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UNIFORM, EQUAL DIVISION, AND OTHER ENVY-FREE RULES BETWEEN THE TWO

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Uniform, equal division, and other envy-free rules between the two*

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Abstract

This paper studies the problem of fairly allocating an amount of a divisible resource when preferences are single-peaked. We characterize the class of *envy-free* and *peak-only* rules and show that the class forms a complete lattice with respect to a dominance relation. We also pin down the subclass of *strategy-proof* rules and show that the subclass also forms a complete lattice. In both cases, the upper bound is the uniform rule, the lower bound is the equal division rule, and any other rule is between the two.

Keywords: Uniform rule, Choice of rules, Lattice, Pareto dominance, Single-peaked preference, Fair allocation.

JEL codes: D71, D63.

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1 Introduction

This paper studies the problem of fairly allocating an amount of a divisible resource among agents whose preferences are single-peaked (Sprumont, 1991). An allocation rule, or simply a *rule*, is a function which maps each single-peaked preference profile to an allocation. The fairness property of the rules we are interested in is *envy-freeness*, which states that, at any chosen allocation, no one should prefer anyone else's consumption to her own (Foley, 1967). The practicality condition we are interested in is *peak-only*, which states that the choice of allocations should only depend on the peaks of preferences. We say "practical," since the user of any *peak-only* rule only needs information on peak amounts of individual preferences, instead of all complicated details.

Our purpose is to study various *envy-free* and *peak-only* rules and to clarify the structure of the set of those rules. We do not impose *efficiency*, although our main results are deeply related to it. The aim is to extract pure implications of *envy-freeness* and *peak-only* as much as possible. However, it will turn out that the absence of *efficiency* does clarify the role of *efficiency* in some existing results in the literature, and in this sense, we are studying *efficiency*.

We have two main theorems. In our first main theorem, we show that a rule is *envy-free* and *peak-only* if and only if it satisfies Kolm's strong fairness condition of *convex envy-freeness* and some mild conditions, and also offer a functional characterization of any such rule. Furthermore, it is proved that the set of these rules forms a complete lattice with respect to a dominance relation. In our second main theorem, we impose *strategy-proofness* on *envy-free* and *peak-only* rules. We then offer a functional characterization of any such rule and prove that the set of these rules also forms a complete lattice with respect to the dominance relation. In both theorems, the unique upper (resp. lower) bound of the dominance relation is the uniform (resp. equal division) rule, and any other rule lies between the two rules. This implies that, in the choice problem of a rule from the set of these rules, there is always the unanimous agreement that the uniform rule is the best and the equal division rule is the worst.

Our work particularly follows the interesting works by Thomson (1994), Chun (2000), and Kesten (2006). Thomson (1994) shows that the uniform rule is the only *efficient*, *envy-free*, and *peak-only* rule, and Chun (2000) shows that the uniform rule is the only *efficient* and *convex envy-free* rule. Our results give some insights into the role of *efficiency* in the list of their axioms, since the results do not rely on *efficiency* with keeping other axioms. Kesten (2006) shows that any *envy-free* and *peak-only* rule is *convex envy-free*. He also points out Paretian dominance relations over the set of *convex envy-free* allocations. Our work can be seen as an extension of his work that offers full characterizations of some *convex envy-free* rules and sets of the rules.[1]

The rest of the paper proceeds as follows: Section 2 offers the model. Section 3 presents main results. Section 4 concludes the paper. Some proofs are relegated to the Appendix.

2 Basic definitions

2.1 Model

Let $N \equiv \{1, 2, \ldots, n\}$ be the finite set of *agents*. There is a fixed amount of an infinitely divisible resource $\Omega > 0$ to be allocated. An *allotment* for $i \in N$ is $x_i \in [0, \Omega]$. An *allocation* is a vector of allotments $x \equiv (x_1, x_2, \ldots, x_n) \in [0, \Omega]^N$ such that $\sum_{i \in N} x_i = \Omega$. Let X be the set of allocations. Given $x \in X$, let $\underline{x} \equiv \min_{i \in N} x_i$ and $\overline{x} \equiv \max_{i \in N} x_i$.

A *single-peaked preference* is a transitive, complete, and continuous binary relation R_i over $[0, \Omega]$ for which there exists a unique point $p_i \in [0, \Omega]$ such that for each $x_i, x_i' \in [0, \Omega]$,

$$[x_i' < x_i \leq p_i \text{ or } p_i \leq x_i < x_i'] \Longrightarrow x_i \; P_i \; x_i',$$

where the symmetric and asymmetric parts of R_i are denoted by I_i and P_i, respectively. The point p_i is called the *peak* of R_i, and the profile of peaks is denoted by $p \equiv (p_1, p_2, \ldots, p_n)$. Let $\mathscr{R}$ be the set of single-peaked preferences and $\mathscr{R}^N$ the set of single-peaked preference profiles $R \equiv (R_1, R_2, \ldots, R_n)$.

2.2 Axioms and rules

A *rule* is a function $f \colon \mathscr{R}^N \to X$ which maps a preference profile $R \in \mathscr{R}^N$ to an allocation $f(R) \equiv (f_1(R), f_2(R), \ldots, f_n(R)) \in X$. Let $\mathscr{F}$ be the set of rules. The following axioms of rules are standard:

- *Efficiency*: An allocation $x \in X$ is *efficient* for $R \in \mathscr{R}^N$ if $\sum_{i \in N} p_i \leq \Omega$ implies $p \leqq x$ and $\sum_{i \in N} p_i \geq \Omega$ implies $p \geqq x$. A rule f is *efficient* if for each $R \in \mathscr{R}^N$, $f(R)$ is *efficient* for R.

[1] We thank the associate editor for raising our attention to Kesten's paper.

- *Envy-freeness* (Foley, 1967): An allocation $x \in X$ is *envy-free* for $R \in \mathscr{R}^N$ if for each $i, j \in N$, $x_i \; R_i \; x_j$. A rule f is *envy-free* if for each $R \in \mathscr{R}^N$, $f(R)$ is *envy-free* for R.

- *Convex envy-freeness* (Kolm, 1973): An allocation $x \in X$ is *convex envy-free* for $R \in \mathscr{R}^N$ if for each $i \in N$ and each $a \in [\underline{x}, \overline{x}]$, $x_i \; R_i \; a$. A rule f is *convex envy-free* if for each $R \in \mathscr{R}^N$, $f(R)$ is *convex envy-free* for R.

- *Peak-only*: For each $R, R' \in \mathscr{R}^N$ with $p = p'$, $f(R) = f(R')$.

- *Strategy-proofness*: For each $R \in \mathscr{R}^N$, each $i \in N$, and each $R'_i \in \mathscr{R}$, $f_i(R) \; R_i \; f_i(R'_i, R_{-i})$.

- *Non-bossiness*: For each $R \in \mathscr{R}^N$, each $i \in N$, and each $R'_i \in \mathscr{R}$, if $f_i(R) = f_i(R'_i, R_{-i})$, then $f(R) = f(R'_i, R_{-i})$.

We also introduce much weaker versions of *strategy-proofness* and *non-bossiness*, which only concern preferences with unchanged peaks. They are also trivially implied by *peak-only*.

- *Strategy-proofness for same peaks*: For each $R \in \mathscr{R}^N$, each $i \in N$, and each $R'_i \in \mathscr{R}$ with $p_i = p'_i$, $f_i(R) \; R_i \; f_i(R'_i, R_{-i})$.

- *Non-bossiness for same peaks*: For each $R \in \mathscr{R}^N$, each $i \in N$, and each $R'_i \in \mathscr{R}$ with $p_i = p'_i$, if $f_i(R) = f_i(R'_i, R_{-i})$, then $f(R) = f(R'_i, R_{-i})$.

Since the seminal works by Benassy (1982) and Sprumont (1991), the following rule has played the central role in the literature.[2] It satisfies all the axioms defined above:

Uniform rule, U: For each $R \in \mathscr{R}^N$ and each $i \in N$,

$$U_i(R) \equiv \begin{cases} \min\{p_i, \lambda\} & \text{if } \sum_{j \in N} p_j \geq \Omega, \\ \max\{p_i, \lambda\} & \text{if } \sum_{j \in N} p_j \leq \Omega, \end{cases}$$

where λ solves $\sum_{j \in N} U_j(R) = \Omega$.

The next rule satisfies all the axioms except for *efficiency*.[3]

Equal division rule, E: For each $R \in \mathscr{R}^N$ and each $i \in N$, $E_i(R) \equiv \Omega/n$.

[2] We refer to Thomson (2005) for a survey on various characterizations of the uniform rule.

[3] This rule is characterized by Bochet and Sakai (2007) on the basis of a strong implementability condition.

2.3 Binary relations

This subsection introduces some standard definitions on binary relations.

Partial ordering. A binary relation $\succsim$ on a set A is a *partial ordering* if it satisfies:

- *Reflexivity*: For each $a \in A$, $a \succsim a$,
- *Transitivity*: For each $a, b, c \in A$, $[a \succsim b$ and $b \succsim c]$ implies $a \succsim c$,
- *Anti-symmetry*: For each $a, b \in A$, $[a \succsim b$ and $b \succsim a]$ implies $a = b$.

Then a pair $(A, \succsim)$ is called a *partially ordered set*.

Linear ordering. A binary relation $\succsim$ on a set A is a *linear ordering* if it is a partial ordering that satisfies:

- *Completeness*: For each $a, b \in A$, $a \succsim b$ or $b \succsim a$.

Then a pair $(A, \succsim)$ is called a *linearly ordered set*.

Lattice theoretic notions. Consider a partial ordering $\succsim$ on a set A.

- *Join*: Given $B \subseteq A$, an element $a \in A$ is the *join* of B for $\succsim$ if it is the least maximal of B according to $\succsim$; that is, (i) for each $b \in B$, $a \succsim b$ and (ii) for each $a' \in A$, $[a' \succsim b$ for each $b \in B]$ implies $a' \succsim a$.
- *Meet*: Similarly, an element $a \in A$ is the *meet* of B for $\succsim$ if it is the greatest minimal of B; that is, (i) for each $b \in B$, $b \succsim a$ and (ii) for each $a' \in A$, $[b \succsim a'$ for each $b \in B]$ implies $a \succsim a'$.
- *Lattice*: A partially ordered set $(A, \succsim)$ is a *lattice* if for each $a, b \in A$, there exist the join and meet of $\{a, b\}$ for $\succsim$.
- *Complete lattice*: A partially ordered set $(A, \succsim)$ is a *complete lattice* if for each $B \subseteq A$, there exist the join and meet of B for $\succsim$.

If they exist, the join and the meet of B are uniquely determined by anti-symmetry of $\succsim$.

Given $Y \subseteq X$ and $R \in \mathscr{R}^N$, the *dominance relation* on Y, $dom[R]$, is defined to be the binary relation on Y such that for each $x, y \in Y$,

$$x \; dom[R] \; y \iff [x_i \; R_i \; y_i \quad \forall i \in N].$$

We shall analyze the order structure of any set $\mathscr{G} \subseteq \mathscr{F}$. The *dominance relation* on $\mathscr{G}$ is denoted by *dom*, which is defined by, for each $f, g \in \mathscr{G}$,

$$f \; dom \; g \iff [f(R) \; dom[R] \; g(R) \;\; \forall R \in \mathscr{R}^N].$$

Note that *dom* is a partial ordering on $\mathscr{G}$. In particular, we denote by $\mathscr{F}^e \subseteq \mathscr{F}$ the set of *envy-free* and *peak-only* rules and by dom^e the dominance relation on $\mathscr{F}^e$, and by $\mathscr{F}^{es} \subseteq \mathscr{F}$ the set of *envy-free*, *peak-only*, and *strategy-proof* rules and by dom^{es} the dominance relation on $\mathscr{F}^{es}$. These notations will appear in the proofs of Theorems 1 and 2.

3 Characterizations

We offer a series of propositions that characterize certain geometric properties or axiomatic relations concerning *convex envy-free* allocations. To do so, it is convenient to denote by $C(R) \subseteq X$ the set of *convex envy-free* allocations for R. These propositions will be finalized into our main theorems.

Proposition 1. *An allocation $x \in X$ is convex envy-free for $R \in \mathscr{R}^N$ if and only if for each $i \in N$,*

$$x_i < p_i \Longrightarrow x_i = \overline{x},$$
$$p_i < x_i \Longrightarrow x_i = \underline{x}.$$

Proof. It is easy to check the "if" part. Let us prove the "only if" part. For each $i \in N$ with $x_i < p_i$, if $x_i < \overline{x}$, then $a \; P_i \; x_i$ for each $a \in (x_i, \overline{x})$, a contradiction to *convex envy-freeness*. Hence, $x_i < p_i$ implies $x_i = \overline{x}$. Similarly, we can show that $p_i < x_i$ implies $x_i = \underline{x}$. □

The *variance function* is a function $var\colon \mathbb{R}^N_+ \to \mathbb{R}_+$ defined by, for each $x \in X$,

$$var(x) \equiv \frac{1}{n} \sum_{i \in N} \left(x_i - \frac{\Omega}{n} \right)^2.$$

The second proposition clarifies how *convex envy-free* allocations can be mutually compared in view of variance or dominance.

Proposition 2. *For every $R \in \mathscr{R}^N$ and every $x, y \in C(R)$,*

$$\underline{x} \leq \underline{y} \iff \overline{y} \leq \overline{x} \iff var(x) \geq var(y) \iff x \; dom[R] \; y; \tag{C1}$$

$$\underline{x} = \underline{y} \iff \overline{x} = \overline{y} \iff var(x) = var(y) \iff x = y; \tag{C2}$$

$$U(R)\ dom[R]\ x\ dom[R]\ E(R). \tag{C3}$$

Proof. See, the Appendix. □

The fact that the variance of the uniform allocation is larger than that of any other *convex envy-free* allocation is first shown by Chun (2000). This is a converse implication to a result by Schummer and Thomson (1997) which states that the variance of the uniform allocation is always smaller than that of any other *efficient* allocation. (C3) is first obtained by Kesten (2006, Proposition 3). Kesten (2006, pp. 199–200) also mentions a procedure that obtains all *convex envy-free* allocations from the equal division allocation, and then points out that all *convex envy-free* allocations are Pareto ranked. Our Proposition 2 can be seen as a completion of their arguments that offers full details of relations among *convex envy-free* allocations.

Proposition 3. *For each $R, R' \in \mathscr{R}^N$ with $p = p'$, $C(R) = C(R')$.*

Proof. It suffices to show that for each $R, R' \in \mathscr{R}^N$ with $p = p'$ and each $x \in C(R)$, we have $x \in C(R')$. To do so, we shall prove that, given any $i \in N$ and any $a \in [\underline{x}, \overline{x}]$, $x_i\ R'_i\ a$. If $x_i = p_i$, then $x_i = p'_i$, so that $x_i\ R'_i\ a$. If $p_i < x_i$, then $p'_i < x_i = \underline{x}$ by Proposition 1, so that $x_i\ R'_i\ a$. If $x_i < p_i$, then $\overline{x} = x_i < p'_i$ by Proposition 1, so that $x_i\ R'_i\ a$. Therefore, $x \in C(R')$ in all cases. □

Proposition 3 does not imply that all *convex envy-free* rules are *peak-only*, since the choice of one allocation from the same set $C(R) = C(R')$ may depend on information other than peaks. However, the next result shows that, under mild conditions, *convex envy-freeness* in fact implies *peak-only*.

Proposition 4. *If a rule is convex envy-free, strategy-proof for same peaks, and non-bossy for same peaks, then it is peak-only.*

Proof. Let f be any *convex envy-free*, *strategy-proof for same peaks*, and *non-bossy for same peaks* rule. By *non-bossiness for same peaks*, it suffices to show that for each $R \in \mathscr{R}^N$, $i \in N$, and each $R'_i \in \mathscr{R}$ with $p_i = p'_i$, we have $f_i(R) = f_i(R'_i, R_{-i})$. If $f_i(R) = p_i$, then by *strategy-proofness for same peaks*, $f_i(R'_i, R_i) = p_i$. Consider the case $f_i(R) \neq f_i(R'_i, R_{-i})$. By *strategy-proofness for same peaks*, either $f_i(R) < p_i < f_i(R'_i, R_{-i})$ or $f_i(R'_i, R_{-i}) < p_i < f_i(R)$. We only consider the subcase $f_i(R) < p_i < f_i(R'_i, R_{-i})$, since the other subcase can be parallely shown. Then Proposition 1 implies

$$\overline{f(R)} = f_i(R) < f_i(R'_i, R_{-i}) = \underline{f(R'_i, R_{-i})},$$

but this contradicts the definition of allocations. □

The next result by Kesten (2006, Proposition 1) is somewhat a converse of Proposition 4. We give a proof for completeness.

Proposition 5. *If a rule is envy-free and peak-only, then it is convex envy-free.*

Proof. Let f be an *envy-free* and *peak-only* rule. Pick any $R \in \mathscr{R}^N$. By Proposition 1, we need to show that $p_i < f_i(R)$ implies $f_i(R) = \underline{f(R)}$ and $f_i(R) < p_i$ implies $f_i(R) = \overline{f(R)}$. In the case $p_i < f_i(R)$, if $f_j(R) < f_i(R)$ for some j, then, whenever R'_i is such that $p'_i = p_i$ and $f_j(R)\ P'_i\ f_i(R)$, i envies j under (R'_i, R_{-i}) by *peak-only*, a contradiction. The parallel proof applies to the case $f_i(R) < p_i$. □

We are now in a position to offer our first main theorem:

Theorem 1. *The following three statements on any rule f are equivalent*:
(i) *f is envy-free and peak-only*;
(ii) *f is convex envy-free, strategy-proof for same peaks, and non-bossy for same peaks*;
(iii) *There exists a function $g\colon [0,\Omega]^N \to X$ such that for each $R \in \mathscr{R}^N$ and each $i \in N$,*

$$f_i(R) = g_i(p), \tag{G1}$$

$$p_i < g_i(p) \Longrightarrow g_i(p) = \underline{g(p)}, \tag{G2}$$

$$g_i(p) < p_i \Longrightarrow g_i(p) = \overline{g(p)}. \tag{G3}$$

Furthermore, the set of these rules is a complete lattice with respect to the dominance relation, whose greatest, least elements are the uniform rule, the equal division rule, respectively.

Proof. The equivalence between (i) and (ii) follows from Propositions 4 and 5. The equivalence between (i) and (iii) follows from Proposition 1.

We next show that $(\mathscr{F}^e, dom^e)$ is a complete lattice. Let $\mathscr{G} \subseteq \mathscr{F}^e$. Define the rule $\bigvee \mathscr{G}$ by, for each $R \in \mathscr{R}^N$,

$$\bigvee \mathscr{G}(R) \equiv x,$$

where x is chosen such that $x \in C(R)$ and

$$\underline{x} = \inf_{g \in \mathscr{G}} \underline{g(R)}.$$

Note that the existence of x follows from the compactness of $C(R)$ and the uniqueness of x follows from Proposition 2. Thus $\bigvee \mathscr{G}$ is well-defined. Obviously, $\bigvee \mathscr{G}$ is the unique least upper bound of $\mathscr{G}$. The unique greatest lower bound of $\mathscr{G}$ can be parallely found. Thus $(\mathscr{F}^e, dom^e)$ is a complete lattice.

The fact that the uniform, the equal division rules are the greatest, least elements of $(\mathscr{F}^e, dom^e)$, respectively, immediately follows from Proposition 2. □

Thomson (1994, Lemma 1) shows that the uniform rule is the only *efficient*, *envy-free*, and *peak-only* rule. The equivalence between (i) and (iii) in Theorem 1 clarifies what happens if *efficiency* is dropped from the list of Thomson's axioms. Theorem 1 also implies that, under *envy-freeness* and *peak-only*, the uniform rule can be selected without caring who gains or loses from the choice of rules, since everyone gains by the use of the uniform rule independent of their preferences.

Given Theorem 1, a natural question is if there is any interesting sublattice. We consider this question for the *strategy-proof* subclass.

Theorem 2. *The following three statements on any rule f are equivalent*:
(i) *f is envy-free, strategy-proof, and peak-only*;
(ii) *f is convex envy-free, strategy-proof, and non-bossy for same peaks*;
(iii) *For every $i \in N$, there exist two functions $\underline{h}_i, \overline{h}_i \colon [0, \Omega]^{N\setminus\{i\}} \to [0, \Omega]$ such that for each $R \in \mathscr{R}^N$,*

$$f_i(R) = \text{med}[p_i, \underline{h}_i(p_{-i}), \overline{h}_i(p_{-i})], \tag{H1}$$

$$p_i < \underline{h}_i(p_{-i}) \Longrightarrow \text{med}[p_i, \underline{h}_i(p_{-i}), \overline{h}_i(p_{-i})] = \min_{j\in N} \text{med}[p_j, \underline{h}_j(p_{-j}), \overline{h}_j(p_{-j})], \tag{H2}$$

$$\underline{h}_i(p_{-i}) < p_i \Longrightarrow \text{med}[p_i, \underline{h}_i(p_{-i}), \overline{h}_i(p_{-i})] = \max_{j\in N} \text{med}[p_j, \underline{h}_j(p_{-j}), \overline{h}_j(p_{-j})]. \tag{H3}$$

Furthermore, the set of these rules is a complete lattice with respect to the dominance relation, whose greatest, least elements are the uniform rule, the equal division rule, respectively.

Proof. The equivalence between (i) and (ii) follows from Propositions 4 and 5. One can easily show that (iii) implies (i).

Let us prove that (i) implies (iii). Pick any *envy-free*, *strategy-proof*, and *peak-only* rule f and let $g \colon [0, \Omega]^N \to X$ be the associated function satisfying the conditions in (iii) of Theorem 1. For each $i \in N$, define two functions $\underline{h}_i, \overline{h}_i \colon [0, \Omega]^{N\setminus\{i\}} \to [0, \Omega]$ by, for every $p_{-i} \in [0, \Omega]^{N\setminus\{i\}}$,

$$\underline{h}_i(p_{-i}) \equiv g_i(0, p_{-i}),$$

$$\overline{h}_i(p_{-i}) \equiv g_i(\Omega, p_{-i}).$$

By *strategy-proofness*, i has no incentive to report peak Ω when her true peak is zero, and hence $\underline{h}_i(p_{-i}) \leq \overline{h}_i(p_{-i})$. Similarly, one can easily show by *strategy-proofness* that $p_i < \underline{h}_i(p_{-i})$ implies $f_i(R) = \underline{h}_i(p_{-i})$ and $\overline{h}_i(p_{-i}) < p_i$ implies $f_i(R) = \overline{h}_i(p_{-i})$. Next, if $\underline{h}_i(p_{-i}) < p_i < \overline{h}_i(p_{-i})$, then by *strategy-proofness* and *peak-only*, $p_i = f_i(R)$. In either case, we obtain $f_i(R) = \text{med}[p_i, \underline{h}_i(p_{-i}), \overline{h}_i(p_{-i})]$, meaning that (H1) holds. Then (H2) and (H3) immediately follow from Theorem 1.

We next establish the complete lattice structure of $\mathscr{F}^{es}$ with respect to the dominance relation. Let $\mathscr{G} \subseteq \mathscr{F}^{es}$. By Theorem 1, there uniquely exist the join and meet of $\mathscr{G}$, $\bigvee \mathscr{G}, \bigwedge \mathscr{G} \in \mathscr{F}^{e}$, respectively. To prove that $(\mathscr{F}^{es}, dom^{es})$ is a complete lattice, it suffices to show that $\bigvee \mathscr{G}$, $\bigwedge \mathscr{G}$ are *strategy-proof*. For each $i \in N$ and each $R \in \mathscr{R}^N$, by definition of the join,

$$\bigvee \mathscr{G}_i(R) \; R_i \; g_i(R) \quad \forall g \in \mathscr{G}. \tag{1}$$

For each $i \in N$, each $R \in \mathscr{R}^N$, and each $R'_i \in \mathscr{R}$, by *strategy-proofness*,

$$g_i(R) \; R_i \; g_i(R'_i, R_{-i}) \quad \forall g \in \mathscr{G}. \tag{2}$$

For each $i \in N$, each $R \in \mathscr{R}^N$, and each $R'_i \in \mathscr{R}$, (1) and (2) together imply

$$\bigvee \mathscr{G}_i(R) \; R_i \; g_i(R'_i, R_{-i}) \quad \forall g \in \mathscr{G},$$

and then by definition of $\bigvee \mathscr{G}_i(R'_i, R_{-i})$ and continuity of R_i,

$$\bigvee \mathscr{G}_i(R) \; R_i \bigvee \mathscr{G}_i(R'_i, R_{-i}).$$

Thus $\bigvee \mathscr{G}$ is *strategy-proof*. We can similarly show that $\bigwedge \mathscr{G}$ is *strategy-proof*, too. Therefore, $(\mathscr{F}^{es}, dom^{es})$ is a complete lattice. □

4 Conclusion

We characterized *envy-free* and *peak-only* rules and clarified the complete lattice structure of the class of these rules. We also imposed *strategy-proofness* to the rules and then identified functional forms of the rules and again found the complete lattice structure of the *strategy-proof* subclass. These results enable us to easily compare any two such rules in view of dominance relations and suggest how strong the position of the uniform rule is and how weak the position of the equal division rule. In general, this kind of easy-to-compare relations is rarely observed, except for two-sided matching problems (e.g., Roth and Sotomayor, 1990). Thus results like ours are rather infrequent. In the theorems, we found the existence of certain functions characterizing rules, but did not clarify concrete forms of the functions. Since they seem to have non-trivial complicated forms, obtaining simpler forms by imposing additional axioms is of interest as a future research.

Appendix: Proof of Proposition 2

The proof proceeds in several lemmas.

Lemma 1. *For each $R \in \mathscr{R}^N$ and each $x \in C(R)$, if $\underline{x} < \overline{x}$, then for each $i \in N$,*

$$x_i = \underline{x} \iff p_i \leq \underline{x}, \tag{3}$$
$$x_i = \overline{x} \iff \overline{x} \leq p_i. \tag{4}$$

Proof. We only prove (3), since (4) can be shown by a parallel way. If $x_i = \underline{x}$ but $\underline{x} < p_i$, then $x_i < p_i$. By Proposition 1, $x_i = \overline{x}$, a contradiction to $\underline{x} < \overline{x}$. Next, if $p_i \leq \underline{x}$ but $\underline{x} < x_i$, then $p_i < x_i$, a contradiction to Proposition 1. □

Lemma 2. *For each $R \in \mathscr{R}^N$, each $x \in C(R)$, and each $i \in N$,*

$$\underline{x} \leq p_i \leq \overline{x} \Longrightarrow p_i = x_i, \tag{5}$$
$$\underline{x} < x_i < \overline{x} \Longrightarrow p_i = x_i. \tag{6}$$

Proof. We only prove (5), since (6) can be shown by a parallel way. By a contraposition argument, suppose that $p_i \neq x_i$. Consider the case $p_i < x_i$. Then by Proposition 1, $x_i = \underline{x}$, so $p_i < \underline{x}$. Next consider the case $x_i < p_i$. Then by Proposition 1, $x_i = \overline{x}$, so $\overline{x} < p_i$. □

Lemma 3. *For each $R \in \mathscr{R}^N$ and each $x, y \in C(R)$, if $\underline{x} = \underline{y}$, then $x = y$.*

Proof. Assume $\underline{x} = \underline{y}$. If $\underline{x} = \overline{x}$ or $\underline{y} = \overline{y}$, then by feasibility, $x = E(R) = y$. Hence, let us consider the case $\underline{x} < \overline{x}$ and $\underline{y} < \overline{y}$. Without loss of generality, we can assume $\overline{x} \leq \overline{y}$. Let

$$N(\underline{x}) \equiv \{i \in N \colon p_i \leq \underline{x}\},$$
$$N(\underline{y}) \equiv \left\{i \in N \colon p_i \leq \underline{y}\right\},$$
$$N(\overline{x}) \equiv \{i \in N \colon \overline{x} \leq p_i\},$$
$$N(\overline{y}) \equiv \{i \in N \colon \overline{y} \leq p_i\}.$$

Note that $N(\overline{y}) \subseteq N(\overline{x})$.

Since $\underline{x} = \underline{y}$ is assumed, we have $N(\underline{x}) = N(\underline{y})$, and by Lemma 1,

$$x_i = \underline{x} = \underline{y} = y_i \quad \forall\, i \in N(\underline{x}). \tag{7}$$

By Lemma 2,

$$x_i = p_i = y_i \;\; \forall\, i \in N \setminus (N(\underline{x}) \cup N(\overline{x})). \tag{8}$$

By Lemmas 1 and 2,

$$x_i = \overline{x} \leq p_i = y_i \;\; \forall\, i \in N(\overline{x}) \setminus N(\overline{y}). \tag{9}$$

By Lemma 1,

$$x_i = \overline{x} \leq \overline{y} = y_i \;\; \forall\, i \in N(\overline{y}). \tag{10}$$

Since $\sum_{i \in N} x_i = \sum_{i \in N} y_i$, (7)–(10) together imply $x = y$. □

Lemma 4. *For each $R \in \mathscr{R}^N$ and each $x, y \in C(R)$, if $\underline{x} < \underline{y}$, then $\overline{y} < \overline{x}$.*

Proof. Suppose, by contradiction, that there exist $R \in \mathscr{R}^N$ and $x, y \in C(R)$ such that $\underline{x} < \underline{y}$ and $\overline{x} \leq \overline{y}$. By feasibility, $\underline{x} < \underline{y} \leq \overline{x} \leq \overline{y}$ and $\underline{y} < \overline{y}$. By Lemmas 1 and 2,

$$\begin{aligned}
p_i \leq \underline{x} &\Longrightarrow x_i = \underline{x} < \underline{y} = y_i, && (11)\\
\underline{x} < p_i \leq \underline{y} &\Longrightarrow x_i = p_i \leq \underline{y} = y_i, && (12)\\
\underline{y} < p_i \leq \overline{x} &\Longrightarrow x_i = p_i = y_i, && (13)\\
\overline{x} < p_i \leq \overline{y} &\Longrightarrow x_i = \overline{x} < p_i = y_i, && (14)\\
\overline{y} < p_i &\Longrightarrow x_i = \overline{x} \leq \overline{y} = y_i. && (15)
\end{aligned}$$

For $j \in N$ such that $x_j = \underline{x}$, Lemma 1 implies $p_j \leq \underline{x}$, so $x_j < y_j$. Hence, (11)–(15) together imply $\sum_{i \in N} x_i < \sum_{i \in N} y_i$, a contradiction. □

Lemma 5. *For each $R \in \mathscr{R}^N$ and each $x, y \in C(R)$, if $\underline{x} < \underline{y}$, then x dom$[R]$ y and not y dom$[R]$ x.*

Proof. Immediately follows from Lemmas 1–4. □

Proof of Proposition 2. (C1) and (C2) immediately follow from Lemmas 1–5. (C3) is a direct consequence from (C1) and (C2). □

References

Benassy, J-P. (1982) *The Economics of Market Disequilibrium*, Academic Press, New York.

Bochet, O. and Sakai, T. (2007) "Secure implementation in allotment economies," mimeo, Maastricht University and Yokohama National University.

Chun, Y. (2000) "Distributional properties of the uniform rule in economies with single-peaked preferences," *Economics Letters* **67**, 23–27.

Foley, D. (1967) "Resource allocation and the public sector," *Yale Economic Essays* **7**, 45–98.

Kesten, O. (2006) "More on the uniform rule: Characterizations without Pareto optimality," *Mathematical Social Sciences* **51**, 192–200.

Kolm, S.C. (1973) "Super-équité," *Kyklos* **26**, 841–843.

Roth, A.E. and Sotomayor, M. (1990) *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*, Cambridge University Press, Cambridge.

Schummer, J. and Thomson, W. (1997) "Two derivations of the uniform rule and an application to bankruptcy," *Economics Letters* **55**, 333–337.

Sprumont, Y. (1991) "The division problem with single-peaked preferences: A characterization of the uniform allocation rule," *Econometrica* **59**, 509–519.

Thomson, W. (1994) "Resource monotonic solutions to the problem of fair division when preferences are single-peaked," *Social Choice and Welfare* **11**, 205–224.

Thomson, W. (2005) *The Theory of Fair Allocation*, Ch. 11, unpublished book manuscript, University of Rochester.