some $i \in \{1, 2\}$. This implies that there exist $i \in \{1, 2\}$ and $m \in \text{supp}(\sigma)$ such that

$$u_i(g^k(m), \theta) > u_i(f(\theta), \theta).$$

We write $m = (m_1, m_2) = ((o_1, x_1), (o_2, x_2))$. If m induces Rule 1, we have $g^k(m) = x_1$ and $x_1 \in \mathcal{A}_{n^k(o_1)}$. This contradicts Property 5 of Condition M. If m induces Rule 2, we have $g^k(m) = x_2$ and $x_2 \in \mathcal{B}_{n^k(o_2)}$. This also contradicts Property 5 of Condition M. If m induces Rule 3, we have $g^k(m) = z(n^k(o_1), n^k(o_2))$. By Property 1 of Condition P, we have $z(n^k(o_1), n^k(o_2)) \in \mathcal{A}_{n^k(o_2)}$. This contradicts Property 5 of Condition M. ■

10.5 Proof of Lemma 6

Let $m_1 = (o_1, x_1)$ denote player 1's a generic message in the mechanism Γ^k. We define the following partition over M_2^k given m_1:

$$\begin{aligned} M_2^1(m_1) &= \left\{ m_2 \in M_2^k | \ (m_1, m_2) \text{ induces Rule } 1 \right\}, \\ M_2^2(m_1) &= \left\{ m_2 \in M_2^k | \ (m_1, m_2) \text{ induces Rule } 2 \right\}, \\ M_2^3(m_1) &= \left\{ m_2 \in M_2^k | \ (m_1, m_2) \text{ induces Rule } 3 \right\}. \end{aligned}$$

By construction, we have $M_2^1(m_1) \cup M_2^2(m_1) \cup M_2^3(m_1) = M_2^k$. Define $m_1^*(m_1) = (o_1, a^{\theta}_{n^k(o_1)})$. When either Rule 2 or Rule 3 is induced, player 1's announcement about alternatives is irrelevant. So, by construction of $m_1^*(m_1)$, we obtain the following property: for any $m_2 \in M_2^2(m_1) \cup M_2^3(m_1)$,

$$g(m_1, m_2) = g(m_1^*(m_1), m_2) \Rightarrow g(m_1^*(m_1), m_2) \sim_1^{\theta} g(m_1, m_2).$$

When (m_1, m_2) induces Rule 1, by its construction, $(m_1^*(m_1), m_2)$ also induces Rule 1. Under Rule 1, we know that player 1's announcement about alternatives solely dictates the outcome. Once again, by construction of m_1^*, along with the fact that we have strict preferences, we obtain the following property: for any $m_2 \in M_2^1(m_1)$,

$$g(m_1, m_2) \neq g(m_1^*, m_2) \Rightarrow g(m_1^*, m_2) \succ_1^\theta g(m_1, m_2).$$

This completes the argument for player 1.

Let $m_2 = (o_2, x_2)$ be a generic message agent 2 sends to the mechanism Γ^k. We define the following partition over M_1^k given m_2:

$$\begin{aligned} M_1^1(m_2) &= \left\{m_1 \in M_1^k |\ (m_1, m_2) \text{ induces Rule 1}\right\}, \\ M_1^2(m_2) &= \left\{m_1 \in M_1^k |\ (m_1, m_2) \text{ induces Rule 2}\right\}, \\ M_1^3(m_2) &= \left\{m_1 \in M_1^k |\ (m_1, m_2) \text{ induces Rule 3}\right\}. \end{aligned}$$

By construction, we have $M_1^1(m_2) \cup M_1^2(m_2) \cup M_1^3(m_2) = M_1^k$. Define $m_2^*(m_2) = (o_2, b^\theta_{n^k(o_2)})$. When either Rule 1 or Rule 3 is induced, player 2's announcement about alternatives is irrelevant. So, by construction of m_2^*, we obtain the following property: for any $m_1 \in M_1^1(m_2) \cup M_1^3(m_2)$,

$$g(m_1, m_2^*(m_2)) = g(m_1, m_2) \Rightarrow g(m_1, m_2^*(m_2)) \sim_2^\theta g(m_1, m_2).$$

When (m_1, m_2) induces Rule 2, by its construction, $(m_1, m_2^*(m_2))$ also induces Rule 2. Under Rule 2, we know that player 2's announcement about alternatives solely dictates the outcome. Once again, by construction of $m_2^*(m_2)$ along with the fact that we have strict preferences, we obtain the following property: for any $m_1 \in M_1^2(m_2)$,

$$g(m_1, m_2^*(m_2)) \neq g(m_1, m_2) \Rightarrow g(m_1, m_2^*(m_2)) \succ_2^\theta g(m_1, m_2).$$

This completes the argument for player 2. ■

10.6 Proof of Lemma 8

Suppose that Condition P+M holds and $f(\theta) \notin Z$. Let $u \in \mathcal{U}^\theta$ and $\sigma = (\sigma_1, \sigma_2) \in NE^*(\theta, u)$ be a compelling mixed strategy Nash equilibrium. For each $i \in \{1, 2\}$ and $j \in \{0, \ldots, J-1\}$, we introduce the following notation: $u_i^a \equiv u_i(a_j^\theta; \theta)$; $u_i^b \equiv u_i(b_j^\theta; \theta)$; $u_i^z \equiv u_i(z(j,j); \theta)$; and $\sigma_i^j \equiv \sum_{x:n^k(x)=j} \sigma_i(x)$. For each $i \in \{1, 2\}$ and $j \in \{0, \ldots, J-1\}$, we define S_i^{max} and S_i^{min} as follows:

$$\begin{aligned} S_i^{max} &\equiv \max\{\sigma_i(\tilde{x}) \in [0,1] |\ \tilde{x} \in \{jk, \ldots, (j+1)k-1\}\}, \\ \text{and } S_i^{min} &\equiv \min\{\sigma_i(\tilde{x}) \in [0,1] |\ \tilde{x} \in \{jk, \ldots, (j+1)k-1\}\}. \end{aligned}$$

Fix $j \in \{0, \ldots, J\}$. If $S_i^{min} = S_i^{max}$ for each $i \in \{1, 2\}$, we have that $\sigma_i(x) = \sigma(x')$ for every x, x' with $n^k(x) = n^k(x') = j$ and $i \in \{1, 2\}$. Thus, for each

$i \in \{1,2\}$ and $j \in \{0,\dots,J\}$, we can set $p_j^i \equiv \sum_{x=jk}^{(j+1)k-1} \sigma_i(x)$. This completes the proof of Lemma 8. Therefore, the rest of the proof is reduced to establishing $S_i^{min} = S_i^{max}$ for each $i \in \{1,2\}$. This will be proved in a series of steps:

Step 8a: Assume that $S_2^{min} \neq S_2^{max}$. Then, for each $\tilde{x} \in \{jk,\dots,(j+1)k-1\}$, $\sigma_2(\tilde{x}) = S_2^{min}$ implies $\sigma_1(\tilde{x}) = 0$.

Proof of Step 8a: Fix $x^{'}$ such that $\sigma_2(x^{'}) = S_2^{min}$ arbitrarily. We claim that there exists $x \neq x^{'}$ such that $m_1(x)$ is a strictly better reply to σ_2 than $m_1(x')$, which implies that $\sigma_1(x^{'}) = 0$. This completes the proof. We show this claim by considering the following two cases: $u_1^a > u_1^b > u_1^z$ or $u_1^a > u_1^z > u_1^b$.

Case 1: $u_1^a > u_1^b > u_1^z$.

Since $S_2^{min} \neq S_2^{max}$, we can pick x such that $\sigma_2(x) = S_2^{max}$. This implies $\sigma_2(x) > \sigma_2(x^{'})$. The expected payoff for agent 1 of sending integer $x^{'}$ against σ_2 in the game $\Gamma^k(\theta)$ is given by

$$U_1(m_1(x^{'}),\sigma_2;\theta) = \sigma_2(x^{'})u_1^a + \sigma_2(\pi^k(x^{'}))u_1^b + \{\sigma_2^j - \sigma_2(x^{'}) - \sigma_2(\pi^k(x^{'}))\}u_1^z + \hat{z}_1^{\sigma_2},$$

where $\hat{z}_1^{\sigma^2}$ denotes the residual utility of agent 1, which depends on σ_2 but remains the same regardless of whether $m_1(x)$ or $m_1(x^{'})$ is sent.

The expected payoff for agent 1 of sending message x against σ_2 in the game $\Gamma^k(\theta)$ is given by

$$U_1(m_1(x),\sigma_2;\theta) = \sigma_2(x)u_1^a + \sigma_2(\pi^k(x))u_1^b + (\sigma_2^j - \sigma_2(x) - \sigma_2(\pi^k(x)))u_1^z + \hat{z}_1^{\sigma_2}.$$

Since $\sigma_2(x) = S_2^{max}$ and $\sigma_2(x^{'}) = S_2^{min}$, we can use the following inequality:

$$\sigma_2(\pi^k(x)) - \sigma_2(\pi^k(x^{'})) \geq -[\sigma_2(x) - \sigma_2(x^{'})].$$

Taking the difference between the two, we compute

$$\begin{aligned}
& U_1(m_1(x),\sigma_2;\theta) - U_1(m_1(x^{'}),\sigma_2;\theta) \\
= \; & [\sigma_2(x) - \sigma_2(x^{'})]u_1^a + [\sigma_2(\pi^k(x)) - \sigma_2(\pi^k(x'))]u_1^b + [\sigma_2(x) - \sigma_2(x^{'}) + \sigma_2(\pi^k(x)) - \sigma_2(\pi^k(x'))]u_1^z \\
= \; & [\sigma_2(x) - \sigma_2(x^{'})](u_1^a - u_1^z) + [\sigma_2(\pi^k(x)) - \sigma_2(\pi^k(x'))](u_1^b - u_1^z) \\
> \; & [\sigma_2(x) - \sigma_2(x^{'})](u_1^a - u_1^z) - [\sigma_2(x) - \sigma_2(x^{'})](u_1^b - u_1^z) \\
& (\because \sigma_2(\pi^k(x)) - \sigma_2(\pi^k(x^{'})) \geq -[\sigma_2(x) - \sigma_2(x^{'})],\; u_1^b > u_1^z) \\
= \; & [\sigma_2(x) - \sigma_2(x^{'})](u_1^a - u_1^b) \\
> \; & 0 \;\; (\because \sigma_2(x) > \sigma_2(x^{'}), u_1^a > u_1^b).
\end{aligned}$$

Case 2: $u_1^a > u_1^z > u_1^b$.

First take x_1 such that $\pi^k(x_1) = x'$. If $\sigma_2(x_1) \neq S_2^{min}$, then we set $x = x_1$. If $x_1 = S_2^{min}$, then take x_2 such that $\pi^k(x_2) = x_1$ and check if $\sigma_2(x_2) = S_2^{min}$. If $\sigma_2(x_2) \neq S_2^{min}$, then we set $x = x_2$. If $\sigma_2(x_2) = S_2^{min}$, we iterate the same argument. As $S_2^{min} \neq S_2^{max}$, by the finiteness of the mechanism Γ, eventually we will find an x such that $\sigma_2(x) \neq S_2^{min}$ but $\sigma_2(\pi^k(x)) = S_2^{min}$. We confirm the following inequalities: Since $\sigma_2(x^{'}) = S_2^{\min}$ and $\sigma_2(x) \neq S_2^{min}$, we have $\sigma_2(x) > \sigma_2(x^{'})$. Moreover, since $\sigma_2(\pi^k(x)) = S_2^{min}$, we have that $\sigma_2(\pi^k(x^{'})) \geq \sigma_2(\pi^k(x))$. We then compute the difference in expected payoffs between these two messages:

$$
\begin{aligned}
& U_1(m_1(x), \sigma_2; \theta) - U_1(m_1(x^{'}), \sigma_2; \theta) \\
= \;& [\sigma_2(x) - \sigma_2(x^{'})]u_1^a + [\sigma_2(\pi^k(x)) - \sigma_2(\pi^k(x'))]u_1^b + [\sigma_2(x) - \sigma_2(x^{'}) + \sigma_2(\pi^k(x)) - \sigma_2(\pi^k(x'))]u_1^z \\
= \;& [\sigma_2(x) - \sigma_2(x^{'})](u_1^a - u_1^z) + [\sigma_2(\pi^k(x')) - \sigma_2(\pi^k(x))](u_1^z - u_1^b) \\
> \;& 0. \\
& (\because \sigma_2(x) > \sigma_2(x^{'}),\ \sigma_2(\pi^k(x^{'})) \geq \sigma_2(\pi^k(x)),\ u_1^a > u_1^z > u_1^b)
\end{aligned}
$$

Considering both Cases 1 and 2, we conclude that $m_1(x)$ is a strictly better reply to σ_2 than $m_1(x')$. This completes the proof. ■

Step 8b: Assume that $S_1^{min} \neq S_1^{max}$. Then, for each $\tilde{x} \in \{jk, \ldots, (j+1)k - 1\}$, $\sigma_1(\tilde{x}) = S_1^{min}$ implies $\sigma_2(\pi^k(\tilde{x})) = 0$.

Proof of Step 8b: Fix $x^{'}$ such that $\sigma_1(x^{'}) = S_1^{min}$ arbitrarily. We claim that there exists $x \neq x^{'}$ such that $m_2(\pi^k(x))$ is a strictly better reply to σ_1 than $m_2(\pi^k(x^{'}))$, which implies that $\sigma_2(\pi^k(x^{'})) = 0$. This completes the proof. We show this claim by considering the following two cases: $u_2^b > u_2^a > u_2^z$ or $u_2^b > u_2^z > u_2^a$.

Case 1: $u_2^b > u_2^a > u_2^z$

Since $S_1^{min} \neq S_1^{max}$, we can pick x such that $\sigma_1(x) = S_1^{max}$. This implies $\sigma_1(x) > \sigma_1(x^{'})$. The expected payoff for agent 2 of sending message $\pi^k(x^{'})$ against σ_1 in the game $\Gamma^k(\theta)$ is given by

$$U_2(\sigma_1, m_2(\pi^k(x^{'})); \theta) = \sigma_1(\pi^k(x^{'}))u_2^a + \sigma_1(x^{'})u_2^b + [\sigma_1^j - \sigma_1(\pi^k(x^{'})) - \sigma_1(x^{'})]u_2^z + \hat{z}_2^{\sigma_1},$$

where $\hat{z}_2^{\sigma_1}$ denotes the residual utility of agent 2, which depends on σ_1 but remains the same regardless of whether $\sigma_2(\pi^k(x))$ or $\sigma_2(\pi^k(x^{'}))$ is played.

The expected payoff for agent 2 of sending message $\pi^k(x)$ against σ_1 in the game $\Gamma^k(\theta)$ is given by

$$U_2(\sigma_1, m_2(\pi^k(x)); \theta) = \sigma_1(\pi^k(x))u_2^a + \sigma_1(x)u_2^b + [\sigma_1^j - \sigma_1(\pi^k(x)) - \sigma_1(x)]u_1^z + \hat{z}_2^{\sigma_1}$$

Since $\sigma_1(x) = S_1^{max}$ and $\sigma_1(x^{'}) = S_1^{min}$, we can use the following inequality:

$$\sigma_1(\pi^k(x)) - \sigma_1(\pi^k(x^{'})) \geq -[\sigma_1(x) - \sigma_1(x^{'})].$$

Taking the difference between the two, we compute

$$\begin{aligned}
& U_2(\sigma_1, m_2(\pi^k(x)); \theta) - U_2(\sigma_1, m_2(\pi^k(x^{'})); \theta) \\
= \;& [\sigma_1(\pi^k(x)) - \sigma_1(\pi^k(x^{'}))]u_2^a + [\sigma_1(x) - \sigma_1(x')]u_2^b + [\sigma_1(\pi^k(x)) - \sigma_1(\pi^k(x^{'})) + \sigma_1(x) - \sigma_1(x')]u_2^z \\
= \;& [\sigma_1(\pi^k(x)) - \sigma_1(\pi^k(x^{'}))](u_2^a - u_2^z) + [\sigma_1(x) - \sigma_1(x')](u_2^b - u_2^z) \\
\geq \;& [\sigma_1(x) - \sigma_1(x^{'})](u_2^b - u_2^z) - [\sigma_1(x) - \sigma_1(x^{'})](u_2^a - u_2^z) \\
& (\because \sigma_1(\pi^k(x)) - \sigma_1(\pi^k(x')) \geq -[\sigma_1(x) - \sigma_1(x^{'})],\ u_2^a > u_2^z) \\
= \;& [\sigma_1(x) - \sigma_1(x^{'})](u_2^b - u_2^a) \\
> \;& 0 \ (\because \sigma_1(x) > \sigma_1(x^{'}),\ u_2^b > u_2^a).
\end{aligned}$$

Case 2: $u_2^b > u_2^z > u_2^a$.

First we take x_1 such that $\pi^k(x_1) = x^{'}$. If $\sigma_1(x_1) \neq S_1^{min}$, we set $x = x_1$. If $\sigma_1(x_1) = S_1^{min}$, we take x_2 such that $\pi^k(x_2) = x_1$. If $\sigma_1(x_2) \neq S_1^{min}$, we set $x = x_2$. If $\sigma_1(x_2) = S_1^{min}$, we repeat the same argument. Since $S_1^{min} \neq S_1^{max}$. we eventually can choose x such that $\sigma_1(x) \neq S_1^{min}$. We confirm the following inequalities: Since $\sigma_1(x^{'}) = S_1^{\min}$ and $\sigma_1(x) \neq S_1^{min}$, we have $\sigma_1(x) > \sigma_1(x^{'})$. Moreover, since $\sigma_1(\pi^k(x)) = S_1^{min}$, we have that $\sigma_1(\pi^k(x^{'})) \geq \sigma_1(\pi^k(x))$. We then compute the difference in expected payoffs between these two messages:

$$\begin{aligned}
& U_2(\sigma_1, m_2(\pi^k(x)); \theta) - U_2(\sigma_1, m_2(\pi^k(x^{'})); \theta) \\
= \;& [\sigma_1(\pi^k(x)) - \sigma_1(\pi^k(x^{'}))]u_2^a + [\sigma_1(x) - \sigma_1(x')]u_2^b + [\sigma_1(\pi^k(x)) - \sigma_1(\pi^k(x^{'})) + \sigma_1(x) - \sigma_1(x')]u_2^z \\
= \;& [\sigma_1(\pi^k(x^{'})) - \sigma_1(\pi^k(x))](u_2^z - u_2^a) + [\sigma_1(x) - \sigma_1(x')](u_2^b - u_2^z) \\
> \;& 0 \ (\because \sigma_1(\pi^k(x^{'})) \geq \sigma_1(\pi^k(x)),\ \sigma_1(x) > \sigma_1(x^{'}),\ u_2^b > u_2^z > u_2^a)
\end{aligned}$$

Considering both Cases 1 and 2, we conclude that $\sigma_2(\pi^k(x))$ is a strictly better reply to σ_1 than $\sigma(\pi^k(x'))$. This completes the proof. ■

Step 8c: $\sigma_1(\tilde{x}) = 0$ for every $\tilde{x} \in \{jk, \ldots, (j+1)k - 1\}$ if and only if $\sigma_2(\tilde{x}) = 0$ for every $\tilde{x} \in \{jk, \ldots, (j+1)k - 1\}$.

Proof of Step 8c: ($\Rightarrow$) Assume that $\sigma_1(\tilde{x}) = 0$ for every $\tilde{x} \in \{jk, \ldots, (j+1)k - 1\}$. Fix $x \in \{jk, \ldots, (j+1)k - 1\}$ arbitrarily. It thus suffices to show $\sigma_2(x) = 0$. The expected payoff for player 2 of sending $m_2(x)$ against σ_1 is given as follows:

$$U_2(\sigma_1, m_2(x); \theta) = \sum_{l \in \{0, \ldots, J\} \setminus \{j\}} \sum_{y: n^k(y) = l} \sigma_1(y) u_2(z(n^k(y), j), \theta).$$

From Property 6 of Condition P+M, we have that $f(\theta) \succ_2^{\theta} z$ for any $z \in Z$. This implies that $u_2(f(\theta);\theta) > U_2(\sigma_1, m_2(x);\theta)$. Suppose by way of contradiction that $\sigma_2(x) > 0$. Since every message in the support of the equilibrium strategy σ_2 must offer the same expected payoff, we have

$$u_2(f(\theta);\theta) > U_2(\sigma_1, m_2(x);\theta) = U_2(\sigma_1, \sigma_2;\theta).$$

This implies that σ is not a compelling equilibrium, which is the desired contradiction. Thus, $\sigma_2(x) = 0$.

($\Leftarrow$) Assume that $\sigma_2(\tilde{x}) = 0$ for every $\tilde{x} \in \{jk, \ldots, (j+1)k-1\}$. Fix $x \in \{jk, \ldots, (j+1)k-1\}$ arbitrarily. It thus suffices to show $\sigma_1(x) = 0$. The expected payoff for player 1 of sending $m_1(x)$ against σ_2 is given as follows:

$$U_1(m_1(x), \sigma_2;\theta) = \sum_{l \in \{0,\ldots,J\}\setminus\{j\}} \sum_{y: n^k(y)=l} \sigma_2(y) u_1(z(j, n^k(y));\theta).$$

From Property 6 of Condition P+M, we have that $f(\theta) \succ_1^{\theta} z$ for any $z \in Z$. This implies that $u_1(f(\theta);\theta) > U_1(m_1(x), \sigma_2;\theta)$. Suppose by way of contradiction that $\sigma_1(x) > 0$. Since every message in the support of the equilibrium strategy σ_1 must offer the same expected payoff, we have

$$u_1(f(\theta);\theta) > U_1(m_1(x), \sigma_2;\theta) = U_1(\sigma_1, \sigma_2;\theta).$$

This implies that σ is not a compelling equilibrium, which is the desired contradiction. Thus, $\sigma_1(x) = 0$. ■

Finally, we shall show that $S_i^{min} = S_i^{max}$ for each $i \in \{1,2\}$. We first claim $S_2^{min} = S_2^{max}$. Assume by way of contradiction that $S_2^{min} \neq S_2^{max}$. We then use Step 8a to conclude that, for each $\tilde{x} \in \{jk, \ldots, (j+1)k-1\}$, $\sigma_2(\tilde{x}) = S_2^{min}$ implies $\sigma_1(\tilde{x}) = 0$. This implies $S_1^{min} = 0$. Since $S_2^{min} \neq S_2^{max}$, there exists $x \in \{jk, \ldots, (j+1)k-1\}$ such that $\sigma_2(x) > 0$. By Step 8c, there also exists $x' \in \{jk, \ldots, (j+1)k-1\}$ such that $\sigma_1(x') > 0$, which further implies $S_1^{min} \neq S_1^{max}$. We next use Step 8b to conclude that, for each $\tilde{x}$, $\sigma_1(\tilde{x}) = S_1^{min}$ implies $\sigma_2(\pi^k(\tilde{x})) = 0$. This implies $S_2^{min} = 0$. Starting from $S_1^{min} = S_2^{min} = 0$, we repeatedly use Steps 8a and 8b so that we are able to conclude that $\sigma_1(\tilde{x}) = 0$ and $\sigma_2(\tilde{x}) = 0$ for each $\tilde{x} \in \{jk, \ldots, (j+1)k-1\}$. This implies that $S_2^{min} = S_2^{max}$, which contradicts the hypothesis that $S_2^{min} \neq S_2^{max}$.

We next claim that $S_1^{min} = S_1^{max}$. Assume, on the contrary, that $S_1^{max} \neq S_1^{min}$. We then use Step 8b to conclude that, for each $\tilde{x} \in \{jk, \ldots, (j+1)k-1\}$, $\sigma_1(\tilde{x}) = S_1^{min}$ implies $\sigma_2(\pi^k(\tilde{x})) = 0$. This implies $S_2^{min} = 0$. Since $S_1^{min} \neq S_1^{max}$, there exists $x \in \{jk, \ldots, (j+1)k-1\}$ such that $\sigma_1(x) > 0$. By Step 8c, there also exists $x' \in \{jk, \ldots, (j+1)k-1\}$ such that $\sigma_2(x') > 0$, which further implies $S_2^{min} \neq S_2^{max}$. However, this contradicts the previously obtained conclusion that $S_2^{min} = S_2^{max}$. We therefore complete the proof of Lemma 8. ■

10.7 Proof of Lemma 9

Fix $u \in \mathcal{U}^\theta$ and $\sigma \in NE^*(\Gamma^k(\theta, u))$. Assume that σ is a compelling mixed strategy equilibrium of the game $\Gamma^k(\theta, u)$. As we know from Lemma 5 that $\Gamma^k = (M^k, g^k)$ pure Nash implements the SCF under Condition P, we have that for each $i \in \{1, 2\}$,

$$\sum_{\tilde{m} \in M} \sigma(\tilde{m}) u_i(g^k(\tilde{m}), \theta) \geq u_i(f(\theta), \theta),$$

with at least one strict inequality for some $i \in \{1, 2\}$. This implies that there exist $i \in \{1, 2\}$ and $m \in \text{supp}(\sigma)$ such that

$$u_i(g^k(m), \theta) > u_i(f(\theta), \theta).$$

We can write $m = (m_1, m_2) = ((o_1, a^\theta_{n^k(o_1)}), (o_2, b^\theta_{n^k(o_2)}))$. By Properties 5 and 6 of Condition M, m induces either Rule 1 or Rule 2 so that there must exist a $j \in \{0, \ldots, J-1\}$ such that $g^k(m) \in \mathcal{A}_j \cup \mathcal{B}_j$. More specifically, if m induces Rule 1, then $g^k(m) = a^\theta_j \in \mathcal{A}_j$ where $j = n^k(o_1)$, while if m induces Rule 2, then $g^k(m) = b^\theta_j \in \mathcal{B}_j$ where $j = n^k(o_2)$. Thus, for some i, we have either $a^\theta_j \succ^\theta_i f(\theta)$ or $b^\theta_j \succ^\theta_i f(\theta)$. This implies four possible different scenarios, with two immediately completing our proof. It remains to show that the last two scenarios, where $a^\theta_j \succ^\theta_2 f(\theta)$ or $b^\theta_j \succ^\theta_1 f(\theta)$, will also result in either $a^\theta_j \succ^\theta_1 f(\theta)$ or $b^\theta_j \succ^\theta_2 f(\theta)$.

Assume first that $a^\theta_j \succ^\theta_2 f(\theta)$ and $f(\theta) \succ^\theta_2 b^\theta_j$ both hold. This implies that $a^\theta_j \neq f(\theta)$ and a^θ_j is $(\mathcal{A}_j, \mathcal{B}_j)$-acceptable at state θ, which would contradict Property 4 of Condition P. Hence, $a^\theta_j \succ^\theta_2 f(\theta)$ implies $b^\theta_j \succ^\theta_2 f(\theta)$ A similar argument holds to show that $b^\theta_j \succ^\theta_1 f(\theta)$ implies $a^\theta_j \succ^\theta_1 f(\theta)$. Thus, we have found a j such that $a^\theta_j \succ^\theta_1 f(\theta)$ or $b^\theta_j \succ^\theta_2 f(\theta)$, completing the proof. ■

10.8 Proof of Lemma 10

We prove this by contradiction. That is, there exists $\varepsilon > 0$ such that for any $k \in \mathbb{N}$, there exist $u \in \mathcal{U}^\varepsilon$ and $\sigma^k \in NE^*(\Gamma^k(\theta, u))$ for which σ^k is a compelling mixed strategy equilibrium of the game $\Gamma^k(\theta, u)$. We fix k large enough so that by our hypothesis, we can fix $u \in \mathcal{U}^\varepsilon$ and a compelling mixed strategy equilibrium $\sigma^k \in NE^*(\Gamma^k(\theta, u))$. When we determine the exact size of k later, we guarantee that such k potentially depends on ε but not on u. Since σ^k is compelling in the game $\Gamma^k(\theta, u)$ and the mechanism Γ^k pure Nash implements the SCF f under Condition P by Lemma 4, we have that, for each $i \in \{1, 2\}$,

$$\sum_{\tilde{m} \in M} \sigma^k(\tilde{m}) u_i(g^k(\tilde{m}), \theta) \geq u_i(f(\theta), \theta),$$

with at least one strict inequality for some $i \in \{1,2\}$. This implies that there exist $i \in \{1,2\}$ and $m \in \text{supp}(\sigma^k)$ such that

$$u_i(g^k(m),\theta) > u_i(f(\theta),\theta).$$

Fix such $i \in \{1,2\}$. We introduce the following partition over $\text{supp}(\sigma^k)$: $\{\{M^+\},\{M^0\},\{M^-\}\} = \text{supp}(\sigma^k)$ such that

$$\begin{aligned}
M^+ &= \{m \in M^k | \ u_i(g^k(m),\theta) > u_i(f(\theta),\theta)\},\\
M^0 &= \{m \in M^k | \ u_i(g^k(m),\theta) = u_i(f(\theta),\theta)\},\\
M^- &= \{m \in M^k | \ u_i(g^k(m),\theta) < u_i(f(\theta),\theta)\}.
\end{aligned}$$

By construction, we have $M^+ \neq \emptyset$. Using the characterization of compelling mixed strategy equilibria in $NE^*(\Gamma^k(\theta,u))$ obtained by Lemmas 8 and 9, σ^k induces Rule 3 with positive probability. By Property 6 of Condition M, we also have $M^- \neq \emptyset$. Define the following notation:

$$\begin{aligned}
u_+ &\equiv \max_{m\in M^+} u_i(g(m),\theta),\\
u_- &\equiv \max_{m\in M^-} u_i(g(m),\theta).
\end{aligned}$$

By construction, we have $u_+ > u_-$ and $u_i(f(\theta),\theta) > u_-$. We now define

$$K \equiv \min\left\{k \in \mathbb{N} \middle| \ k > \frac{2}{\varepsilon}\right\}.$$

We fix $k = K$. Since $\sigma^k \in NE^*(\Gamma^k(\theta,u))$, by the definition of $NE^*(\Gamma^k(\theta,u))$, no agents randomize over alternatives. Since we assume $f(\theta) \notin Z$, by Property 6 of Condition M, we have that $m \in M^+$ only if m induces either Rule 1 or Rule 2. Furthermore, using the characterization of compelling mixed strategy equilibria in $NE^*(\Gamma^k(\theta,u))$ by Lemmas 8 and 9 and the construction of the mechanism Γ^k, we conclude that the probability that σ^k induces messages in M^+ is "at most" $2/k$. Moreover, by the construction of the mechanism Γ^k, we have that $m \in M^-$ if m induces Rule 3. Once again as $f(\theta) \notin Z$, by Property 6 of Condition M and the construction of the mechanism Γ^k, we conclude that the probability that σ^k

induces messages in M^- is “at least” $1-2/k$. Then,

$$\begin{aligned}
U_i(\sigma^k,\theta) &= \sum_{m\in M^+}\sigma^k(m)u_i(g^k(m),\theta)+\sum_{m\in M^0}\sigma^k(m)u_i(g^k(m),\theta)+\sum_{m\in M^-}\sigma^k(m)u_i(g^k(m),\theta)\\
&\le u_+\times\frac{2}{k}+u_-\times\left(1-\frac{2}{k}\right)\\
&\quad \left(\because M^+\neq\emptyset,\ M^-\neq\emptyset,\ \sum_{m\in M^+\cup M^0}\sigma^k(m)u_i(g^k(m),\theta)\le u_+\times(2/k)\ \text{and}\right.\\
&\quad \left.\sum_{m\in M^-}\sigma^k(m)u_i(g^k(m),\theta)\le u_-\times(1-2/k)\right)\\
&= \frac{2}{k}\left(u_+-u_-\right)+u_-,
\end{aligned}$$

which we define as $h(k)$. Since $u\in\mathcal{U}^\varepsilon$, we have

$$\frac{2(u_+-u_-)}{u_i(f(\theta),\theta)-u_-}\le\frac{2}{\varepsilon}<K.$$

As $h(k)$ is strictly decreasing in k, we have

$$h(K)<h\left(\frac{2}{\varepsilon}\right)\le h\left(\frac{2(u_+-u_-)}{u_i(f(\theta),\theta)-u_-}\right)=u_i(f(\theta),\theta).$$

Therefore, when $k=K$, we have

$$U_i(\sigma^K,\theta)\le h(K)<u_i(f(\theta),\theta).$$

This contradicts the hypothesis that σ^K is a compelling mixed strategy equilibrium of the game $\Gamma^K(\theta,u)$. ■

10.9 Proof of Lemma 11

Let $J\equiv|\Theta|-1$ where $|\Theta|$ denotes the number of possible states of the world. We then write $\Theta=\{\theta_j\}_{j=0}^J$ and let $j:\Theta\to\{0,\dots,J\}$ be a bijection. Let $\Gamma=(M,g)$ be a finite mechanism that pure Nash implements f. For each $\theta\in\Theta$, we define $m^\theta=(m_1^\theta,m_2^\theta)$ as a pure strategy Nash equilibrium of the game $\Gamma(\theta)$. The existence of m^θ is guaranteed by our hypothesis that f is pure Nash implementable by the mechanism Γ. Then we can define sets $\mathcal{A}_{j(\theta)},\mathcal{B}_{j(\theta)}$ as follows:

$$\mathcal{A}_{j(\theta)}=\{a\in A\mid\exists\, m_1\in M_1\ \text{such that}\ g(m_1,m_2^\theta)=a\},$$

and

$$\mathcal{B}_{j(\theta)} = \{b \in A \mid \exists\, m_2 \in M_2 \text{ such that } g(m_1^\theta, m_2) = b\}.$$

Let $j^{-1} : \{0, \ldots, J\} \to \Theta$ be the inverse function of j. For each $(j_1, j_2) \in \{0, \ldots, J\} \times \{0, \ldots, J\}$, we define $z(j_1, j_2) = g(m_1^{j^{-1}(j_1)}, m_2^{j^{-1}(j_2)})$.

Fix $\theta \in \Theta$ arbitrarily. It then follows from the construction of the sets $\mathcal{A}_{j(\theta)}, \mathcal{B}_{j(\theta)}$ above that Property 1 of Condition P is satisfied.

We next claim that message profile $m^\theta = (m_1^\theta, m_2^\theta)$ is a pure strategy Nash equilibrium of $\Gamma(\theta)$ if and only if $f(\theta)$ is $(\mathcal{A}_{j(\theta)}, \mathcal{B}_{j(\theta)})$-acceptable at state θ. This claim concludes that Properties 3 and 4 of Condition P hold. To show the only-if part, we assume that m^θ is a Nash equilibrium of the game $\Gamma(\theta)$. Then, there is no message $m_i \in M_i \backslash \{m_i^\theta\}$ such that $g(m_i, m_{-i}^\theta) \succ_i^\theta g(m^\theta)$. Since the SCF f is pure Nash implementable by the mechanism Γ, we have $f(\theta) = g(m^\theta)$ so that $f(\theta)$ is $(\mathcal{A}_{j(\theta)}, \mathcal{B}_{j(\theta)})$-acceptable at state θ. This completes the only-if-part of the claim. To show the if-part of the claim, we rather show its contrapositive form: if m^θ is "not" a pure strategy Nash equilibrium of the game $\Gamma(\theta)$, then $f(\theta)$ is "not" $(\mathcal{A}_{j(\theta)}, \mathcal{B}_{j(\theta)})$-acceptable at state θ. Since m^θ is not a Nash equilibrium of the game $\Gamma(\theta)$, there exist some player $i \in \{1, 2\}$ and a message $m_i' \in M_i$ such that $g(m_i', m_{-i}^\theta) \succ_i^\theta g(m^\theta)$. This implies that we have either $i = 1$ or $i = 2$. If $i = 1$, it follows that $g(m_1', m_2^\theta) \in \mathcal{A}_{j(\theta)}$ such that $g(m_1', m_2^\theta) \succ_1^\theta g(m^\theta)$. Since the SCF f is pure Nash implementable by Γ, $f(\theta)$ is not $(\mathcal{A}_{j(\theta)}, \mathcal{B}_{j(\theta)})$-acceptable at state θ. Instead, if $i = 2$, it follows that $g(m_1^\theta, m_2') \in \mathcal{B}_{j(\theta)}$ such that $g(m_1^\theta, m_2') \succ_1^\theta g(m^\theta)$. Since the SCF f is pure Nash implementable by Γ, $f(\theta)$ is not $(\mathcal{A}_{j(\theta)}, \mathcal{B}_{j(\theta)})$-acceptable at state θ. This completes the proof of the if-part of the claim.

We fix $(j_1, j_2) \in \{0, \ldots, J\} \times \{0, \ldots, J\}$ arbitrarily. To establish Property 2 of Condition P, we assume that $f(\theta) \neq z(j_1, j_2) = g(m_1^{j^{-1}(j_1)}, m_2^{j^{-1}(j_2)})$. It follows from our hypothesis that the SCF f is pure Nash implementable by the mechanism Γ that $(m_1^{j^{-1}(j_1)}, m_2^{j^{-1}(j_2)})$ is "not" a pure strategy Nash equilibrium of the game $\Gamma(\theta)$. This implies that either agent 1 or agent 2 has a profitable unilateral deviation from $(m_1^{j^{-1}(j_1)}, m_2^{j^{-1}(j_2)})$ in the game $\Gamma(\theta)$. Assume first that agent 1 has a profitable deviation from $(m_1^{j^{-1}(j_1)}, m_2^{j^{-1}(j_2)})$. This implies that there exists $m_1' \in M_1$ such that $g(m_1', m_2^{j^{-1}(j_2)}) \succ_1^\theta g(m_1^{j^{-1}(j_1)}, m_2^{j^{-1}(j_2)})$. By construction, we have $g(m_1', m_2^{j^{-1}(j_2)}) \in \mathcal{A}_{j_2}$. Setting $a_{(j_1,j_2)} = g(m_1', m_2^{j^{-1}(j_2)})$, we conclude that Property 2 of Condition P holds. Assume next that agent 2 has a profitable deviation from $(m_1^{j^{-1}(j_1)}, m_2^{j^{-1}(j_2)})$. This implies that there exists $m_2' \in M_2$ such that $g(m_1^{j^{-1}(j_1)}, m_2') \succ_1^\theta g(m_1^{j^{-1}(j_1)}, m_2^{j^{-1}(j_2)})$. By construction, we have $g(m_1^{j^{-1}(j_1)}, m_2') \in \mathcal{B}_{j_1}$. Setting $b_{(j_1,j_2)} = g(m_1^{j^{-1}(j_1)}, m_2')$, we conclude that Property 2 of Condition P holds. This completes the proof of the lemma. ■

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