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SCORING AUCTIONS WITH NON-QUASILINEAR SCORING RULES

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Abstract

In this paper we analyse scoring auctions with general non-quasilinear scoring rules. We assume that cost function of each firm is additively separable in quality and type. In sharp contrast to the recent results in the literature we show the following. (i) Equilibria in scoring auctions can be computed without any endogeneity problems and we get explicit solutions. (ii) We provide a complete characterisation of such equilibria and compare quality, price and expected scores across first-score and second-score auctions. (iii) We show that such properties and rankings depend on the curvature properties of the scoring rule and the distribution function of types.

JEL Classification: D44, H57, L13

1 Introduction

In the modern world, auctions are used to conduct a huge volume of economic transactions. Governments use them to sell treasury bills, foreign exchange, mineral rights including oil fields, and other assets such as firms to be privatized. Government contracts are typically awarded by procurement auctions, which are also often used by firms subcontracting work or buying services and raw materials. In OECD (2011) it is reported that the procurement

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of public services accounts for approximately 17% of GDP of EU countries. Clearly public procurements constitute a significant part of the economic activities in many countries (see Koning and Meerendonk, 2013).

The theory of auction provides the necessary analytical framework to study such procurements. In the canonical model there is one indivisible object up for sale and there are some potential bidders. In any standard auction the object is sold to the highest bidder. In a procurement auction, where the auctioneer is the buyer, the object is sold to the lowest bidder. The payment by each bidder depends on the type of auction used by the seller. There is a huge literature around this model.

It may be noted that the benchmark model of auctions is really a *price-only* auctions. For example, in the traditional theory of standard procurement auctions, the auctioneer cares only about the price of the object, but not the other attributes. However, in many procurement situations, *the buyer cares about attributes other than price* when evaluating the offers submitted by suppliers[1]. Non-monetary attributes that buyers care about include quality, time to completion etc. For example, in the contract for the construction of a new aircraft, the specification of its characteristics is probably as important as its price. Under these circumstances, auctions are usually multidimensional. The essential element of such multi-dimensional auctions is a *scoring rule*. In the *scoring auction*, bidders are asked to submit a set of multidimensional bids that include price and some non-price attributes, such as quality. The bids are then transformed into a score by an ex ante publicly announced scoring rule, and the bidder whose score is the highest is awarded the contract. We now provide a few real life examples of such scoring auctions.

The Department of Defence in USA often relies on competitive source selection to procure weapon systems. Each individual component of a bid of the weapon system is evaluated and assigned a score, these scores are summed to yield a total score, and the firm achieving the highest score wins the contract (see Che, 1993). For highway construction projects, states like Alaska, Colorado, Florida, Michigan, North Carolina, and South Dakota use quality-over-price ratio rules, in which the score is computed based on the quality divided by price. This scoring rule is also extensively used in Japan. Ministry of Land, Infrastructure and Transportation in Japan allocates most of the public construction project contracts through scoring auctions based on quality-over-price ratio rules (see Hanazono, Nakabayashi and Tsuruoka, 2013). In a country like India where fuel costs are very high, airlines greatly value the fuel cost savings. Airline companies in India typically purchase new aircraft after evaluating competing offers (that include price as well as various quality parameters) from big aircraft suppliers like Boeing and Airbus. For example, in 2011 IndiGo, a low-cost Indian airline, received multidimensional bids (price, fuel efficiency of engines etc.) from both Airbus and Boeing. Indigo gave the contract to Airbus and ordered 180 Airbus A320s for a valuation of $15.6 billion.[2] A few years back, Government of India sought multidimensional bids from

[1] "For public funds to be spent efficiently and effectively, value for money is the key principle in public procurement. Low-price auctions have been widespreadly used to allocate contracts as a competitive, transparent, and accountable mechanism. However, costs are not the sole indicator in assessing the best value-for-money contract. More and more procurement buyers, thus, introduce awarding mechanisms with which relevant prices and qualities of proposals in the whole procurement cycle are assessed" (Section 1 in Nakabayashi and Hirose, 2013).

[2] As Airbus offers more fuel-efficient aircraft, in Indian aviation market the demand for its aircarfts is

various companies for renovation of New Delhi's international airport. Finally, GMR won the contract.

Till date, most papers on scoring auctions (except a couple of very recent ones) have dealt only with quasilinear scoring rules. However, in real life, non-quasilinear scoring rules (like the quality-to-price ratio) are often used. As such, in this paper we analyse scoring auctions with non-quasilinear scoring rules. We first proceed to provide a brief literature review.

Relevant Literature Che (1993) is a pioneer in analysing such scoring auctions. In his model both the quality and the bidder's types are single-dimensional, and the scoring rule is quasilinear. Che (1993) computes equilibria in first-score and second-score auctions and also analyses optimal mechanisms when types are identically and independently distributed. Branco (1997) analyses the properties of optimal mechanisms when types are single-dimensional but correlated.

The paper by Asker and Cantillon (2008) deals with multidimensional types in a scoring auction. This paper defines a 'pseudotype' and shows that if the scoring rule is quasilinear and types are independently distributed then every equilibrium in the scoring auction is typewise outcome equivalent to an equilibrium in the scoring auction where suppliers are constrained to bid only on the basis of their pseudotypes.

Asker and Cantillon (2010) analyses optimal mechanisms with one-dimensional quality and two-dimensional discrete types. Nishimura (2012) computes optimal mechanisms with multidimensional quality and single-dimensional types that are identically and independently distributed.

In may be noted that in **all** the above papers the scoring rule in **quasilinear**. Very few papers in the literature deal with general **non-quasilinear** scoring rules. Hanazono, Nakabayashi and Tsuruoka (2013) is an important contribution in this regard. This paper considers a broad class of scoring rules and computes equilibria for first-score and second-score auctions and compares expected scores. Hanazono (2010) provides an example with a specific non-quasilinear scoring rule and a specific cost function[3]. Very recently, in an interesting contribution, Wang and Liu (2014) analyses equilibrium in first-score auctions with a different non-quasilinear scoring rule.

However, it may be noted that in all the above mentioned papers with non-quasilinear scoring rules explicit solution for the equilibrium strategies are not always obtained. For example, in Hanazono, Nakabayashi and Tsuruoka (2013) the choice of 'quality' in equilibrium is endogenous in the 'score' under the general scoring function. Moreover, the comparison of expected scores is based on properties of induced utility whose arguments are implicitly defined.[4]

increasing. Boeing's market share has slumped in the Indian market and Airbus now controls about 73% of the Indian pie. See Keller (2011) and Singhal (2011) for the information regarding the order for Airbus aircrafts placed by some of the Indian airline companies.

[3]This short note is written in Japanese. I am grateful to Masaki Aoyagi for helping me understand the results of this paper.

[4]This paper avoids specific functional forms but instead imposes some restrictions on the induced utility.

Contributions of this paper We take Hanazono et al (2013) as a point of departure and ask the following questions.

Can we get explicit solutions for equilibrium strategies with general non-quasilinear scoring rules? Can we provide a complete characterisation of such equilibria? Also, can we get a ranking of the expected scores (in first-score and second-score auctions) by using the curvature properties of the scoring rule and properties of the distribution function of types? If so, under what conditions can the above be achieved?

We show that all the above can be achieved if the cost function of each firm is ***additively separable*** *in quality and type.* Our computations provide a much simpler way to derive equilibria in scoring auctions without any endogeneity problems. We get ***explicit solutions***. We provide a **complete characterisation** of such equilibria and also provide ranking of the two auction formats. We show that such properties of the equilibria and ranking of expected scores depend on the curvature properties of the scoring rule and properties of the distribution function of type. This stands in contrast to the results derived in Hanazono et al (2013)[5]. We also compute equilibria for the case of multi-dimensional quality and multi-dimensional types. We show that such equilibria are very similar to the case with one-dimensional type and quality. Our approach helps in dealing with most non-quasilinear scoring rules. It essentially complements the one taken in Hanazono et al (2013).

Plan of the paper In section 2 we provide the model of our exercise. In section 3 we compute the equilibria for first-score and second-score auctions. Section 4 provides the equilibrium characterisations. In section 5 we give the main results on the comparison of expected scores. Section 6 extends our model to multi-dimensional quality and types. Lastly, we provide some concluding remarks and possible scope for future research in this area. All proofs are provided in the appendix.

We now proceed to provide the model of our exercise.

2 The Model

A buyer solicits bids from n firms. Each bid, (p, q), specifies an offer of promised quality, q and price, p, at which a fixed quantity of products with the offered level of quality q is delivered. The quantity is normalized to one. For simplicity quality is modelled as a one-dimensional attribute.

A scoring rule is a function $S : \mathbb{R}^2_{++} \longrightarrow \mathbb{R} : (p, q) \longrightarrow S(p, q)$ that associates a score to any potential contract and represents a continuous preference relation over contract characteristics (p, q).

Assumption 1 $S(.)$ is strictly decreasing in p and strictly increasing in q. That is, $S_p < 0$ and $S_q > 0$. We assume that the partial derivatives S_p, S_q, S_{pp}, S_{pq}, S_{qq} exist and they are continuous in all $(p, q) \in \mathbb{R}^2_{++}$.

[5]Unlike Hanazono (2010) and Wang and Liu (2014) we deal with general non-quasilinear scoring rules.

A scoring rule is **quasilinear** if it can be expressed as $\phi(q) - p$ or any monotonic increasing function thereof. For quasilinear rules we must have $S_{pp} = 0$ and $S_{pq} = 0$. For **non-quasi-linear** rules we must have at least one of the following: $S_{pp} \neq 0$ or $S_{pq} \neq 0$.

The auction rules: The buyer awards the contract to a firm whose offer achieves the highest score. This is similar to a standard auction. We consider the following auctions.

1. **First-score auction**: The winning firm's offer is finalised as the contract. This auction rule is a multi-dimensional analogue of the first price auction.

2. **Second-score auction**: Here the winning firm is required to match the highest rejected score. In meeting this score, the firm is free to choose any quality-price combination. This auction rule is a multi-dimensional analogue of the second-price auction.

We provide the following example to illustrate the above two auctions. Let the scoring rule be $S(p, q) = 2q - p$. Suppose two firms A and B offer $(5, 7)$ and $(3, 5)$ as their (p, q) pairs. We have $S(5, 7) = 9$ and $S(3, 5) = 7$. Under both auction formats (first-score and second-score) firm A is declared the winner. The final contract awarded to firm A is $(5, 7)$ under the first-score auction and any (p, q) satisfying $S(p, q) = 7$ under the second-score auction.

The cost to the supplier is $C(q, x)$ where x is the type.

Assumption 2 We assume $C_q > 0$, $C_{qq} \geq 0$ and $C_x > 0$.

Prior to bidding each firm i learns its cost parameter x_i as private information. The buyer and *other* firms (i.e. other than firm i) do not observe x_i but only knows the distribution function of the cost parameter. It is assumed that x_is are identically and independently distributed over $[\underline{x}, \bar{x}]$ where $0 \leq \underline{x} < \bar{x}$.

If supplier i wins the contract, its payoff is $p - C(q, x_i)$.

We now provide our most important assumption which separates our paper from the rest of the papers of this genre.

Assumption 3 *Cost is additively separable in quality and type.*

That is, $C(q, x) = c(q) + \alpha(x)$ where $c'(.) > 0$, $c''(.) \geq 0$, $\alpha(\underline{x}) \geq 0$ and $\alpha'(.) > 0$.

Define $\theta_i = \alpha(x_i)$. Let $\underline{\theta} = \alpha(\underline{x})$ and let $\bar{\theta} = \alpha(\bar{x})$. Clearly, $0 \leq \underline{\theta} < \bar{\theta}$. Since x_is are identically and independently distributed over $[\underline{x}, \bar{x}]$, so are the θ_is over $\left[\underline{\theta}, \bar{\theta}\right]$. Let the distribution function of θ_i be $F(.)$ and the density function be $f(.)$. Note that $f(\theta) \geq 0$ $\forall \theta \in \left[\underline{\theta}, \bar{\theta}\right]$.

We can now write the cost for supplier i as $C(q, \theta_i) = c(q) + \theta_i$, where θ_i is the type of supplier i.

We also assume the following.

Assumption 4

$$-\frac{(S_q)^2}{S_p}S_{pp}+2S_qS_{pq}-S_pS_{qq}-(S_p)^2\,c''(.)<0 \text{ for all } (p,q)\in\mathbb{R}^2_{++}$$

It may also be noted that when $c''(.)>0$ then both for the quasilinear rule $(S(p,q)=\phi(q)-p)$ and the quality-to-price ratio $\left(S(p,q)=\frac{q}{p}\right)$ (which is a non-quasilinear rule) the above is always satisfied.

The following may be noted.

1. The assumption (cost is additively separable in quality and type) is consistent with the set of assumptions in Hanazono et al (2013) and Asker and Cantillon (2008).

2. Additive separability implies $C_{q\theta}(.)=0$. This is different from Che (1993), Branco (1997) and Nishimura (2012).[6]

3. Our cost, $C(q,\theta_i)=c(q)+\theta_i$, can be interpreted in the following way. $c(q)$ is the variable cost and θ_i is the fixed cost of firm i. This means, the variable costs are same across firms but the fixed costs are private information. θ_i can be interpreted to be the inverse of managerial efficiency which is private information to the firm. Higher is θ_i, lower is the managerial efficiency, and consequently, higher will be the cost.

3 Equilibrium in first-score and second-score auctions

We now provide the equilibrium for first-score and second-score auctions. The proofs are given in the appendix.

Proposition 1 In a *first-score* auction there is a symmetric equilibrium where a supplier with type θ chooses $\left(p^I(\theta),q^I(\theta)\right)$. Such $p^I(.)$ and $q^I(.)$ are obtained by solving the following equations:

$$\begin{aligned} -\frac{S_q(.)}{S_p(.)} &= c'(.) \\ p-c(q) &= \theta+\gamma(\theta) \end{aligned}$$

where

$$\gamma(\theta)=\frac{1}{(1-F(\theta))^{n-1}}\int_\theta^{\bar{\theta}}(1-F(t))^{n-1}\,dt$$

[6] In Che (1993) we have $C_{q\theta}(.)>0$ and in Branco (1997) we have $C_{q\theta}<0$. In Nishimura (2012) C_θ has strictly increasing differences in (q,θ).

Proposition 2 In a *second-score* auction there is a weakly dominant strategy equilibrium where a supplier with type θ chooses $\left(p^{II}(\theta), q^{II}(\theta)\right)$. Such $p^{II}(.)$ and $q^{II}(.)$ are obtained by solving the following equations:

$$\begin{aligned} -\frac{S_q(.)}{S_p(.)} &= c'(.) \\ p - c(q) &= \theta \end{aligned}$$

Comment For both first-score and second-score auctions the equation, $-\frac{S_q(.)}{S_p(.)} = c'(.)$, holds true in equilibrium. Note that this equation is independent of θ. From this equation we get the price, p, as a function of the quality, q (say $p = \sigma(q)$). We can then substitute $p = \sigma(q)$ into the next equation to get q as a function of θ. Next, using this we derive p as a function of θ.

In the appendix we provide a proof of the above two propositions. Here we provide a brief sketch of the argument.

First, consider proposition 1. For any quality, q, let $\Psi(s, q)$ be the price required to generate a score of s. That is, $S(\Psi(s, q), q) = s$. Clearly, $\Psi(.)$ is well defined and it is strictly decreasing in s and strictly increasing in q.

Consider any symmetric equilibrium of first-score auction where a bidder with type θ bids (p, q). Let the score generated by such a bid be $S(p, q) = s$. Since $\Psi(.)$ is well defined and is strictly decreasing in s we can think of the equilibrium as where a bidder bids a score s and quality q. The payoff (conditional on winning) with a score s to a bidder with type θ is

$$\Psi(s, q) - c(q) - \theta.$$

In any equilibrium, for any type θ, the quality choice, q, must be such so as to maximise $\Psi(s, q) - c(q) - \theta$. The FOC and SOC for such a maximisation are as follows:

$$\begin{aligned} \Psi_q(.) - c'(.) &= 0 - - - - (1a) \\ \Psi_{qq} - c''(.) &< 0 - - - - (1b) \end{aligned}$$

Note that

$$\Psi_{qq} - c''(.) < 0 \Longleftrightarrow -\frac{(S_q)^2}{S_p} S_{pp} + 2S_q S_{pq} - S_p S_{qq} - (S_p)^2 c''(.) < 0$$

Given our assumption 4, the SOC (which is (1b)) will always be satisfied.

Note that we have the following[7]:

$$\Psi_q(.) = -\frac{S_q(.)}{S_p(.)} \text{ and } \Psi_s(.) = \frac{1}{S_p(.)} - - - - (2)$$

[7] From $S(p, q) - s = 0$ we can implicitly solve for p and then use the implicit function theorem.

Hence we can rewrite (1a) as follows:

$$-\frac{S_q(.)}{S_p(.)} = c'(.) ----(3)$$

Consequently, in any equilibrium (3) will be satisfied. Now let's suppose that all firms other than firm i choose (p, q) according to the equations in proposition 1 (i.e. $-\frac{S_q(.)}{S_p(.)} = c'(.)$ and $p - c(q) = \theta + \gamma(\theta)$). Thereafter, using standard auction theoretic techniques we can show that it is optimal for firm i to choose (p, q) by following the same equations.

Now consider the case of second-score auction (proposition 2). What matters to any firm i is the maximum of scores quoted by *other* firms[8]. Let the maximum of the scores quoted by firms other than i be δ. Now let firm i choose $\left(p^{II}, q^{II}\right)$ by following the two equations in proposition 2 (i.e. $-\frac{S_q(.)}{S_p(.)} = c'(.)$ and $p - c(q) = \theta$) and thereby pick up a score $s = S\left(p^{II}, q^{II}\right)$. Using standard techniques it can be shown that regardless of δ, it is always better for firm i to choose $\left(p^{II}, q^{II}\right)$ by following these two equations.

Several observations can be made.

1. In Che (1993) the scoring rule is quasilinear and he gets explicit solutions for equilibrium strategies. Our equilibrium results are close to Che (1993). In Hanazono et al (2013) the scoring rule is non-quasilinear but explicit solutions for equilibrium strategies are not obtained in general. In most cases the equilibrium strategies are only derived implicitly[9].

2. In our case, the cost function is additively separable in quality and type and we get explicit solutions for equilibrium strategies for both kinds of scoring rules: quasilinear and non-quasilinear. Additive separability of the cost function makes equilibrium computations very simple. This stands in sharp contrast to all the recent papers that deal with non-quasilinear scoring rules.

3. Moreover, our assumptions are also milder and are satisfied a by a large class of scoring rules.

4. When the scoring rule is quasilinear $S_p(.)$ is a constant and S_q is independent of p (since $S_{pp} = S_{qp} = 0$). Note that in any auction the equation $-\frac{S_q(.)}{S_p(.)} = c'(.)$ is satisfied. This means the quality, q, is constant and same for the two auctions.

We illustrate the above two propositions in two examples given below.

[8]Note that in a second-score auction the winning firm is required to match the highest rejected score. In meeting this score, the firm is free to choose any quality-price combination.

[9]Wang and Liu (2014) use a specific scoring rule but here also equilibrium strategies are only derived implicitly.

Example 1 (non-quasilinear scoring rule) Let $S(p,q) = \frac{q}{p}$ and $C(q,\theta) = \frac{1}{2}q^2 + \theta$. Let θ be uniformly distributed over $[1,2]$ and $n = 2$.

In a first-score auction the symmetric equilibrium is as follows.

$$p^I(\theta) = 2 + \theta, \quad q^I(\theta) = \sqrt{2+\theta} \ \forall \theta \in [1,2].$$

In a second-score auction the symmetric equilibrium is as follows.

$$p^{II}(\theta) = 2\theta, \ q^{II}(\theta) = \sqrt{2\theta} \ \forall \theta \in [1,2].$$

Example 2 (quasilinear scoring rule) Let $S(p,q) = q - p$ and $C(q,\theta) = \frac{1}{2}q^2 + \theta$. Let θ be uniformly distributed over $[1,2]$ and $n = 2$.

In a first-score auction the symmetric equilibrium is as follows.

$$p^I(\theta) = \frac{3}{2} + \frac{1}{2}\theta, \quad q^I(\theta) = 1 \ \forall \theta \in [1,2].$$

In a second-score auction the symmetric equilibrium is as follows.

$$p^{II}(\theta) = \frac{1}{2} + \theta, \ q^{II}(\theta) = 1 \ \forall \theta \in [1,2].$$

4 Equilibrium Characterisation

We now provide some properties of the symmetric equilibria that were derived in the previous section. All proofs are given in the appendix. First, we define the following:

$$\begin{aligned} A(p,q) &= -\frac{S_q(p,q)}{S_p(p,q)} S_{pp}(p,q) + S_{qp}(p,q) \\ B(p,q) &= -\frac{S_q(p,q)}{S_p(p,q)} S_{pq}(p,q) + S_p(p,q)\, c''(q) + S_{qq}(p,q) \\ H(p,q) &= S_{pp}(p,q)\left[S_p(p,q)\, c''(q) + S_{qq}(p,q)\right] - \left[S_{qp}(p,q)\right]^2 \end{aligned}$$

Lemma 1 $p^I\left(\bar{\theta}\right) = p^{II}\left(\bar{\theta}\right)$ and $q^I\left(\bar{\theta}\right) = q^{II}\left(\bar{\theta}\right)$.

Comment A firm with the highest type $\left(\bar{\theta}\right)$ quotes the same price and quality across first-score and second-score auctions (lemma 1). This is true regardless of the fact whether the scoring rule is quasilinear or not. Lemma 2 below will be useful in proving some of our results.

Lemma 2 Suppose $A(p,q) \neq 0 \ \forall (p,q) \in \mathbb{R}^2_{++}$.

$$B(p,q) \geq 0 \Longrightarrow A(p,q) < 0.$$

We now proceed to consider scoring rules that are non-quasilinear. Note that for such rules we must have at least one of the following: $S_{pp} \neq 0$, $S_{pq} \neq 0$.

The next proposition compares the equilibrium scores quoted first-score and second-score auctions. Let $S^I(\theta) = S\left(p^I(\theta), q^I(\theta)\right)$ and $S^{II}(\theta) = S^{II}\left(p^{II}(\theta), q^{II}(\theta)\right)$. In the first-score and second-score auctions the equilibrium scores quoted by a firm with type θ is $S^I(\theta)$ and $S^{II}(\theta)$ respectively.

Proposition 3 If $A(p,q) \neq 0 \quad \forall (p,q) \in \mathbb{R}^2_{++}$ then $S^I(\theta) < S^{II}(\theta)\ \forall \theta \in [\underline{\theta}, \bar{\theta})$. Also, $\frac{d}{d\theta}S^I(\theta), \frac{d}{d\theta}S^{II}(\theta) < 0\ \forall \theta \in (\underline{\theta}, \bar{\theta})$.

Comment The above result also holds for quasilinear scoring rules. In a first-score auction (or second-score auction) the price and quality in equilibrium will satisfy the equation, $p - c(q) = \theta + \gamma(\theta)$ (or $p - c(q) = \theta$). We have earlier noted that if the scoring rule is quasilinear ($S_{pp} = S_{qp} = 0$) then quality is constant and same for the two auctions. This means price quoted in a first-score auction will be higher than the price quoted in a second-score auction. Consequently, the score quoted in a first-score auction will be lower than the score quoted in a second-score auction.

The equilibrium score quoted by any type $\theta \in [\underline{\theta}, \bar{\theta})$ is strictly higher in the second-score auction as compared to the equilibrium score in first score-auction. This is analogous to the standard benchmark model where for any particular type, the bid in the second-price auction is always higher than the bid in the first-price auction. Proposition 3 also shows that equilibrium scores are decreasing in type, θ. This means the winner in any auction is the firm with the lowest type (least cost). That is, the symmetric equilibria are always efficient. This is similar to the case where the scoring rule is quasilinear.

Proposition 4 (i) If $A(p,q) > 0 \quad \forall (p,q) \in \mathbb{R}^2_{++}$ then $q^I(\theta) > q^{II}(\theta)\ \forall \theta \in [\underline{\theta}, \bar{\theta})$. Also, $\frac{dq^I(\theta)}{d\theta}, \frac{dq^{II}(\theta)}{d\theta} > 0 \quad \forall \theta \in (\underline{\theta}, \bar{\theta})$.

(ii) If $A(p,q) < 0 \quad \forall (p,q) \in \mathbb{R}^2_{++}$ then $q^I(\theta) < q^{II}(\theta) \quad \forall \theta \in [\underline{\theta}, \bar{\theta})$. Also, $\frac{dq^I(\theta)}{d\theta}, \frac{dq^{II}(\theta)}{d\theta} < 0 \quad \forall \theta \in (\underline{\theta}, \bar{\theta})$.

(iii) If $A(p,q) = 0 \quad \forall (p,q) \in \mathbb{R}^2_{++}$ then $q^I(\theta) = q^{II}(\theta) \quad \forall \theta \in [\underline{\theta}, \bar{\theta})$. Also, $\frac{dq^I(\theta)}{d\theta}, \frac{dq^{II}(\theta)}{d\theta} = 0 \quad \forall \theta \in (\underline{\theta}, \bar{\theta})$.

Comment We now try to provide an intuition behind the above result. It may be noted that $\Psi_{qs} = -\frac{A(.)}{(S_p)^2}$. This implies that Ψ_{qs} has the opposite sign of $A(.)$. We know that for both auctions $\Psi_q(.) = c'(.)$. From this we can derive that $\frac{dq}{ds} = -\frac{\Psi_{qs}}{\Psi_{qq} - c''}$. Since $\Psi_{qq} - c'' < 0$, $\frac{dq}{ds}$ has the same sign as Ψ_{qs}. This means that $\frac{dq}{ds}$ has the opposite sign of $A(.)$ in both auctions. When $A(.) > 0$ then $\frac{dq}{ds} < 0$. Since $S^I(.) < S^{II}(.)$ (see proposition 3) we must have $q^I(.) > q^{II}(.)$. Again, when $A(.) < 0$ then $\frac{dq}{ds} > 0$. Using a similar logic we must have $q^I(.) < q^{II}(.)$.

From proposition 4 it follows that for $\theta < \bar{\theta}$, whether the quality quoted in first-score auction is higher (or lower) than the quality quoted in second-score auction depends crucially on the sign of the term $A(p,q)$. In fact, this term plays a crucial role in determining whether the equilibrium quality quoted in any auction is increasing in θ or not.

However, the next result shows that comparison of price quoted in first-score auction with the one quoted in second-score auction depends crucially on the sign of the term $B(p,q)$. This term also determines whether the equilibrium price quoted in any auction is increasing in θ or not.

We now proceed to state the next result.

Proposition 5 Suppose $A(p,q) \neq 0 \;\forall (p,q) \in \mathbb{R}^2_{++}$.

(i) If $B(p,q) < 0 \;\forall (p,q) \in \mathbb{R}^2_{++}$ then $p^I(\theta) > p^{II}(\theta) \;\; \forall \theta \in \left[\underline{\theta}, \bar{\theta}\right)$. Also, $\frac{dp^I(\theta)}{d\theta}, \; \frac{dp^{II}(\theta)}{d\theta} > 0 \;\forall \theta \in \left(\underline{\theta}, \bar{\theta}\right)$.

(ii) If $B(p,q) > 0 \;\forall (p,q) \in \mathbb{R}^2_{++}$ then $p^I(\theta) < p^{II}(\theta) \;\; \forall \theta \in \left[\underline{\theta}, \bar{\theta}\right)$. Also, $\frac{dp^I(\theta)}{d\theta}, \; \frac{dp^{II}(\theta)}{d\theta} < 0 \;\forall \theta \in \left(\underline{\theta}, \bar{\theta}\right)$.

(iii) If $B(p,q) = 0 \;\forall (p,q) \in \mathbb{R}^2_{++}$ then $p^I(\theta) = p^{II}(\theta) \;\; \forall \theta \in \left[\underline{\theta}, \bar{\theta}\right]$. Also, $\frac{dp^I(\theta)}{d\theta}, \; \frac{dp^{II}(\theta)}{d\theta} = 0 \;\forall \theta \in \left(\underline{\theta}, \bar{\theta}\right)$.

Comment Lemma 2 shows that the signs of $A(.)$ and $B(.)$ are related. From lemma 2 we know that $A(p,q) > 0 \Longrightarrow B(p,q) < 0$. Proposition 4 demonstrates that $A(p,q) > 0 \;\; \Longrightarrow q^I(\theta) > q^{II}(\theta)$. From proposition 5 we get $B(p,q) < 0 \Longrightarrow p^I(\theta) > p^{II}(\theta)$. This clearly means $A(p,q) > 0 \;\; \Longrightarrow q^I(\theta) > q^{II}(\theta)$ and $p^I(\theta) > p^{II}(\theta)$. From proposition 5 we also get that $B(p,q) \geq 0 \Longrightarrow p^I(\theta) \leq p^{II}(\theta)$. Lemma 2 shows that $B(p,q) \geq 0 \Longrightarrow A(p,q) < 0$. Combining this with proposition 4 and lemma 2 we get that $B(p,q) \geq 0 \Longrightarrow p^I(\theta) \leq p^{II}(\theta)$ and $q^I(\theta) < q^{II}(\theta)$.

We now proceed to discuss the impact of increase in n (the number of bidders) on equilibrium quality and price in both auctions. For any given θ, let $q^I(n;\theta)$ and $q^{II}(n;\theta)$ be the quality quoted in first-score and second score auctions respectively when the number of bidders is n. Similarly, for any given θ, let $p^I(n;\theta)$ and $p^{II}(n;\theta)$ be the price quoted in first-score and second-score auctions respectively when the number of bidders is n.

Proposition 6 For all $n > m$

(i) $q^{II}(n;\theta) = q^{II}(m;\theta)$.

(ii) If $A(p,q) > 0 \;\; \forall (p,q) \in \mathbb{R}^2_{++}$ then $q^I(n;\theta) < q^I(m;\theta)$.

(iii) If $A(p,q) < 0 \;\; \forall (p,q) \in \mathbb{R}^2_{++}$ then $q^I(n;\theta) > q^I(m;\theta)$.

Proposition 7 Suppose $A(p,q) \neq 0 \;\forall (p,q) \in \mathbb{R}^2_{++}$. Then for all $n > m$

(i) $p^{II}(n;\theta) = p^{II}(m;\theta)$.

(ii) If $B(p,q) = 0 \;\forall (p,q) \in \mathbb{R}^2_{++}$ then $p^I(n;\theta) = p^I(m;\theta)$.

(iii) If $B(p,q) > 0 \;\forall (p,q) \in \mathbb{R}^2_{++}$ then $p^I(n;\theta) > p^I(m;\theta)$.

(iv) If $B(p,q) < 0 \;\forall (p,q) \in \mathbb{R}^2_{++}$ then $p^I(n;\theta) < p^I(m;\theta)$.

For any given type θ let $S^I(n;\theta)$ and $S^{II}(n;\theta)$ be the scores quoted in equilibrium in first-score and second-score auction respectively when the number of bidders is n. That is, $S^I(n;\theta) = S\left(p^I(n;\theta), q^I(n;\theta)\right)$ and $S^{II}(n;\theta) = S\left(p^{II}(n;\theta), q^{II}(n;\theta)\right)$. The next proposition explores how the equilibrium score quoted changes with an increase in the number of bidders. The proof is in the appendix.

Proposition 8 (i) For all $n > m$, $S^{II}(n;\theta) = S^{II}(m;\theta)$.

(ii) For all $n > m$, $S^I(n;\theta) > S^I(m;\theta)$.

Comment In the second-score auction the quality and price quoted in equilibrium are independent of the number of bidders. Consequently, the score quoted in equilibrium is invariant with respect to the number of bidders. This is similar to the second-price auction in the benchmark model, where, regardless of the number of bidders, all bidders bid their valuations.

In the first-score auction as the competition intensifies (n increases) the score quoted by any type increases. This is in line with the conventional wisdom which suggests that any increase in competition should induce a bidder with type θ to quote a higher score. This is also similar to the first-price auction in the benchmark model where bids increase with the number of bidders.

5 Expected Scores

The previous section provided equilibrium characterisation for first-score and second-score auctions. We now proceed to give our results on expected scores. Before giving our main results we need to discuss some preliminaries on order statistics.

5.1 Order Statistics : some notations and preliminaries

Let $y_1, y_2..y_n$ denote a random sample of size n drawn from $F(.)$. Then $x_1 \leq x_2... \leq x_n$ where $x_i s$ are $y_i s$ arranged in increasing magnitudes, are defined to be the order statistics corresponding to the random sample $y_1, y_2....y_n$.

We would be interested in x_1 (lowest order statistic) and x_2 (second lowest order statistic). The corresponding distribution functions and density functions are $F_1(.)$, $F_2(.)$ and $f_1(.)$, $f_2(.)$. Note that

$$\begin{aligned} F_1(x) &= 1-(1-F(x))^n \text{ and } F_2(x) = 1-(1-F(x))^n - nF(x)(1-F(x))^{n-1} \\ f_1(x) &= n(1-F(x))^{n-1} f(x) \text{ and } f_2(x) = n(n-1)F(x)(1-F(x))^{n-2} f(x) \end{aligned}$$

$$\text{Note that } F_2(x) = F_1(x) - nF(x)(1-F(x))^{n-1}$$

5.2 Comparison of expected scores

In proposition 5 it is shown that $S^I(\theta) < S^{II}(\theta)$ for all $\theta \in [\underline{\theta}, \bar{\theta})$ and both $S^I(\theta)$ and $S^{II}(\theta)$ are strictly decreasing in θ. As noted before, the winner in any auction is the firm with the lowest type.

The following two lemmas will help us in comparing the expected scores across auctions. The proofs appear in the appendix.

Lemma 3 (i) In a first-score auction the expected score is as follows:

$$\begin{aligned}\Sigma^{I} &= \int_{\underline{\theta}}^{\bar{\theta}} S^{I}(\theta) f_{1}(\theta)\, d\theta \\ &= S\left(p^{I}\left(\bar{\theta}\right), q^{I}\left(\bar{\theta}\right)\right) - \int_{\underline{\theta}}^{\bar{\theta}} F_{1}(\theta)\left(1+\gamma'(\theta)\right) S_{p}\left(p^{I}(\theta), q^{I}(\theta)\right) d\theta\end{aligned}$$

(ii) In a second-score auction the expected score is as follows:

$$\begin{aligned}\Sigma^{II} &= \int_{\underline{\theta}}^{\bar{\theta}} S^{II}(\theta) f_{2}(\theta)\, d\theta \\ &= S\left(p^{II}\left(\bar{\theta}\right), q^{II}\left(\bar{\theta}\right)\right) - \int_{\underline{\theta}}^{\bar{\theta}} F_{2}(\theta) S_{p}\left(p^{II}(\theta), q^{II}(\theta)\right) d\theta\end{aligned}$$

Lemma 4

$$\int_{\underline{\theta}}^{\bar{\theta}} F_{1}(\theta)\left(1+\gamma'(\theta)\right) d\theta = \int_{\underline{\theta}}^{\bar{\theta}} F_{2}(\theta)\, d\theta$$

where

$$\gamma(\theta) = \frac{1}{(1-F(\theta))^{n-1}} \int_{\theta}^{\bar{\theta}} (1-F(t))^{n-1}\, dt$$

Comment From lemma 1 we know $p^{I}\left(\bar{\theta}\right) = p^{II}\left(\bar{\theta}\right)$ and $q^{I}\left(\bar{\theta}\right) = q^{II}\left(\bar{\theta}\right)$. This means

$$S\left(p^{I}\left(\bar{\theta}\right), q^{I}\left(\bar{\theta}\right)\right) = S\left(p^{II}\left(\bar{\theta}\right), q^{II}\left(\bar{\theta}\right)\right).$$

Using this and lemma 3 one clearly gets that to compare Σ^{I} and Σ^{II} we need to compare the following terms:

$$\left[\int_{\underline{\theta}}^{\bar{\theta}} F_{1}(\theta)\left(1+\gamma'(\theta)\right) S_{p}\left(p^{I}(\theta), q^{I}(\theta)\right) d\theta\right] \text{ and } \left[\int_{\underline{\theta}}^{\bar{\theta}} F_{2}(\theta) S_{p}\left(p^{II}(\theta), q^{II}(\theta)\right) d\theta\right].$$

Note that if the scoring rule is **quasilinear** (i.e. $S(p,q) = \phi(q) - p$) then $S_{p} = -1$. Hence, from lemma 3 the next result follows.

Proposition 9 If the scoring rule is quasilinear then $\Sigma^{I} = \Sigma^{II}$.

Comment The above result is well known (See Che, 1993, Asker and Cantillon, 2008 and Hanazono et al, 2013). For scoring auctions this is the analogue of revenue equivalence theorem of the canonical model.

Note that we had earlier defined the following: $H(p,q) = S_{pp}\left[S_{p}c'' + S_{qq}\right] - \left(S_{qp}\right)^{2}$.

We now proceed to provide our main results on expected scores when the scoring rules are **non-quasilinear**. We show that such results depend on the curvature properties of the scoring rule and the properties of the distribution function of types. The following result will help us in the ranking of the expected scores. The proof is given in the appendix.

Lemma 5 Suppose $A(p,q) \neq 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$.

(i) If $H(p,q) < 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$ then

$$-S_p\left(p^I(\theta), q^I(\theta)\right) < -S_p\left(p^{II}(\theta), q^{II}(\theta)\right) \;\; \forall \theta \in [\underline{\theta}, \bar{\theta})$$

$$\text{and } \frac{d}{d\theta}\left(-S_p\left(p^I(\theta), q^I(\theta)\right)\right), \; \frac{d}{d\theta}\left(-S_p\left(p^{II}(\theta), q^{II}(\theta)\right)\right) < 0 \; \forall \theta \in (\underline{\theta}, \bar{\theta}).$$

(ii) If $H(p,q) \geq 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$ then

$$-S_p\left(p^I(\theta), q^I(\theta)\right) \geq -S_p\left(p^{II}(\theta), q^{II}(\theta)\right) \;\; \forall \theta \in [\underline{\theta}, \bar{\theta})$$

$$\text{and } \frac{d}{d\theta}\left(-S_p\left(p^I(\theta), q^I(\theta)\right)\right), \; \frac{d}{d\theta}\left(-S_p\left(p^{II}(\theta), q^{II}(\theta)\right)\right) \geq 0 \; \forall \theta \in (\underline{\theta}, \bar{\theta}).$$

Note that for non-quasilinear scoring rules we must have at least one of the following: $S_{pp} \neq 0$, $S_{pq} \neq 0$. Proposition 10 shows that like the quasilinear case, we can have expected score equivalence with non-quasilinear scoring rules.

Proposition 10 If $\forall (p,q) \in \mathbb{R}^2_{++}$ $S_p c'' + S_{qq} = 0$ and $S_{qp} = 0$ then $\Sigma^I = \Sigma^{II}$.

Comment We illustrate proposition 10 with the following example. Let $S(p,q) = 10q - p^2$, $C(q,\theta) = q + \theta$ and θ is uniformly distributed over $[1,2]$. The scoring rule is non-quasilinear and satisfies all our assumptions. Here it can be easily shown that $\Sigma^I = \Sigma^{II}$.

However, when either $S_p c'' + S_{qq} \neq 0$ or $S_{pq} \neq 0$ we do not always have a clear ranking of the two auctions in terms of expected scores. In fact, it can be shown that depending on the distribution function of types we can get different rankings with the same scoring rule and cost function. We illustrate our point with the 'quality over price' scoring rule.[10]

Example 3 Let $S(p,q) = \frac{q}{p}$ and $C(q,\theta) = \frac{1}{2}q^2 + \theta$. Suppose θ be uniformly distributed over $[1,2]$ and $n = 2$. For this distribution we have

$$f_1(\theta) = 2(2-\theta) \quad \text{and } f_2(\theta) = 2(\theta - 1)$$

The equilibria are as follows:

First-score auction:

$$\begin{aligned} \text{price:} \quad & p^I(\theta) = 2 + \theta \\ \text{quality} \;: \quad & q^I(\theta) = \sqrt{2+\theta} \\ \text{score:} \quad & s^I(\theta) = \frac{q^I(\theta)}{p^I(\theta)} = \frac{1}{\sqrt{2+\theta}} \\ \text{Expected score:} \quad & \Sigma^I = \int_1^2 s^I(\theta) f_1(\theta)\, d\theta = 0.548\,72 \end{aligned}$$

[10] I must thank Kasunori Yamada for helping me with the computations using MATLAB.

Second-score auction:

$$\begin{aligned}
\text{price:} &\quad p^{II}(\theta) = 2\theta \\
\text{quality:} &\quad q^{II}(\theta) = \sqrt{2\theta} \\
\text{score:} &\quad s^{II}(\theta) = \frac{q^{II}(\theta)}{p^{II}(\theta)} = \frac{1}{\sqrt{2\theta}} \\
\text{Expected score:} &\quad \Sigma^{II} = \int_1^2 s^{II}(\theta) f_2(\theta)\, d\theta = 0.552\,28
\end{aligned}$$

Example 4 Let $S(p,q) = \frac{q}{p}$ and $C(q,\theta) = \frac{1}{2}q^2 + \theta$. Now suppose $n = 2$ and θ is distributed over $[1.2,\ 1.203731]$ with density $f(x) = 500x^3 - 600$ and distribution function $F(x) = 125x^4 - 600x + \frac{2304}{5}$. For this distribution we have

$$\begin{aligned}
f_1 &= 2\left(-125x^4 + 600x - \frac{2299}{5}\right)\left(500x^3 - 600\right) \text{ and} \\
f_2 &= 2\left(125x^4 - 600x + \frac{2304}{5}\right)\left(500x^3 - 600\right)
\end{aligned}$$

Now the equilibria are as follows:

First-score auction:

$$\begin{aligned}
\text{price:} &\quad p^{I}(\theta) = 2\left(\theta + \frac{25\theta^5 - 300\theta^2 + \frac{45980}{10} - \frac{18197}{100}}{-125\theta^4 + 600\theta - \frac{2299}{5}}\right) \\
\text{quality} &: \quad q^{I}(\theta) = \sqrt{2\left(\theta + \frac{25\theta^5 - 300\theta^2 + \frac{45980}{10} - \frac{18197}{100}}{-125\theta^4 + 600\theta - \frac{2299}{5}}\right)} \\
\text{score:} &\quad s^{I}(\theta) = \frac{q^{I}(\theta)}{p^{I}(\theta)} = \frac{1}{\sqrt{2\left(\theta + \frac{25\theta^5 - 300\theta^2 + \frac{45980}{10} - \frac{18197}{100}}{-125\theta^4 + 600\theta - \frac{2299}{5}}\right)}} \\
\text{Expected score:} &\quad \Sigma^{I} = \int_{1.2}^{1.203731} s^{I}(\theta) f_1(\theta)\, d\theta = 0.6469
\end{aligned}$$

Second-score auction:

$$\begin{aligned}
\text{price:} &\quad p^{II}(\theta) = 2\theta \\
\text{quality:} &\quad q^{II}(\theta) = \sqrt{2\theta} \\
\text{score:} &\quad s^{II}(\theta) = \frac{q^{II}(\theta)}{p^{II}(\theta)} = \frac{1}{\sqrt{2\theta}} \\
\text{Expected score:} &\quad \Sigma^{II} = \int_{1.2}^{1.203731} s^{II}(\theta) f_2(\theta)\, d\theta = 0.6449
\end{aligned}$$

Comment The above examples clearly demonstrate that the distribution of types plays a major role in the ranking of expected scores. Even if the scoring rule and cost functions are the same, the ranking of expected revenues can get reversed if the distribution of types are different. Hence, we need to put restrictions on both the scoring rule and the distribution function to get a ranking of expected scores.

The following proposition provides sufficient conditions under which expected score in a second-score auction is higher than in a first-score auction.

Proposition 11 Suppose $A(p,q) \neq 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$. If $H(p,q) < 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$ and $S^{II}\left(\int_{\underline{\theta}}^{\bar{\theta}} \theta f_2(\theta)\, d\theta\right) > S^I(\underline{\theta})$ then $\Sigma^I < \Sigma^{II}$.

Comment From lemma 3 we know that to compare Σ^I and Σ^{II} we need to compare the following terms:

$$\int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)\left(1+\gamma'(\theta)\right)\left[-S_p\left(p^I(\theta), q^I(\theta)\right)\right] d\theta \text{ and } \int_{\underline{\theta}}^{\bar{\theta}} F_2(\theta)\left[-S_p\left(p^{II}(\theta), q^{II}(\theta)\right)\right] d\theta.$$

From lemma 4 we also know that $\int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)\left(1+\gamma'(\theta)\right) d\theta = \int_{\underline{\theta}}^{\bar{\theta}} F_2(\theta)\, d\theta$. Intuitively this means that if $-S_p\left(p^{II}(\theta), q^{II}(\theta)\right)$ is high enough compared to $-S_p\left(p^I(\theta), q^I(\theta)\right)$ then we should have $\Sigma^I < \Sigma^{II}$. Lemma 5 shows that if $H(p,q) < 0$ then $-S_p\left(p^{II}(\theta), q^{II}(\theta)\right) > -S_p\left(p^I(\theta), q^I(\theta)\right)$. This together with $S^{II}\left(\int_{\underline{\theta}}^{\bar{\theta}} \theta f_2(\theta)\, d\theta\right) > S^I(\underline{\theta})$ ensures that Σ^{II} is strictly higher than Σ^I.

We now proceed to provide an alternative set of sufficient conditions for Σ^{II} to be strictly greater than Σ^I.

Now suppose $f(\theta) > 0 \; \forall \theta \in \left[\underline{\theta}, \bar{\theta}\right]$. This means that $F^{-1}(.)$ is well defined. Let $\hat{\theta} = F^{-1}\left(\frac{1}{n}\right)$. It is easy to check that

$$\begin{aligned} f_2(\theta) &< f_1(\theta) \;\; \forall \theta \in \left(\underline{\theta}, \hat{\theta}\right) \text{ and} \\ f_2(\theta) &> f_1(\theta) \;\; \forall \theta \in \left(\hat{\theta}, \bar{\theta}\right). \end{aligned}$$

Note that

$$\Sigma^{II} = \int_{\underline{\theta}}^{\bar{\theta}} S^{II}(\theta) f_2(\theta)\, d\theta = \int_{\underline{\theta}}^{\hat{\theta}} S^{II}(\theta) f_2(\theta)\, d\theta + \int_{\hat{\theta}}^{\bar{\theta}} S^{II}(\theta) f_2(\theta)\, d\theta$$

Similarly

$$\Sigma^{I} = \int_{\underline{\theta}}^{\bar{\theta}} S^{I}(\theta) f_1(\theta)\, d\theta = \int_{\underline{\theta}}^{\hat{\theta}} S^{I}(\theta) f_1(\theta)\, d\theta + \int_{\hat{\theta}}^{\bar{\theta}} S^{I}(\theta) f_1(\theta)\, d\theta$$

It may be noted that proposition 5 shows $S\left(p^I(\theta), q^I(\theta)\right) < S\left(p^{II}(\theta), q^{II}(\theta)\right) \; \forall \theta \in \left[\underline{\theta}, \bar{\theta}\right)$. This together with the fact that $f_2(\theta) > f_1(\theta) \; \forall \theta \in \left(\hat{\theta}, \bar{\theta}\right)$ means that

$$\int_{\hat{\theta}}^{\bar{\theta}} S^{II}(\theta) f_2(\theta)\, d\theta > \int_{\hat{\theta}}^{\bar{\theta}} S^{I}(\theta) f_1(\theta)\, d\theta.$$

Hence, intuitively if $S^{II}(\theta)$ is high enough for $\theta \in \left(\underline{\theta}, \hat{\theta}\right)$ then Σ^{II} will be higher than Σ^{I}. The next proposition identifies such sufficient conditions.

Proposition 12 For any non-quasilinear scoring rule if

$$\begin{aligned} f(\theta) &> 0 \ \forall\theta \in \left[\underline{\theta}, \bar{\theta}\right] \text{ and} \\ S^{II}\left(\hat{\theta}\right) &\geq \frac{\left[1-\left(\frac{n-1}{n}\right)^{n}\right] S^{I}\left(\underline{\theta}\right) - \left(\frac{n-1}{n}\right)^{n-1} S^{I}\left(\bar{\theta}\right)}{1-\left(\frac{n-1}{n}\right)^{n} - \left(\frac{n-1}{n}\right)^{n-1}} \end{aligned}$$

then $\Sigma^{I} < \Sigma^{II}$.

Comment *Unlike Hanazono et al (2013), for non-quasilinear scoring rules, we get the ranking of expected scores directly from the curvature properties of the scoring rule and properties of the distribution function.*

We now proceed to compute equilibria in first-score and second-score auctions when both quality and types are multidimensional.

6 Extension: Multidimensional quality and multidimensional types

The good is characterized by its price, p, and m non-monetary attributes, $(q_1, q_2, ..q_m) \in \mathbb{R}_{+}^{m}$. We call these attributes as qualities. The scoring rule is $S(p, q_1, q_2, ..q_m)$. We have $S_p(.) < 0$ and $S_{q_i} > 0$. Supplier i's profit from selling the good is given by $p - C(q_1, q_2..q_m, x^i)$, where $x^i = (x_1^i, x_2^i, ..x_n^i) \in \mathbb{R}_{+}^{n}$, is supplier i's type. Types are identically and independently distributed according to the continuous joint density function h with support on $\times_{i=1}^{n}[\underline{x}, \bar{x}]$. Cost is strictly increasing in $q_1, q_2, ..q_m$ and $x_1^i, x_2^i, ..x_n^i$.

We assume **additive separability.** That is,

$$\begin{aligned} C\left(q_1, q_2..q_m, x_1^i, x_2^i, ..x_n^i\right) &= c(q_1, q_2..q_m) + \alpha\left(x_1^i, x_2^i, ..x_n^i\right) \\ \text{with } c_{q_i} &> 0 \text{ and } \alpha_{x_j^i} > 0. \end{aligned}$$

Let $\theta_i = \alpha(x_1^i, x_2^i, ..x_n^i)$.

Remark Clearly the lowest value of θ_i is $\underline{\theta} = \alpha(\underline{x}, \underline{x}....\underline{x})$ and the highest value of θ_i is $\bar{\theta} = \alpha(\bar{x}, \bar{x}, ...\bar{x})$. Since a type, $\left(\theta_1^i, \theta_2^i, ..\theta_n^i\right)$, is distributed according to the continuous joint density function h with support on $\times_{i=1}^{n}\left[\underline{\theta}, \bar{\theta}\right]$; we can think of θ_i as a derived random variable distributed over $\left[\underline{\theta}, \bar{\theta}\right]$ with distribution function $F(.)$ and density function $f(.)$. Both $F(.)$ and $f(.)$ can be computed using standard statistical techniques.

Hence, we can write

$$C\left(q_1, q_2..q_m, x_1^i, x_2^i, ..x_n^i\right) = c(q_1, q_2..q_m) + \theta_i$$

where θ_i is the type of supplier i. θ_is are identically and independently distributed over $\left[\underline{\theta}, \bar{\theta}\right]$ with distribution function $F(.)$ and density function $f(.)$.

Note that additive separability of the cost function helps us to deal with the case of multidimensional types just as a single dimensional type.

The following can be shown. The proofs follow directly from the proofs of propositions 1 and 2.

Proposition 13 (i) In a *first-score* auction there is a symmetric equilibrium where a supplier with type k chooses $p^I(k), q_1^I(k), q_2^I(k), ..q_m^I(k)$. Such $p^I(.)$ and $q_i^I(.)$s are obtained by solving the following equations:

$$
\begin{aligned}
-\frac{S_{q_i}(.)}{S_p(.)} &= c_{q_i}(.) \text{ for all } i \in \{1, 2..m\} \\
& \quad p - c(q_1, q_2, ..q_m) \\
&= \theta + \gamma(\theta)
\end{aligned}
$$

where

$$
\gamma(\theta) = \frac{1}{(1 - F(\theta))^{n-1}} \int_{\theta}^{\bar{\theta}} (1 - F(t))^{n-1} dt
$$

Proposition 14 In a *second-score* auction there is a weakly dominant strategy equilibrium where a supplier with type k chooses $p^I(k), q_1^I(k), q_2^I(k), ..q_m^I(k)$. Such $p^{II}(.)$ and $q_i^{II}(.)$s are obtained by solving the following equations:

$$
\begin{aligned}
-\frac{S_{q_i}(.)}{S_p(.)} &= c_{q_i}(.) \text{ for all } i \in \{1, 2..m\} \\
& \quad p - c(q_1, q_2, ..q_m) = \theta
\end{aligned}
$$

Comment Propositions 13 and 14 provide the equilibrium for first-score and second-score auctions when both quality and type are multidimensional. Charaterisation of equilibrium and ranking of expected scores is left for future research.

7 Conclusion

In this paper we analysed scoring auctions with general non-quasilinear scoring rules. We demonstrated that additive separability of cost functions vastly simplifies the equilibrium computations. Unlike recent papers, we get explicit solutions for the Bayesian-Nash equilibrium without any endogeneity problems. Moreover, we analyse the properties of such equilibria and the ranking of expected scores across first-score and second-score auctions and demonstrate that they depend only on the curvature properties of the scoring rule and distribution function of types. Our approach helps in dealing with most non-quasilinear scoring rules. The following may be noted.

1. We noted that additive separability of cost function implies that multidimensional types can be expressed as a single dimensional type. In this paper we concentrated mainly on single dimensional quality. Characterisation of equilibrium and ranking of expected scores when quality is multidimensional is an open question and is left for future research.

2. Optimal mechanisms (that maximise expected scores) have been derived in the literature for quasi-linear scoring rules (See Che, 1993, Asker Cantillon, 2010 and Nishimura, 2012). However, such optimal mechanisms for general non-quasilinear scoring rules have not been adequately analysed in the literature. This is an open question and is left for future research.

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Appendix

Proof of Proposition 1 In the main body of the paper (see the comment after proposition 2) we have introduced $\Psi(q,s)$ and have given a sketch of the proof. There we have shown that in any equilibrium the following equation (which we now reproduce below) will always be satisfied.

$$-\frac{S_q(.)}{S_p(.)} = c'(.) - - - - (3)$$

We now show that there is a symmetric equilibrium where a bidder with type θ chooses $p^I(\theta)$ and $q^I(\theta)$. Such $p^I(.)$ and $q^I(.)$ are obtained by solving the following equations:

$$\begin{aligned}
-\frac{S_q(.)}{S_p(.)} &= c'(.) - - - - (4a) \\
p - c(q) &= \theta + \gamma(\theta) - - - - (4b) \\
\text{where } \gamma(\theta) &= \frac{1}{(1-F(\theta))^{n-1}} \int_{\theta}^{\bar{\theta}} (1-F(t))^{n-1} dt - - - - (4c)
\end{aligned}$$

First note that (4a) is same as (3) and it is true at any equilibrium. Now we show why (4b) is needed. To do this let's suppose that all firms $j = 2,3..n$ choose $p^I(\theta_j)$ and $q^I(\theta_j)$ according to (4a) and (4b). Then we show that it is optimal for firm 1 to choose the same strategy. Note that from (4b) we have

$$\forall\theta \in \left[\underline{\theta}, \bar{\theta}\right],\ p^I(\theta) - c\left(q^I(\theta)\right) = \theta + \gamma(\theta) - - - - (5)$$

Differentiating both sides of (5) w.r.t. θ we have

$$\begin{aligned}
\forall\theta &\in \left(\underline{\theta}, \bar{\theta}\right),\ \frac{dp^I(\theta)}{d\theta} - c'(q(\theta))\frac{dq^I(\theta)}{d\theta} = 1 + \gamma'(\theta) \\
&= \frac{(n-1) f(\theta)}{(1-F(\theta))^n} \int_{\theta}^{\bar{\theta}} (1-F(t))^{n-1} dt - - - - (6)
\end{aligned}$$

From (6) we clearly have

$$\frac{dp^I(\theta)}{d\theta} - c'(q(\theta))\frac{dq^I(\theta)}{d\theta} > 0 - - - - (7)$$

For any firm $j \in \{2,3..n\}$ the choice of $p^I(\theta_j), q^I(\theta_j)$ leads to score $S\left(p^I(\theta_j), q^I(\theta_j)\right)$. Then we can say that any firm $j \in \{2,3..n\}$ with type θ_j chooses score $S\left(p^I(\theta_j), q^I(\theta_j)\right)$ and quality $q^I(\theta_j)$.

Let

$$\hat{S}(\theta) = S\left(p^I(\theta), q^I(\theta)\right)$$

Then, we have that any firm $j \in \{2, 3..n\}$ with type θ chooses score $\hat{S}(\theta)$ and qualities $q(\theta)$. Now note the following:

$$\begin{aligned}
\frac{d}{d\theta}\,\hat{S}(\theta) &= S_p(.)\,\frac{dp^I(\theta)}{d\theta} + S_q(.)\,\frac{dq^I(\theta)}{d\theta} \\
&= S_p(.)\,\frac{dp^I(\theta)}{d\theta} - C_q(.)\,S_p(.)\,\frac{dq^I(\theta)}{d\theta} \text{ (using (4a))} \\
&= S_p(.)\left[\frac{dp^I(\theta)}{d\theta} - C_q(.)\,\frac{dq^I(\theta)}{d\theta}\right] ----(8)
\end{aligned}$$

From (8) we have

$$\frac{d}{d\theta}\hat{S}(\theta) < 0 \text{ (since } S_p(.) < 0 \text{ and } \frac{dp^I(\theta)}{d\theta} - C_q(.)\,\frac{dq^I(\theta)}{d\theta} > 0 \text{ (from 7)}$$

The above means that for any firm $j \in \{2, 3..n\}$ the score quoted is strictly decreasing in θ. Hence, the scores of firms $2, 3..n$ lie in the interval $\left[\hat{S}(\bar{\theta}), \hat{S}(\underline{\theta})\right]$.

Now take the case of firm 1. It has to choose a score, s_1 and a quality, q, given the choice of firms $2, 3..n$. Clearly $s_1 \in \left[\hat{S}(\bar{\theta}), \hat{S}(\underline{\theta})\right]$. Note that choosing s_1 is equivalent to choosing z s.t. $s_1 = \hat{S}(z)$. Hence, the probability of winning for firm 1 is as follows:

$$\begin{aligned}
&\text{Prob.} \left\{\hat{S}(z) > \max_{j\neq 1}\left(\hat{S}(\theta_j)\right)\right\} \\
= &\text{Prob.} \left\{\hat{S}(z) > \left(\hat{S}\left(\min_{j\neq 1}(\theta_j)\right)\right)\right\} \text{ (since } \hat{S}'(.) < 0) \\
= &\text{Prob.} \left\{z < \min_{j\neq 1}(\theta_j)\right\} ----(9).
\end{aligned}$$

We know that θ is distributed over $[\underline{\theta}, \bar{\theta}]$ with distribution function $F(.)$ and density function $f(.)$. From the basic theory of order statistics (see section 5.1) we also know that the lowest order statistic from among $(n-1)$ i.i.d random variables has a distribution function $G(.) = 1 - (1 - F(.))^{n-1}$. That is, for the random variables $\theta_2, \theta_3..\theta_n$

$$\text{Prob}\left\{\min_{j\neq 1}(\theta_j) < \Sigma\right\} = G(\Sigma) = 1 - (1 - F(\Sigma))^{n-1}.$$

Using (9) we can write

$$\begin{aligned}
&\text{Prob.} \left\{\hat{S}(z) > \max_{j\neq 1}\left(\hat{S}(\theta_j)\right)\right\} \\
= &\text{Prob.} \left\{z < \min_{j\neq 1}(\theta_j)\right\} \\
= &1 - G(z) = (1 - F(z))^{n-1}.
\end{aligned}$$

That is, if firm 1 chooses a score of $s_1 = \hat{S}(z)$ it wins with probability $(1 - F(z))^{n-1}$. Let it choose quality x and let it's type be θ_1. Then, firm 1's cost is $c(x) + \theta_1$. Therefore, firm

1's expected payoff by choosing a score $s_1 = \hat{S}(z)$ and quality x is

$$\pi_1 = (1 - F(z))^{n-1} \left[\Psi\left(x, \hat{S}(z)\right) - c(x) - \theta_1 \right] - - - - (10)$$

Firm 1's choice variables are x and z. Note that from the 1OCs for an optimum we have

$$\frac{\partial \pi_1}{\partial x} = 0 \Longrightarrow \Psi_q\left(x, \hat{S}(z)\right) - c'(x) = 0 - - - - (11)$$

From earlier discussions we know that (11) is equivalent to

$$-\frac{S_q(.)}{S_p(.)} = c'(.) - - - - (12)$$

The above is same as (4a).

We now proceed to deal with the optimal choice of z. It may be noted that

$$\frac{\partial \Psi\left(x, \hat{S}(z)\right)}{\partial z} = \Psi_s\left(x, \hat{S}(z)\right) \hat{S}'(z)$$

By using (2) and (8)

$$\begin{aligned}
\Psi_s\left(x, \hat{S}(z)\right) \hat{S}'(z) &= \frac{1}{S_p(p(z), q(z))} S_p\left(p^I(z), q^I(z)\right) \left[\frac{dp^I(z)}{dz} - c'(q(z)) \frac{dq^I(z)}{dz}\right] \\
&= \frac{dp^I(z)}{dz} - c'(q(z)) \frac{dq^I(z)}{dz} \\
&= 1 + \gamma'(z) \quad = \frac{(n-1) f(z)}{(1 - F(z))^n} \int_z^{\bar{\theta}} (1 - F(t))^{n-1} dt \text{ (from 6)}
\end{aligned}$$

That is,

$$\frac{\partial \Psi\left(x, \hat{S}(z)\right)}{\partial z} = 1 + \gamma'(z) \quad = \frac{(n-1) f(z)}{(1 - F(z))^n} \int_z^{\bar{\theta}} (1 - F(t))^{n-1} dt - - - - (13)$$

Now note that from (10) and (13) we have

$$\begin{aligned}
\frac{\partial}{\partial z}\pi_1 &= -(n-1)(1 - F(z))^{n-2} f(z) \left[\Psi\left(x, \hat{S}(z)\right) - c(x) - \theta_1\right] + (1 - F(z))^{n-1} \frac{\partial \Psi\left(x, \hat{S}(z)\right)}{\partial z} \\
&= (1 - F(z))^{n-2} \begin{bmatrix} -(n-1) f(z) \left\{\Psi\left(x, \hat{S}(z)\right) - c(x) - \theta_1\right\} \\ +(1 - F(z)) \frac{(n-1) f(z)}{(1-F(z))^n} \int_z^{\bar{\theta}} (1 - F(t))^{n-1} dt \end{bmatrix} \\
&= (n-1)(1 - F(z))^{n-2} f(z) \begin{bmatrix} \frac{1}{(1-F(z))^{n-1}} \int_z^{\bar{\theta}} (1 - F(t))^{n-1} dt \\ -\left\{\Psi\left(x, \hat{S}(z)\right) - c(x) - \theta_1\right\} \end{bmatrix}
\end{aligned}$$

From above and using definition of $\gamma(z)$ (see 4c) we get that

$$\frac{\partial}{\partial z}\pi_1 = (n-1)(1-F(z))^{n-2} f(z)\left[\gamma(z) - \left\{\Psi\left(x, \hat{S}(z)\right) - c(x) - \theta_1\right\}\right] \quad ----(14)$$

Note that $(n-1)(1-F(z))^{n-2} f(z) > 0$ for all $z \in (\underline{\theta}, \bar{\theta})$.

Also note that by using (13) we get that

$$\begin{aligned} & \frac{\partial}{\partial z}\left[\gamma(z) - \left\{\Psi\left(x, \hat{S}(z)\right) - c(x) - \theta_1\right\}\right] \\ = & \ \gamma'(z) - \frac{\partial \Psi\left(x, \hat{S}(z)\right)}{\partial z} \\ = & \ \gamma'(z) - [1 + \gamma'(z)] \text{ using (13)} \\ = & \ -1 < 0 \quad ------(15) \end{aligned}$$

From (4b) we know that

$$p^I(\theta) - c\left(q^I(\theta)\right) = \theta + \gamma(\theta)$$

$$\Longrightarrow \Psi\left(q^I(\theta), \hat{S}(\theta)\right) - c\left(q^I(\theta)\right) = \theta + \gamma(\theta) ----(16)$$

We know that firm 1's choice of x is such that (12) (which is same as 4a) is satisfied. Using this fact and (16) we get that

$$\text{if } z = \theta_1 \text{ then } \gamma(z) - \left\{\Psi\left(x, \hat{S}(z)\right) - c(x) - \theta_1\right\} = 0 ----(17)$$

This means (see 14 and 17)

$$\frac{\partial}{\partial z}\pi_1 = 0 \text{ at } z = \theta_1 ----(18)$$

Moreover, from (14), (15) and (18) we clearly get that

$$\begin{aligned} z &< \theta_1 \Longrightarrow \frac{\partial}{\partial z}\pi_1 > 0 \text{ and} \\ z &> \theta_1 \Longrightarrow \frac{\partial}{\partial z}\pi_1 < 0. \quad ----(19) \end{aligned}$$

(18) and (19) implies that $z = \theta_1$ is the optimal choice for firm 1. Therefore firm 1's choice of quality, x and score, $\hat{S}(z)$ must satisfy (12) and (17). This is same as 4a and 4b.

That is, we have proved that in a first-score auction there is a symmetric equilibrium where a bidder with pseudo-type θ chooses $p^I(\theta)$ and $q^I(\theta)$. Such $p^I(.)$ and $q^I(.)$ are obtained by solving the following equations:

$$\begin{aligned}-\frac{S_{q_1}(.)}{S_p(.)} &= c'(.)\\ p - c(q) &= \theta + \frac{1}{(1-F(\theta))^{n-1}} \int_\theta^{\bar{\theta}} (1-F(t))^{n-1}\, dt.\end{aligned}$$

This completes our proof for proposition 1.■

Proof of Proposition 2 We will now show that in a second-score auction the weakly dominant strategy for each firm with type θ is to choose $p(\theta)$ and $q(\theta)$ that are obtained by solving the following equations:

$$\begin{aligned}-\frac{S_q(.)}{S_p(.)} &= c'(.) - - - - (20a)\\ p &= c(q) + \theta - - - - (20b)\end{aligned}$$

Let the score quoted by firm i by following 20a and 20b be s. That is, $s = S(p(\theta), q(\theta))$. It may be recalled from our earlier discussions that (20a) which is same as (3) and it is equivalent to (1a) reproduced below.

$$\Psi_q(.) - c'(.) = 0 - - - - (1a)$$

From (1a) we get q as a function of s. From earlier discussion we know that for any s, the quality choice, q, (as in 1a above) is such so as to maximise $\Psi(q,s) - c(q) - \theta$. Then, using the envelope theorem we get

$$\frac{d}{ds}\left[\Psi(q(s), s) - c(q) - \theta\right] = \Psi_s = \frac{1}{S_p(.)} < 0 \text{ (see (2))} \quad - - - - (21).$$

Now clearly (by using the equivalence of (1a) and (20a),

$$\Psi(q(s), s) = p^{II}(\theta)$$

The above implies from (20b)

$$\Psi(q(s), s) - c(q(s)) - \theta = 0 - - - - (22)$$

Now let firm i follow (20a) and (20b) and thereby pick up a score s. Let the maximum of the scores quoted by firms other than i be δ. Now if $s > \delta$ then by following (20a) and (20b) firm i wins the contract. As per the rules of the second score auction, the winner is required to match the highest rejected score which is δ. In meeting this score, the firm is free to choose any quality-price combination. Clearly, firm i will choose qualities so as to maximise $\Psi(q,\delta) - c(q) - \theta$. Those choice of qualities must satisfy the following equation:

$$\Psi_q(q, \delta) - c'(q) = 0 - - - - (23)$$

The firm's profit by meeting a score δ is therefore $\Psi(q(\delta),\delta) - c(q(\delta)) - \theta$. Since $\delta < s$ and using (21) and (22)

$$\Psi(q(\delta),\delta) - c(q(\delta)) - \theta > \Psi(q(s),s) - c(q(s)) - \theta = 0 - - (24)$$

If firm i decides to pick up any score, $\phi \neq s$ (by choosing (p,q) *other than* as in 20a and 20b),then it would not matter as long as $\phi > \delta$. If $\phi < \delta$, then firm would not win the contract and earn zero payoff. Hence if $s > \delta$ then the firm's best strategy is to quote a score s. Similarly, it can be shown that is $s < \delta$ then also the firm's best strategy is to quote a score s. In other words, choice of s is a weakly dominant strategy.■

Proof of Lemma 1 Note that by using the L'Hospital's rule we get

$$\begin{aligned}
\lim_{\theta \longrightarrow \bar{\theta}} \gamma(\theta) &= \lim_{\theta \longrightarrow \bar{\theta}} \frac{1}{(1-F(\theta))^{n-1}} \int_{\theta}^{\bar{\theta}} (1-F(t))^{n-1} dt \\
&= \lim_{\theta \longrightarrow \bar{\theta}} \frac{\frac{d}{d\theta}\left(\int_{\theta}^{\bar{\theta}} (1-F(t))^{n-1} dt\right)}{\frac{d}{d\theta}(1-F(\theta))^{n-1}} \\
&= \lim_{\theta \longrightarrow \bar{\theta}} \frac{1-F(\theta)}{(n-1) f(\theta)} = 0.
\end{aligned}$$

Hence, using propositions 1 and 2, for the type $\bar{\theta}$, in both first-score and second-score auctions, $p\left(\bar{\theta}\right)$, $q\left(\bar{\theta}\right)$ is obtained by solving the following equations.

$$\begin{aligned}
-\frac{S_q(.)}{S_p(.)} &= c'(.) \\
p &= c(q) + \theta
\end{aligned}$$

This shows that $p^I\left(\bar{\theta}\right) = p^{II}\left(\bar{\theta}\right)$ and $q^I\left(\bar{\theta}\right) = q^{II}\left(\bar{\theta}\right)$. ■

Proof of Lemma 2 Since by assumption S_{pp} and S_{qp} are continuous $\forall (p,q) \in \mathbb{R}^2_{++}$, then $A(p,q) = -\frac{S_q}{S_p} S_{pp} + S_{qp} \neq 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$ implies either (a) $\forall (p,q) \in \mathbb{R}^2_{++} \;\; A(p,q) > 0$ **or** (b) $\forall (p,q) \in \mathbb{R}^2_{++} \;\; A(p,q) < 0$.

It may be noted that

$$\begin{aligned}
B(p,q) &= -\frac{S_q}{S_p} S_{pq} + S_p c'' + S_{qq} = -\frac{1}{S_p}\left[S_q S_{pq} - (S_p)^2 c'' - S_p S_{qq}\right] \\
&< -\frac{1}{S_p}\left[\frac{(S_q)^2}{S_p} S_{pp} - S_q S_{pq}\right] \text{ (assumption 4 of our model)} \\
&= \frac{S_q}{S_p}\left[-\frac{S_q(.)}{S_p(.)} S_{pp}(.) + S_{qp}(.)\right].
\end{aligned}$$

Since $S_p < 0$ and $S_q > 0$ the above means that

$$\begin{array}{rcl} -\dfrac{S_q(.)}{S_p(.)}S_{pp}(.) + S_{qp}(.) & > & 0 \Longrightarrow -\dfrac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq} < 0. \\ & \Longleftrightarrow & \\ -\dfrac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq} & \geq & 0 \Longrightarrow -\dfrac{S_q}{S_p}S_{pp} + S_{qp} < 0. \\ & \Longleftrightarrow & \\ B(p,q) & \geq & A(p,q) < 0 \end{array}$$

Proof of Proposition 3 In equilibrium, in both first-score and second-score auctions the following is true:

$$-\frac{S_q(p,q)}{S_p(p,q)} = c'(q) - - - - (25)$$

From (25) we get p implicitly a function of q. That is, $p = \sigma(q)$ and we have

$$\begin{array}{rcl} -\dfrac{S_q(\sigma(q), q)}{S_p(\sigma(q), q)} - c'(q) & = & 0 \\ & \Longleftrightarrow & \\ S_p(\sigma(q), q)\, c'(q) + S_q(\sigma(q), q) & = & 0 - - - - (26) \end{array}$$

Using the implicit function theorem we get that

$$\sigma'(q) = -\left[\frac{c' S_{pq} + S_p c'' + S_{qq}}{c' S_{pp} + S_{qp}}\right] - - - - (27).$$

Using (25) we have

$$\sigma'(q) = -\left[\frac{-\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq}}{-\frac{S_q}{S_p}S_{pp} + S_{qp}}\right] - - - - (28)$$

Note that $\sigma'(q)$ is well defined since $\forall (p,q) \in \mathbb{R}^2_{++}$, $-\frac{S_q}{S_p}S_{pp} + S_{qp} \neq 0$.

Since by assumption S_{pp} and S_{qp} are continuous $\forall (p,q) \in \mathbb{R}^2_{++}$, then $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} \neq 0$ $\forall (p,q) \in \mathbb{R}^2_{++}$ implies either (a) $\forall (p,q) \in \mathbb{R}^2_{++}$ $A(p,q) > 0$ **or** (b) $\forall (p,q) \in \mathbb{R}^2_{++}$ $A(p,q) < 0$.

Note that for **both** auctions (from (25) using the fact that $c'(.) = -\frac{S_q}{S_p}$)

$$\begin{array}{rcl} \sigma'(q) - c'(q) & = & \sigma'(q) + \dfrac{S_q}{S_p} \\ & = & \dfrac{\left[-\frac{(S_q)^2}{S_p}S_{pp} + 2S_q S_{qp} - S_p S_{qq} - (S_p)^2 c''\right]}{S_p\left[-\frac{S_q}{S_p}S_{pp} + S_{qp}\right]} - - - - (29) \end{array}$$

Note that by assumption the numerator of (31) is strictly negative. Since $S_p < 0$ we have that

$$\begin{aligned} \text{if } \forall (p,q) &\in \mathbb{R}^2_{++},\ A(p,q) = -\frac{S_q}{S_p} S_{pp} + S_{qp} > 0 \text{ then } \sigma'(q) - c'(q) > 0 \text{ and} \\ \text{if } \forall (p,q) &\in \mathbb{R}^2_{++},\ A(p,q) = -\frac{S_q}{S_p} S_{pp} + S_{qp} < 0 \text{ then } \sigma'(q) - c'(q) < 0 \ ----(30) \end{aligned}$$

Now note the following.

$$\begin{aligned} \frac{d}{dq} S(\sigma(q), q) &= S_p \sigma'(q) + S_q \\ &= S_p \sigma'(q) - S_p c'(q) \ \text{(from 25)} \\ &= S_p [\sigma'(q) - c'(q)] ----(31) \end{aligned}$$

From (30) we know that $\sigma'(q) - c'(q)$ has the same sign as $A(p,q) = \left(-\frac{S_q}{S_p} S_{pp} + S_{qp}\right)$. Since $S_p < 0$ from (31) we get that $\frac{d}{dq} S(\sigma(q), q)$ has the opposite sign of $A(p,q)$.

Now suppose $A(p,q) > 0$. This means $\frac{d}{dq} S(\sigma(q), q) < 0$. Since $q^I(\theta) > q^{II}(\theta)$ when $A(p,q) > 0$ we must have $S\left(\sigma\left(q^I(\theta)\right), q^I(\theta)\right) < S\left(\sigma\left(q^{II}(\theta)\right), q^{II}(\theta)\right)$. Now since $p^I(\theta) = \sigma\left(q^I(\theta)\right)$ and $p^{II}(\theta) = \sigma\left(q^{II}(\theta)\right)$ for $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ this implies $S\left(p^I(\theta), q^I(\theta)\right) < S\left(p^{II}(\theta), q^{II}(\theta)\right)$. This means $S^I(\theta) < S^{II}(\theta)$.

Now suppose $A(p,q) = -\frac{S_q}{S_p} S_{pp} + S_{qp} < 0$. This means $\frac{d}{dq} S(\sigma(q), q) > 0$. Since $q^I(\theta) < q^{II}(\theta)$ when $A(p,q) < 0$ we must have $S\left(\sigma\left(q^I(\theta)\right), q^I(\theta)\right) < S\left(\sigma\left(q^{II}(\theta)\right), q^{II}(\theta)\right)$. Now since $p^I(\theta) = \sigma\left(q^I(\theta)\right)$ and $p^{II}(\theta) = \sigma\left(q^{II}(\theta)\right)$ for all $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ this implies $S\left(p^I(\theta), q^I(\theta)\right) < S\left(p^{II}(\theta), q^{II}(\theta)\right)$.

Using (7) and (8) we know that $\frac{d}{d\theta} S\left(p^I(\theta), q^I(\theta)\right) < 0$. Using a exactly similar method we can show that $\frac{d}{d\theta} S\left(p^{II}(\theta), q^{II}(\theta)\right) < 0$.

This completes proof of proposition 3.■

Proof of Proposition 4 (i) and (ii) Note that in equilibrium, in both first-score and second-score auctions $-\frac{S_q(p,q)}{S_p(p,q)} = c'(q)$. As in (25), we get p implicitly a function of q. That is, $p = \sigma(q)$. In a first-score auction we have (see proposition 1)

$$\begin{aligned} p^I - c\left(q^I\right) &= \theta + \gamma(\theta) \\ &\implies \sigma\left(q^I\right) - c\left(q^I\right) = \theta + \gamma(\theta) ----(32) \end{aligned}$$

In second-score auction we have (see proposition 2)

$$\begin{aligned} p^{II} - c\left(q^{II}\right) &= \theta \\ &\implies \sigma\left(q^{II}\right) - c\left(q^{II}\right) = \theta ----(33) \end{aligned}$$

Now using (32) and (33), for any $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ we get

$$\begin{aligned} \sigma\left(q^I(\theta)\right) - c\left(q^I(\theta)\right) &= \theta + \gamma(\theta) \ \text{and} \\ \sigma\left(q^{II}(\theta)\right) - c\left(q^{II}(\theta)\right) &= \theta ----(34) \end{aligned}$$

From (34) it is clear that for any $\theta \in [\underline{\theta}, \bar{\theta})$

$$\sigma\left(q^{I}(\theta)\right) - c\left(q^{I}(\theta)\right) > \sigma\left(q^{II}(\theta)\right) - c\left(q^{II}(\theta)\right) - - - - (34a)$$

Now let $\forall (p,q) \in \mathbb{R}^2_{++}$, $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} > 0$. For any $\theta \in [\underline{\theta}, \bar{\theta})$ if possible let's suppose $q^{I}(\theta) \leq q^{II}(\theta)$. Since from (30) we have $\sigma'(q) - c'(q) > 0$ when $A(p,q) > 0$, we must have $\sigma\left(q^{I}(\theta)\right) - c\left(q^{I}(\theta)\right) \leq \sigma\left(q^{II}(\theta)\right) - c\left(q^{II}(\theta)\right)$. But this contradicts (34a). Hence, when $A(p,q) > 0$ we must have $q^{I}(\theta) > q^{II}(\theta)$.

Now let $\forall (p,q) \in \mathbb{R}^2_{++}$, $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} < 0$. From (30) we have $\sigma'(q) - c'(q) < 0$. Now using a logic similar to the one used in the previous paragraph we get $q^{I}(\theta) < q^{II}(\theta)$.

From (32) we get that in a first-score auction the following is true for all $\theta \in [\underline{\theta}, \bar{\theta}]$

$$\sigma\left(q^{I}(\theta)\right) - c\left(q^{I}(\theta)\right) = \theta + \gamma(\theta) - - - - (35)$$

From (35) we get that for all $\theta \in (\underline{\theta}, \bar{\theta})$ we have

$$\left[\sigma'\left(q^{I}(\theta)\right) - c'\left(q^{I}(\theta)\right)\right] \frac{dq^{I}(\theta)}{d\theta} = 1 + \gamma'(\theta) = \frac{(n-1) f(\theta)}{(1 - F(\theta))^{n}} \int_{\theta}^{\bar{\theta}} (1 - F(t))^{n-1} dt - - - (36).$$

Since $\frac{(n-1)f(\theta)}{(1-F(\theta))^n} \int_{\theta}^{\bar{\theta}} (1 - F(t))^{n-1} dt > 0$ from (36) we get that $\frac{dq^{I}(\theta)}{d\theta}$ has the same sign as $\sigma'\left(q^{I}(\theta)\right) - c'\left(q^{I}(\theta)\right)$. From (32) we know that

$$\begin{aligned} &\text{if } \forall (p,q) \in \mathbb{R}^2_{++}, A(p,q) > 0 \text{ then } \sigma'(.) - c'(.) > 0 \text{ and} \\ &\text{if } \forall (p,q) \in \mathbb{R}^2_{++}, A(p,q) < 0 \text{ then } \sigma'(.) - c'(.) < 0. \end{aligned}$$

This shows that

$$\begin{aligned} &\text{if } \forall (p,q) \in \mathbb{R}^2_{++}, A(p,q) > 0 \text{ then } \frac{dq^{I}(\theta)}{d\theta} > 0 \ \forall \theta \in (\underline{\theta}, \bar{\theta}) \text{ and} \\ &\text{if } \forall (p,q) \in \mathbb{R}^2_{++}, A(p,q) < 0 \text{ then } \frac{dq^{I}(\theta)}{d\theta} < 0 \ \forall \theta \in (\underline{\theta}, \bar{\theta}). \end{aligned}$$

Using a similar logic we can show that in a second-score auction,

$$\begin{aligned} &\text{if } \forall (p,q) \in \mathbb{R}^2_{++}, A(p,q) > 0 \text{ then } \frac{dq^{II}(\theta)}{d\theta} > 0 \ \forall \theta \in (\underline{\theta}, \bar{\theta}) \text{ and} \\ &\text{if } \forall (p,q) \in \mathbb{R}^2_{++}, A(p,q) < 0 \text{ then } \frac{dq^{II}(\theta)}{d\theta} < 0 \ \forall \theta \in (\underline{\theta}, \bar{\theta}) \end{aligned}$$

(iii) Now suppose $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} = 0$ for all $(p,q) \in \mathbb{R}^2_{++}$. Note that from propositions 1 and 2 we get that for both first-score and second-score auctions $S_q + S_p c' = 0$. Differentiating this equation w.r.t θ we get that for both auctions

$$S_{qp}p'(\theta) + S_{qq}q'(\theta) + c'\left[S_{pp}p'(\theta) + S_{pq}q'(\theta)\right] + S_p c'' q'(\theta) = 0 - - - - (37)$$

Since $-\frac{S_q}{S_p} = c'$, by substituting for c' and rearranging terms in (37) we get

$$p'(\theta)\left[-\frac{S_q}{S_p}S_{pp} + S_{qp}\right] + \frac{q'(\theta)}{S_p}\left[S_pS_{qq} - S_qS_{pq} + (S_p)^2 c''\right] = 0$$

Since $-\frac{S_q}{S_p}S_{pp} + S_{qp} = 0$ the above implies that

$$\frac{q'(\theta)}{S_p}\left[S_pS_{qq} - S_qS_{pq} + (S_p)^2 c''\right] = 0 - - - - (37a)$$

From assumption 4 we know $S_pS_{qq} - S_qS_{pq} + (S_p)^2 c'' > -\frac{(S_q)^2}{S_p}S_{pp} + S_qS_{qp} = S_q\left[-\frac{S_q}{S_p}S_{pp} + S_{qp}\right] = 0$ since $-\frac{S_q}{S_p}S_{pp} + S_{qp} = 0$. This means $S_pS_{qq} - S_qS_{pq} + (S_p)^2 c'' > 0$. Since $S_p < 0$ from (37a) we get that for both auctions $q'(\theta) = 0$ for all θ. That is, $\frac{dq^I(\theta)}{d\theta} = \frac{dq^{II}(\theta)}{d\theta} = 0$. This means that for all θ, $q^I(\theta) = q^I(\bar{\theta})$ and $q^{II}(\theta) = q^{II}(\bar{\theta})$. From lemma 1 we know that $q^I(\bar{\theta}) = q^{II}(\bar{\theta})$ and this implies that for all θ, $q^I(\theta) = q^{II}(\theta)$. This completes our proof of proposition 3.■

Proof of Proposition 5 (i) and (ii) Since by assumption S_{pp} and S_{qp} are continuous $\forall (p,q) \in \mathbb{R}^2_{++}$, then $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} \neq 0\ \forall (p,q) \in \mathbb{R}^2_{++}$ implies either (a) $\forall (p,q) \in \mathbb{R}^2_{++}$, $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} > 0$ **or** (b) $\forall (p,q) \in \mathbb{R}^2_{++}$, $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} < 0$.

Now suppose $B(p,q) = -\frac{S_q}{S_p}S_{pq} + S_pc'' + S_{qq} < 0$. Note that $p^I(\theta) = \sigma\left(q^I(\theta)\right)$ and $p^{II}(\theta) = \sigma\left(q^{II}(\theta)\right)$. Also note from (28) when $-\frac{S_q}{S_p}S_{pp} + S_{qp} > 0$ we have that $\sigma'(.) > 0$ and $q^I(\theta) > q^{II}(\theta)$ (shown in proposition 3). Since $q^I(\theta) > q^{II}(\theta)$ and $\sigma'(.) > 0$ we get $\sigma\left(q^I(\theta)\right) > \sigma\left(q^{II}(\theta)\right) \Longrightarrow p^I(\theta) > p^{II}(\theta)$. Again, when $-\frac{S_q}{S_p}S_{pp} + S_{qp} < 0$ we have that $\sigma'(.) < 0$ and $q^I(\theta) < q^{II}(\theta)$. Since $q^I(\theta) < q^{II}(\theta)$ and $\sigma'(.) < 0$ we get $\sigma\left(q^I(\theta)\right) > \sigma\left(q^{II}(\theta)\right) \Longrightarrow p^I(\theta) > p^{II}(\theta)$.

Now suppose $B(p,q) = -\frac{S_q}{S_p}S_{pq} + S_pc'' + S_{qq} > 0$. This implies $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} < 0$ (see lemma 2). From (28) and proposition 3 we know that when $-\frac{S_q}{S_p}S_{pp} + S_{qp} < 0$ we have $\sigma'(.) \geq 0$ and $q^I(\theta) < q^{II}(\theta)$. Since $q^I(\theta) < q^{II}(\theta)$ and $\sigma'(.) \geq 0$ we get $\sigma\left(q^I(\theta)\right) < \sigma\left(q^{II}(\theta)\right) \Longrightarrow p^I(\theta) < p^{II}(\theta)$.

Now since $p^I(\theta) = \sigma\left(q^I(\theta)\right)$ and $p^{II}(\theta) = \sigma\left(q^{II}(\theta)\right)$ for $\theta \in \left[\underline{\theta}, \bar{\theta}\right]$, we get that for all $\theta \in \left(\underline{\theta}, \bar{\theta}\right)$,

$$\frac{dp^I(\theta)}{d\theta} = \sigma'\left(q^I(\theta)\right)\frac{dq^I(\theta)}{d\theta} = -\left[\frac{-\frac{S_q}{S_p}S_{pq} + S_pc'' + S_{qq}}{-\frac{S_q}{S_p}S_{pp} + S_{qp}}\right]\frac{dq^I(\theta)}{d\theta} - - - (38)$$

$$\text{and } \frac{dp^{II}(\theta)}{d\theta} = \sigma'\left(q^{II}(\theta)\right)\frac{dq^{II}(\theta)}{d\theta} = -\left[\frac{-\frac{S_q}{S_p}S_{pq} + S_pc'' + S_{qq}}{-\frac{S_q}{S_p}S_{pp} + S_{qp}}\right]\frac{dq^{II}(\theta)}{d\theta} - - - (38a).$$

Note that from proposition 3 we get that $\frac{dq^I(\theta)}{d\theta}$, $\frac{dq^{II}(\theta)}{d\theta}$ have the same sign as $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp}$. This means $\frac{dp^I(\theta)}{d\theta}$, $\frac{dp^{II}(\theta)}{d\theta}$ has the same sign as $-\left(-\frac{S_q}{S_p}S_{pq} + S_pc'' + S_{qq}\right)$.

This means $B(p,q) = -\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq} < 0$ implies $\frac{dp^I(\theta)}{d\theta}$, $\frac{dp^{II}(\theta)}{d\theta} > 0$. Similarly $B(p,q) = -\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq} > 0$ implies $\frac{dp^I(\theta)}{d\theta}$, $\frac{dp^{II}(\theta)}{d\theta} < 0$.

(iii) Now suppose $B(p,q) = -\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq} = 0$. Using (38) and (38a) we get that $\frac{dp^I(\theta)}{d\theta} = \frac{dp^{II}(\theta)}{d\theta} = 0$. This means that for all θ, $p^I(\theta) = p^I\left(\bar{\theta}\right)$ and $p^{II}(\theta) = p^{II}\left(\bar{\theta}\right)$. From lemma 1 we know that $p^I\left(\bar{\theta}\right) = p^{II}\left(\bar{\theta}\right)$ and this implies that for all θ, $p^I(\theta) = p^{II}(\theta)$. This completes our proof of proposition 3.■

Proof of Proposition 6 (i) Note that from proposition 2 it is clear that $q^{II}(\theta)$ does not depend on n. This means $q^{II}(n;\theta) = q^{II}(m;\theta)$.

(ii) Using (29) and the definition of $\gamma(\theta)$ we know that

$$\sigma\left(q^I(n;\theta)\right) - c\left(q^I(n;\theta)\right) = \theta + \gamma(\theta) = \theta + \int_\theta^{\bar{\theta}} \left(\frac{1-F(t)}{1-F(\theta)}\right)^{n-1} dt - - - - (39)$$

Since $\left(\frac{1-F(t)}{1-F(\theta)}\right) < 1$ for all $t \in \left(\theta, \bar{\theta}\right)$, $\left(\frac{1-F(t)}{1-F(\theta)}\right)^{n-1}$ strictly decreases with an increase in n. That is, $\theta + \gamma(\theta)$ strictly decreases with an increase in n.

Now suppose $\forall (p,q) \in \mathbb{R}^2_{++}$, $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} > 0$. This implies $\sigma'(q) - c'(q) > 0$ (from (30). If possible let $q^I(n;\theta) \geq q^I(m;\theta)$. But this means $\sigma\left(q^I(n;\theta)\right) - c\left(q^I(n;\theta)\right) \geq \sigma\left(q^I(m;\theta)\right) - c\left(q^I(m;\theta)\right)$. But $\theta + \int_\theta^{\bar{\theta}} \left(\frac{1-F(t)}{1-F(\theta)}\right)^{n-1} dt < \theta + \int_\theta^{\bar{\theta}} \left(\frac{1-F(t)}{1-F(\theta)}\right)^{m-1} dt$. But this is a contradiction as we must have $\sigma\left(q^I(n;\theta)\right) - c\left(q^I(n;\theta)\right) = \theta + \int_\theta^{\bar{\theta}} \left(\frac{1-F(t)}{1-F(\theta)}\right)^{n-1} dt$ and $\sigma\left(q^I(m;\theta)\right) - c\left(q^I(m;\theta)\right) = \theta + \int_\theta^{\bar{\theta}} \left(\frac{1-F(t)}{1-F(\theta)}\right)^{m-1} dt$ (from (39). This means if $n > m$ then $q^I(n;\theta) < q^I(m;\theta)$.

(iii) Now suppose $\forall (p,q) \in \mathbb{R}^2_{++}$, $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} < 0$. Using an exactly similar logic as above we can show that if $n > m$ then $q^I(n;\theta) > q^I(m;\theta)$.■

Proof of Proposition 7 (i) Note that from proposition 2 it is clear that $p^{II}(\theta)$ does not depend on n. This means $p^{II}(n;\theta) = p^{II}(m;\theta)$.

(ii) Note that $p^I(n;\theta) = \sigma\left(q^I(n;\theta)\right)$.

Suppose $B(p,q) = -\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq} = 0$, $\forall (p,q) \in \mathbb{R}^2_{++}$. This means $\sigma'(.) = 0$ (using 28). This in turn implies $\sigma\left(q^I(n;\theta)\right) = \sigma\left(q^I(m;\theta)\right)$. But this means $p^I(n;\theta) = p^I(m;\theta)$.

(iii) Now suppose $B(p,q) = -\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq} > 0$, $\forall (p,q) \in \mathbb{R}^2_{++}$. Using lemma 2 this implies $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} < 0$. This means $\sigma'(.) > 0$ (using 28). Since $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} < 0$ we get that if $n > m$ we have $q^I(n;\theta) > q^I(m;\theta)$ (proposition 5). This in turn implies $\sigma\left(q^I(n;\theta)\right) > \sigma\left(q^I(m;\theta)\right)$. This means $p^I(n;\theta) > p^I(m;\theta)$.

(iv) First, suppose $\forall (p,q) \in \mathbb{R}^2_{++}$, $B(p,q) = -\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq} < 0$ and $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} > 0$. This means $\sigma'(.) > 0$ (using 28). Since $-\frac{S_q}{S_p}S_{pp} + S_{qp} > 0$ we get that if $n > m$ we have $q^I(n;\theta) < q^I(m;\theta)$ (proposition 5). This in turn implies $\sigma\left(q^I(n;\theta)\right) < \sigma\left(q^I(m;\theta)\right)$. This means $p^I(n;\theta) < p^I(m;\theta)$.

Now suppose $\forall (p,q) \in \mathbb{R}^2_{++}$, $B(p,q) = -\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq} < 0$ and $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} < 0$. This means $\sigma'(.) < 0$ (using 28). Since $-\frac{S_q}{S_p}S_{pp} + S_{qp} < 0$ we get that if $n > m$ we have $q^I(n;\theta) > q^I(m;\theta)$ (proposition 5). This in turn implies $\sigma\left(q^I(n;\theta)\right) < \sigma\left(q^I(m;\theta)\right)$. This means $p^I(n;\theta) < p^I(m;\theta)$.■

Proof of Proposition 8 (i) From propositions 5 and 6 we get that or all $n > m$ $q^{II}(n;\theta) = q^{II}(m;\theta)$ and $p^{II}(n;\theta) = p^{II}(m;\theta)$. This implies

$$S^{II}(n;\theta) = S\left(p^{II}(n;\theta), q^{II}(n;\theta)\right) = S\left(p^{II}(m;\theta), q^{II}(m;\theta)\right) = S^{II}(m;\theta).$$

(ii) Given any θ, using (4b) and (4c) we have

$$p^I(n;\theta) = c\left(q^I(n;\theta)\right) + \theta + \frac{\int_\theta^{\bar{\theta}} (1 - F(t))^{n-1}\,dt}{(1 - F(\theta))^{n-1}}$$

Differentiating the above w.r.t n we get

$$\frac{\partial}{\partial n}p^I(n;\theta) = c'\left(q^I(n;\theta)\right)\frac{\partial}{\partial n}q^I(n;\theta) + \frac{\partial}{\partial n}\left(\frac{\int_\theta^{\bar{\theta}} (1 - F(t))^{n-1}\,dt}{(1 - F(\theta))^{n-1}}\right) - - - - (40)$$

Note that

$$\frac{\partial}{\partial n}S^I(n;\theta) = \frac{\partial}{\partial n}S\left(p^I(n;\theta), q^I(n;\theta)\right) = S_p(.)\frac{\partial}{\partial n}p^I(n;\theta) + S_q(.)\frac{\partial}{\partial n}q^I(n;\theta)$$

Using (40) the above can be written as

$$\begin{aligned}\frac{\partial}{\partial n}S^I(n;\theta) &= S_p(.)\left[c'\left(q^I(n;\theta)\right)\frac{\partial}{\partial n}q^I(n;\theta) + \frac{\partial}{\partial n}\left(\int_\theta^{\bar{\theta}}\left(\frac{1 - F(t)}{1 - F(\theta)}\right)^{n-1} dt\right)\right] \\ &\quad + S_q(.)\frac{\partial}{\partial n}q^I(n;\theta) - - - - (40a)\end{aligned}$$

Note that in equilibrium $c'\left(q^I(n;\theta)\right) = -\frac{S_q\left(p^I(n;\theta),q^I(n;\theta)\right)}{S_p\left(p^I(n;\theta),q^I(n;\theta)\right)}$ (see 25). Using this in (40a) together with the fact that $S_p < 0$ and $\frac{\partial}{\partial n}\left(\int_\theta^{\bar{\theta}}\left(\frac{1-F(t)}{1-F(\theta)}\right)^{n-1} dt\right) < 0$ we get

$$\frac{\partial}{\partial n}S^I(n;\theta) = S_p(.)\frac{\partial}{\partial n}\left(\int_\theta^{\bar{\theta}}\left(\frac{1 - F(t)}{1 - F(\theta)}\right)^{n-1} dt\right) > 0.$$

This completes proof of proposition 7.■

Proof of lemma 3 In a first-score auction the expected score is as follows:

$$\begin{aligned}\Sigma^I &= \int_{\underline{\theta}}^{\bar{\theta}} S\left(p^I(\theta), q^I(\theta)\right) f_1(\theta)\,d\theta = \int_{\underline{\theta}}^{\bar{\theta}} S\left(p^I(\theta), q^I(\theta)\right) dF_1(\theta)\,d\theta \\ &= \left[S\left(p^I(\theta), q^I(\theta)\right) F_1(\theta)\right]_{\underline{\theta}}^{\bar{\theta}} - \int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)\,dS\left(p^I(\theta), q^I(\theta)\right) d\theta - - - (43)\end{aligned}$$

Note that from (25) we have

$$-\frac{S_q\left(p^I(\theta),q^I(\theta)\right)}{S_p\left(p^I(\theta),q^I(\theta)\right)}=c'\left(q^I(\theta)\right).----(44)$$

Also, from (6) we have

$$\forall\theta\in\left(\underline{\theta},\bar{\theta}\right),\ \frac{dp^I(\theta)}{d\theta}-c'\left(q^I(\theta)\right)\frac{dq^I(\theta)}{d\theta}=1+\gamma'(\theta)----(45)$$

Now we have

$$\begin{aligned} dS\left(p^I(\theta),q^I(\theta)\right) &= S_p\left(p^I(\theta),q^I(\theta)\right)\frac{dp^I(\theta)}{d\theta}+S_q\left(\left(p^I(\theta),q^I(\theta)\right)\right)\frac{dq^I(\theta)}{d\theta} \\ &= S_p\left(p^I(\theta),q^I(\theta)\right)\left[\frac{dp^I(\theta)}{d\theta}-c'\left(q^I(\theta)\right)\frac{dq^I(\theta)}{d\theta}\right]\ \text{(using 44)} \\ &= S_p\left(p^I(\theta),q^I(\theta)\right)\left[1+\gamma'(\theta)\right]\ \text{(using 45)} \end{aligned}$$

Using the above in (43) we get

$$\Sigma^I=S\left(p^I\left(\bar{\theta}\right),q^I\left(\bar{\theta}\right)\right)-\int_{\underline{\theta}}^{\bar{\theta}}F_1(\theta)\left(1+\gamma'(\theta)\right)S_p\left(p^I(\theta),q^I(\theta)\right)d\theta.$$

By a similar logic we can show that

$$\Sigma^{II}=S\left(p^{II}\left(\bar{\theta}\right),q^{II}\left(\bar{\theta}\right)\right)-\int_{\underline{\theta}}^{\bar{\theta}}F_2(\theta)\,S_p\left(p^{II}(\theta),q^{II}(\theta)\right)d\theta.$$

This completes our proof for lemma 3.■

Proof of lemma 4 In the proof of lemma 1 we have shown that

$$\lim_{\theta\longrightarrow\bar{\theta}}\gamma(\theta)=0----(46)$$

Now

$$\int_{\underline{\theta}}^{\bar{\theta}}F_1(\theta)\left(1+\gamma'(\theta)\right)d\theta=\int_{\underline{\theta}}^{\bar{\theta}}F_1(\theta)\,d\theta+\int_{\underline{\theta}}^{\bar{\theta}}F_1(\theta)\,d\gamma(\theta)----(47)$$

Note that

$$\int_{\underline{\theta}}^{\bar{\theta}}F_1(\theta)\,d\gamma(\theta)=\left[F_1(\theta)\,\gamma(\theta)\right]_{\underline{\theta}}^{\bar{\theta}}-\int_{\underline{\theta}}^{\bar{\theta}}\gamma(\theta)\,dF_1(\theta)----(48)$$

Using (46) we know that $\left[F_1(\theta)\,\gamma(\theta)\right]_{\underline{\theta}}^{\bar{\theta}}=0$. Since

$$dF_1(\theta)=f_1(\theta)\,d\theta=n\left(1-F(\theta)\right)^{n-1}f(\theta)\,d\theta \text{ and } \gamma(\theta)=\frac{1}{\left(1-F(\theta)\right)^{n-1}}\int_{\theta}^{\bar{\theta}}\left(1-F(t)\right)^{n-1}dt$$

from (48) we get

$$\int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)\, d\gamma(\theta) = -\int_{\underline{\theta}}^{\bar{\theta}} \frac{1}{(1-F(\theta))^{n-1}} \left(\int_{\theta}^{\bar{\theta}} (1-F(t))^{n-1}\, dt \right) n (1-F(\theta))^{n-1} f(\theta)\, d\theta$$
$$= -\int_{\underline{\theta}}^{\bar{\theta}} \left[\int_{\theta}^{\bar{\theta}} (1-F(t))^{n-1}\, dt \right] n f(\theta)\, d\theta - - - (49)$$

Changing the order of integration in (49) we have

$$\int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)\, d\gamma(\theta) = -n \int_{\underline{\theta}}^{\bar{\theta}} \left[\int_{\underline{\theta}}^{t} f(\theta)\, d\theta \right] (1-F(t))^{n-1}\, dt$$
$$= -n \int_{\underline{\theta}}^{\bar{\theta}} F(t)(1-F(t))^{n-1}\, dt$$
$$= -n \int_{\underline{\theta}}^{\bar{\theta}} F(\theta)(1-F(\theta))^{n-1}\, d\theta - - - - (50)$$

Hence using (50) in (47) we have

$$\int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)(1+\gamma'(\theta))\, d\theta = \int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)\, d\theta - n \int_{\underline{\theta}}^{\bar{\theta}} F(\theta)(1-F(\theta))^{n-1}\, d\theta - - - - (51)$$

Now note that

$$F_2(\theta) = F_1(\theta) - nF(\theta)(1-F(\theta))^{n-1} - - - -(52)$$

Therefore, from (51) and (52) we get

$$\int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)(1+\gamma'(\theta))\, d\theta = \int_{\underline{\theta}}^{\bar{\theta}} F_2(\theta)\, d\theta$$

This completes our proof for lemma 3.■

Proof of lemma 5 Since by assumption S_{pp} and S_{qp} are continuous $\forall (p,q) \in \mathbb{R}^2_{++}$, then $-\frac{S_q}{S_p}S_{pp} + S_{qp} \neq 0\ \forall (p,q) \in \mathbb{R}^2_{++}$ implies either (a) $\forall (p,q) \in \mathbb{R}^2_{++}\ A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} > 0$ **or** (b) $\forall (p,q) \in \mathbb{R}^2_{++}\ A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} < 0$.

Note that

$$\frac{d}{dq}\left[-S_p(\sigma(q), q)\right] = -S_{pp}\sigma'(q) - S_{pq}$$
$$= -S_{pp}\left[-\left[\frac{-\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq}}{-\frac{S_q}{S_p}S_{pp} + S_{qp}}\right]\right] - S_{pq} \text{ (using 28)}$$
$$= \frac{S_{pp}S_p c'' + S_{pp}S_{qq} - (S_{qp})^2}{-\frac{S_q}{S_p}S_{pp} + S_{qp}}$$
$$= \frac{H(p,q)}{A(p,q)} - - - - (53)$$

First, suppose $H(p,q) = S_{pp}S_pc'' + S_{pp}S_{qq} - (S_{qp})^2 < 0$. If $A(p,q) > 0$ then $q^I(\theta) > q^{II}(\theta)$ (from proposition 3). From (53) we have $\frac{d}{dq}[-S_p(\sigma(q),q)] < 0$. This implies $-S_p(\sigma(q^I(\theta)),q^I(\theta)) < -S_p(\sigma(q^{II}(\theta)),q^{II}(\theta))$. If $A(p,q) < 0$ then $q^I(\theta) < q^{II}(\theta)$ (from proposition 3). From (53) we have $\frac{d}{dq}[-S_p(\sigma(q),q)] > 0$. This implies $-S_p(\sigma(q^I(\theta)),q^I(\theta)) < -S_p(\sigma(q^{II}(\theta)),q^{II}(\theta))$. This again means $-S_p(p^I(\theta),q^I(\theta)) < -S_p(p^{II}(\theta),q^{II}(\theta))$.

Now suppose $H(p,q) = S_{pp}S_pc'' + S_{pp}S_{qq} - (S_{qp})^2 \geq 0$. If $A(p,q) > 0$ then $q^I(\theta) > q^{II}(\theta)$ (from proposition 3). From (53) we have $\frac{d}{dq}[-S_p(\sigma(q),q)] \geq 0$. This implies $-S_p(\sigma(q^I(\theta)),q^I(\theta)) \geq -S_p(\sigma(q^{II}(\theta)),q^{II}(\theta))$. This again means $-S_p(p^I(\theta),q^I(\theta)) \geq -S_p(p^{II}(\theta),q^{II}(\theta))$. If $A(p,q) < 0$ then $q^I(\theta) < q^{II}(\theta)$ (from proposition 3). From (53) we have $\frac{d}{dq}[-S_p(\sigma(q),q)] \leq 0$. This implies $-S_p(\sigma(q^I(\theta)),q^I(\theta)) \geq -S_p(\sigma(q^{II}(\theta)),q^{II}(\theta))$. This again means $-S_p(p^I(\theta),q^I(\theta)) \geq -S_p(p^{II}(\theta),q^{II}(\theta))$.

Now note that using (53) we get

$$\begin{aligned}
&\frac{d}{d\theta}[-S_p(\sigma(q(\theta)),q(\theta))] \\
= \; & -S_{pp}\sigma'(q(\theta))\frac{dq(\theta)}{d\theta} - S_{pq}\frac{dq(\theta)}{d\theta} \\
= \; & \frac{\frac{dq(\theta)}{d\theta}}{-\frac{S_q}{S_p}S_{pp} + S_{qp}}\left[S_{pp}S_pc'' + S_{pp}S_{qq} - (S_{qp})^2\right] \\
= \; & \frac{H(p,q)}{A(p,q)}\frac{dq(\theta)}{d\theta} - - - - (54)
\end{aligned}$$

From proposition 3 we know that $\frac{dq^I(\theta)}{d\theta}$ and $\frac{dq^{II}(\theta)}{d\theta}$ have the same sign as $A(p,q)$. This means that for both auction formats

$$\frac{\frac{dq(\theta)}{d\theta}}{A(p,q)} > 0 - - - - (55).$$

Now since $p^I(\theta) = \sigma(q^I(\theta))$ and $p^{II}(\theta) = \sigma(q^{II}(\theta))$ for $\theta \in [\underline{\theta},\bar{\theta}]$ we get that

$$\begin{aligned}
-S_p(\sigma(q^I(\theta)),q^I(\theta)) &= -S_p(p^I(\theta),q^I(\theta)) \\
\text{and} - S_p(\sigma(q^{II}(\theta)),q^{II}(\theta)) &= -S_p(p^{II}(\theta),q^{II}(\theta)).
\end{aligned}$$

Hence, from (54) and (55) we get that if $H(p,q) = S_{pp}S_pc'' + S_{pp}S_{qq} - (S_{qp})^2 < 0$ then $\frac{d}{d\theta}[-S_p(p^I(\theta),q^I(\theta))]$, $\frac{d}{d\theta}[-S_p(p^{II}(\theta),q^{II}(\theta))] < 0$. And if $H(p,q) = S_{pp}S_pc'' + S_{pp}S_{qq} - (S_{qp})^2 \geq 0$ then $\frac{d}{d\theta}[-S_p(p^I(\theta),q^I(\theta))]$, $\frac{d}{d\theta}[-S_p(p^{II}(\theta),q^{II}(\theta))] \geq 0$.■

Proof of Proposition 10 Note that if $S_pc'' + S_{qq} = 0$ and $S_{qp} = 0$ then $B(p,q) = -\frac{S_q}{S_p}S_{pq} + S_pc'' + S_{qq} = 0$. From proposition 4 we know that $B(p,q) = 0$ implies that for all θ, $p^I(\theta) = p^{II}(\theta)$. From lemma 1 we get that this implies $p^I(\theta) = p^{II}(\theta) = p^I(\bar{\theta}) = p^{II}(\bar{\theta})$. Since $S_{pq} = 0$ we must have $S_p(p^I(\theta),q^I(\theta)) = S_p(p^I(\bar{\theta}),q^I(\bar{\theta}))$ and $S_p(p^{II}(\theta),q^{II}(\theta)) = S_p(p^{II}(\bar{\theta}),q^{II}(\bar{\theta}))$ for all θ. That is, for all θ, $S_p(p^I(\theta),q^I(\theta)) = S_p(p^{II}(\theta),q^{II}(\theta)) = S_p(p^I(\bar{\theta}),q^I(\bar{\theta}))$.

Using lemma 3 we know that

$$\begin{aligned}\Sigma^{I} &= S\left(p^{I}\left(\bar{\theta}\right), q^{I}\left(\bar{\theta}\right)\right) - \int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)\left(1+\gamma'(\theta)\right) S_p\left(p^{I}(\theta), q^{I}(\theta)\right) d\theta \\ &= S\left(p^{I}\left(\bar{\theta}\right), q^{I}\left(\bar{\theta}\right)\right) - S_p\left(p^{I}\left(\bar{\theta}\right), q^{I}\left(\bar{\theta}\right)\right)\int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)\left(1+\gamma'(\theta)\right) d\theta - - - - (56)\end{aligned}$$

And

$$\begin{aligned}\Sigma^{II} &= S\left(p^{II}\left(\bar{\theta}\right), q^{II}\left(\bar{\theta}\right)\right) - \int_{\underline{\theta}}^{\bar{\theta}} F_2(\theta) S_p\left(p^{II}(\theta), q^{II}(\theta)\right) d\theta \\ &= S\left(p^{II}\left(\bar{\theta}\right), q^{II}\left(\bar{\theta}\right)\right) - S_p\left(p^{II}\left(\bar{\theta}\right), q^{II}\left(\bar{\theta}\right)\right)\int_{\underline{\theta}}^{\bar{\theta}} F_2(\theta)\, d\theta - - - - (57)\end{aligned}$$

From lemma 1 we know that $p^{I}\left(\bar{\theta}\right) = p^{II}\left(\bar{\theta}\right)$ and $q^{I}\left(\bar{\theta}\right) = q^{II}\left(\bar{\theta}\right)$. From lemma 4 we have $\int_{\underline{\theta}}^{\bar{\theta}} F_2(\theta)\, d\theta = \int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)\left(1+\gamma'(\theta)\right) d\theta$. Combining these with (56) and (57) we get that $\Sigma^{I} = \Sigma^{II}$.■

Proof of Proposition 11 Note that

$$\begin{aligned}\frac{d}{d\theta} S^{II}(\theta) &= \frac{d}{d\theta} S\left(p^{II}(\theta), q^{II}(\theta)\right) \\ &= S_p\left(p^{II}(\theta), q^{II}(\theta)\right)\frac{dp^{II}(\theta)}{d\theta} + S_q\left(p^{II}(\theta), q^{II}(\theta)\right)\frac{dq^{II}(\theta)}{d\theta} \\ &= S_p\left(p^{II}(\theta), q^{II}(\theta)\right)\left[\frac{dp^{II}(\theta)}{d\theta} - c'\left(q^{I}(\theta)\right)\frac{dq^{II}(\theta)}{d\theta}\right] \text{ (using 25)} \\ &= S_p\left(p^{I}(\theta), q^{I}(\theta)\right) \text{ (by using 20b)} - - - - (58)\end{aligned}$$

Also note that from (58) we get

$$\frac{d^2}{d\theta^2} S^{II}(\theta) = \frac{d}{d\theta} S_p\left(p^{I}(\theta), q^{I}(\theta)\right) - - - - (59)$$

Since $S_{pp}S_p c'' + S_{pp}S_{qq} - \left(S_{qp}\right)^2 < 0$ we have $\frac{d}{d\theta} S_p\left(p^{II}(\theta), q^{II}(\theta)\right) > 0$ (using lemma 5). Then, we have from (59)

$$\frac{d^2}{d\theta^2} S^{II}(\theta) > 0 \text{ for all } \theta \in \left(\underline{\theta}, \bar{\theta}\right) - - - - (60)$$

The above means that $S^{II}(\theta)$ is a strictly convex function. Using Jensen's inequality we know that

$$\Sigma^{II} = \int_{\underline{\theta}}^{\bar{\theta}} S^{II}(\theta) f_2(\theta)\, d\theta > S^{II}\left(\int_{\underline{\theta}}^{\bar{\theta}} \theta f_2(\theta)\, d\theta\right) - - - - (61)$$

Now note that since $\frac{d}{d\theta} S^{I}(\theta) < 0$ (proposition 7) we get that

$$\Sigma^{I} = \int_{\underline{\theta}}^{\bar{\theta}} S^{I}(\theta) f_1(\theta)\, d\theta < S^{I}\left(\underline{\theta}\right)\int_{\underline{\theta}}^{\bar{\theta}} f_1(\theta)\, d\theta = S^{I}\left(\underline{\theta}\right) - - - - (62)$$

Since $S^{II}\left(\int_{\underline{\theta}}^{\bar{\theta}} \theta f_2(\theta)\, d\theta\right) > S^{I}\left(\underline{\theta}\right)$ using (61) and (62) we get that $\Sigma^{I} < \Sigma^{II}$.■

Proof of Proposition 12 Note that $\hat{\theta} = F^{-1}\left(\frac{1}{n}\right)$. We have the following:

$$\begin{aligned}\Sigma^{II} &= \int_{\underline{\theta}}^{\bar{\theta}} S^{II}(\theta) f_2(\theta)\, d\theta \\ &= \int_{\underline{\theta}}^{\hat{\theta}} S^{II}(\theta) f_2(\theta)\, d\theta + \int_{\hat{\theta}}^{\bar{\theta}} S^{II}(\theta) f_2(\theta)\, d\theta ----(63)\end{aligned}$$

Similarly

$$\begin{aligned}\Sigma^{I} &= \int_{\underline{\theta}}^{\bar{\theta}} S^{I}(\theta) f_1(\theta)\, d\theta \\ &= \int_{\underline{\theta}}^{\hat{\theta}} S^{I}(\theta) f_1(\theta)\, d\theta + \int_{\hat{\theta}}^{\bar{\theta}} S^{I}(\theta) f_1(\theta)\, d\theta ----(64)\end{aligned}$$

From proposition 5 we know that $S^{II}(\theta) > S^{I}(\theta)$ and both $S^{II}(\theta)$ and $S^{I}(\theta)$ are strictly decreasing in θ. This means

$$\begin{aligned}\int_{\underline{\theta}}^{\hat{\theta}} S^{II}(\theta) f_2(\theta)\, d\theta &> S^{II}\left(\hat{\theta}\right) \int_{\underline{\theta}}^{\hat{\theta}} f_2(\theta)\, d\theta \\ &= S^{II}\left(\hat{\theta}\right) F_2\left(\hat{\theta}\right) ----(65)\end{aligned}$$

and

$$\begin{aligned}&\int_{\hat{\theta}}^{\bar{\theta}} S^{II}(\theta) f_2(\theta)\, d\theta - \int_{\hat{\theta}}^{\bar{\theta}} S^{I}(\theta) f_1(\theta)\, d\theta \\ > &\int_{\hat{\theta}}^{\bar{\theta}} S^{I}(\theta) \left[f_2(\theta) - f_1(\theta)\right] d\theta --(66)\end{aligned}$$

Since $f_2(\theta) - f_1(\theta) > 0$ for all $\theta \in \left(\hat{\theta}, \bar{\theta}\right)$ from (66) we get that

$$\begin{aligned}&\int_{\hat{\theta}}^{\bar{\theta}} S^{II}(\theta) f_2(\theta)\, d\theta - \int_{\hat{\theta}}^{\bar{\theta}} S^{I}(\theta) f_1(\theta)\, d\theta \\ > &\; S^{I}\left(\bar{\theta}\right) \int_{\hat{\theta}}^{\bar{\theta}} \left[f_2(\theta) - f_1(\theta)\right] d\theta \\ = &\; S^{I}\left(\bar{\theta}\right) \left[F_1\left(\hat{\theta}\right) - F_2\left(\hat{\theta}\right)\right] --(67)\end{aligned}$$

Since $S^{I}(\theta)$ is strictly decreasing in θ we also have

$$\begin{aligned}\int_{\underline{\theta}}^{\hat{\theta}} S^{I}(\theta) f_1(\theta)\, d\theta &< S^{I}(\underline{\theta}) \int_{\underline{\theta}}^{\hat{\theta}} f_1(\theta)\, d\theta \\ &= S^{I}(\underline{\theta}) F_1\left(\hat{\theta}\right) ----(68)\end{aligned}$$

From (63)-(68) we get that

$$\begin{aligned}
& \Sigma^{II} - \Sigma^{I} \\
> \; & S^{II}\left(\hat{\theta}\right) F_2\left(\hat{\theta}\right) - S^{I}\left(\underline{\theta}\right) F_1\left(\hat{\theta}\right) + S^{I}\left(\bar{\theta}\right)\left[F_1\left(\hat{\theta}\right) - F_2\left(\hat{\theta}\right)\right] ----(69)
\end{aligned}$$

(78) implies that if

$$\begin{aligned}
S^{II}\left(\hat{\theta}\right) &\geq \frac{S^{I}\left(\underline{\theta}\right) F_1\left(\hat{\theta}\right) - S^{I}\left(\bar{\theta}\right)\left[F_1\left(\hat{\theta}\right) - F_2\left(\hat{\theta}\right)\right]}{F_2\left(\hat{\theta}\right)} \\
\text{then } \Sigma^{II} - \Sigma^{I} &> 0. ----(70)
\end{aligned}$$

From section 5.1 we know that

$$F_1(x) = 1 - (1 - F(x))^n \text{ and } F_2(x) = 1 - (1 - F(x))^n - nF(x)(1 - F(x))^{n-1} ---(71)$$

Since $F\left(\hat{\theta}\right) = \frac{1}{n}$ we can compute $F_1\left(\hat{\theta}\right)$ and $F_2\left(\hat{\theta}\right)$ from (71).

Then we get

$$\begin{aligned}
S^{II}\left(\hat{\theta}\right) &\geq \frac{S^{I}\left(\underline{\theta}\right) F_1\left(\hat{\theta}\right) - S^{I}\left(\bar{\theta}\right)\left[F_1\left(\hat{\theta}\right) - F_2\left(\hat{\theta}\right)\right]}{F_2\left(\hat{\theta}\right)} \\
&\Longleftrightarrow \\
S^{II}\left(\hat{\theta}\right) &\geq \frac{\left[1 - \left(\frac{n-1}{n}\right)^n\right] S^{I}\left(\underline{\theta}\right) - \left(\frac{n-1}{n}\right)^{n-1} S^{I}\left(\bar{\theta}\right)}{1 - \left(\frac{n-1}{n}\right)^n - \left(\frac{n-1}{n}\right)^{n-1}} ----(72)
\end{aligned}$$

Hence, using (70) and (72) we have that if

$$\begin{aligned}
S^{II}\left(\hat{\theta}\right) &\geq \frac{\left[1 - \left(\frac{n-1}{n}\right)^n\right] S^{I}\left(\underline{\theta}\right) - \left(\frac{n-1}{n}\right)^{n-1} S^{I}\left(\bar{\theta}\right)}{1 - \left(\frac{n-1}{n}\right)^n - \left(\frac{n-1}{n}\right)^{n-1}} \\
\text{then } \Sigma^{II} - \Sigma^{I} &> 0.
\end{aligned}$$

This completes proof of proposition 11.■