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A FOUNDATION FOR DOMINANT STRATEGY VOTING MECHANISMS

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Abstract

We study deterministic voting mechanisms by considering an ordinal notion of Bayesian incentive compatibility (OBIC). If the beliefs of agents are independent and *generic*, we show that any OBIC mechanism is dominant strategy incentive compatible under an additional mild requirement. Our result works in a large class of preference domains (that include the unrestricted domain, the single peaked domain, a specific class of single crossing domains) and under a weaker notion of OBIC that we call *locally* OBIC. We also discuss the implications of assuming unanimity on our results.

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1 Introduction

Dominant strategy incentive compatible (DSIC) mechanisms play a central role in the theory of mechanism design. Among other things, such mechanisms enjoy a strong robustness feature due to their prior-free property. Unfortunately, in voting environments, DSIC turns out to be too demanding. In particular, the seminal Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) shows that the only deterministic dominant strategy incentive compatible (DSIC) voting mechanism satisfying unanimity is a dictatorship if there are at least three alternatives. A crucial assumption in this result is that the domain (of preferences of agents) must be unrestricted. A large body of literature in social choice theory has tried to relax the assumption of unrestricted domain in the Gibbard-Satterthwaite theorem, while still working with deterministic DSIC mechanisms. This has led to various (possibility) characterizations of DSIC mechanisms in restricted domains - an excellent survey is Barbera (2010).

A natural question to ask is if there is any loss of generality in not considering a weaker solution concept in such voting environments that can depend on the priors of the agents - the most prominent such solution concept being the Bayes-Nash equilibrium. d'Aspremont and Peleg (1988) introduced the notion of ordinal Bayesian incentive compatibility (OBIC) in this context. OBIC is a natural weakening of DSIC to ordinal voting mechanisms. A voting mechanism is OBIC if for every agent, his interim/expected allocation probability vector from truth-telling first-order-stochastic-dominates any interim allocation probability vector obtained by deviating. This notion is equivalent to requiring that the expected utility from truth-telling must be no less than the expected utility from deviating for all possible utility representation of the ordinal preference of the agent.

Our main result shows that if the priors of the agents are independent and generic, then in a large class of domains, an ordinal deterministic mechanism is OBIC and satisfies *elementary monotonicity* if and only if it is a DSIC mechanism. [1] This result holds even if we weaken OBIC to only allow manipulations of each agent to his *adjacent* preferences - we call this requirement *locally* OBIC (LOBIC). Notice that the result does not require unanimity or any restriction on number of alternatives. Our result establishes a strong robustness property of DSIC mechanisms in voting environments.

We call the domains where our main result holds *local* domains. A domain is local if every *locally* dominant strategy incentive compatible mechanism is a DSIC mechanism. Two recent papers by Carroll (2012) and Sato (2013) give a comprehensive description of domains that are local. Some of the prominent domains identified in these papers are the unrestricted domain, the single peaked domain (Moulin, 1980), and a particular class of single crossing

[1] We discuss elementary monotonicity and generic priors later.

domains (Carroll, 2012; Saporiti, 2009). An implication of our result is that in all these domains, any locally OBIC mechanism (with respect to independent generic prior) satisfying elementary monotonicity must be a DSIC mechanism.

The proof of our main result reveals another surprising fact. In *any* domain, locally OBIC with respect to independent generic priors and elementary monotonicity implies locally DSIC. This shows that if we only care about designing mechanisms that satisfy local incentive constraints, then our main result is true in *any* domain. [2]

Majumdar and Sen (2004) investigated OBIC mechanisms in the unrestricted domain. They assumed that agents have independent priors. If these are uniform priors, then they show that a large class of interesting non-dictatorial social choice functions are OBIC in the unrestricted domain. However, if the priors are *generic*, then the negative result in the Gibbard-Satterthwaite theorem persists: if there are at least three alternatives and priors are generic, then in the unrestricted domain, a deterministic unanimous OBIC mechanism is a dictatorship. We can reinterpret this result as follows. If there are at least three alternatives and the priors are generic, then in the unrestricted domain, a unanimous OBIC mechanism is a *DSIC* mechanism. Our results generalize this interpretation of their result. Notice that our result works in a large class of domains and uses elementary monotonicity while their genericity result is for the unrestricted domain and uses unanimity.

We show that the conclusion in the Majumdar and Sen (2004) result (using unanimity) crucially relies on the unrestricted domain assumption, and it may not extend to restricted domains. In particular, we construct a *non-DSIC* unanimous mechanism that is OBIC with respect to some generic priors when the domain of preferences is restricted to be the single peaked domain - this mechanism works even if there are only two alternatives.

However, we note that such OBIC mechanisms violate elementary monotonicity. Elementary monotonicity is a well-studied axiom in social choice theory (Moulin, 1983). It requires the following. Suppose agent i has a preference P_i where a is ranked just above b and consider a preference profile P_{-i} of other agents such that the mechanism chooses b at the profile (P_i, P_{-i}). Now, suppose agent i reports another preference P_i' where only the positions of a and b are swapped from P_i (maintaining the positions of other alternatives). Elementary monotonicity requires that the outcome at the new profile (P_i', P_{-i}) must be b.

Since our result seems to restore the genericity result of Majumdar and Sen (2004) in a large class of domains by replacing unanimity with elementary monotonicity, it is natural to ask if elementary monotonicity is too strong a condition. As noted in Moulin (1983) and Majumdar and Sen (2004), it is satisfied by many well known social choice functions. [3] For

[2] Carroll (2012) and Sato (2013) contain extensive discussions motivating mechanism design with local incentive compatibility.

[3] We use the terms social choice function and mechanism interchangeably throughout the paper.

instance, the well known scoring rules like the Borda and the plurality rules satisfy elementary monotonicity. Further, in the unrestricted domain with at least three alternatives, we show that LOBIC with generic priors and unanimity imply elementary monotonicity. This leads to an immediate strengthening of the result in Majumdar and Sen (2004) as we can now replace their OBIC requirement by the weaker LOBIC: in the unrestricted domain with at least three alternatives, a unanimous and LOBIC mechanism is a DSIC mechanism if the priors are independent and generic. Further, it shows that our main result using elementary monotonicity is stronger than their result.

In the single peaked domain, we introduce a significantly weaker condition than elementary monotonicity that we call *weak elementary monotonicity*. We show that in that domain, every unanimous and LOBIC (with respect to independent generic priors) mechanism satisfying weak elementary monotonicity must be a DSIC mechanism. This illustrates that in specific restricted domains, we can replace elementary monotonicity in our result by a weaker version of elementary monotonicity if we assume unanimity.

Interestingly, when agents have uniform priors (a non-generic prior), Majumdar and Sen (2004) show that every *neutral* social choice function satisfying elementary monotonicity is OBIC in the unrestricted domain. Since neutrality is also a relatively weak requirement, it shows that a large class of non-DSIC mechanisms are OBIC with uniform priors in the unrestricted domain. [4] Our result shows two very different implications of elementary monotonicity with generic priors and uniform priors.

In summary, our results suggest that as far as ordinal deterministic mechanisms are concerned, there is no loss of generality in focusing attention to DSIC mechanisms (under elementary monotonicity) in many domains. However, the nature of equivalence is nuanced.

Our results parallel recent contributions in the single object auction quasilinear utility models (and some of its extensions) by Manelli and Vincent (2010) and Gershkov et al. (2013) who establish a weaker version of equivalence between Bayesian incentive compatible and DSIC mechanisms. We discuss the precise connection between these results and ours later in the paper.

Our results suggest that if we want to expand the set of incentive compatible mechanisms, then we need to go beyond OBIC deterministic mechanisms, and consider *cardinal* and/or *randomized* voting mechanisms. A small literature (Borgors and Smith, 2014; Kim, 2014) in strategic voting considers the consequences of using cardinal mechanisms with Bayesian incentive compatibility. These papers also consider randomization and work in the unrestricted domain. In such cardinal environment, type of an agent is a von-Neumann-Morgenstern utility function over alternatives. They compare the interim expected utilities of agents from these mechanisms to DSIC mechanisms. While Borgors and Smith (2014) show that such

[4]In Section 6, we illustrate that this result may no longer be true in the restricted domain.

mechanisms improve upon the random cardinal DSIC mechanisms, Kim (2014) shows that such mechanisms improve upon the ordinal BIC mechanisms. [5] Along with these results, our results help to delineate the boundaries of OBIC and DSIC equivalence in voting environments.

OBIC mechanisms have been studied in the context of matching problems in Majumdar (2003); Ehlers and Masso (2007). These papers study the implication of stability and OBIC in two-sided matching problems. The general conclusion in these papers is that stability and OBIC are too restrictive in the two-sided matching problems. Though our main result can be applied in private good allocation problems like the matching problems (by suitably redefining the notations and definitions), elementary monotonicity is a condition used in voting environment. Since we use elementary monotonicity in our main result, and its implications in private good problems is not very clear, we restrict ourselves to voting problems.

The rest of the paper is organized as follows. We introduce our model and assumptions about beliefs in Section 2. In Section 3, we formally state the result of Majumdar and Sen (2004) and show how this result breaks down if we consider a model with two alternatives or restrict the domain to the single peaked domain. Section 4 presents our main result and its proof. In Section 5, we show how our main result changes in specific domains if we restrict attention to unanimous social choice functions. We discuss several extensions and implications of our result in Section 6. Finally, we conclude in Section 7. Appendix A contains all the omitted proofs and Appendix B contains some omitted examples and illustrations.

2 The Model

Let A be a finite set of alternatives and $\mathcal{P}$ be the set of all strict linear orders over A. Let $\mathcal{D} \subseteq \mathcal{P}$ be some subset of strict linear orders. We will refer to $\mathcal{D}$ as the **domain**. There are n agents. The set of agents is denoted by $N = \{1, \ldots, n\}$. The private preference (type) of each agent $i \in N$ is a strict linear order $P_i \in \mathcal{D}$.

A **social choice function** (scf) is a map $f : \mathcal{D}^n \to A$.

Definition 1 *An scf $f : \mathcal{D}^n \to A$ is* **dominant strategy incentive compatible (DSIC)** *if for every $i \in N$, every P_{-i}, and every $P_i \in \mathcal{D}$, there exists no $P'_i \in \mathcal{D}$ such that*

$$f(P'_i, P_{-i}) P_i f(P_i, P_{-i}).$$

[5] Kim (2014) also considers OBIC deterministic mechanisms in the single peaked domain. He shows that such OBIC mechanisms satisfying unanimity must be *peaks only*. However, he has no results on the equivalence between OBIC and DSIC mechanisms.

We now define the notion of ordinal Bayesian incentive compatiblity. For this, we need to define beliefs. A belief for agent i is a mapping $\beta_i : \mathcal{D}^n \to [0,1]$ such that $\sum_{\mathbf{P} \in \mathcal{D}^n} \beta_i(\mathbf{P}) = 1$. Throughout the paper, we will assume that agents have **common belief**, i.e., $\beta_i = \beta_j$ for all $i, j \in N$. Denote this common belief as β. For every agent $i \in N$ and every agent P_{-i}, denote the conditional belief of agent i that others have preferences P_{-i} when he has preference P_i as $\beta(P_{-i}|P_i)$.

We will also assume that beliefs are **independent**, i.e., for every agent $i \in N$, there exists a probability distribution $\mu_i : \mathcal{D} \to [0,1]$ such that for every $P_{-i} \in \mathcal{D}^n$, the conditional belief that agents other than i have preference P_{-i} when i has preference P_i is

$$\beta(P_{-i}|P_i) = \times_{j \neq i} \mu_j(P_j).$$

Since we will only deal with common and independent beliefs, we will assume that the underlying probability distributions of agents are $\{\mu_i\}_{i \in N}$, and refer to them as **priors** of agents. Further, for agent i, the probability that agents other than i have a type profile P_{-i} will be denoted as $\mu(P_{-i}) \equiv \times_{j \neq i} \mu_j(P_j)$.

Given a social choice function f, we can compute the interim allocation probability of each agent from this scf using the priors. For this, consider a profile of priors $\{\mu_i\}_{i \in N}$. For each agent $i \in N$, define $\pi_i^f(a, P_i)$ as the interim allocation probability of the scf f choosing alternative a when agent i reports P_i as his type:

$$\pi_i^f(a, P_i) = \sum_{P_{-i} \in \mathcal{D}^{n-1} : f(P_i, P_{-i}) = a} \mu(P_{-i}).$$

Note that π_i^f depends on the priors, but we have suppressed it to make the notation less complex.

A utility function is a map $u : A \to \mathbb{R}$. A utility function u represents a preference ordering P if for all $a, b \in A$, $u(a) > u(b)$ if and only if aPb. Denote the set of all utility functions representing a preference ordering P as $\mathcal{U}(P)$.

We are now ready to define the notion of ordinal Bayesian incentive compatibility. It was introduced to the literature by d'Aspremont and Peleg (1988).

DEFINITION 2 *An scf f is* **ordinally Bayesian incentive compatible (OBIC)** *with respect to profile of priors $\{\mu_i\}_{i \in N}$ if for every $i \in N$, for every $P_i, P_i' \in \mathcal{D}$, for every $u \in \mathcal{U}(P_i)$, we have*

$$\sum_{a \in A} u(a) \pi_i^f(a, P_i) \geq \sum_{a \in A} u(a) \pi_i^f(a, P_i').$$

An equivalent definition using first order stochastic dominance can also be given. For any alternative $a \in A$ and any $P_i \in \mathcal{D}$, let $B(a, P_i) := \{a\} \cup \{b \in A : bP_ia\}$.

Definition 3 *An scf f is* **ordinally Bayesian incentive compatible (OBIC)** *with respect to profile of priors $\{\mu_i\}_{i\in N}$ if for every $i \in N$, for every $P_i, P_i' \in \mathcal{D}$, and for every $a \in A$, we have*

$$\sum_{b\in B(a,P_i)} \pi_i^f(b, P_i) \geq \sum_{b\in B(a,P_i)} \pi_i^f(b, P_i').$$

OBIC is clearly a weakening of DSIC. A well known fact to note is that if an scf is OBIC with respect to *all* beliefs, then it is DSIC.

2.1 Generic Priors and G-OBIC

We make the following assumption about generic priors in the paper.

Definition 4 *The profile of priors $\{\mu_i\}_{i\in N}$ is* **generic** *if for every $j \in N$ and for every $\mathcal{S}, \mathcal{T} \subseteq \mathcal{D}^{n-1}$ we have*

$$\Big[\sum_{P_{-j}\in\mathcal{S}} \mu(P_{-j}) = \sum_{P_{-j}\in\mathcal{T}} \mu(P_{-j})\Big] \Leftrightarrow [\mathcal{S} = \mathcal{T}].$$

Genericity requires that if we consider an agent i and consider two distinct subsets of profiles of types of agents in $N \setminus \{i\}$, then the probability that agents in $N \setminus \{i\}$ have types in these subsets cannot be the same.

Mathematically, these priors are generic in a topological sense. Denote by Δ the unit simplex of dimension $|\mathcal{D}| - 1$. The set of common independent beliefs is the n-th order Cartesian product of unit simplices Δ, and is given by Δ^n. The set of priors ruled out by genericity are given by equations that define a finite set of hyperplanes in Δ^n. Excluding these hyperplanes excludes only a set of Lebesgue measure zero in Δ^n. Hence, the generic beliefs identified in Definition 4 is a generic (in a topological sense) subset of Δ^n, i.e., a dense subset of the set of all common independent beliefs whose complement has measure zero (Majumdar and Sen, 2004) - to be precise, Majumdar and Sen (2004) show this fact when $\mathcal{D} = \mathcal{P}$, but an identical proof works if $\mathcal{D} \subseteq \mathcal{P}$. Notice that generic priors require that agents must be ex-ante asymmetric, i.e., for any $i, j \in N$, $\mu_i \neq \mu_j$.

If we consider any prior over $\mathcal{D} \subsetneq \mathcal{P}$, it also defines a prior over $\mathcal{P}$. Then, we can imagine Δ to be a unit simplex of dimension $|\mathcal{P}| - 1$. Obviously, such priors define beliefs that have Lebesgue measure zero in this extended simplex. Hence, our set of generic priors is not generic if we extend it to $\mathcal{P}$ and then consider its genericity.

One notable prior that is not generic is the uniform prior.

Definition 5 *A profile of independent priors $\{\mu_i\}_{i\in N}$ are* **uniform** *priors if for every $i \in N$, for every $P_i, P_i' \in \mathcal{D}$, we have*

$$\mu_i(P_i) = \mu_i(P_i').$$

We will discuss the implications of using uniform priors later and assume for the moment generic priors.

Definition 6 *An scf f is* **G-OBIC** *if there exist some profile of generic priors $\{\mu_i\}_{i\in N}$ such that f is OBIC with respect to $\{\mu_i\}_{i\in N}$.*

An important point to remember is that G-OBIC requires OBIC with respect to *some* profile of generic priors (it may be just one profile of generic priors), but *not all* profile of generic priors. For instance, an scf may be OBIC with respect to uniform priors, but it can still be G-OBIC if it is OBIC with respect to some profile of generic priors.

3 Equivalence of G-OBIC and DSIC under Unanimity

We now discuss the implication of G-OBIC along with unanimity. Initially, we discuss this implication in the unrestricted domain. To do so, we need the following standard definition of unanimity. At any preference ordering P, we will denote by $P(k)$ the k-th ranked alternative according to P.

Definition 7 *An scf $f : \mathcal{D}^n \to A$ is* **unanimous** *if for every $\mathbf{P} \equiv (P_1, \ldots, P_n)$ with $P_1(1) = \ldots = P_n(1)$, we have $f(\mathbf{P}) = P_1(1)$.*

An scf f is a dictatorship if there exists an agent (called the dictator) i such that for every $\mathbf{P} \in \mathcal{D}^n$, $f(\mathbf{P}) = P_i(1)$. Gibbard (1973) and Satterthwaite (1975) showed that if $|A| \geq 3$ and $\mathcal{D} = \mathcal{P}$, then an scf is DSIC and unanimous if and only if it is a dictatorship. The following result from Majumdar and Sen (2004) extends the Gibbard-Satterthwaite theorem in the unrestricted domain.

Theorem 1 **(Majumdar and Sen (2004))** *Let $|A| \geq 3$ and $f : \mathcal{P}^n \to A$ be a unanimous scf, where $\mathcal{P}$ is the unrestricted domain. Then, f is G-OBIC if and only if it is DSIC.*

We must point out that the proof in Majumdar and Sen (2004) is direct. Instead of showing that every unanimous and G-OBIC mechanism is DSIC, and then using the Gibbard-Satterthwaite theorem to conclude dictatorship, they directly prove that G-OBIC and unanimity imply dictatorship. Hence, one does not obtain any intuition from their proof whether Theorem 1 will hold in other domains.

We now show how relaxing the assumptions in Theorem 1 alters the equivalence between G-OBIC and DSIC.

3.1 Two Alternatives

We start off by giving an example of an scf with two alternatives and three agents which is unanimous and G-OBIC but not DSIC. This shows that Theorem 1 fails if $|A| = 2$.

EXAMPLE **1** Let $A = \{a, b\}$ and $N = \{1, 2, 3\}$. The scf f we consider is defined below formally. If all the agents have the same top ranked alternative, then f picks that alternative. Else, f picks the alternative which is top ranked for less number of agents.

Formally, given a preference profile $\mathbf{P} \equiv (P_1, P_2, P_3)$, alternative a is the **loser** at $\mathbf{P}$ if $|\{i \in N : bP_ia\}| > |\{i \in N : aP_ib\}|$. Else, b is the loser at $\mathbf{P}$. Since there are odd number of agents, the loser is well defined. We denote the loser at profile $\mathbf{P}$ as $\kappa(\mathbf{P})$.

Now, the scf $\bar{f}$ is defined as follows. For any preference profile $\mathbf{P} \equiv (P_1, P_2, P_3)$, define

$$\bar{f}(\mathbf{P}) = \begin{cases} P_1(1) & \text{if } P_1(1) = P_2(1) = P_3(1) \\ \kappa(\mathbf{P}) & \text{otherwise} \end{cases}$$

Clearly, $\bar{f}$ is unanimous. We next argue that $\bar{f}$ is G-OBIC. Since there are only two alternatives, we only need to show that $\pi_i^{\bar{f}}(P_i(1), P_i) \geq \pi_i^{\bar{f}}(P_i(1), P_i')$ for all $i \in N$, for all P_i, P_i'. For simplicity, for every agent $i \in N$, we will denote the preference ordering where a is top ranked as P_i and the preference ordering where b is top ranked as P_i'. For every $i \in N$, let the probability that agent i has type P_i be p_i and the probability that he has type P_i' be $(1 - p_i)$. Now, we can compute the interim allocation probabilities for agent 1 as follows.

If agent 1 has type P_1 (where a is top ranked), then the probability that b will be the outcome in $\bar{f}$ is the probability that agents 2 and 3 have different types. This probability is exactly $p_2(1 - p_3) + p_3(1 - p_2) = p_2 + p_3 - 2p_2p_3$. Hence, if agent 1 has type P_1, then the probability that a will be the outcome in $\bar{f}$ is $1 - p_2 - p_3 + 2p_2p_3$. An analogous calculation can be used to compute interim allocation probabilities when agent 1 has type P_1'. This is summarized in Table 1.

	P_1	P_1'
$\pi_1^{\bar{f}}(a, \cdot)$	$1 - p_2 - p_3 + 2p_2p_3$	$p_2 + p_3 - 2p_2p_3$
$\pi_1^{\bar{f}}(b, \cdot)$	$p_2 + p_3 - 2p_2p_3$	$1 - p_2 - p_3 + 2p_2p_3$

Table 1: Interim allocation probabilities of agent 1 in $\bar{f}$

The interim allocation probabilities of agents 2 and 3 can be computed in an analogous manner. It is easily seen from Table 1 that OBIC constraints can be satisfied for agent 1 if and only if $1 - p_2 - p_3 + 2p_2p_3 \geq p_2 + p_3 - 2p_2p_3$, which is equivalent to requiring that $(1 - 2p_2)(1 - 2p_3) \geq 0$. Collecting the OBIC constraints for all agents, we can then conclude that $\bar{f}$ is OBIC if and only if priors satisfy

$$(1-2p_2)(1-2p_3) \geq 0$$
$$(1-2p_1)(1-2p_3) \geq 0$$
$$(1-2p_1)(1-2p_2) \geq 0$$

This is possible if and only if either $p_1, p_2, p_3 \in (0, 0.5)$ or $p_1, p_2, p_3 \in (0.5, 1)$. To see why p_1, p_2, p_3 can be picked such that the priors become generic, note that the set of priors satisfying either $p_1, p_2, p_3 \in (0, 0.5)$ or $p_1, p_2, p_3 \in (0.5, 1)$ is a subset of $(0, 1)^3$ with a non-empty interior. Hence, it will have a non-empty intersection with the set of generic priors. [6]

This example also illustrates that Theorem 1 is not true if we do not assume unanimity. This is because even when there are more than three alternatives we can define the scf $\bar{f}$ by considering any two alternatives $a, b \in A$. Such an scf will not be unanimous any more (with range two), but it will be G-OBIC.

3.2 Single Peaked Domain

We now consider the single peaked domain which is a smaller domain than the unrestricted domain. Single peaked domain is an important domain restriction in strategic voting literature with applications in political economy and other disciplines. It is a domain where existence of anonymous, unanimous, and DSIC scfs is guaranteed (Moulin, 1980), and thus, allows one to escape the negative consequences of the Gibbard-Satterthwaite theorem.

In this domain, we give an example of an scf with three alternatives and three agents that is G-OBIC, unanimous, *anonymous* but not DSIC. [7]

The single peaked domain is defined as follows. Let $\succ$ be a strict linear order of the set of alternatives A.

Definition 8 *A preference ordering P is single peaked with respect to $\succ$ if for every $b, c \in A$ with $P(1) \succ b \succ c$ or $c \succ b \succ P(1)$, we have bPc.*

Our scf in this domain is an extension of the scf we considered for the two alternatives case. We assume that $A = \{a, b, c\}$ and $N = \{1, 2, 3\}$ - we will later clarify why at least three agents are required. Suppose the preferences are single peaked with respect to the strict linear order $\succ$ given by $a \succ b \succ c$. The set of all single peaked preference orderings with respect to $\succ$ is denoted by $\mathcal{S}$. For this example, the domain $\mathcal{S}$ is shown in Table 2, where each column is a preference in $\mathcal{S}$. Before defining the scf, we note that a and b are the

[6] For instance, one can verify that $p_1 = 0.49, p_2 = 0.47, p_3 = 0.43$ results in generic priors.

[7] Informally, anonymity requires that if we permute the preferences of agents and consider the new profile of preferences, the outcome at the new profile must be the same as the old one.

a	b	b	c
b	a	c	b
c	c	a	a

Table 2: The single peaked domain

two "left-most" alternatives in $\succ$. Our scf considers the number of agents who prefer a to b and the number of agents who prefer b to a. We say a is a **loser** in $\{a, b\}$ at $\mathbf{P}$ if

$$|\{i \in N : aP_ib\}| < |\{i \in N : bP_ia\}|.$$

Else, we say b is a loser in $\{a, b\}$ at $\mathbf{P}$. We denote by $\kappa^{a,b}(\mathbf{P})$ the loser in $\{a, b\}$ at $\mathbf{P}$.

Now, the scf f^* is defined as follows. For any preference profile $\mathbf{P} \in \mathcal{S}^3$, define

$$f^*(\mathbf{P}) = \begin{cases} P_1(1) & \text{if } P_1(1) = P_2(1) = P_3(1) \\ a & \text{if } \neg(P_1(1) = P_2(1) = P_3(1)) \text{ and } aP_ib \text{ for all } i \in N \\ b & \text{if } \neg(P_1(1) = P_2(1) = P_3(1)) \text{ and } bP_ia \text{ for all } i \in N \\ \kappa^{a,b}(\mathbf{P}) & \text{otherwise,} \end{cases}$$

where we used the notaion $\neg(P_1(1) = P_2(1) = P_3(1))$ to mean that the top ranked alternatives of agents are not the same ("logical not" operation). The following proposition shows that f^* is G-OBIC.

Proposition 1 *The scf f^* is unanimous, anonymous, and G-OBIC, but not DSIC.*

The proof of Proposition 1 is given in Appendix A. It builds on the ideas developed in the previous section for establishing $\bar{f}$ as a G-OBIC scf.

4 Equivalence of Locally G-OBIC and DSIC Mechanisms

Although we constructed a non-DSIC scf f^* in the single peaked domain and showed it to be unanimous and G-OBIC in Proposition 1, we would now like to point out that f^* violates a mild monotonicity property. Consider the profiles of preferences shown in Table 3. By definition, $f^*(P_1, P_2, P_3) = b$ and $f^*(P_1', P_2, P_3) = a$. Notice that the preferences of agents 2 and 3 did not change across the two profiles. Moreover, the only change in agent 1's preference is that a and b swapped their positions: a is top ranked and b second ranked in P_1 and b becomes top ranked and a second ranked P_1'. This means, even though the position of b improves in agent 1's ranking and the scf was picking b in the starting profile, the scf does not pick b any more. We call this violated property elementary monotonicity. We show below that if a unanimous and G-OBIC scf is not DSIC, then it must fail this property in a large class of domains.

P_1	P_2	P_3	P_1'	P_2	P_3
a	a	b	b	a	b
b	b	a	a	b	a
c	c	c	c	c	c

Table 3: Violation of elementary monotonicity by f^*

To define elementary monotonicity formally, we introduce some notation. Consider an agent i and two alternatives $a, b \in A$. Suppose P_i is a preference ordering such that $P_i(k) = a$ and $P_i(k+1) = b$. Now, consider P_i' such that $P_i'(k+1) = a$, $P_i'(k) = b$, and $P_i'(j) = P_i(j)$ for all $j \notin \{k, k+1\}$. In other words, a and b are consecutively ranked in P_i, and P_i' is constructed by swapping only their positions. In this case, we say that P_i' is a (a,b)-*swap* of P_i. Note that if P_i' is an (a,b)-swap of P_i, then the position of b improves from P_i to P_i'. Hence, (a,b)-swap is different from (b,a)-swap.

Definition 9 *An scf f satisfies* **elementary monotonicity** *if for every $i \in N$, every $P_{-i} \in \mathcal{D}^{n-1}$, and every $P_i, P_i' \in \mathcal{D}$ such that P_i' is a (a,b)-swap of P_i for some $a, b \in A$ and $f(P_i, P_{-i}) = b$, we have $f(P_i', P_{-i}) = b$.*

As discussed earlier, a large class of scfs satisfy elementary monotonicity. Moulin (1983) contains an extensive discussion on the scfs that satisfy elementary monotonicity - see also Majumdar and Sen (2004). Notable examples include all the scoring rules (like Borda rule, plurality rule) and all the Condorcet rules (rules that pick the majority winner whenever it exists, e.g., the Kramer rule, the Copeland rule, the Top-cycle rule, the uncovered set rule).

We now define the class of restricted domains we consider. For this, we first introduce the notion of local incentive compatibility.

Definition 10 *An scf f is* **locally dominant strategy incentive compatible (LDSIC)** *if for every $i \in N$, every $P_{-i} \in \mathcal{D}^{n-1}$, and every $P_i \in \mathcal{D}$ there exists no $P_i' \in \mathcal{D}$ such that P_i' is an (a,b)-swap of P_i for some $a, b \in A$ and*

$$f(P_i', P_{-i}) P_i f(P_i, P_{-i}).$$

Local DSIC only prevents manipulations across types which are *swaps* of each other. Using the notion of LDSIC, we now define a class of domains.

Definition 11 *A domain $\mathcal{D}$ is a* **local domain** *if every LDSIC $f : \mathcal{D}^n \to A$ in that domain is also DSIC.*

The role of local domain and related discussions are done after we state our main result. To be able to state our main result, we need to define a local version of OBIC.

Definition 12 *An scf f is* **locally ordinally Bayesian incentive compatible (LOBIC)** *with respect to profile of priors $\{\mu_i\}_{i\in N}$ if for every $i \in N$, for every $P_i, P'_i \in \mathcal{D}$ such that P'_i is an (x,y)-swap of P_i for some $x, y \in A$, and for every $a \in A$, we have*

$$\sum_{b\in B(a,P_i)} \pi_i^f(b, P_i) \geq \sum_{b\in B(a,P_i)} \pi_i^f(b, P'_i).$$

An scf f is **G-LOBIC** *if there exists some profile of generic priors $\{\mu_i\}_{i\in N}$ such that f is LOBIC with respect to $\{\mu_i\}_{i\in N}$.*

Clearly, if f is OBIC, then it is also LOBIC. In general, LOBIC is a very weak incentive compatibility requirement since it requires only a small subset of incentive constraints to hold. Further, LOBIC may not imply OBIC even in the local domain - local domain only requires that LDSIC implies DSIC. We now state the main result of the paper.

Theorem 2 *Suppose $\mathcal{D}$ is a local domain and $f : \mathcal{D}^n \to A$ is an scf on this domain. Then, the following statements are equivalent.*

1. *f is G-LOBIC and satisfies elementary monotonicity.*
2. *f is DSIC.*

Note that Theorem 2 does not assume unanimity. It also does not require any condition on the range of the scf. Before giving the proof of Theorem 2, we make some remarks.

- G-LOBIC and LDSIC. The proof of Theorem 2 reveals another interesting result. In *any* domain $\mathcal{D}$, G-LOBIC and elementary monotonicity are equivalent to LDSIC - see Lemma 2 in the proof of Theorem 2 below. The proof of Theorem 2 then follows by the definition of local domain. Thus, if we use local incentive constraints, then our main result becomes very strong since it works in any domain of preferences.
- Which domains are local? There are two recent papers that identify many interesting local domains. Sato (2013) identifies necessary and sufficient conditions for a domain to be a local domain. One of his sufficient conditions is a *minimal connectedness* property that requires that we should be able to go from one preference ordering to another in a domain by performing swaps in a "minimal" way - we refer the reader to Sato (2013) for details. [8] He shows that this necessary condition is satisfied by the unrestricted domain and the single peaked domain. His condition also holds in some generalizations of the single

[8]The proof of Theorem 4 discusses these conditions in some detail.

Carroll (2012) considers the same question, but his results are slightly general since he allows his scfs to be randomized (Sato (2013), like us, restricts attention to deterministic scfs). He identifies specific domains that are local - they include the unrestricted domain, the single peaked domain, and the *successive single crossing* domain. [9]

These papers show that there are important restricted domains, like the single peaked domain and the single crossing domain, that are local. [10] However, there are local domains not covered in these papers, and our result works in those domains too.

- Elementary monotonicity and uniform priors. Elementary monotonicity plays an important role in the analysis of OBIC scfs with uniform priors and correlated priors. Majumdar and Sen (2004) show that when agents have *uniform* priors, every *neutral* scf satisfying elementary monotonicity is OBIC in the unrestricted domain - we discuss this result further in Section 6. Thus, they show that a large class of scfs that are not DSIC are OBIC with uniform priors in the unrestricted domain. A similar result is shown with correlated priors in Bhargava et al. (2014). Theorem 2 shows that elementary monotonicity brings us back to DSIC scfs under independent generic priors in a large class of domains. We will also show later that in the unrestricted domain, elementary monotonicity is weaker than unanimity for a G-LOBIC scf. Hence, Theorem 2 is a stronger than Theorem 1 in the unrestricted domain.

- Independence of G-LOBIC and elementary monotonicity. The scf f^* discussed in Proposition 1 is an example of an scf that satisfies G-LOBIC but fails elementary monotonicity. Hence, we cannot hope to drop elementary monotonicity in Theorem 2. However, we will show later that we can replace elementary monotonicity with other properties in specific domains. In particular, unanimity and G-LOBIC will imply DSIC in the unrestricted domain and a significantly weaker version of elementary monotonicity along with unanimity will work in the single peaked domain.

4.1 Proof of Theorem 2

The proof is done by establishing two important lemmas. We start by identifying a property that is implied by G-LOBIC.

Definition 13 *An scf f satisfies* **swap monotonicity** *if for every $i \in N$, for every $P_i, P_i' \in \mathcal{D}$, where P_i' is an (a,b)-swap of P_i, we have for every $P_{-i} \in \mathcal{D}^{n-1}$,*

[9] For a definition and analysis of the single crossing domain, see Saporiti (2009). Successive single crossing is a class of single crossing domains identified in Carroll (2012).

[10] A prominent example of a domain that is not local is the multidimensional separable domain studied in Barbera et al. (1993); Le Breton and Sen (1999).

- $f(P_i', P_{-i}) = f(P_i, P_{-i})$ *if* $f(P_i, P_{-i}) \notin \{a, b\}$,
- $f(P_i', P_{-i}) \in \{a, b\}$ *if* $f(P_i, P_{-i}) \in \{a, b\}$.

Our first claim shows the necessity of swap monotonicity.

LEMMA **1** *If an scf is G-LOBIC, then it satisfies swap monotonicity.*

Proof: Let f be an LOBIC scf with respect to independent generic priors $\{\mu_i\}_{i \in N}$. For this, consider agent $i \in N$, and pick two preference orderings P_i and P_i' such that P_i' is an (a, b) swap of P_i. By definition $P_i(k) = a, P_i(k+1) = b$ and $P_i'(k+1) = a, P_i'(k) = b$ for some k and $P_i(j) = P_i'(j)$ for all $j \notin \{k, k+1\}$. We will do the proof in three steps.

STEP 1. Consider an alternative $x \in A \setminus \{a, b\}$ such that $P_i(k') = P_i'(k') = x$, where $k' < k$. We will show that $\{P_{-i} \in \mathcal{D}^{n-1} : f(P_i, P_{-i}) = x\} = \{P_{-i} \in \mathcal{D}^{n-1} : f(P_i', P_{-i}) = x\}$. We do this using induction on k'. If $k' = 1$, by observing that $P_i(k'') = P_i'(k'')$ for all $k'' < k$, LOBIC implies that

$$\sum_{P_{-i}:f(P_i,P_{-i})=P_i(1)} \mu(P_{-i}) \geq \sum_{P_{-i}:f(P_i',P_{-i})=P_i(1)} \mu(P_{-i})$$
$$\sum_{P_{-i}:f(P_i',P_{-i})=P_i'(1)} \mu(P_{-i}) \geq \sum_{P_{-i}:f(P_i,P_{-i})=P_i'(1)} \mu(P_{-i}).$$

Combining these inequalities, we get

$$\sum_{P_{-i}:f(P_i,P_{-i})=P_i(1)} \mu(P_{-i}) = \sum_{P_{-i}:f(P_i',P_{-i})=P_i(1)} \mu(P_{-i}).$$

Since priors are generic, we get that

$$\{P_{-i} : f(P_i, P_{-i}) = P_i(1)\} = \{P_{-i} : f(P_i', P_{-i}) = P_i(1)\}.$$

Now, suppose the claim is true for all $k'' < k'$. Notice that the top k' alternatives in P_i and P_i' are the same - denote this set as B. Now, we apply LOBIC to top k' alternatives in P_i and P_i' to get

$$\sum_{P_{-i}:f(P_i,P_{-i})\in B} \mu_i(P_{-i}) \geq \sum_{P_{-i}:f(P_i',P_{-i})\in B} \mu_i(P_{-i})$$
$$\sum_{P_{-i}:f(P_i',P_{-i})\in B} \mu_i(P_{-i}) \geq \sum_{P_{-i}:f(P_i,P_{-i})\in B} \mu_i(P_{-i}).$$

Using genericity of μ_i gives us

$$\{P_{-i} : f(P_i, P_{-i}) \in B\} = \{P_{-i} : f(P_i', P_{-i}) \in B\}.$$

Using the induction hypothesis, we have for all $k'' < k'$,

$$\{P_{-i} : f(P_i, P_{-i}) = P_i(k'')\} = \{P_{-i} : f(P_i', P_{-i}) = P_i(k'')\}.$$

Hence, we get

$$\{P_{-i} : f(P_i, P_{-i}) = P_i(k')\} = \{P_{-i} : f(P_i', P_{-i}) = P_i(k')\}.$$

Step 2. In this step, we show that $\{P_{-i} : f(P_i, P_{-i}) \in \{a, b\}\} = \{P_{-i} : f(P_i', P_{-i}) \in \{a, b\}\}$. Applying LOBIC, we get

$$\sum_{P_{-i}:f(P_i,P_{-i})\in B(b,P_i)} \mu_i(P_{-i}) \geq \sum_{P_{-i}:f(P_i',P_{-i})\in B(b,P_i)} \mu_i(P_{-i})$$
$$\sum_{P_{-i}:f(P_i',P_{-i})\in B(a,P_i')} \mu_i(P_{-i}) \geq \sum_{P_{-i}:f(P_i,P_{-i})\in B(a,P_i')} \mu_i(P_{-i}).$$

Since $B(b, P_i) = B(a, P_i')$, by genericity we get

$$\{P_{-i} : f(P_i, P_{-i}) \in B(b, P_i)\} = \{P_{-i} : f(P_i', P_{-i}) \in B(b, P_i)\}.$$

By Step 1, this implies that $\{P_{-i} : f(P_i, P_{-i}) \in \{a, b\}\} = \{P_{-i} : f(P_i', P_{-i}) \in \{a, b\}\}$.

Step 3. Consider an alternative $x \in A \setminus \{a, b\}$ such that $P_i(k') = P_i'(k') = x$, where $k' > k + 1$. Using the facts in Steps 1 and 2, we can mimic the method in Step 1 to show that $\{P_{-i} \in \mathcal{D}^{n-1} : f(P_i, P_{-i}) = x\} = \{P_{-i} \in \mathcal{D}^{n-1} : f(P_i', P_{-i}) = x\}$.

Steps 1, 2, and 3 show that f satisfies swap monotonicity. ■

Note that Lemma 1 holds in any arbitrary domain. We now use this to prove the following result.

Lemma **2** *Suppose $\mathcal{D}$ is any domain and $f : \mathcal{D}^n \to A$ is an scf on this domain. Then, the following statements are equivalent.*

1. *f is G-LOBIC and satisfies elementary monotonicity.*
2. *f is LDSIC.*

Proof: First, we show that a LDSIC scf f satisfies elementary monotonicity. To see this, consider $i \in N$ and P_{-i}. Let P_i and P_i' be two preferences in $\mathcal{D}$ such that P_i' is an (a, b)-swap of P_i and $f(P_i, P_{-i}) = b$. Assume for contradiction that $f(P_i', P_{-i}) = c \neq b$. If cP_ib, then i can manipulate from P_i to P_i'. If bP_ic, then, by construction, $bP_i'c$, and again, agent i can manipulate from P_i' to P_i. This is a contradiction.

Further, an LDSIC scf is LOBIC with respect to any priors, and hence, G-LOBIC. Now, we show that if $f : \mathcal{D}^n \to A$ is G-LOBIC and satisfies elementary monotonicity, then it is LDSIC. Fix an agent i and P_{-i}. Now, pick two preference orderings P_i and P'_i such that P'_i is an (a,b) swap of P_i, where $P_i(k) = a, P_i(k+1) = b$ and $P'_i(k+1) = a, P'_i(k) = b$ for some k. Suppose $f(P_i, P_{-i}) = x$ and $f(P'_i, P_{-i}) = y$. Suppose $P_i(k') = x$. If $k' < k$ or $k' > k+1$, then by Lemma 1, we have $y = x$ (swap monotonicity). So, agent i cannot manipulate from P_i to P'_i. If $k' = k$, then $x = a$, and by swap monotonicity $f(P'_i, P_{-i}) \in \{a, b\}$. Since aP_ib, agent i cannot manipulate from P_i to P'_i. The other possibility is $k' = k+1$. In that case, $x = b$, and elementary monotonicity ensures that $f(P'_i, P_{-i}) = b$. Hence, agent i cannot manipulate from P_i to P'_i. This shows that f is LDSIC. ■

Proof of Theorem 2: This follows from Lemma 2 and the definition of a local domain (every LDSIC scf is a DSIC scf).

5 G-LOBIC with Unanimity and DSIC Mechanisms

In this section, we investigate if we can replace/weaken elementary monotonicity in Theorem 2 by other reasonable conditions. One familiar axiom used in strategic voting literature is Pareto efficiency.

Definition 14 *An scf f is* **Pareto efficient** *if for every $\mathbf{P} \in \mathcal{D}^n$ there exists no $b \in A$ such that $bP_if(\mathbf{P})$ for all $i \in N$.*

We first show that if there are two agents, then elementary monotonicity is implied in any domain by G-LOBIC and Pareto efficiency.

Proposition **2** *Suppose $n = 2$. Then, every Pareto efficient and G-LOBIC scf satisfies elementary monotonicity.*

Proof: Let f be a Pareto efficient scf that is LOBIC with respect to generic priors. Pick a profile (P_1, P_2). Suppose $f(P_1, P_2) = b$ and aP_1b. We need to show that if P'_1 is an (a,b)-swap of P_i, then $f(P'_1, P_2) = b$. Since aP_1b and $f(P_1, P_2) = b$, Pareto efficiency implies that bP_2a. But bP'_1a and Pareto efficiency implies that $f(P'_1, P_2) \neq a$. Since f is G-LOBIC, by swap monotonicity (Lemma 1), $f(P'_1, P_2) \in \{a, b\}$, and hence, $f(P'_1, P_2) = b$. ■

Note that Proposition 2 holds in every domain $\mathcal{D}$. This gives us the following simple corollary.

Corollary **1** *Let $n = 2$ and $\mathcal{D}$ be a local domain. Then, every Pareto efficient and G-LOBIC scf $f : \mathcal{D}^n \to A$ is DSIC.*

Proof: This follows from Theorem 2 and Proposition 2. ■

Corollary 1 is not true if $n > 2$ - the scf f^* defined in Proposition 1 is an example of a Pareto efficient scf that is G-LOBIC in the single peaked domain (a local domain) but not DSIC. Corollary 1 also explains why we needed at least three agents for the scf in Proposition 1.

In some specific domains, we can replace Pareto efficiency by unanimity in Proposition 2 and Corollary 1. For instance, in the proofs of Theorems 3 and 4, we show that G-LOBIC and unanimity implies Pareto efficiency in the unrestricted domain and in the single peaked domain respectively. This implies that if there are two agents and the domain is either the unrestricted domain or the single peaked domain, then every unanimous G-LOBIC scf is a DSIC scf. We now discuss the implications of assuming unanimity when $n \geq 3$.

5.1 Unrestricted Domain

We now explore the implication of G-LOBIC along with unanimity when $n \geq 3$. Our first result assumes the unrestricted domain.

THEOREM **3** *Suppose $|A| \geq 3$ and $f : \mathcal{P}^n \to A$ is a G-LOBIC scf, where $\mathcal{P}$ is the unrestricted domain. If f satisfies unanimity, then it satisfies elementary monotonicity. Hence, if f is G-LOBIC and unanimous, then it is DSIC.*

The proof of this theorem is given in Appendix A. Note that Theorem 3 is a strengthening of Theorem 1 since we use G-LOBIC instead of G-OBIC. However, one can still deduce the result of Theorem 3 from existing results in the literature as follows. Carroll (2012) notes that his results also hold if we consider Bayesian incentive compatibility. Since he shows local incentive compatibility implies full incentive compatibility in the unrestricted domain, we can conclude that if f is LOBIC (with respect to *any* prior) then it is OBIC. We can then use Theorem 1 to conclude that if f is G-LOBIC and unanimous, then it must be a dictatorship.

However, we give an independent proof in Appendix A. Our proof establishes directly that in the unrestricted domain, unanimity and G-LOBIC imply elementary monotonicity, and then the result follows by using Theorem 2 and noting that the unrestricted domain is a local domain. Since the proof of Theorem 1 in Majumdar and Sen (2004) directly establishes dictatorship (using induction on the number of agents), our proof provides an alternate and stronger version of their result.

Theorem 3 implies that under G-LOBIC, elementary monotonicity is weaker than unanimity in the unrestricted domain. This implies that our main result in Theorem 2 is a stronger result than Theorem 1.

P_1	P_2	P_3	P_1'	P_2	P_3
a	a	b	b	a	b
b	b	a	a	b	a
c	c	c	c	c	c
d	d	d	d	d	d

Table 4: Weak elementary monotonicity

5.2 Single Peaked Domain

Unfortunately, Theorem 3 heavily relies on the unrestricted domain assumption and Proposition 1 shows that it cannot be extended to restricted domains like the single peaked domain. We are able to show that the conclusions of Theorem 2 can nevertheless be obtained in the single peaked domain by assuming unanimity and a significantly weaker version of elementary monotonicity. We discuss this next.

As before, we will assume that $\mathcal{S}$ is the single peaked domain with respect to an ordering $\succ$ over alternatives in A. We now introduce a weakening of the elementary monotonicity condition.

Definition 15 *An scf $f : \mathcal{S}^n \rightarrow A$ satisfies* **weak elementary monotonicity** *if for every $i \in N$, for every $P_i, P_i' \in \mathcal{S}$ such that P_i' is a $(P_i(1), P_i(2))$-swap of P_i and for every P_{-i} such that $f(P_i, P_{-i}) = P_i(2)$, $P_j \in \{P_i, P_i'\}$ for all $j \neq i$, and $P_j \neq P_k$ for some $j, k \in N \setminus \{i\}$, we have $f(P_i', P_{-i}) = f(P_i, P_{-i})$.*

Weak elementary monotonicity applies to very specific type profiles. First, it requires that P_i' is an (a, b)-swap of P_i, where $P_i(1) = a, P_i(2) = b$, i.e., the swaps happen at the top. Second, it requires that every agent in $N \setminus \{i\}$ must have either P_i or P_i' as his type. Finally, agents in $N \setminus \{i\}$ cannot have the same type, i.e., there is at least one agent who has P_i as his type and at least one agent who has P_i' as his type.

A typical pair of type profiles where weak elementary monotonicity can be applied is shown in Table 4. We assume $A = \{a, b, c, d\}$ and $a \succ b \succ c \succ d$ and $n = 3$. Profiles (P_1, P_2, P_3) and (P_1', P_2, P_3) in Table 4 differ in agent 1's preference and P_1' is a (a, b)-swap of P_i. Also, notice that a and b are neighbors in $\succ$, and $P_2 = P_1, P_3 = P_1'$ (hence, $P_2 \neq P_3$). Weak elementary monotonicity applies to such profiles and requires that if $f(P_1, P_2, P_3) = b$, then $f(P_1', P_2, P_3) = b$.

The following result shows that weak elementary monotonicity can be used with unanimity in the single peaked domain to get a counterpart of Theorem 2.

Theorem 4 *Suppose $f : \mathcal{S}^n \rightarrow A$ is a unanimous scf. Then, the following statements are equivalent.*

1. f *is G-LOBIC and satisfies weak elementary monotonicity.*

2. f *is DSIC.*

The proof of Theorem 4 is in Appendix A. Since weak elementary monotonicity is an extremely weak condition, this result shows how little is required on top of unanimity to get the counterpart of Theorem 3 in the single peaked domain. We believe that in specific restricted domains, we can find similar weakening of elementary monotonicity that may be used in conjunction with unanimity to get a result similar to Theorem 4.

The result in Theorem 4 can also be extended to some extensions of single peaked domain. For instance, Demange (1982) defines a notion of single peakedness on a *tree graph*, which requires single peakedness along paths of a tree graph whose vertices are alternatives. Our result can be easily extended to such a domain.

6 Discussions

We discuss some extensions and implications of our result.

Weaker OBIC-DSIC equivalence. The G-OBIC and DSIC equivalence established in this paper is very strong - under some reasonable assumptions, every G-OBIC mechanism is a DSIC mechanism. However, one could consider the following weaker notion of equivalence.

Definition **16** *A pair of scfs* $f : \mathcal{D}^n \to A$ *and* $f' : \mathcal{D}^n \to A$ *are* **weakly equivalent** *if for every* $i \in N$, *for every* $P_i \in \mathcal{D}$, *and for every* $a \in A$,

$$\pi_i^f(a, P_i) = \pi_i^{f'}(a, P_i).$$

We can then ask if for every OBIC mechanism f, there exists a DSIC mechanism f' that is *weakly equivalent* to f. Of course, one can ask this question under additional conditions like unanimity or elementary monotonicity. Since, this is a weaker requirement, it is plausible that we can get the weak equivalence in a larger class of domains than identified in Theorem 2. However, the answer does not look very clear.

In Appendix B, we show that for the scf f^* considered in Proposition 1, there is no DSIC, unanimous, and anonymous scf that is weakly equivalent to it. The scf f^* is an anonymous and unanimous scf. Further, Proposition 1 showed that f^* is OBIC with respect to a large class of priors (including generic priors). This illustrates that we can have unanimous and anonymous scfs that are OBIC in the single peaked domain for which there are no weakly equivalent DSIC, unanimous, and anonymous scf.

We point out that in cardinal environments with quasilinear preferences over transfers, more specifically, in the single object auction problem and in some specific *one dimensional* mechanism design problems, the weak equivalence question has been addressed in Manelli and Vincent (2010) and Gershkov et al. (2013). They show that for every Bayesian incentive compatible single object auction mechanism, there exists a DSIC single object auction mechanism that generates the same interim allocation probability of winning the object for every agent. [11]

Apart from the weak equivalence notion, there are many other subtle differences between these papers and ours. First, they look at cardinal mechanisms with quasilinearity, whereas ours is a completely ordinal environment without transfer. Second, they allow for mechanisms to be randomized - their result will fail if we restrict attention to only deterministic mechanisms (an example is available upon request). On the other hand, our results are for deterministic mechanisms and we do not know if our result will continue to hold if we consider randomized voting mechanisms. Finally, our results require independent generic priors but the results in these papers only make the independence assumption - see Kushnir (2014) for extension of the results in Manelli and Vincent (2010); Gershkov et al. (2013) to correlated priors.

Chung and Ely (2007); Chen and Li (2014) address a related issue in quasilinear environment with transfers. They consider a mechanism designer who has no specific information about beliefs of agents. They show that if the designer wants to maximize the worst case (over a set of possible beliefs) revenue from the mechanisms, then he should use a dominant strategy mechanism in the single object auction and other specific settings. These results provide a different foundation for dominant strategy mechanisms in their cardinal quasilinear setting than ours - see also Borgors (2013).

Necessary conditions on domains. We do not know if our main result can hold in domains that are not local. We have used the idea of local incentive constraints extensively in our proof. Indeed, Lemma 2 established equivalence between G-LOBIC and G-LDSIC under elementary monotonicity in any arbitrary domain. Using this and the results in Sato (2013), we can easily identify necessary conditions on the domain for the equivalence of G-LOBIC and DSIC. For instance, consider the following condition from Sato (2013).

DEFINITION **17** *A domain $\mathcal{D}$ is* **connected** *if for every $P, P' \in \mathcal{D}$, there exists a sequence of distinct types $(P = P^0, P^1, \dots, P^k, P^{k+1} = P')$ such that for all $j \in \{0, 1, \dots, k\}$, $P^j \in \mathcal{D}$ and P^{j+1} is a (x, y)-swap of P^j for some $x, y \in A$.*

[11] They also consider an even weaker notion of equivalence, where they compare interim expected utilities of agents.

Connectedness requires that we can go from any type in a domain to another type using a sequence of distinct types that are generated by swaps. Sato (2013) gives many examples of domains, which include the unrestricted domain and the single peaked domain, that are connected. He shows that if a domain is a local domain, then it must be connected. We can now prove the following lemma.

LEMMA **3** *Suppose $\mathcal{D}$ is a domain such that every G-LOBIC scf $f : \mathcal{D}^n \to A$ satisfying elementary monotonicity is DSIC. Then, $\mathcal{D}$ is connected.*

Proof: By Lemma 2, any G-LOBIC scf $f : \mathcal{D}^n \to A$ satisfying elementary monotonicity is LDSIC. Since, LDSIC implies DSIC in $\mathcal{D}$, by Sato (2013), $\mathcal{D}$ is connected. ■

Since the separable domains in Le Breton and Sen (1999) are not connected, a consequence of this lemma is that G-LOBIC and elementary monotonicity does not imply DSIC in these domains. We do not have any necessary conditions on domains where G-OBIC along with elementary monotonicity will imply DSIC.

Indirect implications. We can use our results to strengthen some results on DSIC mechanisms. We list some of these results below.

- **Description of mechanisms.** In many domains, the class of DSIC mechanisms have been explicitly characterized. For instance, in the single peaked domain, Moulin (1980) shows that the only DSIC, anonymous and unanimous mechanisms are *generalized median voting* mechanisms. [12] Our Theorem 4 will imply that in the single peaked domain, the only G-OBIC, unanimous, and anonymous mechanisms satisfying weak elementary monotonicity are generalized median voting mechanisms.

  Similarly, Saporiti (2009) shows that in the maximal single crossing domain, the only DSIC, anonymous, and unanimous mechanisms are generalized median voting mechanisms. Since the maximal single crossing domain is a local domain (Carroll, 2012), our Theorem 2 will imply that in the maximal single crossing domain, the only G-OBIC, unanimous, and anonymous mechanisms satisfying elementary monotonicity are generalized median voting mechanisms.

- **Group strategy-proof mechanisms.** Barbera et al. (2010) identify a large class of domains where DSIC imply the stronger property of *group strategy-proofness*. Group strategy-proofness requires that no coalition of agents can manipulate the mechanism.

[12] A generalized median voting scf is defined by locating $n-1$ *phantom peaks* at various alternatives and then taking the median of these peaks and the top ranked alternatives of n agents.

The class of such domains include many local domains (unrestricted domain, single peaked domain). Our results will imply that if any such domain is a local domain, then a G-OBIC mechanism satisfying elementary monotonicity will be a group strategy-proof mechanism.

- **Optimization.** Gershkov et al. (2014) find the expected utility maximizing DSIC, unanimous, and anonymous mechanism in the maximal single crossing domain of Saporiti (2009). Using our Theorem 2, their mechanism will also remain optimal if they search over the set of all G-OBIC mechanisms satisfying elementary monotonicity.

Uniform priors. Though uniform priors are non-generic, they play an important role in decision theory. In their paper, Majumdar and Sen (2004) show that in the unrestricted domain, there are unanimous scfs that are OBIC with respect to uniform priors but not DSIC. They show that any scf satisfying *neutrality* and elementary monotonicity is OBIC with respect to uniform priors. These conditions are mild and hence, cover many scfs.

Unfortunately, this result is no longer true in restricted domains. This is an artifact of the notion of Bayesian incentive compatibility. Suppose we have an OBIC (with respect to uniform priors) scf in a domain $\mathcal{D}$ and we consider the restriction of this scf to a smaller domain $\mathcal{D}'$. The restriction of the original scf need not be OBIC with respect to uniform priors in the smaller domain $\mathcal{D}'$.

Appendix B contains an example of an scf in the unrestricted domain that is OBIC with respect to uniform priors but its restriction to the single peaked domain is no longer OBIC with respect to uniform priors. This illustrates that unlike in the unrestricted domain, elementary monotonicity and neutrality are not sufficient conditions for an scf to be OBIC with uniform priors in restricted domains.

7 Conclusion

We have identified a large class of domains where G-LOBIC and elementary monotonicity are equivalent to DSIC. The equivalence of G-LOBIC and elementary monotonicity with LDSIC works in every domain. We also explored the consequence of using unanimity in place of elementary monotonicity. Unfortunately, except for the unrestricted domain, unanimity is not enough to guarantee similar equivalences. However, in restricted domains like the single peaked domain, we can replace elementary monotonicity by a significantly weaker condition along with unanimity to get our result. Our results are foundations in the sense that they point out an amazing robustness property of DSIC mechanisms in voting environments. To look beyond DSIC mechanisms, our results point to the use of cardinal and/or randomized BIC mechanisms in voting environments.

Appendix A: Omitted Proofs

Proof of Proposition 1

Clearly, f^* is unanimous and anonymous. However, f^* is not DSIC. To see this, consider a profile $\mathbf{P}$ such that for all agents $i \neq 3$, we have $P_i(1) = a$ - note that since a is the leftmost alternative, there is a unique preference ordering where a is top ranked. For agent 3, pick any preference ordering where b is preferred to a. As a result, $f^*(\mathbf{P}) = b$. A possible profile is shown in Table 5 - note that aP_1b. Now, fixing the preference profile of all agents except agent 1, if agent 1 reports a preference ordering P_1' such that $P_1'(1) = b, P_1'(2) = a$, then f^* will choose a - see Table 5. Hence, agent 1 can manipulate.

P_1	P_2	P_3	P_1'	P_2	P_3
a	a	b	b	a	b
b	b	a	a	b	a
c	c	c	c	c	c

Table 5: Failure of DSIC of f^*

However, we show that f^* is G-OBIC. Let the prior of each agent i be given by the map $\mu_i : \mathcal{S} \to (0,1)$. To show that f^* is OBIC with respect to $\{\mu_i\}_{i \in N}$ such that these are generic priors, we will compute the interim allocation probabilities of every agent in $\{1,2,3\}$.

If an agent $i \in \{1,2,3\}$ has type P_i then define

$$O_i^{f^*}(P_i) := \{x \in A : f^*(P_i, P_{-i}) = x \text{ for some } P_{-i}\}.$$

By definition of f^*, for every $i \in \{1,2,3\}$ and for every $P_i \in \mathcal{S}$, $O_i^{f^*}(P_i) \subseteq \{a, b, P_i(1)\}$, and if $P_i(1) \neq c$, then $O_i^{f^*}(P_i) = \{a,b\}$. Let $\bar{P}$ be the unique preference ordering where the leftmost alternative a is top ranked. Denote the probability that agent i has type $\bar{P}$ as $q_i \equiv \mu_i(\bar{P})$. Similarly, denote by $\hat{q}_i \equiv \mu_i(\hat{P})$, where $\hat{P}$ is the unique ordering where alternative c is top ranked. We fix an agent $i \in \{1,2,3\}$ and denote the other two agents in $\{1,2,3\}$ as j and k. We consider three possible cases.

Case 1. Suppose $P_i = \bar{P}$. Note that $O_i^{f^*}(P_i) = \{a,b\}$. Then his interim allocation probability for alternative a can be computed as follows. Note that $f^*(P_i, P_j, P_k) \neq a$ if P_j and P_k are such that either (aP_jb and bP_ka) or (aP_kb and bP_ja). The probability of this event is

$$q_j(1-q_k) + q_k(1-q_j).$$

Here, we used the fact that the probability that agent j has type P_j such that bP_ja is just $(1 - p_j)$ and, similarly, the probability that agent k has type P_k such that bP_ka is $(1 - q_k)$. Since $f^*(P_i, P_j, P_k) \in \{a, b\}$, the interim allocation probability of choosing a at P_i for agent i is

$$1 - q_j - q_k + 2q_jq_k,$$

and the interim allocation probability of choosing b at P_i for agent i is

$$q_j + q_k - 2q_jq_k.$$

Case 2. Suppose P_i is such that $P_i(1) = b$ - this is possible for two preference orderings. Note that $O_i^{f^*}(P_i) = \{a, b\}$. Then, his interim allocation probability for alternative b can be computed as follows. Note that $f^*(P_i, P_j, P_k) \neq b$ if P_j and P_k are such that either (aP_jb and bP_ka) or (aP_kb and bP_ja). The probability of this event is

$$q_j(1 - q_k) + q_k(1 - q_j).$$

Hence, the interim allocation probability of choosing b at P_i for agent i is

$$1 - q_j - q_k + 2q_jq_k.$$

Since $O_i^{f^*}(P_i) = \{a, b\}$, the interim allocation probability of choosing a at P_i for agent i is

$$q_j + q_k - 2q_jq_k.$$

Case 3. Suppose P_i is such that $P_i(1) = c$, i.e., $P_i = \hat{P}$. Then, $O_i^{f^*}(P_i) = \{a, b, c\}$. His interim allocation probability for alternative c can be computed straightforwardly - c is chosen if and only if $P_j(1) = P_k(1) = P_i(1) = c$. This is possible if and only if both agents j and k have the type $\hat{P}$. The probability of this event is $\hat{q}_j\hat{q}_k$. Hence, the interim allocation probability for alternative $P_i(1)$ at P_i for agent i is $\hat{q}_j\hat{q}_k$. Next, the interim allocation probability for agent i for alternative a can be computed as follows. For this, note that cP_ibP_ia. Hence, $f^*(P_i, P_j, P_k) = a$ if P_j and P_k are such that either (aP_jb and bP_ka) or (aP_kb and bP_ja) - notice that these events ensure that tops of all agents are not the same and a is not dominated by b. The probability of this event is

$$q_j(1 - q_k) + q_k(1 - q_j).$$

Hence, the interim allocation probability of choosing a at P_i for agent i is

$$q_j + q_k - 2q_jq_k.$$

Since $O_i^{f^*}(P_i) = \{a, b, P_i(1)\}$, the interim allocation probability of choosing b at P_i for agent i is

$$1 - q_j - q_k + 2q_jq_k - \hat{q}_j\hat{q}_k.$$

We enumerate all the interim allocation probabilities in Table 6 by considering the three cases for an agent $i \in \{1, 2, 3\}$ by denoting the other two agents as j and k.

	Case 1: $P_i = \bar{P}$	Case 2: $P_i(1) = b$	Case 3: $P_i = \hat{P}$
$\pi_i^{f^*}(a, P_i)$	$1 - q_j - q_k + 2q_jq_k$	$q_j + q_k - 2q_jq_k$	$q_j + q_k - 2q_jq_k$
$\pi_i^{f^*}(b, P_i)$	$q_j + q_k - 2q_jq_k$	$1 - q_j - q_k + 2q_jq_k$	$1 - q_j - q_k + 2q_jq_k - \hat{q}_j\hat{q}_k$
$\pi_i^{f^*}(c, P_i)$	0	0	$\hat{q}_j\hat{q}_k$

Table 6: Interim allocation probabilities

Now, notice from Table 6 that if agent i has type P_i such that $P_i(1) = a$, the only OBIC constraint to satisfy is

$$1 - q_j - q_k + 2q_jq_k \geq q_j + q_k - 2q_jq_k.$$

Alternatively, we must have

$$(1 - 2q_j)(1 - 2q_k) \geq 0. \tag{1}$$

This prevents any manipulation of agent i to a type in Case 2 or Case 3. Also, Inequality 1 ensures OBIC constraints when agent i has a type P_i such that $P_i(1) = b$. Finally, Inequality 1 also ensures OBIC constraints when agent i has a type P_i such that $P_i(1) = c$ - this can be verified by checking from Table 6 that the truthtelling lottery first-order stochastic dominates other lotteries as long as Inequality 1 is satisfied.

Hence, f^* is OBIC if and only if the priors of agents 1,2, and 3 satisfy

$$\begin{aligned}(1 - 2q_2)(1 - 2q_3) &\geq 0,\\ (1 - 2q_1)(1 - 2q_3) &\geq 0,\\ (1 - 2q_1)(1 - 2q_2) &\geq 0.\end{aligned}$$

This is satisfied if and only if $q_1, q_2, q_3 \in (0, 0.5)$ or $q_1, q_2, q_3 \in (0.5, 1)$. It also puts no restriction on the probabilities of orderings in $\mathcal{S} \setminus \{\bar{P}\}$. Hence, the set of priors of all agents satisfying $q_1, q_2, q_3 \in (0, 0.5)$ or $q_1, q_2, q_3 \in (0.5, 1)$ is a full dimensional subset of the set of all independent priors. As a result, it must have a non-empty intersection with the set of independent generic priors.

Proof of Theorem 3

We first show that if $f : \mathcal{P}^n \to A$ is G-LOBIC and unanimous, then it is Pareto efficient. Suppose f is G-LOBIC and unanimous but assume for contradiction that it is not Pareto efficient. For this, we consider a profile $\mathbf{P}$ such that $f(\mathbf{P}) = b$ but there exists $a \in A$ such that aP_ib for all $i \in N$. Consider an agent $i \in N$ such that $P_i(k) = a$ and $k \neq 1$. Suppose $P_i(k-1) = x$. Consider P'_i which is a (x, a)-swap of P_i. By swap monotonicity, $f(P'_i, P_{-i}) = b$. We can repeat such swaps to reach a preference ordering P''_i for agent i such that $P''_i(1) = a$ and $f(P''_i, P_{-i}) = b$. Now, we can repeat this procedure for every agent j such that $P_j(k) = a$ and $k \neq 1$ to arrive at a profile $\mathbf{P}''$ such that $f(\mathbf{P}'') = b$. But this will contradict unanimity since $P''_j(1) = a$ for all $j \in N$.

Hence, we show that any $f : \mathcal{P}^n \to A$ that is G-LOBIC and Pareto efficient must satisfy elementary monotonicity. By Theorem 2, we will be done.

To do so, we consider an agent $i \in N$, a preference profile P_{-i} of other agents, and P_i, P'_i such that P'_i is an (a, b)-swap of P_i and $f(P_i, P_{-i}) = b$. The two profiles are shown in Table 7. Notice that there are some agents in P_{-i} who prefer a to b and some prefer b to a. If all the agents in P_{-i} prefer a to b, then the outcome at (P_i, P_{-i}) cannot be b due to Pareto efficiency. Similarly, if all the agents in P_{-i} prefer b to a, the outcome at (P'_i, P_{-i}) cannot be a, and hence, must be b, which gives us the desired result.

Hence, we will assume that there is at least one agent in P_{-i} who prefers a to b and at least one agent who prefers b to a.

P_i	P_{-i}		P'_i	P_{-i}	
$\cdot$	$\ldots$	$\ldots$	$\cdot$	$\ldots$	$\ldots$
$\cdot$	$\ldots$	$\ldots$	$\cdot$	$\ldots$	$\ldots$
a	$b\ldots$	$a\ldots$	b	$b\ldots$	$a\ldots$
b	$\ldots$	$\ldots$	a	$\ldots$	$\ldots$
$\cdot$	$a\ldots$	$b\ldots$	$\cdot$	$a\ldots$	$b\ldots$
$\cdot$	$\ldots$	$\ldots$	$\cdot$	$\ldots$	$\ldots$

Table 7: Profiles (P_i, P_{-i}) and (P'_i, P_{-i})

We will now show that $f(P'_i, P_{-i}) = b$. By swap monotonicity, $f(P'_i, P_{-i}) \in \{a, b\}$. Assume for contradiction that $f(P'_i, P_{-i}) = a$. Now, we do the proof in steps. In the proof, if x and y are consecutive alternatives in an ordering, an (x, y)-swap of this ordering will be referred to as a swap.

STEP 1. We modify the profile (P_i, P_{-i}) to bring one of the alternatives not in $\{a, b\}$ (such an alternative exists since $|A| \geq 3$) just below $\{a, b\}$ for all the agents. Let $x \notin \{a, b\}$ be some alternative. If aP_jx and bP_jx for some $j \in N$, then we can do a series of swaps to lift x up such that it is just below b if aP_jb or just below a if bP_ja (note that none of these swaps will involve b). By swap monotonicity, the outcome at the new profile continues to be b. Using a similar argument, if bP_jx and xP_ja for some $j \in N$, then we can come to a preference ordering where x is just below a maintaining the outcome to be b.

Now, consider $j \in N$, such that xP_jb. If x and b are not consecutive in P_j, then again we can do a series of swaps to come to a preference ordering such that x is just above b (note again that none of these swaps will involve b). By swap monotonicity, the outcome at the new profile continues to be b. Let us denote this new profile by $\bar{\mathbf{P}}$.

So, we have reached a profile, where for every $j \in N$, either x is just above b in $\bar{P}_j$ or [x is just below b if $a\bar{P}_jb$ and x is just below a if $b\bar{P}_ja$]. Now, we pick a $j \in N$ such that x is just above b in $\bar{P}_j$. We do a (x, b)-swap to get a preference ordering $\bar{P}'_j$. By swap monotonicity $f(\bar{P}'_j, \bar{P}_{-j}) \in \{x, b\}$. We repeat this procedure for every agent k such that x is just above b in $\bar{P}_k$. Denote the new profile as $\bar{\mathbf{P}}'$. By repeated application of swap monotonicity, $f(\bar{\mathbf{P}}') \in \{x, b\}$. But $b\bar{P}'_jx$ for all $j \in N$. Hence, by Pareto efficiency, $f(\bar{\mathbf{P}}') = b$.

Now, consider the (a, b)-swap of $\bar{P}'_i$ and denote this preference ordering as $\bar{P}''_i$. Since $f(P'_i, P_{-i}) = a$, an analogous argument will show that $f(\bar{P}''_i, \bar{P}'_{-i}) = a$. The two profiles $(\bar{P}'_i, \bar{P}'_{-i})$ and $(\bar{P}''_i, \bar{P}'_{-i})$ are shown in Table 8.

$\bar{P}'_i$	$\bar{P}'_{-i}$		$\bar{P}''_i$	$\bar{P}'_{-i}$	
$\cdot$	$\dots$	$\dots$	$\cdot$	$\dots$	$\dots$
$\cdot$	$\dots$	$\dots$	$\cdot$	$\dots$	$\dots$
a	$b\dots$	$a\dots$	b	$b\dots$	$a\dots$
b	$\dots$	$\dots$	a	$\dots$	$\dots$
x	$a\dots$	$b\dots$	x	$a\dots$	$b\dots$
$\cdot$	$x\dots$	$x\dots$	$\cdot$	$x\dots$	$x\dots$
$\cdot$	$\dots$	$\dots$	$\cdot$	$\dots$	$\dots$

Table 8: Profiles $(\bar{P}'_i, \bar{P}'_{-i})$ and $(\bar{P}''_i, \bar{P}'_{-i})$

Step 2. In this step, we modify the profile $(\bar{P}'_i, \bar{P}'_{-i})$ in a particular way. First, we look at an agent $j \in N$, such that $a\bar{P}'_jb$ and $b\bar{P}'_jx$. We perform a (b, x) swap for each of these agents. The new profile is shown in Table 9. By swap monotonicity, the outcome at the new profile must be in $\{b, x\}$. But since a is ranked higher than x for all the agents, Pareto efficiency implies the outcome at the new profile must be b.

Step 3. In this step, we modify the profile in Table 9 further. In particular, we lift x just

Agent i	Other agents	
$\cdot$	$\dots$	$\dots$
$\cdot$	$\dots$	$\dots$
a	$b\dots$	$a\dots$
x	$\dots$	$\dots$
b	$a\dots$	$x\dots$
$\cdot$	$x\dots$	$b\dots$
$\cdot$	$\dots$	$\dots$

Table 9: New profile in Step 2

above a. For agent i and for all $j \neq i$ such that x is just below a, this can be done by a (a, x)-swap. For all other agents, this requires a series of swaps - note that these swaps can be done by without involving b. The new profile is shown in Table 10. Since none of the swaps involve b, swap monotonicity implies that the outcome at the new profile remains b.

Agent i	Other agents	
$\cdot$	$\dots$	$\dots$
$\cdot$	$\dots$	$\dots$
x	$b\dots$	$x\dots$
a	$\dots$	$a\dots$
b	$x\dots$	$\dots$
$\cdot$	$a\dots$	$b\dots$
$\cdot$	$\dots$	$\dots$

Table 10: New profile in Step 3

Step 4. In this step, we modify the profile in Step 3 by changing only agent i's preference ordering. We do this by doing an (a, b)-swap of the preference ordering of agent i in the profile shown in Table 10. The new profile is shown in Table 11. By swap monotonicity, the outcome at the new profile is in $\{a, b\}$. But x is better than a for all agents, and hence, Pareto efficiency implies the outcome at the new profile is b.

Step 5. In this step, we modify the profile in Step 4 by changing the preferences of those agents who prefer x to a and a to b (the third column of agents in Table 11). We perform a series of swaps to bring x just one position above b. The new profile is shown in Table 12. By swap monotonicity, the outcome at this profile remains b.

Agent i	Other agents	
$\cdot$	$\dots$	$\dots$
$\cdot$	$\dots$	$\dots$
x	$b\dots$	$x\dots$
b	$\dots$	$a\dots$
a	$x\dots$	$\dots$
$\cdot$	$a\dots$	$b\dots$
$\cdot$	$\dots$	$\dots$

Table 11: New profile in Step 4

Agent i	Other agents	
$\cdot$	$\dots$	$\dots$
$\cdot$	$\dots$	$\dots$
x	$b\dots$	$a\dots$
b	$\dots$	$\dots$
a	$x\dots$	$x\dots$
$\cdot$	$a\dots$	$b\dots$
$\cdot$	$\dots$	$\dots$

Table 12: New profile in Step 5

STEP 6. Now, we perform an (x, b)-swap of preferences of those agents who rank x just above b in the profile in Step 5 - this will be agent i and agents in the third column in Table 12. The new profile is shown in Table 13. By swap monotonicity, the outcome at the new profile is in $\{x, b\}$. But b is preferred to x for all the agents. Hence, Pareto efficiency implies the outcome at the new profile remains b.

Agent i	Other agents	
$\cdot$	$\dots$	$\dots$
$\cdot$	$\dots$	$\dots$
b	$b\dots$	$a\dots$
x	$\dots$	$\dots$
a	$x\dots$	$b\dots$
$\cdot$	$a\dots$	$x\dots$
$\cdot$	$\dots$	$\dots$

Table 13: New profile in Step 6

STEP 7. Finally, we perform a (x, a)-swap for the preferences of all agents in the profile in Step 6 who rank x just above a - this will include agent i and agents in the second column of Table 13. The new profile is shown Table 14. By swap monotonicity, the outcome at this profile remains b.

But the profile shown in Table 14 is exactly the profile $(\bar{P}_i'', \bar{P}_{-i}')$ (see Table 8) and we had assumed that $f(\bar{P}_i'', \bar{P}_{-i}') = a$. This is a contradiction.

Agent i	Other agents	
.	...	...
.	...	...
b	$b \ldots$	$a \ldots$
a	...	...
x	$a \ldots$	$b \ldots$
.	$x \ldots$	$x \ldots$
.	...	...

Table 14: New profile in Step 7

Proof of Theorem 4

Let f be a unanimous scf that is G-LOBIC. The proof goes in many steps.

STEP 1. In this step, we collect some facts about single peaked domain from Sato (2013). These facts will be used in subsequent steps. We have already defined the notion of a connected domain in Definition 17. We now propose a strengthening of this notion.

DEFINITION **18** *A distinct sequence of types* $(P^0, P^1, \ldots, P^k, P^{k+1})$ *is* **without restoration** *if there exists no distinct* $j, j' \in \{0, 1, \ldots, k\}$ *and* $x, y \in A$ *such that* P^{j+1} *is a* (x, y)*-swap of* P^j *and* $P^{j'+1}$ *is a* (y, x)*-swap of* $P^{j'}$*.*

The without restoration property requires that no pair of alternatives is swapped more than once along the sequence. With this notion, we can strengthen the notion of connectedness.

DEFINITION **19** *A domain* $\mathcal{D}$ *is* **connected without restoration** *if for every* $P, P' \in \mathcal{D}$*, there exists a sequence of distinct types* $(P = P^0, P^1, \ldots, P^k, P^{k+1} = P')$ *without restoration such that for all* $j \in \{1, \ldots, k\}$*,* $P^j \in \mathcal{D}$ *and for all* $j \in \{0, 1, \ldots, k\}$*,* P^{j+1} *is a* (x, y)*-swap of* P^j *for some* $x, y \in A$*.*

Sato (2013) shows that if a domain is connected without restoration, then it is a local domain. The unrestricted domain and the single peaked domains are examples of domains that are connected without restoration. We will use this fact in our proofs.

Step 2. In this step, we will prove a claim using the facts in Step 1.

Claim 1 *Suppose $P_i \in \mathcal{S}$ is a preference ordering such that aP_ib. Then, there exists a preference ordering $P_i' \in \mathcal{S}$ such that $P_i'(1) = a$ and $B(b, P_i) = B(b, P_i')$. Moreover, for all $P_{-i} \in \mathcal{S}^{n-1}$ with $f(P_i, P_{-i}) = b$, we have $f(P_i', P_{-i}) = b$.*

Proof: The first part of the claim follows from the single peaked domain - if aP_ib, then we can always lift a to the top and keep all the alternatives that are above b in P_i between a and b and all others below b in the new preference ordering. Let P_i' be such an ordering. By the facts in Step 1, we know that there is a distinct sequence of types $(P_i = P^0, P^1, P^2, \ldots, P^k, P^{k+1} = P_i')$ without restoration such that consecutive types in the sequence are swaps of each other. Since $B(b, P_i) = B(b, P_i')$ and the sequence is without restoration, none of these swaps involve b. By repeatedly applying swap monotonicity along the sequence, we get $f(P_i', P_{-i}) = f(P_i, P_{-i}) = b$. ■

Step 3. In this step, we show that if an scf is G-LOBIC and unanimous, then it must be Pareto efficient. For this consider a profile $\mathbf{P}$ with $f(\mathbf{P}) = b$. Assume for contradiction that there exists $a \neq b$ such that aP_ib for all $i \in N$. By Claim 1, there exists a preference profile $\mathbf{P}'$ such that $f(\mathbf{P}') = b$ and $P_i'(1) = a$ for all $i \in N$. This is a contradiction since unanimity implies that $f(\mathbf{P}') = a$.

Step 4. Now, consider an agent $i \in N$ and $P_{-i} \in \mathcal{S}^{n-1}$. Let $P_i, P_i' \in \mathcal{S}$ be such that P_i' is an (a, b)-swap of P_i and $f(P_i, P_{-i}) = b$. We will show that $f(P_i', P_{-i}) = b$. We consider various cases.

Case 1. Suppose $P_i(k) = a$ and $k > 1$, i.e., the swap from P_i to P_i' is not happening at the top of the preference ordering. Since a and b are consecutively ranked in P_i and P_i' and neither of them are top ranked in P_i and P_i', it must be that a and b are not neighbors (in $\succ$). This is because if a and b are neighbors then they can only be swapped if they are at the top.

Hence, consider a neighbor c of a such that c is between a and b in $\succ$ (i.e., if $a \succ b$, then $a \succ c \succ b$ and if $b \succ a$ then $b \succ c \succ a$). By definition, cP_ia and $cP_i'b$. Further, for any other agent $j \neq i$, there are four possible rankings between a, b, c in P_j: (1) cP_jaP_jb, (2) cP_jbP_ja, (3) bP_jcP_ja, and (4) aP_jcP_jb. The two profiles (P_i, P_{-i}) and (P_i', P_{-i}) are shown in

Table 15. Table 15 shows that there are four group of agents in P_{-i} with different possible rankings between a, b, c. We now modify the profile (P_i, P_{-i}) in a sequence of steps to reach the profile (P'_i, P_{-i}).

P_i	P_{-i}				P'_i	P_{-i}			
$\cdot$	$\dots$	$\dots$	$\dots$	$\dots$	$\cdot$	$\dots$	$\dots$	$\dots$	$\dots$
c	$c\dots$	$c\dots$	$b\dots$	$a\dots$	c	$c\dots$	$c\dots$	$b\dots$	$a\dots$
$\cdot$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
a	$a\dots$	$b\dots$	$c\dots$	$c\dots$	b	$a\dots$	$b\dots$	$c\dots$	$c\dots$
b	$\dots$	$\dots$	$\dots$	$\dots$	a	$\dots$	$\dots$	$\dots$	$\dots$
$\cdot$	$b\dots$	$a\dots$	$a\dots$	$b\dots$	$\cdot$	$b\dots$	$a\dots$	$a\dots$	$b\dots$
$\cdot$	$\dots$	$\dots$	$\dots$	$\dots$	$\cdot$	$\dots$	$\dots$	$\dots$	$\dots$

Table 15: Profiles (P_i, P_{-i}) and (P'_i, P_{-i})

STEP 1a. In this step, we modify the preferences of agents in P_{-i} who rank a better than c better than b. For each such agent j, we construct P'_j such that $P'_j(1) = a, P'_j(2) = c$ and $B(b, P'_j) = B(b, P_j)$. Notice that since a and c are neighbors, single peakedness implies that such a P'_j can be constructed such that it is single peaked. Further, using a reasoning similar to Claim 1, we can argue that we can go from P_j to P'_j using a without restoration sequence and since $B(b, P'_j) = B(b, P_j)$, b will not be involved in any swaps. As a result, the outcome at the new profile will be b. The new profile is shown in Table 16.

P_i	Other	agents		
$\cdot$	$\dots$	$\dots$	$\dots$	$a\dots$
c	$c\dots$	$c\dots$	$b\dots$	$c\dots$
$\cdot$	$\dots$	$\dots$	$\dots$	$\dots$
a	$a\dots$	$b\dots$	$c\dots$	$\dots$
b	$\dots$	$\dots$	$\dots$	$\dots$
$\cdot$	$b\dots$	$a\dots$	$a\dots$	$b\dots$
$\cdot$	$\dots$	$\dots$	$\dots$	$\dots$

Table 16: Profile in Step 1a.

STEP 1b. In this step, we modify the profile in Table 16 as follows. For every agent other than i who rank a at the top, c second, we perform the (a, c)-swap. Notice that this leads to a feasible single peaked preference ordering since a and c are neighbors. The new profile is shown in Table 17. By swap monotonicity, the outcome at the new profile is b.

P_i	Other	agents		
$\cdot$	$\ldots$	$\ldots$	$\ldots$	$c\ldots$
c	$c\ldots$	$c\ldots$	$b\ldots$	$a\ldots$
$\cdot$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
a	$a\ldots$	$b\ldots$	$c\ldots$	$\ldots$
b	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\cdot$	$b\ldots$	$a\ldots$	$a\ldots$	$b\ldots$
$\cdot$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

Table 17: Profile in Step 1b.

STEP 1c. In this step, we perform (a, b)-swap of P_i to reach P'_i. The new profile is shown in Table 18. By swap monotonicity, the outcome at the new profile is in $\{a, b\}$. Note that at the new profile cP_ja for all $j \in N$. Hence, by Pareto efficiency the outcome at the new profile is b.

P'_i	Other	agents		
$\cdot$	$\ldots$	$\ldots$	$\ldots$	$c\ldots$
c	$c\ldots$	$c\ldots$	$b\ldots$	$a\ldots$
$\cdot$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
b	$a\ldots$	$b\ldots$	$c\ldots$	$\ldots$
a	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\cdot$	$b\ldots$	$a\ldots$	$a\ldots$	$b\ldots$
$\cdot$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

Table 18: Profile in Step 1c.

STEP 1d. We can now consider the profile in Table 18 and alter the preferences of last (fifth) column of agents by performing a (c, a)-swap and then doing a sequence of without restoration swaps to go to their preference in Step 1a (see preferences in Table 16). The new profile is shown in Table 19.

Since none of these swaps involve alternative b, the outcome at this new profile is b due to swap monotonicity. But this profile is exactly (P'_i, P_{-i}). Hence, $f(P'_i, P_{-i}) = b$.

CASE 2. The other case is $P_i(k) = a$ and $k = 1$, i.e., the swap from P_i to P'_i is occurring at the top. The profile (P_i, P_{-i}) is shown below in Table 20. We now do the proof in many

P'_i	Other	agents		
$\cdot$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
c	$c\ldots$	$c\ldots$	$b\ldots$	$a\ldots$
$\cdot$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
b	$a\ldots$	$b\ldots$	$c\ldots$	$c\ldots$
a	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\cdot$	$b\ldots$	$a\ldots$	$a\ldots$	$b\ldots$
$\cdot$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

Table 19: Profile in Step 1d.

steps.

P_i	P_{-i}	
a	$\ldots$	$\ldots$
b	$a\ldots$	$b\ldots$
$\cdot$	$\ldots$	$\ldots$
$\cdot$	$b\ldots$	$a\ldots$
$\cdot$	$\ldots$	$\ldots$

Table 20: Profile in Case 2.

Step 2a. Now, consider every agent $j \neq i$ such that aP_jb (agents in second column of Table 20). By Claim 1, we can construct a preference ordering from P_j such that a is top ranked and the outcome remains b. We change the preferences of all the agents in the profile in Table 20 who prefer a to b in this manner to arrive at the new profile. The new profile is shown in Table 21 and the outcome at the new profile is b.

P_i	P_{-i}	
a	$a\ldots$	$\ldots$
b	$\ldots$	$b\ldots$
$\cdot$	$\ldots$	$\ldots$
$\cdot$	$b\ldots$	$a\ldots$
$\cdot$	$\ldots$	$\ldots$

Table 21: Profile in Step 2a.

Step 2b. Now, for every agent $j \neq i$ such that a is top tanked in the profile in Table 20, we consider a preference ordering where a is top ranked and b is second ranked. The new profile is shown in Table 22.

P_i	P_{-i}	
a	$a \ldots$	$\ldots$
b	$b \ldots$	$b \ldots$
$\cdot$	$\ldots$	$\ldots$
$\cdot$	$\ldots$	$a \ldots$
$\cdot$	$\ldots$	$\ldots$

Table 22: Profile in Step 2b.

By Sato (2013), such a preference ordering can be reached by swaps without restoration. Hence, a will not be involved in such swaps. Swap monotonicity implies that the outcome at the new profile is not a.

Step 2c. Now, consider every agent $j \neq i$ such that b is preferred to a in the preference profile in Step 2b (agents in third column of Table 22). By Claim 1, we can construct a preference ordering from this preference ordering such that b is top ranked and the alternatives that were below a do not change. The new profile is shown in Table 23. By definition, we can go to this new profile by doing a sequence of without restoration swaps that do not involve a. Hence, by swap monotonicity, the outcome at the new profile is not a. Then, by Pareto efficiency, the outcome at this new profile must be b.

P_i	P_{-i}	
a	$a \ldots$	$b \ldots$
b	$b \ldots$	$\ldots$
$\cdot$	$\ldots$	$\ldots$
$\cdot$	$\ldots$	$a \ldots$
$\cdot$	$\ldots$	$\ldots$

Table 23: Profile in Step 2c.

Step 2d. In this step, we consider all the agents who have b top-ranked in the profile in Step 2c (agents in the third column in Table 23). For every such agent, we consider another preference ordering where b is top-ranked and a is second ranked. The new profile is shown in Table 24. By Sato (2013), we can go to this new preference ordering by doing a sequence of without restoration swaps. Hence, by swap monotonicity, the outcome at the new profile is b.

P_i	P_{-i}	
a	$a\dots$	$b\dots$
b	$b\dots$	$a\dots$
$\cdot$	$\dots$	$\dots$
$\cdot$	$\dots$	$\dots$
$\cdot$	$\dots$	$\dots$

Table 24: Profile in Step 2d.

Now, consider any agent who ranks a at the top in the profile in Step 2d. If his preference ordering is not the same as P_i, then we can transform it to P_i by a sequence of without restoration swaps. Since this will not involve any swaps of b, by swap monotonicity the outcome will remain b at the new profile. A similar argument can be made to transform the preference ordering of every agent who ranks b at the top to the preference ordering P_i'. As a consequence, the profile in Table 24 can be transformed to a profile where every agent $j \neq i$ has preference ordering P_i or P_i' and the outcome at this profile is b. Denote this profile as $(P_i, \bar{P}_{-i})$.

Now, assume for contradiction $f(P_i', P_{-i}) = a$. We now repeat the above procedure at the profile (P_i', P_{-i}) to arrive at the profile $(P_i', \bar{P}_{-i})$ and the outcome at this profile is a.

If all $j \neq i$ have preference P_i, then unanimity implies $f(P_i, \bar{P}_{-i}) = a$, hence, we reach a contradiction. Similarly, if all $j \neq i$ have preference P_i', then unanimity implies $f(P_i', \bar{P}_{-i}) = b$, and again, we reach a contradiction. Hence, there exist $j, k \neq i$ such that $\bar{P}_j = P_i$ and $\bar{P}_k = P_i'$. Since $f(P_i, \bar{P}_i) = b$, weak elementary monotonicity implies that $f(P_i', \bar{P}_{-i}) = b$. This is a contradiction. This completes the proof.

Appendix B

Failure of Weak Equivalence under Unanimity and Anonymity

Here, we will show that for the scf f^* identified in Proposition 1, there is no DSIC, unanimous, and *anonymous* scf that is weakly equivalent to it for a generic set of priors. Without loss of generality, we show this for $N = \{1, 2, 3\}$.

To remind, an scf is anonymous if by permuting the preferences of agents, the outcome remains the same at the new profile. Hence, f^* is an anonymous scf if the number of agents is three.

We remind some of the notations we used earlier. Suppose the domain is single peaked with respect to $\succ$. As before, we denote the left-most alternative and the second left-most alternative according to $\succ$ as a and b respectively. The unique type where a is top ranked is denoted by $\bar{P}$. The probability that agent $i \in \{1,2,3\}$ has type $\bar{P}$ is denoted as q_i.

For agent 1, the interim allocation probabilities from f^* were computed earlier in the proof of Proposition 1. We will only be interested in the interim allocation probabilities of alternatives a and b when $P_1(1) \in \{a,b\}$. We reproduce that part of the interim allocation probabilities below in Table 25.

	Case 1: $P_1(1) = a$	Case 2: $P_1(1) = b$
$\pi_1^{f^*}(a, P_1)$	$1 - q_2 - q_3 + 2q_2q_3$	$q_2 + q_3 - 2q_2q_3$
$\pi_1^{f^*}(b, P_1)$	$q_2 + q_3 - 2q_2q_3$	$1 - q_2 - q_3 + 2q_2q_2$

Table 25: Interim allocation probabilities of a and b

We also remind that since f^* is OBIC, we must have

$$(1 - 2q_2)(1 - 2q_3) \geq 0.$$

This implies that either $q_2, q_3 \in [0, 0.5]$ or $q_2, q_3 \in [0.5, 1]$.

Now, assume for contradiction that there is a DSIC, unanimous, and anonymous scf f that is weakly equivalent to f^*. By Moulin (1980), f must be a *median voter* scf. To remind, when there are three agents, a median voter scf is defined by locating two *phantom* peaks - denote them by $a_1, a_2 \in A$. At every preference profile $(P_1, P_2, P_3) \in \mathcal{S}^3$, a median voter scf chooses the alternative which is the *median* of $(a_1, a_2, P_1(1), P_2(1), P_3(1))$ according to the ordering $\succ$. We will argue that the location of phantom peaks a_1 and a_2 for f has to be different (i.e., $a_1 \neq a_2$) and both of them have to be located either at a or at b.

Note that if agent 1 has type $\bar{P}$, then for any P_{-1}, the outcome of f^* can only be a or b. Hence, the interim allocation probability of all the alternatives except a and b are zero (see Table 25). This means that for all $c \notin \{a,b\}$,

$$\pi_1^f(c, \bar{P}) = \pi_1^{f^*}(c, \bar{P}) = 0,$$

where the first equality followed from the weak equivalence of f and f^*. Now, if the location of one of the phantom peaks of f is not in $\{a,b\}$, then when agents 2 and 3 have their top ranked alternative outside $\{a,b\}$, because of the median property, the scf will not choose a or b. As a result, we will have $\pi_1^f(c, \bar{P}) > 0$, contradicting the above equation. Hence, the phantom peaks must be located either at a or b.

If both the phantom peaks are located at a, then the median will be a when agent 1 has type $\bar{P}$, irrespective of the type of agents 2 and 3. Hence, $\pi_1^f(a, \bar{P}) = 1$. But, using Table 25 and weak equivalence of f and f^*, we can conclude that $\pi_1^f(a, \bar{P}) = \pi_1^{f^*}(a, \bar{P}) < 1$, which is a contradiction. Hence, both the phantom peaks cannot be at a.

Consider any preference ordering where b is top ranked - denote it by $\hat{P}$. Note that $\pi^f(b,\hat{P}) = \pi_1^{f*}(b,\hat{P}) < 1$ (Table 25). In that case, if both the phantom peaks are located at b, by the median property the scf f will choose b as the outcome for any preference ordering of agents 2 and 3. Hence, $\pi^f(b,\hat{P}) = 1$, which is a contradiction. Hence, we conclude that in the median voter scf f, one of the phantom peaks is located at a and the other one is located at b.

Now, in the median voter scf f, if agent 1 has type $\bar{P}$, then f chooses a if at least one more agent has his top ranked alternative a (this follows from the median property), i.e., at least one more agent has type $\bar{P}$. The probability of this event is exactly $1-(1-q_2)(1-q_3) = q_2+q_3-q_2q_3$. Hence, we have $\pi_1^f(a,\bar{P}) = q_2+q_3-q_2q_3$. But from Table 25, $\pi_1^{f^*}(a,\bar{P}) = 1-q_2-q_3+2q_2q_3$. This implies that $q_2+q_3-q_2q_3 = 1-q_2-q_3+2q_2q_3$, or equivalently,

$$q_2+q_3 = \frac{1+3q_2q_3}{2}. \tag{2}$$

Now, consider the case where agent 1 has a type where his top ranked alternative is b - as before, denote such a type as $\hat{P}$. In this case, f chooses a as the outcome if both agent 2 and agent 3 have their top ranked alternative as a (by the median property), i.e., agent 2 and agent 3 have type $\bar{P}$. The probability of this event is q_2q_3. Hence, $\pi_1^f(a,\hat{P}) = q_2q_3$. But from Table 25, $\pi_1^{f^*}(a,\hat{P}) = q_2+q_3-2q_2q_3$. This implies that $q_2q_3 = q_2+q_3-2q_2q_3$, or equivalently,

$$q_2+q_3 = 3q_2q_3. \tag{3}$$

Equations 2 and 3 give us $q_2q_3 = \frac{1}{3}$ and $q_2+q_3 = 1$. But this has no real solution.

This shows that there is no DSIC, unanimous, and anonymous scf that is weakly equivalent to f^*.

Uniform Priors Example

Suppose $A = \{a,b,c\}$ and $n = 2$. Consider the unrestricted domain consisting of 6 possible types for each agent. The type profiles along with the outcome of an scf $\tilde{f}$ is shown in Table 26. In Table 26, a preference ordering $aPbPc$ is written as abc. Also, the rows of Table are preferences of agent 1 and columns of agent 2. The profiles corresponding to top four rows and first four columns correspond to the single peaked domain (with respect to the ordering $a \succ b \succ c$).

The scf $\tilde{f}$ in Table 26 is unanimous, but not DSIC - e.g., agent 2 can manipulate when his true type is cba and agent 1 reports abc by reporting bca (notice that the manipulation took place inside the single peaked domain, and hence, the scf is not DSIC in the single

	abc	bac	bca	cba	acb	cab
abc	a	a	b	a	a	a
bac	b	b	b	b	a	b
bca	b	b	b	b	b	c
cba	c	b	c	c	c	c
acb	a	a	a	c	a	a
cab	a	c	c	c	c	c

Table 26: An OBIC scf with respect to uniform priors

peaked domain also). However, this is a neutral scf satisfying elementary monotonicity - in fact, $\tilde{f}$ is equivalent to the Borda scoring rule where the top ranked alternative gets score 2, the second ranked alternative gets score 1 and ties are broken in favor of agent 1's top ranked alternative. Hence, it is OBIC with respect to uniform priors in the unrestricted domain (Majumdar and Sen, 2004).

We now consider the single peaked domain and uniform priors in the single peaked domain. Now, we calculate the interim allocation probabilities for agent 2 in the single peaked domain from $\tilde{f}$. They are shown in Table 27. Now, notice that

$$\pi_2^{\tilde{f}}(a, abc) + \pi_2^{\tilde{f}}(b, abc) = \frac{3}{4} < 1 = \pi_2^{\tilde{f}}(a, bac) + \pi_2^{\tilde{f}}(b, bac).$$

This implies that $\tilde{f}$ is not OBIC with respect to uniform priors in the single peaked domain.

	a	b	c
$\pi_2^{\tilde{f}}(\cdot, abc)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$\pi_2^{\tilde{f}}(\cdot, bac)$	$\frac{1}{4}$	$\frac{3}{4}$	0
$\pi_2^{\tilde{f}}(\cdot, bca)$	0	$\frac{3}{4}$	$\frac{1}{4}$
$\pi_2^{\tilde{f}}(\cdot, cba)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Table 27: Interim allocation probability of agent 2 in $\tilde{f}$

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