Discussion Paper No. 645

OPTIMAL SALES SCHEMES AGAINST INTERDEPENDENT BUYERS

Masaki Aoyagi

October 2005

The Institute of Social and Economic Research
Osaka University
6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan

Optimal Sales Schemes against Interdependent Buyers

Masaki Aoyagi*
Osaka University

First draft: October 27, 2005

Abstract

This paper studies a monopoly pricing problem when the seller can also choose the timing of a trade with each buyer endowed with private information about the seller's good. A buyer's valuation of the good is the weighted sum of his and other buyers' private signals, and is affected by the publicly observable outcomes of preceding transactions. We show that it is optimal for the seller to employ a *sequential sales scheme* in which trading with the buyers takes place one by one. Furthermore, when the degree of interdependence differs across buyers, we analyze how the optimal sales scheme orders them, and how it may induce herding among them.
Key words: timing, monopoly pricing, information revelation, linkage principle, social learning.
Journal of Economic Literature Classification Numbers: C72, D82.

*ISER, Osaka University, 6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan.

1 Introduction

When a monopolist seller of a good trades with multiple buyers, he often employs a dynamic sales strategy. That is, instead of serving the entire market at once, we often observe the seller partition the market and serve different segments at different timings. For example, in automobile and electronics industries, firms often launch a new product in one country, introduce the product in another country only after the success in the first country is well publicized.[1] Similar observations hold for movie and other entertainment industries, where success in one market (say in U.S.) is a key to promotion in another.

There is perhaps more than one reason why a seller may employ such a dynamic sales strategy. For example, it may simply be the case that the seller is constrained by his physical resource to a particular segment of the market at one time. In many cases, however, we believe that it is based on more strategic motives. For example, through experimental sales to a small group of consumers, the seller may wish to learn about consumer characteristics or the marketability of his product. He may also target some subset of consumers in the expectation that they provide positive referrals for his product to other consumers through word-of-mouth communication. Marketing theory advocates that when a product is in the introductory stage of its life-cycle, the firm should target a small segment of the market that is willing to accept a new innovation and has an influence on the behavior of other consumers.

In this paper, we take the interpretation that the seller adopts a dynamic strategy in order to take advantage of the interdependence of buyer valuations. More specifically, when buyers' valuations of the seller's good are determined in part by the publicly observable behavior of other buyers, we analyze whether the seller is better off trading with different buyers at different timings. When successful, such a trading strategy can create a chain of positive events: successful transactions with the initial set of buyers raise the valuations of the next line of buyers, success with the latter raises even further the valuation of the buyers to follow, and so on. Once in such a cycle, the seller can continually increase his offer price and raise more revenue than from static sales. Of course, the seller adopting such a scheme also faces the risk of a downward spiral where a failure in the initial markets leads to a sequence of failures in subsequent markets.

In our model of dynamic trading, a seller faces multiple buyers each endowed

[1] One recent example is Toyota's introduction of the Lexus brand to Japan after its well-publicized success in the U.S.

with private information about the seller's good. Each buyer demands one unit of the good, which is produced at no cost to the seller. The private signals are independent across buyers and a buyer's valuation of the seller's good is a weighted sum of all buyers' signals. As in the classical monopoly pricing problem, the seller's trading with each buyer takes the form of a take-it-or-leave-it offer. The outcomes of transactions are publicly observable to subsequent buyers, and form the basis for the expected value of the good to them. Each buyer meets the seller once and leaves the market after accepting or rejecting the offer.

The nature of the problem can be best illustrated in a model where there are only two buyers. The seller can either trade with both at once or trade with one of them first and the other next. In the first scheme, referred to here as a *simultaneous scheme,* the seller provides the buyers no opportunity to learn about each other's private signals. In other words, each buyer's valuation depends only on his own signal and the unconditional expectation of the other buyer's signals. In the second scheme, referred to as a *sequential scheme,* the seller allows the second buyer to infer the private signal of his predecessor: Acceptance by the first buyer raises the second buyer's valuation, while rejection lowers it. It should be noted that the exact amount by which the second buyer's valuation changes depends on the level of the price offer to the first buyer: If the first buyer accepts a high price, then there will be a considerable increase in the second buyer's valuation, while if the first buyer accepts a low price, then the increase in the valuation will be small. In this sense, the seller's price offer to the first buyer determines not only his expected revenue in stage 1, but also the parameters of his maximization problem in stage 2. If the two buyers are not ex ante identical, then the seller must also choose which buyer to serve first. With three or more buyers, the seller's problem is similar but significantly more complex. First, besides sequential and simultaneous schemes, there are a number of intermediate schemes in which trading takes place in multiple stages. Second, the choice of buyers at each stage can be contingent on the history of transactions. For example, the seller may want to trade with either buyer 2 or buyer 3 in stage 2 depending on whether his transaction with buyer 1 in stage 1 is a success or not. Unlike in the two-buyer model, hence, the seller's current price offer may in general affect the contingent choice of buyers in future stages.

The first main conclusion of the paper is that it is optimal for the seller to employ a sequential scheme. The conclusion is based on the construction of a sequential scheme that replicates any given non-sequential scheme. In the two-buyer model, for example, given any simultaneous scheme, we can construct a sequential scheme that raises the same expected revenue. Suppose that the seller orginally employs a simultaneous scheme which offers price x_1 to buyer 1 and x_2 to buyer 2. The alternative sequential scheme offers x_1 to buyer 1 in period 1, and makes contingent offers to buyer 2 in period 2. In particular, the seller offers player 2 a higher price when 1 accepts his offer and a lower price otherwise. The price is adjusted so that buyer 2 accepts the contingent offers with exactly the same probability as he accepts x_2 under the original scheme. The choice of such contingent offers is possible since buyer 2's valuation shifts up or down by a deterministic amount as a result of the period 1 outcome under our assumptions that the valuation function is additive and the private signals are independent. The key is to show that those contingent offers yield the same expected revenue as x_2.

An alternative interpretation of the present model is from the perspective of the seller's information revelation policy. Suppose that the outcome of past transactions is not publicly observable but can be revealed by the seller to each buyer before trading. The sequential scheme of the present model may be identified as the full revelation policy which reveals information about all past transactions to every buyer, while the simultaneous scheme can be identified as the no revelation policy that conceals the information by making price offers non-contingent on past outcomes. In this context, it would be useful to compare the suggested optimality of the full revelation policy in the present paper with the well-known linkage principle in auction theory (Milgrom and Weber (1982)). The principle says that the auctioneer's expected revenue is maximized when he commits to fully revealing his private information before bidding takes place provided that the bidders' private signals are affiliated with one another and that of the auctioneer. The linkage principle also claims the superiority of an English auction, which publicly releases the bidders' private information through their bidding, over a sealed-bid second-price auction. In the sense that the seller's payoff is higher when he uses a mechanism which reveals buyers' private information through their actions, the latter version of the principle bears much resemblance to the conclusion of this paper. It should be noted, however, there is no formal connection between the two, which are based on entirely different logics. To see that direct application of the linkage principle is not possible, it should be pointed out that the principle fails outside the symmetric single-unit auction environment. For example, in the twice-repeated common value auctions, de-Frutos and Rosenthal (1998) show that the auctioneer is better off not revealing bids in the first auction to the bidders in the second auction. The principle may also fail in auctions with asymmetric bidders (Krishna (2002, Ch. 8)), or in multi-unit auctions (Perry and Reny (1999)). The problem of information revelation by a privately informed seller is also studied by Milgrom (1981) and Ottaviani and Prat (2001) in alternative frameworks: For example, Ottaviani and Prat (2001) show in their monopoly pricing model against a single buyer that the optimal strategy of the monopolist is to fully reveal his private information before trading. Aoyagi (2005) points out in a model of a dynamic tournament with a privately informed organizer that the optimal degree of information feedback to the contestants may subtly depend on the parameters of the model.

It is also possible to interpret a sequential sales scheme of the present paper as a generalization of models of social learning as studied by Bikchandhani *et al.* (2000).[2] While models of social learning typically assume correlated private signals about the underlying common value, it is easy to see that cascading does take place in the present environment with independent signals and interdependent valuations. In other words, when the first n buyers all make the same decision, the remaining buyers follow suit irrespective of their signals. The present model departs from the classical setup by making the cost of adopting the alternative (*i.e.,* price of the good) and the buyer sequence both endogenously determined by the revenue maximizing seller.[3] Instead of offering a constant price, for example, the seller will likely raise his price after a series of successful transactions at some price x, but will lower it after a series of failures. Furthermore, with idiosyncratic buyers, the buyer ordering will be determined in a history-contingent manner and is not fixed. As discussed below, we find that the seller's revenue maximizing decisions may actually induce more herding among buyers than in the classical setup.

Given the optimality of a sequential scheme, the second question we address is on the optimal ordering of buyers. Specifically, we suppose that the buyers' private signals have the same distribution, and that their valuation places a single common weight on the signals of all other buyers. The buyers differ from one another only in those weights, which measure how dependent their valuations are on others' information. We provide sufficient conditions under which the optimal sales scheme first trades with the buyer with the smallest weight, then with the buyer with

[2]See also Banerjee (1999).

[3]See Chamley (2004, Chap. 4) for an alternative model of social learning with an endogenously determined price and *ex ante* identical buyers with private information about the common value of the good.

the second smallest weight, and so on until it reaches the last buyer who has the largest weight. According to such a scheme, hence, the buyers who are more heavily influenced by public information about past transactions are placed towards the end of the sequence and given the most opportunity to observe such information. Note also that under the conditions, the optimal buyer sequence is not contingent on the history of transactions.

Because of the technical difficulty, further analysis of the optimal sales scheme is conducted under the additional assumption that the buyers' private signals have a uniform distribution. In this setup, we provide an explicit characterization of the optimal pricing strategy along any fixed sequence of buyers. This characterization can then be used to verify the sufficient conditions for the optimality of the monotone sequence of weights as described above. The analysis also highlights the herd-inducing property of the optimal scheme. Specifically, it can be shown that in the optimal scheme, the probability with which the seller's offer is accepted keeps increasing as long as all the previous transactions are successful, and keeps decreasing as long as all the previous transactions are unsuccessful. Furthermore, the probability that any buyer accepts his offer is maximized when all his predecessors accept their offers, and is minimized when all his predecessors reject the offers. This implies that not only are the buyers who are more heavily influenced by others' behavior placed late in the sequence to observe more information, but also are they induced to take the same action as their predecessors. This finding may be interpreted as providing one further explanation for the frequent occurrence of consumption fads in some markets, and also a theoretical foundation for the aforementioned marketing strategy that first targets the consumers whose behavior influences the decisions of other consumers.

It should be noted that the time dimension introduced in this paper is a device to sort out informational events and does not entail discounting and depreciation. For this reason, we do not discuss issues related to durable good monopoly such as possible delay in transactions associated with the Coase conjecture. Also implicit in our assumption is that the seller cannot gain by creating the scarcity of his good by limiting its supply. Under some parametrization, for example, the seller may be able to raise a higher revenue by producing only half as many units of the good as the number of the buyers and hold an auction. We do not consider such possiblities by supposing that exclusion of any buyers results in lower profits.[4]

[4]That is, the implicit assumption is that the support of buyer valuation is such that selling at

The paper is organized as follows: The next section formulates a model of monopoly. In Section 3, we present an example and some preliminary results that are used extensively in the subsequent analysis. Section 4 proves the optimality of a sequential scheme. In Section 5, we present sufficient conditions for the optimal sequential scheme to entail a monotone ordering of the dependence weights as described above. Section 6 provides an explicit characterization of the optimal sales scheme when the signal distribution is uniform, and demonstrates the herd-generating property of such a scheme. We conclude with a discussion in Section 7.

2 Model

A seller of a good faces the set $I = \{1, \ldots, I\}$ of I buyers each of whom has private information about the valuation of the good.[5] Let s_i denote buyer i's private signal. We assume that $s_1, \ldots, s_n$ are independent and identically distributed over the set $\boldsymbol{R}_+$ of non-negative real numbers. Let μ_i be the mean value of s_i. When $s = (s_1, \ldots, s_n)$ denotes a signal profile, buyer i's valuation of a single unit of the seller's good is given by

$$v_i(s) = c_{i0} + c_{ii}s_i + \sum_{j \neq i} c_{ij}(s_j - \mu_j),$$

where $c_{ii} > 0$, and $c_{ij} \geq 0$ $(j \neq 0,\ i)$. In other words, the valuations are *linearly interdependent,* and buyer i places weight c_{ij} on buyer j's signal.[6] As seen below, subtraction of the mean μ_j from s_j for every $j \neq i$ simplifies the representation of the expected valuation.[7]

We normalize the marginal cost of producing the good to zero, and assume that every buyer demands at most one unit. As discussed in the Introduction, an alternative interpretation of each buyer in this model is that they represent a segment of the market that is difficult to futher break down and consists of individuals who have the same taste and information about the seller's good.

the minimum price to every buyer is more profitable than selling at the maximum price to any proper subset of them.

[5]Note that I represents both the set of buyers and its cardinality.

[6]Note that the linear specification of the valuation function is common in the auction literature since the work of Myerson (1981). It can also be interpreted as an approximation to a more general valuation function.

[7]The optimality of a sequential scheme in Section 4 holds as is without the subtraction of μ_j. For the conclusions in Sections 5 and 6, proper adjustment in the constant term is required.

The seller *trades* with each buyer by offering him a price. The buyer then accepts or rejects the price offer and leaves the market. The price offered to each buyer and their response to the offer are both publicly observable.

In every period, the seller chooses which buyers to make offers and what prices to offer to them as a function of past trades. Formally, denote by $I_t \subset I$ the set of buyers who are made offers in period t. An *outcome* y_t in period t is a partition (A_t, B_t) of the set I_t: A_t represents the set of buyers who have accepted their offers, and B_t represents those who have rejected their offers. For any subset J of buyers, let $Y(J)$ denote the set of possible outcomes from the set J of buyers. In other words, $Y(J)$ consists of all the two-way partitions of the set J including $(J, \emptyset)$ and $(\emptyset, J)$. A *history* of length t consists of the outcomes in periods $1, \dots, t$. Let H_t denote the set of possible histories of length t, and let $H = \cup_{t=0}^{\infty} H_t$ be the set of all possible histories, where H_0 is the singleton set of the null history. Given any history $h \in H$, we denote by $I(h)$ and $U(h) = I \setminus I(h)$ the set of buyers with whom the seller has and has not, respectively, traded along h.

A *sales scheme* of the seller, denoted σ, consists of a pair of mappings $r : H \to 2^I$ and $x = (x_i)_{i \in I} : H \to \boldsymbol{R}_+^I$: $r(h)$ specifies the subset of buyers the seller chooses for trading at histroy h, and $x_i(h)$ specifies the price he offers to buyer i at h. Note in particular that the seller's choice of the buyers in any period is contingent on the history. In a three-buyer model, for example, the seller chooses buyer 1 in period 1, and in period 2, he chooses buyer 2 if buyer 1 accepted his offer, but buyer 3 otherwise, *etc.* It should also be noted that the specification of the price $x_i(h)$ is relevant only for $i \in r(h)$. In order to eliminate the possibility of inaction in any period, we require that r chooses at least one buyer in every period until the list of buyers is exhausted: $r(h) \neq \emptyset$ if $U(h) \neq \emptyset$. This in particular implies that all the trading ends in or before period I. Let Σ be the set of all sales schemes. Two representative classes of sales schemes are the *simultaneous schemes* in which the seller trades with all the buyers at once (*i.e.,* $r(h) = I$ for $h \in H_0$), and the *sequential schemes* in which he trades one by one with each buyer (*i.e.,* $r(h) = \{i\}$ for some $i \in U(h)$ for each $h \in H_{t-1}$ and $t = 1, \dots, I$).

Given a sales scheme σ, let P^σ denote the joint probability distribution of the signal profile s and the history h induced by σ. Let E^σ be the expectation with respect to the distribution P^σ. We use P without the superscript to denote the marginal distribution of s that does not depend on the sales scheme, and E to denote the corresponding expectation. For any history $h \in H$, let

$$V_i^\sigma(s_i \mid h) = E^\sigma[v_i(s_i, \tilde{s}_{-i}) \mid h]$$

be the expected valuation of buyer i with signal s_i given history h. By assumption, it can be explicitly written as

$$V_i^\sigma(s_i \mid h) = c_{i0} + c_{ii}s_i + \sum_{j \in I(h)} c_{ij}\, E^\sigma[\tilde{s}_j - \mu_j \mid h].$$

Note that the summation above is over the set $I(h)$ of buyers who have already traded along h since any other term involves the unconditional expectation of the private signal and hence cancels out. Buyer i with signal s_i accepts the seller's offer x_i at history h if and only if the expected value of the good conditional on h is greater than or equal to x_i: $V_i(s_i \mid h) \geq x_i$. The seller's expected revenue under the sales scheme σ, denoted by $\pi(\sigma)$, is simply the sum of expected payments from the I buyers.

3 Preliminaries

In this section, we first present a simple example to illustrate the problem, and then provide some preliminary results that are key to much of the subsequent analysis.

Consider first the following example. There are two buyers whose private signals s_1 and s_2 both have the uniform distribution over the unit interval $[0, 1]$ with the means $\mu_1 = \mu_2 = 1/2$. Suppose also that their valuation functions are given by

$$v_1(s_1, s_2) = s_1 + c_1\Big(s_2 - \frac{1}{2}\Big) \quad \text{and} \quad v_1(s_1, s_2) = s_2 + c_2\Big(s_1 - \frac{1}{2}\Big),$$

where $0 < c_1 \leq c_2$. It can be seen that this is a special case of the general formulation in the previous section.

When the seller uses the simulataneous sales scheme, he will choose the price offers x_1 and x_2 so as to maximize $x_1 P(\tilde{s}_1 \geq x_1)$ and $x_2 P(\tilde{s}_2 \geq x_2)$, respectively. As is readily verified, the revenue maximizing prices equal $x_1 = x_2 = 1/2$ and the seller's expected payoff equals

$$\pi^0 = \frac{1}{4} \times 2 = \frac{1}{2}.$$

On the other hand, when the seller uses the sequential sales scheme that trades with buyers 1 and 2 in this order, he needs to solve a two-step optimization problem.

Consider first the problem in period 2 given the first period offer $x_1 \in [0,1]$. Let $h_1 = 1$ denote the history corresponding to buyer 1's acceptance, and $h_1 = 0$ denote the histroy corresponding to his rejection. Depending on h_1, buyer 2's valuation function is either

$$V_2(s_2 \mid 1) = s_2 + c_2 E\Big[\tilde{s}_1 - \frac{1}{2} \mid \tilde{s}_1 \geq x_1\Big] = s_2 + c_2\,\frac{x_1}{2},$$

or

$$V_2(s_2 \mid 0) = s_2 + c_2 E\Big[\tilde{s}_1 - \frac{1}{2} \mid \tilde{s}_1 < x_1\Big] = s_2 + c_2\,\frac{x_1 - 1}{2}.$$

The seller also has two prices to consider in period 2: $x_2(1)$ is the price offer to buyer 2 when buyer 1 accepted in period 1, and $x_2(0)$ is the offer to buyer 2 when buyer 1 rejected. The period 2 price offers hence solve

$$x_2(1) \in \arg\max_{x_2} x_2\, P\Big(V_2(\tilde{s}_2 \mid 1) \geq x_2\Big),$$

and

$$x_2(0) \in \arg\max_{x_2} x_2\, P\Big(V_2(\tilde{s}_2 \mid 0) \geq x_2\Big).$$

Upon substituting for V_2, we can solve these problems to obtain

$$x_2(1) = \frac{1}{2}\Big(1 + \frac{c_2}{4}\Big), \quad \text{and} \quad x_2(0) = \frac{1}{2}\Big(1 - \frac{c_2}{4}\Big).$$

Let $\pi_2(x_1 \mid 1)$ and $\pi_2(x_1 \mid 0)$ denote the optimized values of the period 2 expected payoffs after $h_1 = 1$ and $h_1 = 0$, respectively. They are given by

$$\pi_2(x_1 \mid 1) = \frac{1}{4}\Big(1 + \frac{c_2}{2}\,x_1\Big)^2, \quad \text{and} \quad \pi_2(x_1 \mid 0) = \frac{1}{4}\Big(1 + \frac{c_2}{2}\,(x_1 - 1)\Big)^2.$$

Using them, we can also write the seller's period 1 problem as:

$$\max_{x_1} P(\tilde{s}_1 \geq x_1)\Big\{x_1 + \pi_2(x_1 \mid 1)\Big\} + P(\tilde{s}_1 < x_1)\,\pi_2(x_1 \mid 0).$$

Solve this to get

$$x_1 = \frac{1}{2}.$$

These optimal prices can also be obtained from Theorem 6.1 in Section 6. The optimized value of the seller's overall expected payoff under the sequential sales scheme equals

$$\pi^{12} = \frac{1}{2} + \frac{c_2^2}{64}.$$

Likewise, when the seller uses the sequential scheme with the order of buyers 1 and 2 switched, his expected payoff is given by

$$\pi^{21} = \frac{1}{2} + \frac{c_1^2}{64}.$$

Given our assumptions on c_1 and c_2, we hence have the following ordering:

$$\pi^0 < \pi^{21} \leq \pi^{12}.$$

The above inequalities already hint the main conclusions of the paper: A sequential scheme performs better than the simultaneous scheme, and the optimal sequential scheme has an increasing sequence of dependence weights.

In a more general setting, it is not analytically feasible to make direct revenue comparisons of various schemes unlike above. For this reason, we take a different approach to the problem by examining how a local change in the given scheme affects the revenue. Given below is some preliminary analysis of the general model in this direction.

Consider a pair of sales schemes σ and σ', and suppose that a pair of histories h and h' are induced by σ and σ', respectively. Suppose that along these histories, the seller has traded with the same set of buyers with exactly the same outcomes. That is, the set of buyers who have accepted the seller's offers along h is the same as that along h' ($A(h) = A(h')$), and also the set of the buyers who have rejected the offers along h is the same as that along h' ($B(h) = B(h')$). The following lemma states that if, for every one of those buyers, the probability that he would have accepted the offer is the same under both schemes, then so are the valuation functions of subsequent buyers conditional on h and h'. Formally, given any sales scheme $\sigma = (r, x) \in \Sigma$ and any history $h \in H$, let

$$z_i^\sigma(h) = P\Big(V_i^\sigma(\tilde{s}_i \mid h) \geq x_i(h)\Big)$$

be the probability that buyer i accepts the seller's offer $x_i(h)$ given his valuation conditional on history $h \in H$.

Lemma 3.1. *Let $\sigma = (r, x)$ and $\sigma' = (r', x')$ be any sales schemes and h and h' be any histories induced by σ and σ', respectively, with the same set of buyers along them and the same outcomes (i.e., $A(h) = A(h')$ and $B(h) = B(h')$). For any buyer $j \in J \equiv I(h) = I(h')$, let h_j and h'_j denote the truncations of h and h', respectively,* *at which the seller chooses j: $r(h_j) = r'(h'_j) = j$. If $z_j^\sigma(h_j) = z_j^{\sigma'}(h'_j)$ for every $j \in J$, then for any $i \notin J$,*

$$V_i^\sigma(s_i \mid h) = V_i^{\sigma'}(s_i \mid h') \text{ for every } s_i.$$

Proof. Take any $j \in A(h) = A(h')$. Write

$$w_j^\sigma(h_j) = c_{j0} + \sum_{k \in I(h_j)} c_{jk}\, E^\sigma[\tilde{s}_k - \mu_k \mid h_j]$$

for the part of j's valuation under σ that is determined by the history h_j. Likewise, define $w_j^{\sigma'}(h'_j)$ to be the part of j's valuation under σ' that is determined by the history h'_j. Since $V_j^\sigma(s_j \mid h_j) = c_{jj}s_j + w_j^\sigma(h_j)$ and $V_j^{\sigma'}(s_j \mid h'_j) = c_{jj}s_j + w_j^{\sigma'}(h'_j)$, we have by assumption,

$$z_j^\sigma(h_i) = P\Big(c_{jj}\tilde{s}_j \geq x_j(h_j) - w_j^\sigma(h_j)\Big) = P\Big(c_{jj}\tilde{s}_j \geq x'_j(h'_j) - w_j^{\sigma'}(h'_j)\Big) = z_j^{\sigma'}(h'_j).$$

Hence

$$\begin{aligned} E^\sigma[\tilde{s}_j - \mu_j \mid h] &= E[\tilde{s}_j - \mu_j \mid V_j^\sigma(\tilde{s}_j \mid h_j) \geq x_j(h_j)] \\ &= E[\tilde{s}_j - \mu_j \mid c_{jj}\tilde{s}_j \geq x_j(h_j) - w_j^\sigma(h_j)] \\ &= E[\tilde{s}_j - \mu_j \mid c_{jj}\tilde{s}_j \geq x'_j(h'_j) - w_j^{\sigma'}(h'_j)] \\ &= E[\tilde{s}_j - \mu_j \mid V_j^{\sigma'}(\tilde{s}_j \mid h'_j) \geq x'_j(h'_j)] \\ &= E^{\sigma'}[\tilde{s}_j - \mu_j \mid h']. \end{aligned} \tag{1}$$

Likewise, for any $j \in B(h) = B(h')$, we have $E^\sigma[\tilde{s}_j - \mu_j \mid h] = E^{\sigma'}[\tilde{s}_j - \mu_j \mid h']$. Now take any buyer $i \notin J$ who comes after h or h'. Since for any s_i, $V_i^\sigma(s_i \mid h) = c_{i0} + c_{ii}s_i + \sum_{j \in J} c_{ij}\, E^\sigma[\tilde{s}_j - \mu_j \mid h]$ and $V_i^{\sigma'}(s_i \mid h') = c_{i0} + c_{ii}s_i + \sum_{j \in J} c_{ij}\, E^{\sigma'}[\tilde{s}_j - \mu_j \mid h']$, we conclude from the above that $V_i^\sigma(s_i \mid h) = V_i^{\sigma'}(s_i \mid h')$. ∎

Our next observation concerns the expected change in a buyer's valuation as a function of the decision of any other buyer who precedes him. For any sales scheme σ, history h, and buyer i that σ chooses at h, let

$$\begin{aligned} \kappa_i^\sigma(h) &= E[\tilde{s}_i - \mu_i \mid V_i^\sigma(\tilde{s}_i \mid h) \geq x_i(h)], \\ \lambda_i^\sigma(h) &= E[\tilde{s}_i - \mu_i \mid V_i^\sigma(\tilde{s}_i \mid h) < x_i(h)]. \end{aligned} \tag{2}$$

That is, $\kappa_i^\sigma(h)$ denotes the expected value of bidder i's private signal (minus its unconditional mean μ_i) when he accepted the seller's offer $x_i(h)$ at history h. Likewise, $\lambda_i(h)$ is the expected value of his private signal (minus μ_i) when he rejected the offer. It should be noted that for any buyer j that comes after i, $c_{ji}\,\kappa_i^\sigma(h)$ equals the change in his valuation when i accepts the offer, and $c_{ji}\,\lambda_i^\sigma(h)$ equals the change in his valuation when i rejects the offer. The following lemma simply says that conditional on h, the expected change in j's valuation is zero.

Lemma 3.2. *For any $\sigma = (r, x)$, $h \in H$, and $i \in I$ such that $r(h) = i$, $z_i(h)\kappa_i^\sigma(h) + (1 - z_i(h))\,\lambda_i^\sigma(h) = 0$.*

Proof.

$$
\begin{aligned}
& z_i(h)\kappa_i^\sigma(h) + (1 - z_i(h))\,\lambda_i^\sigma(h) \\
&= E\Big[(\tilde{s}_i - \mu_i)\, 1_{\{V_i^\sigma(\tilde{s}_i|h) \geq x_i(h)\}} + (\tilde{s}_i - \mu_i)\, 1_{\{V_i^\sigma(\tilde{s}_i|h) < x_i(h)\}}\Big] \\
&= 0.
\end{aligned}
$$

■

Lemma 3.2 also implies that $\kappa_i^\sigma(h) \geq 0$ and $\lambda_i^\sigma(h) \leq 0$. That is, every acceptance has a positive impact on a subsequent buyer's valuation, while every rejection has a negative impact. As seen in the next section, given any scheme that trades with buyers i and j simultaneously, Lemma 3.2 allows us to construct an alternative scheme that trades with them in sequence, but yields exactly the same revenue.

4 Sequential Sales Scheme

In this section, we show that the seller's expected payoff is maximized when he employs a sequential scheme.

Theorem 4.1. *The seller's expected revenue is maximized when he employs a sequential scheme: There exists a sequential sales scheme σ^* such that $\pi(\sigma^*) = \max_{\sigma\in\Sigma}\pi(\sigma)$.*

Proof. See the Appendix. ■

The proof of the theorem shows that given any non-sequential scheme σ, there exists a sequential scheme that performs at least as well as σ. Suppose for simplicity that σ induces some history $\underline{h} \in H_{n-1}$ ($n \geq 1$) at which it trades with two buyers i

and j. Let $x_j \equiv x_j(\underline{h})$ denote the price offer to buyer j under the original scheme. Consider the following alternative scheme $\sigma^* = (r^*, x^*)$: In period n at history $\underline{h}$, σ^* trades only with buyer i by offering the same price as under σ. In period $n+1$, σ^* trades with buyer j with the price offer adjusted according to the outcome of trade with buyer i. Specifically, the offer to j under σ^* equals $x_j + c_{ji}\,\kappa_i^\sigma(h)$ when buyer i accepted the offer, and it equals $x_j + c_{ji}\,\lambda_i^\sigma(h)$ when buyer i rejected the offer. Since buyer j's valuation changes by either $c_{ji}\,\kappa_i(h)$ or $c_{ji}\,\lambda_i(h)$ as a result of the period n outcome, the probability that he accepts the modified offer after each contingency is equal to the probability that he accepts the offer x_j at $\underline{h}$ under the original scheme. Specifically, if we denote by $(\underline{h}, 1) \in H_n$ the history under σ^* which takes place when i accepts the offer at $\underline{h}$, then

$$\begin{aligned}
z_j^*(\underline{h}, 1) &= P\Big(V_j^{\sigma^*}(\tilde{s}_j \mid \underline{h}, 1) \geq x_j + c_{ji}\,\kappa_i(\underline{h})\Big) \\
&= P\Big(c_{j0} + c_{jj}\tilde{s}_j + \sum_{k \in I(\underline{h})} c_{jk}\, E^\sigma[\tilde{s}_k - \mu_k \mid \underline{h}] + c_{ji}\,\kappa_i(\underline{h}) \geq x_j + c_{ji}\,\kappa_i(\underline{h})\Big) \\
&= P\Big(c_{j0} + c_{jj}\tilde{s}_j + \sum_{k \in I(\underline{h})} c_{jk}\, E^\sigma[\tilde{s}_k - \mu_k \mid \underline{h}] \geq x_j\Big) \\
&= P\Big(V_j^\sigma(\tilde{s}_j \mid \underline{h}) \geq x_j\Big) \\
&= z_j(\underline{h}).
\end{aligned}$$

Likewise, if $(\underline{h}, 0) \in H_n$ denotes the $n+1$-length history under σ^* which takes place when i rejects the offer at $\underline{h}$, then $z_j^*(\underline{h}, 0) = z_j(\underline{h})$. It then follows that the seller's expected revenue from buyer j under σ^* conditional on $\underline{h}$ is computed as

$$\begin{aligned}
&z_i(\underline{h})\, z_j(\underline{h})\, \{x_j + c_{ji}\,\kappa_i^\sigma(\underline{h})\} + (1 - z_i(\underline{h}))\, z_j(\underline{h})\, \{x_j + c_{ji}\,\lambda_i^\sigma(\underline{h})\} \\
&= z_j(\underline{h}) \Big[x_j + c_{ji} \Big\{z_i(\underline{h})\, \kappa_i^\sigma(\underline{h}) + (1 - z_i(\underline{h}))\, \lambda_i^\sigma(\underline{h})\Big\}\Big] \\
&= z_j(\underline{h}) x_j,
\end{aligned}$$

where the second equality follows from Lemma 3.2. Note that $z_j(\underline{h})x_j = z_j(\underline{h})x_j(\underline{h})$ is just the expected revenue from buyer j under the original scheme. It also follows from Lemma 3.1 that the valuation functions of all the buyers that come after $(\underline{h}, 1)$ or $(\underline{h}, 0)$ are the same as those under σ since regardless of i's decision, the probability that j accepts the offer under σ^* is the same under the original scheme. Therefore, if σ^* makes the same price offer to each of those buyers as σ, then the seller's expected revenue from them is just the same. As seen in the formal proof in the Appendix, these arguments generalize to the case where σ chooses more than two buyers at $\underline{h}$. Hence, if σ^* trades with more than one buyer in any period, we can repeatedly apply the above argument to conclude that there is a sequential scheme that yields the same expected revenue as σ.

5 Optimal Buyer Sequence

Given the optimality of sequential sales schemes demonstrated in the previous section, we now turn to the question of how the seller should order the buyers in such schemes. In particular, when the buyers differ in the weights they place on others' signals, how should the seller order them in terms of those weights? In order to focus on this question, we suppose in what follows that each buyer's private signal has a common distribution, and that his valuation $v_i(s)$ given the signal profile $s = (s_1, \ldots, s_I)$ equals

$$v_i(s) = c_0 + s_i + c_i \sum_{j \neq i} (s_j - \mu),$$

where μ is the common mean of s_j. That is, we set $c_{ii} = 1$, $c_{ij} = c_i$ for $j \neq i$, 0, and $c_{i0} = c_0$ in the general formulation of Section 2. Note that c_i is the only source of difference across buyers and is an unambiguous measure of the degree of dependence of buyer i's valuation on others' information.

Given a sequential sales scheme $\sigma = (r, x)$, we redefine $r(h)$ to be the buyer (an element of I) that r chooses at history h. We also express a history induced by σ as a sequence of 0's and 1's: At history $h \in H_{t-1}$, outcome 1 in period t implies that buyer $r(h)$ accepted the seller's offer and outcome 0 implies that he rejected it. For example, $(1, 0) \in H_2$ denotes the history induced by σ in which buyer $r(h_0)$ accepts the offer $x(h_0)$, and then buyer $r(1)$ rejects the offer $x(1)$. Likewise, given any history $h \in H_{t-1}$ induced by σ, $(h, 1)$ and $(h, 0) \in H_t$ represent the histories obtained by appending to h buyer $r(h)$'s acceptance and rejection, respectively, in period t.

Let a sales scheme $\sigma \in \Sigma$ be given. For any history $h \in H_{t-1}$, let

$$\alpha^\sigma(h) = \sum_{j \in I(h)} E^\sigma[\tilde{s}_j - \mu \mid h]$$

be the sum of the expected values of private signals (minus μ) conditional on history h. It can be readily verified that buyer i's valuation function conditional on history

h can be expressed as

$$V_i^\sigma(s_i \mid h) = c_0 + s_i + c_i\alpha^\sigma(h). \tag{3}$$

In the sense that $\alpha^\sigma(h)$ completely determines a buyer's valuation at h, it is referred to as the *state* at h. It should be noted that the transition of the state is described as follows when $r(h) = i$:

$$\alpha^\sigma(h, 1) = \alpha^\sigma(h) + E[\tilde{s}_i - \mu \mid V_i^\sigma(\tilde{s}_i \mid h) \geq x_i(h)] \quad \text{if } i \text{ accepts,}$$

and

$$\alpha^\sigma(h, 0) = \alpha^\sigma(h) + E[\tilde{s}_i - \mu \mid V_i^\sigma(\tilde{s}_i \mid h) < x_i(h)] \quad \text{if } i \text{ rejects.}$$

Since $E[\tilde{s}_i - \mu] = 0$, we have $E[\tilde{s}_i - \mu \mid V_i^\sigma(\tilde{s}_i \mid h) \geq x_i(h)] \geq 0 \geq E[\tilde{s}_i - \mu \mid V_i^\sigma(\tilde{s}_i \mid h) < x_i(h)]$. Hence the state variable goes up with every acceptance and goes down with every rejection. Furthermore, since the initial state is $\alpha_0 = 0$, the state remains positive as long as all previous transactions have been successful, and remains negative as long as they have all failed.

In what follows, we will focus on a class of sequential schemes in which the pricing function x satisfies certain monotonicity conditions. It will be shown in the next section that these conditions are indeed satisfied by the optimal pricing function when the signal distribution is uniform.

Formally, given any history $h \in H_{n-1}$ $(n \leq I-1)$, the selection function r is *non-contingent after* h if the buyers it chooses in periods $n+1, \ldots, I$ are independent of the outcomes in periods $n, \ldots, I-1$, *i.e.,* for any $n+1 \leq t \leq I$, there exists $r_t \in I$ such that for any sequence of outcomes $y_n, \ldots, y_{t-1}$ in periods $n, \ldots, t$,

$$r(h, y_n, \ldots, y_{t-1}) = r_t. \tag{4}$$

The selection function r is *non-contingent* if it is non-contingent after the null history. Intuitively, r is non-contingent if the target buyer in every period is known in advance. It should be noted that every r is non-contingent at any history $h \in H_{I-2}$ since then no matter what happens with buyer $r(h)$ in period $I-1$, the seller always chooses the only remaining buyer in period I.

Given any sales scheme $\sigma \in \Sigma$ and history $h \in H$, recall that $z_i(h) \equiv z_i^\sigma(h)$ denotes the probability that the seller's price offer $x_i(h)$ is accepted by buyer i at history $h \in H$. Define Σ^0 to be the class of sequential sales schemes such that

$$\Sigma^0 = \Big\{\sigma = (r,x) : \text{For any } h \in H \text{ induced by } \sigma, \text{ if } r \text{ is non-contingent after } h,\; r(h) = j, \text{ and } r(h,0) = r(h,1) = i, \text{ then } z_i(h,0) \le z_i(h,1), \text{ and } \alpha^\sigma(h)\Big[z_j(h) - z_j(h)z_i(h,1) - (1 - z_j(h))z_i(h,0)\Big] \ge 0\Big\}. \tag{5}$$

Note that both requirements concern the probabilities of acceptance at histories after which r is non-contingent. The first condition requires that the seller's contingent offer to i have a higher probability of acceptance when j has accepted his offer than when j has rejected it. As for the second requirement, note that $z_j(h)z_i(h,1) + (1 - z_j(h))z_i(h,0)$ is the expected probability conditional on h that buyer i accepts the seller's contingent offers. Hence, it says that the expected probability of acceptance increases as the seller moves from buyer j to buyer i when the state variable $\alpha^\sigma(h)$ is positive, and decreases when $\alpha^\sigma(h)$ is negative.

Now take any pair of buyers i and j such that buyer j is more dependent on others' signals than buyer i: $c_j \ge c_i$. The following lemma states that for any $\sigma \in \Sigma^0$, if r is non-contingent after some history h and trades with buyers j and i in this order at h, then there exists an alternative scheme that switches the order of i and j and yields a higher revenue than σ.

Lemma 5.1. *Let $\sigma = (r,x) \in \Sigma^0$. Suppose σ induces history $\underline{h} \in H_{n-1}$ ($1 \le n \le I-1$) such that r is non-contingent after $\underline{h}$, and*

$$r(\underline{h}) = j \quad \textit{and} \quad r(\underline{h},0) = r(\underline{h},1) = i$$

for some $i \ne j$ such that $c_j \ge c_i$. Then there exists $\sigma^* = (r^*, x^*) \in \Sigma^0$ such that r^* is non-contingent after $\underline{h}$, $r^*(\underline{h}) = i$, $r^*(\underline{h},0) = r^*(\underline{h},1) = j$, and $\pi(\sigma^*) \ge \pi(\sigma)$.

Proof. See the Appendix. ■

Suppose $c_j \ge c_i$. It follows from Lemma 5.1 that revenue improvement is possible if r trades with j and i in this order at $h \in H_{I-2}$ since r is non-contingent at any such h as noted above. Whenever there are two buyers left, then, it is always optimal to trade first with the one with the smaller weight. This suggests that at any history $h \in H_{I-3}$ where three buyers are left, it is optimal to choose r that is non-contingent after h: No matter what the outcome with buyer $r(h)$ in period $I-2$, choose the buyer with the smaller weight in period $I-1$. If r is non-contingent at $h \in H_{I-3}$, however, Lemma 5.1 suggests that revenue improvement is possible if r trades with a buyer with the smallest weight among three first at h. Repeating this argument, we can show that it is optimal to trade with the buyer with the smallest weight in period 1, one with the second smallest weight in period 2, and so on. The following theorem formalizes this argument and states that among the sales schemes in Σ^0, the seller's expected revenue is maximized when he trades with the buyers in the increasing order of their weights c_i.

Theorem 5.2. *Suppose that $c_1 \leq \cdots \leq c_I$. Among the sales schemes in Σ^0, the seller's expected revenue is maximized when he employs a non-contingent scheme that trades with buyer t in period t for every $t \in I$.*

Proof. See the Appendix. ∎

6 Uniform Distribution and Generation of Herds

In this section, we give an explicit characterization of the seller's optimal sales scheme when a buyer's private signal is drawn from a uniform distribution. We will then show that the optimal scheme induces herding among buyers by examining the probability that the seller's offer is accepted along the path of play.

For the characterization of the optimal scheme, we first consider a constrained optimization problem with respect to the probability of acceptance z given the selection function r. We will verify that the solution to this problem satisfies the monotonicity conditions in the definition (5) of Σ^0. This implies that for any scheme $\sigma = (r, z)$ and any history h it induces, if r is non-contingent after h and z maximizes continuation revenue at h, then z satisfies those conditions. This shows that any solution σ to the unconstrained revenue maximization problem belongs to Σ^0. It will then follow from Theorem 5.2 that the optimal σ entails the monotone ordering of the weights c_i.

Formally, suppose that all buyers are identical except for the weights c_i as in the previous section, and that their signals are drawn from the uniform distribution over the unit interval $[0, 1]$. Fix a non-contingent selection function r and consider the constrained optimization problem with respect to the probability of acceptance z given r. With slight abuse of notation, we denote by $r(t) \in I$ the buyer that the selection function r chooses in period $t \in I$. Since a buyer's valuation function at any history is completely determined by the value of α at that history as seen in (3), we use α_{t-1} to denote the state in period t and write $V_{r(t)}(s_{r(t)} \mid \alpha_{t-1})$ for buyer $r(t)$'s valuation in state α_{t-1}. Since the seller's problem in period t is also completely described by α_{t-1}, we will treat the state α_{t-1} as a continuous variable and describe x and the associated probability of acceptance z as functions of α_{t-1}.

Suppose now that the seller chooses his offer $x_{r(t)}$ to buyer $r(t)$ in period t so that it will be accepted with probability z_t when the state is α_{t-1}.[8] It can then be verified from (3) that $x_{r(t)}$ and z_t are related through $x_{r(t)} = 1 - z_t + c_{r(t)}\alpha_{t-1} + c_0$. If buyer $r(t)$ accepts this offer, the updated expected value of $s_{r(t)}$ equals $E[\tilde{s}_{r(t)} \mid V_{r(t)}(\tilde{s}_{r(t)} \mid \alpha_{t-1}) \geq x_{r(t)}]$, or equilivalently, $E[\tilde{s}_{r(t)} \mid \tilde{s}_{r(t)} \geq 1 - z_t]$. We redefine κ and λ as follows: Let $\kappa(z)$ denote the expected value of s_i minus μ conditional on buyer i accepting the seller's offer intended to be accepted with probability z:

$$\kappa(z) = E\big[\tilde{s}_i - \mu \mid \tilde{s}_i \geq 1 - z\big].$$

Likewise, let $\lambda(z)$ denote the expected value of s_i minus μ when buyer i rejects the same offer:

$$\lambda(z) = E\big[\tilde{s}_i - \mu \mid \tilde{s}_i < 1 - z\big].$$

Note that both $\kappa(z)$ and $\lambda(z)$ are independent of the state or the identity of the buyer. The state transition on the path can hence be described as

$$\alpha_t = \begin{cases} \alpha_{t-1} + \kappa(z_t(\alpha_{t-1})) & \text{when buyer } r(t) \text{ accepts,} \\ \alpha_{t-1} + \lambda(z_t(\alpha_{t-1})) & \text{when buyer } r(t) \text{ rejects.} \end{cases}$$

When s_i has the uniform distribution over $[0, 1]$, $\kappa(z)$ and $\lambda(z)$ can be written explicitly as:

$$\kappa(z) = \frac{1-z}{2} \quad \text{and} \quad \lambda(z) = -\frac{z}{2}.$$

Using z and α, we now write down the seller's constrained optimization problem given r. Let $\pi_I(z_I, \alpha_{I-1})$ denote the seller's expected revenue from buyer $r(I)$ in state α_{I-1} when he makes an offer that is accepted with probability z_I. It can be written as

$$\pi_I(z_I, \alpha_{I-1}) = g(z_I) + z_I c_{r(I)} \alpha_{I-1} + c_0 z_I,$$

[8] For simplicity, we use z_t rather than $z_{r(t)}$ to denote the probability of acceptance in period t by buyer $r(t)$.

where $g : [0,1] \to \boldsymbol{R}$ is defined by $g(z) = z(1-z)$. Let $\pi_I^*(\alpha_{I-1})$ denote the maximized value of $\pi_I(z_I, \alpha_{I-1})$:

$$\pi_I^*(\alpha_{I-1}) = \max_{z_I \in [0,1]} \pi_I(z_I, \alpha_{I-1}).$$

For $t = 1, \dots, I-1$, the seller's expected revenue over periods $t, \dots, I$ is recursively defined by

$$\pi_t(z_t, \alpha_{t-1}) = g(z_t) + z_t c_{r(t)} \alpha_{t-1} + c_0 z_t + f_{t+1}(z_t, \alpha_{t-1}),$$

where

$$f_{t+1}(z_t, \alpha_{t-1}) = z_t \, \pi_{t+1}^*\Big(\alpha_{t-1} + \kappa(z_t)\Big) + (1 - z_t) \, \pi_{t+1}^*\Big(\alpha_{t-1} + \lambda(z_t)\Big)$$

is the seller's expected revenue over periods $t+1, \dots, I$ when he chooses z_t in period t, and then follows the optimal course of action in subsequent periods. The optimized value of $\pi_t(z_t, \alpha_{t-1})$ is denoted by

$$\pi_t^*(\alpha_{t-1}) = \max_{z_t \in [0,1]} \pi_t(z_t, \alpha_{t-1}). \tag{6}$$

Let $a_{r(t)}$ and $b_{r(t)}$ be defined by $a_{r(I)} = 1 + c_0$, $b_{r(I)} = c_{r(I)}$, and

$$a_{r(t)} = 1 + \frac{c_0}{1 + \frac{1}{16} \sum_{k=t+1}^{I} b_{r(k)} c_{r(k)}}, \quad \text{and} \quad b_{r(t)} = \frac{c_{r(t)}}{1 + \frac{1}{16} \sum_{k=t+1}^{I} b_{r(k)} c_{r(k)}}$$

for $t = 1, \dots, I-1$. The following theorem explicitly describes the solution to the constrained maximization problem (6) when it has an interior solution for every i.

Theorem 6.1. *Suppose that every s_i has the uniform distribution over $[0,1]$, and that the seller trades with buyer $r(t)$ in period t. If*

$$b_{r(t)} < \frac{2(2 - a_{r(t)})}{I - 1} \quad \textit{for every } t = 1, \dots, I, \tag{7}$$

then the solution to the maximization problem (6) is given by

$$z_t(\alpha_{t-1}) = \frac{1}{2}(a_{r(t)} + b_{r(t)} \alpha_{t-1}) \quad \textit{for every } t = 1, \dots, I. \tag{8}$$

Proof. See the Appendix. ■

The condition (7) guarantees that the solution to (6) is in the interior: $z_t(\alpha_{t-1}) \in (0,1)$ for any α_{t-1} and t. Since $b_{r(t)} \leq c_{r(t)}$, it can be seen that this condition holds when $c_0 < 1$ and all the weights c_i are small. It should be noted however that the total weight $(I-1)\, c_i$ placed by i on others' signals need not be small: For example, when $c_0 = 0$, (7) is sastisfied if $(I-1)\, c_i < 2$ for every $i \in I$.

Suppose that the weights satisfy the following conditions:

$$c_0 = 0, \quad \max_{i \in I} c_i < \frac{2}{I-1}, \tag{9}$$

$$\Big(1 + \frac{1}{16} \sum_{k \neq i,j} c_k^2\Big)\,(c_i - c_j) \leq \frac{c_j^3}{16\Big\{1 + \frac{I-2}{4(I-1)^2}\Big\}} \quad \text{for any } i \neq j. \tag{10}$$

Let σ be any constrained optimal scheme that trades with buyer $r(t)$ in period t. Since $c_0 = 0$ implies $a_{r(t)} = 1$ for every t, (7) holds under (9). Since z_t is an affine function by Theorem 6.1, we have for any α,

$$\begin{aligned} & z_t(\alpha)\, z_{t+1}\Big(\alpha + \kappa(z_t(\alpha))\Big) + (1 - z_t(\alpha))\, z_{t+1}\Big(\alpha + \lambda(z_t(\alpha))\Big) \\ & = z_{t+1}\Big(\alpha + z_t(\alpha)\, \kappa(z_t(\alpha)) + (1 - z_t(\alpha))\, \lambda(z_t(\alpha))\Big) \\ & = z_{t+1}(\alpha), \end{aligned} \tag{11}$$

where the second equality follows from Lemma 3.2. On the other hand, the condition (10) requires that the buyers not be so dissimilar in the sense that c_i and c_j are close to each other. In particular, the following can be verified after some algebra.

Lemma 6.2. *For any permutation $r : I \to I$, if (9) and (10) hold, then*

$$b_{r(t)} \leq b_{r(t+1)} \text{ for every } t = 1, \ldots, I-1. \tag{12}$$

Proof. See the Appendix. ■

It follows from (8) and (12) that

$$\alpha \geq 0 \Rightarrow z_t(\alpha) \leq z_{t+1}(\alpha), \quad \text{and} \quad \alpha \leq 0 \Rightarrow z_t(\alpha) \geq z_{t+1}(\alpha) \tag{13}$$

for $t = 1, \ldots, I-1$. Note that (11) and (13) together imply the second condition for Σ^0 in (5). Since each z_t is also increasing, we can conclude that $\sigma \in \Sigma^0$. Let then $\sigma = (r, z)$ be any (possibly contingent) optimal scheme and h be any history that it induces. If r is non-contingent after h, the above in particular shows that

z restricted to the (non-contingent) subsequence of buyers after h also satifies the monotonicity conditions in (5). This shows that $\sigma \in \Sigma^0$. We hence obtain the following result.

Theorem 6.3. *Suppose that every s_i has the uniform distribution over $[0,1]$, and that (9) and (10) hold. If $c_1 \leq \ldots \leq c_I$, then the seller's expected revenue is maximized when he employs a non-contingent sequential scheme and trades with buyer t in period t (i.e, $r(t) = t$) by making an offer that is accepted with probability z_t given in Theorem 6.1.*

We next use Theorems 6.1 and 6.3 to derive a qualitative characterization of the seller's offers in the optimal scheme. It follows from (13) that if $\alpha_t \geq \max\{0, \alpha_{t-1}\}$, then $z_{t+1}(\alpha_t) \geq z_t(\alpha_t) \geq z_t(\alpha_{t-1})$. In other words, when the seller has a successful transaction with buyer $r(t) = t$ and the resulting state is positive in period $t+1$, the seller's offer to buyer $r(t+1) = t+1$ is more likely to be accepted than his offer to buyer t. Likewise, if $\alpha_t \leq \max\{0, \alpha_{t-1}\}$, then $z_{t+1}(\alpha_t) \leq z_t(\alpha_t) \leq z_t(\alpha_{t-1})$. In other words, when the transaction with buyer t fails and the resulting state is negative in period $t+1$, the seller's offer to buyer $t+1$ is less likely to be accepted than his offer to buyer t. Since the initial state $\alpha_0 = 0$, the state remains positive and increasing along any history in which all the transactions are successful. Hence, the probability of acceptance keeps increasing along any such history. On the other hand, the probability of acceptance keeps declining along the history in which all the transactions have failed. Note also that α_{t-1} is the highest when the buyers $1, \ldots, t-1$ all accept the seller's offers, and is the lowest when they all reject the offers. This, along with the fact that z_t is increasing in α_{t-1}, shows that the probability that buyer t accepts the seller's offer is maximized when all his predecessors accept, and minimized when they all reject. We summarize these observations as a corollary below. Combined with the fact that the buyer who is more heavily influenced by others' behavior are placed towards the end of the sequence, the corollary demonstrates the intrinsic tendency of the optimal sales scheme to induce herding among buyers.

Corollary 6.4. *Suppose that the conditions of Theorem 6.3 hold. In the optimal sales scheme σ that trades with buyer t in period t, the probability of acceptance increases along the history in which all the buyers accept, and decreases along the history in which they all reject. Furthermore, for any t, the probability of acceptance by buyer t is maximized when buyers $1, \ldots, t-1$ all accept, and minimized when they all reject.*

We turn next to the examination of the price path. As seen earlier, the price $x_t(\alpha_{t-1})$ offered to buyer t in state α_{t-1} is related to $z_t(\alpha_{t-1})$ through

$$\begin{aligned} x_t(\alpha_{t-1}) &= 1 - z_t(\alpha_{t-1}) + c_t\alpha_{t-1} \\ &= 1 - \frac{1}{2}(1 + b_t\alpha_{t-1}) + c_t\alpha_{t-1} \\ &= \frac{1}{2} + \left(c_t - \frac{b_t}{2}\right)\alpha_{t-1}. \end{aligned}$$

Likewise, $x_{t+1}(\alpha_t)$ can be expressed as

$$x_{t+1}(\alpha_t) = \frac{1}{2} + \left(c_{t+1} - \frac{b_{t+1}}{2}\right)\alpha_t.$$

It can also be readily verified that the state transition is given by

$$\alpha_t = \begin{cases} \alpha_{t-1} + \frac{1}{4}(1 - b_t\alpha_{t-1}) & \text{when buyer } t \text{ accepts,} \\ \alpha_{t-1} - \frac{1}{4}(1 + b_t\alpha_{t-1}) & \text{when buyer } t \text{ rejects.} \end{cases}$$

The following corollary gives a qualitative characterization of the price path when c_i's are sufficiently small. Its conclusion on the monotone price path conforms to our basic intuition.

Corollary 6.5. *Suppose that the conditions of Theorem 6.3 hold. Let σ be the optimal sales scheme that trades with buyer t in period t. There exists $\epsilon > 0$ such that if $c_I < \epsilon$, then $x_t(\alpha_{t-1}) \leq x_{t+1}(\alpha_t)$ if $\alpha_{t-1} \geq 0$ and buyer t accepts, and $x_t(\alpha_{t-1}) \geq x_{t+1}(\alpha_t)$ if $\alpha_{t-1} \leq 0$ and buyer t rejects. In particular, the price is monotone increasing along the history in which all the buyers accept, and is monotone decreasing along the history in which all the buyers reject.*

Proof. See the Appendix. ■

7 Discussions

As seen in Section 3, it is public information how much a buyer's valuation (as a function of his private signal) shifts up or down as a result of each previous transaction under our assumptions that the buyers' private signals are independent, and that their valuation can be expressed as the weighted sum of those private signals. This property is used extensively in the analysis for the construction of an alternative scheme. We would have a very different problem when the signals are correlated or when the valuation functions are not affine. For example, if the valuation functions have a multiplicative form with respect to the private signals, then it can be seen that the impact of previous transactions on a buyer's valuation varies with his own signal. The same is true when the signals are correlated across buyers. With correlated signals, the seller also faces the learning problem. For example, if the private signals indicate the underlying common value of the good, it may be in the interest of the seller to engage in experimental pricing against the initial set of buyers. These issues significantly complicate the analysis.

As mentioned in the Introduction, an alternative interpretation of the present model is through the seller's information revelation policy. A more direct model of information revelation would be one in which trades take place sequentially but the outcome of each transaction is the seller's private information. In such a setup, the seller's information revelation policy specifies which past outcomes to reveal to each buyer as a function of the history. While this is a possible alternative to the present formulation, we suspect that the analysis of such a model is more involved with its equilibrium delicately dependent on the specification of the buyers' beliefs. From a more practical point of view, the seller may find it difficult to strictly control his sales information when trades take place in sequence.

Appendix

Proof of Theorem 4.1 Fix any sales scheme σ that is not sequential. That is, σ induces a history $\underline{h} \in H_{n-1}$ $(n \geq 1)$ such that $r(\underline{h}) = \{m\} \cup J$ for some $m \in I$ and $J \neq \emptyset$. In other words, according to σ, the seller trades with buyer m and at least one other buyer in period n at history $\underline{h}$. We will construct an alternative scheme σ^* that raises the same expected revenue as σ as follows: The sales scheme σ^* operates in the same way as σ does except when $\underline{h}$ arises. At history $\underline{h}$, σ^* trades only with buyer m with the same offer price as under the original scheme. For simplicity, denote the outcome $y_n \in Y(\{m\})$ in period n from buyer m under σ^* by either 0 or 1: 1 represents the outcome $(\{m\}, \emptyset)$ that buyer m accepts the seller's offer, and 0 represents the outcome $(\emptyset, \{m\})$ that he rejects it. In period $n+1$ at either $(\underline{h}, 1)$ or $(\underline{h}, 0)$, σ^* trades with the buyers in J with the offer prices adjusted according to the outcome in period n. In any subsequent period, the set of buyers and prices specified by σ^* along any history $(\underline{h}, y_n, \ldots, y_{t-1}) \in H_{t-1}$ are the same as those specified by σ along the history $(\underline{h}, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_{t-1}) \in H_{t-2}$, where

$y_n \cup y_{n+1} = (A_n \cup A_{n+1}, B_n \cup B_{n+1})$ is the "union" of two outcomes y_n and y_{n+1}: Those who accept under $y_n \cup y_{n+1}$ are the union of those who accept under y_n and y_{n+1}, and those who reject under $y_n \cup y_{n+1}$ are the union of those who reject under $y_n \cup y_{n+1}$. In other words, σ^* operates just as σ by assuming that the outcomes in periods n and $n+1$ came from the same period. A formal description of σ^* is given as follows:

$$r^*(h) = \begin{cases} \{m\} & \text{if } h = \underline{h}, \\ J & \text{if } h = (\underline{h}, 1) \text{ or } (\underline{h}, 0), \\ r(\underline{h}, y_n \cup y_{n+1}, y_{n+2}, \dots, y_{t-1}) & \\ & \text{if } h = (\underline{h}, y_n, \dots, y_{t-1}) \text{ for some} \\ & y_n, \dots, y_{t-1}\ (t \geq n+2), \\ r(h) & \text{otherwise.} \end{cases} \tag{14}$$

and for any $i \in I$,

$$x_i^*(h) = \begin{cases} x_i(\underline{h}) + c_{im}\kappa_m(\underline{h}) & \text{if } h = (\underline{h}, 1) \\ x_i(\underline{h}) + c_{im}\lambda_m(\underline{h}) & \text{if } h = (\underline{h}, 0) \\ x_i(\underline{h}, y_n \cup y_{n+1}, y_{n+2}, \dots, y_{t-1}) & \\ & \text{if } h = (\underline{h}, y_n, \dots, y_{t-1}) \text{ for some} \\ & y_n, \dots, y_{t-1}\ (t \geq n+2), \\ x_i(h) & \text{otherwise.} \end{cases} \tag{15}$$

In what follows, we will show that σ^* yields the same expected revenue as σ. Since σ is an arbitrary non-sequential scheme, repeated application of this argument shows that for any scheme σ that is not sequential, there exists a sequential scheme that yields the same expected payoff as σ. The desired conclusion would then follow.

For simplicity, denote

$$\begin{aligned} &V_i(s_i \mid h) = V_i^\sigma(s_i \mid h), \quad V_i^*(s_i \mid h) = V_i^{\sigma^*}(s_i \mid h), \\ &\kappa_i(h) = \kappa_i^\sigma(h), \quad \text{and} \quad \lambda_i(h) = \lambda_i^\sigma(h). \end{aligned}$$

Let also $w_i(\underline{h})$ be defined by

$$w_i(\underline{h}) = c_{i0} + \sum_{j \in I_{n-1}} c_{ij} E^\sigma[\tilde{s}_j - \mu_j \mid \underline{h}].$$

Note that

$$V_i(s_i \mid \underline{h}) = V_i^*(s_i \mid \underline{h}) = c_{ii}s_i + w_i(\underline{h}), \tag{16}$$

and for any outcome $y_n \in Y(\{m\}) = \{0, 1\}$ from buyer m in period n,

$$V_i^*(s_i \mid \underline{h}, y_n) = \begin{cases} c_{ii}s_i + w_i(\underline{h}) + c_{im}\kappa_m(\underline{h}) & \text{if } y_n = 1, \\ c_{ii}s_i + w_i(\underline{h}) + c_{im}\lambda_m(\underline{h}) & \text{if } y_n = 0. \end{cases} \tag{17}$$

It hence follows from (15) that for any $i \in J$,

$$\begin{aligned} z_i^*(\underline{h}, y_n) &= P\Big(V_i^*(\tilde{s}_i \mid \underline{h}, y_n) \geq x_i^*(\underline{h}, y_n)\Big) \\ &= P\Big(c_{ii}\tilde{s}_i + w_i(\underline{h}) \geq x_i(\underline{h})\Big) \\ &= P\Big(V_i(\tilde{s}_i \mid \underline{h}) \geq x_i(\underline{h})\Big) \\ &= z_i(\underline{h}). \end{aligned} \tag{18}$$

It then follows from Lemma 3.1 that

$$V_i^*(s_i \mid \underline{h}, y_n, y_{n+1}) = V_i(s_i \mid \underline{h}, y_n \cup y_{n+1}).$$

For any $t \geq n+2$ and any sequence of outcomes $y_n, \ldots, y_{t-1}$ in periods $n, \ldots, t-1$ under σ^*, we will show that a buyer's valuation function $V_i^*(\cdot \mid \underline{h}, y_n, \ldots, y_{t-1})$ in period t induced by σ^* is the same as the valuation function $V_i(\cdot \mid \underline{h}, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_{t-1})$ in period $t-1$ induced by σ. As an induction hypothesis, suppose that

$$V_i^*(s_i \mid \underline{h}, y_n, \ldots, y_{t-1}) = V_i(s_i \mid \underline{h}, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_{t-1})$$

for some $t \geq n+2$. Since

$$x_i^*(\underline{h}, y_n, \ldots, y_{t-1}) = x_i(\underline{h}, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_{t-1})$$

by definition, we have

$$z_i^*(\underline{h}, y_n, \ldots, y_{t-1}) = z_i(\underline{h}, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_{t-1}).$$

Hence, Lemma 3.1 implies that

$$V_i^*(s_i \mid \underline{h}, y_n, \ldots, y_t) = V_i(s_i \mid \underline{h}, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_t).$$

For any $h \in H_{t-1}$, let $\pi_t(h)$ denote the seller's expected revenue in periods $t, \ldots, I$ at history h when he employs the sales scheme σ. Define $\pi_t^*(h)$ similarly for σ^*.

Given the equality of the valuation functions induced by the two schemes as seen above, we have

$$\pi^*_{n+2}(\underline{h}, y_n, y_{n+1}) = \pi_{n+1}(\underline{h}, y_n \cup y_{n+1})$$

for any sequence of outcomes (y_n, y_{n+1}) in periods n and $n+1$ under σ^*. On the other hand,

$$\begin{aligned}\pi^*_n(\underline{h}) = &\sum_{y_n \in Y(\{m\})} \sum_{y_{n+1} \in Y(J)} P^{\sigma^*}(y_n \mid \underline{h})\, P^{\sigma^*}(y_{n+1} \mid \underline{h}, y_n) \\ &\cdot \Big\{ \sum_{i \in A_n} x^*_i(\underline{h}) + \sum_{i \in A_{n+1}} x^*_i(\underline{h}, y_n) + \pi^*_{n+2}(\underline{h}, y_n, y_{n+1}) \Big\}.\end{aligned} \tag{19}$$

Likewise, the expected revenue in period n under σ conditional on $\underline{h}$ can be expressed using $Y(\{m\})$ and $Y(J)$ as:

$$\begin{aligned}\pi_n(\underline{h}) = &\sum_{y_n \in Y(\{m\})} \sum_{y_{n+1} \in Y(J)} P(y_n \mid \underline{h})\, P(y_{n+1} \mid \underline{h}) \\ &\cdot \Big\{ \sum_{i \in A_n} x_i(\underline{h}) + \sum_{i \in A_{n+1}} x_i(\underline{h}) + \pi_{n+1}(\underline{h}, y_n \cup y_{n+1}) \Big\}.\end{aligned} \tag{20}$$

Since σ and σ^* are identical up to and including period $n-1$, we have for $y_n \in Y(\{m\}) = \{0, 1\}$,

$$P^{\sigma^*}(y_n \mid \underline{h}) = P^{\sigma}(y_n \mid \underline{h}). \tag{21}$$

By (18), we also have for any $y_{n+1} \in Y(J)$,

$$\begin{aligned}P^{\sigma^*}(y_{n+1} \mid \underline{h}, y_n) &= \prod_{i \in A_{n+1}} z^*_i(\underline{h}, y_n) \prod_{i \in B_{n+1}} (1 - z^*_i(\underline{h}, y_n)) \\ &= \prod_{i \in A_{n+1}} z_i(\underline{h}) \prod_{i \in B_{n+1}} (1 - z_i(\underline{h})) \\ &= P^{\sigma}(y_{n+1} \mid \underline{h}).\end{aligned} \tag{22}$$

Using (20), (22) and (21), and substituting the definitions of $x^*_i(\underline{h})$ and $x^*_i(\underline{h}, y_n)$, we can rewrite (19) as:

$$\begin{aligned}\pi^*_n(\underline{h}) = \pi_n(\underline{h}) + &\sum_{y_{n+1} \in Y(J)} P^{\sigma}(y_{n+1} \mid \underline{h}) \\ \cdot \sum_{i \in A_{n+1}} &\Big[\sum_{y_n \in Y(\{m\})} P^{\sigma}(y_n \mid \underline{h}) \Big\{ \sum_{j \in A_n} c_{ij} \kappa_j(\underline{h}) + \sum_{j \in B_n} c_{ij} \lambda_j(\underline{h}) \Big\} \Big],\end{aligned} \tag{23}$$

where the order of the summations in the second term is reversed since their ranges are independent of each other. Since $Y(\{m\}) = \{(\emptyset, \{m\}), (\{m\}, \emptyset)\}$, the quantity in the square brackets on the right-hand side of (23) equals

$$\begin{aligned}
&\sum_{y_n \in Y(\{m\})} P(y_n \mid \underline{h}) \Big\{ \sum_{j \in A_n} c_{ij} \kappa_j(\underline{h}) + \sum_{j \in B_n} c_{ij} \lambda_j(\underline{h}) \Big\} \\
&= c_{im} \Big\{ z_m(\underline{h})\, \kappa_m(\underline{h}) + (1 - z_m(\underline{h}))\, \lambda_m(\underline{h}) \Big\} \\
&= 0.
\end{aligned} \tag{24}$$

This completes the proof of the theorem. ■

Proof of Lemma 5.1 Define $\sigma^* = (r^*, x^*)$ as follows:

$$r^*(h) = \begin{cases} i & \text{if } h = \underline{h}, \\ j & \text{if } h = (\underline{h}, 0) \text{ or } (\underline{h}, 1), \\ r(h) & \text{otherwise,} \end{cases}$$

$$x_i^*(h) = \begin{cases} x_j(\underline{h}) - (c_j - c_i)\, \alpha^\sigma(\underline{h}) & \text{if } h = \underline{h}, \\ x_i(h) & \text{otherwise,} \end{cases}$$

$$x_j^*(h) = \begin{cases} x_i(\underline{h}, 0) + (c_j - c_i)\, \{\alpha^\sigma(\underline{h}) + \kappa_j(\underline{h}, 0)\} & \text{if } h = (\underline{h}, 0), \\ x_i(\underline{h}, 1) + (c_j - c_i)\, \{\alpha^\sigma(\underline{h}) + \kappa_j(\underline{h}, 1)\} & \text{if } h = (\underline{h}, 1), \\ x_j(h) & \text{otherwise,} \end{cases}$$

and $x_k^*(h) = x_k(h)$ for any $h \in H$ and $k \neq i,\, j$. Just as in the proof of Theorem 4.1, for any history $h \in H$, denote

$$V_i^*(s_i \mid h) = V_i^{\sigma^*}(s_i \mid h), \quad \text{and} \quad z_i^*(h) = z_i^{\sigma^*}(h).$$

It is clear that r^* is non-contingent after $\underline{h}$. It can also be verified that $z_i^*(\underline{h}) = z_j(\underline{h})$ since $\alpha^\sigma(\underline{h}) = \alpha^{\sigma^*}(\underline{h})$ and hence

$$\begin{aligned}
z_i^*(\underline{h}) &= P\Big(V_i^*(\tilde{s}_i \mid \underline{h}) \geq x_i^*(\underline{h})\Big) \\
&= P\Big(c_0 + \tilde{s}_i + c_i \alpha^{\sigma^*}(\underline{h}) \geq x_j(\underline{h}) - (c_j - c_i)\alpha^\sigma(\underline{h})\Big) \\
&= P\Big(c_0 + \tilde{s}_i + c_j \alpha^\sigma(\underline{h}) \geq x_j(\underline{h})\Big) \\
&= P\Big(V_j(\tilde{s}_j \mid \underline{h}) \geq x_j(\underline{h})\Big) \\
&= z_j(\underline{h}).
\end{aligned}$$

Since $\kappa_i(\underline{h},0)=\kappa_j(\underline{h},0)$ and $\kappa_i(\underline{h},1)=\kappa_j(\underline{h},1)$, it can also be verified that $z_j^*(\underline{h},0)=z_i(\underline{h},0)$ and $z_j^*(\underline{h},1)=z_i(\underline{h},1)$. Lemma 3.1 then implies that for any $k\neq i$, j and any sequence of outcomes (y_n,y_{n+1}) in periods n and $n+1$, $V_k^*(\cdot\mid\underline{h},y_n,y_{n+1})=V_k(\cdot\mid\underline{h},y_n,y_{n+1})$. It follows from this and $x_k^*(h)=x_k(h)$ for any $k\neq i$, j and $h\in H$ that for any $k\neq i$, j and any sequence of outcomes $(y_n,\ldots,y_{t-1})$ in periods $n,\ldots,t-1$,

$$z_k^*(\underline{h},y_n,\ldots,y_{t-1})=z_k(\underline{h},y_n,\ldots,y_{t-1}), \tag{25}$$

and

$$V_k^*(\cdot\mid\underline{h},y_n,\ldots,y_{t-1})=V_k(\cdot\mid\underline{h},y_n,\cdots,y_{t-1}). \tag{26}$$

(25) in particular shows that $\sigma^*\in\Sigma^0$.

Now for any history $h\in H_{t-1}$ and $t\in I$, let $\pi_t(h)$ denote the seller's expected revenue over periods $t,\ldots,I$ at history h under σ. Likewise, let $\pi_t^*(h)$ denote his expected revenue over periods $t,\ldots,I$ at history h under σ^*. By (26), $\pi_{n+2}^*(\underline{h},y_n,y_{n+1})=\pi_{n+2}(\underline{h},y_n,y_{n+1})$ for any sequence of outcomes (y_n,y_{n+1}) in periods n and $n+1$. It hence follows from the definition of σ^* and the above observation that

$$\begin{aligned}
&\pi_n^*(\underline{h})\\
&=z_i^*(\underline{h})\,x_i^*(\underline{h})\\
&+z_i^*(\underline{h})\left[z_j^*(\underline{h},1)\left\{x_j^*(\underline{h},1)+\pi_{n+2}(\underline{h},1,1)\right\}+(1-z_j^*(\underline{h},1))\,\pi_{n+2}(\underline{h},1,0)\right]\\
&+(1-z_i^*(\underline{h}))\left[z_j^*(\underline{h},0)\left\{x_j^*(\underline{h},0)+\pi_{n+2}(\underline{h},0,1)\right\}+(1-z_j^*(\underline{h},0))\,\pi_{n+2}(\underline{h},0,0)\right]\\
&=\pi_n(\underline{h})+(c_j-c_i)\Big[\alpha^\sigma(\underline{h})\Big\{z_j(\underline{h})z_i(\underline{h},1)+(1-z_j(\underline{h}))z_i(\underline{h},0)-z_j(\underline{h})\Big\}\\
&\qquad+\Big\{z_i(\underline{h},1)\,z_j(\underline{h})\kappa_j(\underline{h},1)+z_i(\underline{h},0)\,(1-z_j(\underline{h}))\,\kappa_j(\underline{h},0)\Big\}\Big].
\end{aligned} \tag{27}$$

Since $\sigma\in\Sigma^0$, the first quantity in the square brackets on the right-hand side of (27) is ≥ 0. Furthermore, since $z_i(\underline{h},1)\geq z_i(\underline{h},0)$, the second quantity is greater than or equal to

$$z_i(\underline{h},0)\left\{z_j(\underline{h})\,\kappa_j(\underline{h},1)+(1-z_j(\underline{h}))\,\kappa_j(\underline{h},0)\right\}=0.$$

We hence obtain the desired conclusion that $\pi(\sigma^*)\geq\pi(\sigma)$. ■

Proof of Theorem 5.2 Let $\sigma \in \Sigma^0$ be an arbitrary scheme that is optimal within Σ^0. For any $h \in H_{I-2}$, since $U(h,0) = U(h,1) = \{i\}$ for some $i \in I$, we must have $r(h,0) = r(h,1) = i$. If $r(h) = j > i$, then Lemma 5.1 implies that σ is weakly dominated by an alternative scheme in σ^0 which, after history h, trades with buyer i in period $I-1$ and buyer j in period I. Therefore, we conclude that there exists a revenue maximizing scheme $\sigma \in \Sigma^0$ that satisfies for any $h \in H_{I-2}$, if $U(h) = \{i_1, i_2\}$ for some $i_1 < i_2$, then

$$r(h) = i_1 \quad \text{and} \quad r(h,0) = r(h,1) = i_2.$$

As an induction hypothesis, given t $(2 \leq t \leq I-1)$, suppose that there exists a revenue maximizing scheme $\sigma \in \Sigma^0$ that satisfies for any $h \in H_{I-t}$, if $U(h) = \{i_1, \dots, i_t\}$ for $i_1 < \dots < i_t$, then

$$\begin{aligned} &r(h) = i_1,\ r(h,0) = r(h,1) = i_2,\\ &\dots,\ r(h, \underbrace{0, \dots, 0}_{t-1}) = \dots = r(h, \underbrace{1, \dots, 1}_{t-1}) = i_t. \end{aligned} \tag{28}$$

Take any $h \in H_{I-t-1}$. Since $U(h,0) = U(h,1) = \{i_1, \dots, i_t\}$ for some $i_1 < \dots < i_t$, it follows from the induction hypothesis that

$$\begin{aligned} &r(h,0) = r(h,0) = i_1,\ r(h,0,0) = \dots = r(h,1,1) = i_2,\\ &\dots,\ r(h, \underbrace{0, \dots, 0}_{t}) = \dots = r(h, \underbrace{1, \dots, 1}_{t}) = i_t. \end{aligned} \tag{29}$$

Hence, if $r(h) = j$ for some $j > i_1$, then Lemma 5.1 implies that σ is weakly dominated by an alternative scheme in Σ^0 which, after history h, trades with buyer i_1 in period $t-1$ and buyer j in period t. If $j > i_2$, then the latter scheme is further dominated by a scheme that offers buyer i_2 in period t and buyer j in period $t+1$. Repeating this argument, we can conclude that there exists a revenue maximizing scheme $\sigma \in \Sigma^0$ that satisfies for any $h \in H_{I-t-1}$, if $U(h) = \{i_1, \dots, i_{t+1}\}$ for some $i_1 < \dots < i_{t+1}$, then

$$\begin{aligned} &r(h) = i_1,\ r(h,0) = r(h,1) = i_2,\\ &\dots,\ r(h, \underbrace{0, \dots, 0}_{t}) = \dots = r(h, \underbrace{1, \dots, 1}_{t}) = i_{t+1}. \end{aligned} \tag{30}$$

Therefore, we have advanced the induction step and established that among the optimal schemes within Σ^0, there exists a non-contingent scheme $\sigma \in \Sigma^0$ which trades with buyer t in period t. ■

Proof of Theorem 6.1 Suppose for simplicity that $r(t) = t$ for every $t \in I$. Note that $g(z) = z(1-z)$ for the uniform distribution. Since

$$\frac{\partial \pi_I}{\partial z_I}(z_I, \alpha_{I-1}) = 1 - 2z_I + c_I\alpha_{I-1} + c_0$$

is decreasing in z_I, the first-order condition yields the optimal solution $z_I(\alpha_{I-1}) = \frac{1}{2}(1 + c_0 + c_I\alpha_{I-1})$. The envelope theorem also implies that

$$\frac{\partial \pi_I^*}{\partial \alpha_{I-1}}(\alpha_{I-1}) = c_I z_I(\alpha_{I-1}).$$

As an induction hypothesis, suppose now that (8) holds for $i+1, \dots, I$ $(i \le I-1)$ and that

$$\frac{\partial \pi_{i+1}^*}{\partial \alpha_i}(\alpha_i) = \sum_{j=i+1}^{I} c_j z_j(\alpha_i).$$

The expected revenue function for periods $i, \dots, I$ can be written as

$$\pi_i(z_i, \alpha_{i-1}) = g(z_i) + z_i c_i \alpha_{i-1} + c_0 z_i + f_{i+1}(z_i, \alpha_{i-1}),$$

where

$$f_{i+1}(z_i, \alpha_{i-1}) = z_i\, \pi_{i+1}^*\Big(\alpha_{i-1} + \kappa_i(z_i, 1)\Big) + (1 - z_i)\, \pi_{i+1}^*\Big(\alpha_{i-1} + \kappa_i(z_i, 0)\Big)$$

It follows that

$$\begin{aligned}
\frac{\partial \pi_i}{\partial z_i}(z_i, \alpha_{i-1}) &= 1 - 2z_i + c_0 + c_i\alpha_{i-1} \\
&\quad + \sum_{j=i+1}^{I} c_j \Big\{ \int_{\alpha_{i-1}+\lambda(z_i)}^{\alpha_{i-1}+\kappa(z_i)} z_j(\alpha_i)\, d\alpha_i - \frac{1}{2}\, z_j(\alpha_i) \Big\} \\
&= 1 - 2z_i + c_0 + c_i\alpha_{i-1} + \sum_{j=i+1}^{I} \frac{b_j c_j}{16}\,(1 - 2z_i) \\
&= (1 - 2z_i)\Big(1 + \sum_{j=i+1}^{I} \frac{b_j c_j}{16}\Big) + c_0 + c_i\alpha_{i-1}.
\end{aligned} \tag{31}$$

Since this is decreasing in z_i, the first-order condition yields the optimal solution

$$z_i(\alpha_{i-1}) = \frac{1}{2}\Big(1 + \frac{c_0 + c_i\alpha_{i-1}}{1 + \sum_{j=i+1}^{I} b_j c_j/16}\Big).$$

Furthermore, since each z_i is an affine function,

$$\begin{aligned}
\frac{\partial f_{i+1}}{\partial \alpha_{i-1}}(z_i, \alpha_{i-1}) &= z_i \frac{\partial \pi^*_{i+1}}{\partial \alpha_i}(\alpha_{i-1} + \kappa(z_i)) + (1 - z_i) \frac{\partial \pi^*_{i+1}}{\partial \alpha_i}(\alpha_{i-1} + \lambda(z_i)) \\
&= \sum_{j=i+1}^{I} c_j \Big\{ z_i z_j(\alpha_{i-1} + \kappa(z_i)) + (1 - z_i) z_j(\alpha_{i-1} + \lambda(z_i)) \Big\} \\
&= \sum_{j=i+1}^{I} c_j z_j \Big(\alpha_{i-1} + z_i \kappa(z_i) + (1 - z_i)\lambda(z_i) \Big) \\
&= \frac{\partial \pi^*_{i+1}}{\partial \alpha_i}(\alpha_{i-1}).
\end{aligned} \tag{32}$$

Hence, the envelope theorem implies that

$$\frac{\partial \pi^*_i}{\partial \alpha_{i-1}}(\alpha_{i-1}) = c_i z_i(\alpha_{i-1}) + \frac{\partial \pi^*_{i+1}}{\partial \alpha_i}(\alpha_{i-1}) = \sum_{j=i}^{I} c_j z_j(\alpha_{i-1}).$$

This advances the induction step and completes the proof. ■

Proof of Lemma 6.2 Simple algebra verifies that (12) holds if and only if

$$\Big(1 + \frac{1}{16} \sum_{k=t+2}^{I} b_{r(k)} c_{r(k)}\Big)(c_{r(t)} - c_{r(t+1)}) \le \frac{1}{16} b_{r(t+1)} c^2_{r(t+1)}. \tag{33}$$

Since $b_{r(t)} \le c_{r(t)} \le \frac{2}{I-1}$, we have

$$b_{r(t)} \ge \frac{c_{r(t)}}{1 + \frac{1}{16} \sum_{k=t+1}^{I} c^2_{r(k)}} \ge \frac{c_{r(t)}}{1 + \frac{I-t}{4(I-1)^2}}.$$

Hence, (33) holds if

$$\Big(1 + \frac{1}{16} \sum_{k=t+2}^{I} c^2_{r(k)}\Big)(c_{r(t)} - c_{r(t+1)}) \le \frac{c^3_{r(t+1)}}{16\Big\{1 + \frac{I-t-1}{4(I-1)^2}\Big\}}.$$

Denoting $r(t) = i$ and $r(t+1) = j$, we see that the above is implied by (10). ■

Proof of Corollary 6.5 Suppose first that $\alpha_{t-1} \ge 0$ and that buyer t accepts the seller's offer. In this case, $\alpha_t = \alpha_{t-1} + \frac{1}{4}(1 - b_t \alpha_{t-1}) \ge 0$. Therefore, $x_{t+1}(\alpha_t) \ge x_t(\alpha_{t-1})$ if and only if

$$\Big(c_{t+1} - \frac{b_{t+1}}{2}\Big)\Big\{\alpha_{t-1} + \frac{1}{4}(1 - b_t \alpha_{t-1})\Big\} \ge \Big(c_t - \frac{b_t}{2}\Big)\alpha_{t-1}.$$

Let

$$\kappa = 1 - \frac{1}{2\{1 + \frac{1}{16}(I-1)\epsilon^2\}}.$$

Since $\epsilon \geq c_{t+1} \geq c_t \geq b_t \geq \frac{c_t}{1+\frac{1}{16}(I-t)\epsilon^2} \geq 2c_t(1-\kappa)$, and $c_{t+1} \geq b_{t+1}$, the above inequality is implied by

$$\frac{c_{t+1}}{2}\left\{\alpha_{t-1} + \frac{1}{4}(1-\epsilon\alpha_{t-1})\right\} \geq \kappa\, c_{t+1}\alpha_{t-1}.$$

Dividing both sides by $c_{t+1} > 0$, we can rewrite this as

$$\left(2\kappa - 1 + \frac{\epsilon}{4}\right)\alpha_{t-1} \leq \frac{1}{4}.$$

Since $\kappa \to 1/2$ as $\epsilon \to 0$, this inequality holds for any $\alpha_{t-1} \leq \frac{t-1}{2}$ when $\epsilon > 0$ is sufficiently small.

Suppose next that $\alpha_{t-1} \leq 0$ and that buyer t rejects the seller's offer. We then have $\alpha_t = \alpha_{t-1} - \frac{1}{4}(1 + b_t\alpha_{t-1}) \leq 0$. Therefore, $x_{t+1}(\alpha_t) \leq x_t(\alpha_{t-1})$ if and only if

$$\left(c_{t+1} - \frac{b_{t+1}}{2}\right)\left\{\alpha_{t-1} - \frac{1}{4}(1 + b_t\alpha_{t-1})\right\} \leq \left(c_t - \frac{b_t}{2}\right)\alpha_{t-1}.$$

By the same logic as above, this inequality is implied by

$$\frac{c_{t+1}}{2}\left\{\frac{1}{4}(1+\epsilon\alpha_{t-1}) - \alpha_{t-1}\right\} \geq -\kappa\, c_{t+1}\alpha_{t-1}.$$

We can further rewrite this as

$$\left(2\kappa - 1 + \frac{\epsilon}{4}\right)(-\alpha_{t-1}) \leq \frac{1}{4},$$

which is seen to hold for any $\alpha_{t-1} \geq -\frac{t-1}{2}$ when $\epsilon > 0$ is sufficiently small. ■

References

[1] Aoyagi, M. (2005), "Information feedback in a dynamic tournament," mimeo.

[2] Banerjee, A. (1992), "A simple model of herd behvior," *Quarterly Journal of Economics,* **107**, 797-817.

[3] Chamley, C. P. (2004), *Rational Herds,* Cambridge University Press.

[4] de-Frutos, M.-A. and R. Rosenthal (1998), "On some myths about sequenced common-value auctions," *Games and Economic Behavior,* **23**, 201-21.

[5] Kotlar, P. (1980), *Marketing Management: analysis, planning, and control,* Fourth edition, Prentice Hall.

[6] Krishna, V. (2002), *Auction Theory,* Academic Press: San Diego.

[7] Milgrom, P. R. (1981), "Good news and bad news: Representation theorem with applications," *Bell Journal of Economics,* **21**, 380-391.

[8] Milgrom, P. R. and R. Weber (1982), "A theory of auctions and competitive bidding," *Econometrica,* **50**, 1059-1122.

[9] Myerson, R. (1981), "Optimal auction design," *Mathematics of Operations Research,* **6**, 58-73.

[10] Ottaviani, M. and A. Prat (2001), "The value of public information in monopoly," *Econometrica,* **69**, 1673-83.

[11] Perry, M. and P. J. Reny (1999), "On the failure of the linkage principle in multi-unit auctions," *Econometrica,* **67**, 895-900.