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MATCHING PLATFORMS

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Abstract

A platform matches agents from two sides of a market to create a trading opportunity between them. The agents subscribe to the platform by paying subscription fees which are contingent on their reported private types, and then engage in strategic interactions with their matched partner(s). A *matching mechanism* of the platform specifies the subscription fees as well as the *matching rule* which determines the probability that each type of agent on one side is matched with each type on the other side. We characterize optimal matching mechanisms which induce truthful reporting from the agents and maximize the subscription revenue. We show that the optimal mechanisms for a one-to-one trading platform match do not necessarily entail assortative matching, and may employ an alternative matching rule that maximizes the extraction of informational rents of the higher type. We then study an auction platform that matches each seller to two agents, and show that the optimal mechanism entails the combination of negative and positive assortative matching.

Key words: assortative, random, auction, subscription, revenue maximization, complementarity.
JEL Codes: D42, D47, D62, D82, L12

1 Introduction

Platforms that match agents flourish in modern economies with the development of information technologies. They realize gains from trade and other forms of interactions by providing agents access to each other: an internet auction house matches sellers and buyers who would otherwise not be able to find trading partners, a job matching platform matches firms and workers who would otherwise face under-utilization of their resources, and a crowd-funding platform matches entrepreneurs with investors to create new businesses. While there is now sizable literature on matching platforms, one important aspect of matching platforms yet to be explored concerns the facts that the interactions between their subscribers are often strategic, and that their strategic

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incentives in such interactions are determined by how a match is formed by the platforms. Sellers of items in a trading platform post prices that are optimal given the expected willingness to pay of subscribing buyers for his goods, and bidders in an auction platform choose bids that are optimal given their beliefs about the valuations of other subscribing bidders. Put differently, subscribers to a platform play a game against each other, and the outcome of their interactions is an equilibrium of the game. In particular, when the subscribers are privately informed about their types, they play incomplete information games, and the value of a match to each of them is endogenously determined by their Bayes Nash equilibrium (BNE) payoffs. This uncertainty creates room for the platform to manipulate the subscribers' beliefs as well as the match values. For example, an auction platform may attempt to restrict subscription to high valuation bidders so as to create competition among them and charge a high subscription fee to sellers. The objective of this paper is to study how a matching platform can maximize its subscription revenue by controlling a *matching rule* which determines the subscribers' beliefs in the game played amongst its subscribers. Our approach marks a departure from the literature on a matching platform which assumes that the match value to each subscriber is an exogenous function of their own type as well as those of the matched subscribers.

In our baseline model, a trading platform creates one-to-one matches of sellers and buyers. The sellers have two cost types, whereas the buyers have two valuation types. These types are overlapped so that efficient trading is possible within a match only when it involves a high-valuation buyer or a low-cost seller. The matching mechanism of the platform specifies the matching rule, which determines the probability with which each seller type is matched with each buyer type, and type-contingent subscription fees. Put differently, the matching mechanism is a simultaneous screening device of buyers and sellers based on a matching rule and subscription fees.[1]

We suppose that trade in each match takes the form of a buyer's take-it-or-leave-it offer to a seller, and characterize the incentive compatible matching mechanism that maximizes the platform's subscription revenue. The agents' incentives in the reporting stage depend on both the required transfer as well as the value of the match they expect to obtain upon subscription. Importantly, this match value is the BNE payoff of the game they play, and will depend on the agent's own type as well as their *belief* about the type of the agent they are matched with. In particular, a high-valuation buyer's optimal bid is low if his belief weight on the low-cost seller is above a certain threshold, but it is high otherwise. We show that the matching rule under the optimal mechanism takes one of three forms depending on the proportion of the agents' types in the population. Two of them are positive assortative matching (PAM) that matches a high-valuation buyer with a low-cost seller as much as possible, and random matching (RM) that generates matches according to the proportion of types in the underlying population. Under the third matching rule referred to as B-squeeze matching (BSM), a high-valuation buyer is matched with a low-cost seller more often than a low-valuation buyer is, but not to the maximal extent as under PAM. It instead squeezes the high-valuation buyers by minimizing their informational rents. The suboptimality of PAM for

[1]Screening of subscribers based on both matching and pricing is a common practice. For example, eBay offers sellers and buyers the option "eBay plus" in some countries that their raises visibility to the other side of the market, and a matchmaking platform Vidaselect offers a premium membership which allows men better access to women. One question concerns whether or not potential subscribers know the matching rule adopted by a platform. Casual evidence suggests that potential subscribers to a dating platform have very good ideas about what to expect based on their extensive search for the experiences of past subscribers through various review sites.

some type distributions is in contrast with the finding in the literature, and is a direct consequence of the strategic action choice of the buyers in the game against matched sellers.

To see the effect of the game protocol, we next study the optimal mechanism with seller-offer bargaining, and compare its performance with that under buyer-offer bargaining described above. We find that the mechanism with random matching under buyer-offer bargaining is dominated by the optimal mechanism with PAM under seller-offer bargaining. However, the matching rule of the squeeze type remains optimal when the market has high quality in the sense that the proportions of low-cost sellers and high-valuation buyers are both high. Most surprisingly, even when the proportion of the high-valuation buyers equals that of the low-cost sellers, the platform finds it optimal to create mismatches provided that those types are abundant in the market.

In light of the prevalence of one-to-many matchings in the real world, we then turn to the analysis of an auction platform which matches each seller with two buyers. In this model, each seller has two types which now represent the quality of the good they possess. Each buyer, which is one of two valuation types as in the baseline model, competitively bids for the good of the matched seller upon observing its quality. We consider both the first-price and second-price auction as sales formats, and derive the optimal mechanism. We show that the optimal matching rule depends on the proportion of types in the population, but not on the auction format. Under the optimal mechanism, a high-quality seller is matched with high-valuation buyers more often than a low-quality seller is, and a high-valuation buyer is matched with a high-quality seller more often than a low-valuation buyer is. The optimal matching rule, however, is not PAM in the sense that the probability that a high-quality seller is matched with a pair of high-valuation buyers is not maximized. Instead, we show that the optimal mechanism has an interesting assortative property: First, it entails negative assortative matching (NAM) between two buyers so that a high-valuation buyer is matched with a low-valuation buyer to the maximal extent. Second, subject to the NAM property of matching between buyers, it entails PAM between a seller and a pair of buyers so that a high-quality seller is matched with a high-high buyer pair as much as possible, and then matched with a high-low buyer pair as much as possible. This matching rule is shown to be also first-best efficient and maximize the social surplus from trade.

The paper is organized as follows. In Section 2, we discuss the related literature. Section 3 introduces a model of a trading platform, and Section 4 presents a benchmark case of non-strategic interaction commonly discussed in the literature. A characterization of an optimal matching mechanism is given in Section 5 under the buyer-offer bargaining protocol. Section 6 performs a comparison of optimal mechanisms under seller-offer and buyer-offer protocols, and Section 7 discusses some extensions of the baseline model including the analysis of a fee scheme that is contingent on the outcome of the transaction. Analysis of a model of an auction platform is presented in Section 8. We conclude with a discussion in Section 9.

2 Related Literature

The key component of the matching models in the literature is a *production function,* which determines the value of a match to each member as a function of their types. The critical observation by Becker (1973) is that when the production function is supermodular on the set of agents' type profiles, the match allocation is in the core if it entails positive assortative matching (PAM), whereby the highest type on one side is matched with the highest type on the other side, the second-highest type is matched with the second-highest type, *etc.* Legros & Newman (2002) identify a condition on the production function weaker than supermodularity for PAM to be the core outcome, and Legros & Newman (2007) establish a sufficient condition for PAM to be in the core in an environment without full transferability of payoffs between the matched agents. Shimer & Smith (2000) show that the supermodularity of a production function is necessary but not sufficient for PAM to be the equilibrium outcome of a search model where each agent engages in continuous-time search for his partner.

The use of an exogenously specified production function for each agent is maintained in the literature on platforms that match agents with private types. Damiano & Li (2007) and Hoppe et al. (2011) study matching of agents with heterogeneous quality in two-sided markets when the match quality is the product of the qualities of its members (and hence supermodular), and Gomes & Pavan (2016) study efficient as well as profit-maximizing platforms for non-exclusive many-to-many matching in a two-sided market when the value of a match to an agent is the product of his value type and the average salience type of the matched agents. Board (2009) considers the problem of grouping agents with variable qualities, and identifies a profit maximizing group structure under various assumptions on the form of the production function.[2] In our model, aggregate surplus generated by trading within a match is a function of the types of the matched agents, but the division of surplus is determined endogenously in equilibrium. Specifically, the matched agents play a Bayesian equilibrium of a non-cooperative game, and the distribution of their types in each match is controlled by the matching rule of the platform.

Strategic interactions among subscribing agents are studied by Tamura (2016) and Birge et al. (2019) in models of monopolistic trading platforms. Tamura (2016) considers a platform that matches a single seller with multiple buyers when the seller is privately informed about the quality of his good, and the buyers' private types are affiliated with the quality. Tamura (2016) assumes that the platform offers a single subscription price to each side of the market, and shows that its subscription revenue is higher under the first-price auction than under the second-price auction. Birge et al. (2019) consider a platform that matches sellers and buyers one-to-one under the constraint that some type pairs are not feasible. When the agents' types are public, Birge et al. (2019) show that uniform pricing is suboptimal, and evaluate the optimal subscription revenue from discriminatory pricing under complete information. Unlike these models, our model features screening of privately informed agents through discriminatory pricing and matching. Furthermore, we present explicit characterizations of optimal matching mechanisms in the presence of strategic interactions among subscribers.

It is possible to interpret our model as one of information design by a platform. In the information design literature, a principal controls the type distribution of each player so as to maximize his own payoff. In the Bayesian persuasion model of Kamenica & Gentzkow (2011), for example, a principal controls the distribution of signals about the state of the world that a decision maker observes, and attempts to maximize the probability with which the decision maker chooses the action preferred by the principal. In the multi-player information design model of Bergemann &

[2] Marx & Schummer (2019) focus on the stability of the matching rule when agents have heterogeneous preferences over other agents on the other side of the market, and study the optimal mechanism that offers a single price for each side.

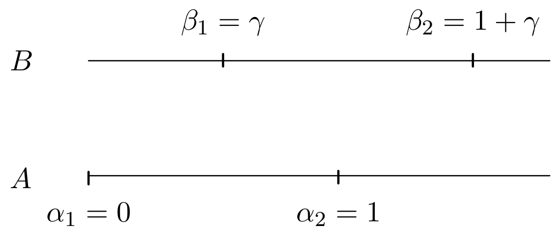

Figure 1: Costs and valuations

Morris (2016), a principal likewise controls the distribution of signals about the state which are privately observed by the players. The principal in this case attempts to induce the (Bayes) correlated equilibrium of the game that is most preferable to him. In the present setting, the matching rule of the platform also controls the type distribution of the agents in the trading game, and is used to induce the Bayesian equilibrium that would maximize the subscription revenue. The key difference from the information design literature is the presence of the information collection process by the platform: The matching rule and subscription fees are chosen so that they induce truth-telling in reporting of private signals by the agents.

3 Model of a Trading Platform

The market consists of two sides A and B as well as a monopolistic provider of a trading platform. The side A is a unit mass of sellers of an indivisible good, and the side B is a unit mass of buyers of the good. Each seller has a single unit of the good and each buyer has a unit demand for the good. An agent on either side has access to another agent on the other side only through subscription to the platform. Specifically, the platform sets fees for subscription, and then forms a one-to-one match between a subscribing seller and a subscribing buyer. A seller's cost of providing the good is denoted α, and a buyer's valuation is denoted β. The types α and β are private information of the agents, and are randomly drawn from the binary sets $A = \{\alpha_1, \alpha_2\}$ and $B = \{\beta_1, \beta_2\}$, respectively.[3] We suppose that the types are overlapped $\alpha_1 < \beta_1 < \alpha_2 < \beta_2$ so that no efficient transaction is feasible when a match involves a type β_1 buyer and a type α_2 seller (Figure 1). For simplicity, we assume that

$$\alpha_1 = 0,\ \alpha_2 = 1,\ \beta_1 = \gamma, \text{ and } \beta_2 = 1 + \gamma \text{ for } \gamma \in (0, 1).^4$$

A seller is type α_i with probability $\lambda_i \in (0, 1)$ and a buyer is type β_i with probability $\mu_i \in (0, 1)$ for $i = 1, 2$. The type realizations are independent across agents so that we may identify λ_i as the proportion of type α_i sellers on side A, and μ_i as the proportion of type β_i buyers on side B.

Once matched, the seller and buyer play a trading game as specified by the platform. Specifically, we suppose that they simultaneously submit bids denoted z_A and z_B for a seller and a buyer,

[3] Note that the symbols A and B are used to denote the sides of the market as well as the sets of types of agents on each side.

[4] The argument goes through with no qualitative change without this simplification.

respectively. Let $z = (z_A, z_B)$ be the bid profile. If $z_B \geq z_A$, then the transaction takes place, and the transaction price is determined by a pricing rule $k : \mathbf{R}^2_+ \to \mathbf{R}_+$ which satisfies $k(z) \in [z_A, z_B]$ for every z such that $z_A \leq z_B$. If $z_A > z_B$, no transaction takes place. Accordingly, under the bid profile $z \in \mathbf{R}^2_+$, the payoff $g_A(z, \alpha)$ of a seller of type α, and the payoff $g_B(z, \beta)$ of a buyer of type β in G are given by

$$g_A(z,\alpha) = \begin{cases} k(z) - \alpha & \text{if } z_A \leq z_B \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad g_B(z,\beta) = \begin{cases} \beta - k(z) & \text{if } z_A \leq z_B \\ 0 & \text{otherwise.} \end{cases}$$

There are two cases of special interest in our analysis: When $k(z) = z_B$ so that the transaction price equals the buyer's bid, the seller has a weakly dominant strategy of bidding his own cost α. This corresponds to the sequential trading game in which the buyer makes a take-it-or-leave-it offer, which the seller chooses to accept or not. When $k(z) = z_A$, on the other hand, the buyer has a weakly dominant strategy of bidding his own valuation, and this corresponds to the game in which the seller makes a take-it-or-leave-it offer.

Note that trade generates surplus unless a high-cost seller (α_2) is matched with a low-valuation buyer (β_1). If we denote by

$$f(\alpha, \beta) = (\beta - \alpha)\, \mathbf{1}_{\{\beta > \alpha\}} \tag{1}$$

the aggregate surplus of trade when a type α seller is matched with a type β buyer, then f is *supermodular* since it satisfies

$$f(\alpha_1, \beta_2) + f(\alpha_2, \beta_1) = \beta_2 - \alpha_1 > (\beta_2 - \alpha_2) + (\beta_1 - \alpha_1) = f(\alpha_2, \beta_2) + f(\alpha_1, \beta_1).^5 \tag{2}$$

Formally, the *matching mechanism* Γ of the platform consists of a matching rule p, a pricing rule k, a transfer rule $t = (t_A, t_B)$, and a strategy profile $\sigma = (\sigma_A, \sigma_B)$ of the trading game as follows. First, the *matching rule* p specifies the distribution of type profiles of matched agents. For each $\alpha \in A$ and $\beta \in B$, $p_{ij} \equiv p(\alpha_i, \beta_j)$ is the proportion of the type pair (α_i, β_j). Denote by P the set of feasible matching rules:

$$P \equiv \{p \in \Delta(A \times B) : \sum_{i=1}^{2} p_{ij} = \mu_j \text{ for } j = 1,\, 2, \text{ and } \sum_{j=1}^{2} p_{ij} = \lambda_i \text{ for } i = 1,\, 2\}.$$

Given the matching rule $p \in P$, we denote by $p_A(\alpha \mid \beta)$ the probability that a buyer with the reported type β is matched to a seller with the reported type α, and by $p_B(\beta \mid \alpha)$ the probability that a seller with the reported type α is matched to a buyer with the reported type β. We have

$$p_B(\beta_j \mid \alpha_i) = \frac{p_{ij}}{\sum_{j'} p_{ij'}} \quad \text{and} \quad p_A(\alpha_i \mid \beta_j) = \frac{p_{ij}}{\sum_{i'} p_{i'j}}.$$

As will be seen, these conditional probabilities determine the agents' incentives in the reporting stage, and are the primary objects of our analysis. Since $p_A(\alpha_1 \mid \beta_1)$ and $p_A(\alpha_1 \mid \beta_2)$ are conditional probabilities, they must satisfy

$$\mu_1 p_A(\alpha_1 \mid \beta_1) + \mu_2 p_A(\alpha_1 \mid \beta_2) = \lambda_1. \tag{3}$$

[5] We consider the partial ordering over the set of type profiles (α, β) that is induced by $\alpha_2 \prec \alpha_1$ and $\beta_1 \prec \beta_2$.

Conversely, as long as $(p_A(\alpha_1 \mid \beta_1), p_A(\alpha_1 \mid \beta_2))$ satisfies (3), we can recover a feasible matching rule $p \in P$ as seen in the following lemma. This allows us to use $(p_A(\alpha_1 \mid \beta_1), p_A(\alpha_1 \mid \beta_2))$ and p exchangeably.

Lemma 1 *For any $(p_A(\alpha_1 \mid \beta_1), p_A(\alpha_1 \mid \beta_2)) \in [0, 1]^2$ that satisfies (3), there exists $p \in P$.*

We refer to (3) as the *Bayes plausibility* condition following Kamenica & Gentzkow (2011), who use the terminology for the corresponding condition in their analysis of Bayesian persuasion.[6]

The pricing rule k and matching rule p together determine the Bayesian game played by a pair of matched agents. The strategy profile $\sigma = (\sigma_A, \sigma_B)$ specified by Γ is a (pure) Bayes Nash equilibrium (BNE) of the incomplete information game that follows truthful reporting by both agents. Since the joint distribution of type profiles equals p in this game, σ satisfies

$$\pi_A(\sigma, \alpha) \equiv \sum_{\beta \in B} p_B(\beta \mid \alpha)\, g_A(\sigma_A(\alpha), \sigma_B(\beta), \alpha) \geq \sum_{\beta \in B} p_B(\beta \mid \alpha)\, g_A(z_A, \sigma_B(\beta), \alpha)$$
$$\text{for any } z_A \geq 0 \text{ and } \alpha \in A,$$

and

$$\pi_B(\sigma, \beta) \equiv \sum_{\alpha \in A} p_A(\alpha \mid \beta)\, g_B(\sigma_A(\alpha), \sigma_B(\beta), \beta) \geq \sum_{\alpha \in A} p_A(\alpha \mid \beta)\, g_B(\sigma_A(\alpha), z_B, \beta)$$
$$\text{for any } z_B \geq 0 \text{ and } \beta \in B.$$

Even when there exist multiple BNE in the trading game on the path of play after truthful reporting, the agents understand that they play σ as suggested by the mechanism.[7]

Finally, the transfer rule $t = (t_A, t_B)$ determines the payment from the agents to the platform: $t_A(\alpha)$ is the payment required from a seller whose reported type is α, and $t_B(\beta)$ is the payment required from a buyer whose reported type is β.

The matching mechanism Γ is *incentive compatible* (IC) if no unilateral deviation in reporting and action choice is profitable:

$$\pi_A(\sigma, \alpha) - t_A(\alpha) \geq \sum_{\beta \in B} p_B(\beta \mid \alpha')\, g_A(z_A, \sigma_B(\beta), \alpha) - t_A(\alpha')$$
$$\text{for every } \alpha,\ \alpha' \in A \text{ and } z_A \geq 0,$$

and

$$\pi_B(\sigma, \beta) - t_B(\beta) \geq \sum_{\alpha \in A} p_A(\alpha \mid \beta')\, g_B(\sigma_A(\alpha), z_B, \beta) - t_B(\beta')$$
$$\text{for every } \beta,\ \beta' \in B \text{ and } z_B \geq 0.$$

[6]It can be readily verified that it is suboptimal for the platform to leave some agents unmatched in our model of a trading platform.

[7]Alternatively, we may assume that the mechanism instructs to each agent on which bid to submit as a function of their reported type. Such instructions are designed so that the agents find it optimal to obey them after truthful reporting.

Γ is *individually rational* (IR) if truthful reporting yields at least as much as the reservation utility, which is normalized to zero:

$$\pi_A(\sigma,\alpha) - t_A(\alpha) \geq 0 \quad \text{for every } \alpha \in A\text{, and}$$
$$\pi_B(\sigma,\beta) - t_B(\beta) \geq 0 \quad \text{for every } \beta \in B.$$

When the mechanism Γ is IC and IR, its *efficiency* is defined by

$$W(\Gamma) = \sum_{i,j} (\beta_j - \alpha_i)\, p_{ij}\, \mathbf{1}_{\{\sigma_A(\alpha_i) \leq \sigma_B(\beta_j)\}}. \tag{4}$$

Two matching rules that play important roles in our analysis are as follows: A matching rule p is (*positively*) *assortative* (PAM) if it matches the low-cost sellers with the high-valuation buyers as much as possible: $p \in \operatorname{argmax}_{\hat{p} \in P}\, \hat{p}_A(\alpha_1 \mid \beta_2)$, or equivalently,

$$(p_A(\alpha_1 \mid \beta_1),\, p_A(\alpha_1 \mid \beta_2)) = \begin{cases} (\frac{\lambda_1 - \mu_2}{\mu_1},\, 1) & \text{if } \lambda_1 > \mu_2 \\ (0,\, \frac{\lambda_1}{\mu_2}) & \text{if } \lambda_1 \leq \mu_2. \end{cases} \tag{5}$$

A matching rule p is *random* (RM) if the matching is independent of the agents' types: $p(\alpha_i, \beta_j) = \lambda_i \mu_j$ $(i,\, j = 1, 2)$, or equivalently,

$$(p_A(\alpha_1 \mid \beta_1),\, p_A(\alpha_1 \mid \beta_2)) = (\lambda_1, \lambda_1).$$

It is intuitive and also can be readily verified that PAM maximizes the aggregate surplus from transactions. This is formally stated in the following proposition.

Proposition 1 *If the matching rule p maximizes $\sum_{i,j} p_{ij}(\beta_j - \alpha_i)\, \mathbf{1}_{\{\beta_j > \alpha_i\}}$, then it is PAM. Furthermore, the maximal social surplus equals*

$$W^* = \begin{cases} \gamma\lambda_1 + \mu_2 & \text{if } \lambda_1 \geq \mu_2, \\ \gamma\mu_2 + \lambda_1 & \text{if } \lambda_1 < \mu_2. \end{cases} \tag{6}$$

4 Non-Strategic Interaction

As discussed in the Introduction, it is standard in the platform literature to exogenously specify the production function for each agent, which determines the value of a match to them as a function of their types. Exogenously specifying a production function amounts to exogenously specifying the outcome of interaction between matched agents. In this section, we replicate such an argument by supposing that each agent non-strategically bids his own type in the trading game. We also assume the pricing rule $k(z_A, z_B) = kz_A + (1-k)z_B$ for some $k \in [0,1]$. The production function for each agent is hence given by

$$f_A(\alpha,\beta) = \begin{cases} (1-k)(\beta - \alpha) & \text{if } \beta > \alpha, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad f_B(\alpha,\beta) = \begin{cases} k(\beta - \alpha) & \text{if } \beta > \alpha, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Since $f_A = (1-k)\, f$ and $f_B = kf$ for the aggregate surplus f defined in (1), both f_A and f_B are supermodular by (2). The following proposition shows that the optimal mechanism under this value specification entails PAM.

Proposition 2 *(Optimal matching under non-strategic interaction) Suppose that the agents' match values are given by (7). If Γ is an optimal mechanism, then p is PAM, and the associated revenue is given by*

$$R(\Gamma) = \begin{cases} \frac{k}{\mu_1}\left\{\gamma(\lambda_1 - \mu_2) + \lambda_2\mu_2\right\} + (1-k)\mu_2 & \text{if } \lambda_1 \geq \mu_2, \\ \frac{1-k}{\lambda_2}\left\{\gamma(\mu_2 - \lambda_1) + \lambda_1\mu_1\right\} + k\lambda_1 & \text{if } \lambda_1 < \mu_2. \end{cases} \tag{8}$$

In particular, when the market is symmetric ($\lambda_1 = \mu_2$), $R(\Gamma) = \lambda_1 = \mu_2$.

Note that the optimality of PAM is consistent with the findings in the literature on profit maximizing platforms under the assumption of a supermodular production function for each agent.

5 Strategic Interaction: Buyer-Offer Bargaining

We now return to our main setup where the matched agents play the Bayes Nash equilibrium (BNE) σ of the trading game. The expected payoff in this BNE will determine their incentives in the reporting stage of the mechanism. We assume in this section that the good is traded at the price suggested by the buyer so that $k(z) = z_B$ for every z. Let $\sigma = (\sigma_A, \sigma_B)$ be a BNE of this game after truthful reporting by both agents. Since it is a weakly dominant strategy for the seller to bid his type in this game, we set $\sigma_A(\alpha) = \alpha$ for every α. The buyer's strategy σ_B is a best response against σ_A. Explicitly, let $z_B^*(\beta, \tilde{p}_A)$ be his optimal bid against σ_A when his true type is β, and his belief about the type of the matched seller is given by $\tilde{p}_A$:

$$z_B^*(\beta, \tilde{p}_A) \in \operatorname{argmax}_{z_B \in \mathbf{R}_+} \sum_{\alpha \in A} \tilde{p}_A(\alpha)\, g_B((\sigma_A(\alpha), z_B), \beta).$$

Explicitly, $z_B^*(\beta, \tilde{p}_A)$ can be written as

$$z_B^*(\beta, \tilde{p}_A) = \begin{cases} \alpha_1 & \text{if } \beta = \beta_1, \text{ or if } \beta = \beta_2 \text{ and } \tilde{p}_A(\alpha_1) \geq \frac{\beta_2 - \alpha_2}{\beta_2 - \alpha_1} = \frac{\gamma}{1+\gamma}, \\ \alpha_2 & \text{if } \beta = \beta_2 \text{ and } \tilde{p}_A(\alpha_1) < \frac{\beta_2 - \alpha_2}{\beta_2 - \alpha_1} = \frac{\gamma}{1+\gamma}. \end{cases}$$

In other words, the only viable bid for the low-valuation buyer β_1 is α_1, whereas the optimal bid for the high valuation buyer β_2 varies with his belief about the type of the matched seller: He either bids α_1 and has the low-cost seller accept it, or bids α_2 and has both seller types accept it. After truthful reporting, his belief is given by $\tilde{p}_A = p_A(\cdot \mid \beta)$ so that the BNE strategy σ_B can be defined by

$$\sigma_B(\beta) = z_B^*(\beta, p_A(\cdot \mid \beta)) \quad \text{for every } \beta.$$

The agents play the BNE $\sigma = (\sigma_A, \sigma_B)$ of the trading game after truthful reporting.

Characterization of an optimal mechanism requires the introduction of another matching rule as follows. Under this matching rule, the distribution of seller types faced by a low-valuation buyer (*i.e.*, $p_A(\cdot \mid \beta_1)$) makes a high-valuation buyer (β_2) exactly indifferent between high (α_2) and low (α_1) bids. Formally, when $(1+\gamma)\lambda_1 + \mu_1 \leq 1 + \gamma$, we define a matching rule p to be *B-squeeze* (BSM) if

$$(p_A(\alpha_1 \mid \beta_1), p_A(\alpha_1 \mid \beta_2)) = \left(\frac{\gamma}{1+\gamma}, \frac{\lambda_1}{\mu_2} - \frac{\mu_1}{\mu_2}\frac{\gamma}{1+\gamma}\right).$$

We can verify that $p_A(\alpha_1 \mid \beta_1) \leq p_A(\alpha_1 \mid \beta_2) \leq 1$ if and only if $(1+\gamma)\lambda_1 + \mu_1 \leq 1+\gamma$, and $\lambda_1 \geq \frac{\gamma}{1+\gamma}$. In other words, the probability that a high-valuation buyer β_2 is matched with a low-cost seller is higher than the probability that a low-valuation buyer is matched with a low-cost seller. The following proposition identifies the optimal mechanism with buyer-offer bargaining.

Proposition 3 *Suppose that Γ is an optimal matching mechanism with buyer-offer bargaining. Then its matching rule p and revenue $R(\Gamma)$ are given as follows.*

a. If $(1+\gamma)\lambda_1 + \mu_1 > 1+\gamma$, then p is PAM and $R(\Gamma) = (1+\gamma)\lambda_1 - \frac{\lambda_1 - \mu_2}{\mu_1}$.

b. If $\mu_1 > \frac{\gamma}{1+\gamma}$, $\lambda_1 + \gamma\mu_1 > \gamma$, and $(1+\gamma)\lambda_1 + \mu_1 \leq 1+\gamma$, then p is PAM and

$$R(\Gamma) = \begin{cases} \frac{\lambda_1 - \mu_2}{\mu_1}\gamma + \mu_2 & \text{if } \lambda_1 \geq \mu_2, \\ (1+\gamma)\lambda_1 - \mu_2\gamma & \text{otherwise.} \end{cases}$$

c. If $\lambda_1 > \frac{\gamma}{1+\gamma}$, $(1+\gamma)\lambda_1 + \mu_1 \leq 1+\gamma$, and $\mu_1 \leq \frac{\gamma}{1+\gamma}$, then p is BSM and $R(\Gamma) = \lambda_1(1+\gamma) - \frac{\gamma}{1+\gamma}$.

d. If $\lambda_1 \leq \frac{\gamma}{1+\gamma}$ and $\lambda_1 + \gamma\mu_1 \leq \gamma$, then p is RM and $R(\Gamma) = \gamma\lambda_1$.

Proposition 3, which is illustrated in Figure 2, shows that revenue maximization does not necessarily imply surplus maximization since PAM is suboptimal for some pair of (λ_1, μ_1). It hence presents a sharp contrast with our finding for the non-strategic benchmark.

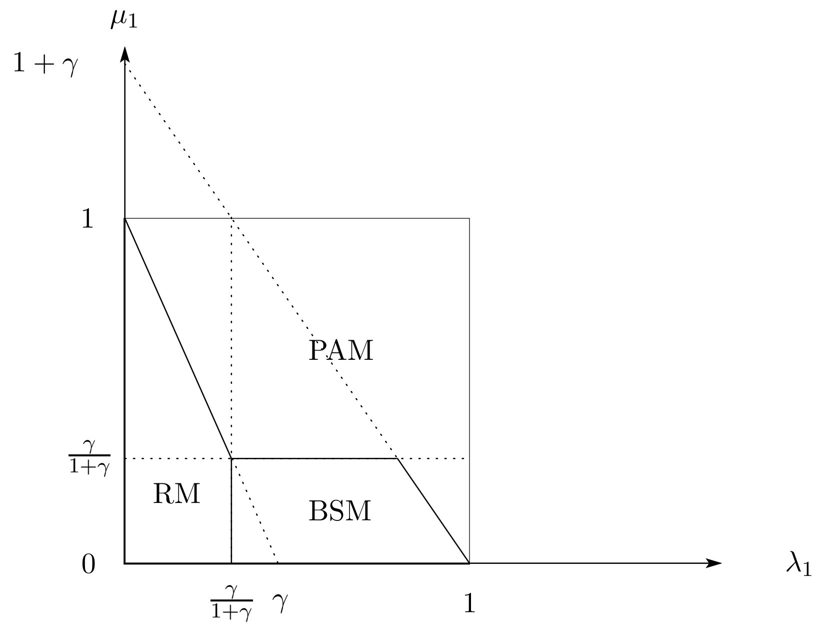

Figure 2: Optimal Matching with Buyer-Offer Bargaining

As mentioned in the Introduction, the suboptimality of PAM is a direct consequence of the strategic interactions between the subscribing agents. To see this, consider first the optimality of B-squeeze matching. As is standard in the screening models, the IR condition for the type β_1 buyers and the IC condition for the type β_2 buyers bind. Hence, the transfer $t_B(\beta_1)$ for β_1 equals his expected surplus from trade: $t_B(\beta_1) = p_A(\alpha_1 \mid \beta_1)(\beta_1 - \alpha_1) = p_A(\alpha_1 \mid \beta_1)\gamma$. On the other hand, the surplus from trade for type β_2 after misreporting is given by $p_A(\alpha_1 \mid \beta_1)(\beta_2 - \alpha_1) = p_A(\alpha_1 \mid \beta_1)(1+\gamma)$ if $p_A(\alpha_1 \mid \beta_1) \geq \frac{\gamma}{1+\gamma}$ (from the offer α_1 accepted only by α_1), and $\beta_2 - \alpha_2 = \gamma$ if $p_A(\alpha_1 \mid \beta_1) \leq \frac{\gamma}{1+\gamma}$ (from the offer α_2 accepted by both seller types). Since β_2's IC condition is binding, his payoff from subscription with truth-telling equals the payoff he would obtain by misreporting:

$$\pi_B(\sigma,\beta_2) - t_B(\beta_2) = \begin{cases} \gamma - t_B(\beta_1) = (1 - p_A(\alpha_1 \mid \beta_1))\gamma & \text{if } p_A(\alpha_1 \mid \beta_1) \leq \frac{\gamma}{1+\gamma}, \\ p_A(\alpha_1 \mid \beta_1)(1+\gamma) - t_B(\beta_1) = p_A(\alpha_1 \mid \beta_1) & \text{if } p_A(\alpha_1 \mid \beta_1) \geq \frac{\gamma}{1+\gamma}. \end{cases}$$

As indicated in the right panel of Figure 3, this surplus is minimized when $p_A(\alpha_1 \mid \beta_1) = \frac{\gamma}{1+\gamma}$. When the proportion μ_2 of type β_2 is sufficiently high in the population (*i.e.*, μ_1 is low), hence, the platform finds it optimal to squeeze their informational rents by setting $p_A(\alpha_1 \mid \beta_1) = \frac{\gamma}{1+\gamma}$ (and then choosing $p_A(\alpha_1 \mid \beta_2)$ to satisfy Bayes plausibility).

Consider next the optimality of RM. When the proportion λ_1 of low-cost sellers (α_1) in the population is low, the probability that type β_2 is matched with α_1 cannot be high, and hence type β_2 should optimally bid α_2 regardless of whether he reports his type truthfully or not. With IC for β_2 binding, however, we have $\pi_B(\sigma,\beta_2) - t_B(\beta_2) = \gamma - t_B(\beta_2) = \gamma - t_B(\beta_1)$ so that the buyer transfer must be independent of the report: $t_B(\beta_1) = t_B(\beta_2)$. This further implies that the matching rule must also be independent of the reported type, and hence that only RM is feasible.

In the non-strategic benchmark of Section 4, the situation is different. As can be readily seen, the surplus of type β_2 in this benchmark is an increasing linear function of $p_A(\alpha_1 \mid \beta_1)$ (the left panel of Figure 3). It then follows that β_2's surplus is minimized when $p_A(\alpha_1 \mid \beta_1)$ is minimized. The optimality of PAM hence follows.

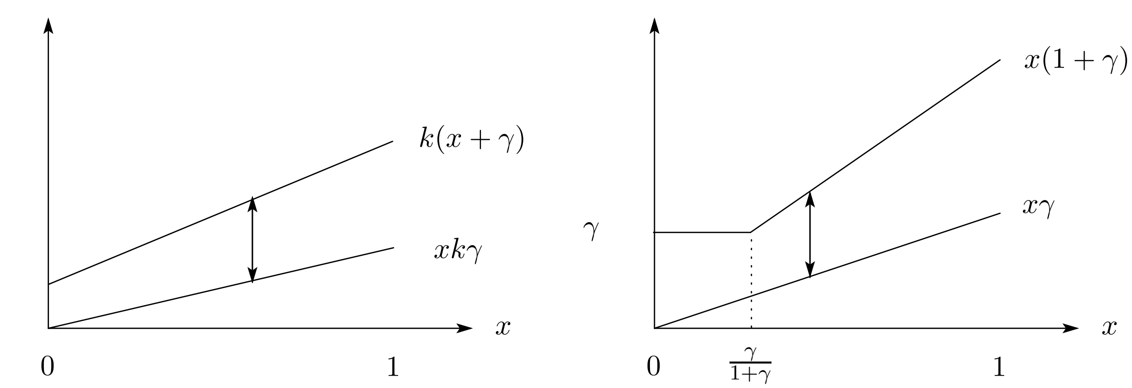

Figure 3: Informational rent of a type β_2 buyer in the non-strategic (left) and strategic (right) interaction: $x = p_A(\alpha_1 \mid \beta_1)$

Does the platform maximize its subscription revenue through the maximization of the surplus from trade between the sellers and buyers? In order to answer this question, we consider the efficient mechanism in the class of IC and IR mechanisms with buyer-offer bargaining. Specifically, the *incentive efficient* mechanism Γ maximizes max $W(\Gamma)$ subject to the IC an IR constraints when the agents play the BNE σ specified above in the buyer-offer bargaining game.

Proposition 4 *Suppose that the matching mechanism Γ maximizes social welfare in the class of IC and IR mechanisms with buyer-offer bargaining. Then the associated matching rule p and the corresponding social welfare are given by*

a. If $\lambda_1 + \gamma\mu_1 > \gamma$ or $\lambda_1 \geq \frac{\gamma}{1+\gamma}$, then p is PAM and $W(\Gamma) = \begin{cases} \gamma\lambda_1 + \mu_2 & \text{if } \lambda_1 \geq \mu_2, \\ (1+\gamma)\,\lambda_1 & \text{if } \lambda_1 < \mu_2. \end{cases}$

b. If $\lambda_1 + \gamma\mu_1 \leq \gamma$ and $\lambda_1 \leq \frac{\gamma}{1+\gamma}$, then p is RM and $W(\Gamma) = \lambda_1(\gamma + \mu_2)$.

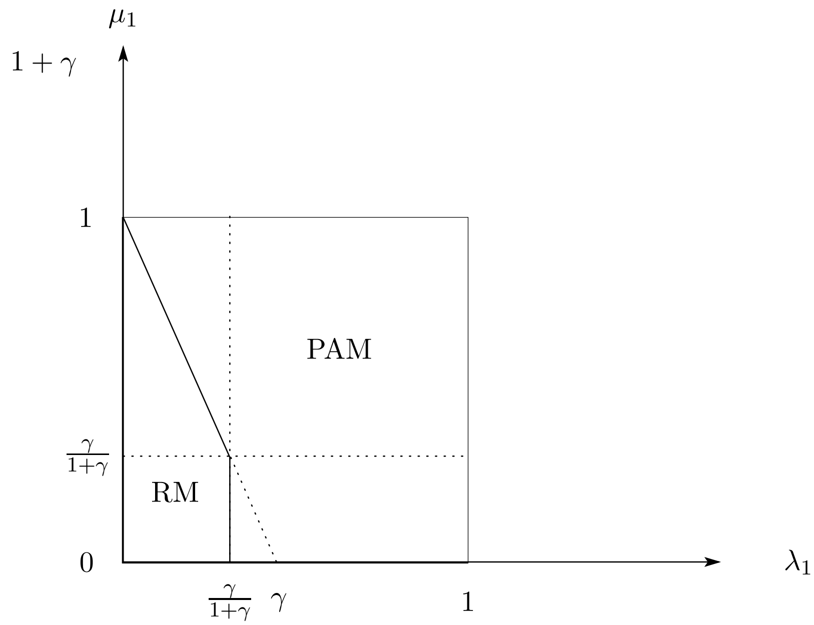

Figure 4: Incentive efficient matching with buyer-offer bargaining

The incentive efficient matching mechanism is illustrated in Figure 4. It can be seen from Propositions 1 and 4 that the incentive efficient mechanism with buyer-offer bargaining is first-best efficient when $\lambda_1 \geq \mu_2$. Comparing Figures 2 and 4, we also see that revenue maximization is equivalent to welfare maximization except when the optimal mechanism entails BSM. Put differently, when the market has a high proportion of high-valuation buyers and a relatively large proportion of low-cost sellers, the subscription revenue is maximized at the expense of social welfare.

6 Seller-Offer versus Buyer-Offer Bargaining

The analysis in the previous section studies the optimal matching mechanism under the pricing rule $k(z) = z_B$ which corresponds to buyer-offer bargaining. A natural question concerns whether or not the platform can do better by having the sellers make offers instead. Given the symmetric nature of the problems, we expect the answer to depend on the proportion of types on each side. We begin by describing the optimal mechanism with seller-offer bargaining, which is expressed formally by setting the pricing rule $k(z) = z_A$. Let $\sigma = (\sigma_A, \sigma_B)$ be the strategy profile of this game in which a buyer plays a weakly dominant strategy of bidding his own type: $\sigma_B(\beta) = \beta$ for every β. Given his belief $\tilde{p}_B$ about a buyer's type, a seller's optimal bid is then given by

$$z_A^*(\alpha, \tilde{p}_B) = \begin{cases} \beta_1 & \text{if } \alpha = \alpha_1 \text{ and } \tilde{p}_B(\beta_2) < \frac{\gamma}{1+\gamma}, \\ \beta_2 & \text{if } \alpha = \alpha_2, \text{ or if } \alpha = \alpha_1 \text{ and } \tilde{p}_B(\beta_2) \geq \frac{\gamma}{1+\gamma}. \end{cases}$$

Again, the only viable bid for the high-cost (α_2) seller is β_2, whereas the optimal bid for the low-cost (α_1) seller is either high or low depending on his belief about the type of the matched buyer. We specify the seller's strategy σ_A by letting $\sigma_A(\alpha) = z_A^*(\alpha, p_B(\cdot \mid \alpha))$ for every α. The assortative and random matching rules are as defined in the previous section.[8] As a counterpart to the B-squeeze matching rule defined in the previous section, when $\lambda_1 + (1+\gamma)\mu_1 \geq 1$, we define a matching rule p to be *S-squeeze* (SSM) if

$$(p_B(\beta_2 \mid \alpha_1), p_B(\beta_2 \mid \alpha_2)) = \Big(\frac{\mu_2}{\lambda_1} - \frac{\lambda_2}{\lambda_1}\frac{\gamma}{1+\gamma}, \frac{\gamma}{1+\gamma}\Big).$$

We can verify that $1 \geq p_B(\beta_2 \mid \alpha_1) \geq p_B(\beta_2 \mid \alpha_2)$ if and only if $\lambda_1 + (1+\gamma)\mu_1 \geq 1$ and $\mu_1 \leq \frac{1}{1+\gamma}$. This rule makes a low-cost seller α_1 exactly indifferent between bidding high β_2 and bidding low β_1 when he faces the type distribution of a buyer intended for a high-cost seller α_2, and minimizes the informational rent of the low-cost seller by the same logic as in the case of B-squeeze matching.

Proposition 5 *Suppose that the mechanism Γ is optimal with seller-offer bargaining. Then its matching rule p and revenue $R(\Gamma)$ are given as follows.*

i. If $\lambda_1 + (1+\gamma)\mu_1 \leq 1$, then p is PAM and $R(\Gamma) = (1+\gamma)\mu_2 - \frac{\mu_2 - \lambda_1}{\lambda_2}$.

ii. If $\lambda_1 \leq \frac{1}{1+\gamma}$, $\gamma\lambda_1 + \mu_1 \leq 1$, and $\lambda_1 + (1+\gamma)\mu_1 > 1$, then p is PAM and

$$R(\Gamma) = \begin{cases} \frac{\mu_2 - \lambda_1}{\lambda_2}\gamma + \lambda_1 & \textit{if } \lambda_1 \leq \mu_2, \\ (1+\gamma)\mu_2 - \lambda_1\gamma & \textit{otherwise.} \end{cases}$$

iii. If $\lambda_1 > \frac{1}{1+\gamma}$, $\lambda_1 + (1+\gamma)\mu_1 > 1$, and $\mu_1 \leq \frac{1}{1+\gamma}$, then p is SSM and $R(\Gamma) = \mu_2(1+\gamma) - \frac{\gamma}{1+\gamma}$.

iv. If $\mu_1 > \frac{1}{1+\gamma}$ and $\gamma\lambda_1 + \mu_1 > \gamma$, then p is RM and $R(\Gamma) = \gamma\mu_2$.

As seen in Figure 5, the optimal configuration with seller-offer bargaining is exactly symmetric to that with buyer-offer bargaining with respect to the diagonal line $\lambda_1 + \mu_1 = 1$ ($\Leftrightarrow \lambda_1 = \mu_2$).

In general, comparison of performance between buyer-offer and seller-offer bargaining is not straightforward.[9] We can however verify that PAM with buyer-offer (resp. seller-offer) bargaining dominates RM with seller-offer (resp. buyer-offer) bargaining.

[8] They can alternatively be defined in terms of $z = p_B(\beta_2 \mid \alpha_1)$ and $w = p_B(\beta_2 \mid \alpha_2)$: p is assortative if

$$(p_B(\beta_2 \mid \alpha_1), p_B(\beta_2 \mid \alpha_2)) = \begin{cases} \left(1, \frac{\mu_2 - \lambda_1}{\lambda_2}\right) & \text{if } \mu_2 > \lambda_1, \\ \left(\frac{\mu_2}{\lambda_1}, 0\right) & \text{if } \mu_2 < \lambda_1, \end{cases}$$

and random if $(p_B(\beta_2 \mid \alpha_1), p_B(\beta_2 \mid \alpha_2)) = (\mu_2, \mu_2)$.

[9] In particular, it is difficult to establish the dominance relationship between PAM with buyer-offer (resp. seller-offer) bargaining and SSM (resp. BSM) with seller-offer (resp. buyer-offer) bargaining.

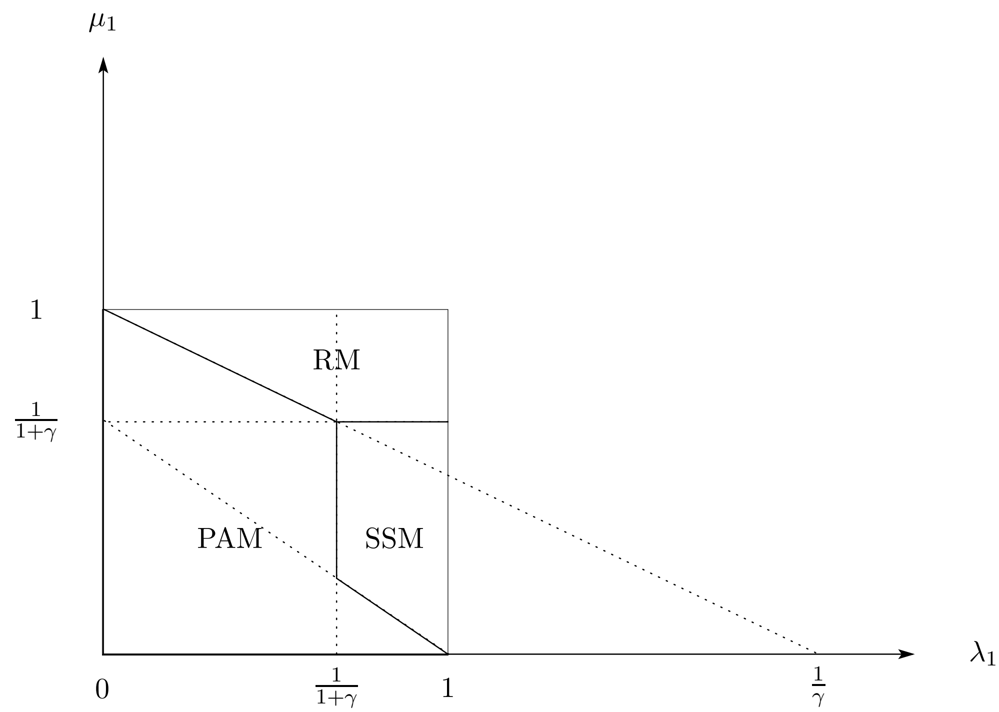

Figure 5: Optimal Matching with Seller-Offer Bargaining

Proposition 6 *Let* Γ *be an optimal mechanism with either seller-offer or buyer-offer bargaining. Then the associated matching rule* p *is either PAM, BSM, or SSM.*

We next consider a market that deviates from symmetry just slightly. We say that side A has higher (resp. lower) quality than side B if the proportion of low cost sellers on side A is higher (resp. lower) than that of high valuation buyers on side B: $\lambda_1 > \mu_2$. The following proposition shows that in a slightly asymmetric market, the optimal mechanism employs a protocol where the side with the lower quality makes an offer. Put differently, it is optimal to have seller-offer bargaining if the proportion of low-cost sellers on side A is lower than the proportion of high-valuation buyers on side B, and vice versa.

Proposition 7 *Take any* $d \in (0, 1)$*. There exists* $\varepsilon > 0$ *such that if* $\|(\lambda_1, \mu_2) - (d, d)\| < \varepsilon$*, then the optimal mechanism* Γ *entails buyer-offer bargaining if* $\lambda_1 > \mu_2$ *and seller-offer bargaining if* $\lambda_1 < \mu_2$*.*

A clear conclusion is possible regarding the comparison of buyer-offer and seller-offer bargaining when the market is symmetric.

Proposition 8 *Suppose that the market is symmetric* ($\lambda_1 = \mu_2$)*. Then the optimal mechanism with buyer-offer bargaining and that with seller-offer bargaining yield the same revenue.*

Proposition 9 *Suppose that the market is symmetric with* $\lambda_1 = \mu_2 = d$*. Let* Γ *be an optimal mechanism with either seller-offer or buyer-offer bargaining. Then it entails PAM and yields* d *if* $d \leq \frac{1}{1+\gamma}$*, and BSM or SSM and yields* $d(1+\gamma) - \frac{\gamma}{1+\gamma}$ *if* $d > \frac{1}{1+\gamma}$*.*

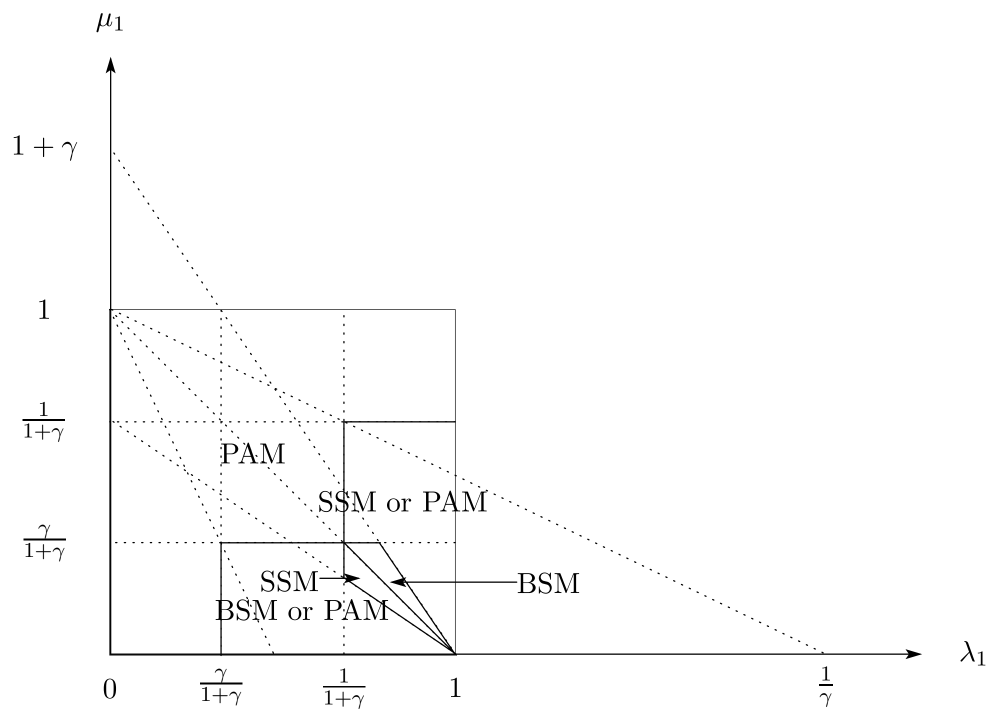

Figure 6: Optimal Matching with Seller-Offer or Buyer-Offer Bargaining

PAM: Positive assortative with buyer- or seller-offer
BSM: B-squeeze with buyer-offer
SSM: S-squeeze with seller-offer

Given that PAM in a symmetric market generates no mismatch by matching every low cost seller with a high valuation buyer and vice versa, the mechanism involving PAM is clearly first-best and also incentive efficient. In this sense, it is striking to observe that it is optimal only when the market quality is below a certain threshold, and is dominated by both BSM and SSM when the quality is above the threshold. In such a market, the creation of a clearly inefficient mismatch generates a higher revenue.

Combining Propositions 2 and 9 reveals another interesting fact. Recall from Proposition 2 that for any value of $d = \lambda_1 = \mu_2$, the platform's optimal revenue under non-strategic interaction equals d (as under PAM in Propositions 9). Hence, the maximal revenue under BSM or SSM under strategic interactions is higher than the maximal revenue with PAM under non-strategic interactions when $d > \frac{1}{1+\gamma}$.

7 Extensions of the Baseline Model

We have so far assumed that the bargaining protocol is essentially a sequential game with either a buyer or seller making a take-it-or-leave-it offer to the other side, and that the latter plays the weakly dominant strategy in the BNE σ specified by the mechanism. What then happens when the mechanism specifies the BNE σ most preferred by the platform? We address this question by supposing that the pricing rule k is given by $k(z) = kz_A + (1-k)z_B$ for some constant $k \in [0,1]$, and that the mechanism specifies a strategy profile $\sigma = (\sigma_A, \sigma_B)$ such that

$$\sigma_A(\alpha) = \begin{cases} \zeta & \text{if } \alpha = \alpha_1, \\ \beta_2 & \text{if } \alpha = \alpha_2, \end{cases} \quad \text{and} \quad \sigma_B(\beta) = \zeta \quad \text{for every } \beta, \tag{9}$$

where $\zeta \in [\alpha_1, \beta_1]$. σ_B is the buyer's best response against σ_A regardless of his belief $\tilde{p}_A$ about α. Furthermore, σ_A is also the seller's best response against σ_B although the high cost type (α_2) will never trade his good under σ. The following proposition shows that when σ is as given in (9) for $\zeta = \alpha_1$, the platform's revenue in the symmetric market equals the first-best level identified in Proposition 1.

Proposition 10 *Suppose that the mechanism Γ entails PAM and the BNE σ in (9) with $\zeta = \alpha_1$. When the market is symmetric, Γ extracts full surplus from the agents and hence is optimal.*

As noted above, σ in Proposition 10 is not the most natural BNE when for example $k = 0$ (buyer-offer bargaining) or $k = 1$ (seller-offer bargaining): When $k = 0$, $\sigma_A(\alpha_2) = \beta_2$ is weakly dominated for the high-cost seller (α_2), and when $k = 1$, $\sigma_B(\beta) = \alpha_1$ is weakly dominated for both buyer types. In other words, it is not possible to replicate such a σ in a buyer-offer or seller-offer game while requiring sequential rationality.

We next consider the possibility that the platform can charge subscription fees contingent on the outcome of the trading game. Specifically, we assume that a subscription fee is contingent on the reported types of the agents and on whether or not the transaction takes place, but not on the transaction price. While such a fee scheme is observed in reality, it requires a different institutional setting from our baseline model. First, the platform needs to monitor whether or not the transaction has actually taken place. In particular, it should prevent secret exchange arrangements outside the system.[10] Second, the platform needs to enforce the payment of the fee even after the transaction. With the requirement of interim individual rationality as assumed elsewhere in the paper, it is clear that the optimal mechanism with outcome-contingent subscription fees generates a weakly higher revenue than the optimal mechanism in the baseline model since information about the transaction can always be ignored. The following proposition shows that the optimal outcome-contingent mechanism with buyer-offer bargaining entails PAM, and strictly dominates the optimal non-PAM mechanism with buyer-offer bargaining in the baseline model for almost every type distribution of agents. Denote by $t_A(\alpha)$ and $t_B(\beta)$ the transfer payments required of a type α seller and a type β buyer, respectively, when there is a successful transaction.[11]

Proposition 11 *Suppose that the transfer can be contingent on the outcome of the transaction. If $\tilde{\Gamma}$ is an optimal mechanism with buyer-offer bargaining that entails no transfer from sellers (*i.e.,

[10] For example, information about the agents' addresses need to be withheld so that physical trading of the good will not be possible until after the payment is made.

[11] Positive subscription fees only in the event of a successful transaction ensure that the mechanism satisfies the stronger requirement of ex post individual rationality. The proposition also assumes that no subscription fee is required from a seller for simplicity. Consideration of a subscription fee for a seller with buyer-offer bargaining complicates the analysis substantially with little added insight.

$t_A(\alpha_1) = t_A(\alpha_2) = 0$), then the associated matching rule is PAM, and the revenue is equal to

$$R(\tilde{\Gamma}) = \begin{cases} \lambda_1(1+\gamma) & \text{if } \frac{\lambda_1}{\mu_2} \leq 1, \\ \lambda_1(1+\gamma) - \frac{\lambda_1 - \mu_2}{\mu_1} & \text{if } \frac{\lambda_1}{\mu_2} > 1. \end{cases} \tag{10}$$

Comparison of Propositions 1 and 11 shows that $\tilde{\Gamma}$ extracts the full social surplus when the market is symmetric (*i.e.,* $\lambda_1 = \mu_2$).[12] The resurgence of PAM as the optimal matching rule in Proposition 11 is a consequence of changes in the agents' strategic incentives induced by the outcome-contingent fees. Specifically, for a high-valuation buyer (β_2), the surplus expected from a high bid α_2 and an aggressive bid α_1 depends on the transfer payment required in the event of a successful transaction. The platform finds it optimal to induce type β_2 to bid α_1 whether he reports his type truthfully or not: When $\lambda_1 \leq \mu_2$, for example, the optimal fee for type β_2 equals $t_B(\beta_2) = \beta_2 - \alpha_1 = 1+\gamma$, inducing type β_2 to bid α_1 after truthful reporting. Misreporting by β_2 is prevented by never matching β_1 with α_1 (*i.e.,* $p_A(\alpha_1 \mid \beta_1) = 0$), and setting $t_B(\beta_1) = \beta_2 - \alpha_2 = \gamma$. Minimization of $p_A(\alpha_1 \mid \beta_1)$ is equivalent to the use of PAM.

8 Auction Platform

In this section, we consider a variation of the baseline model and suppose that two buyers are matched to a single seller and competitively bid for the seller's good. Specifically, side A has a mass of sellers indexed by numbers in $[0, 1]$, whereas side B has a mass of buyers indexed by numbers in $[0, 2]$. Each seller is endowed with a single unit of a good of quality α, which represents his type: The good is either high quality α_2 or low quality α_1.[13] Each buyer has a unit demand for the good, and has type β that reflects his valuation of the good: The type is either high β_2 or low β_1. For a buyer of type β, the value of the good of quality α is given by $v(\alpha, \beta)$. Denote

$$v_{11} = v(\alpha_1, \beta_1), \quad v_{12} = v(\alpha_1, \beta_2), \quad v_{21} = v(\alpha_2, \beta_1), \quad \text{and} \quad v_{22} = v(\alpha_2, \beta_2).$$

We assume that v has increasing differences in the sense that

$$0 \leq \Delta_1 \equiv v_{12} - v_{11} < v_{22} - v_{21} \equiv \Delta_2. \tag{11}$$

Equivalently, the marginal increase in utility in response to an increase in quality is higher for the high-valuation buyer than for the low-valuation buyer. The seller's valuation of the good equals zero regardless of its quality.[14] Each seller is type α_i with probability λ_i and each buyer is type β_i with probability μ_i. The type realizations are independent across agents so that we may again identify λ_i as the proportion of type α_i sellers in side A, and μ_i as the proportion of type β_i buyers on side B. We assume that the quality α of a seller's good is observable to the buyers who are matched with him although it is not directly observable to the platform.

[12] Proposition 16 in the appendix establishes that $\tilde{\Gamma}$ dominates the optimal mechanism Γ of the baseline model.

[13] Uncertainty about the quality of the sellers' good is also assumed in Tamura (2016).

[14] It follows that the aggregate surplus from trade in a match involving seller of type α_i and buyers of types β_j and β_k can be written as $f(\alpha_i, \beta_j, \beta_k) = \max\{v_{ij}, v_{ik}\}$. (11) is not consistent with the supermodularity of f which would require $v_{12} - v_{11} = v_{22} - v_{21} = 0$ when the ordering on the domain of f is induced by $\alpha_1 \prec \alpha_2$ and $\beta_1 \prec \beta_2$.

Matching between a seller and two buyers is implemented through the allocation of buyers to two buyer slots: A buyer with index $k \leq 1$ is allocated to the first slot while a buyer with index $k > 1$ is allocated to the second slot.[15] A matching rule $p = (p_{111}, \ldots, p_{222})$ is a probability distribution over $A \times B^2$: p_{ijk} ($i,\ j,\ k \in \{1,2\}$) is the probability that any given match involves a seller of type α_i along with a buyer of type β_j in the first slot, and a buyer of type β_k in the second slot. We assume that the platform treats the two buyer slots symmetrically:

$$p_{ijk} = p_{ikj} \quad \text{for any } i,\ j,\ k \in \{1,2\}. \tag{12}$$

The matching rule p must also be consistent with the type distribution in the population:

$$\begin{array}{rcl} p_{111} + 2p_{112} + p_{122} = \lambda_1 & \Leftrightarrow & p_{211} + 2p_{212} + p_{222} = \lambda_2, \\ p_{111} + p_{112} + p_{211} + p_{212} = \mu_1 & \Leftrightarrow & p_{121} + p_{122} + p_{221} + p_{222} = \mu_2. \end{array} \tag{13}$$

(13) is a version of the Bayes plausibility conditions when every seller is matched with two buyers. The set P of feasible matching rules is hence given by

$$P = \{p \in A \times B^2 : p \text{ satisfies (12) and (13)}\}.$$

We first suppose that each seller sells his good to the matched buyers through a second-price auction: Buyers submit sealed bids and the bidder with the higher bid wins and pays the price equal to the lower bid. We will later analyze the problem when the transaction is through a first-price auction. In the auction game, buyers choose their bids strategically. When the quality of the seller's good α, let $\sigma_B(\cdot \mid \alpha)$ be the bidding strategy specified by the mechanism. Since it is a weakly dominant strategy for a buyer to bid his true valuation in a second-price auction, we assume that σ_B is given by

$$\sigma_B(\beta \mid \alpha) = v(\alpha, \beta) \quad \text{for every } (\alpha, \beta).^{16}$$

The following proposition identifies the optimal matching mechanism by solving this problem.

Proposition 12 *Suppose that the good is traded through a second-price auction. Then the matching rule p under the optimal mechanism Γ is described as follows.*

	p_{111}	p_{112}	p_{122}	p_{211}	p_{212}	p_{222}
$0 \leq \mu_2 \leq \frac{\lambda_2}{2}$	λ_1	0	0	$\lambda_2 - 2\mu_2$	μ_2	0
$\frac{\lambda_2}{2} < \mu_2 \leq \frac{1}{2}$	$1 - 2\mu_2$	$\mu_2 - \frac{\lambda_2}{2}$	0	0	$\frac{\lambda_2}{2}$	0
$\frac{1}{2} < \mu_2 \leq 1 - \frac{\lambda_1}{2}$	0	$\frac{\lambda_1}{2}$	0	0	$\mu_1 - \frac{\lambda_1}{2}$	$1 - 2\mu_1$
$1 - \frac{\lambda_1}{2} < \mu_2 \leq 1$	0	μ_1	$\lambda_1 - 2\mu_1$	0	0	λ_2

[15] Given the independence of type realizations, we may assume that the type distributions of buyers are the same between the first and second intervals.

[16] Denote by $f_B(\beta_j; \alpha, \beta_k)$ the value of a match to a buyer in this BNE when his own type is β_j, the other bidder's type is β_k, and the seller's type is α. The BNE value of a match to agents is not supermodular. In fact, when a buyer's type changes from β_1 to β_2, the difference in his payoff against (α_2, β_2) is smaller than that against (α_1, β_1):

$$f_B(\beta_1; \alpha_2, \beta_2) - f_B(\beta_1; \alpha_1, \beta_1) = 0 > -(v_{12} - v_{11}) = f_B(\beta_2; \alpha_2, \beta_2) - f_B(\beta_2; \alpha_1, \beta_1).$$

In order to understand the optimal matching rule p in Proposition 12, we consider the probability that an agent of each type is matched with different types of other agents conditional on his own type. Specifically, let $p_{BB}(\beta_j, \beta_k \mid \alpha_i)$ denote the probability that a type α_i seller is matched with a buyer type pair (β_j, β_k), and $p_A(\alpha_i \mid \beta_j)$ denote the probability that a type β_j buyer is matched with a type α_i seller $(i,\, j,\, k = 1, 2)$.[17] We also introduce an ordering over the type profile of buyers such that $(\beta_1, \beta_1) \prec (\beta_1, \beta_2) \sim (\beta_2, \beta_1) \prec (\beta_2, \beta_2)$. The following corollary shows that a high type seller is more likely to be matched with a buyer pair of higher order, and a high type buyer is more likely to be matched with a high type seller, both in the sense of stochastic dominance.

Corollary 1 *Suppose that Γ is the optimal mechanism. Then its matching rule p satisfies*

$$p_{BB}(\cdot \mid \alpha_2) \succ_{\text{FOSD}} p_{BB}(\cdot \mid \alpha_1), \quad \textit{and} \quad p_A(\cdot \mid \beta_2) \succ_{\text{FOSD}} p_A(\cdot \mid \beta_1),$$

where $p_{BB}(\cdot \mid \alpha) = \big(p_{BB}(\beta_1, \beta_1 \mid \alpha),\, 2p_{BB}(\beta_1, \beta_2 \mid \alpha),\, p_{BB}(\beta_2, \beta_2 \mid \alpha)\big)$, and $p_A(\cdot \mid \beta) = \big(p_A(\alpha_1 \mid \beta),\, p_A(\alpha_2 \mid \beta)\big)$.

Despite the above observation, we see from the description of p in Proposition 12 that it is not positively assortative in the sense that a high type is matched with high types to the fullest extent possible. For example, p does not maximize the probability that a high type seller is matched with a pair of high type buyers. However, we make an interesting observation that p is a combination of positive and negative assortative matching as follows. We say that p entails *negative assortative matching* (NAM) *between buyers* if it matches a high type buyer with a low type buyer as much as possible, and vice versa:

$$p \in P^{\text{NAMBB}} \equiv \operatorname{argmax}\{\hat{p}_B(\beta_1 \mid \beta_2) : \hat{p} \in P\},$$

where $p_B(\beta_j \mid \beta_k) = \frac{\sum_i p_{ijk}}{\sum_{i,j'} p_{ij'k}}$ is the probability that a type β_k buyer is matched with a type β_j buyer. Next, p is *PAM between a seller and a buyer pair subject to NAM between buyers* if among those rules in P^{NAMBB}, it first maximizes the probability that a type α_2 seller is matched with the buyer type pair (β_2, β_2), and then maximizes the probability that α_2 is matched with the buyer type pair (β_1, β_2):

$$p \in \operatorname{argmax}\{\hat{p}_{BB}(\beta_1, \beta_2 \mid \alpha_2) : \hat{p} \in P^1\}, \text{ where } P^1 \equiv \operatorname{argmax}\{\hat{p}_{BB}(\beta_2, \beta_2 \mid \alpha_2) : \hat{p} \in P^{\text{NAMBB}}\}.$$

The following proposition shows that the optimal matching rule combines PAM and NAM in the sense described above.

Proposition 13 *The matching rule p in the optimal mechanism Γ in Proposition 12 entails PAM between a seller and a buyer pair subject to NAM between buyers.*

NAM between buyers is interpreted as follows. As before, the IC condition for the type β_2 buyers is binding so that

$$\pi_B(\sigma, \beta_2) - t_B(\beta_2) = p_B(\beta_1 \mid \beta_1)\,(\beta_2 - \beta_1) + p_B(\beta_2 \mid \beta_1)\,\cdot 0 - t_B(\beta_1),$$

[17] $p_{BB}(\beta_j, \beta_k \mid \alpha_i) = \dfrac{p_{ijk}}{\sum_{j',k'} p_{ij'k'}}$ and $p_A(\alpha_i \mid \beta_j) = \dfrac{\sum_k p_{ijk}}{\sum_{i',k} p_{i'j,k}}$.

where the right-hand side is β_2's payoff when he misreports his type as β_1. It follows that β_2's informational rent is minimized when $p_B(\beta_2 \mid \beta_1)$ is maximized as entailed by NAM between buyers.

We next examine the welfare implication of the optimal matching mechanism Γ identified in Proposition 12. Note that the efficiency of Γ is expressed in terms of its matching rule p by

$$W(p) = p_{111}v_{11} + (2p_{112} + p_{122})v_{12} + p_{211}v_{21} + (2p_{212} + p_{222})v_{22}.$$

Proposition 14 *If p is the matching rule in the optimal mechanism Γ in Proposition 12, then $p \in \operatorname{argmax}\{W(\hat{p}) : \hat{p} \in P\}$. It follows that Γ is first-best efficient.*

Having characterized the optimal mechanism under the second-price auction, we now suppose that the platform stipulates the use of the sealed-bid first-price auction for transaction. In this case, we can show by the standard argument that in the BNE of the auction with a type α seller, a low valuation buyer (β_1) bids his value $v_{\alpha 1}$ and a high valuation buyer (β_2) submits a random bid. Specifically, consider the auction game on the path where both buyers have reported their types truthfully so that the joint distribution of type profiles is given by p. The cumulative distribution G_α of β_2's random bid has support $[v_{\alpha 1}, \bar{b}_\alpha]$ for some $\bar{b}_\alpha$. In the Appendix, we show that $\bar{b}_\alpha$ and G_α are given by

$$\bar{b}_\alpha = \Pr(\beta_1 \mid \alpha, \beta_2)v_{\alpha 1} + \Pr(\beta_2 \mid \alpha, \beta_2)v_{\alpha 2},$$

and

$$G_\alpha(b) = \frac{\Pr(\beta_1 \mid \alpha, \beta_2)}{\Pr(\beta_2 \mid \alpha, \beta_2)} \left(\frac{b - v_{\alpha 1}}{v_{\alpha 2} - b}\right) = \frac{p_{\alpha 12}}{p_{\alpha 22}} \left(\frac{b - v_{\alpha 1}}{v_{\alpha 2} - b}\right).$$

Unlike in the case of the second-price auction, a buyer's bidding strategy depends on his belief about the type of the other buyer he is matched with. Specifically, when a buyer's belief places more weight on the other buyer being the high valuation type (β_2), the distribution of his bid is higher in the sense of stochastic dominance. To solve for the optimal matching mechanism, we first note that since a type β_2 bidder is indifferent over bids in the support of G_α, his BNE payoff in the game with a type α seller is given by $\Pr(\beta_1 \mid \alpha, \beta_2)(v_{\alpha 2} - v_{\alpha 1})$, which he would obtain by bidding slightly above $v_{\alpha 1}$. When he misreports, his expected payoff is likewise given by $\Pr(\beta_1 \mid \alpha, \beta_1)(v_{\alpha 2} - v_{\alpha 1})$. For a type β_1 seller, his expected payoff equals zero whether or not he reports truthfully. We can therefore conclude that the IC and IR conditions for both buyer types have exactly the same expressions as (32) and (33) for the second-price auction. On the other hand, a seller's expected revenue after truthful reporting as well as misreporting can be computed from the expected payment by a buyer, and is again shown to be the same as that under the second-price auction. Hence, the IC and IR conditions for both seller types are given by (36) and (37). Taken together, the agents' incentives under the two auction formats are exactly the same, and hence so are the optimal mechanisms in the two cases. The following proposition summarizes this observation.

Proposition 15 *Suppose that the good is traded through the first-price auction. The matching rule p under the optimal matching mechanism Γ is the same as that in Proposition 12 for the second-price auction.*

9 Conclusion

The starting point of our analysis is the observation that few platforms in the real world dictate the terms of trade between their subscribers. When the subscribers play a game against each other, we note that the value of a match created by the platform is endogenously determined by the BNE payoff of the game. Since the BNE depends on the type distribution of the players, there is room for the platform to manipulate the subscribers' beliefs through its matching rule and induce its preferred BNE. Exploration of this possibility provides new insights into the functioning of platforms. In a model of a one-to-one trading platform, we show that the optimal mechanism entails matching rule that is not PAM depending on the proportion of seller and buyer types in the population, and discuss that the optimality of these non-PAM rules is a direct consequence of strategic interactions that are largely ignored in the literature. In a model of an auction platform that matches each seller with two buyers, we show that the optimal mechanism entails a matching rule that combines NAM and PAM, and is first-best efficient.

There are a number of interesting extensions of the model studied in this paper. For example, we may consider a model with continuous type distributions of agents. Although such an extension would bring the model closer to the standard discriminatory pricing models, a clear difficulty is with the numerous variations of feasible matching rules. For example, a matching rule may be of the cutoff type that creates two matching classes: One class consists of agent types above certain thresholds, and the other class consists of agent types below the thresholds. The platform may however choose finer partitions of the type space, and match a pair of segments in non-assortative ways. There are also matching rules that are not based on thresholds and apportion the same type across different match types. In our model of an auction platform, each seller is matched with two buyers. We can alternatively consider a platform that varies the number of buyers matched with each seller depending on their types. We focus on a monopoly platform, and do not discuss how it has acquired the proprietary status in the market. Monopolization can be the outcome of competition, and formal analysis is required on if and how competition among multiple platforms leads to monopolization. An important consideration in modeling such competition is the fact that the externalities between agents are also determined endogenously by equilibrium strategic decisions. This is contrary to the standard exogenous specification of externalities in the literature.

Appendix

For simplicity, we use the following notation in the analysis of the trading platform in the Appendix:

$$\begin{aligned} x &= p_A(\alpha_1 \mid \beta_1), \quad y = p_A(\alpha_1 \mid \beta_2), \\ z &= p_B(\beta_2 \mid \alpha_1), \quad w = p_B(\beta_2 \mid \alpha_2). \end{aligned} \tag{14}$$

Proof of Lemma 1. Take any (x, y) that satisfies the conditions. Define p by

$$\begin{aligned} p(\alpha_1, \beta_1) &= \mu_1 x, & p(\alpha_1, \beta_2) &= \mu_2 y, \\ p(\alpha_2, \beta_1) &= \mu_1(1 - x), & p(\alpha_2, \beta_2) &= \mu_2(1 - y). \end{aligned}$$

We then have $p \in P$ since $\sum_\beta p(\alpha_1, \beta) = \lambda_1$, $\sum_\beta p(\alpha_2, \beta) = 1 - \lambda_1 = \lambda_2$, $\sum_\alpha p(\alpha, \beta_1) = \mu_1$, and $\sum_\alpha p(\alpha, \beta_2) = \mu_2$. Furthermore, these imply that p satisfies (14): $p_A(\alpha_1 \mid \beta_1) = \frac{\mu_1 x}{\mu_1} = x$, $p_A(\alpha_1 \mid \beta_2) = \frac{\mu_2 y}{\mu_2} = y$, $p_A(\alpha_2 \mid \beta_1) = \frac{\mu_1(1-x)}{\mu_1} = 1 - x$, and $p_A(\alpha_2 \mid \beta_2) = \frac{\mu_2(1-y)}{\mu_2} = 1 - y$. ■

Proof of Proposition 1. The mechanism is efficient only if $\alpha < \beta$ implies $\sigma_A(\alpha) \le \sigma_B(\beta)$ so that transaction takes place with probability one between any such pair of agents. In this case, the social welfare W is described as

$$\begin{aligned} W &= \gamma\{p_{11} + p_{22}\} + (1+\gamma)p_{12} \\ &= \gamma\{\mu_1 x + \mu_2(1-y)\} + (1+\gamma)\mu_2 y \\ &= \gamma\mu_1 x + \mu_2 y + \gamma\mu_2. \end{aligned}$$

Since $\frac{\mu_1}{\mu_2} > \frac{\gamma\mu_1}{\mu_2}$, maximization of W with respect to (x, y) subject to the Bayes plausibility condition (3) implies that y should be maximized subject to it. This shows that p is PAM. Substitution of (x, y) for PAM in (5) yields (6). ■

Proof of Proposition 2. For x and y defined in (14), the IC and IR conditions for a type β_1 buyer can be written as:

$$xk(\beta_1 - \alpha_1) - t_B(\beta_1) \ge \max\Big\{0,\ yk(\beta_1 - \alpha_1) - t_B(\beta_1)\Big\},$$

and those for a type β_2 buyer can be written as:

$$yk(\beta_2 - \alpha_1) + (1-y)k(\beta_2 - \alpha_1) - t_B(\beta_2) \ge \max\Big\{0,\ xk(\beta_2 - \alpha_1) + (1-x)k(\beta_2 - \alpha_2) - t_B(\beta_1)\Big\}.$$

Since these imply

$$(y - x)k(\beta_1 - \alpha_1) \le t_B(\beta_2) - t_B(\beta_1) \le (y - x)k(\alpha_2 - \alpha_1),$$

we need $y \ge x$ for the feasibility of the mechanism. In this case, the optimal transfers are given by

$$t_B(\beta_1) = xk\gamma \quad \text{and} \quad t_B(\beta_2) = t_B(\beta_1) + (y - x)k.$$

On the other hand, the IC and IR conditions for a type α_1 seller are given by

$$(1-z)(1-k)(\beta_1-\alpha_1)+z(1-k)(\beta_2-\alpha_1)-t_A(\alpha_1) \ge \max\{0,\ (1-w)(1-k)(\beta_1-\alpha_1)+w(1-k)(\beta_2-\alpha_1)-t_A(\alpha_2)\},$$

and those for a type α_2 seller are given by

$$w(1-k)(\beta_2 - \alpha_2) - t_A(\alpha_2) \ge \max\{0,\ z(1-k)(\beta_2 - \alpha_2) - t_A(\alpha_1)\}.$$

Together, these imply

$$(z - w)(1-k)(\beta_2 - \alpha_2) \le t_A(\alpha_1) - t_A(\alpha_2) \le (z - w)(1-k)(\beta_2 - \beta_1),$$

and hence feasibility requires $z \ge w$, or equivalently, $y \ge \lambda_1$.[18] In this case, the optimal transfers are given by

$$t_A(\alpha_2) = \frac{\mu_2}{\lambda_2}(1-y)(1-k)\gamma \quad \text{and} \quad t_A(\alpha_1) = t_A(\alpha_2) + \left(\frac{\mu_2}{\lambda_1}y - \frac{\mu_2}{\lambda_2}(1-y)\right)(1-k).$$

[18] This holds since $z = \frac{\mu_2}{\lambda_1}y$ and $w = \frac{\mu_2}{\lambda_2}(1-y)$.

It follows that the platform's revenue from both sides of the market under the optimal transfer functions is given by

$$
\begin{aligned}
R(\Gamma) &= w(1-k)\gamma + \lambda_1(z-w)(1-k) + xk\gamma + \mu_2(y-x)k \\
&= \frac{\mu_2}{\lambda_2}(1-y)(1-k)\gamma + \frac{\mu_2}{\lambda_2}(y-\lambda_1)(1-k) + xk\gamma + \mu_2(y-x)k \\
&= k(\gamma-\mu_2)x + \frac{\mu_2}{\lambda_2}\,\{(1-k)(1-\gamma)+k\lambda_2\}\,y + \frac{\mu_2}{\lambda_2}(1-k)(\gamma-\lambda_1).
\end{aligned}
$$

Note that the following relationship holds between the gradient vector (μ_1, μ_2) of the Bayes plausibility condition (3) for x and y, and the gradient vector of R above:

$$
\frac{\mu_1}{\mu_2} > \frac{k(\gamma-\mu_2)}{\frac{\mu_2}{\lambda_2}\{(1-k)(1-\gamma)+k\lambda_2\}}.
$$

This implies that the maximization of R entails the maximization of y subject to Bayes plausibility (3), and the feasibility constraints $y \geq x$ and $y \geq \lambda_1$. Therefore, the optimal matching rule p is assortative. When $\lambda_1 \geq \mu_2$, substitution of $x = \frac{\lambda_1-\mu_2}{\mu_1}$ and $y = 1$ yields the maximized revenue as in the first line of (8), and when $\lambda_1 < \mu_2$, substitution of $x = 0$ and $y = \frac{\lambda_1}{\mu_2}$ yields the maximized revenue as in the second line of (8). ■

Proof of Proposition 3. We proceed by separating cases based on the buyer's belief about the seller's type induced by the matching rule p.

1. The optimal bid for the high-valuation buyer β_2 is α_1 when he has reported type β_2 truthfully, and also when he has misreported his type to be β_1:

   $$z^*_B(\beta_2, p_A(\cdot \mid \beta)) = \alpha_1 \text{ for any } \beta \in B.$$

   This requires that $x, y \geq \frac{\gamma}{1+\gamma}$. In this case, Bayes plausibility implies that the proportion of the low-cost seller must be high in the population:

   $$\lambda_1 = \Pr(\alpha_1) = \mu_1 x + \mu_2 y \geq \frac{\gamma}{1+\gamma}.$$

   The IC and IR conditions for a type β_1 buyer are written as

   $$
   \begin{aligned}
   &p_A(\alpha_1 \mid \beta_1)(\beta_1-\alpha_1) + p_A(\alpha_2 \mid \beta_1)\cdot 0 - t_B(\beta_1) \\
   &\geq \max\{0,\, p_A(\alpha_1 \mid \beta_2)(\beta_1-\alpha_1) + p_A(\alpha_2 \mid \beta_2)\cdot 0 - t_B(\beta_2)\}.
   \end{aligned} \tag{15}
   $$

   Note that the left-hand side is his expected payoff when he reports β_1: The first term corresponds to the event that he is matched against a low-cost seller so that his offer α_1 will be accepted and trade takes place. The second term correspond to the event that he is matched against a high-cost seller so that his offer will be rejected and no trade takes place. The right-hand side is the maximum between the buyer's reservation payoff and his expected payoff when he reports β_2. The IC and IR conditions for a type β_2 buyer are similarly given by

   $$
   \begin{aligned}
   &p_A(\alpha_1 \mid \beta_2)(\beta_2-\alpha_1) + p_A(\alpha_2 \mid \beta_1)\cdot 0 - t_B(\beta_2) \\
   &\geq \max\{0,\, p_A(\alpha_1 \mid \beta_2)(\beta_2-\alpha_1) + p_A(\alpha_2 \mid \beta_2)\cdot 0 - t_B(\beta_1)\}.
   \end{aligned} \tag{16}
   $$

Using the short-hand notation introduced in (14), we can summarize (15) and (16) as

$$\begin{aligned}
&(y-x)(\beta_1-\alpha_1) \le t_B(\beta_2) - t_B(\beta_1) \le (y-x)(\beta_2-\alpha_1),\\
&t_B(\beta_1) \le x(\beta_1-\alpha_1),\\
&t_B(\beta_2) \le y(\beta_2-\alpha_1).
\end{aligned}$$

This is feasible if

$$y = p_A(\alpha_1 \mid \beta_2) \ge p_A(\alpha_1 \mid \beta_1) = x, \tag{17}$$

and the optimal transfer function t_B is given by

$$t_B(\beta_1) = x(\beta_1-\alpha_1) \quad \text{and} \quad t_B(\beta_2) = t_B(\beta_1) + (y-x)(\beta_2-\alpha_1).$$

Turning now to side A, we note that the seller's payoff in G equals zero regardless of his type since both buyer types bid α_1 under q. It follows that the only transfer function t_A that satisfies IC and IR for the seller is given by $t_A(\alpha_1) = t_A(\alpha_2) = 0$. The platform's revenue is then given by

$$\begin{aligned}
R(\Gamma) &= \mu_1 t_B(\beta_1) + \mu_2 t_B(\beta_2)\\
&= x(\beta_1-\alpha_1) + \mu_2(y-x)(\beta_2-\alpha_1)\\
&= x\{\gamma - \mu_2(1+\gamma)\} + y\mu_2(1+\gamma).
\end{aligned}$$

Since R is linear in x and y, comparison of their coefficients against those in the Bayes plausibility condition $\mu_1 x + \mu_2 y = \lambda_1$ determines the optimal matching rule. Specifically, since

$$\frac{\mu_1}{\mu_2} > \frac{\gamma - \mu_2(1+\gamma)}{\mu_2(1+\gamma)} \quad \Leftrightarrow \quad 1+\gamma > \gamma,$$

the optimal p should maximize y subject to the feasibility constraints: $x, y \ge \frac{\gamma}{1+\gamma}$, $y \ge x$ and $\mu_1 x + \mu_2 y = \lambda_1 \ge \frac{\gamma}{1+\gamma}$. As seen in Figure 7, this yields

$$(x,y) = \begin{cases} \left(\frac{\gamma}{1+\gamma}, \frac{\lambda_1}{\mu_2} - \frac{\mu_1}{\mu_2}\frac{\gamma}{1+\gamma}\right) & \text{if } (1+\gamma)\lambda_1 + \mu_1 \le 1+\gamma,\\ \left(\frac{\lambda_1-\mu_2}{\mu_1}, 1\right) & \text{if } (1+\gamma)\lambda_1 + \mu_1 > 1+\gamma. \end{cases}$$

The maximized revenue is given by

$$R^* = \begin{cases} \lambda_1(1+\gamma) - \frac{\gamma}{1+\gamma} & \text{if } (1+\gamma)\lambda_1 + \mu_1 \le 1+\gamma,\\ \gamma\lambda_1 + \frac{\mu_2}{\mu_1}\lambda_2 & \text{if } (1+\gamma)\lambda_1 + \mu_1 > 1+\gamma. \end{cases}$$

2. The optimal bid for the buyer of type β_2 equals α_1 when he reports his type truthfully, but α_2 when he misreports his type to be β_1. This requires $x \le \frac{\gamma}{1+\gamma} \le y$.

   The IC and IR constraints of the type β_2 buyer are given by

   $$y(\beta_2-\alpha_1) - t_B(\beta_2) \ge \max\{0,\ \beta_2-\alpha_2-t_B(\beta_1)\}.$$

   The IC and IR constraints of the type β_1 buyer are given by

   $$x(\beta_1-\alpha_1) - t_B(\beta_1) \ge \max\{0,\ y(\beta_1-\alpha_1) - t_B(\beta_2)\}.$$

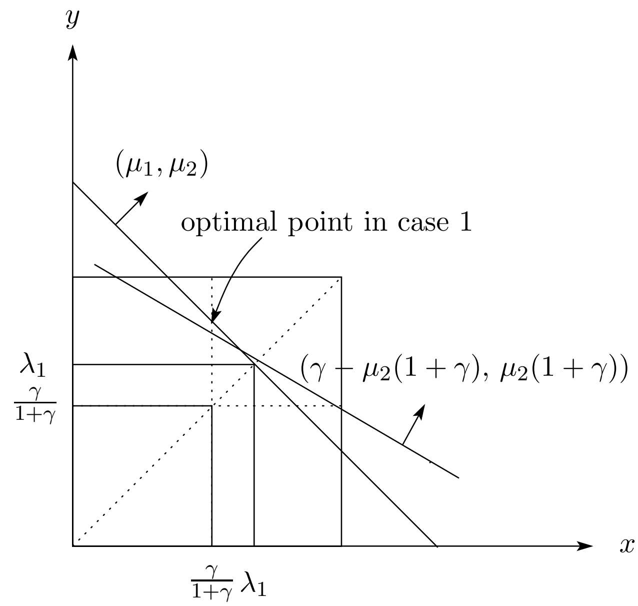

Figure 7: Optimal choice of (x, y) in case 1

These can be summarized as:

$$\begin{aligned}
&(y-x)\gamma \leq t_B(\beta_2) - t_B(\beta_1) \leq y(1+\gamma) - \gamma,\\
&t_B(\beta_1) \leq x\gamma,\\
&t_B(\beta_2) \leq y(1+\gamma).
\end{aligned}$$

For this to be feasible, we need

$$(y-x)\gamma \leq y(1+\gamma) - \gamma \quad \Leftrightarrow \quad \gamma x + y \geq \gamma. \tag{18}$$

Furthermore, there exists (x, y) that satisfies $0 \leq x \leq \frac{\gamma}{1+\gamma} \leq y \leq 1$, $\gamma x + y \geq \gamma$ and Bayes plausibility (3) if and only if

$$\begin{aligned}
&(1+\gamma)\lambda_1 + \mu_1 \leq 1 + \gamma, \text{ and}\\
&\text{either } \lambda_1 \geq \frac{\gamma}{1+\gamma} \text{ or } \lambda_1 + \gamma\mu_1 \geq \gamma.
\end{aligned} \tag{19}$$

The optimal transfer function t_B for the buyer is then given by

$$t_B(\beta_1) = x\gamma \quad \text{and} \quad t_B(\beta_2) = t_B(\beta_1) + y(1+\gamma) - \gamma.$$

The seller's payoff in G equals zero since both buyer types bid α_1 according to q. It follows that the transfer function for the seller equals $t_A(\alpha_1) = t_A(\alpha_2) = 0$, and that the platform's revenue is given by

$$R(\Gamma) = x\gamma + y\mu_2(1+\gamma) - \gamma\mu_2.$$

The optimal matching rule p maximizes R subject to $x \leq \frac{\gamma}{1+\gamma} \leq y$, (18), and Bayes plausibility $\mu_1 x + \mu_2 y = \lambda_1$. In what follows, we separate cases depending on the values of λ_1 and μ_1.

For this, it is useful to note that

$$\mu_1 < \frac{\gamma}{1+\gamma} \quad \Leftrightarrow \quad \gamma > \frac{\mu_1}{\mu_2} \quad \Leftrightarrow \quad \frac{\gamma}{(1+\gamma)\mu_2} > \frac{\mu_1}{\mu_2}, \tag{20}$$

where the second term corresponds to the comparison between the normal vectors of (18) and Bayes plausibility, and the third term corresponds to the comparison between the normal vectors of the revenue function R and Bayes plausibility.

(a) $\lambda_1, \mu_1 < \frac{\gamma}{1+\gamma}$. There is no (x, y) that satisfies $x \leq \frac{\gamma}{1+\gamma} \leq y$, $\gamma x + y \geq \gamma$, and $\mu_1 x + \mu_2 y = \lambda_1$. No p hence satisfies feasibility in this case.

(b) $\mu_1 < \frac{\gamma}{1+\gamma} < \lambda_1$. By (20), x should be as large as possible subject to feasibility, and the optimal matching rule p is such that

$$(x, y) = \left(\frac{\gamma}{1+\gamma}, \frac{\lambda_1}{\mu_2} - \frac{\mu_1}{\mu_2} \frac{\gamma}{1+\gamma} \right).$$

The maximized revenue is given by

$$R^* = (1+\gamma)\lambda_1 - \frac{\gamma}{1+\gamma}.$$

(c) $\mu_1 > \frac{\gamma}{1+\gamma}$. By (20), y should be as large as possible subject to feasibility, and the optimal matching rule p is such that

$$(x, y) = \begin{cases} \left(\frac{\lambda_1 - \mu_2}{\mu_1}, 1 \right) & \text{if } \lambda_1 > \mu_2, \\ \left(0, \frac{\lambda_1}{\mu_2} \right) & \text{if } \lambda_1 < \mu_2. \end{cases}$$

The maximized revenue is give by

$$R^* = \begin{cases} \frac{\lambda_1 - \mu_2}{\mu_1} \gamma + \mu_2 & \text{if } \lambda_1 \geq \mu_2, \\ (1+\gamma)\lambda_1 - \mu_2 \gamma & \text{otherwise.} \end{cases}$$

Figure 8 illustrates the optimal matching rules when $\lambda_1 > \frac{\gamma}{1+\gamma}$.

3. The optimal bid for the type β_2 buyer is α_2 whether he has reported his type truthfully or not. This requires $x, y \leq \frac{\gamma}{1+\gamma}$.

   In this case, Bayes plausibility implies that the proportion of the type α_1 seller is low in the population:

$$\lambda_1 = \mu_1 x + \mu_2 y \leq \frac{\gamma}{1+\gamma}.$$

   The IC and IR constraints for a type β_1 buyer are given by

$$x\gamma - t_B(\beta_1) \geq \max\{0, \, y\gamma - t_B(\beta_2)\},$$

   and those for a type β_2 buyer are given by

$$\gamma - t_B(\beta_2) \geq \max\{0, \gamma - t_B(\beta_1)\}.$$

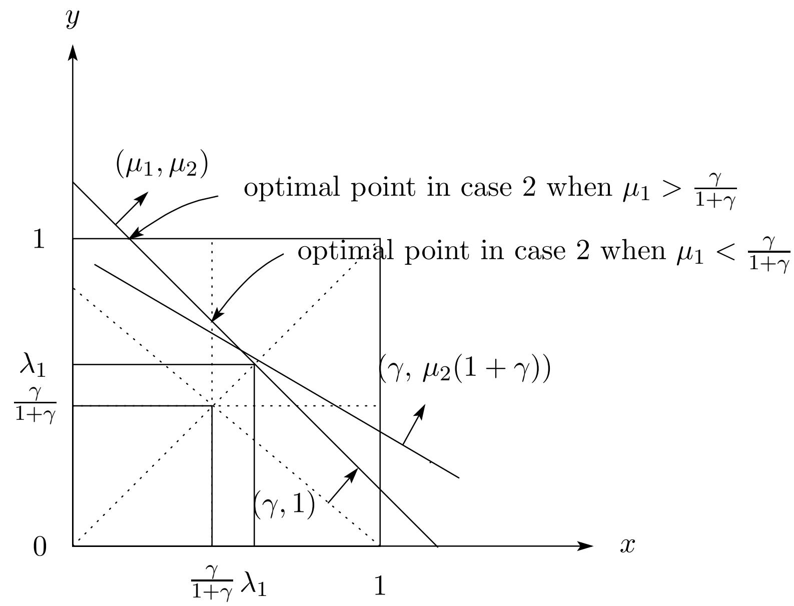

Figure 8: Optimal choice of (x, y) in case 2

These can be summarized as:

$$
\begin{aligned}
&(y - x)\gamma \leq t_B(\beta_2) - t_B(\beta_1) \leq 0, \\
&t_B(\beta_1) \leq x\gamma, \\
&t_B(\beta_2) \leq \gamma.
\end{aligned}
$$

This is hence feasible if $y \leq x$. In this case, the optimal transfer function is given by

$$t_B(\beta_1) = t_B(\beta_2) = x\gamma.$$

On the other hand, the IC and IR constraints for the type α_1 seller are given by

$$p_B(\beta_2 \mid \alpha_1)(\alpha_2 - \alpha_1) - t_A(\alpha_1) \geq \max\{0,\, p_B(\beta_2 \mid \alpha_2)(\alpha_2 - \alpha_1) - t_A(\alpha_2)\},$$

and those for the type α_2 seller are given by

$$-t_A(\alpha_2) \geq \max\{0,\, -t_A(\alpha_1)\}.$$

These can be summarized as

$$
\begin{aligned}
&0 \leq t_A(\alpha_1) - t_A(\alpha_2) \leq \{p_B(\beta_2 \mid \alpha_1) - p_B(\beta_2 \mid \alpha_2)\}(\alpha_2 - \alpha_1), \\
&t_A(\alpha_1) \leq p_B(\beta_2 \mid \alpha_1)(\alpha_2 - \alpha_1), \\
&t_A(\alpha_2) \leq 0.
\end{aligned}
$$

For this to be feasible, we need

$$
\begin{aligned}
p_B(\beta_2 \mid \alpha_1) - p_B(\beta_2 \mid \alpha_2) \geq 0 \quad &\Leftrightarrow \quad \frac{\mu_2}{\lambda_1} p_A(\alpha_1 \mid \beta_2) \geq \frac{\mu_2}{\lambda_2} p_A(\alpha_2 \mid \beta_2) \\
&\Leftrightarrow \quad y \geq \lambda_1.
\end{aligned}
$$

In this case, the optimal transfer function is given by

$$t_A(\alpha_1) = \frac{\mu_2}{\lambda_1\lambda_2}(y - \lambda_1) \quad \text{and} \quad t_A(\alpha_2) = 0.$$

It follows that the platform's revenue equals

$$R(\Gamma) = \gamma x + \frac{\mu_2}{\lambda_2}(y - \lambda_1).$$

The optimal matching rule p maximizes this subject to $x,\ y \leq \frac{\gamma}{1+\gamma}$, $x \geq y \geq \lambda_1$, and $\mu_1 x + \mu_2 y = \lambda_1$. The last two conditions however show that the feasible p is unique and such that $x = y = \lambda_1$. Therefore, the maximized revenue is given by

$$R^* = \lambda_1\gamma.$$

4. The optimal bid for a type β_2 buyer is α_2 when he reports his type truthfully, but α_1 when he misreports. This requires $x \geq \frac{\gamma}{1+\gamma} \geq y$.

   The IC and IR constraints for the type β_1 buyer are given by

$$x\gamma - t_B(\beta_1) \geq \max\{0,\ y\gamma - t_B(\beta_2)\},$$

   and those for the type β_2 buyer are given by

$$\gamma - t_B(\beta_2) \geq \max\{0,\ x(1+\gamma) - t_B(\beta_1)\}.$$

   Since these imply

$$(y - x)\gamma \leq t_B(\beta_2) - t_B(\beta_1) \leq \gamma - x(1+\gamma),$$

   feasibility requires

$$x + \gamma y \leq \gamma.$$

   On the other hand, the IC and IR constraints for the type α_1 seller are given by

$$p_B(\beta_2 \mid \alpha_1)(\alpha_2 - \alpha_1) - t_A(\alpha_1) \geq \max\{0,\ p_B(\beta_2 \mid \alpha_2)(\alpha_2 - \alpha_1) - t_A(\alpha_2)\},$$

   and those for the type α_2 seller are given by

$$-t_A(\alpha_2) \geq \max\{0,\ -t_A(\alpha_1)\}.$$

   These together imply

$$0 \leq t_A(\alpha_1) - t_A(\alpha_2) \leq \{p_B(\beta_2 \mid \alpha_1) - p_B(\beta_2 \mid \alpha_2)\}(\alpha_2 - \alpha_1).$$

   Feasibility requires

$$\begin{aligned} p_B(\beta_2 \mid \alpha_1) \geq p_B(\beta_2 \mid \alpha_2) \quad &\Leftrightarrow \quad \frac{\mu_2}{\lambda_1} p_A(\alpha_1 \mid \beta_2) \geq \frac{\mu_2}{\lambda_2} p_A(\alpha_2 \mid \beta_2) \\ &\Leftrightarrow \quad y \geq \lambda_1. \end{aligned}$$

   Note that $x \geq \frac{\gamma}{1+\gamma} \geq y \geq \lambda_1$ and $\mu_1 x + \mu_2 y = \lambda_1$ imply that $x = y = \lambda_1 = \frac{\gamma}{1+\gamma}$. In other words, feasibility holds only if $\lambda_1 = \frac{\gamma}{1+\gamma}$, and the optimal matching rule p is given by $x = y = \lambda_1$.

Summarizing the four cases above, we can conclude:

- If $\lambda_1 \geq \frac{\gamma}{1+\gamma}$ and $(1+\gamma)\lambda_1 + \mu_1 > 1+\gamma$, then only case 1 is feasible, and the optimal matching in case 1 is PAM. Hence, PAM is optimal.
- If $\lambda_1 \geq \frac{\gamma}{1+\gamma}$ and $(1+\gamma)\lambda_1 + \mu_1 \leq 1+\gamma$, then both cases 1 and 2 are feasible. The optimal matching in case 1 is BSM, whereas the optimal matching in case 2 is PAM if $\mu_1 > \frac{\gamma}{1+\gamma}$, and BSM otherwise. Hence, BSM is optimal if $\mu_1 \leq \frac{\gamma}{1+\gamma}$, and comparison of the revenue shows that PAM is optimal if $\mu_1 > \frac{\gamma}{1+\gamma}$.
- If $\lambda_1 < \frac{\gamma}{1+\gamma}$ and $\lambda_1 + \gamma\mu_1 < \gamma$, then only case 3 is feasible, and the only feasible matching in case 3 is RM. Hence, RM is optimal.
- If $\lambda_1 < \frac{\gamma}{1+\gamma}$ and $\lambda_1 + \gamma\mu_1 \geq \gamma$, then both cases 2 and 3 are feasible. The optimal matching in case 2 is PAM, and optimal matching in case 3 is RM. Comparison of the revenue under these two rules shows that PAM is optimal.

This completes the proof. ■

Proof of Proposition 4. As in the proof of Proposition 3, we separate cases depending on the values of x and y. Note that $(x, y) = \left(\frac{p_{11}}{\mu_1}, \frac{p_{12}}{\mu_2}\right)$.

1. $x,\ y > \frac{\gamma}{1+\gamma}$. In this case, a type β_2 buyer bids α_1 according to the BNE: $\sigma_B(\beta_2) = \alpha_1$. Social welfare is hence given by $W(\Gamma) = \gamma p_{11} + (1+\gamma)p_{12} = \gamma\frac{x}{\mu_1} + (1+\gamma)\frac{y}{\mu_2}$. The proof of Proposition 3 shows that there exists an IC and IR mechanism if and only if $y \geq x$. $W(\Gamma)$ is maximized when y is maximized subject to $y \geq x$ and Bayes plausibility (3). If follows that p is PAM.

2. $x \leq \frac{\gamma}{1+\gamma} \leq y$. A type β_2 buyer bids α_1 according to the BNE, and hence social welfare is again given by $W(\Gamma) = \gamma\frac{x}{\mu_1} + (1+\gamma)\frac{y}{\mu_2}$. The proof of Proposition 3 shows that there exists an IC and IR mechanism if and only if $\gamma x + y \geq \gamma$. The problem hence reduces to:

$$\max_{x,y} \gamma\frac{x}{\mu_1} + (1+\gamma)\frac{y}{\mu_2} \quad \text{subject to } \gamma x + y \geq \gamma,\ x \leq \frac{\gamma}{1+\gamma} \leq y, \text{ and Bayes plausibility (3).}$$

   A feasible (x, y) exists if and only if $(1+\gamma)\lambda_1 + \mu_1 \leq 1+\gamma$, and either $\lambda_1 \geq \frac{\gamma}{1+\gamma}$ or $\lambda_1 + \gamma\mu_1 \geq \gamma$.

   (a) If $\gamma \leq \frac{\lambda_1}{\mu_2}$ ($\Leftrightarrow \lambda_1 + \gamma\mu_1 \geq \gamma$), then PAM satisfies the constraints and maximizes W.

   (b) If $\gamma > \frac{\lambda_1}{\mu_2}$ ($\Leftrightarrow \lambda_1 + \gamma\mu_1 < \gamma$), then a feasible (x, y) exists only if $\lambda_1 \geq \frac{\gamma}{1+\gamma}$. W is maximized when $\gamma x + y = \gamma$. Solving this and (3) simultaneously, we obtain

$$(x, y) = \left(\frac{\gamma\mu_2 - \lambda_1}{\gamma - (1+\gamma)\mu_1}, \frac{\gamma(\lambda_1 - \mu_1)}{\gamma - (1+\gamma)\mu_1}\right), \tag{21}$$

   and

$$W(\Gamma) = \frac{\gamma\{\gamma\mu_2\lambda_1 - \lambda_1\mu_1 + \mu_2(\lambda_1 - \mu_1)\}}{\gamma - (1+\gamma)\mu_1}.$$

3. $x,\ y \leq \frac{\gamma}{1+\gamma}$. The proof of Proposition 3 shows that RM is the only feasible matching rule.

4. $x \geq \frac{\gamma}{1+\gamma} \geq y$. The proof of Proposition 3 shows that there exists no feasible matching rule.

When $(1+\gamma)\lambda_1 + \mu_1 > 1+\gamma$, only case 1 is feasible and PAM is optimal. When $(1+\gamma)\lambda_1 + \mu_1 \leq 1+\gamma$ and $\lambda_1 \geq \frac{\gamma}{1+\gamma}$, cases 1 and 2 are feasible: If $\lambda_1 + \gamma\mu_1 \geq \gamma$ in addition, PAM is optimal in both cases. On the other hand, if $\lambda_1 + \gamma\mu_1 < \gamma$, then either PAM or the matching rule specified in (21) is optimal. Comparison of social welfare associated with each rule shows that PAM is optimal. If $\lambda < \frac{\gamma}{1+\gamma}$ and $\lambda_1 + \gamma\mu_1 \geq \gamma$, then cases 2 and 3 are feasible: PAM is optimal in case 2 and RM is optimal in case 3. Comparison of social welfare in each case shows that PAM is optimal. If $\lambda < \frac{\gamma}{1+\gamma}$ and $\lambda_1 + \gamma\mu_1 < \gamma$, then only case 3 is feasible and RM is optimal. ∎

Proof of Proposition 6. We show that if RM is optimal with seller-offer bargaining for (λ_1, μ_1), then it is dominated by PAM with buyer-offer bargaining. By Proposition 5, RM is optimal with seller-offer bargaining when (λ_1, μ_1) satisfies $\mu_1 > \frac{1}{1+\gamma}$ and $\gamma\lambda_1 + \mu_1 > \gamma$, and yields

$$\gamma\mu_2.$$

Furthermore, Figure 5 shows that any such (λ_1, μ_1) satisfies $\lambda_1 \geq 1 - \mu_1 = \mu_2$, and Figure 2 shows that PAM with buyer-offer bargaining is feasible whenever RM is optimal with seller-offer bargaining. By Proposition 3, we can evaluate the revenue raised by PAM with buyer-offer bargaining as follows:

1. If $(1+\gamma)\lambda_1 + \mu_1 > 1+\gamma$, then the revenue equals

$$(1+\gamma)\lambda_1 - \frac{\lambda_1 - \mu_2}{\mu_1} > (1+\gamma)\lambda_1 - \lambda_1 = \gamma\lambda_1 \geq \gamma\mu_2,$$

   where the first inequality follows since $\frac{\lambda_1 - \mu_2}{\mu_1} < \lambda_1 \Leftrightarrow \lambda_1 < 1$.

2. If $\mu_1 > \frac{\gamma}{1+\gamma}$, $\lambda_1 + \gamma\mu_1 > \gamma$, $(1+\gamma)\lambda_1 + \mu_1 \leq 1+\gamma$, and $\lambda_1 \geq \mu_2$, then the revenue equals

$$\frac{\lambda_1 - \mu_2}{\mu_1} + \mu_2 \geq \mu_2 > \gamma\mu_2.$$

In both cases, hence, PAM with buyer-offer bargaining yields a higher revenue than RM with seller-offer bargaining. A similar argument shows that RM with buyer-offer bargaining is dominated by PAM with seller-offer bargaining. ∎

Proof of Proposition 7.

We will specifically show the following.

- PAM with buyer-offer bargaining if $\frac{1}{1+\gamma} > \lambda_1 > \mu_2$,
- BSM with buyer-offer bargaining if $\lambda_1 > \mu_2 > \frac{1}{1+\gamma}$,
- PAM with seller-offer bargaining if $\frac{1}{1+\gamma} > \mu_2 > \lambda_1$,
- SSM with seller-offer bargaining if $\mu_2 > \lambda_1 > \frac{1}{1+\gamma}$.

1. First fix $d < \frac{1}{1+\gamma}$. If $\|(\lambda_1, \mu_2) - (d, d)\| < \varepsilon$ for a sufficiently small $\varepsilon > 0$, then Figures 2 and 5 show that at (λ_1, μ_1), B-squeeze matching is optimal with buyer-offer bargaining, and S-squeeze matching is optimal with seller-offer bargaining. The former yields $\lambda_1(1+\gamma) - \frac{\gamma}{1+\gamma}$ in revenue, whereas the latter yields $\mu_2(1+\gamma) - \frac{\gamma}{1+\gamma}$. It follows that B-squeeze matching with buyer offer is optimal if $\lambda_1 > \mu_2$ and S-squeeze matching with seller offer is optimal if $\lambda_1 < \mu_2$.

2. Next fix $d < \frac{1}{1+\gamma}$. If $\|(\lambda_1, \mu_2) - (d, d)\| < \varepsilon$ for a sufficiently small $\varepsilon > 0$, then Figures 2 and 5 again show that at (λ_1, μ_1), PAM is optimal with both buyer-offer and seller-offer bargaining. If $\lambda_1 > \mu_2$, then buyer-offer yields $\frac{\lambda_1 - \mu_2}{\mu_1}\gamma + \mu_2$ in revenue and seller-offer yields $(1+\gamma)\mu_2 - \lambda_1\gamma$. The former dominates the latter since $\frac{\lambda_1-\mu_2}{\mu_1}\gamma + \mu_2 > (1+\gamma)\mu_2 - \lambda_1\gamma \Leftrightarrow 1 + \mu_1 > 0$. If $\lambda_1 < \mu_2$, a similar argument shows that PAM with seller-offer bargaining is optimal.

3. Finally, fix $d = \frac{1}{1+\gamma}$ and suppose that $\|(\lambda_1, \mu_2) - (d, d)\| < \varepsilon$ for a sufficiently small $\varepsilon > 0$ and that $\lambda_1 > \mu_2$. If λ_1, $\mu_2 > d$, then we have the same situation as case 1 above. If $\lambda_1 < d$ and $\mu_2 < d$, then we have the same situation as case 2 above. If $\lambda_1 > d$ and $\mu_2 < d$, then S-squeeze matching is optimal with seller-offer bargaining and PAM is optimal with buyer-offer bargaining. The former yields $\mu_2(1+\gamma) - \frac{\gamma}{1+\gamma}$ in revenue and the latter yields $\frac{\lambda_1-\mu_2}{\mu_1}\gamma + \mu_2$. The latter dominates the former since
$$\begin{aligned}\left\{\frac{\lambda_1 - \mu_2}{\mu_1}\gamma + \mu_2\right\} - \left\{\mu_2(1+\gamma) - \frac{\gamma}{1+\gamma}\right\} &= \gamma\left\{\frac{\lambda_1 - \mu_2}{\mu_1} - \mu_2 + \frac{1}{1+\gamma}\right\} \\ &> \gamma(d - \mu_2) > 0.\end{aligned}$$
A similar argument proves that PAM with seller-offer bargaining is optimal when $\lambda_1 < \mu_2$.

■

Proof of Proposition 10. The IC and IR conditions for β_1 are given by

$$x(\beta_1 - \zeta) - t_B(\beta_1) \geq \max\{0,\, y(\beta_1 - \zeta) - t_B(\beta_2)\},$$

and since $x \geq r$, those for β_2 are given by

$$y(\beta_2 - \zeta) - t_B(\beta_2) \geq \max\{0,\, x(\beta_2 - \zeta) - t_B(\beta_1)\}.$$

For this to be feasible, we need

$$(y - x)(\beta_1 - \zeta) \leq (y - x)(\beta_2 - \zeta) \quad \Leftrightarrow \quad y \geq x. \tag{22}$$

We can verify that the IR condition for β_1 and the IC condition for β_2 bind so that $t_B(\beta_1) = x\gamma$ and $t_B(\beta_2) = t_B(\beta_1) + (y - x)\gamma$. On the other hand, the IC and IR conditions for α_1 are given by

$$\zeta - \alpha_1 - t_A(\alpha_1) \geq \max\{0,\, \zeta - \alpha_1 - t_A(\alpha_2)\},$$

and those for α_2 are given by

$$0 - t_A(\alpha_2) \geq \max\{0,\, 0 - t_A(\alpha_2)\}.$$

We obtain from these the optimal transfer function for the seller:

$$t_A(\alpha_1) = t_A(\alpha_2) = 0.$$

It follows that the platform's revenue is given by

$$R = x(\beta_1 - \zeta) + \mu_2(y - x)(\beta_2 - \zeta) = \{\beta_1 - \zeta - \mu_2(\beta_2 - \zeta)\}x + \mu_2(\beta_2 - \zeta)y. \tag{23}$$

Since this is decreasing in ζ, we set $\zeta = \alpha_1$. Furthermore, comparing the gradient vector of R with the normal vector of the Bayes plausibility condition, we see that PAM is optimal since

$$\frac{\mu_1}{\mu_2} > \frac{\beta_1 - \zeta - \mu_2(\beta_2 - \zeta)}{\mu_2(\beta_2 - \zeta)} \quad \Leftrightarrow \quad \beta_2 > \beta_1. \tag{24}$$

When $\lambda_1 \geq \mu_2$, substitution of $(x, y) = (\frac{\lambda_1 - \mu_2}{\mu_1}, 1)$ yields

$$R^* = \gamma\lambda_1 + \frac{\lambda_2\mu_2}{\mu_1},$$

and when $\lambda_1 < \mu_2$, substitution of $(x, y) = (1, \frac{\lambda_1}{\mu_2})$ yield

$$R^* = \lambda_1(1 + \gamma).$$

Note from Proposition 1 that when the market is symmetric ($\lambda_1 = \mu_2$), the maximized revenue above equals the maximal social surplus. It follows that this mechanism is optimal among all possible mechanisms under symmetry. ■

Proof of Proposition 11. Since the expected payoff of a type α_2 seller in the trading game equals zero regardless of his report, and since $t_A(\alpha_1) = t_A(\alpha_2) = 0$, the IC and IR conditions of type α_1 always hold with equality. Let

$$k_1 = \frac{\gamma - t_B(\beta_1)}{1 + \gamma - t_B(\beta_1)} \quad \text{and} \quad k_2 = \frac{\gamma - t_B(\beta_2)}{1 + \gamma - t_B(\beta_2)}.$$[19]

The optimal bid for a type β_2 buyer equals α_1 if $y = p_A(\alpha_1 \mid \beta_2) \geq k_2$ when he reports his type truthfully, and if $x = p_A(\alpha_1 \mid \beta_1) \geq k_1$ when he misreports his type. Note also that

$$\begin{aligned} x \leq k_1 \quad &\Leftrightarrow \quad t_B(\beta_1) \leq \gamma - \tfrac{x}{1-x}, \\ y \leq k_2 \quad &\Leftrightarrow \quad t_B(\beta_2) \leq \gamma - \tfrac{y}{1-y}. \end{aligned}$$

For any values of x and y, the IC and IR conditions of a type β_1 buyer are given by:

$$x\{\gamma - t_B(\beta_1)\} \geq \max\{0, y\{\gamma - t_B(\beta_2)\}\}. \tag{25}$$

a) A type β_2 buyer optimally bids α_2 whether he has reported his type truthfully or not. This requires $x \leq k_1$ and $y \leq k_2$. The IC and IR conditions of type β_2 are given by

$$\gamma - t_B(\beta_2) \geq \max\{0, \gamma - t_B(\beta_1)\}. \tag{26}$$

[19] Let $k_2 = 0$ if $1 + \gamma - t_B(\beta_2) = 0$.

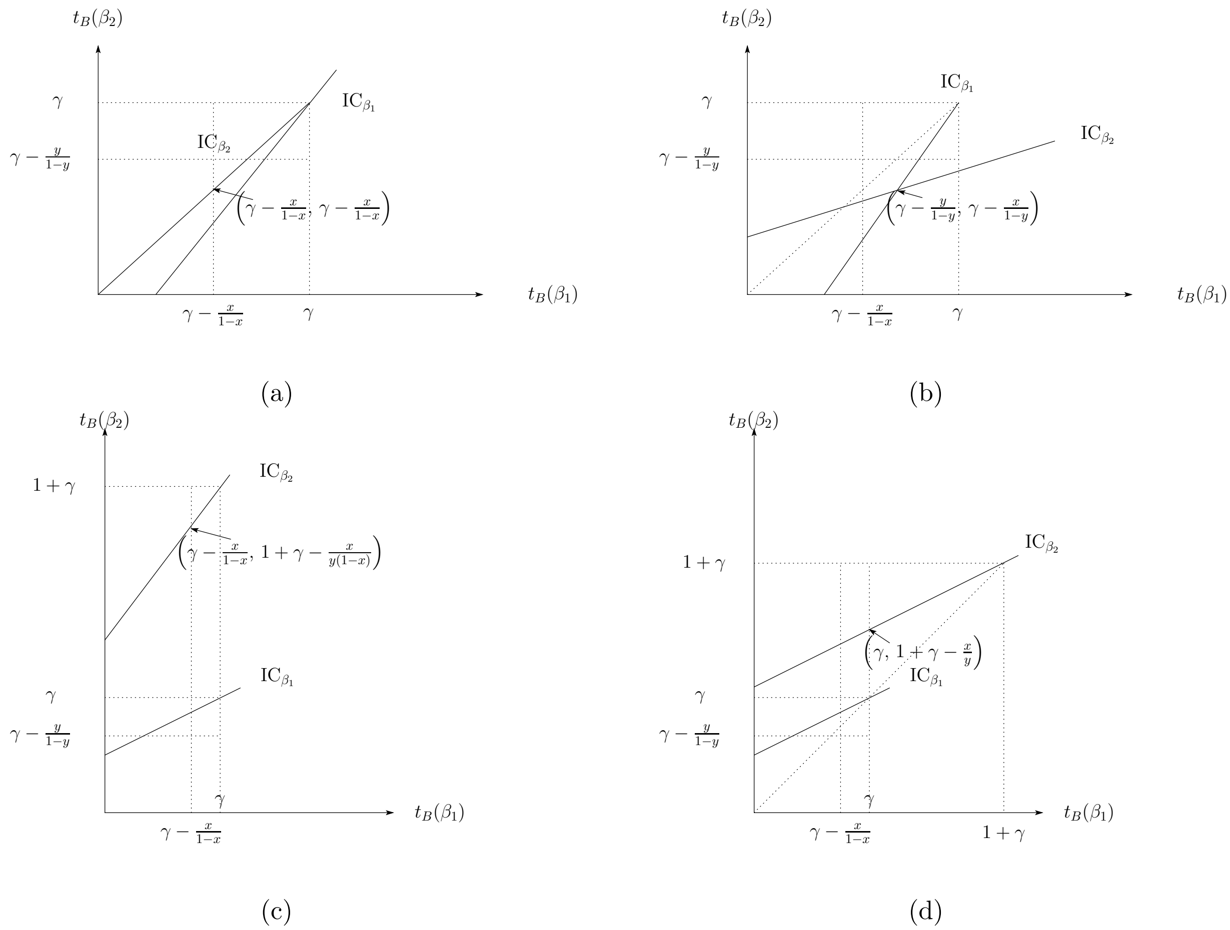

Figure 9: Optimal transfer $(t_B(\beta_1), t_B(\beta_2))$: (a) $x \le k_1$, $y \le k_2$, (b) $x > k_1$, $y \le k_2$, (c) $x \le k_1$, $y > k_2$, (d) $x > k_1$, $y > k_2$.

(25) and (27) together show that $y \leq x$. As seen in Figure 9, the optimal transfer in this case is given by

$$t_B(\beta_1) = t_B(\beta_2) = \gamma - \frac{x}{1-x}.$$

On the other hand, the IC and IR conditions of a type α_1 seller are given by

$$z(1 - t_A(\alpha_1)) \geq \max\{0,\, w(1 - t_A(\alpha_2))\}. \tag{27}$$

Substitution of $t_A(\alpha_1) = t_A(\alpha_2) = 0$ yields $z \geq w$, which in turn leads to

$$p_B(\beta_2 \mid \alpha_1) \geq p_B(\beta_2 \mid \alpha_2) \Leftrightarrow \frac{\mu_2}{\lambda_1} p_A(\alpha_1 \mid \beta_2) \geq \frac{\mu_2}{\lambda_2} p_A(\alpha_2 \mid \beta_2) \Leftrightarrow y \geq \lambda_1.$$

Along with $x \geq y$ above and Bayesian plausibility, this implies RM: $x = y = \lambda_1$. It follows that the maximized revenue of the platform in this case is given by

$$R^* = \mu_1 x t_B(\beta_1) + \mu_2 t_B(\beta_2) = (\lambda_1\mu_1 + \mu_2)\left(\gamma - \frac{\lambda_1}{1-\lambda_1}\right).$$

b) A type β_2 buyer optimally bids α_2 when he has reported his type truthfully, but α_1 when he has misreported his type. This requires $x \geq k_1$ and $y \leq k_2$. The IC and IR conditions of type β_2 are given by

$$\gamma - t_B(\beta_2) \geq \max\{0,\, x\{1 + \gamma - t_B(\beta_1)\}\}. \tag{28}$$

Since $t_B(\beta_1) \geq \gamma - \frac{x}{1-x}$, $t_B(\beta_2) \leq \gamma - \frac{y}{1-y}$, and $yt_B(\beta_2) - xt_B(\beta_1) \geq \gamma(y - x)$ by (25), a feasible transfer $(t_B(\beta_1), t_B(\beta_2))$ exists only if

$$y\left(\gamma - \frac{y}{1-y}\right) - x\left(\gamma - \frac{x}{1-x}\right) \geq \gamma(y - x) \quad \Leftrightarrow \quad (y - x)\{1 - (1-x)(1-y)\} \leq 0.$$

Since $(1-x)(1-y) < 1$ by (3), we must have $y \leq x$. In this case, the optimal transfer is given by $(t_B(\beta_1), t_B(\beta_2)) = \left(\gamma - \frac{y}{1-y},\, \gamma - \frac{x}{1-y}\right)$, which satisfies the IC conditions of β_1 and β_2 in (25) and (28) with equality. It also satisfies the IR conditions of both types, as well as $t_B(\beta_1) > \gamma - \frac{x}{1-x}$ and $t_B(\beta_2) \leq \gamma - \frac{y}{1-y}$.

On the other hand, the IC and IR conditions of a type α_1 seller are the same as in case (a), and reduce to $z \geq w$. This coupled with $x \geq y$ implies RM: $x = y = \lambda_1$. It follows that the maximized revenue of the platform in this case is again given by

$$R^* = \mu_1 x t_B(\beta_1) + \mu_2 t_B(\beta_2) = (\lambda_1\mu_1 + \mu_2)\left(\gamma - \frac{\lambda_1}{1-\lambda_1}\right).$$

c) A type β_2 buyer optimally bids α_1 when he has reported his type truthfully, but α_2 when he has misreported his type. This requires $x \leq k_1$ and $y \geq k_2$. The IC and IR conditions of β_2 are given by

$$y\{1 + \gamma - t_B(\beta_2)\} \geq \max\{0,\, \gamma - t_B(\beta_1)\}. \tag{29}$$

Since $t_B(\beta_1) \leq \gamma - \frac{x}{1-x}$, $t_B(\beta_2) \geq \gamma - \frac{y}{1-y}$, and $yt_B(\beta_2) - t_B(\beta_1) \leq y(1+\gamma) - \gamma$ by (29), a feasible transfer $(t_B(\beta_1), t_B(\beta_2))$ exists only if

$$y\left(\gamma - \frac{y}{1-y}\right) - \left(\gamma - \frac{x}{1-x}\right) \leq y(1+\gamma) - \gamma \quad \Leftrightarrow \quad x \leq y.$$

In this case, the optimal transfer is given by $(t_B(\beta_1), t_B(\beta_2)) = \left(\gamma - \frac{x}{1-x},\, 1+\gamma - \frac{x}{y(1-x)}\right)$, which satisfies type β_2's IC condition with equality and also $x = k_1$. On the other hand, the IC and IR conditions of a type α_1 seller always hold with equality. Substituting $y = \frac{\lambda_1 - \mu_1 x}{\mu_2}$ from (3), we can write the platform's expected revenue as:

$$R(\tilde{\Gamma}) = \mu_1 x \left(\gamma - \frac{x}{1-x}\right) + \mu_2 \left(\frac{\lambda_1 - \mu_1 x}{\mu_2}\right) \left(1 + \gamma - \frac{\mu_2 x}{(\lambda_1 - \mu_1 x)(1-x)}\right),$$

which is strictly decreasing in x. This implies that the optimal matching rule in this case is PAM, and the maximized revenue is given by

$$R^* = \begin{cases} \lambda_1(1+\gamma) & \text{if } \frac{\lambda_1}{\mu_2} \leq 1, \\ \lambda_1(1+\gamma) - \frac{\lambda_1 - \mu_2}{\lambda_2} & \text{if } \frac{\lambda_1}{\mu_2} > 1. \end{cases}$$

d) A type β_2 buyer optimally bids α_1 whether he has reported his type truthfully or not. This requires $x \geq k_1$ and $y \geq k_2$. The IC and IR conditions of β_2 are given by

$$y\{1 + \gamma - t_B(\beta_2)\} \geq \max\{0,\, x\{1 + \gamma - t_B(\beta_1)\}\}. \tag{30}$$

(25) and (30) together imply

$$(y - x)\gamma \leq y t_B(\beta_2) - x t_B(\beta_1) \leq (y - x)(1 + \gamma),$$

so that $y \geq x$. In this case, the optimal transfer is given by $(t_B(\beta_1), t_B(\beta_2)) = \left(\gamma,\, 1 + \gamma - \frac{x}{y}\right)$, which satisfies type β_2's IC condition and type β_1's IR condition both with equality. On the other hand, the IC and IR conditions of a type α_1 seller always hold. The expected revenue of the platform then equals

$$R(\tilde{\Gamma}) = \mu_1 x t_B(\beta_1) + \mu_2 y t_B(\beta_2) = \mu_2(y - x)(1 + \gamma) + x\gamma.$$

By substituting $y = -\frac{\mu_1}{\mu_2}x + \frac{\lambda_1}{\mu_2}$, we can rewrite this as

$$R(\tilde{\Gamma}) = (1 + \gamma)\left\{\frac{\gamma}{1 + \gamma} - 1\right\} x + \lambda_1(\beta_2 - \alpha_1), \tag{31}$$

which is a decreasing function of x. Hence, the optimal matching rule is PAM, and the maximized revenue is given by

$$R^* = \begin{cases} \lambda_1(1+\gamma) & \text{if } \frac{\lambda_1}{\mu_2} \leq 1, \\ \lambda_1(1+\gamma) - \frac{\lambda_1 - \mu_2}{\mu_1} & \text{if } \frac{\lambda_1}{\mu_2} > 1. \end{cases}$$

Comparison of the maximized revenue in the above four cases shows that the optimal mechanism $\tilde{\Gamma}$ is one described in case (d), which entails PAM, and transfer given by

$$(t_B(\beta_1), t_B(\beta_2)) = \begin{cases} (\gamma,\, 1 + \gamma) & \text{if } \frac{\lambda_1}{\mu_2} \leq 1, \\ \left(\gamma,\, 1 + \gamma - \frac{\lambda_1 - \mu_2}{\mu_1}\right) & \text{if } \frac{\lambda_1}{\mu_2} > 1. \end{cases}$$

This mechanism induces a type β_2 buyer to bid α_1 after both truthful and untruthful reporting, and yields the expected revenue as described in (10). ■

Proposition 16 *The optimal outcome-contingent mechanism $\tilde{\Gamma}$ with buyer-offer bargaining dominates the optimal mechanism Γ with buyer-offer bargaining in the baseline model. In particular, the dominance is strict when Γ entails non-PAM matching rules for almost every type distribution.*

Proof. The claim is established in each case below.

1. $\lambda_1 + \mu_1 > 1$ and $(1+\gamma)\lambda_1 + \mu_1 > 1 + \gamma$.

$$R(\Gamma) = \lambda_1(1+\gamma) - \frac{\lambda_1 - \mu_2}{\mu_1} = R(\tilde{\Gamma}).$$

2. $\lambda_1 + \mu_1 > 1$, $(1+\gamma)\lambda_1 + \mu_1 \leq 1 + \gamma$ and $\mu_1 > \frac{\gamma}{1+\gamma}$.

$$R(\Gamma) = \frac{\lambda_1 - \mu_2}{\mu_1}\gamma + \mu_2 < R(\tilde{\Gamma}) = \lambda_1(1+\gamma) - \frac{\lambda_1 - \mu_2}{\mu_1}.$$

3. $\lambda_1 + \mu_1 > 1$, $(1+\gamma)\lambda_1 + \mu_1 \leq 1 + \gamma$ and $\mu_1 \leq \frac{\gamma}{1+\gamma}$.

$$R(\Gamma) = \lambda_1(1+\gamma) - \frac{\gamma}{1+\gamma} \leq R(\tilde{\Gamma}) = \lambda_1(1+\gamma) - \frac{\lambda_1 - \mu_2}{\mu_1},$$

where the equality holds if and only if $(1+\gamma)\lambda_1 + \mu_1 = 1 + \gamma$.

4. $\lambda_1 + \mu_1 \leq 1$, $\lambda_1 > \frac{\gamma}{1+\gamma}$, and $\mu_1 \leq \frac{\gamma}{1+\gamma}$.

$$R(\Gamma) = \lambda_1(1+\gamma) - \frac{\gamma}{1+\gamma} < R(\tilde{\Gamma}) = \lambda_1(1+\gamma).$$

5. $\lambda_1 + \mu_1 \leq 1$, $\lambda_1 + \gamma\mu_1 > \gamma$, and $\mu_1 > \frac{\gamma}{1+\gamma}$.

$$R(\Gamma) = \lambda_1(1+\gamma) - \mu_2\gamma < R(\tilde{\Gamma}) = \lambda_1(1+\gamma).$$

6. $\lambda_1 \leq \frac{\gamma}{1+\gamma}$, and $\lambda_1 + \gamma\mu_1 \leq \gamma$.

$$R(\Gamma) = \lambda_1\gamma < R(\tilde{\Gamma}) = \lambda_1(1+\gamma).$$

■

Proof of Proposition 12. A buyer's IC and IR conditions are then given as follows. For type β_1,

$$0 - t_B(\beta_1) \geq \max\left\{0 - t_B(\beta_2),\, 0\right\}, \tag{32}$$

and for type β_2,

$$\begin{aligned} &\Pr(\alpha_1, \beta_1 \mid \beta_2)\,(v_{12} - v_{11}) + \Pr(\alpha_2, \beta_1 \mid \beta_2)(v_{22} - v_{21}) - t_B(\beta_2) \\ &\geq \max\left\{\Pr(\alpha_1, \beta_1 \mid \beta_1)(v_{12} - v_{11}) + \Pr(\alpha_2, \beta_1 \mid \beta_1)(v_{22} - v_{21}) - t_B(\beta_1),\, 0\right\}. \end{aligned} \tag{33}$$

We have from (32) and (33) that

$$
\begin{aligned}
0 \le t_B(\beta_2) - t_B(\beta_1) &\le \{\Pr(\alpha_1, \beta_1 \mid \beta_2) - \Pr(\alpha_1, \beta_1 \mid \beta_1)\}\, \Delta_1 \\
&\quad + \{\Pr(\alpha_2, \beta_1 \mid \beta_2) - \Pr(\alpha_2, \beta_1 \mid \beta_1)\}\Delta_2 \\
&= \left(\frac{p_{112}}{\mu_2} - \frac{p_{111}}{\mu_1}\right) \Delta_1 + \left(\frac{p_{212}}{\mu_2} - \frac{p_{211}}{\mu_1}\right) \Delta_2.
\end{aligned}
$$

For the feasibility of these conditions, we hence need

$$
\left(\frac{p_{112}}{\mu_2} - \frac{p_{111}}{\mu_1}\right) \Delta_1 + \left(\frac{p_{212}}{\mu_2} - \frac{p_{211}}{\mu_1}\right) \Delta_2 \ge 0. \tag{34}
$$

We can also show that the IR condition for the low type (*i.e.,* β_1) and the IC condition for the high type (*i.e.,* β_2) bind. Hence, when (34) holds, the optimal transfer from the buyer is given by

$$
\begin{aligned}
t_B(\beta_1) &= 0, \\
t_B(\beta_2) &= \left(\frac{p_{112}}{\mu_2} - \frac{p_{111}}{\mu_1}\right) \Delta_1 + \left(\frac{p_{212}}{\mu_2} - \frac{p_{211}}{\mu_1}\right) \Delta_2.
\end{aligned} \tag{35}
$$

Turning now to the seller side, recall that their types are observable by the matched buyers. Hence, the incentive compatibility and individual rationality conditions for type α_1 are given by

$$
\begin{aligned}
&\{1 - \Pr(\beta_2, \beta_2 \mid \alpha_1)\}v_{11} + \Pr(\beta_2, \beta_2 \mid \alpha_1)v_{12} - t_A(\alpha_1) \\
&\ge \max\left\{\{1 - \Pr(\beta_2, \beta_2 \mid \alpha_2)\}v_{11} + \Pr(\beta_2, \beta_2 \mid \alpha_2)v_{12} - t_A(\alpha_2),\, 0\right\},
\end{aligned} \tag{36}
$$

and those for type α_2 are given by

$$
\begin{aligned}
&\{1 - \Pr(\beta_2, \beta_2 \mid \alpha_2)\}v_{21} + \Pr(\beta_2, \beta_2 \mid \alpha_2)v_{22} - t_A(\alpha_2) \\
&\ge \max\left\{\{1 - \Pr(\beta_2, \beta_2 \mid \alpha_1)\}v_{21} + \Pr(\beta_2, \beta_2 \mid \alpha_1)v_{22} - t_A(\alpha_1),\, 0\right\}.
\end{aligned} \tag{37}
$$

(36) and (37) together imply

$$
\begin{aligned}
\{\Pr(\beta_2, \beta_2 \mid \alpha_2) - \Pr(\beta_2, \beta_2 \mid \alpha_1)\}\Delta_1 &\le t_A(\alpha_2) - t_A(\alpha_1) \\
&\le \{\Pr(\beta_2, \beta_2 \mid \alpha_2) - \Pr(\beta_2, \beta_2 \mid \alpha_1)\}\Delta_2,
\end{aligned}
$$

which is equivalent to

$$
\left(\frac{p_{222}}{\lambda_2} - \frac{p_{122}}{\lambda_1}\right) \Delta_1 \le t_A(\alpha_2) - t_A(\alpha_1) \le \left(\frac{p_{222}}{\lambda_2} - \frac{p_{122}}{\lambda_1}\right) \Delta_2.
$$

Since $\Delta_2 > \Delta_1$ by our assumption (11), this implies that the following feasibility condition must hold:

$$
\frac{p_{222}}{\lambda_2} - \frac{p_{122}}{\lambda_1} \ge 0. \tag{38}
$$

Again, the IR condition for the low type (*i.e.,* α_1) and the IC condition for the high type (*i.e.,* α_2) bind. Hence, when (38) holds, the optimal transfer from the seller is given by

$$
\begin{aligned}
t_A(\alpha_1) &= v_{11} + \frac{p_{122}}{\lambda_1} \Delta_1, \\
t_A(\alpha_2) &= t_A(\alpha_1) + \left(\frac{p_{222}}{\lambda_2} - \frac{p_{122}}{\lambda_1}\right) \Delta_2.
\end{aligned} \tag{39}
$$

(35) and (39) yield the maximal payoff for the platform given the matching rule p:

$$\begin{aligned} R(\Gamma) &= \lambda_1 t_A(\alpha_1) + \lambda_2 t_A(\alpha_2) + 2\left\{\mu_1 t_B(\beta_1) + \mu_2 t_B(\beta_2)\right\} \\ &= v_{11} + \frac{p_{122}}{\lambda_1}\,\Delta_1 + \lambda_2\left(\frac{p_{222}}{\lambda_2} - \frac{p_{122}}{\lambda_1}\right)\Delta_2 \\ &+ 2\mu_2\left(\frac{p_{112}}{\mu_2} - \frac{p_{111}}{\mu_1}\right)\Delta_1 + 2\mu_2\left(\frac{p_{212}}{\mu_2} - \frac{p_{211}}{\mu_1}\right)\Delta_2. \end{aligned} \tag{40}$$

The optimal matching rule $p = (p_{111}, \ldots, p_{222})$ is one that solves

$$\max\Big\{R(p) \,:\, p \in P \text{ satisfies (34) and (38)}\Big\}.$$

Bayes plausibility (13) allows us to express p_{111}, p_{211} and p_{212} in terms of p_{112}, p_{122} and p_{222} as:

$$\begin{cases} p_{111} = \lambda_1 - p_{122} - 2p_{112}, \\ p_{211} = \lambda_2 - 2\mu_2 + 2p_{122} + 2p_{112} + p_{222}, \\ p_{212} = \mu_2 - p_{222} - p_{122} - p_{112}. \end{cases} \tag{41}$$

We then rewrite the feasibility condition (34) and the platform's payoff (40) in terms of $(p_{112}, p_{122}, p_{222})$:

$$\{\mu_2(\Delta_2 - \Delta_1) + \Delta_2\}\,p_{122} + (1+\mu_2)(\Delta_2 - \Delta_1)\,p_{112} + \Delta_2 p_{222} \le \mu_2\{(\lambda_1 + \mu_2)\Delta_2 - \lambda_1\Delta_1\}, \tag{42}$$

and

$$\begin{aligned} R(\Gamma) &= v_{11} - \frac{2\mu_2}{\mu_1}\,\lambda_1\Delta_1 + 2\mu_2\left(1 - \frac{\lambda_2 - 2\mu_2}{\mu_1}\right)\Delta_2 \\ &- \left\{\left(\frac{\lambda_2}{\lambda_1} + 2 + \frac{4\mu_2}{\mu_1}\right)\Delta_2 - \left(\frac{1}{\lambda_1} + \frac{2\mu_2}{\mu_1}\right)\Delta_1\right\} p_{122} \\ &- 2\left(1 + \frac{2\mu_2}{\mu_1}\right)(\Delta_2 - \Delta_1)\,p_{112} \\ &- \left(1 + \frac{2\mu_2}{\mu_1}\right)\Delta_2\,p_{222}. \end{aligned}$$

Writing $\kappa = 1 + \frac{2\mu_2}{\mu_1}$, we see that this simplifies to

$$\begin{aligned} R(\Gamma) &= v_{11} - (\kappa - 1)\lambda_1\Delta_1 + 2\mu_2\left(\kappa - \frac{\lambda_2}{\mu_1}\right)\Delta_2 \\ &- \left\{\kappa\Delta_2 + \left(\kappa + \frac{\lambda_2}{\lambda_1}\right)(\Delta_2 - \Delta_1)\right\} p_{122} - 2\kappa(\Delta_2 - \Delta_1)\,p_{112} - \kappa\Delta_2\,p_{222}. \end{aligned} \tag{43}$$

Figure 10 illustrates the feasible combinations of (p_{122}, p_{222}) for $p_{112} < \mu_2 - \frac{\lambda_2}{2}$.

1. $\mu_2 \le \frac{\lambda_2}{2}$. Let $(p_{112}, p_{122}, p_{222}) = (0, 0, 0)$. It clearly maximizes the platform's payoff (43) subject to $(p_{112}, p_{122}, p_{222}) \ge (0, 0, 0)$. It satisfies (38) and (42) and hence is feasible. By (41), we have

$$(p_{111}, p_{211}, p_{212}) = (\lambda_1,\ \lambda_2 - 2\mu_2,\ \mu_2).$$

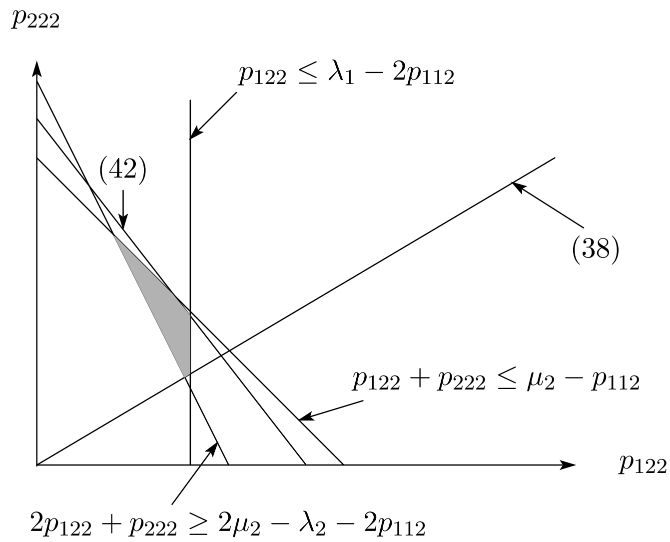

Figure 10: Feasible combinations of (p_{122}, p_{222}) (shaded area) when $p_{112} < \mu_2 - \frac{\lambda_2}{2}$.

$$\begin{aligned} p_{111} \ge 0 &\quad \Leftrightarrow \quad p_{122} \le \lambda_1 - 2p_{112}, \\ p_{211} \ge 0 &\quad \Leftrightarrow \quad 2p_{122} + p_{222} \ge 2\mu_2 - \lambda_2 - 2p_{112}, \\ p_{212} \ge 0 &\quad \Leftrightarrow \quad p_{222} + p_{122} \le \mu_2 - p_{112}. \end{aligned}$$

2. $\mu_2 > \frac{\lambda_2}{2}$. Since the platform's payoff (43) is decreasing in p_{112}, p_{122} and p_{222}, if $(p_{112}, p_{122}, p_{222})$ is optimal, then it satisfies the constraint $2p_{112} + 2p_{122} + p_{222} \ge 2\mu_2 - \lambda_2 > 0$ with equality. Substituting $p_{222} = 2\mu_2 - \lambda_2 - 2p_{112} - 2p_{122}$ into (43), we obtain

$$\begin{aligned} R(\Gamma) &= v_{11} - \frac{2\mu_2}{\mu_1}\lambda_1\Delta_1 + 2\mu_2\left(1 - \frac{\lambda_2 - 2\mu_2}{\mu_1}\right)\Delta_2 \\ &\quad - \left\{\kappa\Delta_2 + \left(\kappa + \frac{\lambda_2}{\lambda_1}\right)(\Delta_2 - \Delta_1)\right\} p_{122} - 2\kappa(\Delta_2 - \Delta_1)\, p_{112} \\ &\quad - \kappa\Delta_2\,(2\mu_2 - \lambda_2 - 2p_{112} - 2p_{122}) \\ &= v_{11} - \frac{2\mu_2}{\mu_1}\lambda_1\Delta_1 + 2\mu_2\left(1 - \frac{\lambda_2 - 2\mu_2}{\mu_1}\right)\Delta_2 - \kappa\Delta_2(2\mu_2 - \lambda_2) \\ &\quad + \left\{\kappa\Delta_1 - \frac{\lambda_2}{\lambda_1}(\Delta_2 - \Delta_1)\right\} p_{122} + 2\kappa\Delta_1 p_{112}. \end{aligned} \tag{44}$$

There are three subcases to consider.

(a) $\frac{\lambda_2}{2} < \mu_2 \le \frac{1}{2}$.
Let $(p_{112}, p_{122}) = \left(\mu_2 - \frac{\lambda_2}{2},\, 0\right)$. Since $2\kappa\Delta_1 > \kappa\Delta_1 - \frac{\lambda_2}{\lambda_1}(\Delta_2 - \Delta_1)$, this maximizes (44) subject to the constraints $(p_{112}, p_{122}) \ge (0, 0)$ and $p_{112} + p_{122} \le \mu_2 - \frac{\lambda_2}{2}$ ($\Leftrightarrow p_{222} \ge 0$). We then have $p_{222} = 2\mu_2 - \lambda_2 - 2p_{112} - 2p_{122} = 0$, and also by (41),

$$(p_{111}, p_{211}, p_{212}) = \left(1 - 2\mu_2,\, 0,\, \frac{\lambda_2}{2}\right).$$

This p clearly satisfies (38). To see that it also satisfies (42), note that

$$\begin{aligned}
(42) \quad \Leftrightarrow \quad & (1+\mu_2)(\Delta_2-\Delta_1)\left(\mu_2-\frac{\lambda_2}{2}\right) \le \mu_2\{\lambda_1(\Delta_2-\Delta_1)+\mu_2\Delta_2\} \\
\Leftrightarrow \quad & \left\{(1+\mu_2)\left(\mu_2-\frac{\lambda_2}{2}\right)-\mu_2\lambda_1\right\}(\Delta_2-\Delta_1) \le \mu_2^2\Delta_2 \\
\Leftarrow \quad & (1+\mu_2)\left(\mu_2-\frac{\lambda_2}{2}\right)-\mu_2\lambda_1 \le \mu_2^2 \\
\Leftrightarrow \quad & \mu_2^2-\frac{\lambda_2\mu_1}{2} \le \mu_2^2.
\end{aligned}$$

(b) $\frac{1}{2} < \mu_2 \le 1-\frac{\lambda_1}{2}$. We let $(p_{112}, p_{122}) = (\frac{\lambda_1}{2}, 0)$. This maximizes platform's payoff (44) subject to $2p_{112}+p_{122} \le \lambda_1$ ($\Leftrightarrow p_{111} \ge 0$). We have $p_{222} = 2\mu_2-\lambda_2-2p_{112}-2p_{122} = 2\mu_2-1$, and hence from (41),

$$(p_{111}, p_{211}, p_{212}) = \left(0,\, 0,\, \mu_1-\frac{\lambda_1}{2}\right).$$

This p satisfies (38), and also (42) since

$$\begin{aligned}
(42) \quad \Leftrightarrow \quad & (1+\mu_2)\frac{\lambda_1}{2}(\Delta_2-\Delta_1)+\Delta_2(2\mu_2-1) \le \mu_2\{\lambda_1(\Delta_2-\Delta_1)+\mu_2\Delta_2\} \\
\Leftrightarrow \quad & \left\{\frac{\lambda_1}{2}(1+\mu_2)-\mu_2\lambda_1\right\}(\Delta_2-\Delta_1) \le \Delta_2\mu_1^2 \\
\Leftrightarrow \quad & \frac{\lambda_1}{2}(\Delta_2-\Delta_1) \le \Delta_2\mu_1 \\
\Leftarrow \quad & 1-\frac{\lambda_1}{2} \ge \mu_2.
\end{aligned}$$

(c) $\mu_2 > 1-\frac{\lambda_1}{2}$. Substituting $p_{222} = 2\mu_2-\lambda_2-2p_{112}-2p_{122}$ into the condition $p_{212} \ge 0$ in (41), we obtain $p_{112}+p_{122} \le \mu_2-p_{222} = -\mu_2+\lambda_2+2p_{112}+2p_{122}$, or equivalently, $p_{122} \ge \mu_2-\lambda_2-p_{112}$. This combined with $p_{122} \le \lambda_1-p_{112}$ in (41) yields

$$p_{112} \le \mu_1.$$

We let $(p_{112}, p_{122}) = (\mu_1, \lambda_1-2\mu_1)$. This maximizes the platform's payoff (44) subject to $2p_{112}+p_{122} \le \lambda_1$ ($\Leftrightarrow p_{111} \ge 0$) and $p_{112} \le \mu_1$. We then have $p_{222} = 2\mu_2-\lambda_2-2p_{112}-2p_{122} = \lambda_2$, and hence from (41),

$$(p_{111}, p_{211}, p_{212}) = (0,\, 0,\, 0).$$

This p satisfies (38) since $\frac{p_{222}}{\lambda_2}-\frac{p_{122}}{\lambda_1} = 1-\frac{\lambda_1-2\mu_1}{\lambda_1} > 0$. To see that it also satisfies (42), note that

$$\begin{aligned}
(42) \quad \Leftrightarrow \quad & \{\mu_2(|Dt_2-\Delta_1)+\Delta_2\}(\lambda_1-2\mu_1)+(1+\mu_2)(\Delta_2-\Delta_1)\mu_1+\Delta_2\lambda_2 \\
& \quad \le \mu_2\{\lambda_1(\Delta_2-\Delta_1)+\mu_2\Delta_2 \\
\Leftrightarrow \quad & \mu_1^2(\Delta_2-\Delta_1)-\mu_1^2\Delta_2 \le 0.
\end{aligned}$$

This completes the proof. ■

Proof of Corollary 1. Substitution of the values from Proposition 12 yields the following probability distributions. It is clear that the dominance relations hold in each case.

1. $\mu_2 \in \left[0,\, \frac{\lambda_2}{2}\right]$.

$$(p(\beta_1,\beta_1 \mid \alpha), 2p(\beta_1,\beta_2 \mid \alpha), p(\beta_2,\beta_2 \mid \alpha) = \begin{cases} (1,0,0) & \text{if } \alpha = \alpha_1, \\ \left(1 - \frac{2\mu_2}{\lambda_2},\, \frac{2\mu_2}{\lambda_2},\, 0\right) & \text{if } \alpha = \alpha_2, \end{cases}$$

and

$$(p(\alpha_1 \mid \beta), p(\alpha_2 \mid \beta)) = \begin{cases} \left(\frac{\lambda_1}{\mu_1}, 1 - \frac{\lambda_1}{\mu_1}\right) & \text{if } \beta = \beta_1, \\ (0,1) & \text{if } \beta = \beta_2. \end{cases}$$

2. $\mu_2 \in \left(\frac{\lambda_2}{2},\, \frac{1}{2}\right]$.

$$(p(\beta_1,\beta_1 \mid \alpha), 2p(\beta_1,\beta_2 \mid \alpha), p(\beta_2,\beta_2 \mid \alpha) = \begin{cases} \left(\frac{1-2\mu_2}{\lambda_1},\, \frac{2\mu_2-\lambda_2}{\lambda_1},\, 0\right) & \text{if } \alpha = \alpha_1, \\ (0,1,0) & \text{if } \alpha = \alpha_2, \end{cases}$$

and

$$(p(\alpha_1 \mid \beta), p(\alpha_2 \mid \beta)) = \begin{cases} \left(1 - \frac{\lambda_2}{2\mu_1},\, \frac{\lambda_2}{2\mu_1}\right) & \text{if } \beta = \beta_1, \\ \left(1 - \frac{\lambda_2}{2\mu_2},\, \frac{\lambda_2}{2\mu_2}\right) & \text{if } \beta = \beta_2. \end{cases}$$

3. $\mu_2 \in \left(\frac{1}{2},\, 1 - \frac{\lambda_1}{2}\right]$.

$$(p(\beta_1,\beta_1 \mid \alpha), 2p(\beta_1,\beta_2 \mid \alpha), p(\beta_2,\beta_2 \mid \alpha) = \begin{cases} (0,1,0) & \text{if } \alpha = \alpha_1, \\ \left(0,\, \frac{2\mu_1-\lambda_1}{\lambda_2},\, \frac{1-2\mu_1}{\lambda_2}\right) & \text{if } \alpha = \alpha_2, \end{cases}$$

and

$$(p(\alpha_1 \mid \beta), p(\alpha_2 \mid \beta)) = \begin{cases} \left(\frac{\lambda_1}{2\mu_1},\, 1 - \frac{\lambda_1}{2\mu_1}\right) & \text{if } \beta = \beta_1, \\ \left(\frac{\lambda_1}{2\mu_2},\, 1 - \frac{\lambda_1}{2\mu_2}\right) & \text{if } \beta = \beta_2. \end{cases}$$

4. $\mu_2 \in \left(1 - \frac{\lambda_1}{2},\, 1\right]$.

$$(p(\beta_1,\beta_1 \mid \alpha), 2p(\beta_1,\beta_2 \mid \alpha), p(\beta_2,\beta_2 \mid \alpha) = \begin{cases} \left(0,\, \frac{2\mu_1}{\lambda_1},\, 1 - \frac{2\mu_1}{\lambda_1}\right) & \text{if } \alpha = \alpha_1, \\ (0,0,1) & \text{if } \alpha = \alpha_2, \end{cases}$$

and

$$(p(\alpha_1 \mid \beta), p(\alpha_2 \mid \beta)) = \begin{cases} (1,0) & \text{if } \beta = \beta_1, \\ \left(1 - \frac{\lambda_2}{\mu_2},\, \frac{\lambda_2}{\mu_2}\right) & \text{if } \beta = \beta_2. \end{cases}$$

■

Proof of Proposition 13. The NAM between buyers implies that

$$p(\beta_1 \mid \beta_1) = \begin{cases} 0 & \text{if } \mu_2 \geq \mu_1, \\ 1 - \frac{\mu_2}{\mu_1} & \text{if } \mu_2 < \mu_1, \end{cases} \quad \text{and} \quad p(\beta_2 \mid \beta_2) = \begin{cases} 1 - \frac{\mu_1}{\mu_2} & \text{if } \mu_2 \geq \mu_1, \\ 0 & \text{if } \mu_2 < \mu_1. \end{cases}$$

It follows that

$$p(\beta_1, \beta_1) = \begin{cases} 0 & \text{if } \mu_2 \geq \frac{1}{2}, \\ 1 - 2\mu_2 & \text{if } \mu_2 < \frac{1}{2}, \end{cases} \qquad p(\beta_2, \beta_2) = \begin{cases} 1 - 2\mu_1 & \text{if } \mu_2 \geq \frac{1}{2}, \\ 0 & \text{if } \mu_2 < \frac{1}{2}, \end{cases}$$

and

$$2p(\beta_1, \beta_2) = \begin{cases} 2\mu_1 & \text{if } \mu_2 \geq \frac{1}{2}, \\ 2\mu_2 & \text{if } \mu_2 < \frac{1}{2}. \end{cases}$$

PAM between a seller and a buyer pair then implies the following for the probability of buyer type profiles matched with a high type seller: When $\mu_2 \geq \frac{1}{2}$, $p(\beta_1, \beta_1 \mid \alpha_2) = 0$,

$$2p(\beta_1, \beta_2 \mid \alpha_2) = \begin{cases} 1 - \frac{1-2\mu_1}{\lambda_2} & \text{if } 1 - 2\mu_1 < \lambda_2, \\ 0 & \text{if } 1 - 2\mu_1 \geq \lambda_2, \end{cases}$$

and

$$p(\beta_2, \beta_2 \mid \alpha_2) = \begin{cases} \frac{1-2\mu_1}{\lambda_2} & \text{if } 1 - 2\mu_1 < \lambda_2, \\ 1 & \text{if } 1 - 2\mu_1 \geq \lambda_2. \end{cases}$$

On the other hand, when $\mu_2 \leq \frac{1}{2}$, $p(\beta_2, \beta_2 \mid \alpha_2) = 0$,

$$p(\beta_1, \beta_1 \mid \alpha_2) = \begin{cases} 1 - \frac{2\mu_2}{\lambda_2} & \text{if } 2\mu_2 < \lambda_2, \\ 0 & \text{if } 2\mu_2 \geq \lambda_2, \end{cases} \quad \text{and} \quad 2p(\beta_1, \beta_2 \mid \alpha_2) = \begin{cases} \frac{2\mu_2}{\lambda_2} & \text{if } 2\mu_2 < \lambda_2, \\ 1 & \text{if } 2\mu_2 \geq \lambda_2, \end{cases}$$

To summarize, we have

$$(p_{211}, p_{212}, p_{222}) = \begin{cases} (\lambda_2 - 2\mu_2, \mu_2, 0) & \text{if } \mu_2 < \frac{\lambda_2}{2}, \\ (0, \frac{\lambda_2}{2}, 0) & \text{if } \frac{\lambda_2}{2} \leq \mu_2 < \frac{1}{2}, \\ (0, \mu_1 - \frac{\lambda_1}{2}, 1 - 2\mu_1) & \text{if } \frac{1}{2} \leq \mu_2 < 1 - \frac{\lambda_1}{2}, \\ (0, 0, \lambda_2) & \text{if } \mu_2 \geq 1 - \frac{\lambda_1}{2}, \end{cases}$$

Likewise, the probability of buyer type profiles matched with a low type seller (α_1) is given by

$$(p_{111}, p_{112}, p_{122}) = \begin{cases} (\lambda_1, 0, 0) & \text{if } \mu_2 < \frac{\lambda_2}{2}, \\ (1 - 2\mu_2, \mu_2 - \frac{\lambda_2}{2}, 0) & \text{if } \frac{\lambda_2}{2} \leq \mu_2 < \frac{1}{2}, \\ (0, \frac{\lambda_1}{2}, 0) & \text{if } \frac{1}{2} \leq \mu_2 < 1 - \frac{\lambda_1}{2}, \\ (0, \mu_1, \lambda_1 - 2\mu_1) & \text{if } \mu_2 \geq 1 - \frac{\lambda_1}{2}, \end{cases}$$

This p is identical to that described in Proposition 12. ■

Proof of Proposition 14. Using (41), we can rewrite W in terms of $(p_{112}, p_{122}, p_{222})$ as

$$W(p) = \lambda_1 v_{11} + \lambda_2 v_{21} + 2\mu_2\Delta_2 - 2(\Delta_2 - \Delta_1)p_{112} - (2\Delta_2 - \Delta_1)p_{122} - \Delta_2 p_{222}. \tag{45}$$

We will identify the socially efficient matching rule p, which solves the following problem.

$$\max_{p_{112}, p_{122}, p_{222}} W(p) \quad \text{subject to} \begin{cases} p_{122} \leq \lambda_1 - 2p_{112}, \\ 2p_{112} + 2p_{122} + p_{222} \geq 2\mu_2 - \lambda_2, \\ p_{112} + p_{122} + p_{222} \leq \mu_2, \\ p_{112},\, p_{122},\, p_{222} \geq 0. \end{cases} \tag{46}$$

As in the proof of Proposition 12, we proceed by separating cases as follows:

1. $\mu_2 < \frac{\lambda_2}{2}$. Set $(p_{112}, p_{122}, p_{222}) = (0, 0, 0)$. This clearly maximizes $W(p)$ in (45) subject to $(p_{112}, p_{122}, p_{222}) \geq (0, 0, 0)$. We can also verify that it satisfies other constraints in (46). Substituting this back into (41), we obtain $p_{111} = \lambda_1$, $p_{211} = \lambda_2 - 2\mu_2$, and $p_{212} = \mu_2$.

2. $\mu_2 \geq \frac{\lambda_2}{2}$. In this case, the constraint $2p_{112} + 2p_{122} + p_{222} \geq 2\mu_2 - \lambda_2$ should hold with equality since $W(p)$ in (45) is decreasing in the three variables. Hence, we substitute $p_{222} = 2\mu_2 - \lambda_2 - 2p_{112} - 2p_{122}$ into $W(p)$ to rewrite the maximization problem as:

$$\max_{p_{112}, p_{122}} \lambda_1 v_{11} + \lambda_2 v_{22} + \Delta_1(2p_{112} + p_{122}) \quad \text{subject to} \begin{cases} \mu_2 - \lambda_2 \leq p_{112} + p_{122} \leq \mu_2 - \frac{\lambda_2}{2}, \\ 2p_{112} + p_{122} \leq \lambda_1, \\ p_{112},\, p_{122} \geq 0. \end{cases}$$

   (a) $\mu_2 \leq \frac{1}{2}$. Since $\mu_2 - \frac{\lambda_2}{2} \leq \frac{\lambda_1}{2}$, the constraint $p_{112} + p_{122} \leq \mu_2 - \frac{\lambda_2}{2}$ holds with equality. The optimal p is then given by $p_{112} = \mu_2 - \frac{\lambda_2}{2}$ and $p_{122} = p_{222} = 0$. Furthermore, $p_{111} = 1 - 2\mu_1$, $p_{211} = 0$ and $p_{212} = \frac{\lambda_2}{2}$.

   (b) $\frac{1}{2} < \mu_2 \leq 1 - \frac{\lambda_1}{2}$. If we choose $(p_{112}, p_{122}) = (\frac{\lambda_1}{2}, 0)$, then it maximizes $W(p)$ subject to $2p_{112} + p_{122} = \lambda_1$. It also satisfies the other constraints. Hence, we can take p such that $p_{112} = \frac{\lambda_1}{2}$, $p_{122} = p_{111} = p_{211} = 0$, $p_{212} = \mu_1 - \frac{\lambda_1}{2}$, and $p_{222} = 2\mu_2 - 1$.

   (c) $\mu_2 > 1 - \frac{\lambda_1}{2}$. If we choose $(p_{112}, p_{122}) = (\mu_1, \lambda_1 - 2\mu_1)$, then it maximizes $W(p)$ subject to $p_{112} + p_{122} = \mu_2 - \lambda_2$ and $2p_{112} + p_{122} = \lambda_1$. Hence, we can take p such that $p_{112} = \mu_1$, $p_{122} = \lambda_1 - 2\mu_1$, $p_{111} = p_{211} = p_{212} = 0$, and $p_{222} = \lambda_2$.

■

Proof of Proposition 15.

It is useful to analyze the buyers' problem in two interim stages: In the reporting stage, a buyer only knows his own valuation type, whereas in the bidding stage, a buyer also knows the quality of the good sold by the matched seller.

First, consider the bidding stage on the path after truthful reporting by both buyers. The auction game is symmetric between the two buyers since $p_{\alpha 12} = p_{\alpha 21}$ for each α, and hence has a symmetric BNE in which the low valuation buyer (β_1) bids $v_{\alpha 1}$ whereas the high valuation buyer (β_2) chooses his bid according to some distribution $G_\alpha(b)$ with support $[v_{\alpha 1}, \overline{b}_\alpha]$ for some $\overline{b}_\alpha > v_{\alpha 1}$.

Call this strategy σ_B. Against σ_B played by the other buyer, when the high valuation buyer β_2 chooses bid $b \in [v_{\alpha 1}, \overline{b}_\alpha]$, his expected payoff is given by

$$(v_{\alpha 2} - b)\left\{\Pr(\beta_1 \mid \alpha, \beta_2) + \Pr(\beta_2 \mid \alpha, \beta_2) G_\alpha(b)\right\}.$$

Since type β_2 is indifferent over bids in the support of G_α,

$$(v_{\alpha 2} - b)\left(\Pr(\beta_1 \mid \alpha, \beta_2) + \Pr(\beta_2 \mid \alpha, \beta_2) G_\alpha(b)\right) = (v_{\alpha 2} - \overline{b}_\alpha). \tag{47}$$

When $b = v_{\alpha 1}$, we have $(v_{\alpha 2} - \overline{b}_\alpha) = (v_{\alpha 2} - v_{\alpha 1}) \Pr(\beta_1 \mid \alpha, \beta_2)$, which yields

$$\overline{b}_\alpha = \Pr(\beta_1 \mid \alpha, \beta_2) v_{\alpha 1} + \Pr(\beta_2 \mid \alpha, \beta_2) v_{\alpha 2},$$

and

$$G_\alpha(b) = \frac{\Pr(\beta_1 \mid \alpha, \beta_2)}{\Pr(\beta_2 \mid \alpha, \beta_2)} \left(\frac{b - v_{\alpha 1}}{v_{\alpha 2} - b}\right) = \frac{p_{\alpha 12}}{p_{\alpha 22}} \left(\frac{b - v_{\alpha 1}}{v_{\alpha 2} - b}\right).$$

Hence, the BNE payoff to the type β_2 buyer in the auction game on the path after truthful reporting equals

$$v_{\alpha 2} - \overline{b}_\alpha = \Pr(\beta_1 \mid \alpha, \beta_2)(v_{\alpha 2} - v_{\alpha 1}). \tag{48}$$

It follows that the type β_2's expected payoff in the reporting stage from truthful reporting equals

$$\begin{aligned} &\Pr(\alpha_1 \mid \beta_2) \Pr(\beta_1 \mid \alpha_1, \beta_2)(v_{12} - v_{11}) + \Pr(\alpha_2 \mid \beta_2) \Pr(\beta_1 \mid \alpha_2, \beta_2)(v_{22} - v_{21}) - t_B(\beta_2) \\ &= \Pr(\alpha_1, \beta_1 \mid \beta_2)(v_{12} - v_{11}) + \Pr(\alpha_2, \beta_1 \mid \beta_2)(v_{22} - v_{21}) - t_B(\beta_2). \end{aligned} \tag{49}$$

Consider now the auction game that follows when a buyer unilaterally misreports his type. If the buyer is the low valuation type (β_1), it is weakly dominant for him to bid $v_{\alpha 1}$, and his expected payoff equals zero. If the buyer is the high valuation type (β_2), his payoff from bidding $b \in [v_{\alpha 1}, \overline{b}_\alpha]$ equals

$$\begin{aligned} &(v_{\alpha 2} - b)\left\{\Pr(\beta_1 \mid \alpha, \beta_1) + \Pr(\beta_2 \mid \alpha, \beta_1) G_\alpha(b)\right\} \\ &= (v_{\alpha 2} - b)\left\{\frac{p_{\alpha 11}}{\Pr(\alpha, \beta_1)} + \frac{p_{\alpha 21}}{\Pr(\alpha, \beta_1)} G_\alpha(b)\right\} \\ &= (v_{\alpha 2} - b)\frac{\Pr(\alpha, \beta_2)}{\Pr(\alpha, \beta_1)}\left[\frac{p_{\alpha 11}}{\Pr(\alpha, \beta_2)} + \frac{p_{\alpha 21}}{\Pr(\alpha, \beta_2)} G_\alpha(b)\right] \\ &= \frac{\Pr(\alpha, \beta_2)}{\Pr(\alpha, \beta_1)}(v_{\alpha 2} - \overline{b}_\alpha). \end{aligned}$$

where the last equality follows from (47). Using (48), we can further rewrite this as

$$\frac{\Pr(\alpha, \beta_2)}{\Pr(\alpha, \beta_1)} \Pr(\beta_1 \mid \alpha, \beta_2)(v_{\alpha 2} - v_{\alpha 1}) = \Pr(\beta_1 \mid \alpha, \beta_1)(v_{\alpha 2} - v_{\alpha 1}). \tag{50}$$

Hence, type β_2's expected payoff in the reporting stage from unilateral misreporting is given by

$$\begin{aligned} &\Pr(\alpha_1 \mid \beta_1) \Pr(\beta_1 \mid \alpha_1, \beta_1)(v_{12} - v_{11}) + \Pr(\alpha_2 \mid \beta_1) \Pr(\beta_1 \mid \alpha_2, \beta_1)(v_{22} - v_{21}) - t_B(\beta_1) \\ &= \Pr(\alpha_1, \beta_1 \mid \beta_1)(v_{12} - v_{11}) + \Pr(\alpha_2, \beta_1 \mid \beta_1)(v_{22} - v_{21}) - t_B(\beta_1). \end{aligned} \tag{51}$$

Combining (49) and (51), we see that the IC and IR conditions for type β_2 are just the same as those for the second-price auction. On the other hand, since the expected payoff of type β_1 equals

0 after truthful reporting as well as after misreporting, his IC and IR conditions are again the same as those for the second-price auction given in (32) and (33).

For the checking of the seller's incentive in reporting, we first compute the expected payment by each buyer type in the bidding stage. When the seller is type α, the expected payment by a type β_1 buyer equals

$$\Pr(\beta_1 \mid \alpha, \beta_1)\frac{1}{2}v_{\alpha 1},$$

and that by a type β_2 buyer equals

$$\int_{v_{\alpha 1}}^{\overline{b}_\alpha} b\left[\Pr(\beta_1 \mid \alpha, \beta_2) + \Pr(\beta_2 \mid \alpha, \beta_2)G_\alpha(b)\right] dG_\alpha(b).$$

Using (47) and $(v_{\alpha 2} - \overline{b}_\alpha) = (v_{\alpha 2} - v_{\alpha 1}) \Pr(\beta_1 \mid \alpha, \beta_2)$, we can rewrite this as

$$\begin{aligned}
&\int_{v_{\alpha 1}}^{\overline{b}_\alpha} b\left[\Pr(\beta_1 \mid \alpha, \beta_2) + \Pr(\beta_2 \mid \alpha, \beta_2)G_\alpha(b)\right] dG_\alpha(b) \\
&= v_{\alpha 2} \Pr(\beta_2 \mid \alpha, \beta_2) \int_{v_{\alpha 1}}^{\overline{b}_\alpha} G_\alpha(b) dG_\alpha(b) + \Pr(\beta_1 \mid \alpha, \beta_2)v_{\alpha 1} \\
&= \Pr(\beta_2 \mid \alpha, \beta_2)\frac{1}{2}v_{\alpha 2} + \Pr(\beta_1 \mid \alpha, \beta_2)v_{\alpha 1}.
\end{aligned}$$

Hence, when the type α seller reports his type truthfully, the payment he can expect from a single buyer is

$$\begin{aligned}
&\Pr(\beta_1 \mid \alpha) \Pr(\beta_1 \mid \alpha, \beta_1) \frac{1}{2}v_{\alpha 1} + \Pr(\beta_2 \mid \alpha) \left[\Pr(\beta_2 \mid \alpha, \beta_2) \frac{1}{2}v_{\alpha 2} + \Pr(\beta_1 \mid \alpha, \beta_2)\, v_{\alpha 1}\right] \\
&= \Pr(\beta_1, \beta_1 \mid \alpha) \frac{1}{2}v_{\alpha 1} + \Pr(\beta_1, \beta_2 \mid \alpha)\, v_{\alpha 1} + \Pr(\beta_2, \beta_2 \mid \alpha) \frac{1}{2}v_{\alpha 2}.
\end{aligned}$$

The seller's expected revenue from two buyers when he reports his type truthfully is then given by

$$\Pr(\beta_1, \beta_1 \mid \alpha)v_{\alpha 1} + 2\Pr(\beta_1, \beta_2 \mid \alpha)v_{\alpha 1} + \Pr(\beta_2, \beta_2 \mid \alpha)v_{\alpha 2}.$$

On the other hand, when the seller misreports his type, it will only change the probability that he will be matched with each buyer type since his quality is observed by the buyers. It follows that the seller's IC and IR conditions are just the same as those for the second-price auction given in (36) and (37). ■

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