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SECURE IMPLEMENTATION IN SHAPLEY-SCARF HOUSING MARKETS

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Abstract

This paper considers the object allocation problem introduced by Shapley and Scarf (1974). We study secure implementation (Saijo, Sjöström, and Yamato, 2007), that is, double implementation in dominant strategy and Nash equilibria. We prove that (i) an *individually rational* solution is securely implementable if and only if it is the no-trade solution, (ii) a *neutral* solution is securely implementable if and only if it is a serial dictatorship, and (iii) an *efficient* solution is securely implementable if and only if it is a sequential dictatorship. Furthermore, we provide a complete characterization of securely implementable solutions in the two-agent case.

Keywords: Secure implementation; Sequential dictatorship; Strict core; Strategy-proofness, Shapley-Scarf housing markets.

JEL codes: C72; C78; D61; D63; D71.

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1 Introduction

We consider the object allocation problem introduced by Shapley and Scarf (1974) with strict preferences. There is a group of agents, each of whom initially owns one object.[1] A solution reallocates the objects with the condition that each agent consumes one and only one object. Important real-life examples of this model are the assignment of campus housing to students (Abdulkadiroğlu and Sönmez, 1999; Chen and Sönmez, 2002, 2004; and Sönmez and Ünver, 2005) and kidney exchange (Roth, Sönmez, and Ünver, 2004).

In this context, the "strict core solution" is a central one since it satisfies various desirable properties. Some characterizations of the solution can be found in Ma (1994), Svensson (1999), Takamiya (2001), and Miyagawa (2002). Furthermore, the solution is dominant strategy implementable (Mizukami and Wakayama, 2007) and Nash implementable when there are at least three agents (Sönmez, 1996). However, these results do not guarantee that the solution is *securely implementable* (Saijo, Sjöström, and Yamato, 2007); note that here, the notion of implementation signifies double implementation in the two equilibrium concepts. Thus, it is natural to raise the following question: Can the strict core solution be securely implemented? In fact, the answer to this question is no (Saijo, Sjöström, and Yamato, 2007). Based on the result, this paper seeks solutions that can be securely implemented in our model.

Our main results consist of two parts. We first focus on the two-agent case. In this case, we provide a complete characterization of securely implementable solutions; a solution is securely implementable if and only if it is either a constant solution or a "serial dictatorship." By a serial dictatorship, we mean that one agent chooses her best object from among the set of objects, then the second agent chooses his best object from among the set of remaining objects, then the third agent chooses, and so on; the order in which agents make their choices is fixed in advance.

Next, we consider the general case where there are more than two agents. In contrast to the two-agent case, it is hard to characterize the class of securely implementable solutions in the general case. Thus, in the general case, we then pin down smaller classes of securely implementable solutions by adding some properties.

[1] In this paper, the sets of agents and objects are fixed. Some studies consider object allocation problems where either the set of agents or the set of objects varies; for instance, Ergin (2000), Ehlers, Klaus, and Pápai (2002), and Ehlers and Klaus (2003a) consider house allocation problems where each agent consumes at most one object, and Klaus and Miyagawa (2001) and Ehlers and Klaus (2003b) consider multiple assignment problems where agents may consume more than one object.

First, we show that the "no-trade solution" is the unique securely implementable one that satisfies *individual rationality* (no agent is worse off after trading with other agents). The no-trade solution is the one that selects the initial endowments for each preference profile. Second, we prove that a securely implementable solution satisfies *neutrality* (symmetric treatment of objects) if and only if it is a serial dictatorship. Finally, we establish that an *efficient* solution is securely implementable if and only if it is a "sequential dictatorship." For any sequential dictatorship, there exists the first dictator in every preference profile. However, in contrast to serial dictatorships, in the sequential dictatorship, the second agent, who chooses his best object from among the set of remaining objects, is decided by the choice of the first dictator. Similarly, the third agent is decided by the choices of the previous agents, and so on. As far as we know, ours is the first result that characterizes the class of sequential dictatorships in Shapley-Scarf housing markets.

Our model has a close relationship with multiple assignment problems. Klaus and Miyagawa (2001) show that serial dictatorships are the only ones that satisfy *efficiency* and *strategy-proofness* in the two-agent case. In the general case, Pápai (2001) and Ehlers and Klaus (2003b) characterize sequential dictatorships by means of *efficiency*, *strategy-proofness*, and *non-bossiness*. Their characterizations still hold even if *strategy-proofness* and *non-bossiness* are replaced by secure implementability. On the other hand, it should be noted that the results of Klaus and Miyagawa (2001), Pápai (2001), and Ehlers and Klaus (2003b) do not hold in our model. This is because the strict core solution satisfies *efficiency*, *strategy-proofness*, and *non-bossiness*. Therefore, results in multiple assignment problems cannot directly apply to our model.

When monetary transfers are admissible, Fujinaka and Wakayama (2008) show that constant solutions are the only ones that are securely implementable. This means that many solutions including "fixed-price core solutions" (Miyagawa, 2001) that satisfy many desirable properties are not securely implementable when monetary transfers are allowed.

The rest of the paper is organized as follows: Section 2 provides basic notation and definitions. Section 3 addresses the two-agent case. Section 4 analyzes the general case. Section 5 discusses our results. Section 6 concludes the paper. Appendix A contains the proofs of the results omitted from the main text. Appendix B proves that the strict core solution is dominant strategy implemented by its associated direct revelation mechanism; further, the strict core solution is not Nash implementable in the case of two agents.

2 Preliminaries

2.1 The model

We denote the set of *agents* by $N = \{1, 2, \ldots, n\}$, where $2 \leq n < +\infty$. Each agent $i \in N$ owns one object, denoted by i. Thus, N also stands for the set of *objects*.

Each agent $i \in N$ has a complete and transitive binary relation $\succsim_i$ over N, i.e., a *preference relation*. We denote the associated strict preference relation by $\succ_i$ and indifferent relation by $\sim_i$. We assume that all preferences are *strict*; i.e., for each $h, k \in N$, if $h \sim_i k$, then $h = k$. Let $\mathscr{P}$ denote the set of all strict preferences. A *preference profile* is a list of preferences $\succsim \equiv (\succsim_1, \succsim_2, \ldots, \succsim_n) \in \mathscr{P}^N$. We often denote $N \setminus \{i\}$ by "$-i$." With this notation, $(\succsim'_i, \succsim_{-i})$ is the preference profile where agent i has $\succsim'_i$ and agent $j \neq i$ has $\succsim_j$. Similarly, given $S \subseteq N$, we denote $N \setminus S$ by "$-S$," and $(\succsim'_S, \succsim_{-S})$ is the preference profile where each agent $i \in S$ has $\succsim'_i$ and each agent $i \notin S$ has $\succsim_i$. We often represent $\succsim_i$ by an ordered list of objects as follows:

$$\succsim_i\colon h_1, h_2, h_{3,} \ldots$$

This means that agent i prefers object h_1 the most; further, i prefers h_1 to h_2, h_2 to h_3, and so on.

An *allocation* is a bijection $x\colon N \to N$. Let $x(i)$ denote the object allocated to agent $i \in N$. For convenience, we use the notation x_i instead of $x(i)$. Let X be the set of allocations.

2.2 Solutions

A *solution* is a function $f\colon \mathscr{P}^N \to X$ that associates an allocation $x \in X$ with each preference profile $\succsim \in \mathscr{P}^N$. Let $f_i(\succsim)$ denote the object allocated to agent i at $\succsim$.

Let $x, y \in X$ and $S \subseteq N$ with $S \neq \emptyset$. Then, x *weakly dominates* y via S at $\succsim \in \mathscr{P}^N$ if $S = \bigcup_{i \in S}\{x_i\}$, and $x_i \succsim_i y_i$ for each $i \in S$ and $x_j \succ_j y_j$ for some $j \in S$. The *strict core* for $\succsim \in \mathscr{P}^N$ is the set of all allocations that are not weakly dominated by any other allocation at $\succsim \in \mathscr{P}^N$. The *strict core solution* is the solution $C\colon \mathscr{P}^N \to X$ such that for each $\succsim \in \mathscr{P}^N$, $C(\succsim)$ is the strict core for $\succsim$.[2]

A solution f is *constant* if there exists $x \in X$ such that for each $\succsim \in \mathscr{P}^N$,

[2]Under strict preferences, the strict core is a singleton for every preference profile (Roth and Postlewaite, 1977). Thus, the strict core solution C is well-defined.

$f(\succsim) = x$. In particular, we term the constant solution that selects the initial endowments for each preference profile as the *no-trade solution*.

A *permutation* π on N is a bijection $\pi\colon N \to N$. Let Π^N denote the set of all permutations on N. Given that $i \in N$ and $S \subseteq N$, let $b(\succsim_i, S)$ be agent i's most preferred object under $\succsim_i$ in S, i.e., $b(\succsim_i, S) \in S$ and for each $h \in S$, $b(\succsim_i, S) \succsim_i h$. A solution f is a *sequential choice function* if for each $\succsim \in \mathscr{P}^N$, there exists a permutation $\pi_{\succsim} \in \Pi^N$ such that

$$
\begin{aligned}
f_{\pi_{\succsim}(1)}(\succsim) &= b(\succsim_{\pi_{\succsim}(1)}, N);\\
f_{\pi_{\succsim}(2)}(\succsim) &= b(\succsim_{\pi_{\succsim}(2)}, N \setminus \{f_{\pi_{\succsim}(1)}(\succsim)\});\\
f_{\pi_{\succsim}(3)}(\succsim) &= b(\succsim_{\pi_{\succsim}(3)}, N \setminus [\{f_{\pi_{\succsim}(1)}(\succsim)\} \cup \{f_{\pi_{\succsim}(2)}(\succsim)\}]);\\
&\vdots\\
f_{\pi_{\succsim}(n)}(\succsim) &= b\left(\succsim_{\pi_{\succsim}(n)}, N \Bigg\backslash \left[\bigcup_{i=1}^{n-1} \left\{f_{\pi_{\succsim}(i)}(\succsim)\right\}\right]\right).
\end{aligned}
$$

We then say that $\pi_{\succsim}(i)$ is the *i-th dictator* at $\succsim$.

The class of sequential dictatorships is a subclass of sequential choice functions. For any sequential dictatorship, there exists a unique first dictator who chooses her best object in every preference profile. However, the second dictator, who chooses his best object from among the set of remaining objects, is decided by the choice of the first dictator. Similarly, the next dictator is decided by the choices of the previous dictators. Formally, a solution f is a *sequential dictatorship* if it is a sequential choice function that satisfies the following properties: for each $\succsim, \succsim' \in \mathscr{P}^N$, (i) $\pi_{\succsim}(1) = \pi_{\succsim'}(1)$ and (ii) for each $j \in N \setminus \{1\}$, if $\pi_{\succsim}(i) = \pi_{\succsim'}(i)$ and $f_{\pi_{\succsim}(i)}(\succsim) = f_{\pi_{\succsim'}(i)}(\succsim')$ for each $i \in \{1, 2, \ldots, j-1\}$, then $\pi_{\succsim}(j) = \pi_{\succsim'}(j)$.

The class of serial dictatorship is a subclass of sequential dictatorships. For any serial dictatorship, the order in which an agent chooses an object from the set of remaining objects is fixed. That is, the order does not depend on the choices of the previous dictators. Formally, a solution f is a *serial dictatorship* if it is a sequential dictatorship and there exists $\bar{\pi} \in \Pi^N$ such that for each $\succsim \in \mathscr{P}^N$, $\pi_{\succsim} = \bar{\pi}$.

2.3 Axioms and secure implementation

In this subsection, we first define a number of basic axioms. The first axiom is a voluntary participation condition, according to which no agent receives an object that she considers worse than her endowment.

Individual rationality: For each $\succsim \in \mathscr{P}^N$ and each $i \in N$, $f_i(\succsim) \succsim_i i$.

The next axiom says that it is impossible to make an agent better off without making someone else worse off.

Efficiency: For each $\succsim \in \mathscr{P}^N$, there does not exist $x \in X$ such that $x_i \succsim_i f_i(\succsim)$ for each $i \in N$ and $x_j \succ_j f_j(\succsim)$ for some $j \in N$.

The next axiom states that a solution is defined independently of the names of the objects. For each $\succsim \in \mathscr{P}^N$ and each $\pi \in \Pi^N$, let $T(\succsim, \pi)$ be a preference profile $\succsim'$ such that for each $i, j, k \in N$,

$$j \succsim_i k \iff \pi(j) \succsim'_i \pi(k).$$

Neutrality: For each $\succsim \in \mathscr{P}^N$, each $\pi \in \Pi^N$, and each $i \in N$, $f_i(T(\succsim, \pi)) = \pi(f_i(\succsim))$.

The last axiom states that no agent can obtain a benefit by misrepresenting her preferences.

Strategy-proofness: For each $\succsim \in \mathscr{P}^N$, each $i \in N$, and each $\succsim'_i \in \mathscr{P}$, $f_i(\succsim) \succsim_i f_i(\succsim'_i, \succsim_{-i})$.

The strict core solution is the central solution in our model. The reason for this is that the strict core solution is the only one that satisfies *strategy-proofness*, *efficiency*, and *individual rationality* (Ma, 1994).[3] Moreover, the solution is dominant strategy implementable (See Appendix B) and Nash implementable when there are at least three agents (Sönmez, 1996).

Saijo, Sjöström, and Yamato (2007) say that a solution is *securely implementable* if there exists a mechanism that doubly implements the solution in dominant strategy and Nash equilibria.[4] They provide a characterization of the class in the abstract setting:[5]

Proposition 1 (Saijo, Sjöström, and Yamato, 2007). *A solution is securely implementable if and only if it satisfies* strategy-proofness *and the* rectangular property.

[3]The strict core solution satisfies not only *strategy-proofness* (Roth, 1982) but also *coalitional strategy-proofness* (Bird, 1984). Other studies on *coalitional strategy-proofness* are, for example, Takamiya (2001) and Ehlers (2002). See those for the definition of *coalitional strategy-proofness*.

[4]See Saijo, Sjöström, and Yamato (2007) for the formal definition of secure implementation.

[5]Mizukami and Wakayama (2008) provide an alternative characterization of securely implementable solutions.

Rectangular property: For each $\succsim, \succsim' \in \mathscr{P}^N$, if $f_i(\succsim') = f_i(\succsim_i, \succsim'_{-i})$ for each $i \in N$, then $f(\succsim) = f(\succsim')$.

Note that the *rectangular property* implies *non-bossiness*, according to which when each agent unilaterally changes her preference report, she cannot influence the total allocation without changing her own consumption.

Non-bossiness: For each $\succsim \in \mathscr{P}^N$, each $i \in N$, and each $\succsim'_i \in \mathscr{P}$, if $f_i(\succsim) = f_i(\succsim'_i, \succsim_{-i})$, then $f(\succsim) = f(\succsim'_i, \succsim_{-i})$.

Fact 1. *If f satisfies the rectangular property, then it satisfies non-bossiness.*

Since the strict core solution is both dominant strategy implementable and Nash implementable, one might conjecture that it is securely implementable. However, Saijo, Sjöström, and Yamato (2004) point out that the strict core solution is *not* securely implementable.[6] To see this, consider the following example:

Example 1. Suppose that $N = \{1, 2, 3\}$. Let $\succsim \in \mathscr{P}^N$ and $\succsim'_1, \succsim'_2 \in \mathscr{P}$ be such that

$$\begin{aligned} &\succsim_1 : 1, 2, 3; \qquad &&\succsim'_1 : 2, 1, 3; \\ &\succsim_2 : 1, 2, 3; &&\succsim'_2 : 2, 1, 3; \\ &\succsim_3 : 3, 2, 1. \end{aligned}$$

Then,

$$\begin{aligned} C(\succsim_1, \succsim_2, \succsim_3) = C(\succsim_1, \succsim'_2, \succsim_3) = C(\succsim'_1, \succsim'_2, \succsim_3) = (1, 2, 3); \\ C(\succsim'_1, \succsim_2, \succsim_3) = (2, 1, 3). \end{aligned}$$

Since $C(\succsim_1, \succsim'_2, \succsim_3) = C(\succsim'_1, \succsim'_2, \succsim_3)$ and $C(\succsim_1, \succsim'_2, \succsim_3) = C(\succsim_1, \succsim_2, \succsim_3)$, the *rectangular property* requires that $C(\succsim_1, \succsim'_2, \succsim_3) = C(\succsim'_1, \succsim_2, \succsim_3)$. However, since $C(\succsim_1, \succsim'_2, \succsim_3) \neq C(\succsim'_1, \succsim_2, \succsim_3)$, the strict core solution violates the *rectangular property* and is thus not securely implementable.[7] ■

Thus, this paper seeks to identify which solutions are securely implementable.

[6]Saijo, Sjöström, and Yamato (2004) illustrate this for the two-agent case. However, as we show in Appendix B, the strict core solution is not even Nash implementable in the two-agent case. On the other hand, it is Nash implementable when there are at least three agents. Thus, it is not obvious whether the strict core solution is securely implementable when there are three or more agents.

[7]This can be directly derived from Theorem 2.

3 The two-agent case

In this section, we consider the two-agent case. For each $i \in N$, let

$$\succsim_i^{12}: 1, 2;$$
$$\succsim_i^{21}: 2, 1.$$

Proposition 2 provides a complete characterization of the class of solutions satisfying *strategy-proofness* and the *rectangular property* in the two-agent case.

Proposition 2. *Assume $n = 2$. A solution f satisfies strategy-proofness and the rectangular property if and only if it is either a constant solution or a serial dictatorship.*

Proof. It is easy to verify the "if" part. We prove the "only if" part below. Let f be a solution satisfying the two axioms. We now discuss the following two cases:

Case 1: $\boldsymbol{f(\succsim_1^{12}, \succsim_2^{12}) = (1, 2)}$. If $f(\succsim_1^{12}, \succsim_2^{21}) = (2, 1)$, then $f_2(\succsim_1^{12}, \succsim_2^{21}) \succ_2^{12} f_2(\succsim_1^{12}, \succsim_2^{12})$, which is in violation of *strategy-proofness*. Therefore, $f(\succsim_1^{12}, \succsim_2^{21}) = (1, 2)$.

We first consider the case $f(\succsim_1^{21}, \succsim_2^{12}) = (1, 2)$. By the *rectangular property*, $f(\succsim_1^{21}, \succsim_2^{21}) = (1, 2)$. Hence, f is constant.

Next, we consider the case $f(\succsim_1^{21}, \succsim_2^{12}) = (2, 1)$. If $f(\succsim_1^{21}, \succsim_2^{21}) = (1, 2)$, then, by the *rectangular property*, $f(\succsim_1^{21}, \succsim_2^{12}) = (1, 2)$. This is a contradiction. Therefore, $f(\succsim_1^{21}, \succsim_2^{21}) = (2, 1)$. Then, $f_1(\succsim) = b(\succsim_1, N)$ for each $\succsim \in \mathscr{P}^N$. This implies that f is a serial dictatorship.

Case 2: $\boldsymbol{f(\succsim_1^{12}, \succsim_2^{12}) = (2, 1)}$. By an argument similar to that in Case 1, we have that f is either a constant solution or a serial dictatorship. □

The two axioms in Proposition 2 are independent. It is easily verifiable that the strict core solution satisfies *strategy-proofness* but violates the *rectangular property*. The following solution satisfies the *rectangular property* but violates *strategy-proofness*: for each $\succsim \in \mathscr{P}^N$,

$$f(\succsim) = \begin{cases} (2, 1) & \text{if } \succsim = (\succsim_1^{12}, \succsim_2^{21}); \\ C(\succsim) & \text{otherwise.} \end{cases}$$

By Proposition 1, we immediately obtain the characterization of the class of securely implementable solutions in the two-agent case.

Theorem 1. *Assume $n = 2$. A solution f is securely implementable if and only if it is either a constant solution or a serial dictatorship.*

Considering other axioms, we obtain the following corollary:

Corollary 1. *Assume $n = 2$.*

1. *An individually rational solution f is securely implementable if and only if it is the no-trade solution.*

2. *A neutral solution f is securely implementable if and only if it is a serial dictatorship.*

3. *An efficient solution f is securely implementable if and only if it is a serial dictatorship.*

4 The general case

In contrast to the two-agent case, in the general case where there are more than two agents, there exists a securely implementable solution other than constant solutions and serial dictatorships. To verify this, consider the following example:

Example 2. Let $N = \{1, 2, 3\}$. Let f be a solution satisfying the following: for each $\succsim \in \mathscr{P}^N$,

$$f(\succsim) = \begin{cases} (2, 1, 3) & \text{if } 1 \succ_2 3; \\ (2, 3, 1) & \text{if } 3 \succ_2 1. \end{cases}$$

It is easy to see that the solution is securely implementable. ■

It would be expected that there are a lot of securely implementable solutions in the general case. In fact, as we will see later, in the general case, there are several different types of securely implementable solutions. Thus, the main purpose of this section is to characterize the class of securely implementable solutions satisfying a certain property.

4.1 Individual rationality and neutrality

This subsection first considers the class of securely implementable solutions that satisfy *individual rationality*. The next proposition would be helpful in characterizing the class.

Proposition 3. *A solution f satisfies individual rationality and the rectangular property if and only if it is the no-trade solution.*

Proof. Since the "if" part is obvious, it will suffice to show the "only if" part. Let f be a solution satisfying the two axioms. Let $\succsim' \in \mathscr{P}^N$ be such that for each $i \in N$, $b(\succsim'_i, N) = i$. By *individual rationality*, $f_i(\succsim') = i$ for each $i \in N$. Let $\succsim \in \mathscr{P}^N$. Then, *individual rationality* implies that $f_i(\succsim_i, \succsim'_{-i}) = i$ for each $i \in N$. Hence, by the *rectangular property*, $f(\succsim') = f(\succsim)$. This implies that f is the no-trade solution. $\square$

It is easy to check that none of the axioms in Proposition 3 are redundant. The strict core solution satisfies *individual rationality* but violates the *rectangular property.* A constant solution that is not the no-trade solution satisfies the *rectangular property* but violates *individual rationality.*

Interestingly, Proposition 3 enables us to pin down the class of securely implementable solutions satisfying *individual rationality* without using *strategy-proofness.* Thus, we immediately obtain the following result.

Theorem 2. *An individually rational solution f is securely implementable if and only if it is the no-trade solution.*

Next, we consider the class of securely implementable solutions that satisfy *neutrality.* Svensson (1999) establishes that a solution is *strategy-proof*, *non-bossy*, and *neutral* if and only if it is a serial dictatorship.[8] From the logical relationship between the *rectangular property* and *non-bossiness*, we obtain the following result:

Theorem 3. *A neutral solution is securely implementable if and only if it is a serial dictatorship.*

4.2 Efficiency

In this subsection, we characterize the class of securely implementable solutions that satisfy *efficiency.* We first provide a characterization of the class of solutions that satisfy *strategy-proofness*, the *rectangular property*, and *efficiency.*

Proposition 4. *A solution f satisfies* strategy-proofness, *the* rectangular property, *and* efficiency *if and only if it is a sequential dictatorship.*

[8]Svensson (1999) considers a situation where the total number of objects is at least as great as the number of agents. Therefore, Theorem 3 holds in this situation.

Proof. **The "if" part.** Let f be a sequential dictatorship. Since it is obvious that f satisfies *efficiency*, we show that f satisfies *strategy-proofness* and the *rectangular property*.

- *Strategy-proofness*: Let $\succsim \in \mathscr{P}^N$, $j \in N$, and $\succsim'_j \in \mathscr{P}$ be such that $\pi_{\succsim}(k) = j$. First, let $k = 1$. Then, obviously, j cannot manipulate at $\succsim$. Next, let $k = 2$. Since the first dictator $\pi_{\succsim}(1)$ $(= \pi_{(\succsim'_j, \succsim_{-j})}(1))$ reveals the same preference $\succsim_{\pi_{\succsim}(1)}$ at both $\succsim$ and $(\succsim'_j, \succsim_{-j})$, it holds that $f_{\pi_{\succsim}(1)}(\succsim) = f_{\pi_{\succsim}(1)}(\succsim'_j, \succsim_{-j})$. Therefore, $\pi_{\succsim}(2) = \pi_{(\succsim'_j, \succsim_{-j})}(2) = j$. Since $f_j(\succsim) = b\left(\succsim_j, N \Big\backslash \left\{f_{\pi_{\succsim}(1)}(\succsim)\right\}\right)$ and $f_j(\succsim'_j, \succsim_{-j}) = b\left(\succsim'_j, N \Big\backslash \left\{f_{\pi_{\succsim}(1)}(\succsim'_j, \succsim_{-j})\right\}\right) = b\left(\succsim'_j, N \Big\backslash \left\{f_{\pi_{\succsim}(1)}(\succsim)\right\}\right)$, agent j cannot manipulate at $\succsim$. Repeating a similar argument for each $k \in \{3, 4, \ldots, n\}$, we can establish that f is *strategy-proof*.

- *Rectangular property*: Let $\succsim, \succsim' \in \mathscr{P}^N$ be such that for all $i \in N$, $f_i(\succsim') = f_i(\succsim_i, \succsim'_{-i})$. Without loss of generality, we assume that $\pi_{\succsim'}(i) = i$ for each $i \in N$. First, let us consider agent 1. Note that $\pi_{\succsim'}(1) = \pi_{(\succsim_1, \succsim'_{-1})}(1) = \pi_{\succsim}(1) = 1$. Then, $f_1(\succsim') = f_1(\succsim_1, \succsim'_{-1})$ implies $b(\succsim'_1, N) = b(\succsim_1, N)$. Therefore,

$$f_1(\succsim') = b(\succsim'_1, N) = b(\succsim_1, N) = f_1(\succsim). \tag{1}$$

Next, let us consider agent 2, who is the second dictator at $\succsim'$. Since agent 1 reveals the same preference relation $\succsim'_1$ at both $\succsim'$ and $(\succsim_2, \succsim'_{-2})$, $f_1(\succsim') = f_1(\succsim_2, \succsim'_{-2})$. Thus, $\pi_{\succsim'}(2) = \pi_{(\succsim_2, \succsim'_{-2})}(2) = 2$. Then, $f_1(\succsim') = f_1(\succsim_2, \succsim'_{-2})$ and $f_2(\succsim') = f_2(\succsim_2, \succsim'_{-2})$ together imply that

$$b\left(\succsim'_2, N \setminus \{f_1(\succsim')\}\right) = b\left(\succsim_2, N \setminus \{f_1(\succsim')\}\right). \tag{2}$$

Furthermore, by (1), $\pi_{\succsim'}(2) = \pi_{\succsim}(2) = 2$. Therefore, (1) and (2) together imply that

$$f_2(\succsim') = b\left(\succsim'_2, N \setminus \{f_1(\succsim')\}\right) = b\left(\succsim_2, N \setminus \{f_1(\succsim)\}\right) = f_2(\succsim).$$

Iterating a similar augment for the other agents in N yields $f(\succsim') = f(\succsim)$.

The "only if" part. Let f be a solution satisfying the three axioms. We begin by proving that there exists the first dictator. For each $i \in N$, let $\hat{\succsim}_i$ be such that

$$\hat{\succsim}_i\colon n, n-1, \ldots, k+1, k, k-1, \ldots, 2, 1.$$

Let $\hat{\succsim} \equiv (\hat{\succsim}_1, \hat{\succsim}_2, \ldots, \hat{\succsim}_n)$. Without loss of generality, assume that for each $i \in N$, $f_i(\hat{\succsim}) = i$. For each $k \in N$, let $N_k \equiv \{1, 2, \ldots, k\}$. We establish the following claim:

Claim 1. *For each $k \in N$ and each $\succsim_{N_k} \in \mathscr{P}^{N_k}$,*

$$f_i(\succsim_{N_k}, \hat{\succsim}_{-N_k}) = i \quad \forall\, i \in N \setminus N_k;$$
$$f_k(\succsim_{N_k}, \hat{\succsim}_{-N_k}) = b(\succsim_k, N_k).$$

The proof for Claim 1 can be found in Appendix A. When $k = n$, Claim 1 implies that for each $\succsim \in \mathscr{P}^N$, $f_n(\succsim) = b(\succsim_n, N)$. Therefore, agent n is the first dictator.

Now, we show that f is a sequential dictatorship. Since agent n is the first dictator, we can set $\pi_{\succsim}(1) = n$ and $f_n(\succsim) = b(\succsim_n, N)$ for each $\succsim \in \mathscr{P}^N$. We will now establish the following claim:

Claim 2. *For each $\succsim \in \mathscr{P}^N$ and each $\succsim'_{-n} \in \mathscr{P}^{N \setminus \{n\}}$, if for each $i \in N \setminus \{n\}$,*

$$c \succsim_i d \iff c \succsim'_i d \quad \forall\, c, d \in N \setminus \{b(\succsim_n, N)\}, \tag{3}$$

then $f(\succsim) = f(\succsim_n, \succsim'_{-n})$.

The proof for Claim 2 can be found in Appendix A. Pick any $\succsim \in \mathscr{P}^N$. Let $a \equiv b(\succsim_n, N)$. Let $\mathscr{P}|_{N\setminus\{a\}}$ denote the set of all strict preferences $\succsim_i|_{N\setminus\{a\}}$ over $N \setminus \{a\}$. Then, let $f^a \colon \left(\mathscr{P}|_{N\setminus\{a\}}\right)^{N\setminus\{n\}} \to N \setminus \{a\}$ be a solution such that for each $i \in N \setminus \{n\}$, $f^a_i(\succsim|_{N\setminus\{a\}}) = f_i(\succsim)$ where for each $i \in N$, $b(\succsim_i, N) = a$ and for each $i \in N \setminus \{n\}$,

$$c \succsim_i|_{N\setminus\{a\}} d \iff c \succsim_i d \quad \forall\, c, d \in N \setminus \{a\}.$$

Since f satisfies *strategy-proofness*, the *rectangular property*, and *efficiency*, f^a also satisfies the three axioms. Therefore, by adopting an argument similar to that for proving that there is the first dictator of f, we can prove that there is a dictator of f^a. Let $j(a)$ be the dictator of f^a. Then,

$$f_{j(a)}(\succsim) = f_{j(a)}(\succsim_n, \succsim'_{-n}) = f^a_{j(a)}(\succsim|_{N\setminus\{a\}}) = b(\succsim_{j(a)}|_{N\setminus\{a\}}, N \setminus \{a\}) = b(\succsim_{j(a)}, N \setminus \{a\}),$$

where for each $i \in N \setminus \{n\}$, $\succsim'_i$ is a preference relation such that $b(\succsim'_i, N) = a$ and (3) holds; the first equation follows from Claim 2. Hence, we observe that for each $\succsim \in \mathscr{P}^N$, if $b(\succsim_n, N) = a$, then $\pi_{\succsim}(2) = j(a)$ and $f_{j(a)}(\succsim) = b(\succsim_{j(a)}, N \setminus \{a\})$.

By repeating a similar argument, we can establish that f is a sequential dictatorship. □

Remark. We can see that the proof of Proposition 4, particularly Claim 1, does not work in a situation where the *null object*, which means "not receiving any real object," may be preferred to a real object. Therefore, we cannot apply Proposition 4 to such a situation. On the other hand, the proof can be extended, in a straightforward way, to a situation where every real object is preferred to the null object.

It is easy to verify that the "only if" part of Proposition 4 does not hold when any of the three axioms—*efficiency*, *strategy-proofness*, and the *rectangular property*—is dropped. The strict core solution satisfies *efficiency* and *strategy-proofness* but violates the *rectangular property*. The no-trade solution satisfies *strategy-proofness*, and the *rectangular property* but violates *efficiency*. Finally, the following solution satisfies *efficiency* and the *rectangular property* but violates *strategy-proofness*: let f be a sequential choice solution such that for each $\succsim \in \mathscr{P}^{\{1,2,3\}}$,

$$(\pi_{\succsim}(1), \pi_{\succsim}(2), \pi_{\succsim}(3)) = \begin{cases} (1,2,3) & \text{if } b(\succsim_i, N) = b(\succsim_j, N) \quad \forall\, i, j \in N; \\ (2,3,1) & \text{otherwise.} \end{cases}$$

The following result is a characterization of securely implementable solutions satisfying *efficiency* and follows easily from Proposition 4.

Theorem 4. *An efficient solution f is securely implementable if and only if it is a sequential dictatorship.*

It is well-known that *strategy-proofness* together with *non-bossiness* implies *efficiency* as long as no alternative is excluded in advance (Takamiya, 2001); this is an axiom called *ontoness*. This axiom can be expressed as follows:

***Ontoness*:** For each $x \in X$, there exists $\succsim \in \mathscr{P}^N$ such that $f(\succsim) = x$.

Since *ontoness* is a necessary condition for *efficiency*, *ontoness* deems a minimal efficiency condition. Then, we have the following corollary:

Corollary 2. *An onto solution f is securely implementable if and only if it is a sequential dictatorship.*

5 Discussions

5.1 Other securely implementable solutions

Thus far, we have considered securely implementable solutions satisfying certain properties in the general case. Now, we present other securely implementable solutions in the general case.

Example 2 (continued). It can easily be verified that f is securely implementable but satisfies none of the other axioms. ∎

Example 3. Let $N = \{1, 2, 3, 4\}$. Let f be a solution satisfying the following: for each $\succsim \in \mathscr{P}^N$,

$$\begin{aligned}
f_1(\succsim) &= b(\succsim_1, \{1, 2, 3\});\\
f_2(\succsim) &= b(\succsim_2, N \setminus \{f_1(\succsim)\});\\
f_3(\succsim) &= b(\succsim_3, N \setminus \{f_1(\succsim), f_2(\succsim)\});\\
f_4(\succsim) &= N \setminus \{f_1(\succsim), f_2(\succsim), f_3(\succsim)\}.
\end{aligned}$$

This solution is securely implementable but satisfies none of the other axioms. ∎

It follows from Examples 2 and 3 that the class of securely implementable solutions is expected to be of complicated form. Thus, the characterization of the class of securely implementable solutions remains for future research.

5.2 Reallocation-proofness and anonymity

Pápai (2000) studies solutions that are robust to pairwise manipulations through reallocations of assignments. Such a robustness is formalized by a requirement that rules out the possibility that any two agents can gain by swapping objects after reporting dishonestly. This can be expressed as follows:

Reallocation-proofness: There does not exist $\succsim \in \mathscr{P}^N$, $i, j \in N$, and $\succsim'_{\{i,j\}} \in \mathscr{P}^{\{i,j\}}$ such that (i) $f_j(\succsim'_{\{i,j\}}, \succsim_{-\{i,j\}}) \succsim_i f_i(\succsim)$, (ii) $f_i(\succsim'_{\{i,j\}}, \succsim_{-\{i,j\}}) \succ_j f_j(\succsim)$, and (iii) $f_h(\succsim) = f_h(\succsim'_h, \succsim_{-h}) \neq f_h(\succsim'_{\{i,j\}}, \succsim_{-\{i,j\}})$ for $h = i, j$.

Pápai (2000) discusses *reallocation-proofness* as well as *strategy-proofness*, *efficiency*, and *non-bossiness*. She establishes that a solution satisfies the four axioms if and only if it is a *hierarchical exchange solution*.[9] It immediately follows from the definitions that the *rectangular property* implies *reallocation-proofness*. Therefore, Theorem 4 implies that any hierarchical exchange solution other than sequential dictatorships is *not* securely implementable.

The next axiom, which is first introduced by Miyagawa (2002), is related to fairness. It states that a solution does not depend on the names of agents and objects. Given that $\succsim \in \mathscr{P}^N$ and $\pi \in \Pi^N$, let $\hat{T}(\succsim, \pi)$ be the preference profile $\succsim'$ defined by the condition that for each $i, j, k \in N$,

$$j \succsim_i k \iff \pi(j) \succsim'_{\pi(i)} \pi(k).$$

***Anonymity*:** For each $\succsim \in \mathscr{P}^N$, each $\pi \in \Pi^N$, and each $i \in N$, $f_{\pi(i)}(\hat{T}(\succsim, \pi)) = \pi(f_i(\succsim))$.

In the two-agent case, we obtain the following result from Theorem 2:

Theorem 5. *Assume $n = 2$. An anonymous solution f is securely implementable if and only if it is constant.*

In the general case, characterizing the class of *anonymous* and securely implementable solutions is still an open question. We point out that the class in the case $n \geq 4$ may be substantially different from that in the case $n = 3$. To verify this, we define the *modified strict core solution* C^m, which is proposed by Miyagawa (2002): for each $\succsim \in \mathscr{P}^N$, $C^m(\succsim) = C(\succsim^*)$, where $\succsim^*$ is a preference profile such that for each $i \in N$, $\succsim_i^*$ is identical to $\succsim_i$ except for the agent's initial endowment; further, the endowment ranking is worst at $\succsim_i^*$. It can easily be verified that the modified strict core solution satisfies the *rectangular property* in the case $n = 3$ although the solution violates it in the case $n \geq 4$. This hints toward a difference between the characterization results.

5.3 Coalitional stability

As shown in Section 2, the strict core solution is not securely implementable. One way to avoid this result is to consider an equilibrium concept related to coalitional stability instead of Nash equilibrium. This approach is adopted from Bochet and Sakai (2007), who study secure implementation in allotment economies. In our model, Takamiya (2009) shows that the strict core solution is implemented by its

[9] See Pápai (2000) for the formal definition of a hierarchical exchange solution.

associated direct revelation mechanism in strict strong Nash equilibria. Thus, it is doubly implemented through dominant strategy and strict strong Nash equilibria.[10] However, the characterization of the class of solutions that can be doubly implemented through dominant strategy equilibrium and an equilibrium notion related to coalitional stability remains for future research.

6 Conclusion

We succeeded in classifying the class of securely implementable solutions satisfying a certain property such as *individual rationality*, *neutrality*, and *efficiency* in Shapley-Scarf housing markets. This paper discussed a deterministic object allocation model and proved that a serial dictatorship is securely implementable but the strict core solution is not. On the other hand, in a random allocation model, two solutions related to a serial dictatorship and the strict core solution are equivalent: Abdulkadiroğlu and Sönmez (1998) establish the equivalence between the random serial dictatorship and the core solution from random endowment.[11] The examination of whether or not the solution is securely implementable and the identification of the securely implementable solutions in the random allocation model are interesting issues left for future research.

A Appendix: Proofs of claims

Before proving claims, we define *monotonicity* (Maskin, 1999) and provide a useful fact. We denote by $L(h, \succsim_i) \equiv \{k \in N\colon h \succsim_i k\}$ agent i's *lower contour set* of object $h \in N$ at $\succsim_i \in \mathscr{P}$.

***Monotonicity*:** For each $\succsim, \succsim' \in \mathscr{P}^N$, if $L(f_i(\succsim), \succsim_i) \subseteq L(f_i(\succsim), \succsim'_i)$ for each $i \in N$, then $f(\succsim) = f(\succsim')$.

Fact 2. *If f satisfies both strategy-proofness and the rectangular property, then it satisfies monotonicity.*

Proof. It follows from Fact 1 and Theorem 4.12 in Takamiya (2001). □

[10]Wako (1999) establishes that the strict core solution is strong Nash implementable by constructing a "natural" mechanism. However, the mechanism does not implement the solution via dominant strategy equilibria.

[11]See Abdulkadiroğlu and Sönmez (1998) for the formal definitions of the two solutions.

A.1 Proof of Claim 1

We now prove this claim by using an induction argument.

- *Basic step*: When $k = 1$, the claim holds: Pick any $\succsim_1 \in \mathscr{P}$. Note that $f_1(\hat{\succsim}) = 1$. Since $L(1, \hat{\succsim}_1) = \{1\}$, $L(1, \hat{\succsim}_1) \subseteq L(1, \succsim_1)$. Thus, by *monotonicity* (Fact 2), $f(\succsim_1, \hat{\succsim}_{-1}) = f(\hat{\succsim})$. Therefore,

$$f_i(\succsim_{N_1}, \hat{\succsim}_{-N_1}) = i \quad \forall\, i \in N \setminus N_1;$$
$$f_1(\succsim_{N_1}, \hat{\succsim}_{-N_1}) = 1 = b(\succsim_1, N_1).$$

- *Induction hypothesis*: When $k = \ell - 1$, it holds that for each $\succsim_{N_{\ell-1}} \in \mathscr{P}^{N_{\ell-1}}$,

$$f_i(\succsim_{N_{\ell-1}}, \hat{\succsim}_{-N_{\ell-1}}) = i \quad \forall\, i \in N \setminus N_{\ell-1}; \tag{4}$$
$$f_{\ell-1}(\succsim_{N_{\ell-1}}, \hat{\succsim}_{-N_{\ell-1}}) = b(\succsim_{\ell-1}, N_{\ell-1}). \tag{5}$$

- *Induction step*: Let $k = \ell$. We will now prove the three steps.

Step 1: For each $\boldsymbol{\succsim_{N_\ell} \in \mathscr{P}^{N_\ell}}$, $\boldsymbol{f_\ell(\succsim_{N_\ell}, \hat{\succsim}_{-N_\ell}) = b(\succsim_\ell, N_\ell)}$.

Let $\succsim_{N_\ell} \in \mathscr{P}^{N_\ell}$. Furthermore, let $x \equiv f(\succsim_{N_\ell}, \hat{\succsim}_{-N_\ell})$, $y \equiv f(\succsim_{N_{\ell-1}}, \hat{\succsim}_{-N_{\ell-1}})$, $b \equiv b(\succsim_\ell, N_\ell)$, and $s \equiv b(\succsim_\ell, N_\ell \setminus \{b\})$. Note that by the induction hypothesis, $y_\ell = \ell$. Moreover, since $y_\ell = \ell \mathrel{\hat{\succsim}_\ell} x_\ell$ by *strategy-proofness*, then $x_\ell \in N_\ell$. There are three cases:

Case 1-1: $\boldsymbol{b = \ell}$. Since $y_\ell = \ell$ and $N_\ell = L(\ell, \hat{\succsim}_\ell) \subseteq L(\ell, \succsim_\ell)$, by *monotonicity* (Fact 2), $f_\ell(\succsim_{N_\ell}, \hat{\succsim}_{-N_\ell}) = \ell = b$.

Case 1-2: $\boldsymbol{s = \ell}$. By *strategy-proofness*, $x_\ell \succsim_\ell \ell = y_\ell$. This and $x_\ell \in N_\ell$ together imply that either $x_\ell = b$ or $x_\ell = \ell$. Suppose that $x_\ell = \ell$. By the induction hypothesis, (4) implies that

$$\bigcup_{i \in N_{\ell-1}} \{y_i\} = N_{\ell-1}. \tag{6}$$

Since $b \neq s = \ell$, then $b \in N_{\ell-1}$. Therefore, by (6), there exists $j \in N_{\ell-1}$ such that

$$y_j = b. \tag{7}$$

Define $\bar{\succsim}_{N_{\ell-1}}$ as follows:

P1. $\bar{\succsim}_j: \ell, b, \ldots;$

P2. For each $i \in N_{\ell-1} \setminus \{j\}$, $\bar{\succsim}_i: y_i, \ldots$

Since $y_\ell = \ell = x_\ell$,

$$y_\ell = x_\ell = f_\ell(\succsim_{N_\ell}, \hat{\succsim}_{-N_\ell}). \tag{8}$$

For each $i \in N_{\ell-1} \setminus \{j\}$, $L(y_i, \succsim_i) \subset N = L(y_i, \bar{\succsim}_i)$. Thus, by *monotonicity* (Fact 2),

$$y = f(\succsim_{N_{\ell-1}\setminus\{i\}}, \bar{\succsim}_i, \hat{\succsim}_{-N_{\ell-1}}). \tag{9}$$

By the induction hypothesis, $f_\ell(\succsim_{N_{\ell-1}\setminus\{j\}}, \bar{\succsim}_j, \hat{\succsim}_{-N_{\ell-1}}) = \ell$, which implies

$$f_j(\succsim_{N_{\ell-1}\setminus\{j\}}, \bar{\succsim}_j, \hat{\succsim}_{-N_{\ell-1}}) \neq \ell.$$

Then, by (7) and *strategy-proofness*,

$$y_j = b = f_j(\succsim_{N_{\ell-1}\setminus\{j\}}, \bar{\succsim}_j, \hat{\succsim}_{-N_{\ell-1}}). \tag{10}$$

Then (8), (9), and (10) together imply that by the *rectangular property*,

$$f(\bar{\succsim}_{N_{\ell-1}}, \succsim_\ell, \hat{\succsim}_{-N_\ell}) = y. \tag{11}$$

Thus,

$$\begin{aligned} b \succ_\ell \ell &= f_\ell(\bar{\succsim}_{N_{\ell-1}}, \succsim_\ell, \hat{\succsim}_{-N_\ell}); \\ \ell \mathrel{\bar{\succ}_j} b &= f_j(\bar{\succsim}_{N_{\ell-1}}, \succsim_\ell, \hat{\succsim}_{-N_\ell}), \end{aligned}$$

which is a contradiction to *efficiency*. Hence, $x_\ell = b$.

Case 1-3: $b \neq \ell$ and $s \neq \ell$. Pick any $\succsim'_\ell$ such that $b(\succsim'_\ell, N) = b$ and $b(\succsim'_\ell, N \setminus \{b\}) = \ell$. Then, by Case 1-2, $f_\ell(\succsim_{N_{\ell-1}}, \succsim'_\ell, \hat{\succsim}_{-N_\ell}) = b$. Since $x_\ell \in N_\ell$, by *strategy-proofness*, $x_\ell = b$.

Step 2: For each $\succsim_{N_\ell} \in \mathscr{P}^{N_\ell}$ and each $i \in N_\ell$, $f_i(\succsim_{N_\ell}, \hat{\succsim}_{-N_\ell}) \in N_\ell$.

Pick any $\succsim_{N_\ell} \in \mathscr{P}^{N_\ell}$. Let $x \equiv f(\succsim_{N_\ell}, \hat{\succsim}_{-N_\ell})$ and $b \equiv b(\succsim_\ell, N_\ell)$. By Step 1, $x_\ell = b$. Let $\succsim'_\ell$ be such that $b' \equiv b(\succsim'_\ell, N_\ell) = b$ and $L(b', \succsim'_\ell) = N_\ell$. Let $\succsim'_{N_{\ell-1}}$ be such that for each $i \in N_{\ell-1}$, $b(\succsim'_i, N) = b'$ and the ordering other than b' is the same as that of $\succsim_i$. Let $x' \equiv f(\succsim'_{N_\ell}, \hat{\succsim}_{-N_\ell})$. By Step 1, $x'_\ell = b'$.

We first show that $x'_i \in N_\ell$ for each $i \in N_\ell$. Suppose, by contradiction, that there exist $j \in N_{\ell-1}$ such that $x'_j \notin N_\ell$. Then,

$$x'_j \succ'_\ell b';$$
$$b' \succ'_j x'_j,$$

which is a contradiction to *efficiency*. Thus, $x'_i \in N_\ell$ for each $i \in N_\ell$.

Next, we show that $x = x'$. By Step 1, $x'_i \neq b'$ for each $i \in N_{\ell-1}$. Therefore, for each $i \in N_{\ell-1}$, by the definition of $\succsim'_i$, we have either $L(x'_i, \succsim_i) = L(x'_i, \succsim'_i)$ or $L(x'_i, \succsim_i) = L(x'_i, \succsim'_i) \cup \{b'\}$. This implies that $L(x'_i, \succsim'_i) \subseteq L(x'_i, \succsim_i)$ for each $i \in N_{\ell-1}$. By *monotonicity* (Fact 2), for each $i \in N_{\ell-1}$,

$$x' = f(\succsim'_{N_\ell \setminus \{i\}}, \succsim_i, \hat{\succsim}_{-N_\ell}). \tag{12}$$

Furthermore, by Step 1,

$$x'_\ell = b' = b = f_\ell(\succsim'_{N_{\ell-1}}, \succsim_\ell, \hat{\succsim}_{-N_\ell}). \tag{13}$$

By (12) and (13), the *rectangular property* implies that $x = x'$.

Hence, we obtain $x_i = x'_i \in N_\ell$ for each $i \in N_\ell$.

Step 3: For each $\boldsymbol{\succsim_{N_\ell} \in \mathscr{P}^{N_\ell}}$ and each $\boldsymbol{i \in N \setminus N_\ell}$, $\boldsymbol{f_i(\succsim_{N_\ell}, \hat{\succsim}_{-N_\ell}) = i}$.

Suppose, by contradiction, that there exist $\succsim_{N_\ell} \in \mathscr{P}^{N_\ell}$ and $i \in N \setminus N_\ell$ such that

$$f_i(\succsim_{N_\ell}, \hat{\succsim}_{-N_\ell}) \neq i. \tag{14}$$

We now construct the preference profile $\succsim^*_{N_\ell}$ as follows:

P*1. $\succsim^*_\ell :\ n, n-1, \ldots, \ell+2, \ell+1, b(\succsim_\ell, N_\ell), \ldots;$

P*2. For each $j \in N_{\ell-1}$, $b(\succsim^*_j, N) = f_j(\succsim_{N_\ell}, \hat{\succsim}_{-N_\ell})$.

By Step 1,

$$f_\ell(\succsim_{N_{\ell-1}}, \succsim^*_\ell, \hat{\succsim}_{-N_\ell}) = b(\succsim^*_\ell, N_\ell) = b(\succsim_\ell, N_\ell) = f_\ell(\succsim_{N_\ell}, \hat{\succsim}_{-N_\ell}). \tag{15}$$

For each $j \in N_{\ell-1}$, by *strategy-proofness*,

$$f_j(\succsim_{N_\ell \setminus \{j\}}, \succsim^*_j, \hat{\succsim}_{-N_\ell}) = f_j(\succsim_{N_\ell}, \hat{\succsim}_{-N_\ell}). \tag{16}$$

By (15) and (16), the *rectangular property* implies that

$$f(\succsim^*_{N_\ell}, \hat{\succsim}_{-N_\ell}) = f(\succsim_{N_\ell}, \hat{\succsim}_{-N_\ell}). \tag{17}$$

Then, (14) and (17) together imply that $f_i(\succsim^*_{N_\ell}, \hat{\succsim}_{-N_\ell}) \neq i$. Note that by Step 2, we have

$$\bigcup_{k \in N \setminus N_\ell} \left\{ f_k(\succsim^*_{N_\ell}, \hat{\succsim}_{-N_\ell}) \right\} = N \setminus N_\ell.$$

Therefore, there exists $j \in N \setminus N_\ell$ such that $j > f_j(\succsim^*_{N_\ell}, \hat{\succsim}_{-N_\ell})$. Now, let $\succsim^{**}_j$ be such that

$$\succsim^{**}_j : n, n-1, \ldots, j+1, j, b(\succsim^*_\ell, N_\ell), f_j(\succsim^*_{N_\ell}, \hat{\succsim}_{-N_\ell}), \ldots$$

Let $x^* \equiv f(\succsim^*_{N_\ell}, \succsim^{**}_j, \hat{\succsim}_{-(N_\ell \cup \{j\})})$ and $y^* \equiv f(\succsim^*_{N_\ell}, \hat{\succsim}_{-N_\ell})$. By *strategy-proofness*, $x^*_j \succsim^{**}_j y^*_j$. If $x^*_j \geq j$, then $x^*_j \hat{\succ}_j y^*_j$, which is a contradiction to *strategy-proofness*. Therefore, we have either $x^*_j = b(\succsim^*_\ell, N_\ell)$ or $x^*_j = y^*_j$.

Case 3-1: $x^*_j = y^*_j$. By *non-bossiness* (Fact 1), $x^* = y^*$. Then, since $y^*_j \in N \setminus N_\ell$ and $y^*_\ell = b(\succsim^*_\ell, N_\ell)$ by Step 1,

$$b(\succsim^*_\ell, N_\ell) \succ^{**}_j y^*_j = x^*_j;$$
$$y^*_j \succ^*_\ell b(\succsim^*_\ell, N_\ell) = x^*_\ell,$$

which is a contradiction to *efficiency*.

Case 3-2: $x^*_j = b(\succsim^*_\ell, N_\ell)$. By *efficiency*, $x^*_\ell \succ^*_\ell b(\succsim^*_\ell, N_\ell)$; otherwise, we define the allocation z as follows:

Z1. For each $k \in N_{\ell-1}$, $z_k = b(\succsim^*_k, N)$;

Z2. $z_j = b(\succsim^*_\ell, N_\ell)$;

Z3. For each $k \in \{h \in N \setminus (N_\ell \cup \{j\}) : x^*_h \in N \setminus N_\ell\}$, $z_k = x^*_k$.

Z1, P*2, and (17) together imply that $z_k = y_k$ for each $k \in N_{\ell-1}$. Step 1 implies that $z_j = y_\ell$. Thus, by Step 2, $\bigcup_{k \in N_{\ell-1} \cup \{j\}} \{z_k\} = N_\ell$, which implies that $z_k \in N \setminus N_\ell$ for each $k \in N \setminus (N_{\ell-1} \cup \{j\})$. Therefore, by Z1, Z2, and Z3,

$$z_\ell \succ^*_\ell b(\succsim^*_\ell, N_\ell) \succ^*_\ell x^*_\ell;$$

$$\begin{aligned}
&z_j \sim_j^{**} x_j^*;\\
&z_k \mathrel{\hat{\succsim}_k} x_k^* \quad \forall\, k \in N \setminus (N_\ell \cup \{j\})\,;\\
&z_h \succsim_h^* x_h^* \quad \forall\, h \in N_{\ell-1}.
\end{aligned}$$

Then the allocation z Pareto dominates x^*, which is a contradiction to *efficiency.*

Thus, $x_\ell^* \in N \setminus N_\ell$. Then, $L(x_\ell^*, \succsim_\ell^*) \subseteq L(x_\ell^*, \hat{\succsim}_\ell)$. Therefore, by *monotonicity* (Fact 2),

$$x^* = f(\succsim_{N_{\ell-1}}^*, \succsim_j^{**}, \hat{\succsim}_{-(N_{\ell-1}\cup\{j\})}). \tag{18}$$

By the induction hypothesis, $j = f_j(\succsim_{N_{\ell-1}}^*, \hat{\succsim}_{-N_{\ell-1}})$. Then, $L(j, \hat{\succsim}_j) \subseteq L(j, \succsim_j^{**})$. Therefore, by *monotonicity* (Fact 2),

$$f(\succsim_{N_{\ell-1}}^*, \succsim_j^{**}, \hat{\succsim}_{-(N_{\ell-1}\cup\{j\})}) = f(\succsim_{N_{\ell-1}}^*, \hat{\succsim}_{-N_{\ell-1}}). \tag{19}$$

Then, (18) and (19) together imply that $x^* = f(\succsim_{N_{\ell-1}}^*, \hat{\succsim}_{-N_{\ell-1}})$. This is a contradiction to $x_j^* = b(\succsim_\ell^*, N_\ell) \neq j = f_j(\succsim_{N_{\ell-1}}^*, \hat{\succsim}_{-N_\ell})$. □

A.2 Proof of Claim 2

Pick any $\succsim \in \mathscr{P}^N$. Let $\succsim'_{-n} \in \mathscr{P}^{N\setminus\{n\}}$ be such that (3) holds. Let us consider $\succsim''_{-n} \in \mathscr{P}^{N\setminus\{n\}}$ such that for each $i \in N\setminus\{n\}$, $b(\succsim''_i, N) = b(\succsim_n, N)$ and the ordering other than $b(\succsim_n, N)$ is the same as that of both $\succsim_i$ and $\succsim'_i$. Note that since agent n is the first dictator, $f_n(\succsim_n, \succsim''_{-n}) = b(\succsim_n, N)$. Therefore, $f_i(\succsim_n, \succsim''_{-n}) \neq b(\succsim_n, N)$ for each $i \in N \setminus \{n\}$. These imply that for each $i \in N \setminus \{n\}$,

$$\begin{aligned}
L(f_i(\succsim_n, \succsim''_{-n}), \succsim''_i) \subseteq L(f_i(\succsim_n, \succsim''_{-n}), \succsim_i);\\
L(f_i(\succsim_n, \succsim''_{-n}), \succsim''_i) \subseteq L(f_i(\succsim_n, \succsim''_{-n}), \succsim'_i).
\end{aligned}$$

By *monotonicity* (Fact 2), $f(\succsim_n, \succsim_{-n}) = f(\succsim_n, \succsim''_{-n}) = f(\succsim_n, \succsim'_{-n})$. □

B Appendix: Implementability of the strict core solution

B.1 Dominant strategy implementation

To the best of our knowledge, no one has previously attempted to explicitly identify whether the strict core solution is dominant strategy implementable. Thus, below, we show that the strict core solution is dominant strategy implementable by exploiting the result of Mizukami and Wakayama (2007). They show that if a solution satisfies *strategy-proofness* and *quasi-strong-non-bossiness*, then it is dominant strategy implemented by its associated direct revelation mechanism (see Theorem 2 in Mizukami and Wakayama, 2007).

***Quasi-strong-non-bossiness*:** For each $\succsim \in \mathscr{P}^N$, each $i \in N$, and each $\succsim_i' \in \mathscr{P}$, if $f_i(\succsim_i, \succsim''_{-i}) \sim_i f_i(\succsim_i', \succsim''_{-i})$ for each $\succsim''_{-i} \in \mathscr{P}^{N\setminus\{i\}}$, then $f(\succsim) = f(\succsim_i', \succsim_{-i})$.

Proposition 5. *The strict core solution is dominant strategy implemented by its associated direct revelation mechanism.*

Proof. It suffices to show that the strict core solution C satisfies *quasi-strong-non-bossiness*. Let $\succsim \in \mathscr{P}^N$, $i \in N$, and $\succsim_i' \in \mathscr{P}$ be such that $C_i(\succsim_i, \succsim''_{-i}) \sim_i C_i(\succsim_i', \succsim''_{-i})$ for each $\succsim''_{-i} \in \mathscr{P}^{N\setminus\{i\}}$. Since preferences are strict, $C_i(\succsim_i, \succsim''_{-i}) = C_i(\succsim_i', \succsim''_{-i})$ for each $\succsim''_{-i} \in \mathscr{P}^{N\setminus\{i\}}$. Thus, $C_i(\succsim) = C_i(\succsim_i', \succsim_{-i})$. Since C satisfies *non-bossiness*, $C(\succsim) = C(\succsim_i', \succsim_{-i})$. $\square$

B.2 Nash implementation in the case of two agents

Although the strict core solution is Nash implementable in the case where there are at least three agents, it is *not* Nash implementable in the two-agent case.

Proposition 6. *Assume $n = 2$. The strict core solution is not implementable in Nash equilibria.*

Proof. Suppose, by contradiction, that the strict core solution is Nash implementable. Then, there is a mechanism $\Gamma = (M, g)$ implementing the strict core solution C in Nash equilibria. Here, $M \equiv M_1 \times M_2$ is a message space and $g\colon M \to X$ is an outcome function. Given that $i \in N$ and $m \in M$, let $g_i(m)$ denote agent i's consumption associated with a message profile m. For each $\succsim \in \mathscr{P}^N$, let $\mathbf{NE}(\succsim, \Gamma) \subseteq M$ be the set of all Nash equilibria of the mechanism Γ at $\succsim \in \mathscr{P}^N$ and $g(\mathbf{NE}(\succsim, \Gamma))$ be the set of the Nash equilibrium outcomes of Γ at $\succsim$. Since the mechanism $\Gamma = (M, g)$ implements C,

$$g(\mathbf{NE}((\succsim_1^{12}, \succsim_2^{12}), \Gamma)) = (1,2) = C(\succsim_1^{12}, \succsim_2^{12}); \tag{20}$$
$$g(\mathbf{NE}((\succsim_1^{12}, \succsim_2^{21}), \Gamma)) = (1,2) = C(\succsim_1^{12}, \succsim_2^{21});$$
$$g(\mathbf{NE}((\succsim_1^{21}, \succsim_2^{12}), \Gamma)) = (2,1) = C(\succsim_1^{21}, \succsim_2^{12});$$
$$g(\mathbf{NE}((\succsim_1^{21}, \succsim_2^{21}), \Gamma)) = (1,2) = C(\succsim_1^{21}, \succsim_2^{21}). \tag{21}$$

Now, let $m^{21,21} \in \mathbf{NE}((\succsim_1^{21}, \succsim_2^{21}), \Gamma)$. Then, by (21),

$$1 = g_1(\mathbf{NE}((\succsim_1^{21}, \succsim_2^{21}), \Gamma)) = g_1(m^{21,21}) \succsim_1^{21} g_1(m_1', m_2^{21,21}) \quad \forall\, m_1' \in M_1,$$

which implies that

$$g_1(m_1', m_2^{21,21}) = 1 \quad \forall\, m_1' \in M_1. \tag{22}$$

Next, let $m^{12,12} \in \mathbf{NE}((\succsim_1^{12}, \succsim_2^{12}), \Gamma)$. Then, by (20),

$$2 = g_2(\mathbf{NE}((\succsim_1^{12}, \succsim_2^{12}), \Gamma)) = g_2(m^{12,12}) \succsim_2^{12} g_2(m_1^{12,12}, m_2') \quad \forall\, m_2' \in M_2,$$

which implies that

$$g_2(m_1^{12,12}, m_2') = 2 \quad \forall\, m_2' \in M_2. \tag{23}$$

Then, (22) and (23) together imply that

$$1 = g_1(m_1^{12,12}, m_2^{21,21}) \succsim_1^{21} g_1(m_1', m_2^{21,21}) \quad \forall\, m_1' \in M_1;$$
$$2 = g_2(m_1^{12,12}, m_2^{21,21}) \succsim_2^{12} g_2(m_1^{12,12}, m_2') \quad \forall\, m_2' \in M_2.$$

This implies that $(m_1^{12,12}, m_2^{21,21}) \in \mathbf{NE}((\succsim_1^{21}, \succsim_2^{12}), \Gamma)$. However, $g(m_1^{12,12}, m_2^{21,21}) = (1,2) \neq (2,1) = g(\mathbf{NE}((\succsim_1^{21}, \succsim_2^{12}), \Gamma))$, which is a contradiction. $\square$

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