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SELLING ORDER IN A SEQUENTIAL AUCTION

Hikmet Gunay
Xin Meng
Victor Perez

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The Institute of Social and Economic Research
Osaka University
6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan

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Hikmet Gunay*

Department of Economics, University of Manitoba, Winnipeg, MB, R3T 5V5, Canada

Xin Meng

CIBO, Dongbei University of Finance and Economics, Dalian, 116025, China

Victor Perez

Instituto de Matematica Pura e Aplicada, IMPA, Rio de Janeiro, Brazil

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Abstract

In a second-price sequential auction with both global and local bidders, we explore the optimal order for selling heterogeneous goods to maximize efficiency or revenue. Our findings indicate that selling the good with very small variance (almost-zero variance) first yields higher revenue, while selling it second results in an efficient outcome with probability almost 1. We link the optimal selling order to the likelihood of various inefficient outcomes. Specifically, selling the good with small variance first increases the probability of ex-post loss for the global bidder, boosting the seller's revenue at the expense of overall social welfare.

JEL Codes: D44, D82
Keywords: Sequential Auctions, Multi-dimensional values, Simulations

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1 Introduction

There are some auctions such as highway procurement auctions in which heterogenous synergistic goods are sold to global and local bidders via sequential auctions (e.g., De Silva 2005). In such sequential auctions, the aim of the auction can be to generate higher revenues or obtain efficient outcomes depending on whether governments, non-profit organizations, or firms are conducting the auction. Therefore, it should be better understood how to reach these different goals in an auction by studying its different aspects. One such aspect is the selling order of goods, which the auction literature neglected due to the assumption of identical goods. If goods are ex-ante identical and bidder numbers are equal, selling order does not matter. However, if the goods are heterogenous, the selling order should matter. In this paper, we analyze selling which of the heterogenous goods first (rather than second) generates higher revenue or social welfare in a second-price sequential auction to fill the gap in the literature.

We show that the selling order for higher revenue or efficiency is determined by the variance of the (valuation of) goods.[1] Specifically, if all bidders are more likely to have very close valuations, that is, one good has a variance approaching to zero, then selling that good first results in higher revenue, but selling it second results in a more efficient outcome. Why do we get this result? As it is well-known in the sequential auction literature (e.g., Gunay and Meng 2022), the global bidder bids over its stand alone valuation in the first auction hoping to win the second good and enjoy the synergy regardless of the variance of the good. When the low variance good is sold first, the global bidder wins it with almost probability 1 as they bid over the stand alone valuation. Moreover, they win it over their stand-alone valuation with some positive probability as his and the local bidders' valuations are close. While this creates a high enough probability of an ex-post loss for the global bidder, it also

[1]In the highway-construction procurement auctions, some projects have high variance according to civil engineers. A private conversation with a civil engineer, Mr. Mert Gulcat, revealed that BidBid-Sur Highway project in Oman had a high variance due to the difficulty in assessing the cost of the project. However, some other highway projects have low variance especially when the terrain that the highway will be constructed is flat and hence, the cost of the project can be calculated easily.

increases the auction revenue at the expense of both the global bidder's profits and the social welfare.

We show that if the small variance good is sold second, then the global bidder bids (almost) truthfully and hence the outcome is (almost) efficient as all bidders bid (almost) truthfully. We note that, in the limit when the variance is zero, selling order would not matter in the sense that any kind of selling ordering will result in an efficient outcome. However, the probability of inefficient outcomes do not approach to zero as the variance approaches to zero, when the small variance good is sold first [2] but it does if it is sold second.

While using ex-ante expected revenue and welfare is calculated in part of the auction literature when bidders are (ex-ante) identical, it is not possible in our model as the bidders are heterogenous and the bid of the global bidder is complicated. Moreover, the sequential auction literature does not calculate the probability of ex-post loss for the global bidder as it is either impossible or extremely complicated. To overcome this problem, we do some simulations to quantify the probability of ex-post loss for the global bidder in different selling order of goods by using ex-post valuations (papers using such simulation methods include Krishna and Rosenthal, 1996; Meng and Gunay, 2017; Gunay and Meng, 2022). If the low variance good is sold second, the ex-post loss probability approaches to zero as the variance of the good decreases to zero. However, if it is sold first, the ex-post loss probability stays high. While the theoretical results for the revenue and efficiency in different selling orders are valid when one good has an almost zero variance, the simulations show that the variance need not be too small to get those results.

This result has a policy implication. To make the bidders' valuations closer on one good, the seller might reveal the information on that good or allow the bidders inspect it (but not the other good). Then set the order of selling as discussed in our paper depending on whether their aim is an efficient outcome or a higher revenue.[3]

Elmaghraby (2003) studies the implications of selling order on efficiency with capacity

[2] While the probability of inefficiency is positive, the amount of inefficiency is very small; hence the amount of inefficiency approaches to 0.

[3] Though we note that this is not a mechanism design paper and it compares two selling order.

constrained bidders on procurement auctions as this is relevant to businesses. Meng and Gunay (2022) also study how the selling order of ex-ante identical goods affect the efficiency when the number of local bidders bidding for each good is different. Most other papers in the literature have not studied the selling order as they have assumed (ex-ante) identical goods, hence, the ordering would not matter (Jeitschko and Wolfstetter, 2002; De Silva, 2005; Leufkens et. al., 2010; Ghosh and Liu, 2019; Ghosh and Liu, 2021); or the goods are sold via simultaneous ascending auction (Meng and Gunay, 2017), and thus, the order of selling goods has no impact on revenue and welfare. One exception is Benoit and Krishna (2001). They find that selling the more valuable good first generates more revenue in a common value model with budget constrained (global) bidders in a **complete information** game. Our model is a private value model without "budget constrained bidders" assumptions.

A novel feature of our paper is linking the inefficiency outcomes to the revenue of the sequential auctions by using the probability theory. Namely, when global bidder wins an auction by making a loss, which is an inefficient outcome, this has a revenue increasing effect. However, there are two other inefficient outcomes where the global bidder wins one good with profit or when the local bidders won both goods when the global bidder is supposed to win them for efficiency. These inefficient outcomes have generally revenue-lowering effects. Therefore, when we have more of the first type of inefficiency, we get a revenue increasing and efficiency decreasing outcomes. The order of selling goods determine which of these inefficiencies disappear (with almost probability one) when the variance of one good is approaching to zero. We show this in our proofs to get our results.

Next, we set up our model and show our theoretical results. Then, we present our simulation results which shows that our results are valid even when the variance of a good is not too small. We conclude the paper with conclusion and discussion section.

2 The Model

Two goods, A and B, are sold in a second-price sequential auction. The goods have zero value to the seller. There is one risk-neutral global bidder, G, who bids for both goods, and enjoy a synergy of $\theta > 0$ if wins both goods.[4] There are also $N > 0$ risk neutral local bidders bidding for good $i = A, B$. There are $N + 1$ independent draws from the distribution function F_i determines the private valuation, v_{Gi}, for the global bidder, and v_{ki}, for each local bidder, $k = 1, 2.., N$, and $i = A, B$. The distribution function F_i, has a twice differentiable density function $f_i > 0$ on the interval $(0, 1]$ with $f_i(0) \geq 0$.[5] We will sometimes denote by $\hat{F} : \mathbb{R}_+ \to \mathbb{R}$ the extension of the distribution function F given by $\hat{F}(t) = F(t)$ for all $t \in [0, 1]$ and $\hat{F}(t) = 1$ for $t > 1$.

We use symmetric subgame perfect Bayesian equilibrium. The equilibrium strategy for local bidders is bidding their valuations truthfully in both auctions (in weakly undominated strategies). The global bidder's equilibrium strategy in the second auction is bidding her marginal valuation truthfully; hence, she bids $v_{Gj} + \theta$ if won good i in the first auction, and bid v_{Gj} otherwise, where $i, j = A, B$ and $i \neq j$.

To derive the global bidder's equilibrium strategy in the first auction for good i, we have to write the expected payoff given the sequential rationality and then maximize it. To calculate the expected payoff, we need to know the (expected) price the global bidder will pay, if wins the goods. Becuase this is a second-price auction, the global bidder pays the maximum of the local bidders' valuations (as they bid truthfully). Let $b_i = max\{v_{ki}\}, k = 1, 2.., N$ denote the maximum valuation of local bidders for good $i = A, B$. Since each local bidders' valution is a private information, we need the distribution function for b_i, which is $G_i(.) = [F_i(.)]^N$ for $i = A, B$. Now, we can write the expected payoff for the global bidder when she bids b.

$$Max_b \int_0^b (v_{Gi} - b_i) dG_i(b_i) + Pr(b > b_i) \int_0^{min\{v_{Gj}+\theta,1\}} (v_{Gj} + \theta - b_j) dG_j(b_j) + Pr(b < b_i) \int_0^{v_{Gj}} (v_{Gj} - b_j) dG_j(b_j) \quad (1)$$

[4]Assuming one global bidder when a bidder has *multi-dimensional* valuations is not uncommon in the literature (see Meng and Gunay(2017), Goeree and Lien (2014), Albano et al. (2006), Kagel and Levin (2005)) since the equilibrium strategy for multiple global bidders have not been calculated unless they have single types.

[5]The model is similar to the corresponding model of Gunay and Meng (2022) as both papers deal with selling order. However, the current model will let one of the goods variance approach to zero later in the model.

The first integral is the expected profit from winning i in the first auction. If the global bidder wins the first auction, she will get her stand-alone valuation v_{Gi} and pay the price of b_i. Since we do not know the price b_i, we have to use its density function dG_i. However, since calculations are done conditional on the global bidder is winning, we are only interested in how b_i is distributed in $[0, b]$. Hence, the lower limit and the upper limit of the integral are 0 and b, respectively. If, $b_i > b$, then the global bidder loses the first auction and gets 0 payoff. Therefore we do not write this part of the integral in the equation.

The second integral is the expected profit from winning j after winning i. This case can happen with probability $Pr(b > b_i)$, which means that the global bidder's bid were the highest bid in the first auction. Then, by sequential rationality, the global bidder bids $v_{Gj} + \theta$. If she wins the auction, she pays b_j for the good j. We should calculate how b_j is distributed between $[0, min\{v_{Gj} + \theta, 1\}]$ by using the density function dG_j.[6] Hence, the lower and upper limit of the integrals are 0 and $min\{v_{Gj} + \theta, 1\}]$, respectively. If the global bidder loses the auction, she gets 0 payoff so we do not write this case.

The third integral is the expected profit from winning j only in the second leg of the auction. This case happens only if the global bidder's bid in the first auction was lower than the local bidders' valuations/bids. This happens with probability $Pr(b < b_i)$. Then, by sequential rationality, the global bidder bids v_{Gj}. If she wins, she pays b_j. We should calculate how b_j is distributed on $[0, v_{Gj}]$ by using its density function dG_j. Hence, the lower and upper limit of the integrals are 0 and, v_{Gj}, respectively. If b_j is between $[v_{Gj}, 1]$, then the global bidder loses the second leg of the auction and gets 0. Hence, we do not need to write this case in the equation.

Equation 2 is the first order condition that calculates the optimal bid b_{ij} in the ij auction where good i is sold first and good j is sold second. The payoff from winning the first auction for the global bidder is the left hand side when he pays (the optimal bid) b_{ij}. The payoff from losing the first auction is the right hand side. Hence, the optimal bid b_{ij} is the (highest) price that the global bidder is willing to pay to win the first auction, or equivalently, he is

[6] b_j cannot be greater than 1 as its distributed up to 1. We could write this equation by using $\hat{G}$ (rather than G), then we would not have to use a min function.

indifferent between losing and winning the first auction at the price b_{ij} in the ij auction.

$$\underbrace{(v_{Gi} - b_{ij}) + \int_0^{min\{v_j+\theta,1\}} (v_{Gj} + \theta - b_j) dG_j(b_j)}_{\text{Expected profit from winning the first auction when price is } b_{ij}} = \underbrace{\int_0^{v_j} (v_{Gj} - b_j) dG_j(b_j)}_{\text{and losing the first auction}} \quad (2)$$

By using integration by parts and equation 2, we derive the global bidder's equilibrium bid.[7]

Proposition 1 *The global bidder's equilibrium bid, b_{ij} in the first auction for good i is*

$$b_{ij}(v_{Gi}, v_{Gj}, N) = v_{Gi} + \int_{v_{Gj}}^{v_{Gj}+\theta} \hat{G}_j(b, N) db$$

A few observations based on proposition 1. First, the global bidder bids over her stand-alone valuation which exposes her to the ex-post loss as well known in the literature (e.g. Krishna and Rosenthal, 1996). Second, when $v_{GA} + v_{GB} + \theta \geq 2$, the global bidder bids above 1 in both auctions, and wins both goods.[8] Third, the global bidder bids such that $v_{Gi} \leq b_{ij} \leq v_{Gi} + \theta$ since the integral in proposition 1 is between 0 and θ given $0 \leq \hat{G}_j(.) \leq 1$. Let us summarize some of the results in a corollary below.

Corollary 2

$$v_{Gi} \leq b_{ij}(v_{Gi}, v_{Gj}, N) \leq v_{Gi} + \theta$$

Now, we can present the optimal bid as the variance of one good, good j, is approaching to 0. Let P denote the probability.

Proposition 3 *a) Let F_j be a probability distribution with mean μ_j and variance σ_j^2. When $\sigma_j^2 \to 0$ and $\mu_j \to \mu$, then the optimal bid b_{ij} converges to*

$$\hat{b_{ij}} = \begin{cases} v_{Gi} + \theta & : \mu \leq v_{Gj} \\ v_{Gi} + v_{Gj} + \theta - \mu & : v_{Gj} < \mu < v_{Gj} + \theta \\ v_{Gi} & : v_{Gj} + \theta \leq \mu \end{cases}$$

[7] See the discussion paper of Gunay and Meng (2017) for the proof.

[8] Since $v_{Gi} + v_{Gj} + \theta \geq 2$ implies $v_{Gj} + \theta \geq 2 - v_{Gi} \geq 1$. But then since $\hat{G}_j(t) = 1$ for $1 \leq t \leq v_{Gj} + \theta$ then we have $b_{ij} \geq v_{Gi} + (v_{Gj} + \theta - 1) \geq 1$, which guarantees that global bidder wins the first and then the second auction.

Moreover, when $\mu_j \to \mu$ *and* $\sigma_j^2 \to 0$ *we have that* $P(\hat{b}_{ij} = v_{Gi} + \theta) \to 1$

b) As $\mu_j \to \mu$ *and* $Var(F_j) \to 0$, $Prob(b_{ji} > \max_{1 \leq k \leq N}\{v_{jk}\}) \to 1$.

Proposition $3a$ says that as the variance of good j approaches to 0, then the global bidder's bid in the i leg of the ij auction approaches (from the left) to $v_{Gi} + \theta$. Becasue the global bidder knows that if she wins good i in the first auction, then the probability that she will win good j approaches to 1. As she will bid $v_{Gj} + \theta$ when the local bidders' valuations are very close to v_{Gj}, she will win the j auction and pay only v_{Gj} with almost probability 1. But then the value of good i becomes (almost) $v_{Gi} + \theta$ for the global bidder in the first leg of the auction. The global bidder is actually bidding (almost) truthfully in both leg of the auction in this case! Proposition $3b$ says that the global bidder wins the good j in the ji auction as $Var(F_j) \to 0$. The reason is that the global and local bidders have very close valuations (around μ) for good j but the global bidder bids over its stand alone valuation by Proposition 1; hence, the global bidder wins the good with almost probability 1.

Now we can prove our main result. Let R_{ij} and SW_{ij} denote the (ex-post) revenue and social welfare, respectively, in the ij auction with $i, j = A, B$ and $i \neq j$.

Proposition 4 *Assume that* $N_A = N_B \geq 1$. *If* $Var(F_B)$ *approaches to zero, then we have that the probability that* $R_{AB} \leq R_{BA}$ *and* $SW_{BA} \leq SW_{AB}$ *converges to 1. That is,* $P(R_{AB} \leq R_{BA}) \to 1$ *and* $P(SW_{BA} \leq SW_{AB}) \to 1$.

Proposition 4 states that, selling the good B first - which its variance is approaching to zero- gives a higher revenue than selling good A first. However, the social welfare will be less. On the other hand, selling good B second gives a higher social welfare than selling good A second. We have already explained the latter part above as the global bidder bids truthfully in this case. The first part mainly occurs from inefficient cases in which the global bidder makes an ex-post loss. And this inefficiency survives even when the variance of B goes to zero in a BA auction but disappears in an AB auction.

Corollary 5 *In an AB auction, local bidders win both auctions when it is efficent for them to win and* $v_{LA} \geq v_{GA} + \theta$. *However, for the same valuations, in the BA auction, as the* *variance of good B goes to zero, the global bidder wins good B but loses good A to the local bidders. This increases the revenue compared to the AB auction, but results in an inefficient outcome.*

The proof of proposition 4, under Case III.a proves the corollary. Let us give an example. In a BA auction, when the global bidder wins B in a case $v_{GB} < p_B < b_{BA}$, it will make a potential loss. When the global bidder bids for good A and loses, it will be an ex-post loss and this can happen if $v_{GA} + \theta < v_{LA}$.[9] For such cases, the revenue in the BA auction is $R_{BA} = p_B + v_{GA} + \theta$.

Now, imagine selling good A first and good B second for the ex-post valuations in the example above. The local bidder wins good A since global bidder bids $b_{AB} < v_{GA} + \theta < v_{LA}$. Then global bidder bids also loses good B since $v_{GB} < p_B$. In this case, revenue $R_{AB} = b_{AB} + v_{GB}$. If we compare the revenues $R_{AB} = b_{AB} + v_{GB} < R_{BA} = p_B + v_{GA} + \theta$. The inequality holds since $b_{AB} < v_{GA} + \theta$ and $v_{GB} < p_B$.

One might think that this is just an example but as the variance of good B approaches to 0, this case still survives. The reason is that even if v_{GB} and p_b gets closer to each other, it is still a positive probability to get $v_{GB} < p_B$. Since the variance of good A is not small, it is also a positive probability to have $b_{AB} < v_{GA} + \theta < v_{LA}$ (assuming θ is not too big) and we note that $b_{AB} \to v_{GA} + \theta$ from the left.

While there are cases in which $R_{AB} > R_{BA}$, we show that these cases disappear as variance of good B approaches to 0 in our proofs.

3 Simulations

Given the complexities of multi-unit valuations, calculating ex-ante revenue, social welfare, and the probability of inefficient allocations proves to be quite challenging. To address this, we employ simulations based on ex-post valuations to compare AB and BA auction formats in terms of revenue, social welfare, and the likelihood of loss for the global bidder. Previous

[9]It canalso happen even if this inequality is reversed but both terms are close to each other

studies, such as those by Krishna and Rosenthal (1996) and Meng and Gunay (2017), have also utilized simulations in this context.[10]

To conduct the simulation in MATLAB, we draw valuations for each bidder from specified distributions for goods A and B. Using Proposition 1, we calculate the equilibrium bidding prices, b_{AB} and b_{BA}, for the AB and BA auctions, respectively. With complete knowledge of all valuations and equilibrium bids, we can determine the winners of the auctions, the prices paid for each set of valuations, and assess whether the outcomes are efficient or inefficient in both auction formats. This process is repeated 50,000 times to compute average revenue and efficiency.[11] To calculate the probability of inefficiency, we divide the number of inefficient outcomes by 50,000. Table 1 presents all possible outcomes along with the corresponding revenue and welfare for both AB and BA auctions, aiding in our ex-post calculations. The table identifies four types of inefficiencies involving a single local bidder: two cases where the global bidder wins one or both goods at an ex-post loss (rows 2 and 4), one case where the global bidder wins one good inefficiently but with a profit (row 6), and another case where the local bidders win both goods inefficiently (row 8).

	License i won by	License j won by	Global bidder makes	Allocation is	Revenue is	Welfare is
1.	Global Bidder	Global Bidder	Profit	Efficient	$b_i + b_j$	$v_{Gi} + v_{Gj} + \theta$
2.	Global Bidder	Global Bidder	Loss	Inefficient	$b_i + b_j$	$v_{Gi} + v_{Gj} + \theta$
3.	Global Bidder	Local Bidder j	Profit	Efficient	$b_i + v_{Gj} + \theta$	$v_{Gi} + b_j$
4.	Global Bidder	Local Bidder j	Loss	Inefficient	$b_i + v_{Gj} + \theta$	$v_{Gi} + b_j$
5.	Local Bidder i	Global Bidder	Profit	Efficient	$b_{ij} + b_j$	$b_i + v_{Gj}$
6.	Local Bidder i	Global Bidder	Profit	Inefficient	$b_{ij} + b_j$	$b_i + v_{Gj}$
7.	Local Bidder i	Local Bidder j	Zero Profit	Efficient	$b_{ij} + v_j$	$b_i + b_j$
8.	Local Bidder i	Local Bidder j	Zero Profit	Inefficient	$b_{ij} + v_j$	$b_i + b_j$

Table 1: All possible outcomes in an ij auction when $N_i = N_j = 1$

In the simulations, we examined three different synergy levels: 0.2, 0.5, and 0.8. For Good A's valuations, we employed a uniform distribution, while for Good B's valuations, we utilized various beta distributions, with parameters (α, β) set to $(3, 3)$, $(10, 10)$, $(50, 50)$, $(100, 100)$, and $(500, 500)$. All of these distributions share the same mean of 0.5. As shown

[10] Krishna and Rosenthal (1996) employ simulations except for uniform distributions, focusing on a single type for the global bidder.

[11] We run it for 20,000 times only for the beta distribution ($\alpha = \beta = 500$) as the code was running very slow for this distribution.

Table 2: The Variance of Beta Distributions Used in Simulations

$alpha$	$beta$	$Var(B)$	$Var(A)$
1	1	0.08333	0.08333
3	3	0.03571	0.08333
10	10	0.01191	0.08333
50	50	0.00248	0.08333
100	100	0.00124	0.08333
500	500	0.00025	0.08333

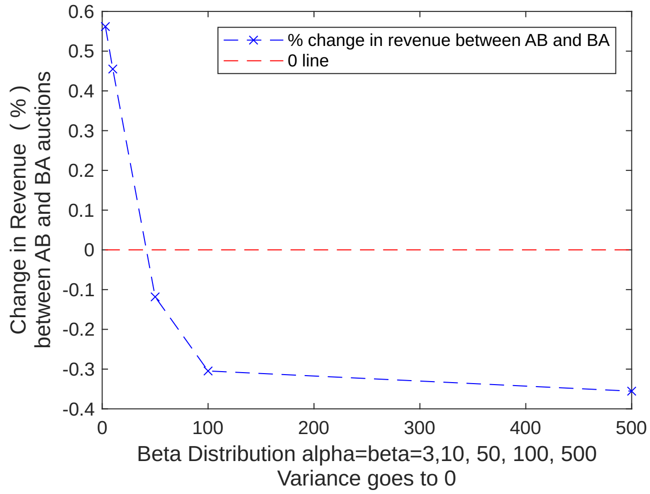

Figure 1: Revenue Comparison for $N = 2$, for $\theta = 0.5$

in Table 2, the variance of each beta distribution decreases as α and β increase.[12]

Simulation results are presented in Table 3 for $N_A = N_B = 1$ and in Table 5 for $N_A = N_B = 2$, which can be found in the Appendix. In what follows, we will mainly discuss the results for $N_A = N_B = 2$ and $\theta = 0.5$ with the help of the figures.

Figure 1 compares revenue as the variance of good B decreases. As the variance of good B decreases, the revenue from the BA auction ultimately surpasses that of the AB auction. This finding is crucial as it illustrates that the conclusions drawn in Proposition 4 do not necessitate the variance to be very close to zero.

Now, we will explain why we get this result by focusing on Figure 2. The primary observation in Figure 2 is that as the variance of good B decreases, the probability of ex-post loss

[12]Additionally, we explored the beta distribution with $\alpha = 1$ and β values of $3, 4, 5, 8, 20, 50, 100,$ and 1000, which yielded qualitatively similar results. For these distributions, both the mean and variance approached zero.

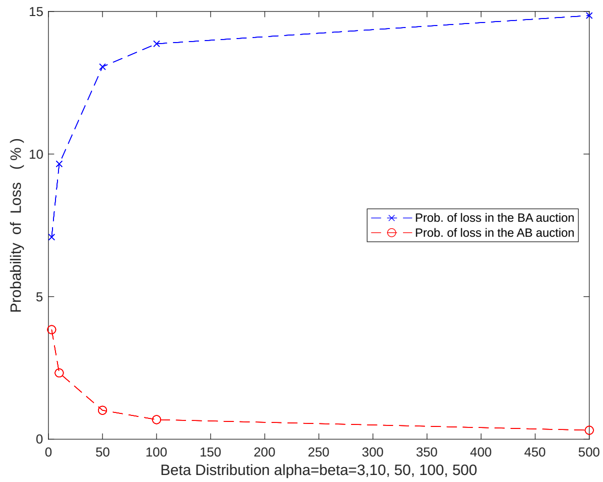

Figure 2: Probability for expost lost for $\theta = 0.5$ for $N = 2$.

in the BA auction does not approach zero, whereas in the AB auction, it does. Additionally, while other types of inefficiencies vanish (as Table 3 and Table 5 in the Appendix show), the inefficiency associated with the global bidder winning at a loss persists in the BA auction. Consequently, the BA auction is more inefficient overall and generates higher revenue, but this comes at the expense of the global bidder.

In Figure 3 , we compare social welfare across various auction formats. Our findings indicate that the social welfare associated with the BA auction is lower, even when the variance of good B is not close to zero. We note that in the figure Social Welfare of BA auction is lower for all distribtuions we have used. This is not contradicting our theoretical result which is only valid when the variance is close to zero.

4 Conclusion

In this paper, we demonstrate that the order in which goods are sold in a second-price sequential auction significantly affects revenue, welfare, and the likelihood of inefficient allocations. Our findings reveal that selling a good with nearly zero variance first yields higher revenue with a probability approaching 1, while selling it second results in an efficient outcome with a similar likelihood. We establish a connection between the auction's inefficient outcomes

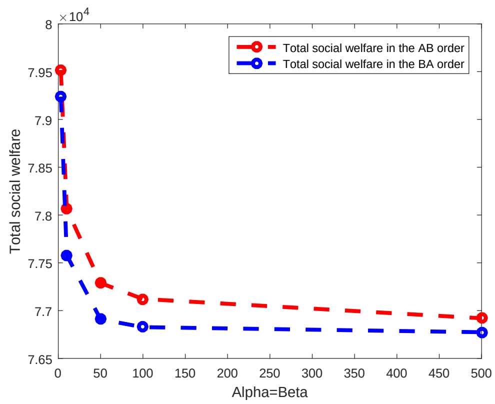

Figure 3: Social Welfare Comparison for $\theta = 0.5$ for $N = 2$.

and its revenue—a novel contribution to the literature on sequential auctions, to the best of our knowledge. When the global bidder bids beyond their standalone valuation, they face a substantial risk of incurring a loss if the nearly zero variance good is sold first.

Additionally, we believe that the code developed for this study can be a valuable resource for policymakers and firms conducting auctions. Our simulation results indicate that the variance of goods does not need to be excessively small to achieve these outcomes.

We hope this paper encourages further research into the optimal selling order of goods, particularly in scenarios where heterogeneity manifests in various forms beyond just variance.

5 Appendix-Proofs

Proof of proposition 1

We take the derivative of equation 1 by using the fact that $Pr(b > b_i) = G_i(b)$. After equating the derivative to zero, and cancelling $g(b)$ from the equation, we get equation 2

$$b \equiv b_{ij} = v_{Gi} + \int_0^{min\{v_{Gj}+\theta,1\}} (v_{Gj} + \theta - b_j)dG_j(b_j) - \int_0^{v_{Gj}} (v_{Gj} - b_j)dG_j(b_j)$$

We can re-write this by using integration by parts by setting $dv = dG_j(b_j)$ in both integrals. For instance, the first integral becomes

$$[(v_{Gj} + \theta - b_j)G_j(b_j)]|_{b_j=0}^{b_j=min\{v_{Gj}+\theta,1\}} + \int_0^{min\{v_{Gj}+\theta,1\}} G_j(b_j)db_j = \int_0^{v_{Gj}+\theta} \hat{G}_j(b_j)db_j$$

Where the equality holds since when $min\{v_{Gj} + \theta, 1\} = v_{Gj} + \theta$ the first term vanishes and when $min\{v_{Gj} + \theta, 1\} = 1$, $G_j(1) = 1$ and $v_{Gj} + \theta - 1 = \int_1^{v_{Gj}+\theta} \hat{G}_j(b)db$. Similarly, the second integral becomes $\int_0^{v_{Gj}} \hat{G}_j(b_j)db_j$ and substituting into the first order condition we get the result:

$$b \equiv b_{ij} = v_{Gi} + \int_0^{v_{Gj}+\theta} \hat{G}_j(b_j)db_j - \int_0^{v_{Gj}} \hat{G}_j(b_j)db_j = v_{Gi} + \int_{v_{Gj}}^{v_{Gj}+\theta} \hat{G}_j(b_j)db_j$$

Finally, we show that the SOC is satisfied.

$FOC = [v_{Gi} - b + \int_0^{min\{v_{Gj}+\theta,1\}} (v_{Gj} + \theta - b_j)dG_j(b_j) - \int_0^{v_{Gj}} (v_{Gj} - b_j)dG_j(b_j)]g_i(b)$
$= [v_{Gi} - b + \int_{v_{Gj}}^{v_{Gj}+\theta} \hat{G}_j(b_j)d(b_j)]g_i(b)$

$SOC = [v_{Gi} - b + \int_{v_{Gj}}^{v_{Gj}+\theta} \hat{G}_j(b_j)d(b_j)]g_i'(b) - g_i(b) < 0$, since $[v_{Gi} - b + \int_{v_{Gj}}^{v_{Gj}+\theta} G_j d(b_j)] = 0$. Since there is a unique equilibrium b_{ij}, and SOC is negative at b_{ij}, SOC is satisfied.

■

Proof of corollary 2: Since $0 \le \hat{G}_j(t) \le 1$, then $0 \le \int_{v_{Gj}}^{v_{Gj}+\theta} \hat{G}_j d(b_j) \le \theta$ and inmediately from proposition 1 we get

$$v_{Gi} \le b_{ij} = v_{Gi} + \int_{v_{Gj}}^{v_{Gj}+\theta} \hat{G}_j d(b_j) \le v_{Gi} + \theta$$

■

Proof of Proposition 3: Let X_n be a sequence of random variables distributed according to the cdf F_n with mean μ_n and variance σ_n^2 and let $Y_n = \mu_n$ and $Y = \mu$ be constant random variables. Then,

$$\|X_n - Y_n\|_{L^2} = (\textstyle\int \mid X_n - Y_n \mid^2 dP)^{1/2} = (\int \mid X_n - \mu_n \mid^2 dP)^{1/2} = Var(X_n)^{1/2} = \sigma_n$$

$$\|Y_n - Y\|_{L^2} = (\textstyle\int \mid Y_n - Y \mid^2 dP)^{1/2} = (\int \mid \mu_n - \mu \mid^2 dP)^{1/2} = \mid \mu_n - \mu \mid$$

$$\Rightarrow \|X_n - Y\|_{L^2} \le \|X_n - Y_n\|_{L^2} + \|Y_n - Y\|_{L^2} = \sigma_n + \mid \mu_n - \mu \mid$$

Hence, if as $n \to +\infty$ we have $\sigma_n \to 0$ and $\mu_n \to \mu$, then X_n converge to Y in the L^2 norm and therefore it also converges in distribution, i.e., for all $t \in [0, 1]$, $t \ne \mu$:

$$F_n(t) \to F_Y(t) = 1_{[\mu,1]}(t) = \begin{cases} 0 : 0 \le t < \mu \\ 1 : \mu \le t \le 1 \end{cases}$$

Since we have $\mu_j \to \mu$ and $Var(F_j) \to 0$ then $F_j \to 1_{[\mu,1]}$ and then $G_j(t) = (F_j)^N \to 1_{[\mu,1]}^N(t) = 1_{[\mu,1]}(t)$ for all $t \ne \mu$, $t \in [0, 1]$. Moreover, since by convention for any distribution F its extension has $\hat{F}(t) = 1$ for $t > 1$ we have $\hat{G}_j(t) \to 1_{\{[\mu,+\infty)\}}(t)$ for all $t \ge 0$, $t \ne \mu$. Now, using the dominated convergence theorem we have:

$$b_{ij} = v_{Gi} + \int_{v_{Gj}}^{v_{Gj}+\theta} \hat{G}_j(t)dt \to \hat{b}_{ij} \equiv v_{Gi} + \int_{v_{Gj}}^{v_{Gj}+\theta} 1_{[\mu,+\infty)}(t)dt$$

and evaluating the last integral we have that the optimal bid converges in the limit to:

$$\hat{b}_{ij} = \begin{cases} v_{Gi} + \theta & : \mu \le v_{Gj} \\ v_{Gi} + v_{Gj} + \theta - \mu & : v_{Gj} < \mu < v_{Gj} + \theta \\ v_{Gi} & : v_{Gj} + \theta \le \mu \end{cases}$$

Now let us show that the last two cases tend to disappear in the limit. That is $P(\hat{b}_{ij} = v_{Gi} + \theta) \to 1$. Observe that $P(\hat{b}_{ij} = v_{Gi} + \theta) = P(v_{Gj} \geq \mu) = 1 - P(v_{Gj} < \mu)$. We will prove that this last probability converges to zero in the limit.[13]

Since v_{Gj} is distributed according to F_j we showed that it is converging in L^2 to the constant random variable $Y = \mu$ and hence it also converges in probability, i.e., for every $k \in \mathbb{N}$ we have $P(\mid \mu - v_{Gj} \mid > \frac{1}{k}) \to 0$. Moreover, $\{v_{Gj} < \mu - \frac{1}{k}\} \uparrow \{v_{Gj} < \mu\}$ implies that for any $\epsilon > 0$ there is a $k_0 \in \mathbb{N}$ s.t.

$$P(v_{Gj} < \mu) \leq P(v_{Gj} < \mu - \frac{1}{k_0}) + \epsilon \leq P(\mid \mu - v_{Gj} \mid > \frac{1}{k_0}) + \epsilon$$

Then, taking limsup as $\mu_j \to \mu$ and $\sigma_j \to 0$ and using the convergence in probability, we have that for every $\epsilon > 0$:

$$0 \leq \underset{\mu_j \to \mu, \sigma_j \to 0}{limsup} P(v_{Gj} < \mu) \leq \epsilon$$

which guarantees that the limsup is indeed a limit and equals zero.

b) By proposition 1 we have $b_{ji} = v_{Gj} + \int_{v_{Gi}}^{v_{Gi}+\theta} \hat{G}_i(b_i)db_i$. We know that v_{Gj} is distributed according to F_j and $\max_{1 \leq k \leq N}\{v_{kj}\}$ is distributed according to $G_j = F_j^N$. Moreover, as $\mu_j \to \mu$ and $\sigma_j \to 0$ we have seen that v_{Gj} converges in probability to $Y = \mu$ and $\max_{1 \leq k \leq N}\{v_{kj}\}$ converges in distribution to Y. Since Y is a constant random variable then $\max_{1 \leq k \leq N}\{v_{kj}\}$ also converges in probability to Y. [14] Hence, since $\alpha = \int_{v_{Gi}}^{v_{Gi}+\theta} \hat{G}_i(b_i)db_i > 0$ we have:

$$\begin{aligned} P(b_{ji} \leq \max_{1 \leq k \leq N}\{v_{kj}\}) &= P(\alpha \leq \max_{1 \leq k \leq N}\{v_{kj}\} - v_{Gj}) \leq P(\alpha \leq \mid \max_{1 \leq k \leq N}\{v_{kj}\} - v_{Gj} \mid) \\ &\leq P(\tfrac{\alpha}{2} \leq \mid \max_{1 \leq k \leq N}\{v_{kj}\} - Y \mid) + P(\tfrac{\alpha}{2} \leq \mid Y - v_{Gj} \mid) \end{aligned}$$

and as $\mu_j \to \mu$ and $\sigma_j \to 0$ the last bound goes to zero by convergence in probability. Hence, $P(b_{ji} > \max_{1 \leq k \leq N}\{v_{kj}\}) \to 1$

■

Proof of Proposition 4:

[13] When computing its optimal bid, the global bidder knows its own valuation v_{Gj} which is deterministic. From an outside perspective, such as the perspective of the auctioneer, however, v_{Gj} is a random variable with distribution F_j and it makes sense to examine its behaviour in the limit as $\mu_j \to \mu$ and $\sigma_j \to 0$

[14] This would not be true in general if the limit Y were not a constant random variable. See Cinlar (2011, chapter III: section 5) and in particular remarks b) and c) on page 110.

For $j = A, B$, let's define $v_{Lj} = \max\{v_{ij} : 1 \leq i \leq N_j\}$ the maximum valuation for good j among the local bidders and $v_{L_2 j} = \max[\{v_{ij} : 1 \leq i \leq N_j\} - \{v_{Lj}\}]$ the second maximum valuation for good j among local bidders. By definition, $v_{Lj} \geq v_{L_2 j}$ always. Given any realization of the bidders valuations, $MSW = \max\{v_{LA} + v_{GB}, v_{GA} + v_{LB}, v_{LA} + v_{LB}, v_{GA} + v_{GB} + \theta\}$ gives the maximum social welfare that can be achieved. Hence, according to where this maximum is achieved it will be efficient that: I) A local bidder wins the A auction and the global wins the B auction or II) The global bidder wins the A auction and a local the B auction or, III) the Local bidders wins both auctions or, IV) the global bidder wins both auctions.

We will use this classification of the possible realizations of the bidders valuation to show that only the cases where $R_{AB} \leq R_{BA}$ and $SW_{BA} \leq SW_{AB}$ do not disappear as the variance goes to zero.

Case I: $MSW = v_{LA} + v_{GB}$ which implies:

$$v_{LA} + v_{GB} \geq v_{GA} + v_{GB} + \theta \Rightarrow v_{LA} \geq v_{GA} + \theta \tag{3}$$

$$v_{LA} + v_{GB} \geq v_{LA} + v_{LB} \Rightarrow v_{GB} \geq v_{LB} \tag{4}$$

In the AB auction, by corollary 2, $b_{AB} \leq v_{GA} + \theta$. Hence, a local bidder wins the A auction by (3) and the global bidder wins B by (4). Thus, the AB auction produces an efficient result. In the AB auction, the revenue is $R_{AB} = \max\{b_{AB}, v_{L_2 A}\} + v_{LB}$ and the social welfare is $SW_{AB} = v_{LA} + v_{GB}$.

In the BA auction, by corollary 2, $b_{BA} \geq v_{GB}$. Hence, the global bidder wins the B auction by (4). In the A leg of the BA auction, the global bidder will bid $v_{GA} + \theta$ and a local bidder will win by (3). Therefore, the BA auction still produces the efficient outcome $SW_{BA} = v_{LA} + v_{GB}$ but now the revenue is $R_{BA} = \max\{v_{GA} + \theta, v_{L_2 A}\} + v_{LB}$.

In summary, in this case both auctions are equally efficient $SW_{AB} = v_{LA} + v_{GB} = SW_{BA}$. On the other hand, $R_{AB} = \max\{b_{AB}, v_{L_2 A}\} + v_{LB} \leq \max\{v_{GA} + \theta, v_{L_2 A}\} + v_{LB} = R_{BA}$ since $b_{AB} < v_{GA} + \theta$ by Corollary 2. So the BA auction produces an ex-post greater revenue.

Case II: $MSW = v_{GA} + v_{LB}$ which implies:

$$v_{GA} + v_{LB} \geq v_{GA} + v_{GB} + \theta \Rightarrow v_{LB} \geq v_{GB} + \theta \tag{5}$$

$$v_{GA} + v_{LB} \geq v_{LA} + v_{LB} \Rightarrow v_{GA} \geq v_{LA} \tag{6}$$

This case tends to disappear as $Var(F_B) \to 0$. Indeed, the probability that (5) happens goes to zero: $P(v_{LB} \geq v_{GB} + \theta) \leq P(\mid v_{LB} - v_{GB} \mid \geq \theta) \to 0$. The convergence to zero follows from $\theta > 0$ and the fact that as $Var(F_B) \to 0$ and $\mu_{F_B} \to \mu$, v_{LB} and v_{GB} are both converging in probability to μ according to the proof of proposition 3b.

Case III: $MSW = v_{LA} + v_{LB}$ which implies:

$$v_{LA} + v_{LB} \geq v_{GA} + v_{LB} \Rightarrow v_{LA} \geq v_{GA} \tag{7}$$

$$v_{LA} + v_{LB} \geq v_{LA} + v_{GB} \Rightarrow v_{LB} \geq v_{GB} \tag{8}$$

Unlike the previous cases, we can't determine the outcomes of the AB and BA auctions from (7) and (8). There are two main subcases.

III.a) Assume $v_{LA} \geq v_{GA} + \theta$. This implies that, in the AB auction, a local bidder wins A since by corollary 2, $b_{AB} \leq v_{GA} + \theta$. A local bidder also wins B by (8). Hence, the AB auction is efficient $SW_{AB} = v_{LA} + v_{LB}$ and generates a revenue of $R_{AB} = \max\{b_{AB}, v_{L_2A}\} + \max\{v_{GB}, v_{L_2B}\}$.

The result of the BA auction has two subcases. When $b_{BA} < v_{LB}$, a local bidder wins B. However, by proposition 3b, this subcase tends to disappear in the limit[15]. Hence, we can assume that $b_{BA} \geq v_{LB}$. Then the global bidder wins B and bids $v_{GA} + \theta$ in the A auction. Thus, the local bidder wins auction A by assumption. In this case, the BA auction is inefficient with $SW_{BA} = v_{GB} + v_{LA}$ and generates revenue $R_{BA} = v_{LB} + \max\{v_{GA} + \theta, v_{L_2A}\}$.

Thus, the revenue comparison between the two auctions is $R_{AB} = \max\{b_{AB}, v_{L_2A}\} + \max\{v_{GB}, v_{L_2B}\} < v_{LB} + \max\{v_{GA} + \theta, v_{L_2A}\} = R_{BA}$, by (8) and corollary 2 ($b_{AB} < v_{GA} + \theta$). The social welfare comparison between the two auctions is $SW_{AB} = v_{LA} + v_{LB} \geq SW_{BA} = v_{LA} + v_{GB}$ by (8).

[15] That is, as $Var(F_B) \to 0$ and $\mu_B \to \mu$, $P(b_{BA} < v_{LB}) \to 0$

III.b). Assume, $v_{LA} < v_{GA} + \theta$. Since, $b_{AB} \to v_{GA} + \theta$ by proposition 3a[16], then, $v_{LA} < b_{AB}$ when $Var(F_B)$ is sufficiently small.[17] Then, in the AB auction the global bidder wins A and bids $v_{GB} + \theta$ in the second leg of the auction. From here, there are two subcases. If $v_{LB} \geq v_{GB} + \theta$ the local bidder wins B. However, as above $P(v_{LB} - v_{GB} \geq \theta) \to 0$ and this subcase disappears as $Var(F_B) \to 0$. The other subcase $v_{LB} < v_{GB} + \theta$ implies that the global bidder also wins the B leg of the AB auction. In summary, in the AB auction the revenue is $R_{AB} = v_{LA} + v_{LB}$ and the social welfare is $SW_{AB} = v_{GA} + v_{GB} + \theta$.

In the BA auction, the case where the global bidder loses the B leg of the auction tends to disappear in the limit according to proposition 3b. Hence, we may assume $b_{BA} > v_{LB}$. The global bidder wins B and bids $v_{GA} + \theta$ in the A auction. Since $v_{LA} < v_{GA} + \theta$ by assumption, then the global bidder wins both goods inefficiently in the BA auction. The revenue and social welfare are equal to the ones in the AB auction: $R_{BA} = v_{LA} + v_{LB} = R_{BA}$ and $SW_{BA} = v_{GA} + v_{GB} + \theta = SW_{AB}$.

Case IV: $MSW = v_{GA} + v_{GB} + \theta$. Since this is the last case we can assume that the maximum is unique. If the maximum is not unique we are within one of the previous 3 cases. Hence, this case implies:

$$v_{GA} + v_{GB} + \theta > v_{GA} + v_{LB} \Rightarrow v_{GB} + \theta > v_{LB} \tag{9}$$

$$v_{GA} + v_{GB} + \theta > v_{LA} + v_{GB} \Rightarrow v_{GA} + \theta > v_{LA} \tag{10}$$

Since, according to proposition 3a, $b_{AB} \to v_{GA} + \theta$ then, by (10), $b_{AB} > v_{LA}$ when $Var(F_B)$ is sufficiently small. In the AB auction, this means that the global bidder wins A and bids $v_{GB} + \theta$ for B. By (9), the global bidder also wins B. Hence, the revenue in the AB auction is $R_{AB} = v_{LA} + v_{LB}$ and the result is efficient $SW_{AB} = v_{GA} + v_{GB} + \theta$.

In the BA auction, the case where the global bidder loses the B leg of the auction tends to disappear in the limit according to proposition 3b. Hence, we may assume $b_{BA} > v_{LB}$. The global bidder wins B and bids $v_{GA} + \theta$ in the A auction. Then by (10), the global

[16]Strictly speaking, b_{AB} may be converging to others values but those cases tend to disappear in the limit according to proposition 3a.

[17]Technically we require both $Var(F_B)$ and $\mid \mu_B - \mu \mid$ to be sufficiently small.

bidder also wins A. Therefore, we have the same revenue and welfare as in the AB auction: $R_{BA} = R_{AB}$ and $SW_{BA} = SW_{AB}$.

We now summarize all the analysis up to this point. We proved that if $Var(F_B)$ is sufficiently small, $R_{AB} \leq R_{BA}$ and $SW_{AB} \geq SW_{BA}$ for all cases except those that tend to disappear when $Var(F_B) \to 0$. Then $P(R_{AB} \leq R_{BA}, SW_{AB} \geq SW_{BA}) \to 1$ as $Var(F_B) \to 0$.
■

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6 APPENDIX TABLES

	Synergy with $\theta = 0.2$ $N_A = N_B = 1$	Synergy with $\theta = 0.5$ $N_A = N_B = 1$	Synergy with $\theta = 0.8$ $N_A = N_B = 1$
	F_A =**Uniform Distr.with** $\alpha = \beta = 1$;		F_B = **Beta Distr. with** $\alpha = \beta = 3$;
$p_{AB}^{Exp.Loss}\%$	2.5840	3.2460	1.9320
$p_{AB}^{Ineff.}\%$	2.3120	2.3280	1.3060
$p_{BA}^{Exp.Loss}\%$	4.0340	6.2420	3.7800
$p_{BA}^{Ineff.}\%$	3.5840	3.4660	0.8020
$\triangle R\%$	0.0016	-0.3976	-0.5563
$\triangle SW\%$	0.1016	0.3753	0.1864
	F_A =**Uniform Distr.with** $\alpha = \beta = 1$;		F_B = **Beta Distr. with** $\alpha = \beta = 10$;
$p_{AB}^{Exp.Loss}\%$	1.1000	1.5220	0.7920
$p_{AB}^{Ineff.}\%$	1.8600	1.2640	0.5880
$p_{BA}^{Exp.Loss}\%$	6.8420	7.7320	2.8040
$p_{BA}^{Ineff.}\%$	4.7540	1.3680	0.0440
$\triangle R\%$	-0.0696	-0.8363	-0.5024
$\triangle SW\%$	0.3049	0.4748	0.1222
	F_A =**Uniform Distr.with** $\alpha = \beta = 1$;		F_B = **Beta Distr. with** $\alpha = \beta = 50$;
$p_{AB}^{Exp.Loss}\%$	1.0320	0.6100	0.3160
$p_{AB}^{Ineff.}\%$	1.0020	0.6040	0.2300
$p_{BA}^{Exp.Loss}\%$	11.6180	7.5620	1.7400
$p_{BA}^{Ineff.}\%$	2.7780	0.0740	0.000
$\triangle R\%$	-0.2826	-0.7601	-0.1656
$\triangle SW\%$	0.4477	0.2502	0.0405
	F_A =**Uniform Distr.with** $\alpha = \beta = 1$;		F_B = **Beta Distr. with** $\alpha = \beta = 100$;
$p_{AB}^{Exp.Loss}\%$	0.6960	0.4600	0.1540
$p_{AB}^{Ineff.}\%$	0.6620	0.4020	0.1640
$p_{BA}^{Exp.Loss}\%$	13.4760	7.4100	1.4620
$p_{BA}^{Ineff.}\%$	1.5640	0.0000	0.0000
$\triangle R\%$	-0.3796	-0.5691	-0.1033
$\triangle SW\%$	0.4057	0.1757	0.0262
	F_A =**Uniform Distr.with** $\alpha = \beta = 1$;		F_B = **Beta Distr. with** $\alpha = \beta = 500$;
$p_{AB}^{Exp.Loss}\%$	0.2880	0.1740	0.0760
$p_{AB}^{Ineff.}\%$	0.2700	0.1460	0.0740
$p_{BA}^{Exp.Loss}\%$	15.7300	6.5440	1.1300
$p_{BA}^{Ineff.}\%$	0.2600	0.0000	0.0000
$\triangle R\%$	-0.4231	-0.2349	-0.0383
$\triangle SW\%$	0.2181	0.0743	0.0100

$p_{ij}^{Exp.Loss}$=100*(Sum of outcomes in row 2 and 4 in Table 1)/50000. Probability of the global bidder having ex-post loss.
$p_{ij}^{Ineff.}$=100*(Sum of outcomes in row 6 and 8 in Table 1)/50000. Probability of all inefficient allocations.
$\triangle R\% = 100 * (R_{AB} - R_{BA})/R_{BA}$. Percentage change in revenue between AB and BA auctions.
$\triangle SW\% = 100 * (SW_{AB} - SW_{BA})/SW_{BA}$. Percentage change in social welfare (SW) between AB and BA auctions.

Table 3: Simulation Results for F_A Uniform Distribution when $N_i = N_j = 1$

	Synergy with $\theta = 0.2$ $N_A = N_B = 2$	**Synergy with** $\theta = 0.5$ $N_A = N_B = 2$	**Synergy with** $\theta = 0.8$ $N_A = N_B = 2$
$F_A =$ **Uniform Distr.with** $\alpha = \beta = 1$;		$F_B =$ **Beta Distr. with** $\alpha = \beta = 3$;	
$p_{AB}^{Exp.Loss}\%$	2.3540	3.8420	3.2740
$p_{AB}^{Ineff.}\%$	2.3700	3.7520	2.4860
$p_{BA}^{Exp.Loss}\%$	3.6720	7.0860	6.0160
$p_{BA}^{Ineff.}\%$	3.7720	5.3060	2.5860
$\triangle R\%$	0.1514	0.5619	0.1496
$\triangle SW\%$	0.0882	0.3519	0.3134
$F_A =$ **Uniform Distr.with** $\alpha = \beta = 1$;		$F_B =$ **Beta Distr. with** $\alpha = \beta = 10$;	
$p_{AB}^{Exp.Loss}\%$	1.9880	2.3240	1.5540
$p_{AB}^{Ineff.}\%$	2.0660	2.0980	1.2520
$p_{BA}^{Exp.Loss}\%$	6.4000	9.6540	5.7600
$p_{BA}^{Ineff.}\%$	5.7540	3.7520	0.7980
$\triangle R\%$	0.3432	0.4550	-0.3464
$\triangle SW\%$	0.2913	0.6302	0.2954
$F_A =$ **Uniform Distr.with** $\alpha = \beta = 1$;		$F_B =$ **Beta Distr. with** $\alpha = \beta = 50$;	
$p_{AB}^{Exp.Loss}\%$	1.1840	1.0140	0.5860
$p_{AB}^{Ineff.}\%$	1.1820	0.9060	0.4200
$p_{BA}^{Exp.Loss}\%$	12.1660	13.0600	4.3080
$p_{BA}^{Ineff.}\%$	4.5100	1.2660	0.0240
$\triangle R\%$	0.4125	-0.1183	-0.3000
$\triangle SW\%$	0.5073	0.4925	0.1223
$F_A =$ **Uniform Distr.with** $\alpha = \beta = 1$;		$F_B =$ **Beta Distr. with** $\alpha = \beta = 100$;	
$p_{AB}^{Exp.Loss}\%$	0.8640	0.6860	0.3640
$p_{AB}^{Ineff.}\%$	0.7740	0.6340	0.3020
$p_{BA}^{Exp.Loss}\%$	14.9260	13.8660	3.8000
$p_{BA}^{Ineff.}\%$	3.2220	0.6420	0.0000
$\triangle R\%$	0.3591	-0.3047	-0.2091
$\triangle SW\%$	0.4938	0.3832	0.0816
$F_A =$ **Uniform Distr.with** $\alpha = \beta = 1$;		$F_B =$ **Beta Distr. with** $\alpha = \beta = 500$;	
$p_{AB}^{Exp.Loss}\%$	0.3700	0.3140	0.1520
$p_{AB}^{Ineff.}\%$	0.3280	0.2840	0.1540
$p_{BA}^{Exp.Loss}\%$	21.0640	14.8560	2.9500
$p_{BA}^{Ineff.}\%$	1.2420	0.0520	0.0000
$\triangle R\%$	0.1402	-0.3555	-0.0794
$\triangle SW\%$	0.3347	0.1909	0.0319

$p_{ij}^{Exp.Loss}$=100*(Sum of outcomes in row 2 and 4 in Table 1)/50000. Probability of the global bidder having ex-post loss.
$p_{ij}^{Ineff.}$=100*(Sum of outcomes in row 6 and 8 in Table 1)/50000. Probability of all inefficient allocations.
$\triangle R\% = 100 * (R_{AB} - R_{BA})/R_{BA}$. Percentage change in revenue between AB and BA auctions.
$\triangle SW\% = 100 * (SW_{AB} - SW_{BA})/SW_{BA}$. Percentage change in social welfare (SW) between AB and BA auctions.

Table 4: Simulation Results for F_A Uniform Distribution when $N_i = N_j = 2$. Here, we assume $v_{1i} > v_{2i} > ...$

	License i won by	License j won by	Global bidder makes	Allocation is	Revenue is	Welfare is
1.	Global Bidder	Global Bidder	Profit	Efficient	$v_{1i} + v_{1j}$	$v_{Gi} + v_{Gj} + \theta$
2.	Global Bidder	Global Bidder	Loss	Inefficient	$v_{1i} + v_{1j}$	$v_{Gi} + v_{Gj} + \theta$
3.	Global Bidder	Local Bidder j	Profit	Efficient	$v_{1i} + v_{Gj} + \theta$	$v_{Gi} + v_{1j}$
4.	Global Bidder	Local Bidder j	Loss	Inefficient	$v_{1i} + v_{Gj} + \theta$	$v_{Gi} + v_{1j}$
5.	Global Bidder	Local Bidder j	Profit	Efficient	$v_{1i} + v_{2j}$	$v_{Gi} + v_{1j}$
6.	Global Bidder	Local Bidder j	Loss	Inefficient	$v_{1i} + v_{2j}$	$v_{Gi} + v_{1j}$
7.	Local Bidder i	Global Bidder	Profit	Efficient	$b_{ij} + v_{1j}$	$v_{1i} + v_{Gj}$
8.	Local Bidder i	Global Bidder	Profit	Inefficient	$b_{ij} + v_{1j}$	$v_{1i} + v_{Gj}$
9.	Local Bidder i	Global Bidder	Profit	Efficient	$v_{2i} + v_{1j}$	$v_{1i} + v_{Gj}$
10.	Local Bidder i	Global Bidder	Profit	Inefficient	$v_{2i} + v_{1j}$	$v_{1i} + v_{Gj}$
11.	Local Bidder i	Local Bidder j	Zero Profit	Efficient	$b_{ij} + v_{Gj}$	$v_{1i} + v_{1j}$
12.	Local Bidder i	Local Bidder j	Zero Profit	Inefficient	$b_{ij} + v_{Gj}$	$v_{1i} + v_{1j}$
13.	Local Bidder i	Local Bidder j	Zero Profit	Efficient	$b_{ij} + v_{2j}$	$v_{1i} + v_{1j}$
14.	Local Bidder i	Local Bidder j	Zero Profit	Inefficient	$b_{ij} + v_{2j}$	$v_{1i} + v_{1j}$
15.	Local Bidder i	Local Bidder j	Zero Profit	Efficient	$v_{2i} + v_{Gj}$	$v_{1i} + v_{1j}$
16.	Local Bidder i	Local Bidder j	Zero Profit	Inefficient	$v_{2i} + v_{Gj}$	$v_{1i} + v_{1j}$
17.	Local Bidder i	Local Bidder j	Zero Profit	Efficient	$v_{2i} + v_{2j}$	$v_{1i} + v_{1j}$
18.	Local Bidder i	Local Bidder j	Zero Profit	Inefficient	$v_{2i} + v_{2j}$	$v_{1i} + v_{1j}$

Table 5: All possible outcomes in an ij auction when $N_i = N_j = 2$