Discussion Paper No. 643

IRREVERSIBILITIES AND THE OPTIMAL TIMING OF ENVIRONMENTAL POLICY UNDER KNIGHTIAN UNCERTAINTY

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October 2005

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This Version: September 27, 2005

Abstract

In this paper, we consider a problem in environmental policy design by applying optimal stopping rules. The purpose of this paper is to analyze the optimal timings at which the government should adopt environmental policies to deal with increases in greenhouse gas concentrations and to reduce emissions of SO_2 or CO_2 under the continuous-time Knightian uncertainty. Furthermore, we analyze the effects of increases in Knightian uncertainty on optimal environmental policies and the reservation value.

Journal of Economic Literature Classification Numbers: D81, H23, L51, Q23

Key Words: Knightian Uncertainty, Environmental Policy, Irreversibilities

*I would like to thank participants at the 2005 Annual Meeting of the Japanese Economic Association (Chuo University), and especially Ken-Ichi Akao for his comments and discussions. Financial support from the 21st century COE program (Osaka University), Nomura Foundation for Social Science, and the Japan Securities Scholarship Foundation is greatly acknowledged.

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1. Introduction

We have been facing with the problem of global warming. In order to deal with the threat to the environment, the government should adopt environmental policies to reduce emissions of SO_2 or CO_2 and not to increase greenhouse gas concentrations. In this paper, we consider a problem in environmental policy design within the framework of optimal timing problems. The purpose of this paper is to derive optimal timings at which the government should adopt environmental policies to deal with increases in greenhouse gas concentrations and to reduce emissions of SO_2 or CO_2 under the *continuous-time Knightian uncertainty.* Furthermore, we analyze the effects of increases in *Knightian uncertainty* on optimal environmental policies and the reservation value.

Usually, such a problem is analyzed based on the cost-benefit analysis. However, this standard approach does not consider three significant characteristics of environmental problems, that is, *uncertainty*, *irreversibility*, and *the flexibility in deciding the timing of adopting environmental policies.* We briefly explain these notions before we go into details. First, there exist *economic uncertainty* over future costs and benefits of adopting environmental policies and *ecological uncertainty* over the evolution of ecological systems. We do not know exactly the effect of adopting environmental policies nor know the economic damages caused by increases in average temperature. Second, there exist, at least two *irreversibilities* to be considered in environmental problems. The one is the irreversibility with respect to environmental damage. For instance, emissions of CO_2 will increase greenhouse gas concentrations, which is considered to lead to grobal warming and damage the ecosystems. The other is the irreversibility with respect to economic damage. For example, the installation of "scrubbers" by truck companies will be sunk costs on society. Third, the adoption of environmental policies is *not now-or-never decisions.* That is, the government has the option to postpone the adoption of policies and can wait for the arrival of new information, for example, some data about global warming, or some innovation about scrubbers, which will enable us to put off adopting environmental policies and to avoid imposing sunk costs on society.

As we have already mentioned, we have to consider two kinds of *uncertainties* that play important parts in environmental policy design. The one is *economic uncertainty*, that is, the uncertainty over future costs and benefits of adopting environmental policies to reduce emissions of SO_2 or CO_2. When it comes to global warming, we would not grasp the resulting cost to society, or we would not exactly predict how the increase in temperature would affect agricultural outputs, or ecological systems, even if how large average temperature is expected to rise. The other is *ecological uncertainty*, that is, the uncertainty over the evolution of ecological systems. We could not exactly predict how further increase in greenhouse gas concentrations would affect average temperature in the future, or how much today's level of greenhouse gas concentrations could be reduced by government's policies.

We also have to take care of at least two different kinds of irreversibilities that have the opposite characteristics. First, environmental policies that aim at reducing environmental damage impose *sunk cost* on society. For example, consider the situations in which companies owning coal-burning utilities might be forced to install scrubbers, or the situations in which they might have to scrap existing facilities and invest in more efficient machines. These sunk costs generate an opportunity cost of adopting policies immediately, rather than waiting for the arrival of new information about environmental damage and their economic concequences, for example, the information about innovations of new technologies in the near future that might enable us to remove sulfur more cheaply and efficiently. Traditional cost and benefit analysis based on the net present value approach ignores this opportunity cost, which should be considered carefully.

The second irreversibility is the one with respect to environmental damage. For example, environmental pollutants are not easily removable from the atmosphere; even if the government adopts severe policies to reduce greenhouse gas emissions, it would take many years to reduce greenhouse gas concentrations in the atmosphere. The damage to environmental systems caused by higher global warming cannot be easily reversible. Contrary to the first irreversibility, this implies that adopting environmental policies right now rather than waiting has *sunk benefit*, a negative opportunity cost. Again, the standard cost-benefit analysis based on the net present value approach does not consider *opportunity benefit*.

In the literature of decision making under *non-deterministic* situation, decision maker's beliefs are captured by a *single probability measure* over the state of the world, which is categorized as *risk*. Recently, new approaches have been proposed and axiomatized by Gilboa and Schmeidler (1989) in which decision maker's beliefs are captured by not a *single* but *multiple* *probability measures* over the state of the world, which is categorized as *Knightian uncertainty* or *uncertainty* in the literature.[1] This paper adapts a model of the continuous-time Knightian uncertainty proposed by Chen and Epstein (2002), which is a counterpart of the static framework developed by Gilboa and Schmeidler (1989). We show that while an increase in risk does not affect the value of adopting environmental policies, an increase in Knightian uncertainty induces a *decrease* in the value of adopting environmental policies. This result implies that if the government is uncertainty averse, then she underestimates the value of adopting environmental policies since she makes decision on the worst case scenario. Our paper is related to Pindyck (2000, 2002). He shows that an increase in risk induces an increase in the *reservation value* above which environmental policies are immediately adopted and below which environmental policies are never adopted. This paper shows that an increase in Knightian uncertainty induces a decrease in the reservation value under some condition, which is a stark contrast to Pindyck (2000, 2002).

The organization of this paper is as follows. Section 2 provides continuous-time models under risk and Knightian uncertainty. In order to analyze the value of adopting environmental policies under the continuous-time Knightian uncertainty in the later sections, mathematical definitions and results are provided. Section 3 derives the value of adopting environmental policies under the continuous-time Knightian uncertainty. Section 4 provides the further characterization of the value of adopting policies, and derives the value of optimal environmental policies under the continuous-time Knightian uncertainty. Section 5 provides sensitivity analyses, which are main results of this paper.

2. The Value of Environmental Policies under the Continuous-Time Knightian Uncertainty

In this section, we provide continuous-time models under risk and Knightian uncertainty. At first, definitions and mathematical notions are in order.

2.1 A General Continuous-Time Model under Risk

Let $(\Omega, \mathcal{F}_T, P)$ be a probability space, and let $(B_t)_{0\leq t\leq T}$ be a standard Brownian motion with

[1] For example, see Knight (1921), Gilboa (1987), Schmeidler (1989) or Gilboa and Schmeidler (1989).

respect to P.[2] We consider the standard filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ for a standard Brownian motion $(B_t)_{0\leq t\leq T}$.[3] Let $(M_t)_{0\leq t\leq T}$ be a stochastic process of the stock of an environmental pollutant (for example, CO_2 concentrations in the atmosphere) and let E_t be the evolution of the rate of emission of the pollutant at time t. We assume that the evolution of the stock of the environmental pollutant $(M_t)_{0\leq t\leq T}$ follows a controlled arithmetic Brownian motion,

$$dM_t = (\beta E_t - \delta M_t)dt + \sigma_M dB_t^M, \tag{1}$$

where $\beta \in (0, 1]$ denotes the absorption rate of the environmental pollutant, $\delta \in [0, 1]$ denotes the natural decay rate of the stock of the environmental pollutant over time, and σ_M is a constant real number. In other words, a fraction β of the emission E_t goes into the atmosphere, and a fraction δ of the environmental pollutant M_t on the atmosphere diffuses into the ocean, and the forests. In this paper, we ignore the stochastic fluctuation of M_t for simplicity. Thus, equation (1) reduces to the following,

$$dM_t = (\beta E_t - \delta M_t)dt. \tag{2}$$

We assume that E_t stays at the constant initial level E_0, until an environmental policy is adopted.

We assume that economic uncertainty follows a geometric Brownian motion,

$$dX_t = \alpha X_t dt + \sigma X_t dB_t, \tag{3}$$

where α and σ are constant real numbers. This stochastic process $(X_t)_{0\leq t\leq T}$ is assumed to capture economic uncertainty over the future costs and benefits of policy adoptions. Changes in X_t over time might reflect changes in technologies. For instance, if M is SO_2 concentrations, then changes in X_t might reflect the innovation of technologies that would drastically reduce the social cost of M, or population increase that would raise the social cost. Without loss of generality, it is assumed that $\sigma > 0$.

[2] A stochastic process $(B_t)_{0\leq t\leq T}$ is a *standard Brownian motion* if it is a continuous and adapted process on $(\Omega, \mathcal{F}_T, P)$ with properties that $B_0 = 0$ a.s. and for $0 \leq s < T$, the increment $B_T - B_s$ is independent of $\mathcal{F}_s$ and is normally distributed with mean zero and variance $T - s$. For definitions and notions of stochastic differenital equations, see Karatzas and Shreve (1991) or Protter (2004).

[3] A stochastic process $(\mathcal{F}_t)_{0\leq t\leq T}$ is a *standard filtration* for a Brownian motion (B_t) if for each $t \geq 0$, $\mathcal{F}_t$ is the smallest σ-algebra that contains the σ-algebra generated by $(B_s)_{0\leq s\leq t}$ and all P-null sets.

We assume that the flow of social cost associated with the stock variable M_t, $C(M_t, X_t)$ is linear in M_t,[4] that is

$$C(X_t, M_t) = -X_t M_t. \tag{4}$$

The value at t of adopting environmental policies with T an expiration time is

$$W(X_t, M_t, t) \equiv E^P \left[\int_t^T e^{-r(s-t)} C(X_s, M_s) ds \middle| \mathcal{F}_t \right],$$

where $r > 0$ is the discount rate, and $E^P[\cdot | \mathcal{F}_t]$ is the expectation with respect to P conditioned on $\mathcal{F}_t$.

The government's problem is formulated to decide the time when she will incur the social cost of adopting policies. The optimal time can be considered to be the solution to the optimal stopping problem of finding an $(\mathcal{F}_t)$-stopping time, $t' \in [0, T]$ that maximizes the value of adopting policies at period 0

$$E^P \left[\int_0^T e^{-rs} C(X_s, M_s) ds - e^{-rt'} K \middle| \mathcal{F}_0 \right]. \tag{5}$$

It is also assumed that $r > \alpha$.[5]

The value at t of optimal environmental policies V_t, is defined by

$$V_t \equiv \max_{t' \in [t,T]} E^P \left[\int_t^T e^{-r(s-t)} C(X_s, M_s) ds - e^{-r(t'-t)} K \middle| \mathcal{F}_t \right]. \tag{6}$$

We can show that[6]

$$\begin{aligned} & W(X_t, M_t, t) \\ = \; & -X_t M_t \int_t^T \exp\left(-(r + \delta - \alpha)(s - t)\right) ds \\ = \; & -\frac{X_t M_t}{r + \delta - \alpha} \left(1 - \exp\left(-(r + \delta - \alpha)(T - t)\right)\right). \end{aligned} \tag{7}$$

The value of adopting policies $W(X_t, M_t, t)$ equals

$$W_t \equiv W(X_t, M_t) = -\frac{X_t M_t}{r + \delta - \alpha} \tag{8}$$

[4]This assumption makes the optimal policy independent of M_t.
[5]In the later sections, we impose a strong assumption that $r > \alpha + \sigma\kappa$ for $\kappa > 0$, which implies $r > \alpha$.
[6]See Appendix in details.

as T goes to ∞. Since this model is stationary, the value of optimal environmental policies V_t is shown to depend only on X_t and M_t, not to depend directly on t, and satisfies the following Hamilton-Jacobi-Bellman equation,

$$V(X_t, M_t) = \max \left\{ W_t - K,\ -X_t M_t dt + E^P \left[dV_t | \mathcal{F}_t \right] + V(X_t, M_t) - rV(X_t, M_t)dt \right\}. \tag{9}$$

Proof. See Appendix. □

It can be shown that the optimal strategy is to stop and adopt the environmental policy right now if $X_t \geq X^*$ and to continue if $X_t < X^*$, where X^* is the *reservation value*. From the Hamilton-Jacobi-Bellman equation above, it follows that in the continuation region,

$$-X_t M_t dt + E^P \left[dV_t | \mathcal{F}_t \right] = rV(X_t, M_t)dt. \tag{10}$$

Note that the left-hand side of this equation is the social cost associated with the stock of the environmental pollutant plus the government's expected gain of having the rights to implement environmental policies, and the right-hand side is the opportunity cost measured in terms of government's discount rate. By applying Ito's lemma to V_t and some calculations, it follows that

$$\begin{aligned} dV_t &= \frac{\partial V_t}{\partial X_t} dX_t + \frac{\partial V_t}{\partial M_t} dM_t + \frac{1}{2} \frac{\partial^2 V_t}{\partial X_t^2} dX_t^2 \\ &= \frac{\partial V_t}{\partial M_t} (\beta E_0 - \delta M_t) dt + \frac{\partial V_t}{\partial X_t} (\alpha X_t dt + \sigma X_t dB_t) + \frac{1}{2} \frac{\partial^2 V_t}{\partial X_t^2} \sigma^2 X_t^2 dt. \end{aligned}$$

Combining two equations implies that

$$E^P[dV_t] = \frac{\partial V_t}{\partial M_t} (\beta E_0 - \delta M_t) dt + \frac{\partial V_t}{\partial X_t} \alpha X_t dt + \frac{1}{2} \frac{\partial^2 V_t}{\partial X_t^2} \sigma^2 X_t^2 dt. \tag{11}$$

Thus, in the continuation region,

$$rV_t = -X_t M_t + \frac{\partial V_t}{\partial M_t} (\beta E_0 - \delta M_t) + \frac{\partial V_t}{\partial X_t} \alpha X_t + \frac{1}{2} \frac{\partial^2 V_t}{\partial X_t^2} \sigma^2 X_t^2. \tag{12}$$

We solve this differential equation under three boundary conditions,

$$V_t(0, M_t) = 0, \tag{13}$$

$$V_t(X^*, M_t) = -\frac{X^* M_t}{r + \delta - \alpha} - K, \text{ and} \tag{14}$$

$$\frac{\partial V_t}{\partial X_t}(X^*, M_t) = \frac{\partial W_t}{\partial X_t}(X^*, M_t), \tag{15}$$

where X^* is the critical value of X at or above which environmental policies should be adopted. Condition (13) reflects the fact that if X_t is always zero, then the flow of the social cost associated with the stock variable $C(X_t, M_t)$ is zero. Thus the value of optimal policies will remain to be zero. Condition (14) that follows from (9) is called the *value matching condition*; when $X_t = X^*$ and the government exercises her option to adopt policies, it incurs a sunk cost K and obtains the net payoff. Condition (15) is called the *smooth pasting condition*; if adopting policies at X^* is critical, then the derivative of the value function must be continuous.[7] We guess the solution to this equation as follows:

$$V_t = AX_t^\gamma + BX_tM_t + DX_t,$$

where A, B and D are some constants. Then

$$\begin{aligned}
&\frac{1}{2}\sigma^2 X_t^2 A\gamma(\gamma-1)X_t^{\gamma-2} + \alpha\left(A\gamma X_t^\gamma + BM_tX_t + DX_t\right) \\
&-r\left(AX_t^\gamma + BX_tM_t + DX_t\right) - X_tM_t + (\beta E_0 - \delta M_t)BX_t = 0. \\
\Leftrightarrow\quad & AX_t^\gamma\left(\frac{1}{2}\sigma^2\gamma(\gamma-1) + \alpha\gamma - r\right) + ((\alpha - r - \delta)B - 1)X_tM_t + (\beta BE_0 + D(\alpha - r))X_t = 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{2}\sigma^2\gamma(\gamma-1) + \alpha\gamma - r = 0 \\
&(\alpha - r - \delta)B - 1 = 0 \Leftrightarrow B = -\frac{1}{r+\delta-\alpha} \\
&\beta BE_0 + D(\alpha - r) = 0 \Leftrightarrow D = -\frac{\beta E_0}{(r-\alpha)(r+\delta-\alpha)}.
\end{aligned}$$

The value A remains to be determined. By the boundary conditions, the negative part of the solution to $(1/2)\sigma^2\gamma(\gamma-1) + \alpha\gamma - r = 0$ is ruled out.[8] Note that

$$\begin{aligned}
\gamma &= \frac{-(\alpha - \sigma^2/2) + \sqrt{(\alpha-\sigma^2/2)^2 + 2r\sigma^2}}{\sigma^2} \\
&> \frac{-(\alpha - \sigma^2/2) + \sqrt{(\alpha-\sigma^2/2)^2 + 2\alpha\sigma^2}}{\sigma^2}
\end{aligned}$$

[7] For the value matching condition and the smooth pasting condition, see Dixit and Pindyck (1994, Chapter 4), and Dixit (1993).

[8] See Appendix.

$$
\begin{aligned}
&= \frac{-(\alpha - \sigma^2/2) + \sqrt{(\alpha + \sigma^2/2)^2}}{\sigma^2} \\
&= \frac{-(\alpha - \sigma^2/2) + (\alpha + \sigma^2/2)}{\sigma^2} = 1. \qquad (16)
\end{aligned}
$$

Then

$$
\begin{aligned}
A &= \left(\frac{K}{\gamma - 1}\right)^{1-\gamma} \gamma^{-\gamma} \left(\frac{\beta E_0}{(r-\alpha)(r+\delta-\alpha)}\right)^{\gamma} \\
X^* &= \left(\frac{\gamma K}{\gamma - 1}\right) \left(\frac{(r-\alpha)(r+\delta-\alpha)}{\beta E_0}\right).
\end{aligned}
$$

Thus

$$
V_t = \begin{cases} \left(\frac{K}{\gamma - 1}\right)^{1-\gamma} \gamma^{-\gamma} \left(\frac{\beta E_0}{(r-\alpha)(r+\delta-\alpha)}\right)^{\gamma} X_t^{\gamma} - \frac{X_t M_t}{r+\delta-\alpha} - \frac{\beta E_0 X_t}{(r-\alpha)(r+\delta-\alpha)} & \text{if} \quad X_t < X^* \\ W_t - K & \text{if} \quad X_t \geq X^*. \end{cases}
$$

The value function in the continuation region consists of three components. The first term is the value of the option to adopt environmental policies. The second term is the present value of the flow of social cost from the current stock of the pollutant. The third is the present value of the flow of social cost from the emission E_0. The value function in the stopping region consists of two terms: the value of adopting environmental policies defined by (8) plus the direct cost resulting from adopting environmental policies.

2.2 Continuous-Time Knightian Uncertainty

In this subsection, we provide a model of continuous-time Knightian uncertainty, which is first proposed by Chen and Epstein (2002).

2.2.1 Densitiy Generators, Girsanov's Theorem

Let $\mathcal{L}$ be the set of real-valued, measurable,[9] and $(\mathcal{F}_t)$-adapted[10] stochastic process on $(\Omega, \mathcal{F}_T, P)$ with an index set $[0, T]$ and let $\mathcal{L}^2$ be a subset of $\mathcal{L}$ that is defined by

$$
\mathcal{L}^2 = \left\{ (\theta_t)_{0 \leq t \leq T} \in \mathcal{L} \,\middle|\, \int_0^T \theta_t^2 dt < +\infty \ P\text{-a.s.} \right\}.
$$

[9] A stochastic process $(X_t)_{0\leq t\leq T}$ on $(\Omega, \mathcal{F}_T, P)$ is *measurable* if for every $A \in \mathcal{B}(\mathbb{R})$, $\{(t, \omega) | X_t(\omega) \in A\}$ belongs to the product σ-algebra $(\mathcal{B}([0, T]) \otimes \mathcal{F}_T)$, in other words, a function $X : [0, T] \times \Omega \to \mathbb{R}$ is $(\mathcal{B}([0, T]) \otimes \mathcal{F}_T)$-measurable. For example, see Karatzas and Shreve (1991, p.3).

[10] A stochastic process $(X_t)_{0\leq t\leq T}$ on $(\Omega, \mathcal{F}_T, P)$ is *adapted* to the filtration $(\mathcal{F}_t)$ if X_t is $\mathcal{F}_t$-measurable for each t. For example, see Karatzas and Shreve (1991, p.4).

Given $\theta = (\theta_t) \in \mathcal{L}^2$, define a stochastic process $(z_t^\theta)_{0\le t\le T}$ by[11]

$$(\forall t \in [0,T]) \;\; z_t^\theta = \exp\left(-\frac{1}{2}\int_0^t \theta_s^2 ds - \int_0^t \theta_s dB_s\right). \tag{17}$$

A stochastic process $(\theta_t) \in \mathcal{L}^2$ is called a *density generator* if (z_t^θ) is $(\mathcal{F}_t)$-martingale. Novikov's condition is one of the sufficient conditions for (z_t^θ) to be $(\mathcal{F}_t)$-martingale: [12]

$$E^P\left[\exp\left(\frac{1}{2}\int_0^T \theta_s^2 ds\right)\right] < +\infty.$$

Let θ be a density generator and define Q^θ by

$$(\forall A \in \mathcal{F}_T) \;\; Q^\theta(A) = \int_\Omega z_T^\theta(\omega)\chi_A(\omega)\, dP(\omega) = E^P[\chi_A z_T^\theta]. \tag{18}$$

Since (z_t^θ) is $(\mathcal{F}_t)$-martingale, $Q^\theta(\Omega) = E^P(z_T^\theta) = z_0^\theta = 1$. Thus, Q^θ is a probability measure, [13] and it is absolutely continuous with respect to P. [14] Furthermore, since z_T^θ is strictly positive, P is also absolutely continuous with respect to Q^θ. Thus, Q^θ is equivalent to P.[15] Conversely, any probability measures equivalent to P can be obained by a density generator in this way.[16]

Let Θ be a set of density generators. For such a set Θ, define the set of probability measures, $\mathcal{P}^\Theta$ on $(\Omega, \mathcal{F}_T)$, generated by Θ, by

$$\mathcal{P}^\Theta = \left\{ Q^\theta \;\middle|\; \theta \in \Theta \right\}, \tag{19}$$

where Q^θ is derived from P according to (18). In this paper, decision maker's beliefs are captured by not a single probability measure, but the set of probability measures equivalent to a probability measure P.

For any $\theta \in \Theta$, a stochastic process $(B_t^\theta)_{0\le t\le T}$ defined by[17]

$$(\forall t \in [0,T]) \;\; B_t^\theta = B_t + \int_0^t \theta_s ds$$

[11] By Ito's lemma, we can define $(z_t^\theta)_{0\le t\le T}$ as a unique solution to the stochastic differential equation: $dz_t^\theta = -z_t^\theta \theta_t dB_t$ with $z_0^\theta = 1$.

[12] For example, see Karatzas and Shreve (1991, p.199, Corollary 5.13).

[13] The countable additivity of Q^θ on $(\Omega, \mathcal{F}_T)$ is easy to show.

[14] A probability measure Q on $(\Omega, \mathcal{F})$ is *absolutely continuous* with respect to a probability measure P if for any A such that $P(A) = 0$, $Q(A) = 0$.

[15] This argument draws on the following result: If Y is a nonnegative random variable with $E^P[Y] = 1$, then we can create a new probability measure Q from the old probability measure P by defining $Q(A) = E^P[1_A Y]$ for all $A \in \mathcal{F}_T$.

[16] For example, see Duffie (1996).

[17] Equivalently, we can rewrite this as follows: $(\forall t \in [0,T])\; dB_t^\theta = dB_t + \theta_t dt$.

is a standard Brownian motion with respect to $(\mathcal{F}_t)$ on $(\Omega, \mathcal{F}_T, Q^\theta)$. This follows from Girsanov's Theorem.[18]

2.2.2 A Set of Stochastic Differential Equations

By Girsanov's theorem, the stochastic differential equation capturing economic uncertainty turns out to be

$$\begin{aligned} dX_t &= \alpha X_t dt + \sigma X_t dB_t \\ &= \alpha X_t dt + \sigma X_t (dB_t^\theta - \theta_t dt) \\ &= (\alpha - \sigma\theta_t) X_t dt + \sigma X_t dB_t^\theta \end{aligned} \tag{20}$$

for any $\theta \in \Theta$. Note that the change of measure formula by way of Girsanov's Theorem does not affect the volatility term. By (20), and by applying Ito's lemma to $\ln X_t$ by considering Q^θ as the true probability measure,

$$(\forall t \in [0,T]) \quad X_t = X_0 \exp\left((\alpha - (1/2)\sigma^2)t - \sigma \int_0^t \theta_s ds + \sigma B_t^\theta \right). \tag{21}$$

2.2.3 Rectangularity, Strong Rectangularity, i.i.d. Uncertainty, and κ-ignorance

Now we define three classes of density generators for the later analyses in this paper. A set of density generators, $\Theta^{K_t(\omega)}$, is *rectangular* if there exists a set-valued stochastic process $(K_t)_{0 \leq t \leq T}$ such that

$$\Theta^{K_t(\omega)} = \left\{ (\theta_t) \in \mathcal{L}^2 \,|\, \theta_t(\omega) \in K_t(\omega) \;\; (m \otimes P)\text{-a.s.} \right\}, \tag{22}$$

and, there exists a compact subset $\mathcal{K}$ of $\mathbb{R}$ such that for each t, $K_t : \Omega \twoheadrightarrow \mathcal{K}$ is compact-valued and convex-valued, the correspondence $(t,\omega) \twoheadrightarrow K_t(\omega)$, when restricted to $[0,s] \times \Omega$, is $\mathcal{B}([0,s]) \otimes \mathcal{F}_s$-measurable for any $0 < s \leq T$,[19] and $0 \in K_t(\omega)\,(m \otimes P)$-a.s., where m is the Lebesgue measure restricted on $\mathcal{B}([0,T])$.

A set of density generators Θ^{K_t} is *strongly rectangular* if there exist a nonempty compact subset[20] $\mathcal{K}$ of $\mathbb{R}$ and a compact-valued, convex-valued, measurable correspondence[21] $K : [0,T] \twoheadrightarrow$

[18] See Karatzas and Shreve (1991, p.191), for example.

[19] That is, $\{(t,\omega) \in [0,s] \times \Omega \,|\, K_t(\omega) \cap U \neq \emptyset\} \in \mathcal{B}([0,s]) \otimes \mathcal{F}_s$ for any open set U. See Aliprantis and Border (1994).

[20] This assumption corresponds to *uniform boundedness* in Chen and Epstein (2002). Under this assumption, we can show that any $\theta \in \Theta^{K_t}$ satisfies Novikov's condition.

[21] See Chen and Epstein (2002) in details.

$\mathcal{K}$ such that $0 \in K_t$ and

$$\Theta^{K_t} = \left\{ (\theta_t) \in \mathcal{L}^2 \,|\, \theta_t(\omega) \in K_t \;\; (m \otimes P)\text{-a.s.} \right\}, \tag{23}$$

where m is the Lebesgue measure restricted on $\mathcal{B}([0,T])$. Any element in Θ^{K_t} satisfies Novikov's condition, which follows since K_t is a subset of a compact subset $\mathcal{K}$ of $\mathbb{R}$ for all t. Note that the set K_t is independent of a state ω, that is, non-stochastic, contrary to the set $K_t(\omega)$ in (22).

Next, we consider a special case of strongly rectangular sets in which K_t is independent of time t. The uncertainty characterized by Θ^K is *i.i.d. uncertainty* if there exists a compact subset K of $\mathbb{R}$ such that $0 \in K$ and

$$\Theta^{K} = \left\{ (\theta_t) \in \mathcal{L}^2 \,|\, \theta_t(\omega) \in K \; (m \otimes P)\text{-a.s.} \right\}.$$

Note that the set Θ^K is independent of the state and time.

Finally, we consider a special case of the i.i.d. uncertainty Θ^K, where the set K is specified as

$$K = [-\kappa, \kappa],$$

where $\kappa > 0$. This type of uncertainty is called the *κ-ignorance*. The real number κ is considered to be a degree of Knightian uncertainty because the larger κ is, the larger the set of probability measures is.

In order to prove Proposition 1, we adapt the theory of support functions along with Chen and Epstein (2002). Define

$$e_t(\sigma)(\omega) \equiv \max_{x \in K_t(\omega)} \sigma x \text{ for } \sigma \in \mathbb{R}_{++}.$$

The strict positivity of σ for the definition of the support function is assumed in order to define θ_t^* and $(\theta_t)_*$ below. Since $K_t(\omega)$ is convex-valued, we can adapt the theory of support functions. See Rockafeller (1970) in details. In this paper, further restriction is imposed on the support function. Define

$$e_t(\sigma) \equiv \max_{x \in K_t} \sigma x \quad \text{for } \sigma \in \mathbb{R}_{++}. \tag{24}$$

Recall that K_t is convex-valued. For this support function, define

$$(\forall t \in [0,T]) \quad \theta_t^* \equiv \operatorname{argmax}\{\sigma x \,|\, x \in K_t\} = \{\max K_t\}. \tag{25}$$

The equality holds since $\sigma > 0$ and K_t is compact-valued. Note that (θ_t^*) is a degenetate measurable process, and $(\theta_t^*) \in \mathcal{L}$. Thus $(\theta_t^*) \in \Theta^{K_t}$. Furthermore, if the κ-ignorance is assumed, then $\theta_t^* = \kappa$ since $\theta_t^* = \max[-\kappa, \kappa]$. In this case, the support function turns out to be

$$e(\sigma) = \max_{x \in K} \sigma x = \sigma\kappa \quad \text{for } \sigma \in \mathbb{R}_{++},$$

where $K = [-\kappa, \kappa]$. Note that θ_t^* and $e(\sigma)$ are independent of the state and time in case of the κ-ignorance. For the analyses in the later section, we define $(\theta_t)_*$ by

$$(\forall t \in [0,T]) \quad (\theta_t)_* \equiv \operatorname{argmin}\{\sigma x \,|\, x \in K_t\} = \{\min K_t\}.$$

This is a counterpart of Equation (25). Note that $(\theta_t)_* = -\kappa$ under the κ-ignorance since $(\theta_t)_* = \min[-\kappa, \kappa]$.

3. The Value of Adopting Policies under Strong Rectangularity

In this section, we derive the value of adopting environmental policies W_t by assuming the strong rectangularity.

We assume that the government is uncertainty averse. In other words, her beliefs are captured by the set of probability measures $\mathcal{P}^\Theta$, (19), and she maximizes the infimum of expected returns over P^Θ. Furthermore, we impose the strong rectangularity on Θ, which implies that Θ is equal to Θ^{K_t}. Thus, the value at t of adopting environmental policies with T an expiration time is

$$W(X_t, M_t, t) \equiv \inf_{Q \in \mathcal{P}^\Theta} E^Q \left[\int_t^T e^{-r(s-t)} C(X_s, M_s) ds \middle| \mathcal{F}_t \right],$$

where $C(X_t, M_t)$ is defined by (4)

Proposition 1. *Suppose that the government is uncertainty averse, and her beliefs are characterized by Θ^{K_t}, where Θ^{K_t} is a strongly rectangular set of density generators defined by* (23) *for some* (K_t). *Then the value of adopting environmental policies is provided by*

$$W(X_t, M_t, t) = -\int_t^T X_t M_t \exp\left(-(r + \delta - \alpha)(s - t) - \int_t^s \sigma(\theta_h)_* dh \right) ds, \tag{26}$$

where $(\theta_t)_*$ *is defined by*

$$(\forall t \in [0, T]) \quad (\theta_t)_* \equiv \operatorname{argmin}\{\sigma x \,|\, x \in K_t\} = \{\min K_t\}.$$

Proof. See Appendix. □

4. The Optimal Environmental Policy under Strong Rectangularity

In this section, we derive the values of adopting environmental policies and optimal environmental policies under the i.i.d. uncertainty and the infinite time horizon in addition to the strong rectangularity.

4.1 The Value of Optimal Environmental Policies under the Strong Rectangularity

The optimal time is the solution to the optimal stopping problem of finding an $(\mathcal{F}_t)$-stopping time, $t' \in [0, T]$ that maximizes the value of adopting policies at period 0

$$\min_{Q \in \mathcal{P}^\Theta} E^Q \left[\int_0^T e^{-rs} C(X_s, M_s) ds - e^{-rt'} K \,\middle|\, \mathcal{F}_0 \right].$$

Thus, the value at t of optimal environmental policies V_t, is defined by

$$V_t \equiv \max_{t' \in [t,T]} \min_{Q \in \mathcal{P}^\Theta} E^Q \left[\int_t^T e^{-r(s-t)} C(X_s, M_s) ds - e^{-r(t'-t)} K \,\middle|\, \mathcal{F}_t \right], \tag{27}$$

where $C(X_t, M_t)$ is defined by (4). Appendix shows that V_t is a solution to the following Hamilton-Jacobi-Bellman equation,

$$V_t = \max \left\{ W_t - K, -X_t M_t dt + \min_{Q \in \mathcal{P}^\Theta} E^Q [dV_t \,|\, \mathcal{F}_t] + V_t - rV_t dt \right\}, \tag{28}$$

where W_t is defined below by (32). Further assumptions enable us to solve this type of the Hamilton-Jacobi-Bellman equation, otherwise difficult to solve analytically. We discuss this topic in the next two subsections.

4.2 The Value of Adopting Policies under the i.i.d. Uncertainty and the Infinite-Time Horizon

In this subsection, we derive the value of adopting policies by assuming the i.i.d. uncertainty, the infinite-time horizon and no-existence of expiration date.

Under the assumption of the i.i.d. uncertainty, the support function (24) reduces to

$$e(\sigma) = \max_{x \in K} \sigma x \ \text{ for } \sigma \in \mathbb{R}_{++}, \tag{29}$$

and

$$\theta^* = \text{argmax}\,\{\sigma x |\, x \in K\} = \max K. \tag{30}$$

Furthermore, $\theta_* = \text{argmin}\,\{\sigma x |\, x \in K\} = \min K$. Note that θ^* and θ_* are independent of time and the state. Under the i.i.d. uncertainty, (26) reduces to

$$\begin{aligned} W(X_t, M_t, t) &= -\int_t^T X_t M_t \exp\left(-(r + \delta - \alpha + \sigma\theta_*)(s - t)\right) ds \\ &= -\int_t^T X_t M_t \exp(-\lambda(s - t)) ds \\ &= -\frac{X_t M_t}{\lambda} \left(1 - \exp(-\lambda(T - t))\right), \end{aligned} \tag{31}$$

where $\lambda \equiv r + \delta - \alpha + \sigma\theta_*$.

Thus, by assuming that there exists no expiration date, together with the i.i.d. uncertainty and the infinite-time horizon, the value of adopting environmental policies reduces to

$$W_t \equiv W(X_t, M_t) = -\frac{X_t M_t}{\lambda} \tag{32}$$

as T goes to ∞, since $r > \alpha$, $\sigma > 0$, $\delta > 0$, and $0 \in K$, which implies the positivity of λ.

4.3 The Value of Optimal Environmental Policies under i.i.d. uncertainty and Infinite Horizon

In this subsection, we derive the value of optimal environmental policies V_t under the i.i.d. uncertainty and the infinite-time horizon. In order to solve the HJB equation (28) analytically, we assume that the continuous-time Knightian uncertainty is independent of time, in other words, it is characterized by the i.i.d. uncertainty, the planning horizon is infinite, and there exists no expiration date,[22] which implies that V_t depends on X_t and M_t, not on time t directly.

[22] We impose three assumptions, that is, the i.i.d. uncertainty, the infinite time horizon, and no-existence of expiration date. See discussions in Nishimura and Ozaki (2003).

Thus we can write $V_t = V(X_t, M_t)$ for some function $V : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$. Then the Hamilton-Jacobi-Bellman equation (28) turns out to be

$$\begin{aligned} & V(X_t, M_t) \\ = \quad & \max\left\{W_t - K, -X_t M_t dt + \min_{Q\in\mathcal{P}^\Theta} E^Q\left[dV_t \,|\, \mathcal{F}_t\right] + V(X_t, M_t) - rV(X_t, M_t)dt\right\}, \end{aligned} \tag{33}$$

where $V : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$. From the Hamilton-Jacobi-Bellman equation (33), it follows that

$$-X_t M_t dt + \min_{Q\in\mathcal{P}^\Theta} E^Q\left[dV_t \,|\, \mathcal{F}_t\right] = rV(X_t, M_t)dt,$$

in the continuation region. The left-hand side of this equation is the social cost assosiated with the stocks of environmental pollutants plus the government's expected minimum gain of having the rights to carry out environmental policies, and the right-hand side is the opportunity cost measured in terms of government's discount rate. If $\mathcal{P}^\Theta$ is singleton, then this equation reduces to (10).

In the continuation region, it is shown that[23]

$$\min_{Q\in\mathcal{P}^\Theta} E^Q\left[dV_t|\mathcal{F}_t\right] = \frac{\partial V_t}{\partial X_t}(\alpha - \sigma\theta_*)X_t dt + \frac{1}{2}\frac{\partial^2 V_t}{\partial X_t^2}\sigma^2 X_t^2 dt + \frac{\partial V_t}{\partial M_t}\left(\beta E_0 - \delta M_t\right) dt.$$

Thus in the continuation region, it follows that

$$\frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 V_t}{\partial X_t^2} + (\alpha - \sigma\theta_*)X_t \frac{\partial V_t}{\partial X_t} - rV_t + \frac{\partial V_t}{\partial M_t}(\beta E_0 - \delta M_t) - X_t M_t = 0.$$

We solve this differential equation with the following boundary conditions,

$$V_t(0, M_t) = 0, \tag{34}$$

$$V_t(X^*, M_t) = -\frac{X^* M_t}{\lambda} - K, \text{ and} \tag{35}$$

$$\frac{\partial V_t}{\partial X_t}(X^*, M_t) = \frac{\partial W_t}{\partial X_t}(X^*, M_t), \tag{36}$$

where X^* is the critical value of X at or above which environmental policies should be adopted. The three conditions have been already explained within the framework of risk. Condition (34) reflects the fact that if X is always zero, then the flow of the social cost associated with the stock

[23] In order to derive this equation, we assume that $\partial V/\partial X_t$ is negative in the continuation region, and V is twice differentiable in the continuation region. These two assumptions actually hold. We verify these assumptions in Appendix.

variable $C(X_t, M_t)$ is zero. Thus the value of optimal policies will remain to be zero. Condition (35) is the value matching condition, and condition (36) is the smooth pasting condition. By solving the differential equation with the three boundary conditions, we obtain the following optimal strategy,

$$V_t = \begin{cases} AX_t^{\gamma} - \dfrac{X_t M_t}{r+\delta-(\alpha-\sigma\theta_*)} - \dfrac{\beta E_0 X_t}{(r-(\alpha-\sigma\theta_*))(r+\delta-(\alpha-\sigma\theta_*))} & \text{if} \quad X_t < X^* \\ W_t - K & \text{if} \quad X_t \geq X^*, \end{cases}$$

where

$$A = \left(\frac{K}{\gamma-1}\right)^{1-\gamma} \gamma^{-\gamma} \left(\frac{\beta E_0}{(r-(\alpha-\sigma\theta_*))(r+\delta-(\alpha-\sigma\theta_*))}\right)^{\gamma}$$

$$X^* = \left(\frac{\gamma K}{\gamma-1}\right)\left(\frac{(r-(\alpha-\sigma\theta_*))(r+\delta-(\alpha-\sigma\theta_*))}{\beta E_0}\right)$$

$$\gamma = \frac{-(\alpha-\sigma\theta_*-\sigma^2/2)+\sqrt{(\alpha-\sigma\theta_*-\sigma^2/2)^2+2r\sigma^2}}{\sigma^2}.$$

It can be shown that $\gamma > 1$ (see Appendix).

The value function in the continuation region consists of three components. The first term is the value of the option to adopt the environmental policy. The second term is the present value of the flow of social cost from the current stock of pollutants. The third is the present value of the flow of social cost from the emission E_0. The value function in the stopping region consists of two terms: the value of adopting environmental policies defined by (32) plus the direct cost resulting from adopting the environmental policy.

5. Sensitivity Analyses

In this section, we analyze the effects of increases in risk and Knightian uncertainty on the value of adopting environmental policies and the value of optimal environmental policies. We also analyze the effects of increases in risk and Knightian uncertainty on the optimal timing of adopting environmental policies.

5.1 An Increase in Risk (the case of no-Knightian uncertainty)

In this subsection, we consider the case of no-Knightian uncertainty. We show that an increase in risk does not affect the value of adopting policies W_t, and an increase in risk does induce increases in the value of optimal environmental policies V_t in the stopping region and the reservation value X^*.

Proposition 2. *In the case of no Knightian uncertainty, an increase in risk induces no change in the value of adopting policies W_t, an increase in risk induces an increase in the value of optimal environmental policies V_t in the stopping region, and an increase in risk induces an increase in the reservation value X^*.*

Proof. See Appendix. □

The first claim in this proposition states that an increase in risk does not have any effect on the value of adopting environmental policies. The third claim implies that the more risk there exists over the future social cost of pollutants, the greater is the incentive to wait rather than to adopt environmental policies immediately. Note that while the second and third claims are presented in Pindyck (2000), the first claim is not.

5.2 An Increase in Knightian uncertainty (the case of κ ignorance)

In this subsection, we analyze the effects of Knightian uncertainty on the value of adopting policies and the reservation value. We show that an increase in Knightian uncertainty induces a decrease in the value of adopting policies, and induces a decrease in the reservation value under some condition. Recall that in the case of no-Knightian uncertainty, an increase in risk does not affect the value of adopting policies W_t, and an increase in risk does induce an increase in the reservation value X^*. Finally, we provide the following proposition.

Proposition 3. *We assume the same conditions in Proposition 2, and assume the κ-ignorance. Then, an increase in Knightian uncertainty induces a decrease in the value of adopting policies W_t, and an increase in Knightian uncertainty induces a decrease in the reservation value X^* for parameters that make the absolute value of $\partial\gamma/\partial\kappa$ sufficiently small.*

Proof. See Appendix. □

The first claim in this proposition states that an increase in Knightian uncertainty has the negative effect on the value of adopting environmental policies. This result implies that if the government has imprecise knowledge about the state of the world, then she underestimates the value of adopting environmental policies since she makes decisions on the worst case scenario. This is a stark contrast to the first claim in Proposition 2. Contrary to the first claim in this proposition, which holds without any restrictions, the second claim in this proposition holds for sets of parameters that make the absolute value of $\partial\gamma/\partial\kappa$ sufficiently small.

Appenix 1: Derivations of Mathematical Results

Derivation of (8)**.**

At first, we derive the value at t of adopting policies as follows:

$$
\begin{aligned}
& W(X_t, M_t, t) \\
\equiv\ & E^P\left[\int_t^T e^{-r(s-t)} C(M_s, X_s) ds \,\middle|\, \mathcal{F}_t\right] \\
=\ & E^P\left[-\int_t^T X_s M_s e^{-r(s-t)} ds \,\middle|\, \mathcal{F}_t\right] \\
=\ & -\int_t^T E^P\left[e^{-r(s-t)} X_t \exp\left((\alpha - 1/2\sigma^2)(s-t) + \sigma(B_s - B_t)\right) M_s \,\middle|\, \mathcal{F}_t\right] ds \\
=\ & \cdots\cdots \\
=\ & -\int_t^T X_t M_s \exp\left(-(r-\alpha)(s-t)\right) ds \\
=\ & -\int_t^T X_t \left(-\mu + (M_t + \mu)\exp\left(-\delta(s-t)\right)\right) \exp\left(-(r-\alpha)(s-t)\right) ds \\
=\ & \mu X_t \int_t^T \exp\left(-(r-\alpha)(s-t)\right) ds - X_t M_t \int_t^T \exp\left(-(r+\delta-\alpha)(s-t)\right) ds \\
& -\mu X_t \int_t^T \exp\left(-\delta(s-t)\right) \exp\left(-(r-\alpha)(s-t)\right) ds \\
=\ & \frac{\mu X_t}{r-\alpha}\left(1 - e^{-(r-\alpha)(T-t)}\right) - \frac{X_t M_t}{r+\delta-\alpha}\left(1 - e^{-(r+\delta-\alpha)(T-t)}\right) - \frac{\mu X_t}{r+\delta-\alpha}\left(1 - e^{-(r+\delta-\alpha)(T-t)}\right),
\end{aligned}
$$

which goes to

$$
-\frac{X_t M_t}{r+\delta-\alpha} - \frac{\beta E_0 X_t}{(r-\alpha)(r+\delta-\alpha)} \tag{37}
$$

as $T \to \infty$, where the second equality follows from Fubini's Theorem and solving the stochastic differential equation $dX_t = \alpha X_t dt + \sigma X_t dB_t$,[24] and the fifth equality follows from solving the ordinal differential equation $dM_t/dt = \beta E_0 - \sigma M_t$.[25] By letting $E_0 = 0$, we obtain the value of adopting policies $W(X_t, M_t) = -X_t M_t/(r + \delta - \alpha)$. Thus, the stationarity is obtained. □

Derivation of HJB in case of risk

[24]The solution to $dX_t = \alpha X_t dt + \sigma X_t dB_t$ is $X_s = X_t \exp\left((\alpha - 1/2\sigma^2)(s-t) + \sigma(B_s - B_t)\right)$ for all $s \geq t$.

[25]The solution to $dM_t/dt = \beta E_0 - \delta M_t = -\delta(M_t + \mu)$ is $M_s = -\mu + (M_t + \mu)e^{-\delta(s-t)}$ for all $s \geq t$, where $\mu \equiv -(\beta/\delta)E_0$.

The Hamilton-Jacobi-Bellman equation follows since

$$
\begin{aligned}
& V_t \\
= \; & \max_{t' \in [t,T]} E^P \left[\int_t^{t'} e^{-r(s-t)} C(X_s, M_s) ds + \int_{t'}^{T} e^{-r(s-t)} C(X_s, M_s) ds - e^{-r(t'-t)} K \middle| \mathcal{F}_t \right] \\
= \; & \max \left\{ E^P \left[\int_t^T e^{-r(s-t)} C(X_s, M_s) ds \middle| \mathcal{F}_t \right] - K, \; -X_t M_t dt + \right. \\
& \left. \max_{t' \in [t+dt,T]} E^P \left[\int_t^{t'} e^{-r(s-t)} C(X_s, M_s) ds + \int_{t'}^{T} e^{-r(s-t)} C(X_s, M_s) ds - e^{-r(t'-t)} K \middle| \mathcal{F}_t \right] \right\} \\
= \; & \max \left\{ W_t - K, \; -X_t M_t dt + \right. \\
& \left. \max_{t' \in [t+dt,T]} E^P \left[\int_t^{t'} e^{-r(s-t)} C(X_s, M_s) ds + \int_{t'}^{T} e^{-r(s-t)} C(X_s, M_s) ds - e^{-r(t'-t)} K \middle| \mathcal{F}_t \right] \right\} \\
= \; & \max \left\{ W_t - K, \; -X_t M_t dt + \right. \\
& \left. e^{-rdt} \max_{t' \in [t+dt,T]} E^P \left[E^P \left[\int_t^T e^{-r(s-t-dt)} C(X_s, M_s) ds - e^{-r(t'-t-dt)} K \middle| \mathcal{F}_{t+dt} \right] \middle| \mathcal{F}_t \right] \right\} \\
= \; & \max \left\{ W_t - K, \; -X_t M_t dt + e^{-rdt} \, E^P \left[V_{t+dt} \middle| \mathcal{F}_t \right] \right\} \\
= \; & \max \left\{ W_t - K, \; -X_t M_t dt + (1 - rdt) \left(E^P [dV_t | \mathcal{F}_t] + V_t \right) \right\} \\
= \; & \max \left\{ W_t - K, \; -X_t M_t dt + E^P [dV_t | \mathcal{F}_t] + V_t - rV_t dt \right\},
\end{aligned}
$$

where the first equality follows from the definition of V_t, the third equality follows from the definition of W_t, the fourth holds by the law of iterated integrals, the fifth follows from the definition of V_t, the sixth holds by approximating e^{-rdt} by $(1 - rdt)$, and the last equality holds by eliminating higher order terms than dt. □

Proof of Proposition 1

In order to prove Proposition 1, we have to show the next lemma.

Lemma 1. *For any $s \geq t$ and for any $\theta \in \Theta^{K_t}$,*

$$
\begin{aligned}
& E^{Q^\theta} \left[\exp \left(- \int_t^s \sigma \theta_h dh + \sigma \left(B_s^\theta - B_t^\theta \right) \right) \middle| \mathcal{F}_t \right] \\
\leq \; & E^{Q^{\theta_*}} \left[\exp \left(- \int_t^s \sigma (\theta_h)_* dh + \sigma \left(B_s^{\theta_*} - B_t^{\theta_*} \right) \right) \middle| \mathcal{F}_t \right],
\end{aligned}
$$

where $\theta_* \equiv \operatorname{argmin} \{ \sigma x | x \in K \} = \min K$.

Proof. Let $s \geq t$ and let $\theta \in \Theta^{K_t}$. Then

$$(\forall \omega) \quad \exp\left(-\int_t^s \sigma\theta_r dr + \sigma\left(B_s^\theta - B_t^\theta\right)\right) \leq \exp\left(-\int_t^s \sigma(\theta_r)_* dr + \sigma\left(B_s^\theta - B_t^\theta\right)\right).$$

Thus

$$\begin{aligned}
& E^{Q^\theta}\left[\exp\left(-\int_t^s \sigma\theta_r dr + \sigma\left(B_s^\theta - B_t^\theta\right)\right) \middle| \mathcal{F}_t\right] \\
\leq\ & E^{Q^\theta}\left[\exp\left(-\int_t^s \sigma(\theta_r)_* dr + \sigma\left(B_s^\theta - B_t^\theta\right)\right) \middle| \mathcal{F}_t\right] \\
=\ & \exp\left(-\int_t^s \sigma(\theta_r)_* dr\right) \exp\left(\frac{1}{2}\sigma^2(s-t)\right) \\
=\ & E^{Q^{\theta_*}}\left[\exp\left(-\int_t^s \sigma(\theta_r)_* dr + \sigma\left(B_s^{\theta_*} - B_t^{\theta_*}\right)\right) \middle| \mathcal{F}_t\right],
\end{aligned}$$

where the inequality follows from the monotonicity of conditional expectation.[26] □

Now we are in a position to prove Proposition 1.

Proof of Proposition 1.

$$\begin{aligned}
& W(X_t, M_t) \\
=\ & \inf_{Q \in \mathcal{P}^\Theta} E^Q\left[\int_t^T e^{-r(s-t)} C(X_s, M_s) ds \middle| \mathcal{F}_t\right] \\
=\ & \inf_{\theta \in \Theta} E^{Q^\theta}\left[-\int_t^T e^{-r(s-t)} X_s M_s ds \middle| \mathcal{F}_t\right] \\
=\ & \inf_{\theta \in \Theta} \int_t^T E^{Q^\theta}\left[-X_s M_s e^{-r(s-t)} \middle| \mathcal{F}_t\right] ds \\
=\ & \inf_{\theta \in \Theta} -\int_t^T X_t M_s E^{Q^\theta}\left[\exp(-r(s-t)) X_t \exp\left((\alpha - \sigma^2/2)(s-t) - \sigma\int_t^s \theta_h dh + \sigma(B_s^\theta - B_t^\theta)\right) \middle| \mathcal{F}_t\right] ds \\
=\ & \inf_{\theta \in \Theta} -\int_t^T X_t M_s \exp\left((\alpha - \sigma^2/2 - r)(s-t)\right) E^{Q^\theta}\left[\exp\left(-\sigma\int_t^s \theta_h dh + \sigma(B_s^\theta - B_t^\theta)\right) \middle| \mathcal{F}_t\right] ds \\
=\ & -\int_t^T X_t M_s \exp\left((\alpha - \sigma^2/2 - r)(s-t)\right) E^{Q^{\theta_*}}\left[\exp\left(-\sigma\int_t^s (\theta_h)_* dh + \sigma(B_s^{\theta_*} - B_t^{\theta_*})\right) \middle| \mathcal{F}_t\right] ds \\
=\ & -\int_t^T X_t M_s \exp\left((\alpha - \sigma^2/2 - r)(s-t) - \int_t^s \sigma(\theta_h)_* dh\right) E^{Q^{\theta_*}}\left[\exp\sigma\left(B_s^{\theta_*} - B_t^{\theta_*}\right) \middle| \mathcal{F}_t\right] ds \\
=\ & -\int_t^T X_t M_s \exp\left((\alpha - \sigma^2/2 - r)(s-t) - \int_t^s \sigma(\theta_h)_* dh\right) \exp\left(\sigma^2(s-t)/2\right) ds \\
=\ & -\int_t^T X_t M_s \exp\left(-(r-\alpha)(s-t) - \int_t^s \sigma(\theta_h)_* dh\right) ds
\end{aligned}$$

[26] For example, see Billingsley (1995), p.447.

$$
\begin{aligned}
&= \int_t^T X_t M_t \exp\left(-\delta(s-t)\right) \exp\left(-(r-\alpha)(s-t) - \int_t^s \sigma(\theta_h)_* dh\right) ds \\
&= -\int_t^T X_t M_t \exp\left(-(r+\delta-\alpha)(s-t) - \int_t^s \sigma(\theta_h)_* dh\right) ds,
\end{aligned}
$$

where the second equality holds by (19), the third equality holds by Fubini's theorem for conditional expectation,[27] the fourth equality holds by (21), the sixth equality holds by Lemma 1, the seventh equality follows from the fact that (θ_*) is a degenerate stochastic process, the eighth equality holds by the fact that $(B_t^{\theta_*})$ is a Brownian motion with respect to Q^{θ_*}, and the tenth equality follows since $M_s = -\mu + (M_t + \mu)e^{-\delta(s-t)}$ for all $s \geq t$, where $\mu = -(\beta/\delta)E_0$. Thus the proof is completed. □

Derivation of HJB in case of Knightian Uncertainty

$$
\begin{aligned}
& V_t \\
= \; & \max_{t' \in [t,T]} \min_{Q \in \mathcal{P}^\Theta} E^Q \left[\int_t^T e^{-r(s-t)} C(X_s, M_s) ds - e^{-r(t'-t)} K \,\middle|\, \mathcal{F}_t \right] \\
= \; & \max \left\{ \min_{Q \in \mathcal{P}^\Theta} E^Q \left[\int_t^T e^{-r(s-t)} C(X_s, M_s) ds \,\middle|\, \mathcal{F}_t \right] - K, \right. \\
& \left. -X_t M_t dt + \max_{t' \geq t+dt} \min_{Q \in \mathcal{P}^\Theta} E^Q \left[\int_t^T e^{-r(s-t)} C(X_s, M_s) ds - e^{-r(t'-t)} K \,\middle|\, \mathcal{F}_t \right] \right\} \\
= \; & \max \{ W_t - K, \\
& \left. -X_t M_t dt + \max_{t' \geq t+dt} \min_{Q \in \mathcal{P}^\Theta} E^Q \left[\int_t^T e^{-r(s-t)} C(X_s, M_s) ds - e^{-r(t'-t)} K \,\middle|\, \mathcal{F}_t \right] \right\} \\
= \; & \max \{ W_t - K, \\
& \left. -X_t M_t dt + \max_{t' \geq t+dt} \min_{\theta \in \Theta} E^{Q^\theta} \left[\int_t^T e^{-r(s-t)} C(X_s, M_s) ds - e^{-r(t'-t)} K \,\middle|\, \mathcal{F}_t \right] \right\} \\
= \; & \max \{ W_t - K, \; -X_t M_t dt \\
& \left. + e^{-rdt} \max_{t' \geq t+dt} \min_{\theta \in \Theta} E^{Q^\theta} \left[E^{Q^\theta} \left[\int_t^T e^{-r(s-t-dt)} C(X_s, M_s) ds - e^{-r(t'-t-dt)} K \,\middle|\, \mathcal{F}_{t+dt} \right] \,\middle|\, \mathcal{F}_t \right] \right\} \\
= \; & \max \{ W_t - K, \; -X_t M_t dt \\
& \left. + e^{-rdt} \max_{t' \geq t+dt} \min_{\theta \in \Theta} E^{Q^\theta} \left[\min_{\theta' \in \Theta} E^{Q^{\theta'}} \left[\int_t^T e^{-r(s-t-dt)} C(X_s, M_s) ds - e^{-r(t'-t-dt)} K \,\middle|\, \mathcal{F}_{t+dt} \right] \,\middle|\, \mathcal{F}_t \right] \right\} \\
= \; & \max \{ W_t - K, \; -X_t M_t dt
\end{aligned}
$$

[27] See Ethier and Kurtz (1986) and Nishimura and Ozaki (2003) in details.

$$
\begin{aligned}
&+e^{-rdt}\min_{\theta\in\Theta}E^{Q^{\theta}}\left[\max_{t'\geq t+dt}\min_{\theta'\in\Theta}E^{Q^{\theta'}}\left[\int_t^T e^{-r(s-t-dt)}C(X_s,M_s)ds-e^{-r(t'-t-dt)}K\,\middle|\,\mathcal{F}_{t+dt}\right]\,\middle|\,\mathcal{F}_t\right]\Bigg\}\\
=&\ \max\left\{W_t-K,\ -X_tM_tdt+e^{-rdt}\min_{\theta\in\Theta}E^{Q^{\theta}}\left[V_{t+dt}|\,\mathcal{F}_t\right]\right\}\\
=&\ \max\left\{W_t-K,\ -X_tM_tdt+(1-rdt)\left(\min_{\theta\in\Theta}E^{Q^{\theta}}[dV_t|\,\mathcal{F}_t]+V_t\right)\right\}\\
=&\ \max\left\{W_t-K,\ -X_tM_tdt+\min_{\theta\in\Theta}E^{Q^{\theta}}[dV_t|\,\mathcal{F}_t]+V_t-rV_tdt\right\},
\end{aligned}
$$

where the first equality follows from the definition of V_t, the third equality follows from (32), the fourth follows from the definition of $\mathcal{P}^{\Theta}$, the fifth holds by the law of iterated integrals, the sixth follows from the strong rectangularity (41), the eighth follows from the definition of V_t, the ninth holds by approximating e^{-rdt} by $(1-rdt)$, and the last equality holds by eliminating higher order terms than dt. □

Derivation of V_t under Knightian uncertainty

By Ito's lemma,

$$
\begin{aligned}
dV_t &= \frac{\partial V_t}{\partial X_t}dX_t+\frac{1}{2}\frac{\partial^2 V_t}{\partial X_t^2}dX_t^2+\frac{\partial V_t}{\partial M_t}dM_t\\
&= \frac{\partial V_t}{\partial X_t}\left((\alpha-\sigma\theta_t)X_tdt+\sigma X_tdB_t^{\theta}\right)+\frac{1}{2}\frac{\partial^2 V_t}{\partial X_t^2}\sigma^2X_t^2dt+\frac{\partial V_t}{\partial M_t}\left(\beta E_0-\delta M_t\right)dt.
\end{aligned}
$$

Thus we can show that

$$
\begin{aligned}
&\min_{Q\in\mathcal{P}^{\Theta}}E^{Q}\left[dV_t|\mathcal{F}_t\right]\\
=&\ \min_{Q\in\Theta^K}E^{Q}\left[dV_t|\mathcal{F}_t\right]\\
=&\ \min_{\theta\in\Theta^K}E^{Q^{\theta}}\left[\frac{\partial V_t}{\partial X_t}\left((\alpha-\sigma\theta_t)X_tdt+\sigma X_tdB_t^{\theta}\right)+\frac{1}{2}\frac{\partial^2 V_t}{\partial X_t^2}\sigma^2X_t^2dt+\frac{\partial V_t}{\partial M_t}\left(\beta E_0-\delta M_t\right)dt\,\middle|\,\mathcal{F}_t\right]\\
=&\ \min_{\theta\in\Theta^K}\frac{\partial V_t}{\partial X_t}(\alpha-\sigma\theta_t)X_tdt+\frac{1}{2}\frac{\partial^2 V_t}{\partial X_t^2}\sigma^2X_t^2dt+\frac{\partial V_t}{\partial M_t}\left(\beta E_0-\delta M_t\right)dt\\
=&\ \frac{\partial V_t}{\partial X_t}\max_{\theta\in\Theta^K}(\alpha-\sigma\theta_t)X_tdt+\frac{1}{2}\frac{\partial^2 V_t}{\partial X_t^2}\sigma^2X_t^2dt+\frac{\partial V_t}{\partial M_t}\left(\beta E_0-\delta M_t\right)dt\\
=&\ \frac{\partial V_t}{\partial X_t}\left(\alpha+\max_{\theta\in\Theta^K}(-\sigma\theta_t)\right)X_tdt+\frac{1}{2}\frac{\partial^2 V_t}{\partial X_t^2}\sigma^2X_t^2dt+\frac{\partial V_t}{\partial M_t}\left(\beta E_0-\delta M_t\right)dt\\
=&\ \frac{\partial V_t}{\partial X_t}\left(\alpha-\min_{\theta\in\Theta^K}(\sigma\theta_t)\right)X_tdt+\frac{1}{2}\frac{\partial^2 V_t}{\partial X_t^2}\sigma^2X_t^2dt+\frac{\partial V_t}{\partial M_t}\left(\beta E_0-\delta M_t\right)dt\\
=&\ \frac{\partial V_t}{\partial X_t}(\alpha-\sigma\theta_*)X_tdt+\frac{1}{2}\frac{\partial^2 V_t}{\partial X_t^2}\sigma^2X_t^2dt+\frac{\partial V_t}{\partial M_t}\left(\beta E_0-\delta M_t\right)dt,
\end{aligned}
$$

where the first equality follows from the assumption of the i.i.d. uncertainty, the fourth equality holds by the negativity of $\partial V_t/\partial X_t$ in the continuation region, and the last equality follows from the definition of θ_*.

Therefore, in the continuation region, it follows that

$$\frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 V_t}{\partial X_t^2} + (\alpha - \sigma\theta_*)X_t \frac{\partial V_t}{\partial X_t} - rV_t + \frac{\partial V_t}{\partial M_t}(\beta E_0 - \delta M_t) - X_t M_t = 0,$$

with the following boundary conditions:

$$\begin{aligned}
&V_t(0, M_t) = 0, \\
&V_t(X^*, M_t) = W_t - K, \text{ and} \\
&\frac{\partial V_t}{\partial X_t}(X^*, M_t) = \frac{\partial W_t}{\partial X_t}(X^*, M_t).
\end{aligned}$$

We guess the solution to this equation as follows:

$$V_t = AX_t^\gamma + BX_t M_t + DX_t,$$

where A, B and D are some constants. Then

$$\begin{aligned}
&\frac{1}{2}\sigma^2 X_t^2 A\gamma(\gamma-1)X_t^{\gamma-2} + (\alpha - \sigma\theta_*)\left(A\gamma X_t^\gamma + BM_t X_t + DX_t\right) \\
&-r\left(AX_t^\gamma + BX_t M_t + DX_t\right) - X_t M_t + (\beta E_0 - \delta M_t)BX_t = 0. \\
\Leftrightarrow\quad & AX_t^\gamma\left(\frac{1}{2}\sigma^2\gamma(\gamma-1) + (\alpha - \sigma\theta_*)\gamma - r\right) \\
&+((\alpha - \sigma\theta_* - r - \delta)B - 1)X_t M_t + (\beta B E_0 + D(\alpha - \sigma\theta_* - r))X_t = 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{2}\sigma^2\gamma(\gamma-1) + (\alpha - \sigma\theta_*)\gamma - r = 0 \\
&(\alpha - \sigma\theta_* - r - \delta)B - 1 = 0 \Leftrightarrow B = -\frac{1}{r + \delta - \alpha + \sigma\theta_*} \\
&\beta B E_0 + D(\alpha - \sigma\theta_* - r) = 0 \Leftrightarrow D = -\frac{\beta E_0}{(r - (\alpha - \sigma\theta_*))(r + \delta - (\alpha - \sigma\theta_*))}.
\end{aligned}$$

The value A remains to be determined. By the boundary conditions, the negative part of the solution to $(1/2)\sigma^2\gamma(\gamma-1) + (\alpha - \sigma\theta_*)\gamma - r = 0$ is ruled out. Note that

$$\gamma \quad = \quad \frac{-(\alpha - \sigma\theta_* - \sigma^2/2) + \sqrt{(\alpha - \sigma\theta_* - \sigma^2/2)^2 + 2r\sigma^2}}{\sigma^2}$$

$$
\begin{aligned}
&> \frac{-(\alpha-\sigma\theta_*-\sigma^2/2)+\sqrt{(\alpha-\sigma\theta_*-\sigma^2/2)^2+2(\alpha-\sigma\theta_*)\sigma^2}}{\sigma^2} \\
&= \frac{-(\alpha-\sigma\theta_*-\sigma^2/2)+|\alpha-\sigma\theta_*+\sigma^2/2|}{\sigma^2} \\
&= 1, \qquad (38)
\end{aligned}
$$

where the inequality holds by the assumption that $r > \alpha + \sigma\kappa$, and the last equality holds since α, $\sigma > 0$ and $\theta_* = -\kappa < 0$, which implies that $|\alpha - \sigma\theta_* + \sigma^2/2| = \alpha - \sigma\theta_* + \sigma^2/2$. Then

$$
\begin{aligned}
A &= \left(\frac{K}{\gamma-1}\right)^{1-\gamma} \gamma^{-\gamma} \left(\frac{\beta E_0}{(r-(\alpha-\sigma\theta_*))(r+\delta-(\alpha-\sigma\theta_*))}\right)^{\gamma} \\
X^* &= \left(\frac{\gamma K}{\gamma-1}\right)\left(\frac{(r-(\alpha-\sigma\theta_*))(r+\delta-(\alpha-\sigma\theta_*))}{\beta E_0}\right).
\end{aligned}
$$

Thus

$$
V_t = \begin{cases} AX_t^{\gamma} - \dfrac{X_t M_t}{r+\delta-(\alpha-\sigma\theta_*)} - \dfrac{\beta E_0 X_t}{(r-(\alpha-\sigma\theta_*))(r+\delta-(\alpha-\sigma\theta_*))} & \text{if} \quad X_t < X^* \\ W_t - K & \text{if} \quad X_t \geq X^*. \end{cases}
$$

Therefore, the derivation is completed. □

Proof of $\partial\gamma/\partial\sigma^2 < 0$ and $\partial\gamma/\partial\kappa < 0$.

Let

$$
Q_1(\gamma) = \frac{1}{2}\sigma^2\gamma(\gamma-1) + \alpha\gamma - r.
$$

Note that $Q_1(1) = \alpha - r < 0$, $Q_1(0) = -r < 0$, and $\gamma_1 > 1$ by (16), where γ_1 is the positive part of this quadratic equation. By differentiating this quadratic equation totally with respect to σ^2, it follows that

$$
\frac{\partial Q_1}{\partial \gamma_1}\frac{\partial \gamma_1}{\partial \sigma^2} + \frac{\partial Q_1}{\partial \sigma^2} = 0.
$$

$\partial Q_1/\partial\sigma^2 = (1/2)\gamma(\gamma-1) > 0$ and $\partial Q_1/\partial\gamma_1 > 0$ at $\gamma_1 > 1$ imply

$$
\frac{\partial \gamma_1}{\partial \sigma^2} < 0. \qquad (39)
$$

Let

$$Q_2(\gamma) \equiv \frac{1}{2}\sigma^2\gamma(\gamma-1) + (\alpha+\sigma\kappa)\gamma - r = 0.$$

Note that $Q_2(1) = \alpha - r + \sigma\kappa < 0$,[28] $Q_2(0) = -r < 0$, and $\gamma_1 > 1$ by (38), where γ_1 is the positive part of this quadratic equation. By differentiating this quardratic equation totally with respect to κ, it follows that

$$\frac{\partial Q_2}{\partial \gamma_1}\frac{\partial \gamma_1}{\partial \kappa} + \frac{\partial Q_2}{\partial \kappa} = 0.$$

$\partial Q_2/\partial\kappa = \sigma\gamma > 0$ at $\gamma_1 > 1$ and $\partial Q_2/\partial\gamma_1 > 0$ at γ_1 imply

$$\frac{\partial \gamma_1}{\partial \kappa} < 0. \tag{40}$$

Thus, the proof is completed. □

Proof of the negativity of $\partial V_t/\partial X_t$

In order to show that $\partial V_t/\partial X_t < 0$ in the continuation region, it suffices to show that in the continuation region,

$$A\gamma X_t^{\gamma-1} - \frac{\beta E_0}{(r-(\alpha+\sigma\kappa))(r+\delta-(\alpha+\sigma\kappa))} - \frac{M_t}{r+\delta-(\alpha+\sigma\kappa)} < 0,$$

where A is defined by

$$A = \left(\frac{K}{\gamma-1}\right)^{1-\gamma}\gamma^{-\gamma}\left(\frac{\beta E_0}{(r-(\alpha-\sigma\theta_*))(r+\delta-(\alpha-\sigma\theta_*))}\right)^{\gamma}.$$

The inequality is proved as follows:

$$\begin{aligned}
& A\gamma X_t^{\gamma-1} - \frac{\beta E_0}{(r-(\alpha+\sigma\kappa))(r+\delta-(\alpha+\sigma\kappa))} - \frac{M_t}{r+\delta-(\alpha+\sigma\kappa)} \\
< \;& A\gamma (X^*)^{\gamma-1} - \frac{\beta E_0}{(r-(\alpha+\sigma\kappa))(r+\delta-(\alpha+\sigma\kappa))} - \frac{M_t}{r+\delta-(\alpha+\sigma\kappa)} \\
= \;& -\frac{M_t}{r+\delta-(\alpha+\sigma\kappa)} < 0
\end{aligned}$$

for any $X_t < X^*$, $\gamma > 1$, $\delta > 0$, and $r > \alpha + \sigma\kappa$. Furthermore,

$$\frac{\partial^2 V_t}{\partial X_t^2} = A\gamma(\gamma-1)X_t^{\gamma-2} > 0$$

for any $\gamma > 1$, $\delta > 0$, and $r > \alpha+\sigma\kappa$. Thus, the proof is completed. □

[28] We need to assume that $r > \alpha + \sigma\kappa$.

Proof of Proposition 2

By (8), it follows that $\partial W_t/\partial\sigma^2 = 0$.

Define V_t' as follows:

$$V_t' \equiv \left(\frac{K}{\gamma-1}\right)^{1-\gamma} \gamma^{-\gamma} \left(\frac{\beta E_0}{(r-\alpha)(r+\delta-\alpha)}\right)^{\gamma} X_t^{\gamma}.$$

By taking the logarithm of both sides of the equation and differentiating both sides of the equation with respect to γ, it follows that

$$\begin{aligned}
&\frac{\partial(\ln V_t')}{\partial\gamma} \\
= &\ -(\ln K - \ln(\gamma-1)) + (1-\gamma)(-\frac{1}{\gamma-1}) - \ln\gamma - \gamma\frac{1}{\gamma} + \ln\left(\frac{\beta E_0}{(r-\alpha)(r+\delta-\alpha)}\right) + \ln X_t \\
< &\ \ln\left(\frac{\gamma-1}{K\gamma}\right) + \ln\left(\frac{\beta E_0}{(r-\alpha)(r+\delta-\alpha)}\right) + \ln X_t^* \\
= &\ 0.
\end{aligned}$$

Recall that $\partial\gamma/\partial\sigma^2 < 0$ by (39). Thus

$$\frac{\partial V}{\partial\sigma^2} = \frac{\partial V}{\partial\gamma}\frac{\partial\gamma}{\partial\sigma^2} > 0.$$

Moreover, it follows that

$$\frac{\partial X^*}{\partial\gamma} = -\frac{K}{(1-\gamma)^2}\left(\frac{(r-\alpha)(r+\delta-\alpha)}{\beta E_0}\right) < 0.$$

Thus

$$\frac{\partial X^*}{\partial\sigma^2} = \frac{\partial X^*}{\partial\gamma}\frac{\partial\gamma}{\partial\sigma^2} > 0,$$

which completes the proof. □

Proof of Proposition 3

It follows from (32) that $\partial W_t/\partial\kappa < 0$.

By differentiating X^* with respect to κ, it follows that

$$\frac{\partial X^*}{\partial\kappa}$$

$$= \quad -\frac{K}{\beta E_0(\gamma-1)}\left(\frac{(r-(\alpha+\sigma\kappa))(r+\delta-(\alpha+\sigma\kappa))}{\gamma-1}\frac{\partial\gamma}{\partial\kappa}+2\sigma\kappa\left(2r+\delta-2(\alpha+\sigma\kappa)\right)\right).$$

Thus, $\partial X^*/\partial\kappa < 0$ if

$$(0>)\,\frac{\partial\gamma}{\partial\kappa} > -\frac{2\sigma\gamma(\gamma-1)(2r+\delta-2(\alpha+\sigma\kappa))}{(r-(\alpha+\sigma\kappa))(r+\delta-(\alpha+\sigma\kappa))}.$$

Note that the negativity of $\partial\gamma/\partial\kappa$ follows from (40). □

Appendix 2

Let $0 \leq s \leq t \leq T$, let x be an $\mathcal{F}_T$-measurable function, and let Θ be rectangular. Then,

$$\begin{aligned}\min_{\theta\in\Theta} E^{Q^\theta}\left[x|\,\mathcal{F}_s\right] &= \min_{\theta\in\Theta} E^{Q^\theta}\left[E^{Q^\theta}\left[x|\mathcal{F}_t\right]\,\middle|\,\mathcal{F}_s\right] \\ &= \min_{\theta\in\Theta} E^{Q^\theta}\left[\min_{\theta'\in\Theta} E^{Q^{\theta'}}\left[x|\mathcal{F}_t\right]\,\middle|\,\mathcal{F}_s\right], \qquad (41)\end{aligned}$$

where the first equality follows from the law of iterated integral. The second equality follows from Lemma 4 below. See Nishimura and Ozaki (2003) in details.

Let θ be a density generator, let (z_t^θ) be defined by

$$(\forall t\in[0,T])\ z_t^\theta = \exp\left(-\frac{1}{2}\int_0^t \theta_s^2 ds - \int_0^t \theta_s dB_s\right),$$

and define the measure Q_t^θ by

$$(\forall t\in[0,T])(\forall A\in\mathcal{F}_T)\ Q_t^\theta(A) = \int_A z_t^\theta dP.$$

Lemma 2. *Q_t^θ is a probability measure satisfying*

$$(\forall A\in\mathcal{F}_t)\ \ Q_t^\theta(A) = Q^\theta(A),$$

where Q^θ is defined by

$$(\forall A\in\mathcal{F}_T)\ Q^\theta(A) = \int_A z_T^\theta(\omega)dP(\omega).$$

Lemma 3. *Let $0\leq s\leq t\leq T$ and let x be an $\mathcal{F}_t$-measurable function. Then $E^{Q^\theta}[x|\,F_s]$ depends only on $(\theta_u)_{s\leq u<t}$.*

These lemmas together with the assumption that Θ is strongly rectangular prove the following lemma.

Lemma 4. *Let $0 \leq s \leq t \leq T$ and let x be an $\mathcal{F}_t$-measurable function. Also assume that Θ is strongly rectangular. Then under the assumption that the minima exist,*

$$\min_{\theta \in \Theta} E^{Q^\theta} \left[E^{Q^\theta} \left[x \,|\, \mathcal{F}_t \right] \,\middle|\, \mathcal{F}_s \right] = \min_{\theta \in \Theta} E^{Q^\theta} \left[\min_{\theta' \in \Theta} E^{Q^{\theta'}} \left[x \,|\, \mathcal{F}_t \right] \,\middle|\, \mathcal{F}_s \right].$$

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