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A CHARACTERIZATION OF THE MINIMUM PRICE WALRASIAN RULE WITH RESERVE PRICES FOR AN ARBITRARY NUMBER OF AGENTS AND OBJECTS

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A Characterization of the Minimum Price Walrasian Rule with Reserve Prices for an Arbitrary Number of Agents and Objects *†

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Abstract

We consider the economy consisting of n agents and m heterogenous objects where the seller benefits v from objects. Our study focuses on the multi-object allocation problem with monetary transfers where each agent obtains at most one object (unit-demand). In the situation with arbitrary n, m and v, we show that the minimum price Walrasian rule with reserve prices adjusted to v on the classical domain is the only rule satisfying four desirable properties; efficiency, strategy-proofness, individual rationality and no-subsidy. Our result is an extension of that of Morimoto and Serizawa (2015), and so we can consider more general situation than them. Moreover, we characterize the minimum price Walrasian rules by efficiency, strategy-proofness and two-sided individual rationality.

JEL classification: D82, D47, D63.

Keywords: Multi-object allocation problem, Strategy-proofness, Efficiency, Minimum price Walrasian rule, Non-quasi-linear preference, Heterogeneous objects, Reserve prices.

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†Independently, Wakabayashi and Serizawa have studied the rules with reserve prices and Sakai and Serizawa (2021) have studied the rules for an arbitrary number of agents and objects. Our paper combines their works because there are many things in common.

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1 Introduction

Auctions are popular methods to allocate public assets efficiently. Recent examples includes spectrum license auctions, vehicle ownership auctions, land auctions, etc. An important features of such auctions is that several objects are sold simultaneously, which promotes the efficiency of allocations. Another feature is that winning prices are extremely high.[1] This second feature causes nonnegligible income effects of bidders or faces then with nonlinear borrowing costs. These factors make bidders' quasi-linear preferences implausible. Quasi-linearity is a standard assumption of auction theory, and it simplifies the analysis. In contrast, non-quasi-linear preferences complicate designs of auction rule for efficient allocations. We analyze efficient multi-object auction rules in non-quasi-linear environments.

In the non-quasi-linear environments, one of prominent rules is the minimum price Walrasian (MPW) rule (Demange and Gale, 1985). In the settings where bidders have unit-demand preferences, the MPW rule satisfies not only efficiency, but also strategy-proofness, individual rationality and no subsidy. *Strategy-proofnesss* is an incentive compatibility property that achieves efficient allocations in dominant-strategy equilibria. *Individual rationality* is a property to encourage voluntary participations of bidders. *No subsidy* is a property that the payment of each agents is always nonnegative. Moreover, in cases where the number of agents is greater than objects and the seller benefits nothing from objects, the MPW rule is the unique rule satisfying efficiency, strategy-proofness, individual rationality and no-subsidy (Morimoto and Serizawa, 2015).

This result implies the distinguished theoretical merit of the MPW rule. However, it often happens that the seller fails to invite an enough number of bidders, and consequently the number of agents is equal to or less than objects. Besides typically, objects to be auctioned are previously utilized by public or private sectors for different purposes. It goes without saying that lands and spectrum frequency licenses are such examples. Those sectors have benefitted from auctioned objects, and their benefits should be taken into account to allocate objects efficiently. These factors violate the assumption of Morimoto and Serizawa (2015) and make their result inapplicable. Thus, this article investigates whether a similar result holds in cases where the number of agents is not necessarily more than objects, and the seller may benefits from objects to be auctioned.

In our model, there are n bidders (hereafter "agents") and m heterogenous objects. Each agent obtains at most one object (unit-demand) and pays to the seller. The benefits from the previous utilizations of objects are counted as the seller's benefits, i.e., the seller benefits $v^x \geq 0$ from each object x. His *net revenue* is the sum of agents' payments minus the sum of the benefits of the objects sold to agents. An allocation is *efficient* if no allocation can increase seller's net revenue without worsening off agents' welfare.

[1] For example, in the 3G Spectrum license auction in U.K. (2000), the total revenue for five licenses amounted to £22.5 billion, which is approximately 2.5% of the GDP of U.K. in 2000. See Binmore and Klemperer (2002) for the details.

An (*allocation*) *rule* determines, for each preference profile, the object each agent receives and how much each agent pays. We mainly focus on the four desirable properties, efficiency, strategy-proofness, individual rationality and no subsidy, but we also consider *two-sided individual rationality*; it requires in addition to individual rationality that each agent pay at least the benefit the seller enjoys from the object the agent receives.

For each preference profile, *Walrasian equilibrium with reserve prices* exists (Alkan and Gale, 1990), and the set of Walrasian prices has a lattice structure, so that there is the MPW equilibrium with reserve prices (Demange and Gale, 1985). The MPW rule with reserve prices is the rule which assigns to each preference profile the MPW equilibrium with reserve prices. When reserve prices are adjusted to $v = (v^1, \ldots, v^m)$, the MPW rule with the reserve prices satisfies efficiency in this setting. (Proposition 1)

Extending Morimoto and Serizawa's (2015) result, we show that *the minimum price Walrasian rule with reserve prices equal to* $v = (v^1, \ldots, v^m)$ *is the only rule satisfying efficiency, strategy-proofness, individual rationality and no subsidy* (Theorem 1). We also show that *the minimum price Walrasian rule with reserve prices equal to* $v = (v^1, \ldots, v^m)$ *is the only rule satisfying efficiency, strategy-proofness, and two-sided individual rationality* (Theorem 2).

We emphasize that reserve prices equal to $v = (v^1, \ldots, v^m)$ in Theorem 1 do not directly follow from efficiency and the seller's benefits from objects. Efficiency and seller's benefits from objects, only when combined with strategy-proofness, individual rationality and no subsidy, imply reserve prices equal to $v = (v^1, \ldots, v^m)$.[2] Although it is inevitable to take the seller's benefits into account for practical applications, its consequence is not straightforward. This article analyzes such factors and establishes results that can be applied to more general environments than the previous literature.

We also emphasize that although our results are the extensions of Morimoto and Serizawa's (2015), there are several points in which their proof fails to work in our model.[3] Such points necessitates the develop of our own proof techniques to establish our results, and makes our extensions far from trivial.

This article is organized as follows. Section 2 introduces the model and basic concepts and checks the properties of minimum price Walrasian rules. Our results are in Section 3. We discuss the challenging points in the proofs in Section 4. Section 5 provides proofs. Section 6 discusses related literatures, and Section 7 concludes.

2 The model

Let $N \equiv \{1, \cdots, n\}$ be the set of agents (or buyers) and $M \equiv \{1, \cdots, m\}$ be the set of indivisible and heterogeneous objects. Not consuming an object in M is called consuming the "null object." Let $L \equiv M \cup \{0\} = \{0, 1, \ldots, m\}$, where 0 denotes the null object.

[2] We demonstrate this point by an example in Section 3.

[3] See Subsection 4.1 for the detailed explanation.

Note that we impose no constraint on n and m. We assume that each agent consumes at most one object (unit demand). A typical **(consumption) bundle** for agent i is a pair $z_i \equiv (x_i, t_i) \in L \times \mathbb{R}$: agent i receives object x_i and pays t_i. Let $\mathbf{0} \equiv (0,0) \in L \times \mathbb{R}$.

Each agent has a complete and transitive preference relation R_i over $L \times \mathbb{R}$. Let I_i and P_i be the indifference relation and strict preference relation associated with R_i. A typical class of preferences is denoted by $\mathcal{R}$. We call $\mathcal{R}^n$ a **domain**. We introduce some properties of preferences.

Continuity: For each $z_i \in L \times \mathbb{R}$, the sets $\{z_i' \in L \times \mathbb{R} : z_i' \, R_i \, z_i\}$ and $\{z_i' \in L \times \mathbb{R} : z_i \, R_i \, z_i'\}$ are closed.

Finite compensation: For each $(a,t) \in L \times \mathbb{R}$ and each $b \in L$, there exist $t', t'' \in \mathbb{R}$ such that $(b, t') \; R_i \; (a,t)$ and $(a,t) \; R_i \; (b, t'')$.

Money monotonicity: For each $a \in L$ and each $t, t' \in \mathbb{R}$, if $t < t'$, then $(a,t) \; P_i \; (a,t')$.

Object Monotonicity: For each $a \in M$ and each $t \in \mathbb{R}$, $(a,t) \; P_i \; (0,t)$.

Definition 1. A preference $R_i \in \mathcal{R}$ is **classical** if it satisfies continuity, finite compensation, money monotonicity and object monotonicity.

Let $\mathcal{R}^C$ be the set of all classical preferences. Classical preference R_i implies that all objects are goods for agent i. On the other hand, the preferences in Definition 2 below allow for an agent to have some bads.

Definition 2. A preference $R_i \in \mathcal{R}$ is **extended** if it satisfies continuity, finite compensation and money monotonicity.

Let $\mathcal{R}^E$ be the set of all extended preferences. Note that $\mathcal{R}^C \subsetneq \mathcal{R}^E$. We assume that $\mathcal{R} \subseteq \mathcal{R}^E$.

A **preference profile** is a list of preferences $R \equiv (R_1, \cdots, R_n)$. Given $i \in N$ and $N' \subseteq N$, let $R_{-i} \equiv (R_j)_{j \neq i}$ and $R_{-N'} \equiv (R_j)_{j \in N \setminus N'}$.

An **object allocation** is an n-tuple $x \equiv (x_1, \ldots, x_n) \in L^n$ such that for each $i, j \in N$, if $x_i = x_j$, then $x_i = x_j = 0$, which means that each real object is assigned to at most one agent. Let X be the set of all object allocations. A **(feasible) allocation** is an n-tuple $z \equiv (z_1, \ldots, z_n) = ((x_1, t_1), \ldots, (x_n, t_n)) \in (L \times \mathbb{R})^n$ with $(x_1, \ldots, x_n) \in X$. Let Z be the set of all feasible allocations. We also write an allocation as $z = (x, t)$. We denote the object allocation and payments at $z' \in Z$ by $x' = (x_1', \ldots, x_n')$ and $t' = (t_1', \ldots, t_n')$.

In this model, the seller benefits $v^a \in \mathbb{R}_+$ from each object $a \in M$. Let $v \equiv (v^1, \ldots, v^m) \in \mathbb{R}_+^m$ and $v^0 = 0$. Then, given $z \in Z$, seller's net revenue is denoted by $\sum_{i \in N} (t_i - v^{x_i})$. We assume that v is common knowledge among agents.

An allocation $z' \in Z$ **(Pareto-)dominates** $z \in Z$ for $R \in \mathcal{R}$ if (i) for each $i \in N$, $z'_i \, R_i \, z_i$ and (ii) $\sum_{i \in N} \left(t'_i - v^{x'_i}\right) > \sum_{i \in N} \left(t_i - v^{x_i}\right)$. An allocation $z \in Z$ is **(Pareto-)efficient** for $R \in \mathcal{R}^n$ if there is no allocation that dominates z.[4]

An **(allocation) rule** is a mapping $f = (x,t) \colon \mathcal{R}^n \to Z$. Given a rule $f = (x,t)$ and a preference profile $R \in \mathcal{R}^n$, we denote agent i's assigned object and payment by $f_i(R) = (x_i(R), t_i(R)) \in L \times \mathbb{R}$. Also, we write $f(R) = (f_1(R), \cdots, f_n(R)) \in Z$.

We introduce some desirable properties of allocation rules.

(Pareto-)Efficiency: For each $R \in \mathcal{R}^n$, $f(R)$ is efficient for R.

Strategy-proofness: For each $R \in \mathcal{R}^n$, each $i \in N$ and each $R'_i \in \mathcal{R}$, $f_i(R) \, R_i \, f_i(R'_i, R_{-i})$.

(Buyer-sided) individual rationality: For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R) \, R_i \, \mathbf{0}$.

No-subsidy: For each $R \in \mathcal{R}$ and each $i \in N$, $t_i(R) \geq 0$.

Seller-sided individual rationality: For each $R \in \mathcal{R}^n$ and each $i \in N$, $t_i(R) \geq v^{x_i(R)}$.

Two-sided individual rationality: For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R) \, R_i \, \mathbf{0}$ and $t_i(R) \geq v^{x_i(R)}$.

A **price (vector)** is an m-tuple $p \equiv (p^1, \ldots, p^m) \in \mathbb{R}^m_+$. Given $p \equiv (p^1, \ldots, p^m) \in \mathbb{R}^m_+$, we abuse notation and let p denote the $(m+1)$-tuple $(p^0, p^1, \ldots, p^m)$, where $p^0 = 0$ is the price of the null object, when it causes no confusion. Given $i \in N$, $R_i \in \mathcal{R}$ and $p \in \mathbb{R}^m_+$, agent i's **demand set** $D(R_i, p)$ is the set of his most preferred objects among $\{(0,0), (1,p^1), \ldots, (m,p^m)\}$, that is,

$$D(R_i, p) \equiv \left\{a \in L : \forall b \in L, (a, p^a) \, R_i \, (b, p^b)\right\}.$$

Next, we define the concept of Walrasian equilibrium with a reserve price (vector) $r \equiv (r^1, \ldots, r^m) \in \mathbb{R}^m_+$, where $r^0 = 0$. It is a pair of an allocation and a price vector such that each agent receives an object he demands and pays its price, the price of each object is not less than its reserve price, and the price of an unassigned object is equal to the reserve price of it. Given $r \equiv (r^1, \ldots, r^m) \in \mathbb{R}^m_+$, the set of all price vectors such that the price of each object is not less than its reserve price is denoted by $\mathbb{R}^m_{r+}$, that is,

$$\mathbb{R}^m_{r+} \equiv \{p \in \mathbb{R}^m_+ : \forall a \in M, p^a \geq r^a\}.$$

Definition 3. Given $R \in \mathcal{R}^n$ and $r \in \mathbb{R}^m_+$, a pair $(z,p) = ((x_i, t_i)_{i \in N}, (p^a)_{a \in M}) \in Z \times \mathbb{R}^m_{r+}$ is a **Walrasian equilibrium with (a reserve price) *r* for *R*** if

(WE-i) for each $i \in N$, $x_i \in D(R_i, p)$ and $t_i = p^{x_i}$,

[4] This condition is equivalent to the following: there is no $z' \in Z$ such that (i) for each $i \in N$, $z'_i \, R_i \, z_i$, (ii) there is $j \in N$ such that $z'_j \, P_j \, z_j$ and (iii) $\sum_{i \in N}(t'_i - v^{x'_i}) \geq \sum_{i \in N}(t_i - v^{x_i})$.

(WE-ii) for each $a \in M \setminus \{x_i\}_{i \in N}$, $p^a = r^a$.

Given $R \in \mathcal{R}^n$ and $r \in \mathbb{R}^m_+$, let $W(R, r)$ be the set of Walrasian equilibria with r for R, and define

$$Z(R, r) \equiv \{z \in Z : \exists p \in \mathbb{R}^m_{r+}, (z, p) \in W(R, r)\}$$

and

$$P(R, r) \equiv \left\{p \in \mathbb{R}^m_{r+} : \exists z \in Z, (z, p) \in W(R, r)\right\}.$$

Fact 1 (Alkan and Gale, 1990)**.** For each $R \in \mathcal{R}^n$ and each $r \in \mathbb{R}^m_+$, there is a Walrasian equilibrium, that is, $W(R, r) \neq \emptyset$.

By Fact 1, $Z(R, r) \neq \emptyset$ and $P(R, r) \neq \emptyset$ for each $R \in \mathcal{R}^n$ and each $r \in \mathbb{R}^m_+$.

Fact 2 (Demange and Gale, 1985)**.** For each $R \in \mathcal{R}^n$ and $r \in \mathbb{R}^m_+$, there is $p \in P(R, r)$ such that for each $p' \in P(R, r)$, $p \leq p'$.[5]

Fact 2 shows the existence of the minimum Walrasian equilibrium price. Given $R \in \mathcal{R}^n$ and $r \in \mathbb{R}^m_+$, we denote the **minimum Walrasian equilibrium price with $\boldsymbol{r}$ for $\boldsymbol{R}$** by $p_{\min}(R, r)$, and we define the set of the minimum price Walrasian allocations with r for R by

$$Z_{\min}(R, r) \equiv \{z \in Z : (z, p_{\min}(R, r)) \in W(R, r)\}.$$

A set of objects is overdemanded if the number of agents who demand only objects in the set is larger than the number of objects in the set.

Definition 4. Given $p \in \mathbb{R}^m_{r+}$ and $R \in \mathcal{R}^n$, $M' \subseteq M$ is **(weakly) overdemanded at $\boldsymbol{p}$ for $\boldsymbol{R}$** if

$$|\{i \in N : D(R_i, p) \subseteq M'\}| (\geq) > |M'|.$$

A set of objects is underdemended if the number of agents who demand at least one object in the set is smaller than the number of objects in the set.

Definition 5. Given $p \in \mathbb{R}^m_{r+}$ and $R \in \mathcal{R}^n$, $M' \subseteq M$ is **(weakly) underdemanded at $\boldsymbol{p}$ for $\boldsymbol{R}$** if

$$[\forall a \in M', p^a > r^a] \text{ and } |\{i \in N : D(R_i, p) \cap M' \neq \emptyset\}| (\leq) < |M'|.$$

If objects are (weakly) underdemanded, by decreasing prices of these objects, we can balance demand and supply. However, if the price of some object is equal to the reserve price of it, then we cannot decrease the price of it any more. Hence, in the definition of (weak) underdemand, we don't consider such objects.

By the above two definitions, we characterize the minimum Walrasian equilibrium price.

[5] $p \leq p'$ means that $p^a \leq p'^a$ for each $a \in M$.

Fact 3 (Morimoto and Serizawa, 2015)**.** [6] Let $n, m \in \mathbb{N}$ and $\mathcal{R} = \mathcal{R}^E$. Then, for each $R \in \mathcal{R}^n$, each $r \in \mathbb{R}^m_+$ and each $p \in \mathbb{R}^m_{r+}$, $p = p_{\min}(R, r)$ if and only if no set is overdemanded at p for R and no set is weakly underdemanded at p for R, that is, for each $M' \subseteq M$,

(i) $|\{i \in N : D(R_i, p) \subseteq M'\}| \leq |M'|$,

(ii) $[\forall a \in M', p^a > r^a] \Longrightarrow |\{i \in N : D(R_i, p) \cap M' \neq \emptyset\}| > |M'|$.

An allocation rule is a minimum price Walrasian rule with reserve prices if for each preference profile, the outcome of the rule is in the set of minimum price Walrasian allocations with reserve prices.

Definition 6. Given $r \in \mathbb{R}^m_+$, an allocation rule f is a **minimum price Walrasian rule with (a reserve price) $\boldsymbol{r}$** if for each $R \in \mathcal{R}^n$, $f(R) \in Z_{\min}(R, r)$.

We discuss the properties of the minimum price Walrasian rule with reserve prices.[7] For each reserve prices, the minimum price Walrasian rule with reserve prices satisfies (i) strategy-proofness, (ii) individual rationality and (iii) no-subsidy.

Fact 4. Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Then, for each $r \in \mathbb{R}^m_+$, The minimum price Walrasian rule with r on $\mathcal{R}^n$ satisfies (i) strategy-proofness (Demange and Gale, 1985),[8] (ii) individual rationality, and (iii) no-subsidy.

Note that a reserve price vector with which the minimum price Walrasian rule is associated are not necessarily equal to seller's benefits. Thus, the minimum price Walrasian rule with a reserve price is not necessarily efficient or seller-sided individual rational. Proposition 1 shows that (i) it is efficient if and only if the reserve price of each object is equal to the benefit of it, and (ii) it is seller-sided individual rational if and only if the reserve price of each object is larger than or equal to the benefit of it.

Proposition 1. Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Let $r \in \mathbb{R}^m_+$. Then, the following statements hold.

(i) The minimum price Walrasian rule with r on $\mathcal{R}^n$ satisfies efficiency if and only if $r = v$.

(ii) The minimum price Walrasian rule with r on $\mathcal{R}^n$ satisfies seller-sided individual rationality if and only if $r \geq v$.

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[6] Precisely, Morimoto and Serizawa (2015) shows the above statement in the only case of $r = (0, \ldots, 0)$. However, the proof in the case of $r \geq (0, \ldots, 0)$ is almost same as their proof. The proof of Fact 3 is given in Appendix.

[7] The proofs of Fact 4 is given in Appendix.

[8] Precisely, they show that the minimum price Walrasian rule f is *group strategy-proof*: that is, for each $R \in \mathcal{R}^n$ and each $N' \subseteq N$, there is no $R'_{N'} \in \mathcal{R}^{|N'|}$ such that for each $i \in N$, $f_i(R'_{N'}, R_{-N'}) \, P_i \, f_i(R)$.

Note that in a competitive market, the seller sells each object if and only if its price is greater than the benefit from the object. Thus, the "if" part of Proposition 1 (i) is essentially First Welfare Theorem. On the other hand, the "only if" part of Proposition 1 (i) is not straightforward. The discrepancies between the reserve prices of objects and the benefits from them, if exist, might distort allocations and cause inefficiency. However, since objects are indivisible, small discrepancies may not distort object allocations but only change payments, keeping allocations efficient. Thus, the "only if" part of Proposition 1 (i) holds for the minimum price Walrasian rules with reserve prices, but not for a fixed preference profile.

The "if" part of Proposition 1 (ii) is straightforward, but the "only if" part of Proposition 1 (ii) also holds only for the minimum price Walrasian rules with reserve prices, but not for a fixed preference profile.

The formal proof of Proposition 1 is relegated to Section 5.

3 Characterizations

In this section, we give two characterizations. Theorem 1 says that the minimum price Walrasian rule with the reserve prices equal to seller's benefits is the unique rule satisfying efficiency, strategy-proofness, individual rationality and no-subsidy.

Theorem 1. Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Then, a rule f on $\mathcal{R}^n$ satisfies efficiency, strategy-proofness, individual rationality and no-subsidy if and only if it is a minimum price Walrasian rule with the reserve price $r = v$.

Morimoto and Serizawa (2015) assume that $m < n$ and $r = v = (0, \dots, 0)$, and show that a rule f on the classical domain $\mathcal{R}^n$ satisfies efficiency, strategy-proofness, individual rationality and no-subsidy if and only if it is a minimum price Walrasian rule. Our result generalizes their result for any $n, m \in \mathbb{N}$ and $v \in \mathbb{R}^m_+$.

We emphasize that $r = v$ is not straightforward from the properties of Theorem 1. Proposition 1 says that seller-sided individual rationality implies $r \geq v$ (Proposition 1-ii), but Theorem 1 does not assume seller-sided individual rationality. Proposition 1 also says that $r = v$ is necessary for a minimum price Walrasian rule with a reserve price r to be efficient (Proposition 1-i), but it is not true for non-Walrasian rules. Example 1 below demonstrates this point.

Example 1 (Efficiency and no-subsidy)**.** Let $N = \{1, 2\}$, $M = \{a, b\}$ and $v \in \mathbb{R}^2_+$ with $v^a = v^b = 2\varepsilon > 0$. Let $R = (R_1, R_2) \in \mathcal{R}^2$ be such that for each $i \in N$, $(0, -\varepsilon)\, I_i\, (a, \varepsilon)\, I_i\, (b, \varepsilon)$.

Let f be such that $f(R) = ((a, \varepsilon), (b, \varepsilon))$ and for each $R' \in \mathcal{R}^2 \setminus \{R\}$, $f(R') \in Z(R', v)$. Then, f satisfies efficiency and no-subsidy, but not $r = v$.

In Theorem 1, $r = v$ does not follow from efficiency only, but together with strategy-proofness, individual rationality and no-subsidy. Thus, it is not trivial to obtain $r = v$ from the properties of Theorem 1.

The proposition below says that if the rule satisfies efficiency and strategy-proofness, then no-subsidy ensures that each agent pays at least the benefit the seller enjoys from the object the agent receives.

Proposition 2. Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. If a rule f on $\mathcal{R}^n$ satisfies efficiency, strategy-proofness and no-subsidy, then f satisfies seller-sided individual rationality.

Note that in Example 1, by $t_1(R) = \varepsilon < v^{x_1(R)}$ and $t_2(R) = \varepsilon < v^{x_2(R)}$, f violates seller-sided individual rationality. Thus, this example also demonstrates that if strategy-proofness is dropped, then Proposition 2 dose not hold.

The following example says that if efficiency is dropped, then Proposition 2 dose not hold.[9]

Example 2 (Strategy-proofness and no-subsidy)**.** Let $n = m \in \mathbb{N}$ and $v \in \mathbb{R}^m_+$ such that for each $a \in M$, $v^a > 0$.

Let f be such that for each $R \in \mathcal{R}^n$ and $i \in N$, $f_i(R) = (i, 0)$. Then, f satisfies strategy-proofness and no-subsidy, but violates seller-sided individual rationality.

Since for each $a \in L$, v^a is nonnegative, seller-sided individual rationality implies no-subsidy. Hence, by Proposition 2, no-subsidy is equivalent to seller-sided individual rationality if the rule satisfies efficiency and strategy-proofness. By adding buyer-sided individual rationality, we get the following theorem.

Theorem 2. Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Then, a rule f on $\mathcal{R}^n$ satisfies efficiency, strategy-proofness and two-sided individual rationality if and only if it is a minimum price Walrasian rule with $r = v$.

The following examples show the independence of each properties.

Example 3 (Dropping efficiency)**.** Let $v \in \mathbb{R}^m_+$ and $r \in \mathbb{R}^m_+$ be such that $r \geq v$ and $r \neq v$. Then, the minimum price Walrasian rule with r satisfies strategy-proofness, individual rationality and no-subsidy (or seller-sided individual rationality), but violates efficiency (by Proposition 1).

Example 4 (Dropping strategy-proofness)**.** Let $v \in \mathbb{R}^m_+$. Let $r = v$. Then, the maximum price Walrasian rule with r satisfies efficiency, individual rationality and no-subsidy (or seller-sided individual rationality), but violates strategy-proofness (Demange and Gale, 1985).

[9] We omit the counterexample dropping no-subsidy, since it is obvious that if the rule dose not satisfy no-subsidy, then it also dose not satisfy seller-sided individual rationality.

Given $R \in \mathbb{R}$, $r \in \mathbb{R}^m_+$ and $e \in \mathbb{R}$, $(z, p) \in Z \times \mathbb{R}^m_{r+}$ is a **Walrasian equilibrium with a reserve price $\boldsymbol{r}$ and a entry fee $\boldsymbol{e}$** if (WE-i*) for each $i \in N$, $x_i \in \{a \in L : \forall b \in L, (a, p^a + e) \, R_i \, (b, p^b + e)\}$ and $t_i = p^{x_i} + e$, and (WE-ii) for each $a \in M \setminus \{x_i\}_{i \in N}$, $p^{x_i} = r^{x_i}$ (Morimoto and Serizawa, 2015). Note that if $r = v$, for each $e \in \mathbb{R}$, the minimum price Walrasian rule with r and e satisfies efficiency and strategy-proofness.[10]

Example 5 (Dropping individual rationality)**.** Let $v \in \mathbb{R}^m_+$. Let $r = v$ and $e > 0$. Then, the minimum price Walrasian rule with r and e satisfies efficiency, strategy-proofness and no-subsidy (or seller-sided individual rationality), but violates individual rationality.

Example 6 (Dropping no-subsidy or seller-sided individual rationality)**.** Let $v \in \mathbb{R}^m_+$. Let $r = v$ and $e < -\max_{a \in L} v^a$ (or $e < 0$). Then, the minimum price Walrasian rule with r and e satisfies efficiency, strategy-proofness and individual rationality, but violates no-subsidy (or seller-sided individual rationality).

4 Challenging points in the proofs

By Fact 4 and Proposition 1, in order to prove Theorem 1, it suffices to show that if an allocation rule on $\mathcal{R}^n$ satisfies efficiency, strategy-proofness, individual rationality and no-subsidy, then it is the minimum price Walrasian rule with reserve prices equal to seller's benefits. To prove this, we show the following propositions.

Proposition 3 says that if f satisfies the four axioms, then for each agent, the outcome of f is at least as good as the minimum price Walrasian allocation with reserve prices equal to seller's benefits.

Proposition 3. Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Assume that f satisfies efficiency, strategy-proofness and individual rationality and no-subsidy, and let $R \in \mathcal{R}^n$ and $z \in Z_{\min}(R, v)$. Then, for each $i \in N$, $f_i(R) \, R_i \, z_i$.

Proposition 4 says that if f satisfies the four axioms, then for each agent, his payment is larger than or equal to the minimum Walrasian equilibrium price with reserve prices equal to seller's benefits of the object he receives.

Proposition 4. Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Assume that f satisfies Pareto-efficiency, strategy-proofness and individual rationality and no-subsidy, and let $R \in \mathcal{R}^n$ and $p = p_{\min}(R, v)$. Then, for each $i \in N$, $t_i(R) \geq p^{x_i(R)}$.

Figure 1 illustrates Proposition 3 and 4. In Figure 1, the vertical line corresponds to objects and the horizontal line corresponds to payments. A point in each horizontal line is a consumption bundle of the object. Payments are expressed by the distance from the vertical line. Note that the minus distance means that the agent receives money. The red

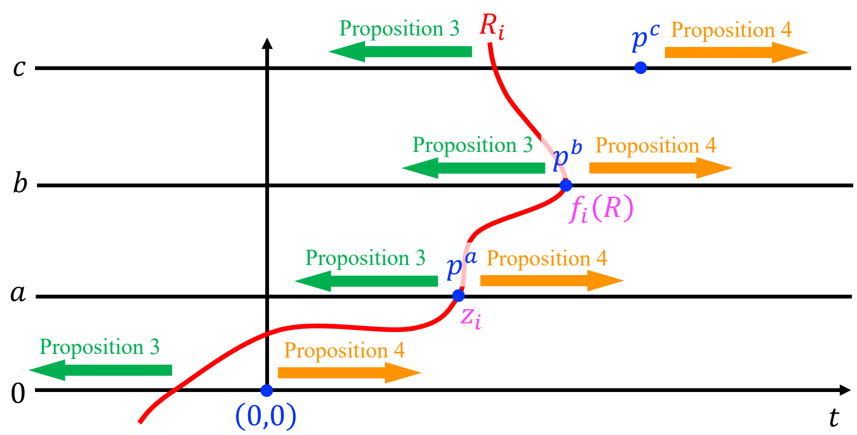

Figure 1: Illustration of Proposition 3 and 4

curve in this figure is the indifference curve of R_i. By money monotonicity, bundles in the left hand side of the indifference curve are preferred to those in the right hand side of it.

Hence, by Proposition 3, $f_i(R)$ must be in the left hand side of the difference curve which is thorough z_i, where z_i is the minimum Walrasian equilibrium allocation of agent i. This relation is expressed by green arrows in this figure. On the other hand, by Proposition 4, the payment should be larger than or equal to the price of the object he receives. This relation is expressed by orange arrows. By these two relations, $f_i(R)$ must be (a, p^a) or (b, p^b) in this figure, and so (WE-i) holds for agent i. (WE-ii) also holds because if there exist some unassigned object a with $p^a > v^a$ under f, then z dominates f for R since a is assigned to some agent under z, which is a contradiction.[11] From this illustration, we can see that Proposition 3 and Proposition 4 prove our theorem.

Morimoto and Serizawa (2015) also use the parallel results of Proposition 3 and 4 to prove that the allocation rule satisfying our four properties is a minimum price Walrasian rule. Although we owe them this basic structure of proof, there are several points that make their proof inapplicable to our model. We explain such points to clarify the novelties of our proofs. To distinguish Morimoto and Serizawa's lemmas from ours, we attach the superscript $*$ to their lemmas. For example, Lemma 1^* is Lemma 1 of Morimoto and Serizawa (2015).

[10]The proof is given in the same way of Fact 4 and Proposition 1.

[11]See the formal proof in Subsection 5.8.

4.1 Challenging points in the proofs of Proposition 3

Lemma 10* of Morimoto and Serizawa (2015) provides a sufficient condition for an agent to get an object. It says that given $i \in N$ and $a \in M$, if i's compensated valuation of $b \in M \setminus \{a\}$ from $\mathbf{0}$ is smaller than the mth highest valuation among $N \setminus \{i\}$, and i's compensated valuation of a from $\mathbf{0}$ is larger than the highest valuation among $N \setminus \{i\}$, then i gets object a.[12] This sufficient condition is a key to prove Proposition 3 since the proof of Proposition 3 repeatedly uses the condition.

Lemma 10* is in turn based on Lemma 4* of Morimoto and Serizawa (2015), which says that all objects are assigned to some agent under the rule satisfying efficiency, individual rationality and no-subsidy for losers.[13] If an agent satisfies the sufficient condition of Lemma 10* but dose not get the object, Lemma 4* implies that some other agent get the object. The proof of Lemma 10* uses this fact to derive a contradiction. Lemma 4* is simple and intuitive, but is a basis of the proof of Proposition 3.

However, in our model, Lemma 4* dose not hold. To see this point, consider Case (i) $m > n$, or Case (ii) $m \leq n$ but the reserve price of some object is so high that no agent demand it. In Case (i), it is obvious that Lemma 4* dose not hold. Case (ii) happens for $R \in \mathcal{R}^n$ and $a \in M$ such that for each $i \in N$, $V_i(a; \mathbf{0}) < v^a$, which implies $\mathbf{0}\, P_i\, (a, v^a)$. If some object a is assigned to some agent i, then by Proposition 2, $\mathbf{0}\, P_i\, (a, v^a)\, R_i\, f_i(R)$, but this violates individual rationality. Thus, Lemma 4* dose not hold in Case (ii). The fact that there are unassigned objects in Cases (i) and (ii) is a hurdle to apply the proof of Morimoto and Serizawa (2015) to our model.

To overcome this hurdle, we need to come up with new proof techniques that are applicable whether the all objects are assigned to agents or not. In Subsection 5.4, we introduce "$(a, t)^\varepsilon$-favoring preferences." This type of preferences plays important roles in our proof.

4.2 Challenging points in the proofs of Proposition 4

In the proof of Proposition 4, Morimoto and Serizawa (2015) use "z-indifferent preferences." A *z-indifferent preference* R_i is a preference such that for each $j, k \in N$, $z_j\, I_i\, z_k$. Their Lemma 11* says that the minimum price does not change after replacing some agents' preferences with z-indifferent preferences.[14] This lemma is repeatedly used to pin

[12] **Lemma 10*.** Let f satisfy efficiency, strategy-proofness, individual rationality and no-subsidy for losers. Let $R \in \mathcal{R}^n$, $a \in M$, $i \in N$ and $z \in Z$ such that $\forall j \in N, z_j\, R_j\, \mathbf{0}$. Assume (a) $\forall b \in M \setminus \{a\}, V_i(b; \mathbf{0}) < V_k(b; \mathbf{0})$, where k has the mth highest valuation among $\{V_j(b; \mathbf{0}) : j \in N \setminus \{i\}\}$, (b) $\forall j \in N \setminus \{i\}, f_j(R)\, R_j\, z_j$, and (c) $V_i(a; \mathbf{0}) > \max\{V_j(b; \mathbf{0}) : j \in N \setminus \{i\}\}$. Then, $x_i(R) = a$.

[13] **Lemma 4*.** Let f satisfy efficiency, individual rationality and no-subsidy for losers. Then, $\forall R \in \mathcal{R}^n$ and $\forall a \in M$, $\exists i \in N$ with $x_i(R) = a$.

[14] **Lemma 11*.** Let f satisfy efficiency, strategy-proofness, individual rationality and no-subsidy for losers. Let $R \in \mathcal{R}^n$ and $(z, p) \in W_{\min}(R, v)$. Let $N' \subseteq N$, $R_{N'} \in \mathcal{R}_v^I(z)^{|N'|}$ and $R' \equiv (R'_{N'}, R_{-N'})$, where $\mathcal{R}_v^I(z)$ is the set of z-indifferent preferences. Then, (a) $(z, p) \in W_{\min}(R', v)$ and (b) $\forall i \in N$, $f_i(R')\, R'_i\, z_i$.

down the allocation of the rule satisfying strategy-proofness, efficiency, individual rationality and no-subsidy for losers. Accordingly, z-indifferent preferences play essential parts of their proof of Proposition 4.

However, in our model, there are cases where z-indifferent preferences are not available. In our model, as discussed above, some objects may be unassigned, or even if all objects objects are assigned, some object's price may be zero. For an allocation z where the price of an object x_i is zero, z-indifferent preferences violate object monotonicity since $(x_i, 0) = z_i \, I_i \, z_j = (0, 0)$. This fact makes us dispense with z-indifferent preferences in our proof of Proposition 4. Thus, instead of z-indifferent preferences, we introduce "p-indifferent preferences" in Subsection 5.6. These preferences enable us to establish a result (Lemma 6 in Subsection 5.6) similar to Lemma 11*.

Lemma 12* plays an important role in Morimoto and Serizawa's (2015) proof of Proposition 4.[15] It implies that if Proposition 4 does not hold, then there is an allocation that Pareto-dominates the allocation of the rule satisfying strategy-proofness, efficiency, individual rationality and no-subsidy for losers. Similar arguments are applied to derive contradictions in several points in their proofs. Thus, Lemma 12* is an essential part of Morimoto and Serizawa's (2015) proof. However in our model, Lemma 12* does not hold.[16] It is challenging to establish Proposition 4 via a different proof route without Lemma 12*.

It is also worth mentioning that our proof of Proposition 4 is more straightforward than the proof of Morimoto and Serizawa (2015). In their proof, in order to derive a contradiction, they use z-indifferent preferences at first and after this, they replace z-indifferent preferences with positive income effect for the null object.[17] This procedure is very complicated. In fact, the related proposition in their paper, Proposition 3*, requires very intricate conditions.[18] We simplify this complicated procedure by taking p-indifferent with positive income effect for reserve-priced objects preferences[19] at first in our proof. This new route of the proof is also our novelty.

[15] **Lemma 12***. Let f satisfy efficiency, strategy-proofness, individual rationality and no-subsidy for losers. Let $R \in \mathcal{R}^n$ and $(z^*, p) \in W_{\min}(R, v)$. Let $N' \subseteq N$ with $1 \leq |N'|$, $R'_{N'} \in \mathcal{R}^I(z^*)^{|N'|}$ and $R' \equiv (R'_{N'}, R_{-N'})$. Assume that (12-i) $\forall i \in N \setminus N'$, $x_i(R') \neq 0 \Longrightarrow t_i(R') \geq p_i^x(R')$, and (12-ii) $\forall j \in N'$, $x_j(R') \neq 0$. Then, $\exists \{i_k\}_{k=1}^K \subseteq N$ such that (i) $x_{i_1}(R') = 0$, (ii) $\forall k \in \{2, \ldots, K\}$, $x_{i_k}(R') \neq 0$, (iii) $\forall k \in \{1, \ldots, K-1\}$, $i_k \in N \setminus N'$ and $i_K \in N'$, and (iv) $\forall k \in \{1, \ldots, K-1\}$, $\{x_{i_k}(R'), x_{i_{k+1}}(R')\} \subseteq D(R_{i_k}, p)$.

[16] The counterexample is given in Appendix.

[17] $R_i \in \mathcal{R}^I(z)$ such that $\forall (a, t) \in M \times \mathbb{R}_+$ with $t < p^a$, $-V_i(0; (a, t)) < p^a - t$.

[18] **Proposition 3***. Let f satisfy efficiency, strategy-proofness, individual rationality and no-subsidy for losers. Let $R \in \mathcal{R}^n$, $(z, p) \in W_{\min}(R, v)$ and $N' \subseteq N$. Assume that $\forall \bar{R}_{N'} \in \mathcal{R}_v^I(z)^{|N'|}$, $\forall i \in N \setminus N'$ and $\forall a \in M$, $x_i(\bar{R}_{N'}, R_{-N'}) = a \Rightarrow t_i(\bar{R}_{N'}, R_{-N'}) \geq p^a$. Then, $\forall R'_{N'} \in \mathcal{R}_v^I(z)^{|N'|}$, $\forall i \in N'$ and $\forall a \in M$, $x_i(R'_{N'}, R_{-N'}) = a \Rightarrow t_i(R'_{N'}, R_{-N'}) \geq p^a$.

[19] See the formal definition in Definition 11 in Subsection 5.6.

5 Proofs

Let $n, m \in \mathbb{N}$ be any numbers and $v \in \mathbb{R}^m_+$ be any benefits of the seller. In this section, we assume $\mathcal{R} = \mathcal{R}^C$. We give the proofs of Proposition 1-4 and Theorem 1.

5.1 Preliminary results

Given $R_i \in \mathcal{R}^E$, $a \in L$ and $(b,t) \in L \times \mathbb{R}$, $V_i(a;(b,t))$ is the **compensated valuation of $\boldsymbol{a}$ from $\boldsymbol{(b,t)}$ for $\boldsymbol{R_i}$** which is defined by $(a, V_i(a;(b,t)))\, I_i\, (b,t)$. The compensated valuation for R'_i is denoted by V'_i.

Fact 5 (Lemma 5 in Morimoto and Serizawa, 2015)**.** Let $R \in \mathcal{R}^n$ and $z \in Z$. For each $i, j \in N$, if $t_i + t_j < V_i(x_j; z_i) + V_j(x_i; z_j)$, then there exists $z' \in Z$ that dominates z for R.

Definition 7. Given $(a,t) \in M \times \mathbb{R}_+$, a preference $R_i \in \mathcal{R}$ is **$\boldsymbol{(a,t)}$-favoring** if for each $b \in L \setminus \{a\}$, $V_i(b;(a,t)) < 0$.

Given $(a,t) \in M \times \mathbb{R}_+$, let $\mathcal{R}^F((a,t))$ be the set of all (a,t)-favoring preferences. Note that $\mathcal{R}^F((a,t)) \subsetneq \mathcal{R}^C$.

Fact 6 (Lemma 8 in Morimoto and Serizawa, 2015)**.** Let f satisfy strategy-proofness and individual rationality and no-subsidy. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $x_i(R) \neq 0$. Then, for each $R'_i \in \mathcal{R}^F(f_i(R))$, $f_i(R'_i, R_{-i}) = f_i(R)$.

5.2 Proof of Proposition 1

Proposition 1. Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Let $r \in \mathbb{R}^m_+$. Then, the following statements hold.

(i) The minimum price Walrasian rule with r on $\mathcal{R}^n$ satisfies efficiency if and only if $r = v$.

(ii) The minimum price Walrasian rule with r on $\mathcal{R}^n$ satisfies seller-sided individual rationality if and only if $r \geq v$.

Proof. Let f be the minimum price Walrasian rule with r. First, we show (i).

If. Assume $r = v$. Let $R \in \mathbb{R}^n$, $z \equiv f(R)$ and $p = p_{\min}(R, r)$. Suppose to the contrary that there is some $z' \in Z$ such that (i) for each $i \in N$, $z'_i\, R_i\, z_i$, and (ii) $\sum_{i \in N}(t'_i - v^{x'_i}) > \sum_{i \in N}(t_i - v^{x_i})$. Then, for each $i \in N$,

$$(x'_i, t'_i) = z'_i \underset{\text{(i)}}{R_i} z_i = (x_i, p^{x_i}) \underset{x_i \in D(R_i, p)}{R_i} (x'_i, p^{x'_i}),$$

which implies $t'_i \leq p^{x'_i}$. Hence,

$$\sum_{i \in N}(p^{x'_i} - v^{x'_i}) \underset{p^{x'_i} \geq t'_i}{\geq} \sum_{i \in N}(t'_i - v^{x'_i}) \underset{\text{(ii)}}{>} \sum_{i \in N}(t_i - v^{x_i}) \underset{\text{(WE-i)}}{=} \sum_{i \in N}(p^{x_i} - v^{x_i}). \tag{1}$$

Also, by $\{x_i\}_{i \in N} \supseteq \{a \in M : p^a > r^a\}$, $\sum_{i \in N}(p^{x_i} - r^{x_i}) = \max_{x'' \in X} \sum_{i \in N}(p^{x''_i} - r^{x''_i})$. Hence,

$$\sum_{i \in N}(p^{x_i} - v^{x_i}) \underset{r=v}{=} \sum_{i \in N}(p^{x_i} - r^{x_i}) \geq \sum_{i \in N}(p^{x'_i} - r^{x'_i}) \underset{r=v}{=} \sum_{i \in N}(p^{x'_i} - v^{x'_i}).$$

However, this inequality contradicts (1).

Only if. Assume that f satisfies efficiency. We show $r = v$ in the following two steps.

Step 1 $\boldsymbol{r \geq v}$. Suppose to the contrary that there is some $a \in M$ with $r^a < v^a$. Let $\varepsilon \in (0, \frac{v^a - r^a}{2})$. Let $R \in \left(\mathcal{R}^Q\right)^n$ be such that for each $i \in N$ and each $b \in M \setminus \{a\}$, $(a, r^a + 2\varepsilon) \, I_i \, (b, r^b + \varepsilon) \, I_i \, \mathbf{0}$ and $p = p_{\min}(R, r)$.

First, we show that there is some $j \in N$ with $x_j(R) = a$. Suppose to the contrary that for each $i \in N$, $x_i(R) \neq a$. Then, by (WE-ii), $p^a = r^a$. Let $i \in N$. Note that by $x_i(R) \neq a$, $D(R_i, p) \setminus \{a\} \neq \emptyset$. By the definition of $R_i \in \mathcal{R}^Q$ and $p \geq r$, for each $b \in M \setminus \{a\}$,

$$(a, p^a) = (a, r^a) \, P_i \begin{cases} (a, r^a + \varepsilon) \, I_i \, (b, r^b) \, R_i \, (b, p^b) \\ (a, r^a + 2\varepsilon) \, I_i \, \mathbf{0} \end{cases}.$$

Hence, $D(R_i, p) = \{a\}$, but this contradicts $D(R_i, p) \setminus \{a\} \neq \emptyset$. Thus, there is some $j \in N$ with $x_j(R) = a$.

Next, we derive a contradiction. By $R_j \in \mathcal{R}^Q$,

$$(a, r^a + 2\varepsilon) \, I_j \, \mathbf{0} \Longleftrightarrow (a, p^a) \, I_j \, (0, p^a - r^a - 2\varepsilon).$$

Also, by the definition of ε,

$$2\varepsilon < v^a - r^a \Longleftrightarrow -v^a < -r^a - 2\varepsilon \Longleftrightarrow p^a - v^a < p^a - r^a - 2\varepsilon.$$

Let $z' \in Z$ be such that $z'_j = (0, p^a - r^a - 2\varepsilon)$ and for each $i \in N \setminus \{j\}$, $z'_j = f_j(R)$. Then, by $f_j(R) \, I_j \, z'_j$ and $t_j(R) - v^{x_j(R)} < t'_j - v^{x'_j}$, z' dominates $f(R)$ for R. However, this is a contradiction. **(End of Step 1)**

Step 2 $\boldsymbol{r = v}$. By Step 1, $v \leq r$. Suppose to the contrary that there is some $a \in M$ with $v^a < r^a$. Let $M^0 \equiv \{b \in M : r^b = 0\}$, $\underline{r} \equiv \min\{r^b\}_{b \in M \setminus M^0}$ and $\varepsilon \in (0, \min\{\underline{r}, \frac{r^a - v^a}{2}\})$. Let $R \in \left(R^Q\right)^n$ be such that for each $i \in N$ and each $b \in M \setminus \{a\}$, $(a, v^a + 2\varepsilon) \, I_i \, (b, \varepsilon) \, I_i \, \mathbf{0}$ and

$p = p_{\min}(R, r)$. Note that $a \in M \setminus M^0$ and $\{x_i(R)\}_{i \in N} \subseteq M^0 \cup \{0\}$ since $0 < v^a + 2\varepsilon < r^a$ and for each $b \in M \setminus M^0$, $\varepsilon < r^b$.

First, we show that for each $i \in N$, $x_i(R) \neq 0$. Suppose to the contrary that there is some $j \in N$ with $x_j(R) = 0$. Then, by $f_j(R) = \mathbf{0}\, I_j\, (a, v^a + 2\varepsilon)$ and $t_j(R) - v^{x_j(R)} = 0 < 2\varepsilon = (v^a + 2\varepsilon) - v^a$, $((a, v^a + 2\varepsilon), f_{-j}(R))$ dominates $f(R)$ for R. However, this is a contradiction. Thus, for each $i \in N$, $x_i(R) \neq 0$.

Next, we derive a contradiction. By $\{x_i(R)\}_{i \in N} \subseteq M^0$, there is some $j \in N$ and $b \in M^0$ with $x_j(R) = b$. By $R_j \in \mathcal{R}^Q$,

$$(b, \varepsilon)\, I_j\, (a, v^a + 2\varepsilon) \Longleftrightarrow (b, p^b)\, I_j\, (a, v^a + \varepsilon + p^b).$$

Also, by $0 \leq v^b \leq r^b$ (Step 1) and $b \in M^0$,

$$p^b - v^b \underset{v^b = r^b = 0}{=} p^b \underset{0 < \varepsilon}{<} p^b + \varepsilon = (v^a + \varepsilon + p^b) - v^a.$$

Let $z' \in Z$ be such that $z'_j = (a, v^a + \varepsilon + p^b)$ and for each $i \in N \setminus \{j\}$, $z'_i = f_i(R)$. Then, by $f_j(R)\, I_j\, z'_j$ and $t_j(R) - v^{x_j(R)} < t'_j - v^{x'_j}$, z' dominates $f(R)$ for R. However, this is a contradiction. Thus, $r = v$ holds.

Next, we show (ii).

If. Assume $r \geq v$. Let $R \in \mathcal{R}$ and $i \in N$. By $f(R) \in Z_{\min}(R, r)$, there is some $p \in \mathbb{R}^m_{r+}$ with $(f(R), p) \in W_{\min}(R, r)$. By $t_i(R) = p^{x_i(R)}$, $p \in \mathbb{R}^m_{r+}$ and $r \geq v$, $t_i(R) \geq r^{x_i(R)} \geq v^{x_i(R)}$.

Only if. Assume that f satisfies seller-sided individual rationality. Suppose to the contrary that there is some $a \in M$ with $r^a < v^a$. Let $\varepsilon \in (0, \frac{v^a - r^a}{2})$. Let $R \in \left(\mathcal{R}^Q\right)^n$ be such that for each $i \in N$ and each $b \in M \setminus \{a\}$, $(a, r^a + 2\varepsilon)\, I_i\, (b, r^b + \varepsilon)\, I_i\, \mathbf{0}$ and $p = p_{\min}(R, r)$.

First, we show that there is some $j \in N$ with $x_j(R) = a$. Suppose to the contrary that for each $i \in N$, $x_i(R) \neq a$. Then, by (WE-ii), $p^a = r^a$. Let $i \in N$. Note that by $x_i(R) \neq a$, $D(R_i, p) \setminus \{a\} \neq \emptyset$. By $R_i \in \mathcal{R}^Q$ and $p \geq r$, for each $b \in M \setminus \{a\}$,

$$(a, p^a) = (a, r^a)\, P_i \begin{cases} (a, r^a + \varepsilon)\, I_i\, (b, r^a)\, R_i\, (b, p^b) \\ (a, r^a + 2\varepsilon)\, I_i\, \mathbf{0} \end{cases}.$$

Hence, $D(R_i, p) = \{a\}$, but this contradicts $D(R_i, p) \setminus \{a\} \neq \emptyset$. Thus, there is some $j \in N$ with $x_j(R) = a$.

Next, we derive a contradiction. By $x_j(R) = a$, $a \in D(R_j, p)$. Then,

$$(a, p^a) \underset{a \in D(R_j, p)}{R_j} \mathbf{0} \underset{\text{def. of } R}{I_j} (a, r^a + 2\varepsilon) \underset{r^a + 2\varepsilon < v^a}{P_j} (a, v^a).$$

Thus, $t_j(R) = p^a < v^a = v^{x_j(R)}$, but this contradicts seller-sided individual rationality. □

5.3 Proof of Proposition 2

Proposition 2. Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. If a rule f on $\mathcal{R}^n$ satisfies efficiency, strategy-proofness and no-subsidy, then f satisfies seller-sided individual rationality.

Proof. Let f satisfy efficiency, strategy-proofness and no-subsidy. Let $R \in \mathcal{R}^n$ and $i \in N$. We show $t_i(R) \geq v^{x_i(R)}$. Suppose to the contrary that $t_i(R) < v^{x_i(R)}$. By $t_i(R) < v^{x_i(R)}$ and no-subsidy, $f_i(R) \in M \times \mathbb{R}_+$. Let $R'_i \in \mathcal{R}^F(f_i(R))$ be such that $-V'_i(0; f_i(R)) < v^{x_i(R)} - t_i(R)$. Then, by Fact 6, $f_i(R'_i, R_{-i}) = f_i(R)$. Let $z' \in Z$ be such that $z'_i \equiv (0, V'_i(0; f_i(R'_i, R_{-i})))$ and for each $j \in N \setminus \{i\}$, $z'_j \equiv f_j(R'_i, R_{-i})$. Then, $z'_i \, I'_i \, f_i(R'_i, R_{-i})$ and for each $j \in N \setminus \{i\}$, $z'_j \, I_j \, f_j(R'_i, R_{-i})$. Also,

$$\begin{aligned}
\sum_{j \in N} \left(t'_j - v^{x'_j} \right) &= V'_i(0; f_i(R'_i, R_{-i})) + \sum_{j \in N \setminus \{i\}} \left(t_j(R'_i, R_{-i}) - v^{x_j(R'_i, R_{-i})} \right) \\
&> \left(t_i(R'_i, R_{-i}) - v^{x_i(R'_i, R_{-i})} \right) + \sum_{j \in N \setminus \{i\}} \left(t_j(R'_i, R_{-i}) - v^{x_j(R'_i, R_{-i})} \right) \quad \text{by def. of } R'_i \\
&= \sum_{j \in N} \left(t_j(R'_i, R_{-i}) - v^{x_j(R'_i, R_{-i})} \right).
\end{aligned}$$

However, these equations contradict that $f(R'_i, R_{-i})$ is efficient for (R'_i, R_{-i}). Therefore, we have $t_i(R) \geq v^{x_i(R)}$. □

5.4 Preliminary results for Proposition 3

Definition 8. Given $(a, t) \in M \times \mathbb{R}_+$ and $\varepsilon \in \mathbb{R}_{++}$, a preference $R_i \in \mathcal{R}$ is **$(a, t)^\varepsilon$-favoring** if

(i) R_i is (a, t)-favoring,

(ii) $V_i(a; \mathbf{0}) = t + 2\varepsilon$,

(iii) for each $b \in M \setminus \{a\}$, $V_i(b; \mathbf{0}) = \varepsilon$.

Given $(a, t) \in M \times \mathbb{R}_+$ and $\varepsilon \in \mathbb{R}_{++}$, let $\mathcal{R}^F((a, t); \varepsilon)$ be the set of all $(a, t)^\varepsilon$-favoring preferences. Note that $\mathcal{R}^F((a, t); \varepsilon) \subsetneq \mathcal{R}^F((a, t)) \subsetneq \mathcal{R}^C$.

Figure 2 illustrates the $(a, t)^\varepsilon$-favoring preference. R_i in Figure 2 is the $(a, t)^\varepsilon$-favoring preference because it satisfies that (i) $V_i(0; (a, t)) = -2\varepsilon < 0$ and $V_i(b; (a, t)) = -\varepsilon < 0$ (R_i is (a, t)-favoring), (ii) $V_i(a; \mathbf{0}) = t + 2\varepsilon$, and (iii) $V_i(b; \mathbf{0}) = \varepsilon$.

Definition 9. A preference $R_i \in \mathcal{R}$ is **quasi-linear** if for each $(a, t), (b, t') \in L \times \mathbb{R}$ and each $\delta \in \mathbb{R}$,

$$(a, t) \, I_i \, (b, t') \Longleftrightarrow (a, t - \delta) \, I_i \, (b, t' - \delta).$$

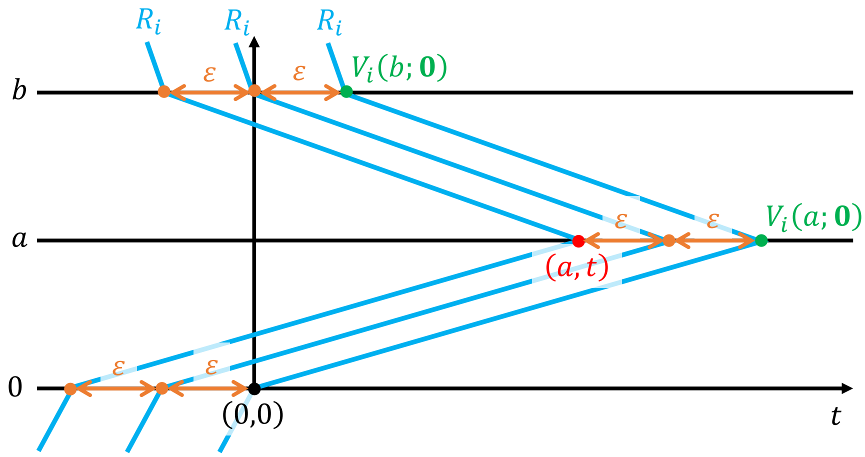

Figure 2: Illustration of the $(a,t)^{\varepsilon}$-favoring preference.

Let $\mathcal{R}^Q$ be the set of all quasi-linear preferences. Note that $\mathcal{R}^Q \subsetneq \mathcal{R}^C$. Also, note that the indifference curves of a quasi-linear preference are parallel as in Figure 2. Given $(a,t) \in M \times \mathbb{R}_+$ and $\varepsilon \in \mathbb{R}_{++}$, a quasi-linear preference R_i is uniquely determined if we define $\mathbf{0}\, I_i\, (a, t+2\varepsilon)$ and $\mathbf{0}\, I_i\, (b,\varepsilon)$ for each $b \in M \setminus \{a\}$. This is because the above relations determines one indifference curve thorough $\mathbf{0}$ and any other indifference curves are parallel from the original indifference curve. Thus, we can get the following remark.

Remark 1. For each $(a,t) \in M \times \mathbb{R}_+$ and each $\varepsilon \in \mathbb{R}_{++}$, a $(a,t)^{\varepsilon}$-favoring and quasi-linear preference R_i uniquely exists.

Given $(a,t) \in M \times \mathbb{R}_+$ and $\varepsilon \in \mathbb{R}_{++}$, the $(a,t)^{\varepsilon}$-favoring and quasi-linear preference is denoted by $R^Q((a,t);\varepsilon)$. Note that R_i in Figure 2 is $R^Q((a,t);\varepsilon)$.

Lemma 1 says that if an agent has $R^Q(z_i;\varepsilon_i)$ under some conditions, then he never gets x_i.

Lemma 1. Let f satisfy strategy-proofness, individual rationality and no-subsidy. Let $R \in \mathcal{R}^n$, $i \in N$ and $z_i \in M \times \mathbb{R}_+$ be such that $z_i\, P_i\, f_i(R)$. Let $\varepsilon_i \in (0, \frac{1}{2}(V_i(x_i; f_i(R)) - t_i))$ and $R'_i \equiv R^Q(z_i;\varepsilon_i)$. Then, $x_i(R'_i, R_{-i}) \neq x_i$.

Proof. Note that by $z_i\, P_i\, f_i(R)$, $t_i < V_i(x_i; f_i(R))$, and so we can pick $\varepsilon_i \in (0, \frac{1}{2}(V_i(x_i; f_i(R)) - t_i))$. Suppose to the contrary that $x_i(R'_i, R_{-i}) = x_i$. Then,

$$t_i(R'_i, R_{-i}) \underset{f_i(R'_i,R_{-i})\, R'_i\, \mathbf{0}}{\leq} V'_i(x_i(R'_i, R_{-i}); \mathbf{0}) \underset{x_i(R'_i,R_{-i})=x_i}{=} V'_i(x_i;\mathbf{0}) \underset{\text{(ii) in def. 8}}{=} t_i + 2\varepsilon_i \underset{\text{def. of } \varepsilon_i}{<} V_i(x_i; f_i(R)).$$

By $x_i(R'_i, R_{-i}) = x_i$, $t_i(R'_i, R_{-i}) < V_i(x_i; f_i(R))$ implies $f_i(R'_i, R_{-i})\, P_i\, f_i(R)$. However, this relation contradicts strategy-proofness. Hence, $x_i(R'_i, R_{-i}) \neq x_i$. $\square$

Lemma 2 describes how f allocates objects and payments if some agents have $R^Q(z_i; \varepsilon_i)$.

Lemma 2. Let f satisfy Pareto-efficiency, strategy-proofness and individual rationality and no-subsidy. Let $N' \subseteq N$, $z \in Z$, $R \in \mathcal{R}^n$ and $(\varepsilon_i)_{i \in N'} \in \mathbb{R}^{|N'|}_{++}$ be such that for each $i \in N'$, $x_i \neq 0$, $t_i \geq v^{x_i}$ and $R_i = R^Q(z_i; \varepsilon_i)$. Then, for each $i \in N'$, there exists some $j \in N$ such that

(i) $x_j(R) = x_i$,

(ii) if $j \neq i$, then $t_j(R) \geq t_i + \varepsilon_i$.

Proof. Let $i \in N'$.

(i) Suppose to the contrary that for each $j \in N$, $x_j(R) \neq x_i$. Let

$$\delta_i \equiv \begin{cases} \varepsilon_i & \text{if } x_i(R) \neq 0 \\ 2\varepsilon_i & \text{if } x_i(R) = 0 \end{cases}.$$

Then, by $x_i(R) \neq x_i$ and $R_i = R^Q(z_i; \varepsilon_i)$, $(x_i, t_i + \delta_i) \, I_i \, (x_i(R), 0)$. Moreover, by $R_i = R^Q(z_i; \varepsilon_i)$,

$$(x_i, t_i + \delta_i + t_i(R)) \, I_i \, (x_i(R), t_i(R)).$$

Also,

$$(t_i + \delta_i + t_i(R)) - v^{x_i} \underset{t_i \geq v^{x_i}}{\geq} \delta_i + t_i(R) \underset{\delta_i > 0}{>} t_i(R) \underset{v^{x_i(R)} \geq 0}{\geq} t_i(R) - v^{x_i(R)}.$$

Then, since for each $j \in N$, $x_j(R) \neq x_i$, $((x_i, t_i + \delta_i + t_i(R)), f_{-i}(R))$ dominates $f(R)$ for R. However, this is a contradiction. Hence, there exists some $j \in N$ such that $x_j(R) = x_i$.

(ii) Let $j \in N$ be such that $x_j(R) = x_i$ and $j \neq i$. Suppose to the contrary that $t_j(R) < t_i + \varepsilon_i$. Let $R'_j \in \mathcal{R}^F(f_j(R))$ be such that for each $a \in L \setminus \{x_i\}$, $-V'_j(a; f_j(R)) < t_i + \varepsilon_i - t_j(R)$. Let $R' \equiv (R'_j, R_{-j})$. Then, by Fact 6, $f_j(R') = f_j(R)$, which implies $x_i(R') \neq x_i$. Thus, we get $-V'_j(x_i(R'); f_j(R')) < t_i + \varepsilon_i - t_j(R')$. Let

$$\delta'_i \equiv \begin{cases} \varepsilon_i & \text{if } x_i(R') \neq 0 \\ 2\varepsilon_i & \text{if } x_i(R') = 0 \end{cases}.$$

By $x_i(R') \neq x_i$ and $R_i = R^Q(z_i; \varepsilon_i)$, $(x_i, t_i + \delta'_i) \, I_i \, (x_i(R'), 0)$ if and only if $(x_i, t_i + \delta'_i + t_i(R')) \, I_i \, (x_i(R'), t_i(R'))$, which implies $V_i(x_j(R'); f_i(R')) = t_i + \delta'_i + t_i(R')$. Then, by $-V'_j(x_i(R'); f_j(R')) < t_i + \varepsilon_i - t_j(R')$ and $V_i(x_j(R'); f_i(R')) = t_i + \delta'_i + t_i(R') \geq t_i + \varepsilon_i + t_i(R')$,

$$\begin{aligned} & V'_j(x_i(R'); f_j(R')) + V_i(x_j(R'); f_i(R')) \\ > & -(t_i + \varepsilon_i) + t_j(R') + t_i + \varepsilon_i + t_i(R') \\ = & t_j(R') + t_i(R'). \end{aligned}$$

By Fact 5, there exists some $z \in Z$ that dominates $f(R')$ for R'. However, this is a contradiction. Hence, $t_j(R) \geq t_i + \varepsilon_i$. □

5.5 Proof of Proposition 3

Proposition 3. Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Assume that f satisfies efficiency, strategy-proofness and individual rationality and no-subsidy, and let $R \in \mathcal{R}^n$ and $z \in Z_{\min}(R, v)$. Then, for each $i \in N$, $f_i(R)\, R_i\, z_i$.

Proof. Suppose to the contrary that for some $i \in N$, $z_i\, P_i\, f_i(R)$. Without loss of generality, let $i = 1$. Let $\varepsilon_0 \equiv \frac{1}{2}(V_1(x_1; f_1(R)) - t_1)$. Then, we show the following claim by induction. The last condition (iii) derives a contradiction.

Claim 1. For each $k \in \{1, \ldots, n\}$, there exist $N(k) \equiv \{1, \ldots, k\} \subseteq N$, $(\varepsilon_i)_{i \in N(k)} \in \mathbb{R}^k_{++}$ and $R^{(k)} \equiv (R'_{N(k)}, R_{-N(k)}) \in \mathcal{R}^n$ such that

(i) for each $i \in N(k)$, $x_i \neq 0$,

(ii) for each $i \in N(k)$, $0 < \varepsilon_i < \min\{\varepsilon_{i-1}, \frac{1}{2}(V_i(x_i; f_i(R^{(i-1)})) - t_i)\}$ and $R'_i = R^Q(z_i; \varepsilon_i)$,

(iii) $x_k(R^{(k)}) \notin \{x_i\}_{i \in N(k)}$.

Induction Base. Let $k = 1$.

(i) By $z_1\, P_1\, f_1(R)$, if $x_1 = 0$, $\mathbf{0}\, P_i\, f_1(R)$. However, this contradicts individual rationality. Hence, $x_1 \neq 0$.

(ii) By $z_1\, P_1\, f_1(R)$, we can pick $\varepsilon_1 \in (0, \frac{1}{2}(V_1(x_1; f_1(R)) - t_1))$, and let $R'_1 \equiv R^Q(z_1; \varepsilon_1)$.

(iii) By (i), (ii) and Lemma 1, $x_1(R^{(1)}) \neq x_1$.

Induction Hypothesis. Let $s \in \{1, \ldots, n-1\}$. Assume that there exist $N(s) \equiv \{1, \ldots, s\} \subseteq N$, $(\varepsilon_i)_{i \in N(s)} \in \mathbb{R}^s_{++}$ and $R^{(s)} \equiv (R'_{N(s)}, R_{-N(s)}) \in \mathcal{R}^n$ such that

(i-s) for each $i \in N(s)$, $x_i \neq 0$,

(ii-s) for each $i \in N(s)$, $0 < \varepsilon_i < \min\{\varepsilon_{i-1}, \frac{1}{2}(V_i(x_i; f_i(R^{(i-1)})) - t_i)\}$ and $R'_i = R^Q(z_i; \varepsilon_i)$,

(iii-s) $x_s(R^{(s)}) \notin \{x_i\}_{i \in N(s)}$.

Induction Argument. We consider the case $s+1$. By (i-s), $(t_i)_{i \in N(s)} \geq (v^{x_i})_{i \in N(s)}$, (ii-s) and Lemma 2 (i), for each $i \in N(s)$, there exists some $j \in N$ such that $x_j(R^{(s)}) = x_i$. In particular, by (iii-s), there exists some $k \in N \setminus N(s)$ such that $x_k(R^{(s)}) \in \{x_i\}_{i \in N(s)}$. Without loss of generality, let $k \equiv s+1$. Moreover, let $l \in N(s)$ be such that $x_{s+1}(R^{(s)}) = x_l$. By $s + 1 \neq l$ and Lemma 2 (ii), $t_{s+1}(R^{(s)}) \geq t_l + \varepsilon_l > t_l$. Then,

$$z_{s+1} \underset{\text{(WE-i)}}{R_{s+1}} z_l \underset{t_l < t_{s+1}(R^{(s)})}{P_{s+1}} (x_l, t_{s+1}(R^s)) = f_{s+1}(R^{(s)}).$$

(i) By individual rationality and $z_{s+1}\, P_{s+1}\, f_{s+1}(R^{(s)})$, $x_{s+1} \neq 0$.

(ii) By (i) and $z_{s+1}\, P_{s+1}\, f_{s+1}(R^{(s)})$, we can pick ε_{s+1} such that

$$0 < \varepsilon_{s+1} < \min\left\{\varepsilon_s, \frac{1}{2}(V_{s+1}(x_{s+1}; f_{s+1}(R^{(s)})) - t_{s+1})\right\},$$

and let $R'_{s+1} = R^Q(z_{s+1}; \varepsilon_{s+1})$.

(iii) By (i), (ii) and Lemma 1, $x_{s+1}(R^{(s+1)}) \neq x_{s+1}$. Suppose to the contrary that $x_{s+1}(R^{(s+1)}) \in \{x_i\}_{i \in N(s)}$. Let $l' \in N(s)$ be such that $x_{s+1}(R^{(s+1)}) = x_{l'}$. Then, by $x_{s+1} \neq x_{s+1}(R^{(s+1)}) = x_{l'}$, $s+1 \neq l'$. By $s+1 \neq l'$ and Lemma 2 (ii), $t_{s+1}(R^{(s+1)}) \geq t_{l'} + \varepsilon_{l'}$. By $t_{l'} \geq 0$ and $\varepsilon_{s+1} < \varepsilon_s < \cdots < \varepsilon_{l'+1} < \varepsilon_{l'}$, $\varepsilon_{s+1} + 0 < \varepsilon_{l'} + t_{l'} \leq t_{s+1}(R^{(s+1)})$. Then,

$$\mathbf{0} \underset{x_{l'} \neq x_{s+1},\, \text{(iii) in def. 8}}{I'_{s+1}} (x_{l'}, \varepsilon_{s+1}) \underset{\varepsilon_{s+1} < t_{s+1}(R^{(s+1)})}{P'_{s+1}} (x_{l'}, t_{s+1}(R^{(s+1)})) \underset{x_{l'} = x_{s+1}(R^{(s+1)})}{=} f_{s+1}(R^{(s+1)}).$$

However, this contradicts individual rationality. Hence, $x_{s+1}(R^{(s+1)}) \notin \{x_i\}_{i \in N(s)}$, and so $x_{s+1}(R^{(s+1)}) \notin \{x_i\}_{i \in N(s+1)}$. **(End of Claim 1)**

Let $k = n$. Then, by Claim 1 (i), $(t_i)_{i \in N} \geq (v^{x_i})_{i \in N}$, Claim 1 (ii) and Lemma 2 (i), for each $i \in N$, there exists some $j \in N$ such that $x_j(R^{(n)}) = x_i$, which implies $\{x_j(R^{(n)})\}_{j \in N} = \{x_i\}_{i \in N}$. Hence, there exists some $i \in N$ such that $x_n(R^{(n)}) = x_i$. However, this contradicts Claim 1 (iii). □

5.6 Preliminary results for Proposition 4

Lemma 3 says that if an agent's payment is larger than or equal to the minimum Walrasian price with v of the object he receives, then $f_i(R)$ satisfies (WE-i) for him.

Lemma 3. Let f satisfy efficiency, strategy-proofness, individual rationality and no-subsidy. Let $R \in \mathcal{R}^n$ and $p = p_{\min}(R, v)$. For each $i \in N$, if $t_i(R) \geq p^{x_i(R)}$, then $x_i(R) \in D(R_i, p)$ and $t_i(R) = p^{x_i(R)}$.

Proof. Let $i \in N$ be such that $t_i(R) \geq p^{x_i(R)}$. By $p = p_{\min}(R, v)$, there exists some $z \in Z$ such that $(z, p) \in W_{\min}(R, v)$. Then,

$$(x_i(R), p^{x_i(R)}) \underset{p^{x_i(R)} \leq t_i(R)}{R_i} (x_i(R), t_i(R)) = f_i(R) \underset{\text{Prop. 3}}{R_i} z_i \underset{\text{(WE-i)}}{R_i} (x_i(R), p^{x_i(R)}).$$

Hence, we obtain

$$(x_i(R), p^{x_i(R)})\, I_i\, f_i(R)\, I_i\, z_i,$$

which implies that $x_i(R) \in D(R_i, p)$ and $t_i(R) = p^{x_i(R)}$. □

Given $N' \subseteq N$, Lemma 4 describes a sufficient condition for constructing a sequence of agents such that (a) it starts at the agent who gets an object a with $p^a = v^a$, (b) an

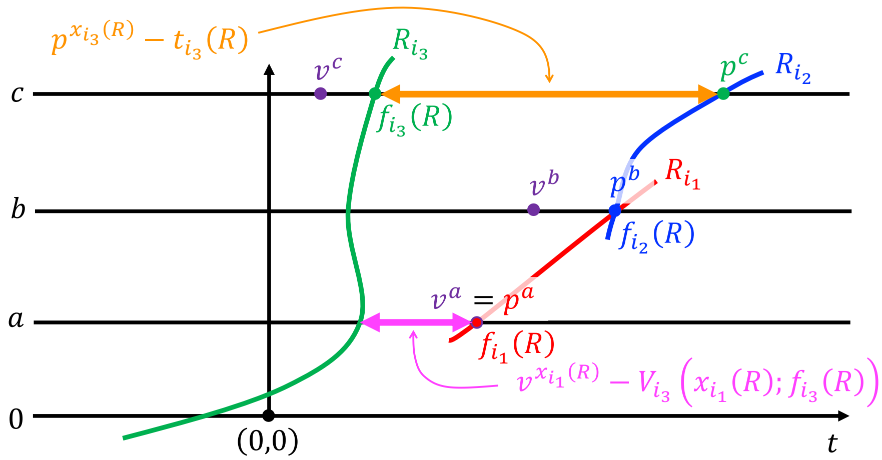

Figure 3: Illustration of the sequence.

agent in the sequence demands the object of his next agent, and (c) the final agent must be in N'. This lemma is closely related to Lemma 5.

Figure 3 illustrates the sequence in Lemma 4. In this figure, there exists a sequence $\{i_1, i_2, i_3\}$ such that (a) i_1 gets object $a = x_{i_1}(R)$ with $p^a = v^a$, (b) i_1 demands $b = x_{i_2}(R)$ and i_2 demands $c = x_{i_3}(R)$, and (c) $i_3 \in N'$.

Lemma 4. Let f satisfy efficiency, strategy-proofness, individual rationality and no-subsidy. Let $R \in \mathcal{R}^n$, $p = p_{\min}(R, v)$ and $N' \subseteq N$. If

(i) for some $j \in N'$, $p^{x_j(R)} > v^{x_j(R)}$,

(ii) for each $i \in N \setminus N'$, $t_i(R) \geq p^{x_i(R)}$,

then there exists $\{i_k\}_{k=1}^{K} \subseteq N$ with $K \geq 2$ such that

(a) $p^{x_{i_1}(R)} = v^{x_{i_1}(R)}$,

(b) for each $k \in \{1, \ldots, K-1\}$, $\{x_{i_k}(R), x_{i_{k+1}}(R)\} \subseteq D(R_{i_k}, p)$ and $t_{i_k}(R) = p^{x_{i_k}(R)}$,

(c) $i_K \in N'$.

Proof. Assume that (i) and (ii) hold. First, we construct $\{j_k\}_{k=1}^{K'} \subseteq N$ with $K' \geq 2$ as follows.

Step 1: By (i), we can pick $j_1 \in N'$ such that $p^{x_{j_1}(R)} > v^{x_{j_1}(R)}$ and go to the next step.

Step $s \geq 2$: Since for each $k \in \{1, \ldots, s-1\}$, $p^{x_{j_k}(R)} > v^{x_{j_k}(R)}$ and $\{x_{j_1}(R), \ldots, x_{j_{s-1}}(R)\}$

is not weakly underdemanded at p for R (ii, Fact 3), there exists some $j_s \in N \setminus \{j_k\}_{k=1}^{s-1}$ such that $D(R_{j_s}, p) \cap \{x_{j_1}(R), \ldots, x_{j_{s-1}}(R)\} \neq \emptyset$. If $p^{x_{j_s}(R)} = v^{x_{j_s}(R)}$, we stop this process. Otherwise, we go to the next step.

Since $\{a \in M : p^a > v^a\}$ is not weakly underdemanded at p for R,

$$|N| \geq |\{i \in N : D(R_i, p) \cap \{a \in M : p^a > v^a\} \neq \emptyset\}| > |\{a \in M : p^a > v^a\}\}|.$$

Thus, there exists some $l \in N$ such that $p^{x_l(R)} = v^{x_l(R)}$. Hence, the above process finishes in the finite time. As the result, we get $\{j_k\}_{k=1}^{K'} \subseteq N$ with $K' \geq 2$ such that

(K'-i) $j_1 \in N'$,

(K'-ii) for each $k \in \{2, \ldots, K'\}$, $D(R_{j_k}, p) \cap \{x_{j_1}(R), \ldots, x_{j_{k-1}}(R)\} \neq \emptyset$,

(K'-iii) $p^{x_{j_{K'}}(R)} = v^{x_{j_{K'}}(R)}$.

Next, we construct $\{i_k\}_{k=1}^{K} \subseteq \{j_k\}_{k=1}^{K'}$ with $K \geq 2$ as follows.

Step 1: Let $i_1 \equiv j_{K'}$. By (K'-ii), $D(R_{i_1}, p) \cap \{x_{j_1}(R), \ldots, x_{j_{K'-1}}(R)\} \neq \emptyset$. Hence, there exists some $i_2 \in \{j_1, \ldots, j_{K'-1}\}$ such that $x_{i_2}(R) \in D(R_{i_1}, p)$. If $i_2 \in N'$, we stop this process. Otherwise, we go to the next step.

Step $s \geq 2$: $i_s \in \{j_k\}_{k=1}^{K'}$ is determined by the previous step. By (K'-ii), $D(R_{i_s}, p) \cap \{x_{j_1}(R), \ldots, x_{j_{K''-1}}(R)\} \neq \emptyset$, where $j_{K''} = i_s$. Hence, there exists some $i_{s+1} \in \{j_1, \ldots, j_{K''-1}\}$ such that $x_{i_{s+1}}(R) \in D(R_{i_s}, p)$. If $i_{s+1} \in N'$, we stop this process. Otherwise, we go to the next step.

By $j_1 \in N'$, $\{j_k\}_{k=1}^{K'} \cap N' \neq \emptyset$. Thus, since the number of agents is finite and the left agents in $\{j_k\}_{k=1}^{K'}$ strictly decrease step by step, the above process finishes in the finite time.[20]

Hence, we get $\{i_k\}_{k=1}^{K} \subseteq \{j_k\}_{k=1}^{K'}$ with $K \geq 2$ such that

(K-i) $p^{x_{i_1}(R)} = v^{x_{i_1}(R)}$,

(K-ii) for each $k \in \{1, \ldots, K-1\}$, $x_{i_{k+1}} \in D(R_{i_k}, p)$,

(K-iii) for each $k \in \{2, \ldots, K-1\}$, $i_k \in N \setminus N'$,

(K-iv) $i_K \in N'$.

[20]More precisely, the proof is as follows. By $j_1 \in N'$, $\{j_k\}_{k=1}^{K'} \cap N' \neq \emptyset$. Let $I(l) \equiv \{j_k : i_l = j_{k'} \text{ and } k < k'\}$. Then, by the construction of the sequence, $I(l+1) \subsetneq I(l) \subsetneq \cdots \subsetneq I(1) \subsetneq \{j_k\}_{k=1}^{K'}$. By $\{j_k\}_{k=1}^{K'} \cap N' \neq \emptyset$, $I(l+1) \subsetneq I(l)$ and finiteness of agents, the above process finishes in the finite time.

(a) By (K-i), $p^{x_{i_1}(R)} = v^{x_{i_1}(R)}$.

(b) By Proposition 2 and (K-i), $t_{i_1}(R) \geq v^{x_{i_1}(R)} = p^{x_{i_1}(R)}$. Thus, by (ii) and ($K$-iii), for each $k \in \{1, \ldots, K-1\}$, $t_{i_k}(R) \geq p^{x_{i_k}(R)}$. Hence, by Lemma 3 and (K-ii), for each $k \in \{1, \ldots, K-1\}$, $\{x_{i_k}(R), x_{i_{k+1}}(R)\} \subseteq D(R_{i_k}, p)$ and $t_{i_k}(R) = p^{x_{i_k}(R)}$.

(c) By (K-iii), $i_K \in N'$. □

Lemma 5 says that if there exists a sequence of agents satisfying (a) it starts at the agent who gets an object a with $p^a = v^a$, (b) an agent in the sequence demands the object of his next agent, and (c) the final agent can move a with small compensation, then we can construct an allocation that dominates the allocation rule outcome.

Figure 3 also illustrates the sequence in Lemma 5. In this figure, $\{i_1, i_2, i_3\}$ satisfies (a) i_1 gets object $a = x_{i_1}(R)$ with $p^a = v^a$, (b) i_1 demands $b = x_{i_2}(R)$ and i_2 demands $c = x_{i_3}(R)$, and (c) $r^a - V_{i_3}(a; f_{i_3}(R)) < p^{x_{i_3}(R)} - t_{i_3}(R)$, and so $((b, p^b), (c, p^c), (a, V_{i_3}(a; f_{i_3}(R))))$ dominates $((a, p^a), (b, p^b), (c, t_{i_3}(R)))$.

Lemma 5. Let $R \in \mathcal{R}^n$ and $p = p_{\min}(R, v)$. If there exists $\{i_k\}_{k=1}^K \subseteq N$ with $K \geq 2$ such that

(i) $p^{x_{i_1}(R)} = v^{x_{i_1}(R)}$,

(ii) for each $k \in \{1, \ldots, K-1\}$, $\{x_{i_k}(R), x_{i_{k+1}}(R)\} \subseteq D(R_{i_k}, p)$ and $t_{i_k}(R) = p^{x_{i_k}(R)}$,

(iii) $v^{x_{i_1}(R)} - V_{i_K}(x_{i_1}(R); f_{i_K}(R)) < p^{x_{i_K}(R)} - t_{i_K}(R)$,

then there exists some $z' \in Z$ that dominates $f(R)$ for R.

Proof. Assume that there exists $\{i_k\}_{k=1}^K \subseteq$ with $K \geq 2$ which satisfies (i), (ii) and (iii). Let $N^- \equiv N \setminus \{i_k\}_{k=1}^K$. Let $z' \in Z$ be such that

(a) for each $k \in \{1, \ldots, K-1\}$, $z'_{i_k} \equiv \left(x_{i_{k+1}}(R), p^{x_{i_{k+1}}(R)}\right)$,

(b) $z'_{i_K} \equiv (x_{i_1}(R), V_{i_K}(x_{i_1}(R); f_{i_K}(R)))$,

(c) for each $j \in N^-$, $z'_j \equiv f_j(R)$.

Then, by (ii) and (a), for each $k \in \{1, \ldots, K-1\}$, $z'_{i_k} \, I_{i_k} \, f_{i_k}(R)$. Thus, by (b) and (c), for each $j \in N$, $z'_j \, I_j \, f_j(R)$. Also,

$$\sum_{j \in N} (t'_j - v^{x'_j})$$

$$= \sum_{k=1}^{K-1} \left(p^{x_{i_{k+1}}(R)} - v^{x_{i_{k+1}}(R)}\right) + \left(V_{i_K}(x_{i_1}(R); f_{i_K}(R)) - v^{x_{i_1}(R)}\right) + \sum_{j \in N^-} \left(t_j(R) - v^{x_j(R)}\right) \qquad \text{by (a, b, c)}$$

$$
\begin{aligned}
&> \sum_{k=1}^{K-1} \left(p^{x_{i_{k+1}}(R)} - v^{x_{i_{k+1}}(R)}\right) + \left(t_{i_K}(R) - p^{x_{i_K}(R)}\right) + \sum_{j \in N^-} \left(t_j(R) - v^{x_j(R)}\right) && \text{by (iii)} \\
&= \sum_{k=1}^{K-2} \left(p^{x_{i_{k+1}}(R)} - v^{x_{i_{k+1}}(R)}\right) + \left(t_{i_K}(R) - v^{x_{i_K}(R)}\right) + \sum_{j \in N^-} \left(t_j(R) - v^{x_j(R)}\right) \\
&= \left(p^{x_{i_1}(R)} - v^{x_{i_1}(R)}\right) + \sum_{k=2}^{K-1} \left(p^{x_{i_k}(R)} - v^{x_{i_k}(R)}\right) + \left(t_{i_K}(R) - v^{x_{i_K}(R)}\right) + \sum_{j \in N^-} \left(t_j(R) - v^{x_j(R)}\right) && \text{by (i)} \\
&= \sum_{k=1}^{K} \left(t_{i_k}(R) - v^{x_{i_k}(R)}\right) + \sum_{j \in N^-} \left(t_j(R) - v^{x_j(R)}\right) && \text{by (ii)} \\
&= \sum_{j \in N} \left(t_j(R) - v^{x_j(R)}\right).
\end{aligned}
$$

Hence, z' dominates $f(R)$ for R. □

Definition 10. Given $r \in \mathbb{R}^m_+$ and $p \in \mathbb{R}^m_{r+}$, $R_i \in \mathcal{R}$ is ***p*-indifferent (for *r*)** if

(i) $[\forall a \in M, p^a > 0] \Longrightarrow [\forall a, b \in L, (a, p^a)\, I_i\, (b, p^b)]$,

(ii) $[\exists a \in M, p^a = 0] \Longrightarrow [\forall a, b \in M, (a, p^a)\, I_i\, (b, p^b)]$.

As we discussed in Subsection 4.2, we have to consider two cases: the prices of all objects are strictly larger than zero; and there is an object whose price is zero. Given $r \in \mathbb{R}^m_+$ and $p \in \mathbb{R}^m_{r+}$, let $\mathcal{R}^I_r(p)$ be the set of all p-indifferent preferences. Note that $\mathcal{R}^I_r(p) \subsetneq \mathcal{R}^C$.

Figure 4 illustrates p-indifferent preferences. In this figure, there are two prices p and p' with $(p^a, p^b, p^c) > (0,0,0)$ and $p'^c = 0$. Also, R_i is a p-indifferent preference and R'_i is a p'-indifferent preference. By (i), the indifference curve of R_i in this figure goes thorough $(0,0)$, (a, p^a), (b, p^b) and (c, p^c). On the other hand, by (ii), the indifference curve of R'_i goes thorough (a, p'^a), (b, p'^b) and (c, p'^c), but not $(0,0)$. This is because if $(c, 0) = (c, p'^c)\, I'_i\, (0,0)$, then this relation violates object monotonicity.

Lemma 6 says that even if agents' preferences are replaced by p-indifferent preferences for any r, the minimum price with r is unchanged.

Lemma 6. Let $R \in \mathcal{R}^n$, $r \in \mathbb{R}^m_+$ and $p = p_{\min}(R, r)$. Let $N' \subseteq N$, $R'_{N'} \in \mathcal{R}^I_r(p)^{|N'|}$ and $R' \equiv (R'_{N'}, R_{-N'})$. Then, $p = p_{\min}(R', r)$.

Proof. It suffices to show (i) and (ii) of Fact 3. Let $M' \subseteq M$.

(i) We consider the following three cases.

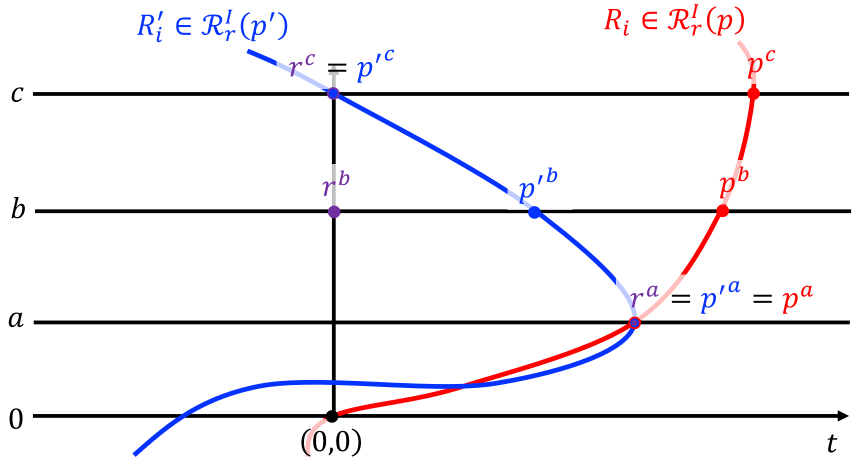

Figure 4: Illustration of p-indifferent preferences.

Case 1 $M' \subsetneq M$. By $R'_{N'} \in \mathcal{R}^I_r(p)^{|N'|}$, for each $i \in N'$, $M \subseteq D(R'_i, p)$. Thus, by $M' \subsetneq M$, for each $i \in N'$, $D(R'_i, p) \nsubseteq M'$. Since M' is not overdemanded at p for R,

$$|\{i \in N : D(R'_i, p) \subseteq M'\}| \leq |\{i \in N : D(R_i, p) \subseteq M'\}| \leq |M'|.$$

Case 2 $M' = M$ and $\forall a \in M$, $p^a > 0$. For each $i \in N'$, by $D(R'_i, p) = L$, $D(R'_i, p) \nsubseteq M$. Thus, since M is not overdemanded at p for R,

$$|\{i \in N : D(R'_i, p) \subseteq M\}| \leq |\{i \in N : D(R_i, p) \subseteq M\}| \leq |M|.$$

Case 3 $M' = M$ and $\exists a \in M$, $p^a = 0$. By object monotonicity, for each $i \in N$, $(a, p^a) = (a, 0)\, P_i\, \mathbf{0}$, so that $D(R_i, p) \subseteq M$. Since M is not overdemanded at p for R, $|N| = |\{i \in N : D(R_i, p) \subseteq M\}| \leq |M|$. Thus, we have

$$|\{i \in N : D(R'_i, p) \subseteq M\}| \leq |N| = |\{i \in N : D(R_i, p) \subseteq M\}| \leq |M|.$$

(ii) Assume that for each $a \in M'$, $p^a > r^a$. By $R'_{N'} \in \mathcal{R}^I_r(p)^{|N'|}$, for each $i \in N'$, $M \subseteq D(R'_i, p)$. Thus, for each $i \in N'$, $D(R'_i, p) \cap M' \neq \emptyset$. Since M' is not weakly underdemanded at p for R,

$$|\{i \in N : D(R'_i, p) \cap M' \neq \emptyset\}| \geq |\{i \in N : D(R_i, p) \cap M' \neq \emptyset\}| > |M'|.$$

Therefore, $p = p_{\min}(R', r)$. $\square$

Given $r \in \mathbb{R}^m_+$ and $p \in \mathbb{R}^m_{r+}$, an object $a \in M$ is **reserve-priced object** if $p^a = r^a$. Let $M^-_r(p) \equiv \{a \in M : p^a = r^a\}$ be the set of all reserve-priced objects at p, and let $L^-_r(p) \equiv M^-_r(p) \cup \{0\}$ and $M^+_r(p) \equiv M \setminus M^-_r(p)$.

A good has positive income effect if the object is more preferred against other objects as income increases, or equivalently as payments decreases.[21] If $(a, p^a)\, I_i\, (b, p^b)$ and for $\delta > 0$, $(a, p^a - \delta)\, P_i\, (b, p^b - \delta)$, then the preference exhibits positive income effect for a against b. The definition below applies this idea to p-indifferent preferences with respect to reserve-priced objects.

Definition 11. Given $r \in \mathbb{R}^m_+$ and $p \in \mathbb{R}^m_{r+}$, a p-indifferent preference $R_i \in \mathcal{R}^I_r(p)$ exhibits **positive income effect for reserve-priced objects** if

$$\text{(i) } [\forall a \in M, p^a > 0] \Longrightarrow \begin{bmatrix} \forall a \in L^-_r(p), \text{ and } \forall (b,t) \in M^+_r(p) \times \mathbb{R}_+ \text{ with } t < p^b, \\ r^a - V_i(a; (b,t)) < p^b - t \end{bmatrix},$$

$$\text{(ii) } [\exists a \in M, p^a = 0] \Longrightarrow \begin{bmatrix} \forall a \in M^-_r(p), \text{ and } \forall (b,t) \in M^+_r(p) \times \mathbb{R}_+ \text{ with } t < p^b, \\ r^a - V_i(a; (b,t)) < p^b - t \end{bmatrix}$$

To elucidate the condition (i) above, let $R_i \in \mathcal{R}^I_r(p)$, $a \in L^-_r(p)$, and $(b,t) \in M^+_r(p) \times \mathbb{R}_+$ with $t < p^b$. By $R_i \in \mathcal{R}^I_r(p)$, $(a, p^a)\, I_i\, (b, p^b)$. Let $\delta \equiv p^b - t$. Then, the inequality $r^a - V_i(a; (b,t)) < p^b - t$ implies $(a, p^a - \delta)\, P_i\, (b, p^b - \delta)$. Thus, as income increases by δ, a is preferred to b. The condition (ii) is similarly elucidated.

Given $r \in \mathbb{R}^m_+$ and $p \in \mathbb{R}^m_{r+}$, let $\mathcal{R}^{I+}_r(p)$ be the set of all p-indifferent preferences exhibiting positive income effects for reserve-priced objects. Note that $\mathcal{R}^{I+}_r(p) \subsetneq \mathcal{R}^I_r(p) \subsetneq \mathcal{R}^C$.

In Figure 5, Condition (i) is illustrated for price p. Note that $p^a > 0$, $p^b > 0$ and $p^c > 0$, and that $a, 0 \in L^-_r(p)$, $b, c \in M^+_r(p)$, and $t < p^b$. It holds that $r^a - V_i(a; (b,t)) < p^b - t$, $r^0 - V_i(0; (b,t)) < p^b - t$, and so $R_i \in \mathcal{R}^{I+}_r(p)$. Condition (ii) is similarly illustrated for price p'. Note that $p'^a > 0$, $p'^b > 0$ and $p'^c = 0$, and that $a, c \in M^-_r(p')$, $b \in M^+_r(p')$, and $t' < p'^b$. It holds that $r^a - V_i(a; (b,t')) < p'^b - t$, $r^c - V_i(c; (b,t')) < p'^b - t'$, and so $R'_i \in \mathcal{R}^{I+}_r(p')$.

Lemma 7 says that after replacing some agents' preferences with p-indifference preferences exhibiting positive income effect for reserve-priced objects, where reserve prices are equal to seller's benefits, if the payment of each agent whose preference is not replaced is larger than or equal to the price of the object he receives, then the payment of each agent whose preference is replaced is also larger than or equal to the price of the object he receives.

Lemma 7. Let f satisfy Pareto-efficiency, strategy-proofness, individual rationality and no-subsidy. Let $R \in \mathcal{R}^n$ and $p = p_{\min}(R, v)$. Let $N' \subseteq N$, $R'_{N'} \in \mathcal{R}^{I+}_v(p)^{|N'|}$ and $R' \equiv (R'_{N'}, R_{-N'})$. If for each $i \in N \setminus N'$, $t_i(R') \geq p^{x_i(R')}$, then for each $i \in N'$, $t_i(R') \geq p^{x_i(R')}$.

[21] Although income is not modeled explicitly, the zero payment corresponds to the endowed income. When an agent's income increases by $\delta > 0$, then his payment for each object decreases by δ.

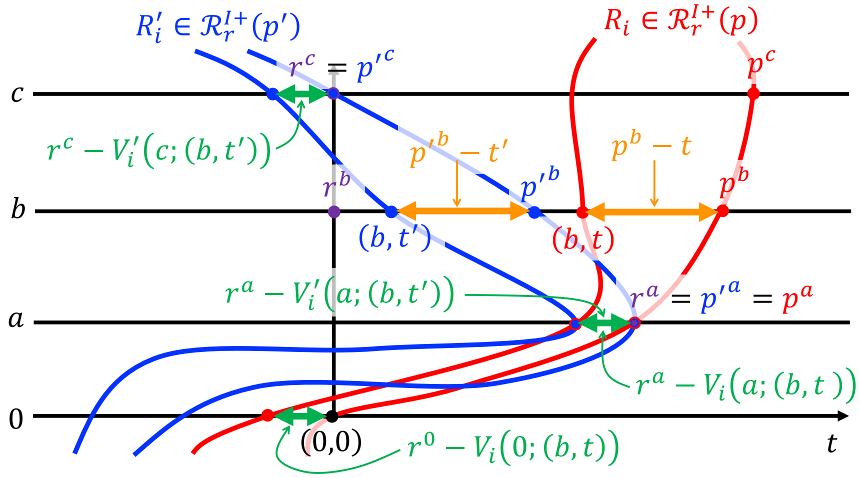

Figure 5: Illustration of p-indifferent preferences exhibiting positive income effect for reserve-priced objects.

Proof. Assume that for each $i \in N \setminus N'$, $t_i(R') \geq p^{x_i(R')}$. Note that by $R'_{N'} \in \mathcal{R}_v^{I+}(p)^{|N'|} \subseteq \mathcal{R}_v^I(p)^{|N'|}$ and Lemma 6, $p = p_{\min}(R', v)$.

Suppose to the contrary that for some $j \in N'$, $t_j(R') < p^{x_j(R')}$. By Proposition 2, $v^{x_j(R')} \leq t_j(R') < p^{x_j(R')}$. Thus, since for each $i \in N \setminus N'$, $t_i(R') \geq p^{x_i(R')}$, by $v^{x_j(R')} < p^{x_j(R')}$ and Lemma 4, there exists $\{i_k\}_{k=1}^{K'} \subseteq N$ with $K' \geq 2$ which satisfies the following conditions:

(a) $p^{x_{i_1}(R')} = v^{x_{i_1}(R')}$,

(b) for each $k \in \{1, \ldots, K'-1\}$, $\{x_{i_k}(R'), x_{i_{k+1}}(R')\} \subseteq D(R'_{i_k}, p)$ and $t_{i_k}(R') = p^{x_{i_k}(R')}$,

(c) $i_{K'} \in N'$.

We construct $\{i_k\}_{k=1}^{K}$ from $\{i_k\}_{k=1}^{K'}$ as follows: If $t_{i_{K'}}(R') < p^{x_{i_{K'}}(R')}$, let $K \equiv K'$ and $\{i_k\}_{k=1}^{K} \equiv \{i_k\}_{k=1}^{K'}$. If $t_{i_{K'}}(R') \geq p^{x_{i_{K'}}(R')}$, let $K \equiv K'+1$, $i_K = j$ and $\{i_k\}_{k=1}^{K} \equiv \{i_k\}_{k=1}^{K'} \cup \{i_K\}$. Note that if $t_{i_{K'}}(R') \geq p^{x_{i_{K'}}(R')}$, then by $t_j(R') < p^{x_{i_K}(R')}$, $i_{K'} \neq j = i_K$.

We show that $\{i_k\}_{k=1}^{K}$ satisfies the following conditions:

(i) $p^{x_{i_1}(R')} = v^{x_{i_1}(R')}$,

(ii) for each $k \in \{1, \ldots, K-1\}$, $\{x_{i_k}(R'), x_{i_{k+1}}(R')\} \subseteq D(R'_{i_k}, p)$ and $t_{i_k}(R') = p^{x_{i_k}(R')}$,

(iii) $i_K \in N'$ and $t_{i_K}(R') < p^{x_{i_K}(R')}$.

(i): It follows from (a).
(ii): If $t_{i_{K'}}(R') < p^{x_{i_{K'}}(R')}$, then (ii) follows from (b). Thus, let $t_{i_{K'}}(R') \geq p^{x_{i_{K'}}(R')}$. Note that by $v^{x_j(R')} < p^{x_j(R')}$, $x_{i_K}(R') = x_j(R') \in M$. Thus, by $i_{K'} = i_{K-1}$, Lemma 3 and $R'_{i_{K'}} \in \mathcal{R}^{I+}_v(p) \subseteq \mathcal{R}^I_v(p)$, it holds that $\{x_{i_{K-1}}(R'), x_{i_K}(R')\} \subseteq D(R'_{i_{K-1}}, p)$ and $t_{i_{K-1}}(R') = p^{x_{i_{K-1}}(R')}$. Thus, (ii) also holds.
(iii): If $t_{i_{K'}}(R') < p^{x_{i_{K'}}(R')}$, then (iii) follows from (c). Thus, let $t_{i_{K'}}(R') \geq p^{x_{i_{K'}}(R')}$. By $t_j(R') < p^{x_j(R')}$ and $i_K = j$, $t_{i_K}(R') < p^{x_{i_K}(R')}$. Thus, by $i_K = j \in N'$, (iii) holds.

Finally, in order to derive a contradiction, we apply Lemma 5 to $\{i_k\}_{k=1}^K$, which concludes that there is an allocation Pareto-dominating $f(R')$ for R'. Note that by (i) and (ii), to apply Lemma 5, we only need to show

$$v^{x_{i_1}(R')} - V'_{i_K}(x_{i_1}(R'); f_{i_K}(R')) < p^{x_{i_K}(R')} - t_{i_K}(R'). \tag{2}$$

Note that by (i), (iii) and Proposition 2, $x_{i_1}(R') \in L^-_v(p)$, $x_{i_K}(R') \in M^+_v(p)$ and $t_{i_K}(R') < p^{x_{i_K}(R')}$.

If for each $a \in M$, $p^a > 0$, then by $R'_{i_K} \in \mathcal{R}^{I+}_v(p)$, (2) holds obviously. Hence, we assume that for some $a \in M$, $p^a = 0$. Then,

$$f_{i_1}(R') \underset{\text{(ii)}}{=} \left(x_{i_1}(R'), p^{x_{i_1}(R')}\right) \underset{x_{i_1}(R') \in D(R'_{i_1}, p)}{R'_{i_1}} (a, p^a) = (a, 0) \underset{\text{object monotonicity}}{P'_{i_1}} (0, 0).$$

Thus, if $x_{i_1}(R') = 0$, then by money monotonicity, $t_{i_1}(R') < 0$, contradicting no-subsidy. Hence, $x_{i_1}(R') \neq 0$. By $x_{i_1}(R') \in L^-_v(p)$, this implies $x_{i_1}(R') \in M^-_v(p)$. Thus, by $x_{i_K}(R') \in M^+_v(p)$, $t_{i_K}(R') < p^{x_{i_K}(R')}$ and $R'_{i_K} \in \mathcal{R}^{I+}_v(p)$, (2) holds. □

5.7 Proof of Proposition 4

Proposition 4. Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Assume that f satisfies Pareto-efficiency, strategy-proofness and individual rationality and no-subsidy, and let $R \in \mathcal{R}^n$ and $p = p_{\min}(R, v)$. Then, for each $i \in N$, $t_i(R) \geq p^{x_i(R)}$.

Proof. Let $R' \in \mathcal{R}^{I+}_v(p)^n$. We prove the following claim by induction.

Claim 2. For each $S \subseteq N$ and each $i \in N$, $t_i(R_S, R'_{-S}) \geq p^{x_i(R_S, R'_{-S})}$, where $R'_{-S} = (R'_i)_{i \in N \setminus S}$.

Induction Base. Let $j \in N$ and $S = \{j\}$. By Lemma 7, it suffices to show that $t_j(R_j, R'_{-j}) \geq p^{x_j(R_j, R'_{-j})}$. Suppose to the contrary that $t_j(R_j, R'_{-j}) < p^{x_j(R_j, R'_{-j})}$.

If $x_j(R_j, R'_{-j}) = 0$, then $t_j(R_j, R'_{-j}) < p^{x_j(R_j, R'_{-j})} = 0$. However, this contradicts no-subsidy. Hence, $x_j(R_j, R'_{-j}) \neq 0$.

By Lemma 7, for each $i \in N$, $t_i(R') \geq p^{x_i(R')}$. Thus, by $p = p_{\min}(R', v)$ and Lemma 3, for each $i \in N$, $t_i(R') = p^{x_i(R')}$. In particular, $t_j(R') = p^{x_j(R')}$. Then,

$$f_j(R_j, R'_{-j}) \, P'_j \left(x_j(R_j, R'_{-j}), p^{x_j(R_j, R'_{-j})}\right) \qquad \text{by } t_j(R_j, R'_{-j}) < p^{x_j(R_j, R'_{-j})}$$

$$
\begin{aligned}
& R'_j \left(x_j(R'), p^{x_j(R')} \right) && \text{by } x_j(R_j, R'_{-j}) \neq 0 \text{ and def. 10} \\
& = f_j(R') && \text{by } t_j(R') = p^{x_j(R')}.
\end{aligned}
$$

Thus, $f_j(R_j, R'_{-j})\, P'_j\, f_j(R')$, but this contradicts strategy-proofness. Hence, $t_j(R_j, R'_{-j}) \geq p^{x_j(R_j, R'_{-j})}$.

Induction Hypothesis. Let $n' \leq n$. Assume that for each S' with $|S'| \leq n' - 1$ and each $i \in N$, $t_i(R_{S'}, R'_{-S'}) \geq p^{x_i(R_{S'}, R'_{-S'})}$.

Induction Argument. Let $S \subseteq N$ be such that $|S| = n'$. By Lemma 7, it suffices to show that for each $i \in S$, $t_i(R_S, R'_{-S}) \geq p^{x_i(R_S, R'_{-S})}$. Suppose to the contrary that for some $k \in S$, $t_k(R_S, R'_{-S}) < p^{x_k(R_S, R'_{-S})}$.

Note that $x_k(R_S, R'_{-S}) \neq 0$. Let $S' \equiv S \setminus \{k\}$. By $|S'| = n' - 1$ and the Induction Hypothesis, for each $i \in N$, $t_i(R_{S'}, R'_{-S'}) \geq p^{x_i(R_{S'}, R'_{-S'})}$. Then, by $p = p_{\min}(R_{S'}, R'_{-S'}, v)$ and Lemma 3, for each $i \in N$, $t_i(R_{S'}, R'_{-S'}) = p^{x_i(R_{S'}, R'_{-S'})}$. In particular, $t_k(R_{S'}, R'_{-S'}) = p^{x_k(R_{S'}, R'_{-S'})}$. Then, we have

$$
\begin{aligned}
f_k(R_S, R'_{-S})\, P'_k & \left(x_k(R_S, R'_{-S}), p^{x_k(R_S, R'_{-S})} \right) && \text{by } t_k(R_S, R'_{-S}) < p^{x_k(R_S, R'_{-S})} \\
& R'_k \left(x_k(R_{S'}, R'_{-S'}), p^{x_k(R_{S'}, R'_{-S'})} \right) && \text{by } x_k(R_S, R'_{-S}) \neq 0 \text{ and def. 10} \\
& = f_k(R_{S'}, R'_{-S'}) && \text{by } t_k(R_{S'}, R'_{-S'}) = p^{x_k(R_{S'}, R'_{-S'})},
\end{aligned}
$$

which implies $f_k(R_S, R'_{-S})\, P'_k\, f_k(R_{S'}, R'_{-S'})$, but this contradicts strategy-proofness. Hence, for each $i \in S$, $t_i(R_S, R'_{-S}) \geq p^{x_i(R_S, R'_{-S})}$. **(End of Claim 2)**

Let $S = N$, then for each $i \in N$, $t_i(R) \geq p^{x_i(R)}$. □

5.8 Proof of Theorem 1

Theorem 1. Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Then, a rule f on $\mathcal{R}^n$ satisfies efficiency, strategy-proofness, individual rationality and no-subsidy if and only if it is a minimum price Walrasian rule with $r = v$.

Proof. Let f satisfy efficiency, strategy-proofness, individual rationality and no-subsidy. By Fact 4, it suffices to show that f is a minimum price Walrasian rule. Let $R \in \mathcal{R}^n$ and $p = p_{\min}(R, v)$. First, we show that $(f(R), p)$ satisfies (WE-i). By Lemma 3 and Proposition 4, $x_i(R) \in D(R_i, p)$ and $t_i(R) = p^{x_i(R)}$. Hence, $(f(R), p)$ satisfies (WE-i).

Next, we show that $(f(R), p)$ satisfies (WE-ii). Suppose to the contrary that for some $a \in M \setminus \{x_i(R)\}_{i \in N}$, $p^a > v^a$.

By $p = p_{\min}(R, v)$, there exists some $z \in Z$ such that $(z, p) \in W_{\min}(R, v)$. For each $i \in N$, by $\{x_i, x_i(R)\} \subseteq D(R_i, p)$, $t_i = p^{x_i}$ and $t_i(R) = p^{x_i(R)}$, we have $z_i \, I_i \, f_i(R)$. Let $M^+ \equiv \{b \in M : p^b > v^b\}$. Then,

$$
\begin{aligned}
\sum_{i \in N}(t_i - v^{x_i}) &= \sum_{b \in M^+}(p^b - v^b) && \text{by } (z,p) \in W_{\min}(R,v) \\
&> \sum_{b \in M^+ \setminus \{a\}}(p^b - v^b) && \text{by } p^a - v^a > 0 \\
&\geq \sum_{i \in N}(t_i(R) - r^{x_i(R)}). && \text{by } M^+ \setminus \{a\} \supseteq M^+ \cap \{x_i(R)\}_{i \in N}
\end{aligned}
$$

Hence, z dominates $f(R)$ for R. However, this contradicts efficiency. Thus, $(f(R), p)$ satisfies (WE-ii).

Therefore, $(f(R), p) \in W(R, v)$, and by $p = p_{\min}(R, v)$, $f(R) \in Z_{\min}(R, v)$. Hence, f is a minimum price Walrasian rule with $r = v$. □

6 Related literatures

In the seminal work of Myerson (1981), the reserve price has an important role in characterizing optimal auctions. In a symmetric environment where there is a single object and preferences are quasi-linear, the Vickrey auction rule with a suitably set reserve price maximizes seller revenues. Since his article, a vast number of articles analyzes optimal auctions in environments of single object and quasi-linear preferences. However, there are several strands of literature analyzing auction rules of multiple objects in environments where preferences are non-quasi-linear. We discuss such strands of literature.

The first strand of the literature we discuss analyzes efficient auction rules of homogeneous goods for non-quasi-linear but unit-demand preferences. In the cases where objects are homogeneous, Saitoh and Serizawa (2008), and Sakai (2008) characterize the generalized Vickrey rule by efficiency, strategy-proofness, individual rationality and no-subsidy.

The second strand of the literature extends the setting of the first strand to heterogeneous objects. Morimoto and Serizawa (2015) show that in cases where the number of agents is greater than objects, the MPW rule is the unique rule satisfying efficiency, strategy-proofness, individual rationality and no-subsidy. Zhou and Serizawa (2018) maintain the assumption of unit-demand, but focus on the special class of preferences, "the common-tiered domains." It says that objects are partitioned into several tiers, and if objects are equally priced, agents prefer an object in a higher tier to one in a lower tier. They show that when the tier including nth highest objects is singleton, the MPW rule is the only rule satisfying the above four properties on the common-tiered domains.

The third strand of the literature analyze efficient auction rules with non-quasi-linear preferences admitting multi-demand. Kazumura and Serizawa (2016) study classes of preferences that include unit-demand preferences and additionally includes at least one multi-demand preference, and show that no rule satisfies the four properties on such a domain. Malik and Mishra (2021) study the special classes of preferences, "dichotomous" domains. A preference is *dichotomous* if there is a set of objects such that the valuations of its supersets are constant and the valuations of other sets are zero. A *dichotomous domain* includes all such dichotomous preferences for a given set of objects. They show that no rule satisfies the four properties on a dichotomous domain, but that the generalized Vickrey rule is the only rule satisfying the four properties on a class of dichotomous preferences exhibiting positive income effects.

This strand includes Baisa (2020). He assumes that objects are homogeneous, and preferences are non-quasi-linear and multi-demand, and shows that on the class of preferences exhibiting decreasing marginal valuations, positive income effect, and single-crossing property, if the preferences are parametrized by one dimensional types, there is a rule satisfying efficiency, strategy-proofness, individual rationality and no-subsidy, but that if types of preferences are multi-dimensional, no rule satisfies these properties.

The above three strands of literature on efficient auction rules in non-quasi-linear environments takes no account of the seller's benefits from objects to be auctioned, and excludes reserve prices. Our article is different from these strands of literature in this point.

The fourth strand of literature works on optimal auctions for non-quasi-linear preferences. On the unit-demand setting, Kazumura et al. (2020) and Sakai and Serizawa (2020) show that the MPW rule maximizes ex-post revenue among the class of auction rules satisfying strategy-proofness, individual rationality, no-subsidy, non-wastefulness and equal treatment of equals, and such a revenue maximizing rule is unique. These works also excludes reserve prices by non-wastefulness, which means that no agent prefers his own bundle to unassigned object with no payment.

The fifth strand works on efficient auction rules with reserve prices. Sakai (2013) studies strategy-proof auction in the single object setting on the quasi-linear domain, and assumes "non-imposition," which requires that the payment of the agent with zero valuation be zero. He shows that the allocation rule satisfies weak efficiency, strategy-proofness and non-imposition if and only if it is the Vickrey auction rule with a reserve price or no-trade rule, where no-trade rule is the rule such that for each preference profile, each agent gets no object and pays nothing.

Andersson and Svensson (2014) study a housing allocation model where preferences are non-quasi-linear and unit demand, and rents are bounded not only below by reserve prices but also above by price ceilings by governments. They introduce "rationing price equilibrium (RPE)," which a hybrid of Walrasian equilibrium and a rationing mechanism with fixed prices for a given priority structure. A RPE is not Pareto-efficient, but constrained efficient for a given priority structure. They show that the minimum RPE price uniquely exists, and that the minimum RPE mechanism is group strategy-proof. Our result is different from this strand in that we derive reserve prices from the seller's benefits from objects.

7 Conclusion

We extended the result of Morimoto and Serizawa (2015) to the settings where there is an arbitrary number of agents and objects, and the seller may benefit from objects to be auctioned, and showed 1) the minimum price Walrasian rule with reserve prices set equal to the benefits the seller enjoys from objects is a unique rule satisfying efficiency, strategy-proofness, individual rationality and no-subsidy, and 2) it is also a unique rule satisfying efficiency, strategy-proofness, and two-sided individual rationality. Our result demonstrates that the minimum price Walrasian rule has distinguished theoretical merits and applicabilities in a variety of environments.

Appendix

A The counterexample of Lemma 12*

We give Lemma 12*.

Lemma 12*. Let f satisfy efficiency, strategy-proofness, individual rationality and no-subsidy. Let $R \in \mathcal{R}^n$ and $(z^*, p) \in W_{\min}(R, v)$. Let $N' \subseteq N$ with $1 \leq |N'|$, $R'_{N'} \in \mathcal{R}^I(z^*)^{|N'|}$ and $R' \equiv (R'_{N'}, R_{-N'})$. Assume that

(12-i) $\forall i \in N \setminus N'$, $x_i(R') \neq 0 \Longrightarrow t_i(R') \geq p^{x_i(R')}$, and

(12-ii) $\forall j \in N'$, $x_j(R') \neq 0$.

Then, $\exists \{i_k\}_{k=1}^K \subseteq N$ such that

(i) $p^{x_{i_1}(R')} = v^{x_{i_1}(R')}$,

(ii) $\forall k \in \{2, \ldots, K\}$, $x_{i_k}(R') \neq 0$,

(iii) $\forall k \in \{1, \ldots, K-1\}$, $i_k \in N \setminus N'$ and $i_K \in N'$,

(iv) $\forall k \in \{1, \ldots, K-1\}$, $\{x_{i_k}(R'), x_{i_{k+1}}(R')\} \subseteq D(R_{i_k}, p)$.

We give the following counterexample of Lemma 12*.

Example 7. Let $N = \{1, 2\}$, $M = \{a, b\}$, $v^a = 0$ and $v^b > 0$. Let $R = (R_1, R_2)$ be such that $(a, v^a)\, P_1\, (b, v^b)\, P_1\, \mathbf{0}$ and $(a, v^a)\, I_2\, (b, v^b)\, P_2\, \mathbf{0}$. Then, $(z, p) = (((a, p^a), (b, p^b)), (v^a, v^b))$ is the only minimum price Walrasian equilibrium for R. Let $N' = \{2\}$, $R'_2 = R_2$ and $R' = (R_1, R'_2)$. Note that $R'_2 \in \mathcal{R}^I_v(p)$.

By $p = v$ and Proposition 2, $t_1(R') \geq v^{x_1(R')} = p^{x_1(R')}$. Hence, (12-i) holds. If $x_2(R') = 0$, by individual rationality and no-subsidy, $f_2(R') = \mathbf{0}$. This implies $z_2 = (b, v^b)\, P'_2\, \mathbf{0} = f_2(R')$, contradicting Proposition 3. Hence, $x_2(R') \neq 0$, and so (12-ii) holds.

However, there is no sequence $\{i_1, i_2\}$ satisfying (i)-(iv). In fact, by (iii), $i_1 = 1$ but $D(R_{i_1}, p) = \{a\}$ violates (iv).

B Preliminary results for Fact 3

Fact 7 (Hall, 1935)**.** For each $i \in N$, let $D_i \subseteq M$. Then, there exists $x' \in X$ such that for each $i \in N$, $x'_i \in D_i$ if and only if for each $N' \subseteq N$, $|N'| \leq \left|\bigcup_{i \in N'} D_i\right|$.

Proof. **"Only if."** Assume that there exists $x' \in X$ such that for each $i \in N$, $x'_i \in D_i$. Let $N' \subseteq N$. We show that $|N'| \leq |\bigcup_{i \in N'} D_i|$. Since for each $i, j \in N'$ with $i \neq j$, $x'_i \neq x'_j$, we have $|N'| = |\{x'_i\}_{i \in N'}|$. Also, since for each $i \in N'$, $x'_i \in D_i$, we get $\{x'_i\}_{i \in N'} \subseteq \bigcup_{i \in N'} D_i$, and so $|\{x'_i\}_{i \in N'}| \leq |\bigcup_{i \in N'} D_i|$. Combining these two equations, we obtain $|N'| \leq |\bigcup_{i \in N'} D_i|$.

"If." Assume that for each $N' \subseteq N$, $|N'| \leq \left|\bigcup_{i \in N'} D_i\right|$. We prove by induction for the number of agents. Hence, we can arbitrarily pick the set of objects. This set of objects is denoted by $M' \equiv \{1, \ldots, m'\}$ and m' is not always equal to m. Then, for each $i \in N$, let $D'_i \subseteq M'$. In the following Induction Base and Induction Hypothesis step, we consider M' instead of M. Futhermore, for each $S \subseteq N$, let $X_S \equiv \{x_S \in L^{|S|} \mid \forall i, j \in S, x_i = x_j \Rightarrow x_i = x_j = 0\}$.

Induction Base. Pick $i \in N$. Assume that $|\{i\}| \leq |D'_i|$. Then, since $|\{i\}| = 1$, there exists some $x'_i \in D'_i$.

Induction Hypothesis. Assume that for each $S' \subsetneq N$, if for each $S'' \subseteq S'$, $|S''| \leq |\bigcup_{i \in S''} D'_i|$, then there exists $x'_{S'} \in X_{S'}$ such that for each $i \in S'$, $x'_i \in D'_i$.

Induction Argument. Let $S \subseteq N$. Assume that for each $S' \subseteq S$, $|S'| \leq \left|\bigcup_{i \in S'} D_i\right|$. We consider the following two cases.

Case 1 $\forall \boldsymbol{S'} \subsetneq \boldsymbol{S}$, $|\boldsymbol{S'}| + \mathbf{1} \leq |\bigcup_{\boldsymbol{i \in S'}} \boldsymbol{D_i}|$. We pick an arbitrary agent $i \in S$. Without loss of generality, let $1 \equiv i$. By $|\{1\}| \leq |D_1|$, there exists some $a \in D_1$. Let $x'_1 \equiv a$, $S' \equiv S \setminus \{1\}$ and $M' \equiv M \setminus \{x'_1\}$. For each $i \in S'$, let $D'_i \equiv D_i \setminus \{x'_1\}$. Then, for each $i \in S'$, $D'_i \subseteq M'$. Let $S'' \subseteq S'$. We show that $|S''| \leq |\bigcup_{i \in S''} D'_i|$ in order to use the Induction Hypothesis. By the assumption of Case 1 and $S'' \subseteq S' \subsetneq S \subseteq N$,

$$|S''| + 1 \leq \left|\bigcup_{i \in S''} D_i\right|. \tag{3}$$

Moreover, since for each $i \in S''$, $D'_i \equiv D_i \setminus \{x'_1\}$, we have $|\bigcup_{i \in S''} D_i| = |\bigcup_{i \in S''} D'_i|$ or $|\bigcup_{i \in S''} D_i| = |\bigcup_{i \in S''} D'_i| + 1$, which implies that

$$\left|\bigcup_{i \in S''} D_i\right| \leq \left|\bigcup_{i \in S''} D'_i\right| + 1. \tag{4}$$

Thus, by (3) and (4), we get $|S''| \leq |\bigcup_{i \in S''} D'_i|$. Hence, by $S' \subsetneq S \subseteq N$ and the Induction Hypothesis, there exists some $x'_{S'} \in X_{S'}$ such that for each $i \in S' \equiv S \setminus \{1\}$, $x'_i \in D'_i \equiv D_i \setminus \{x'_1\}$. Let $x'_S \equiv (x'_1, x'_{S'})$, then $x'_S \in X_S$ and for each $i \in S$, $x'_i \in D_i$.

Case 2 $\boldsymbol{\exists S' \subsetneq S, |S'| = |\bigcup_{i \in S'} D_i|}$**.** Let $S' \subsetneq N$ be such that $|S'| = |\bigcup_{i \in S'} D_i|$. Since $S' \subsetneq S$ and for each $S'' \subseteq S'$, $|S''| \leq |\bigcup_{i \in S''} D_i|$, by the Induction Hypothesis, there exists some $x'_{S'} \in X_{S'}$ such that for each $i \in S'$, $x'_i \in D_i$. Let $T \equiv S \setminus S'$ and $M' \equiv M \setminus \{x'_i\}_{i \in S'}$. Moreover, for each $i \in S$, let $D'_i \equiv D_i \setminus \{x'_i\}_{i \in S'}$. Then, for each $i \in T$, $D'_i \subseteq M'$.

Let $T' \subseteq T$. We show that $|T'| \leq |\bigcup_{i \in T'} D'_i|$. By $|S'| = |\bigcup_{i \in S'} D_i|$,

$$\bigcup_{i \in S'} D_i = \{x'_i\}_{i \in S'}.$$

Then, we have

$$\begin{aligned} \bigcup_{i \in T' \cup S'} D_i &= \left(\bigcup_{i \in T'} D_i\right) \cup \left(\bigcup_{i \in S'} D_i\right) \\ &= \left(\bigcup_{i \in T'} D'_i\right) \cup \{x'_i\}_{i \in S'}. \end{aligned}$$

Thus, by $\left\{\bigcup_{i \in T'} D'_i\right\} \cap \{x'_i\}_{i \in S'} = \emptyset$ and $|\{x'_i\}_{i \in S'}| = |S'|$, we obtain

$$\left|\bigcup_{i \in T' \cup S'} D_i\right| = \left|\bigcup_{i \in T'} D'_i\right| + |S'|. \tag{5}$$

Furthermore, by $T' \cap S' = \emptyset$ and $T' \cup S' \subseteq S$,

$$|T'| + |S'| = |T' \cup S'| \leq \left|\bigcup_{i \in T' \cup S'} D_i\right| \tag{6}$$

Hence, by (5) and (6), we have $|T'| \leq |\bigcup_{i \in T'} D'_i|$. Then, by $T \subsetneq S \subseteq N$ and the Induction Hypothesis, there exists some $x'_T \in X_T$ such that for each $i \in T \equiv S \setminus S'$, $x'_i \in D'_i \equiv D_i \setminus \{x'_i\}_{i \in S'}$. Let $x'_S \equiv (x'_{S'}, x'_T)$, then $x'_S \in X_S$ and for each $i \in S$, $x'_i \in D_i$.

Let $S = N$, then we complete the proof. $\square$

Let $N' \subseteq N$ and $M' \subseteq M$. Sometimes, we consider an economy $E' \equiv (N', M')$. Then, the any notation A in the original economy $E \equiv (N, M)$ is replaced by $A^{E'}$ or $A_{E'}$ in the economy $E' \equiv (N', M')$. N' or M' is also used instead of E'. For example, given $R_{N'} \in \mathcal{R}^{|N'|}$, the set of Walrasian equilibria in the economy $E' \equiv (N', M')$ is denoted by $W^{E'}(R_{N'}, r^{M'})$.

Fact 8 (Mishra and Talman, 2010)**.** Let $n, m \in \mathbb{N}$ and $\mathcal{R} = \mathcal{R}^E$. Then, for each $R \in \mathcal{R}^n$, each $r \in \mathbb{R}^m_+$ and each $p \in \mathbb{R}^m_{r+}$, $p \in P(R, r)$ if and only if no set is overdemanded at p for R and no set is underdemanded at p for R, that is, for each $M' \subseteq M$,

(i) $|\{i \in N : D(R_i, p) \subseteq M'\}| \leq |M'|$,

(ii) $[\forall a \in M', p^a > r^a] \Longrightarrow |\{i \in N : D(R_i, p) \cap M' \neq \emptyset\}| \geq |M'|$.

Proof. Let $R \in \mathcal{R}^n$, $r \in \mathbb{R}^m_+$ and $p \in \mathbb{R}^m_{r+}$.

"Only If." Assume that $p \in P(R, r)$. Then, there exists some $z \in Z$ such that $(z, p) \in W(R, r)$. We show that (i) and (ii) hold. Let $M' \subseteq M$.

(i) Let $N' \equiv \{i \in N : D(R_i, p) \subseteq M'\}$. Since for each $i \in N'$, $x_i \in D(R_i, p) \subseteq M'$, we have $\{x_i\}_{i \in N'} \subseteq M'$ and so

$$|\{x_i\}_{i \in N'}| \leq |M'|. \tag{7}$$

Since for each $i, j \in N'$ with $i \neq j$, $x_i \neq x_j$, we get

$$|N'| = |\{x_i\}_{i \in N'}|. \tag{8}$$

By (7) and (8), we obtain $|N'| \leq |M'|$, that is, $|\{i \in N : D(R_i, p) \subseteq M'\}| \leq |M'|$.

(ii) Assume that for each $a \in M'$, $p^a > r^a$. Let $N' \equiv \{i \in N : D(R_i, p) \cap M' \neq \emptyset\}$. Suppose to the contrary that $|N'| < |M'|$. Then, there exists some $b \in M'$ such that $b \notin \{x_i\}_{i \in N'}$. Moreover, since for each $i \in N \setminus N'$, $D(R_i, p) \cap M' = \emptyset$, $b \notin \{x_i\}_{i \in N \setminus N'}$. Hence, by $b \notin \{x_i\}_{i \in N'}$ and $b \notin \{x_i\}_{i \in N \setminus N'}$, $b \in M \setminus \{x_i\}_{i \in N}$. Then, by (WE-ii), $p^b = r^b$, but this contradicts $p^b > r^b$. Thus, we obtain $|N'| \geq |M'|$, that is, $|\{i \in N : D(R_i, p) \cap M' \neq \emptyset\}| \geq |M'|$.

"If." Assume that for each $M' \subseteq M$, (i) and (ii) hold. We show that there exists some $z \in Z$ such that $(z, p) \in W(R, r)$, which implies that $p \in P(R, r)$.

Let $N' \equiv \{i \in N : D(R_i, p) \subseteq M\}$ and $E' \equiv (N', M)$. Assume that there exists $z'_{N'} \in Z_{N'}$ be such that $(z'_{N'}, p) \in W^{E'}(R_{N'}, r)$. Let $z \in Z$ be such that for each $i \in N'$, $z_i \equiv z'_i$ and for each $i \in N \setminus N'$, $z_i \equiv \mathbf{0}$. Then, since for each $i \in N \setminus N'$, $0 \in D(R_i, p)$ and $(z'_{N'}, p) \in W^{E'}(R_{N'}, r)$, $(z, p) \in W(R, r)$. Hence, we only consider the case that for each $i \in N$, $D(R_i, p) \subseteq M$.

Let $Z^* \equiv \{z \in Z : \forall i \in N, x_i \in D(R_i, p) \text{ and } t_i = p^{x_i}\}$. We show that there exists some $z \in Z^*$ such that $(z, p) \in W(R, r)$ in the following two steps.

Step 1 $\boldsymbol{Z^* \neq \emptyset}$. Let $N' \subseteq N$ and $M' \equiv \bigcup_{i \in N'} D(R_i, p)$. We show that $|N'| \leq |M'|$ to use Fact 7. Since for each $i \in N'$, $D(R_i, p) \subseteq M'$, we have $N' \subseteq \{i \in N : D(R_i, p) \subseteq M'\}$, and so

$$|N'| \leq |\{i \in N : D(R_i, p) \subseteq M'\}|. \tag{9}$$

Since M' is not overdemanded at p for R,

$$|\{i \in N : D(R_i, p) \subseteq M'\}| \leq |M'|. \tag{10}$$

By (9) and (10), we get $|N'| \leq |\bigcup_{i \in N'} D(R_i, p)|$. Then, by Fact 7, there exists some $x' \in X$ such that for each $i \in N$, $x'_i \in D(R_i, p)$. For each $i \in N$, let $t'_i \equiv p^{x'_i}$, then $z' \in Z^*$. Thus, $Z^* \neq \emptyset$. **(End of Step 1)**

Step 2 There exists some $\boldsymbol{z \in Z^*}$ such that $\boldsymbol{(z, p) \in W(R, r)}$. Since for each $z' \in Z^*$, (z', p) satisfies (WE-i), we only show that there exists $z \in Z^*$ such that (z, p) satisfies (WE-ii).

Let $M^+ \equiv \{a \in M : p^a > r^a\}$ and let

$$z \in \underset{z' \in Z^*}{\arg\max} |\{a \in M^+ : \exists i \in N, x'_i = a\}|.$$

Suppose to the contrary that for some $b \in M \setminus \{x_i\}_{i \in N}$, $p^b > r^b$.

First, we construct $\{j_k\}_{k=1}^{K'} \subseteq N$ as follows. Let $x_{j_0} \equiv b$. Since $\{x_{j_0}\}$ is not underdemanded at p for R, there exists $j_1 \in N$ such that $D(R_{j_1}, p) \cap \{x_{j_0}\} \neq \emptyset$. If $p^{x_{j_1}} = r^{x_{j_1}}$, then we stop this process and get a sequence $\{j_1\}$. If $p^{x_{j_1}} > r^{x_{j_1}}$, since $\{x_{j_0}, x_{j_1}\}$ is not underdemanded at p for R, we can pick $j_2 \in N \setminus \{j_1\}$ such that $D(R_{j_2}, p) \cap \{x_{j_0}, x_{j_1}\} \neq \emptyset$. We repeat this process and stop if $p^{x_{j_t}} = r^{x_{j_t}}$ in some step t. Note that by $N \supseteq \{i \in N : D(R_i, p) \cap M^+ \neq \emptyset\}$ and $|\{i \in N : D(R_i, p) \cap M^+ \neq \emptyset\}| \geq |M^+|$, we have $|N| \geq |M^+|$. Then, by $b \in M^+$, $b \notin \{x_i\}_{i \in N}$ and $|N| \geq |M^+|$, there exists some $j \in N$ such that $p^{x_j} = r^{x_j}$. Hence, the above process finishes in the finite time and we can get $\{j_k\}_{k=1}^{K'} \subseteq N$ such that for each $k \in \{1, \ldots, K'\}$, $D(R_{j_k}, p) \cap \{x_{j_0}, \ldots, x_{j_{k-1}}\}$ and $p^{x_{j_{K'}}} = r^{x_{j_{K'}}}$.

Next, we construct a sequence $\{i_k\}_{k=1}^{K} \subseteq N$ as follows. Let $i_1 \equiv j_{K'}$. If $x_{j_0} \in D(R_{i_1}, p)$, then we stop this process and get a sequence $\{i_1\}$. If $x_{j_0} \notin D(R_{i_1}, p)$, by $D(R_{i_1}, p) \cap \{x_{j_0}, x_{j_1}, \ldots, x_{j_{K'-1}}\} \neq \emptyset$, we can pick $i_2 \in \{j_1, \ldots, j_{K'-1}\}$ such that $x_{i_2} \in D(R_{i_1}, p)$. We repeat this process and stop if $x_{j_0} \in D(R_{i_t}, p)$ in some step t. By $x_{j_0} \in D(R_{j_1}, p)$, this process finishes in the finite time and we can get $\{i_k\}_{k=1}^{K} \subseteq N$ such that $p^{x_{i_1}} = r^{x_{i_1}}$ and for each $k \in \{1, \ldots, K\}$, $\{x_{i_k}, x_{i_{k+1}}\} \subseteq D(R_{i_k}, p)$, where $x_{i_{K+1}} \equiv x_{j_0}$.

Finally, we derive a contradiction. Let $z' \in Z$ be such that for each $k \in \{1, \ldots, K\}$, $z'_{i_k} \equiv (x_{i_{k+1}}, p^{x_{i_{k+1}}})$ and for each $i \in N \setminus \{i_k\}_{k=1}^{K}$, $z'_i \equiv z_i$. Then, since for each $i \in N$, $x'_i \in D(R_i, p)$ and $t'_i = p^{x'_i}$, we have $z' \in Z^*$. Also, since $p^{x_{i_1}} = r^{x_{i_1}}$ and for each $k \in$

$\{2, \ldots, K+1\}$, $p^{x_{i_k}} > r^{x_{i_k}}$, $|\{a \in M^+ : \exists i \in N, x'_i = a\}| = |\{a \in M^+ : \exists i \in N, x_i = a\}|+1$. However, this equation contradicts $z \in \arg\max_{z'' \in Z^*} |\{a \in M^+ : \exists i \in N, x''_i = a\}|$. Hence, for each $b \in M \setminus \{x_i\}_{i \in N}$, $p^b = r^b$. Therefore, $(z, p) \in W(R, r)$, which implies $p \in P(R, r)$. □

C Proof of Fact 3

Given $i \in N$, $R_i \in \mathcal{R}$ and $d_i \in \mathbb{R}$, the ***d_i*-truncation of *R_i*** is the preference R'_i such that for each $z_i \in M \times \mathbb{R}$, $V'_i(0; z_i) = V_i(0; z_i) + d_i$. Note that $R'_i \in \mathcal{R}^E$ and for each $z_i, z'_i \in M \times \mathbb{R}$, $z_i \, R'_i \, z'_i$ if and only if $z_i \, R_i \, z'_i$.[22]

Fact 3 (Morimoto and Serizawa, 2015)**.** Let $n, m \in \mathbb{N}$ and $\mathcal{R} = \mathcal{R}^E$. Then, for each $R \in \mathcal{R}^n$, each $r \in \mathbb{R}^m_+$ and each $p \in \mathbb{R}^m_{r+}$, $p = p_{\min}(R, r)$ if and only if no set is overdemanded at p for R and no set is weakly underdemanded at p for R, that is, for each $M' \subseteq M$,

(i) $|\{i \in N : D(R_i, p) \subseteq M'\}| \leq |M'|$,

(ii) $[\forall a \in M', p^a > r^a] \Longrightarrow |\{i \in N : D(R_i, p) \cap M' \neq \emptyset\}| > |M'|$.

Proof. Let $R \in \mathcal{R}^n$, $r \in \mathbb{R}^m_+$ and $p \in \mathbb{R}^m_{r+}$.

"If." Assume that for each $M' \subseteq M$, (i) and (ii) hold. Since no set is weakly underdemanded at p for R, no set is underdemanded at p for R. Thus, by Fact 8, $p \in P(R, r)$. We show that $p = p_{\min}(R, r)$. Suppose to the contrary that there exists some $q \in P(R, r)$ such that $q \leq p$ and $q \neq p$.

Let $M' \equiv \{a \in M : q^a < p^a\}$ and $N' \equiv \{i \in N : D(R_i, p) \cap M' \neq \emptyset\}$. First, we show that $|M'| < |N'|$. Note that $M' \neq \emptyset$. For each $a \in M'$, by $r^a \leq q^a$, $r^a < p^a$. Thus, since M' is not weakly underdemanded at p for R, $|M'| < |N'|$.

Next, we show that $N' \subseteq \{i \in N : D(R_i, q) \subseteq M'\}$. Let $i \in N'$ and $a \in D(R_i, q)$. By $D(R_i, p) \cap M' \neq \emptyset$, there exists some $b \in D(R_i, p) \cap M'$. Then,

$$(a, q^a) \underset{a \in D(R_i, q)}{R_i} (b, q^b) \underset{q^b < p^b}{P_i} (b, p^b) \underset{b \in D(R_i, p)}{R_i} (a, p^a).$$

Thus, $q^a < p^a$, which implies $a \in M'$. Hence, we obtain $D(R_i, q) \subseteq M'$, and so $N' \subseteq \{i \in N : D(R_i, q) \subseteq M'\}$.

Finally, we derive a contradiction. Note that by $q \in P(R, r)$ and Fact 8, M' is not overdemanded at q for R. By $N' \subseteq \{i \in N : D(R_i, q) \subseteq M'\}$, $|N'| \leq |\{i \in N : D(R_i, q) \subseteq M'\}|$. Then, by $|M'| < |N'|$, $|M'| < |\{i \in N : D(R_i, q) \subseteq M'\}|$. However, this inequality

[22] In fact, for each $z_i, z'_i \in M \times \mathbb{R}$

$$z_i \, R'_i \, z'_i \Longleftrightarrow V'_i(0; z_i) \geq V'_i(0; z'_i) \Longleftrightarrow V_i(0; z_i) + d_i \geq V_i(0; z'_i) + d_i \Longleftrightarrow V_i(0; z_i) \geq V_i(0; z'_i) \Longleftrightarrow z_i \, R_i \, z'_i.$$

contradicts that M' is not overdemanded at q for R. Hence, $p = p_{\min}(R, r)$.

"Only if." Assume that $p = p_{\min}(R, r)$. By $p \in P(R, r)$ and Fact 8, no set is overdemanded at p for R and no set is underdemanded at p for R. Hence, we only show that no set is weakly underdemanded at p for R. Suppose to the contrary that there exists some $M' \subseteq M$ such that for each $a \in M'$, $p^a > r^a$ and $|\{i \in N : D(R_i, p) \cap M' \neq \emptyset\}| \leq |M'|$. Without loss of generality, we assume that M' is minimal, that is, for each $M'' \subsetneq M'$, it is not weakly underdemanded at p for R. Let $N' \equiv \{i \in N : D(R_i, p) \cap M' \neq \emptyset\}$. Since M' is not underdemanded ($|N'| \geq |M'|$) but weakly underdemanded ($|N'| \leq |M'|$) at p for R, we have $|N'| = |M'|$. Without loss of generality, let $M' \equiv \{1, \ldots, m'\}$ and $N' \equiv \{1, \ldots, m'\}$. By $p = p_{\min}(R, r)$, there exists some $z \in Z$ such that $(z, p) \in W_{\min}(R, r)$.

Step 1 $\forall i \in N'$, $x_i \in M'$. Suppose to the contrary that for some $i \in N'$, $x_i \notin M'$. Then, by $|N'| = |M'|$, there exists some $a \in M'$ such that $a \notin \{x_i\}_{i \in N'}$. Moreover, since for each $i \in N \setminus N'$, $D(R_i, p) \cap M' = \emptyset$ and $a \in M'$, $a \notin \{x_i\}_{i \in N \setminus N'}$. Hence, $a \in M \setminus \{x_i\}_{i \in N}$ and, by $a \in M'$, $p^a > r^a$, but this contradicts (WE-ii). Hence, for each $i \in N'$, $x_i \in M'$. **(End of Step 1)**

For each $a \in M'$, let $u^a \equiv \max(\{V_j(a; z_j) : j \in N \setminus N'\} \cup \{r^a\})$.

Step 2 $\forall a \in M'$, $u^a < p^a$. Suppose to the contrary that for some $a \in M'$, $u^a \geq p^a$. Then, by $p^a > r^a$, $u^a > r^a$. Hence, there exists some $j \in N \setminus N'$ such that $V_j(a; z_j) = u^a$. Thus, by $p^a \leq u^a$, $p^a \leq V_j(a; z_j)$, which implies that $(a, p^a) \, R_j \, z_j$. Then, by $x_j \in D(R_j, p)$, $z_j \, R_j \, (a, p^a) \, R_j \, z_j$ and so $z_j \, I_j \, (a, p^a)$. Hence, we get $a \in D(R_j, p)$. However, this contradicts $j \in N \setminus N'$, that is, $D(R_j, p) \cap M' = \emptyset$. Therefore, for each $a \in M'$, $u^a < p^a$. **(End of Step 2)**

By Step 2, we can let $R_0 \in \mathcal{R}^E$ be such that for each $a \in M'$, $V_0(a; \mathbf{0}) \in (u^a, p^a)$. Note that for each $a \in M'$, by $r^a \leq u^a$, $r^a < V_0(a; \mathbf{0}) < p^a$. We consider the economy $E' \equiv (N'', M')$, where $N'' \equiv N' \cup \{0\}$. Let $z_0 \equiv \mathbf{0}$ and $z_{N''} \equiv (z_0, z_{N'})$.

Step 3 $(z_{N''}, p^{M'}) \in W^{E'}_{\min}(R_{N''}, r^{M'})$.

Step 3.1 $(z_{N''}, p^{M'}) \in W^{E'}(R_{N''}, r^{M'})$. First, we consider an agent $i \in N'$. By Step 1, $x_i \in M'$. Moreover, by $x_i \in D(R_i, p)$, for each $a \in M' \subseteq M$, $(x_i, p^{x_i}) \, R_i \, (a, p^a)$. Hence, $x_i \in D(R_i, p^{M'})$. Next, we consider the agent 0. For each $a \in M'$, by $V_0(a; \mathbf{0}) < p^a$, $\mathbf{0} \, P_0 \, (a, p^a)$. Hence, $D(R_0, p^{M'}) = \{0\}$, and so $x_0 \in D(R_0, p^{M'})$. Thus, since for each $i \in N''$, $t_i = p^{x_i}$, $(z_{N''}, p^{M'})$ satisfies (WE-i). Since for each $i \in N'$, $x_i \in M'$ (Step 1) and $|N'| = |M'|$, $M' \setminus \{x_i\}_{i \in N'} = \emptyset$. Hence, $(z_{N''}, p^{M'})$ satisfies (WE-ii). Therefore, $(z_{N''}, p^{M'}) \in W^{E'}(R_{N''}, r^{M'})$. **(End of Step 3.1)**

Let $(\tilde{z}_{N''}, \tilde{p}^{M'}) \in W^{E'}_{\min}(R_{N''}, r^{M'})$. Let $M^- \equiv \{a \in M' : \tilde{p}^a < p^a\}$ and $N^- \equiv \{i \in N' : D(R_i, p^{M'}) \cap M^- \neq \emptyset\}$. We show that $M^- = \emptyset$. Suppose to the contrary that $M^- \neq \emptyset$.

Step 3.2 $\forall i \in N^-$, $\tilde{x}_i \in M^-$. Let $i \in N^-$. By $D(R_i, p^{M'}) \cap M^- \neq \emptyset$, there exists some $a \in D(R_i, p^{M'})$ with $\tilde{p}^a < p^a$. Then,

$$(\tilde{x}_i, \tilde{p}^{\tilde{x}_i}) \underset{\tilde{x}_i \in D(R_i, \tilde{p}^{M'})}{R_i} (a, \tilde{p}^a) \underset{\tilde{p}^a < p^a}{P_i} (a, p^a) \underset{a \in D(R_i, p^{M'})}{R_i} (\tilde{x}_i, p^{\tilde{x}_i}),$$

which implies that $\tilde{p}^{\tilde{x}_i} < p^{\tilde{x}_i}$, and so $\tilde{x}_i \in M^-$. Hence, for each $i \in N^-$, $\tilde{x}_i \in M^-$. **(End of Step 3.2)**

Step 3.3 $N^- = N'$ and $M^- = M'$. Note that $N^- \equiv \{i \in N' : D(R_i, p^{M'}) \cap M^- \neq \emptyset\}$. First, we show that $N^- = \{i \in N : D(R_i, p) \cap M^- \neq \emptyset\}$. For each $i \in N \setminus N'$, by $D(R_i, p) \cap M' = \emptyset$ and $M^- \subseteq M'$, $D(R_i, p) \cap M^- = \emptyset$. Thus, $\{i \in N : D(R_i, p) \cap M^- \neq \emptyset\} = \{i \in N' : D(R_i, p) \cap M^- \neq \emptyset\}$. Hence, it suffices to show that $N^- = \{i \in N' : D(R_i, p) \cap M^- \neq \emptyset\}$.

Let $j \in N^-$. By $D(R_j, p^{M'}) \cap M^- \neq \emptyset$, there exists some $a \in D(R_j, p^{M'}) \cap M^-$. Also, by $j \in N^- \subseteq N'$, $D(R_j, p) \cap M' \neq \emptyset$. Thus, there exists some $b \in D(R_j, p) \cap M'$. Then,

$$(a, p^a) \underset{a \in D(R_j, p^{M'}),\, b \in M'}{R_j} (b, p^b) \underset{b \in D(R_j, p)}{R_j} (a, p^a),$$

which implies that $(a, p^a)\, I_j\, (b, p^b)$. Thus, $a \in D(R_j, p)$. By $a \in M^-$, $a \in D(R_j, p) \cap M^-$. Hence, $j \in \{i \in N' : D(R_i, p) \cap M^- \neq \emptyset\}$, and so $N^- \subseteq \{i \in N' : D(R_i, p) \cap M^- \neq \emptyset\}$.

Let $j \in \{i \in N' : D(R_i, p) \cap M^- \neq \emptyset\}$. By $D(R_j, p) \cap M^- \neq \emptyset$, there exists some $a \in D(R_j, p) \cap M^-$. By Step 3.1, $x_j \in D(R_j, p^{M'})$. By $M^- \subseteq M'$, $a \in M'$ Then,

$$(a, p^a) \underset{a \in D(R_j, p)}{R_j} (x_j, p^{x_j}) \underset{x_j \in D(R_j, p^{M'}),\, a \in M'}{R_j} (a, p^a),$$

which implies that $(a, p^a)\, I_j\, (x_j, p^{x_j})$. Thus, $a \in D(R_j, p^{M'})$. By $a \in M^-$, $a \in D(R_j, p^{M'}) \cap M^-$. Hence, $j \in N^-$, and so $\{i \in N' : D(R_i, p) \cap M^- \neq \emptyset\} \subseteq N^-$. Therefore, $N^- = \{i \in N : D(R_i, p) \cap M^- \neq \emptyset\}$.

Next, we show that $M^- = M'$. Note that by Step 3.2, $|N^-| \leq |M^-|$. Suppose to the contrary that $M^- \subsetneq M'$. Since M' is minimal, M^- is not weakly underdemanded at p for R, that is, $|\{i \in N : D(R_i, p) \cap M^- \neq \emptyset\}| > |M^-|$. Then, by $N^- = \{i \in N : D(R_i, p) \cap M^- \neq \emptyset\}$, we have $|N^-| > |M^-|$, but this contradicts to $|N^-| \leq |M^-|$. Hence, $M^- = M'$. Also, by $N^- = \{i \in N : D(R_i, p) \cap M^- \neq \emptyset\}$ and $M^- = M'$, we obtain $N^- = N'$. **(End of Step 3.3)**

Step 3.4 $\forall a \in M'$, $u^a < \tilde{p}^a$. Suppose to the contrary that there exists some $a \in M'$ such that $\tilde{p}^a \leq u^a$. Note that, by Step 3.2, Step 3.3 and $|N'| = |M'|$, $\tilde{x}_0 = 0$. By

$\tilde{p}^a \leq u^a < V_0(a;\mathbf{0})$, $(a,\tilde{p}^a)\, P_0\, \mathbf{0} = \tilde{z}_0$. However, this contradicts $\tilde{x}_0 \in D(R_0, \tilde{p}^{M'})$. Hence, for each $a \in M'$, $u^a < \tilde{p}^a$. **(End of Step 3.4)**

Let $(\bar{z},\bar{p}) \in Z \times \mathbb{R}^m_{r+}$ be such that $\bar{z}_{N'} \equiv \tilde{z}_{N'}$, $\bar{z}_{-N'} \equiv z_{-N'}$, $\bar{p}^{M'} \equiv \tilde{p}^{M'}$ and $\bar{p}^{-M'} \equiv p^{-M'}$. Note that for each $i \in N$, $\bar{t}_i = \bar{p}^{\bar{x}_i}$ since for each $i \in N'$, $\tilde{x}_i \in M'$ and for each $i \in N \setminus N'$, $x_i \in M \setminus M'$ (by $D(R_i,p) \cap M' = \emptyset$).

Step 3.5 $(\bar{z},\bar{p}) \in W(R,r)$. First, we show that $(\bar{z},\bar{p})$ satisfies (WE-i). Let $i \in N'$. Note that $\bar{x}_i = \tilde{x}_i$ and $\bar{t}_i = \tilde{p}^{\tilde{x}_i}$, and by $x_i \in M' = M^-$ (Step 1 and Step 3.3), $\tilde{p}^{x_i} < p^{x_i}$. If $a \in M'$,

$$(\bar{x}_i, \bar{p}^{\bar{x}_i}) = (\tilde{x}_i, \tilde{p}^{\tilde{x}_i}) \underset{\tilde{x}_i \in D(R_i,\tilde{p}^{M'})}{R_i} (a, \tilde{p}^a) \underset{a \in M'}{=} (a, \bar{p}^a).$$

If $a \notin M'$,

$$(\bar{x}_i, \bar{p}^{\bar{x}_i}) = (\tilde{x}_i, \tilde{p}^{\tilde{x}_i}) \underset{\tilde{x}_i \in D(R_i,\tilde{p}^{M'})}{R_i} (x_i, \tilde{p}^{x_i}) \underset{\tilde{p}^{x_i} < p^{x_i}}{P_i} (x_i, p^{x_i}) \underset{x_i \in D(R_i,p)}{R_i} (a, p^a) \underset{a \notin M'}{=} (a, \bar{p}^a).$$

Hence, for each $a \in M$, $(\bar{x}_i, \bar{p}^{\bar{x}_i})\, R_i\, (a, \bar{p}^a)$ and so $\bar{x}_i \in D(R_i, \bar{p})$.

Let $i \in N \setminus N'$. Note that for each $a \in M'$, by $u^a \equiv \max\{\{V_j(a;z_j) : j \in N \setminus N'\} \cup \{r^a\}\}$ and Step 3.4, $V_i(a;z_i) \leq u^a < \tilde{p}^a$. If $a \in M'$,

$$(\bar{x}_i, \bar{p}^{\bar{x}_i}) = (x_i, p^{x_i}) \underset{V_i(a;z_i) < \tilde{p}^a}{P_i} (a, \tilde{p}^a) \underset{a \in M'}{=} (a, \bar{p}^a).$$

If $a \notin M'$,

$$(\bar{x}_i, \bar{p}^{\bar{x}_i}) = (x_i, p^{x_i}) \underset{x_i \in D(R_i,p)}{R_i} (a, p^a) \underset{a \notin M'}{=} (a, \bar{p}^a).$$

Hence, for each $a \in M$, $(\bar{x}_i, \bar{p}^{\bar{x}_i})\, R_i\, (a, \bar{p}^a)$, and so $\bar{x}_i \in D(R_i, \bar{p})$. Therefore, $(\bar{z},\bar{p})$ satisfies (WE-i).

Next, we show that $(\bar{z},\bar{p})$ satisfies (WE-ii). Suppose to the contrary that for some $a \in M \setminus \{\bar{x}_i\}_{i \in N}$, $\bar{p}^a > r^a$. If $a \in M'$, by $\{\tilde{x}_i\}_{i \in N'} = \{\bar{x}_i\}_{i \in N'} \subseteq \{\bar{x}_i\}_{i \in N}$, $\tilde{x}_0 = 0$ and $\bar{p}^a = \tilde{p}^a$, $a \in M \setminus \{\tilde{x}_i\}_{i \in N''}$ and $\tilde{p}^a > r^a$. However, this contradicts $(\tilde{z}_{N''}, \tilde{p}^{M'}) \in W^{E'}_{\min}(R_{N''}, r^{M'})$. If $a \notin M'$, by $\{x_i\}_{i \in N \setminus N'} = \{\bar{x}_i\}_{i \in N \setminus N'}$, $\{x_i\}_{i \in N'} = M'$ (Step 1) and $\bar{p}^a = p^a$, $a \in M \setminus \{x_i\}_{i \in N}$ and $p^a > r^a$. However, this contradicts $(z,p) \in W_{\min}(R,r)$. Thus, for each $a \in M \setminus \{\bar{x}_i\}_{i \in N}$, $\bar{p}^a = r^a$. Hence, $(\bar{z},\bar{p}) \in W(R,r)$. **(End of Step 3.5)**

By Step 3.5 and $M^- \neq \emptyset$, $\bar{p} \in P(R,r)$ and for some $a \in M^- = M'$, $\bar{p}^a = \tilde{p}^a < p^a$. However, this contradicts $p = p_{\min}(R,r)$. Hence, $(z_{N''}, p^{M'}) \in W^{E'}_{\min}(R_{N''}, r^{M'})$. **(End of Step 3)**

Let $\pi = \{i(k)\}_{k=1}^{m'}$ be a permutation of N' and let Π be the set of permutations. Given, $\pi = \{i(k)\}_{k=1}^{m'} \in \Pi$, let $Q(\pi)$ be the set of $q^{M'} \in \mathbb{R}^{m'}_{r+}$ with $q^0 = 0$ such that for each $k \in \{1,\ldots,m'\}$, $q^{x_{i(k)}} \leq V_{i(k-1)}(x_{i(k)}; (x_{i(k-1)}, q^{x_{i(k-1)}}))$, where $i(0) \equiv 0$ and so

$x_{i(0)} = 0$. In particular, given $\pi = \{i(k)\}_{k=1}^{m'} \in \Pi$, let $q^{M'}(\pi) \in Q(\pi)$ be an price vector such that the above inequality holds with equality, that is, for each $k \in \{1, \dots, m'\}$, $q^{x_{i(k)}}(\pi) = V_{i(k-1)}(x_{i(k)}; (x_{i(k-1)}, q^{x_{i(k-1)}}(\pi)))$. Let $Q \equiv \bigcup_{\pi \in \Pi} Q(\pi)$.

Step 4 There exists some $\boldsymbol{b < p^{x_1}}$ such that $\boldsymbol{\forall q^{M'} \in Q}$, $\boldsymbol{q^{x_1} < b}$.

Let $\pi = \{i(k)\}_{k=1}^{m'} \in \Pi$ and $\hat{q}^{M'} = q^{M'}(\pi)$.

Step 4.1 $\boldsymbol{\forall q^{M'} \in Q(\pi)}$ and $\boldsymbol{\forall k \in \{1, \dots, m'\}}$, $\boldsymbol{q^{x_{i(k)}} \leq \hat{q}^{x_{i(k)}}}$. Let $q^{M'} \in Q(\pi)$. By $x_{i(0)} = 0$, $q^{x_{i(0)}} = \hat{q}^{x_{i(0)}} = 0$. Then,

$$q^{x_{i(1)}} \underset{\text{by def. of } q^{M'}}{\leq} V_{i(0)}(x_{i(1)}; (x_{i(0)}, q^{x_{i(0)}})) \underset{q^{x_{i(0)}} = \hat{q}^{x_{i(0)}}}{=} V_{i(0)}(x_{i(1)}; (x_{i(0)}, \hat{q}^{x_{i(0)}})) \underset{\text{by def. of } \hat{q}^{M'}}{=} \hat{q}^{x_{i(1)}}.$$

Let $s \in \{1, \dots, m'-1\}$. Assume that $q^{x_{i(s)}} \leq \hat{q}^{x_{i(s)}}$. Then,

$$q^{x_{i(s+1)}} \underset{\text{by def. of } q^{M'}}{\leq} V_{i(s)}(x_{i(s+1)}; (x_{i(s)}, q^{x_{i(s)}})) \underset{q^{x_{i(s)}} \leq \hat{q}^{x_{i(s)}}}{\leq} V_{i(s)}(x_{i(s+1)}; (x_{i(s)}, \hat{q}^{x_{i(s)}})) \underset{\text{by def. of } \hat{q}^{M'}}{=} \hat{q}^{x_{i(s+1)}}.$$

Thus, for each $k \in \{1, \dots, m'\}$, $q^{x_{i(k)}} \leq \hat{q}^{x_{i(k)}}$. **(End of Step 4.1)**

Step 4.2 $\boldsymbol{\forall k \in \{1, \dots, m'\}}$, $\boldsymbol{\hat{q}^{x_{i(k)}} < p^{x_{i(k)}}}$. Note that by $x_{i(1)} \in M'$ (Step 1), $V_0(x_{i(1)}; \mathbf{0}) < p^{x_{i(k)}}$. Then,

$$\hat{q}^{x_{i(1)}} \underset{\text{by def. of } \hat{q}^{M'}}{=} V_{i(0)}(x_{i(1)}; (x_{i(0)}, \hat{q}^{x_{i(0)}})) \underset{i(0)=0}{=} V_0(x_{i(1)}; \mathbf{0}) < p^{x_{i(1)}}.$$

Let $s \in \{1, \dots, m'-1\}$. Assume that $\hat{q}^{x_{i(s)}} < p^{x_{i(s)}}$. Note that by $\hat{q}^{x_{i(s+1)}} = V_{i(s)}(x_{i(s+1)}; (x_{i(s)}, \hat{q}^{x_{i(s)}}))$, $(x_{i(s+1)}, \hat{q}^{x_{i(s+1)}}) \, I_{i(s)} \, (x_{i(s)}, \hat{q}^{x_{i(s)}})$. Then,

$$(x_{i(s+1)}, \hat{q}^{x_{i(s+1)}}) \, I_{i(s)} \, (x_{i(s)}, \hat{q}^{x_{i(s)}}) \underset{\hat{q}^{x_{i(s)}} < p^{x_{i(s)}}}{P_{i(s)}} (x_{i(s)}, p^{x_{i(s)}}) \underset{x_{i(s)} \in D(R_{i(s)}, p)}{R_{i(s)}} (x_{i(s+1)}, p^{x_{i(s+1)}}),$$

which implies $\hat{q}^{x_{i(s+1)}} < p^{x_{i(s+1)}}$. Thus, for each $k \in \{1, \dots, m'\}$, $\hat{q}^{x_{i(k)}} < p^{x_{i(k)}}$. **(End of Step 4.2)**

By Step 4.2, for each $\pi \in \Pi$, $q^{x_1}(\pi) < p^{x_1}$. Since the number of permutations is finite $(m'!)$, $\max_{\pi \in \Pi} q^{x_1}(\pi)$ exists, and $\max_{\pi \in \Pi} q^{x_1}(\pi) < p^{x_1}$ holds. Note that by Step 4.1, for each $q^{M'} \in Q$, $q^{x_1} \leq \max_{\pi \in \Pi} q^{x_1}(\pi)$. Let $b \in \mathbb{R}$ be such that $\max_{\pi \in \Pi} q^{x_1}(\pi) < b < p^{x_1}$. Then, for each $q^{M'} \in Q$, $q^{x_1} < b < p^{x_1}$. **(End of Step 4)**

Let $R'_1 \in \mathcal{R}^E$, be a d_1-truncation of R_1 such that $b < V'_1(x_1; \mathbf{0}) < p^{x_1}$ and let $(\hat{z}_{N''}, \hat{p}^{M'}) \in W^{E'}_{\min}(R'_1, R_{N'' \setminus \{1\}}, r^{M'})$.

Step 5 $\boldsymbol{\hat{x}_1 \neq 0}$. Suppose to the contrary that $\hat{x}_1 = 0$. Then, we show that the following claim by induction and the last condition derives a contradiction.

Claim 3. There is some $\{i(k)\}_{k=1}^{m'} \in \Pi$ such that for each $k \in \{1, \ldots, m'\}$, (a) $\hat{p}^{\hat{x}_{i(k-1)}} < p^{\hat{x}_{i(k-1)}}$, (b) $\hat{x}_{i(k-1)} = x_{i(k)}$ and (c) $i(k) \neq 1$.

Induction Base. We find $i(1) \in N'$ satisfying (a) to (c). First, we show that $\hat{x}_{i(0)} \neq 0$. Suppose to the contrary that $\hat{x}_{i(0)} = 0$. Then, by $\hat{x}_1 = 0$ and (WE-ii), there is some $a \in M' \setminus \{\hat{x}_i\}_{i \in N''}$ with $\hat{p}^a = r^a$. By $r^a < V_{i(0)}(a; \mathbf{0})$, $(a, \hat{p}^a)\, P_{i(0)}\, \mathbf{0} = \hat{z}_{i(0)}$. However, this contradicts $\hat{x}_{i(0)} \in D(R_{i(0)}, \hat{p}^{M'})$. Hence, $\hat{x}_{i(0)} \neq 0$.

(a) Note that by $\hat{x}_{i(0)} \in M'$, $V_{i(0)}(\hat{x}_{i(0)}; \mathbf{0}) < p^{\hat{x}_{i(0)}}$. Then,

$$(\hat{x}_{i(0)}, \hat{p}^{\hat{x}_{i(0)}}) \underset{\hat{x}_{i(0)} \in D(R_{i(0)}, \hat{p}^{M'})}{R_{i(0)}} \mathbf{0} \underset{V_{i(0)}(\hat{x}_{i(0)}; \mathbf{0}) < p^{\hat{x}_{i(0)}}}{P_{i(0)}} (\hat{x}_{i(0)}, p^{\hat{x}_{i(0)}}),$$

which implies that $\hat{p}^{\hat{x}_{i(0)}} < p^{\hat{x}_{i(0)}}$.

(b) Note that by Step 1 and $|N'| = |M'|$, $M' = \{x_i\}_{i \in N'}$. By $\hat{x}_{i(0)} \in M' = \{x_i\}_{i \in N'}$, there is some $i(1) \in N'$ such that $\hat{x}_{i(0)} = x_{i(1)}$.

(c) Suppose to the contrary that $i(1) = 1$. Let $\pi = \{j(k)\}_{k=1}^{m'} \in \Pi$ and $q^{M'} \in \mathbb{R}_{r+}^{m'}$ be such that $j(1) = i(1)$ and $q^{x_{j(1)}} = \hat{p}^{x_{j(1)}}$, and for each $k \in \{2, \ldots, m'\}$, $j(k) \equiv \min N' \setminus \{j(s)\}_{s=1}^{k-1}$ and $q^{x_{j(k)}} = V_{j(k-1)}(x_{j(k)}; (x_{j(k-1)}, q^{x_{j(k-1)}}))$, where $j(0) = 0$. Then, by $\hat{x}_{j(0)} = x_{j(1)} \in D(R_{j(0)}, \hat{p}^{M'})$ and $x_{j(0)} = 0$, $q^{x_{j(1)}} = \hat{p}^{x_{j(1)}} \leq V_{j(0)}(x_{j(1)}; (x_{j(0)}, \hat{p}^{x_{j(0)}}))$, and so $q^{M'} \in Q(\pi) \subseteq Q$. Thus, by Step 4, $q^{x_1} < b$. Then, by $\hat{p}^{x_1} = q^{x_1} < b < V'_1(x_1; \mathbf{0})$, $(x_1, \hat{p}^{x_1})\, P'_1\, \mathbf{0} = \hat{z}_1$. However, this contradicts $\hat{x}_1 \in D(R'_1, \hat{p}^{M'})$. Hence, $i(1) \neq 1$.

Induction Hypothesis. Let $s \in \{1, \ldots, m'-1\}$. Assume that there is $\{i(k)\}_{k=1}^{s}$ such that for each $s' \in \{1, \ldots, s\}$, (a-s) $\hat{p}^{\hat{x}_{i(s'-1)}} < p^{\hat{x}_{i(s'-1)}}$, (b-s) $\hat{x}_{i(s'-1)} = x_{i(s')}$ and (c-s) $i(s') \neq 1$.

Induction Argument. (a) By (a-s), it suffices to show that $\hat{p}^{\hat{x}_{i(s)}} < p^{\hat{x}_{i(s)}}$. By $\hat{x}_{i(s-1)} = x_{i(s)}$ and $\hat{p}^{\hat{x}_{i(s-1)}} < p^{\hat{x}_{i(s-1)}}$, we have $\hat{p}^{x_{i(s)}} < p^{x_{i(s)}}$. Then,

$$(\hat{x}_{i(s)}, \hat{p}^{\hat{x}_{i(s)}}) \underset{\hat{x}_{i(s)} \in D(R_{i(s)}, \hat{p}^{M'})}{R_{i(s)}} (x_{i(s)}, \hat{p}^{x_{i(s)}}) \underset{\hat{p}^{x_{i(s)}} < p^{x_{i(s)}}}{P_{i(s)}} (x_{i(s)}, p^{x_{i(s)}}) \underset{x_{i(s)} \in D(R_{i(s)}, p)}{R_{i(s)}} (\hat{x}_{i(s)}, p^{\hat{x}_{i(s)}}),$$

which implies $\hat{p}^{\hat{x}_{i(s)}} < p^{\hat{x}_{i(s)}}$.

(b) By $\hat{p}^{\hat{x}_{i(s)}} < p^{\hat{x}_{i(s)}}$ and $\hat{p}^0 = p^0 = 0$, $\hat{x}_{i(s)} \neq 0$. Hence, by $\{x_i\}_{i \in N'} = M'$, there is some $i(s+1) \in N' \setminus \{i(s')\}_{s'=1}^{s}$ such that $\hat{x}_{i(s)} = x_{i(s+1)}$.

(c) Suppose to the contrary that $i(s+1) = 1$. Let $\pi = \{j(s')\}_{s'=1}^{m'} \in \Pi$ and $q^{M'} \in \mathbb{R}_{r+}^{m'}$ be such that for each $s' \in \{1, \ldots, s+1\}$ $j(s') = i(s')$ and $q^{x_{j(s')}} = \hat{p}^{x_{j(s')}}$, and for each $s' \in \{s+2, \ldots, m'\}$, $j(s') \equiv \min N' \setminus \{j(s'')\}_{s''=1}^{s'-1}$ and $q^{x_{j(s')}} = V_{j(s'-1)}(x_{j(s')}; (x_{j(s'-1)}, q^{x_{j(s'-1)}}))$. Then, for each $s' \in \{1, \ldots, s+1\}$, by $\hat{x}_{j(s'-1)} = x_{j(s')} \in D(R_{j(s'-1)}, \hat{p}^{M'})$, $q^{x_{j(s')}} = \hat{p}^{x_{j(s')}} \leq V_{j(s'-1)}(x_{j(s')}; (x_{j(s'-1)}, \hat{p}^{x_{j(s'-1)}}))$, and so $q^{M'} \in Q(\pi) \subseteq Q$. Thus, by Step 4, $q^{x_1} < b$. By $\hat{p}^{x_1} = q^{x_1} < b < V'_1(x_1; \mathbf{0})$, $(x_1, \hat{p}^{x_1})\, P'_1\, \mathbf{0} = \hat{z}_1$. However, this contradicts $\hat{x}_1 \in D(R'_1, \hat{p}^{M'})$. Hence, $i(s+1) \neq 1$. **(End of Claim 3)**

By Claim 3, we get $\{i(k)\}_{k=1}^{m'} \in \Pi$ such that for each $k \in \{1, \dots, m'\}$, $\hat{p}^{\hat{x}_{i(k-1)}} < p^{\hat{x}_{i(k-1)}}$, $\hat{x}_{i(k-1)} = x_{i(k)}$ and $i(k) \neq 1$. Thus, we have $1 \notin \{i(k)\}_{k=1}^{m'}$. However, this contradicts $\{i(k)\}_{k=1}^{m'} = N'$. Hence, we obtain $\hat{x}_1 \neq 0$. **(End of Step 5)**

Step 6 $(\hat{z}_{N''}, \hat{p}^{M'}) \in W^{E'}(R_{N''}, r^{M'})$. By $(\hat{z}_{N''}, \hat{p}^{M'}) \in W^{E'}(R'_1, R_{N''\setminus\{1\}}, r^{M'})$, for each $i \in N'' \setminus \{1\}$, $\hat{x}_i \in D(R_i, \hat{p}^{M'})$ and $\hat{t}_i = \hat{p}^{\hat{x}_i}$. Since $\hat{t}_1 = \hat{p}^{\hat{x}_1}$ and (WE-ii) does not depend on preferences, it suffices to show that $\hat{x}_1 \in D(R_1, \hat{p}^{M'})$.

First, we show that $d_1 > 0$. By $(x_1, p^{x_1})\, R_1\, \mathbf{0}$, $V_1(0; (x_1, p^{x_1})) \leq 0$. Also, by $V'_1(x_1; \mathbf{0}) < p^{x_1}$, $\mathbf{0}\, P'_1\, (x_1, p^{x_1})$, and so $0 < V'_1(0; (x_1, p^{x_1}))$. Hence, we get $V_1(0; (x_1, p^{x_1})) < V'_1(0; (x_1, p^{x_1}))$. Then, by $V'_1(0; (x_1, p^{x_1})) = V_1(0; (x_1, p^{x_1})) + d_1$, we obtain $d_1 > 0$.

Next, we show that $\hat{x}_1 \in D(R_1, \hat{p}^{M'})$. By $\hat{x}_1 \neq 0$ (Step 5), for each $a \in M'$, $(\hat{x}_1, \hat{p}^{\hat{x}_1})\, R'_1\, (a, \hat{p}^a)$ if and only if $(\hat{x}_1, \hat{p}^{\hat{x}_1})\, R_1\, (a, \hat{p}^a)$. Moreover, by $\hat{x}_1 \in D(R'_1, \hat{p}^{M'})$ and $d_1 > 0$, $V'_1(0; (\hat{x}_1, \hat{p}^{\hat{x}_1})) \leq 0 < d_1$. Then, by $V'_1(0; (\hat{x}_1, \hat{p}^{\hat{x}_1})) = V_1(0; (\hat{x}_1, \hat{p}^{\hat{x}_1})) + d_1$, $V_1(0; (\hat{x}_1, \hat{p}^{\hat{x}_1})) < 0$, and so $(\hat{x}_1, \hat{p}^{\hat{x}_1})\, P_1\, \mathbf{0}$. Hence, $\hat{x}_1 \in D(R_1, \hat{p}^{M'})$. Therefore, $(\hat{z}_{N''}, \hat{p}^{M'}) \in W^{E'}(R_{N''}, r^{M'})$. **(End of Step 6)**

By $p^{M'} = p^{E'}_{\min}(R_{N''}, r^{M'})$ (Step 3) and $\hat{p}^{M'} \in P^{E'}(R_{N''}, r^{M'})$ (Step 6), $p^{M'} \leq \hat{p}^{M'}$. In particular, we have $p^{\hat{x}_1} \leq \hat{p}^{\hat{x}_1}$. Also, by $\hat{x}_1 \in D(R'_1, \hat{p}^{M'})$ and $V'_1(x_1; \mathbf{0}) < p^{x_1}$, we get $(\hat{x}_1, \hat{p}^{\hat{x}_1})\, R'_1\, \mathbf{0}\, P'_1\, (x_1, p^{x_1})$. Thus, by $\hat{x}_1 \neq 0$, $x_1 \neq 0$ and $(\hat{x}_1, \hat{p}^{\hat{x}_1})\, P'_1\, (x_1, p^{x_1})$, $(\hat{x}_1, \hat{p}^{\hat{x}_1})\, P_1\, (x_1, p^{x_1})$. Then, by $x_1 \in D(R_1, p^{M'})$, $(\hat{x}_1, \hat{p}^{\hat{x}_1})\, P_1\, (x_1, p^{x_1})\, R_1\, (\hat{x}_1, p^{\hat{x}_1})$, which implies $\hat{p}^{\hat{x}_1} < p^{\hat{x}_1}$. However, this inequality contradicts $p^{\hat{x}_1} \leq \hat{p}^{\hat{x}_1}$. Therefore, M' is not weakly underdemanded at p for R. □

D Proof of Fact 4

Fact 4. Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Then, for each $r \in \mathbb{R}^m_+$, The minimum price Walrasian rule with r on $\mathcal{R}^n$ satisfies (i) strategy-proofness (Demange and Gale, 1985), (ii) individual rationality, and (iii) no-subsidy.

Proof. Let $r \in \mathbb{R}^m_+$ and f be the minimum price Walrasian rule with r. Also, let $R \in \mathcal{R}$ and $i \in N$.

(i) Suppose to the contrary that there is some $R'_i \in \mathcal{R}$ such that $f_i(R'_i, R_{-i})\, P_i\, f_i(R)$. Let $z \equiv f(R)$ and $z' \equiv f(R'_i, R_{-i})$. Then, by $z \in Z_{\min}(R, r)$ and $z' \in Z_{\min}(R'_i, R_{-i}, r)$, there exist $p, p' \in \mathbb{R}^m_{r+}$ such that $(z, p) \in W_{\min}(R, r)$ and $(z', p') \in W_{\min}(R'_i, R_{-i}, r)$.

First, we show that $p'^{x'_i} < p^{x'_i}$. Note that $z'_i = f_i(R'_i, R_{-i})\, P_i\, f_i(R) = z_i$. Then,

$$(x'_i, p'^{x'_i}) = z'_i\, P_i\, z_i = (x_i, p^{x_i}) \underset{x_i \in D(R_i, p)}{R_i} (x'_i, p^{x'_i}).$$

Hence, we get $p'^{x'_i} < p^{x'_i}$.

Next, we derive a contradiction. By $r^{x'_i} \leq p'^{x'_i} < p^{x'_i}$ and Fact 3 (ii), $|\{i \in N : D(R_i, p) \cap \{x'_i\} \neq \emptyset\}| > |\{x_i\}|$. Hence, there is some $j_1 \in N \setminus \{i\}$ such that $D(R_{j_1}, p) \cap \{x'_i\} \neq \emptyset$.

Then,

$$(x'_{j_1}, p'^{x'_{j_1}}) \underset{x'_{j_1} \in D(R_{j_1}, p')}{R_{j_1}} (x'_i, p'^{x'_i}) \underset{p'^{x'_i} < p^{x'_i}}{P_{j_1}} (x'_i, p^{x'_i}) \underset{x'_i \in D(R_{j_1}, p)}{R_{j_1}} (x'_{j_1}, p^{x'_{j_1}}).$$

Hence, we get $p'^{x'_{j_1}} < p^{x'_{j_1}}$. Then, by $r^{x'_{j_1}} \leq p'^{x'_{j_1}} < p^{x'_{j_1}}$ and Fact 3 (ii), $|\{j \in N : D(R_j) \cap \{x_i, x'_{j_1}\} \neq \emptyset\}| > |\{x_i, x'_{j_1}\}|$. Thus, there is some $j_2 \in N \setminus \{i, j_1\}$ such that $D(R_{j_2}, p) \cap \{x'_i, x'_{j_1}\} \neq \emptyset$. Let $a \in \{x'_i, x'_{j_1}\}$ be such that $a \in D(R_{j_2}, p)$. Then,

$$(x'_{j_2}, p'^{x'_{j_2}}) \underset{x'_{j_2} \in D(R_{j_2}, p')}{R_{j_2}} (a, p'^a) \underset{p'^a < p^a}{P_{j_2}} (a, p^a) \underset{a \in D(R_{j_2}, p)}{R_{j_2}} (x'_{j_2}, p^{x'_{j_2}}).$$

Hence, we get $p'^{x'_{j_2}} < p^{x'_{j_2}}$. Repeating this argument, we can get $\{j_k\}_{k=1}^K \subseteq N$ such that $\{j_k\}_{k=1}^K = N \setminus \{i\}$ and for each $k \in \{1, \ldots, K\}$, $p'^{x'_{j_k}} < p^{x'_{j_k}}$. Then, for each $i \in N$, $r^{x'_i} < p^{x'_i}$, which implies that $|N| \leq |\{b \in M : p^b > r^b\}|$. However, this inequality contradicts that $\{b \in M : p^b > r^b\}$ is not weakly underdemanded at p for R, that is, $|N| \geq |\{j \in N : D(R_j, p) \cap \{b \in M : p^b > r^b\} \neq \emptyset\}| > |\{b \in M : p^b > r^b\}|$.

(ii) By $f(R) \in Z_{\min}(R, r)$, there is some $p \in \mathbb{R}^m_{r+}$ with $(f(R), p) \in W_{\min}(R, r)$. By $x_i(R) \in D(R_i, p)$, $f_i(R) = (x_i(R), p^{x_i(R)}) \, R_i \, \mathbf{0}$.

(iii) By $f(R) \in Z_{\min}(R, r)$, there is some $p \in \mathbb{R}^m_{r+}$ with $(f(R), p) \in W_{\min}(R, r)$. By $t_i(R) = p^{x_i(R)}$, $p \in \mathbb{R}^m_{r+}$ and $r \in \mathbb{R}^m_+$, $t_i(R) = p^{x_i(R)} \geq r^{x_i(R)} \geq 0$. □

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