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REVISITING CES UTILITY FUNCTIONS FOR DISTRIBUTIONAL PREFERENCES: DO PEOPLE FACE THE EQUALITY–EFFICIENCY TRADE-OFF?

Keigo Inukai
Yuta Shimodaira
Kohei Shiozawa

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The Institute of Social and Economic Research
Osaka University
6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan

Revisiting CES utility functions for distributional preferences: Do people face the equality–efficiency trade-off?

Keigo Inukai, Yuta Shimodaira, and Kohei Shiozawa*

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Abstract

While the constant elasticity of substitution (CES) utility function is widely used in the modified dictator game, a well-known shortcoming is that the distribution parameter loses its interpretation in the limit of the substitution parameter. This paper shows that previously proposed solutions for this problem necessarily have a similar mathematical shortcoming. Hence, we propose a new CES specification to address this problem and demonstrate its usefulness. Our results clarified empirical inconsistencies in the conventional analysis of measuring subjects' preferences. In particular, the conventional interpretation of the substitution parameter, "the equality–efficiency trade-off," is not based on subjects' behavior.

JEL classification: C91, D63, D64, D91

Keywords: Distributional preferences, Dictator game, Constant elasticity of substitution (CES), Equality, Efficiency

*Inukai: Faculty of Economics, Meiji Gakuin University, 1-2-37 Shirokanedai, Minato, Tokyo, 108-8636, Japan (email: inukai@eco.meijigakuin.ac.jp). Shimodaira: Institute of Social and Economic Research, Osaka University, 6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan (email: shimodaira@iser.osaka-u.ac.jp). Shiozawa: Faculty of Economics, Takasaki City University of Economics, 1300 Kaminamie-machi, Takasaki, Gunma, 370-0801, Japan (email: shiozawa@tcue.ac.jp). The authors are grateful to Taisuke Imai, Nobuyuki Hanaki, Takeshi Murooka, Masaru Sasaki, Shachar Kariv, and participants of 2021 ESA Global Online Around-the-Clock Meetings and the 24th Experimental Social Science Conference of Japan for helpful discussions. We gratefully acknowledge the financial support from a grant-in-aid for scientific research (KAKENHI; grant numbers: 17H04780, 18K19954, 19J12097, 19K21701, 19K23195, 20H05631, and 21H05070) from the Japan Society for the Promotion of Science. OnLine English (`www.oleng.com.au`) edited the manuscript to improve its readability.

1 Introduction

Modeling individual preferences is a central issue in understanding other-regarding behaviors observed in experiments. One of the most employed modeling strategies is an outcome-based model that focuses on the distribution of payoffs between the individual and the other (Fehr and Schmidt, 2006, Section 3.1; Cooper and Kagel, 2017, Section II). In particular, in modified dictator game experiments, the constant elasticity of substitution (CES) utility function of Arrow et al. (1961) has been the standard specification (Andreoni and Miller, 2002; Fisman et al., 2007).

The CES utility function encompasses typical forms of other-regarding preferences, such as Rawlsian inequality aversion, altruistic, and selfish preferences, by continuously using two parameters α—distribution parameter—and ρ—substitution parameter. Parameter α controls the relative importance of one's own and others' payoffs, and ρ controls the elasticity of substitution between these payoffs. These features help us study the heterogeneous nature of individual preferences concerning "equality" and "efficiency."[1]

Many papers have documented the heterogeneity of preferences and their association with other variables using this method (Andreoni and Miller, 2002; Fisman et al., 2007, 2014, 2015a,b, 2023; Li et al., 2017, 2022). It has also been utilized to study other aspects of other-regarding preferences with individual-level analysis (Andreoni, 2007; Korenok et al., 2013; Hong et al., 2015; Müller, 2019; Robson, 2021) and group-level analysis (Jakiela, 2013; Porter and Adams, 2016; Erkut, 2022; Brown et al., 2019). In other words, the CES utility function has been an essential component of experiments used as a measurement device for other-regarding preferences (Guala, 2008; Guala and Mittone, 2010; Fehr and Charness, 2023).

Despite its appeal, the CES utility function is known to have a theoretical difficulty. That is, in the limit of ρ corresponding to the Rawlsian maximin-type form (i.e., the Leontief function form), α becomes invalid in identifying differences in behavior and thus loses its natural interpretation. In the context of other-regarding preferences, this is a severe problem. As demonstrated in the present paper, this feature makes our measurement of the central concern, i.e., selfishness, equality-orientation, or generosity of behavior, incoherent and unstable. Furthermore, in the context of preference modeling in general, improving such aspects would be desirable.[2]

[1]The efficiency of distribution is an essential element of other-regarding preferences (Charness and Rabin, 2002; Engelmann and Strobel, 2004, 2006; Fehr et al., 2006; Fehr and Schmidt, 2006, Section 4.2; Cooper and Kagel, 2017, Section II.C).

[2]In this paper, we do not commit to what preference really is. However, we do commit that an essential purpose of our modeling practice is to explain behavior by a model of preferences (Guala, 2019, Section 4).

However, few studies have addressed this problem in this context.[3] Thöni (2015) suggested this ineffectiveness as an essential issue and proposed a variant of the CES formulation, but the issue itself has yet to be explored in detail. Moreover, as presented in Section 2, even Thöni (2015)'s solution suffers from essentially the same problem as the standard CES function. Becker et al. (2015) mentioned the problem but did not explore it in detail. The authors made an effort to enhance the analysis by combining two estimation methods. However, their solution, while resourceful, is somewhat ad hoc as it relies on a deterministic threshold value, a factor that was reportedly derived from some graphical inspections. Gauriot et al. (2020) discussed the problem in conventional group-level analysis with the CES utility function or equivalent functional systems. The authors proposed a novel utility function to generalize the CES function. However, its theoretical properties are less known than other well-studied CES variants.[4]

In this paper, we address this parameter validity problem of the CES utility function. First, we show that previously proposed variants of the CES function cannot solve this problem satisfactorily. To show this, we formulated a generalized class of CES utility functions that encompasses several CES specifications mentioned in Thöni (2015), such as Arrow et al. (1961), Senhadji (1997), and Thöni (2015)'s proposal itself. We then prove that a similar parameter validity problem, the invalidity of α in representing behavior, occurs more generally in this broad class.

Therefore, to circumvent the problem, we formulate the *extended CES utility function*. Intuitively, it combines the standard formulation of Arrow et al. (1961) and the variant of Senhadji (1997) to compensate for each other's disadvantages. We show that the novel formulation does not belong to the abovementioned class of CES functions and circumvents the problem. In other words, its distribution parameter, α, is always valid and consistently has straightforward interpretation in the limit of ρ. As a result, α describes our central concern, i.e., selfishness, equality-orientation, or generosity of subjects' behavior, more consistently than other CES specifications. Hence, it is a more appropriate formulation to measure preference heterogeneity.

Furthermore, we formulated a statistical model using the extended CES utility function.

Hence, if we say "heterogeneity of preferences," it means "heterogeneity of preferences and/or behavior."

[3] Note that, in macroeconomics, this problem has been recognized from the beginning in the context of production analysis (Arrow et al., 1961, p.231). It has been studied as a problem of normalization in the CES production function (Klump et al., 2012, Section 2; Embrey, 2019, Section 4).

[4] The authors proposed a generalization with two curvature parameters for the individual's and the other's payoffs. However, as some numerical simulations easily show, their solution makes what α is expected to represent more ambiguous. For example, it is possible to move the two curvature parameters, keeping $\alpha = 1/2$ fixed, to represent behaviors ranging from always spending equally to always allocating the most to oneself.

In the process, we argue that the other parameter, ρ, has a similar shortcoming to that of α and propose to solve it by changing the scale of ρ to another scale, τ. It describes subjects' efficiency-orientation more proportionally than the original scale.

We empirically demonstrate the appropriateness of our formulation and clarify the problem of the standard CES formulation using data from two experiments, Fisman et al. (2007) and Fisman et al. (2023).

The results clarify that the consequence of the invalidity of α in the standard CES function is not negligible. More precisely, in the standard CES model, some slight differences in subjects' mean expenditure shares may lead to significant differences in the estimate of α, and vice versa. This means that the estimate is inconsistent with subjects' behavior and makes the estimate unstable. Such inconsistency and instability that stem from the experimental design itself are not desirable features for scientific measurement (Tal, 2020, Section 7.1), and in particular, for an economic experiment used as a measurement device (Guala, 2008, Section 8). Moreover, we show that similar inconsistency and instability occur in the other parameter, ρ.

By contrast, the extended CES model mitigates such shortcomings. In the extended CES model, α and τ describe the two aspects of subjects' behavior separately and more consistently: α represents the other-regarding spectrum (selfishness, equality-orientation, or generosity), and τ represents efficiency-orientation. Notably, the consistency of α and τ casts doubt on the conventional interpretation of ρ—"the equality–efficiency trade-off." We argue that this interpretation is not objective because it does not reflect the nature of what is being measured (i.e., preferences). Hence, this interpretation is inappropriate for studying preferences. To sum up, the extended CES model should help strengthen the conclusions of applied studies by reducing estimate instability and enhance our understanding of distributional preferences by avoiding inconsistent interpretations.

In addition to the empirical applications, we conducted a parameter recovery simulation to examine the statistical behavior of the model (Wilson and Collins, 2019). Based on the results, we present specific points that should be considered by researchers using this method.

The remainder of the paper is organized as follows. Section 2 provides theoretical results regarding the shortcomings of previously proposed CES functions and describes our model and its advantages. Section 3 empirically explores the problems of the standard CES model and demonstrates the usefulness of our model. Section 4 presents the parameter recovery simulation. Section 5 concludes.

2 Theory

In this paper, we study CES utility models for distributional preferences in a modified dictator game (Fisman et al., 2007). In this methodology, each subject is asked to allocate monetary payoffs between themselves and another anonymous subject. Each allocation decision is made by choosing a pair of monetary payoffs (x_1, x_2) from the budget line $p_1x_1 + p_2x_2 = 1$, where x_1 and x_2 are the payoffs to the self (the subject) and the other (an anonymous subject), respectively. The ratio p_2/p_1 is the price of giving. Each subject makes 50 allocation decisions defined by different prices of giving. The experimental data are analyzed using the utility maximization model. We assume that there is a utility function $u(x_1, x_2)$ and that a payoff allocation is generated from the following maximization problem:

$$\begin{aligned} \max_{x_1,x_2} \quad & u(x_1, x_2) \\ \text{s.t.} \quad & p_1x_1 + p_2x_2 \leqq 1 \\ & x_1, x_2 \geqq 0. \end{aligned} \tag{1}$$

A crucial methodological decision is the specification of the parametric class of the utility function.

2.1 Utility function

We first describe two well-known formulations and their shortcomings. A usual choice has been the standard CES function of Arrow et al. (1961). It is defined as follows:

$$u(x_1, x_2) = (\alpha x_1^{-\rho} + (1-\alpha)x_2^{-\rho})^{-\frac{1}{\rho}} \tag{2}$$

where $\alpha \in (0, 1)$ is the *distribution parameter* and $\rho \in (-1, \infty) \setminus \{0\}$ is the *substitution parameter*. α controls the relative importance of the payoffs. ρ controls the curvature of the indifference curves and characterizes the *elasticity of substitution* $\sigma \in (0, \infty)$ as $\sigma = 1/(\rho + 1)$.

The standard CES utility function has three typical forms depending on the limit of the substitution parameter. Specifically, the utility function (2) tends to the *symmetric* Leontief utility function $u(x_1, x_2) = \min\{x_1, x_2\}$ as $\rho \to \infty$, the Cobb–Douglas utility function $u(x_1, x_2) = x_1^{\alpha}x_2^{1-\alpha}$ as $\rho \to 0$, and the linear utility function $u(x_1, x_2) = \alpha x_1 + (1-\alpha)x_2$ as $\rho \to -1$. Figure 1 visualizes these three limits.

It is well-known that special care must be taken when using this function for economic modeling (Arrow et al., 1961, p.231). As in the first case visualized in the left panel of Figure 1, the distribution parameter, α, becomes less effective at describing individual

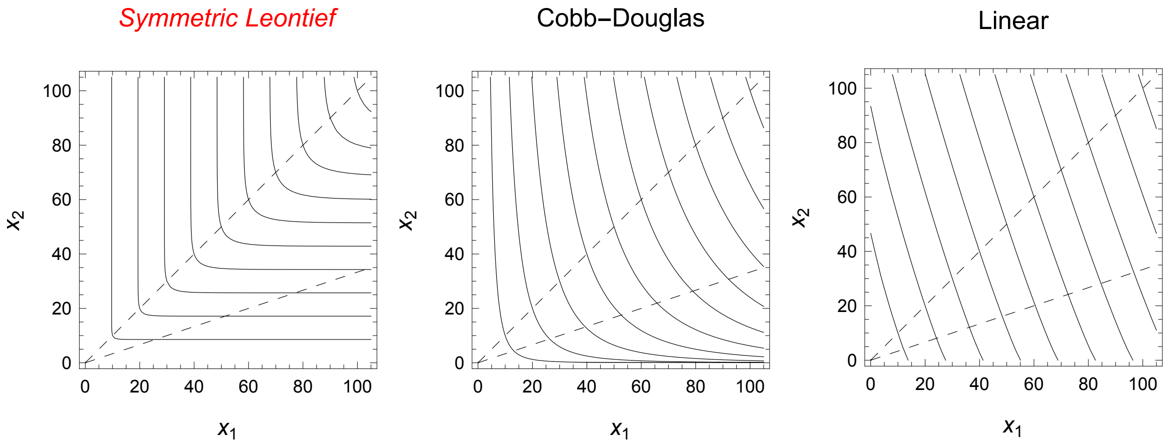

Figure 1: Indifference curves of the standard CES function of Arrow et al. (1961) with $\alpha = 0.75$. *Notes:* Each panel depicts an approximation of a limiting case: $\sigma = 0.1$, $\sigma = 1-10^{-3}$, and $\sigma = 10$ are depicted from left to right. Dashed lines depict equations $x_1 = x_2$ and $x_1/\alpha = x_2/(1-\alpha)$. Note that the Leontief case on the left approximates the *symmetric* Leontief function, while we set the distribution parameter *asymmetric*.

behavior as ρ approaches ∞. This can be an essential problem when studying altruistic preferences (Thöni, 2015). It means that the behavioral tendency to give some of one's money to others may not be represented by the parameter, even if it is present in observed behaviors. Indeed, as presented in Section 3, a non-negligible proportion of estimation results are affected by this feature.

Fortunately, a variant of the CES function overcomes this shortcoming. In Senhadji (1997), the variant was defined as follows:

$$u(x_1, x_2) = (\alpha^{1+\rho} x_1^{-\rho} + (1-\alpha)^{1+\rho} x_2^{-\rho})^{-\frac{1}{\rho}} \tag{3}$$

where $\alpha \in (0,1)$, $\rho \in (-1, \infty) \setminus \{0\}$. Figure 2 visualizes three limits of this formulation. This variant tends to a Leontief utility function $u(x_1, x_2) = \min\{x_1/\alpha, x_2/(1-\alpha)\}$ as $\rho \to \infty$. Moreover, for $\rho \to 0$, the limit is $u(x_1, x_2) = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}\ x_1^{\alpha} x_2^{1-\alpha}$, which is a strictly increasing transformation of the Cobb–Douglas utility function, and hence it is equivalent to the limit of the standard CES function.[5] Therefore, we can circumvent the well-known deficit of the standard CES function. However, unfortunately, this variant similarly has the essential problem in studying altruistic preferences. It tends to the *symmetric* linear utility function $u(x_1, x_2) = x_1 + x_2$ as $\rho \to -1$, as visualized in the right

[5]We say two utility functions are *equivalent* if one of them is a strictly increasing transformation of the other. Note that if this is the case, both represent the same preference relation and hence have the same optimal solution for the utility maximization problem (Mas-Colell et al., 1995, p.9).

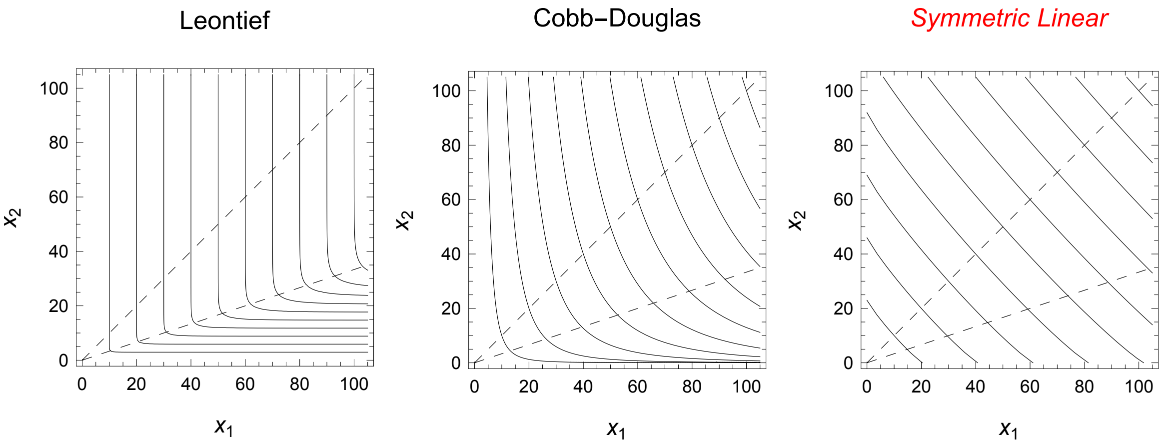

Figure 2: Indifference curves of the CES variant of Senhadji (1997) with $\alpha = 0.75$. *Notes:* As in Figure 1, cases $\sigma = 0.1$, $\sigma = 1 - 10^{-3}$, and $\sigma = 10$ are depicted from left to right. Note that the linear case on the right approximates the *symmetric* linear function while we set the distribution parameter *asymmetric*.

panel of Figure 2. Hence, α becomes ineffective as ρ approaches 1.

In fact, this shortcoming shared by the two CES specifications is a particular case of a more general common mathematical structure. To see this, let us devise a generalized form of the CES specifications. Consider an arbitrary affine function $f(\rho) = a\rho + b$. We generalize the two CES specifications as follows.[6]

$$u(x_1, x_2) = \left(\alpha^{f(\rho)} x_1^{-\rho} + (1-\alpha)^{f(\rho)} x_2^{-\rho}\right)^{-\frac{1}{\rho}} \tag{4}$$

where $\alpha \in (0,1)$ and $\rho \in (-1, \infty) \setminus \{0\}$. Then, this utility function can have *at most two* limiting forms that are equivalent to the general forms of the Leontief, Cobb–Douglas, or linear utility functions. The following result states this fact. The proof is in Appendix A.

Theorem 1. *Let $f(\rho) = a\rho + b$ for some real numbers a and b, and $\alpha \in (0,1)$. The utility function* (4) *cannot have three limiting forms that are equivalent to* $\min\{x_1/\alpha, x_2/(1-\alpha)\}$, $x_1^{\alpha} x_2^{1-\alpha}$, *and* $\alpha x_1 + (1-\alpha)x_2$ *as* $\rho \to \infty, 0$, *and* -1, *respectively.*

Therefore, to circumvent this general shortcoming, we need to formulate a utility function (4) from a wider class beyond the affine specification, $f(\rho) = a\rho + b$. Hence, we propose the following non-affine specification: $f(\rho) = \max\{0, \rho\} + 1$. It is equivalent to a class of utility functions formulated by combining the abovementioned two specifications

[6] Note that this form also generalizes two other well-known variants of Thöni (2015) and David and Van de Klundert (1965).

as:

$$u(x_1, x_2) = \begin{cases} (\alpha x_1^{-\rho} + (1-\alpha)x_2^{-\rho})^{-\frac{1}{\rho}} & \text{if } \rho < 0 \\ (\alpha^{1+\rho} x_1^{-\rho} + (1-\alpha)^{1+\rho} x_2^{-\rho})^{-\frac{1}{\rho}} & \text{if } \rho > 0 \end{cases} \tag{5}$$

where $\alpha \in (0,1)$ and $\rho \in (-1, \infty) \setminus \{0\}$. We call this the *extended CES utility function.*

Intuitive validity of this formulation is clear. As mentioned above, each of the two conventional formulations has its advantages and disadvantages in the mutually opposite limits of ρ. Therefore, if we combine the two properly, then we can utilize the advantages of the two while avoiding the disadvantages. Formally, it circumvents the abovementioned shortcoming as follows. The proof is in Appendix A.

Theorem 2. *Let* $\alpha \in (0,1)$ *and* $f(\rho) = \max\{0, \rho\} + 1$ *for any* $\rho \in (-1, \infty) \setminus \{0\}$*. Then, utility function* (4) *has the following limits:*

$$\lim_{\rho \to -1} u(x_1, x_2) = \alpha x_1 + (1-\alpha) x_2 \tag{6}$$

$$\lim_{\rho \to 0^-} u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha} \tag{7}$$

$$\lim_{\rho \to 0^+} u(x_1, x_2) = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} x_1^{\alpha} x_2^{1-\alpha} \tag{8}$$

$$\lim_{\rho \to \infty} u(x_1, x_2) = \min\left\{\frac{x_1}{\alpha}, \frac{x_2}{1-\alpha}\right\} \tag{9}$$

Note that although the two one-sided limits do not coincide, both are equivalent to the general Cobb–Douglas utility function $x_1^{\alpha} x_2^{1-\alpha}$. Hence, if we define the combined formulation (5) with the case $\rho = 0$ by $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, then it has a demand function that is continuous with respect to ρ, including the case $\rho \to 0$ (Proposition 1 in Appendix A and subsequent remarks). Clearly, this demand function tends to that of the general form of the linear, Cobb–Douglas, and Leontief utility functions. Hence, unlike all of the CES variants in the class (4) with $f(\rho) = a\rho + b$, the behavioral interpretation for α is standard and clear in three limiting cases.[7] First, in the linear function case, α characterizes the degenerate allocation for the self as $p_1 x_1 = 1$ if $p_1/p_2 \leqq \alpha/(1-\alpha)$ and $p_1 x_1 = 0$ otherwise. Second, in the Cobb–Douglas function case, it characterizes the ratio of expenditure shares, i.e., $p_1 x_1 / p_2 x_2 = \alpha/(1-\alpha)$. Finally, in the Leontief function case, it characterizes the ratio of payoffs as $x_1/x_2 = \alpha/(1-\alpha)$.

[7] In Thöni (2015), another variant was proposed as a remedy for the shortcomings of (2) and (3). However, as implied by Theorem 1, it has another undesirable consequence. It does not coincide with the classical Cobb–Douglas utility function $x_1^{\alpha} x_2^{1-\alpha}$ when $\rho \to 0$. Hence, we adopt our combined formulation rather than Thöni (2015)'s proposal to maintain the classical interpretation of α in the Cobb–Douglas case, which is the very reason why it is called the distribution parameter (Arrow et al., 1961, Footnote 10). See Appendix A.1 for more details.

A limitation of this formulation should also be noted. As can easily be seen from the definition of the utility function (5), in the purely selfish case, i.e., $\alpha \to 1$, the substitution parameter, ρ, becomes meaningless.[8] This limitation is not the same problem as the meaninglessness of α that we argued above. Indeed, if $\alpha \to 1$, then the idea behind the substitution parameter itself is meaningless because purely selfish behavior cannot be sensitive to price changes in principle. By contrast, if $\rho \to \infty$, then the idea behind the distribution parameter *can* be meaningful because perfectly complementary behavior *can* be asymmetric, as shown in Figure 2 by Senhadji (1997)'s formulation.[9] The meaninglessness of ρ in the limit of α is not a problem posed only by our extension, but is a drawback associated with the use of CES-type parameterization in general. In the following, we also report the empirical implications of this fact to clarify the limitation.

2.2 Statistical model

We formulate a Bayesian statistical model to estimate the extended CES utility function. It is an extension of the standard method based on the expenditure share form of the demand function with an additive error.

2.2.1 Demand function

For simplicity, we restrict the utility maximization problem as follows (Appendix A.2). Let $c > 0$ be an arbitrarily small number. Consider a utility maximization problem specified by replacing the non-negativity constraint in (1) by $x_1, x_2 \geqq c$. Because c can be arbitrarily small, it is unlikely that this modification affects the conclusions of a given study in this context. Then, the optimal expenditure share is derived as follows:

$$s^* = p_1 x_1^* = \begin{cases} 1 - p_2 c & \text{if } \left(\frac{\alpha}{1-\alpha}\right)^{f(\rho)} \geqq \frac{p_1}{p_2}\left(\frac{1-p_2 c}{p_1 c}\right)^{\rho+1}, \\ p_1 c & \text{if } \left(\frac{\alpha}{1-\alpha}\right)^{f(\rho)} \leqq \frac{p_1}{p_2}\left(\frac{p_2 c}{1-p_1 c}\right)^{\rho+1}, \\ \dfrac{\left(\frac{\alpha}{1-\alpha}\right)^{\frac{f(\rho)}{\rho+1}}}{\left(\frac{\alpha}{1-\alpha}\right)^{\frac{f(\rho)}{\rho+1}} + \left(\frac{p_2}{p_1}\right)^{\frac{\rho}{\rho+1}}} & \text{otherwise} \end{cases} \tag{10}$$

[8]Although the exact opposite case ($\alpha = 0$) raises similar issues, here we only discuss $\alpha = 1$, which typically appears in the context of altruistic preferences.

[9]Moreover, if we argue that it is desirable to accept the meaninglessness of α when $\rho = \infty$ based on the prescription of a specific utility function (the standard CES specification), then we can also argue that α should also be meaningless when $\rho = -1$ as prescribed by another specific utility function (Senhadji (1997)'s formulation). Similarly, if we argue that the meaningfulness of α is desirable when $\rho = -1$ based on the standard CES specification, then α should also be meaningful when $\rho = \infty$ based on Senhadji (1997)'s formulation. We believe (and demonstrate in the present paper) that the latter argument is more beneficial for our scientific understanding of subjects' preferences.

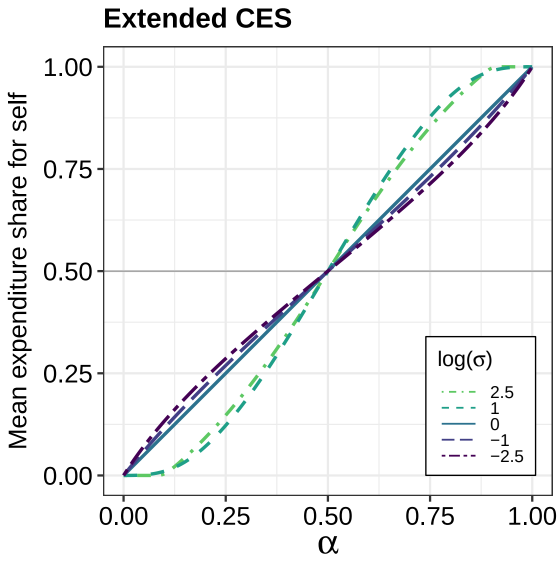

Figure 3: Mean expenditure share of the extended CES function. *Notes:* Each line shows mean expenditure shares with a fixed ρ and various α where $\rho = 1/e^{\log(\sigma)} - 1$ and $\log(\sigma) \in \{2.5, 1, 0, -1, -2.5\}$. We used $\log(\sigma)$ to symmetrically present the limiting cases (Figure 5). For each pair of parameters, expenditure shares are calculated by (10) with $f(\rho) = \max\{0, \rho\} + 1$ and a fixed set of 50 price systems, and we calculated the sample mean of these shares.

where $\alpha \in (0, 1)$, $\rho \in (-1, \infty)$, and $f(\rho) = \max\{0, \rho\} + 1$. The main part is the interior solution presented in the last case. It can be expressed more intuitively using the elasticity of substitution parameter σ as follows.[10]

$$\frac{\left(\frac{\alpha}{1-\alpha}\right)^{\frac{f(\rho)}{\rho+1}}}{\left(\frac{\alpha}{1-\alpha}\right)^{\frac{f(\rho)}{\rho+1}} + \left(\frac{p_2}{p_1}\right)^{\frac{\rho}{\rho+1}}} = \begin{cases} \frac{\left(\frac{\alpha}{1-\alpha}\right)^{\sigma}}{\left(\frac{\alpha}{1-\alpha}\right)^{\sigma} + \left(\frac{p_2}{p_1}\right)^{1-\sigma}} & \text{if } \sigma \geqq 1 \\ \frac{\left(\frac{\alpha}{1-\alpha}\right)}{\left(\frac{\alpha}{1-\alpha}\right) + \left(\frac{p_2}{p_1}\right)^{1-\sigma}} & \text{if } \sigma < 1 \end{cases} \tag{11}$$

where $\sigma = 1/(\rho + 1) \in (0, \infty)$. In this equation, the first case is the optimal expenditure share of the standard CES function, and the second case is that of Senhadji's variant. Figure 3 visualizes the optimal expenditure share.

Figure 3 shows that the mean expenditure share is roughly proportional to the distribution parameter, α, regardless of the value of the substitution parameter, ρ.[11] Because the mean expenditure share is a frequently used and straightforward measure to summarize subjects' distributional preferences, especially subjects' selfishness, equality-orientation, or generosity, mutual consistency with it can be a normative requirement to

[10] In the following, we mainly use the elasticity of substitution parameter, σ. The Leontief, Cobb–Douglas, and linear cases correspond to $\sigma \to 0, 1$, and ∞, respectively (Figure 5).

[11] Here, ρ is transformed into $\log(\sigma)$ for symmetric presentation. See Figure 5.

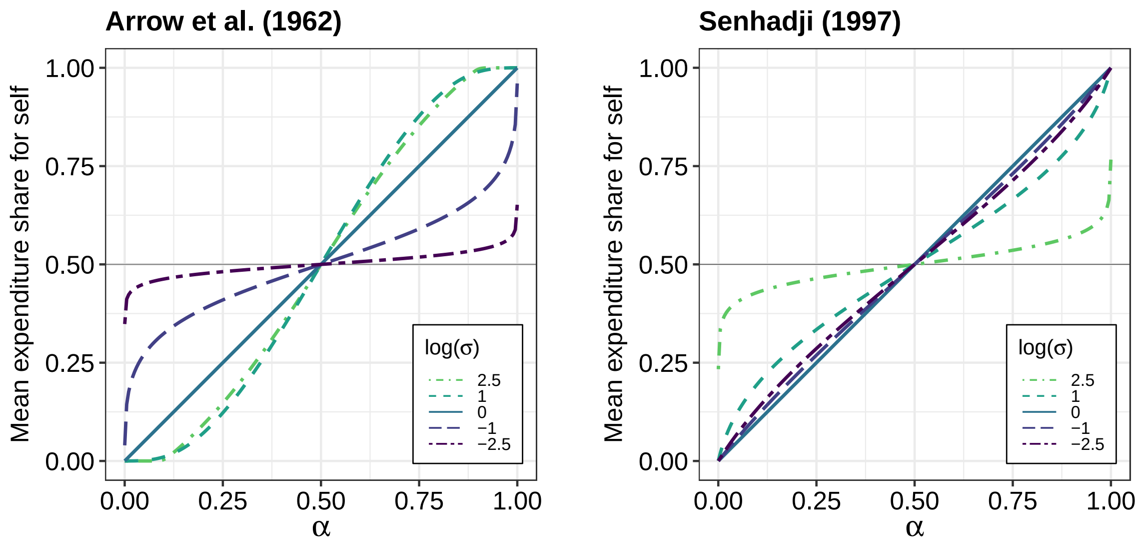

Figure 4: The expenditure share of the CES functions. *Notes:* Each line is obtained by the same calculation method as in Figure 3. **Left**: The standard CES of Arrow et al. (1961), i.e., we used the equation (10) with $f(\rho) = 1$. **Right**: The CES variant of Senhadji (1997), i.e., we used (10) with $f(\rho) = 1 + \rho$.

"ensure that measurement outcomes can be reasonably attributed to the measured object rather than to some artifact of the measuring instrument, environment or model" (Tal, 2020, p.33).

By contrast, the standard CES function of Arrow et al. (1961) is not mutually consistent with the mean expenditure share. As shown in the left panel of Figure 4, α is insensitive to the described behavior for some values of ρ. For example, when ρ roughly corresponds to the Leontief function (the case $\log(\sigma) = -2.5$), for a vast majority of α, the demand function describes behavior similar to that of the symmetric Leontief function (with a mean share of around 0.5). Moreover, intermediate behaviors, such as one with a mean share of around 0.75, are stuck in a narrow region of α at the right end. Therefore, the described behavior changes highly nonlinearly as $\alpha \to 1$. Hence, it is difficult to infer behavioral variation from α. On the one hand, a significant difference between $\alpha = 0.25$ (a "generous" value) and $\alpha = 0.75$ (a "selfish" value) means very slight behavioral change. On the other hand, a slight difference between $\alpha = 0.95$ and $\alpha = 0.975$ means a significant shift in behavior. The CES variant of Senhadji (1997) also has this undesirable nonlinearity on the opposite side of the substitution parameter (the right panel of Figure 4).[12] The

[12]We also show similar visualizations for the CES variants of Thöni (2015) and David and Van de Klundert (1965) in Figure 22 in Appendix B. In particular, that for Thöni (2015) shows a shortcoming in the Cobb–Douglas case, i.e., the distribution parameter α is not equal to the mean share of the behavior,

extended CES function clearly mitigates such undesirable features.

Here, note that our primary focus is not on predictive accuracy or goodness of fit but on the validity of the scale of α as a measure of preferences. Indeed, a choice of these CES variants (i.e., the choice of $f(\rho)$) amounts to a choice of a specific scale of α, as suggested by the abovementioned figures and formally shown by Proposition 2 in Appendix A.2. Therefore, our primary focus is not predictive accuracy because models can fit exactly the same data with different values of α, as demonstrated empirically in Section 3.

In addition, note that our point is relevant for *group-level* analysis. Indeed, our argument so far does not depend on the number of allocation decisions. Hence, even if we have a large dataset of pooled allocation decisions, the consequence of the invalidity of α applies. To make matters worse, in a group-level analysis, we have only two or three estimates in the study. Therefore, the invalidity of α can significantly change the study's conclusion in a group-level analysis, even more so than in an individual-level analysis.

2.2.2 Residual error term distribution

A truncated normal distribution was used for the residual error term ϵ. It is assumed that an observation of the expenditure share s is distributed on the unit interval in proportion to the normal distribution with location parameter s^* and scale parameter σ_ϵ, i.e., $s = s^* + \epsilon$, where $\epsilon \in [-s^*, 1 - s^*]$ is distributed in proportion to the normal distribution with mean 0 and variance σ_ϵ^2. This formulation leads to a natural interpretation that the expenditure share of an allocation decision is likely to be around the theoretical expenditure share s^*.[13]

2.2.3 Prior distributions for parameters α, σ, and σ_ϵ

For the distribution parameter, α, the uniform distribution on the unit interval is used as a non-informative prior distribution.

For the elasticity of substitution parameter, σ, we specified a prior distribution on $\log(\sigma)$, rather than on σ, to symmetrically treat the three limiting cases. The symmetry is visible if we see the parameters in the $(\sigma, \rho + 1)$ plane (Figure 5). In the original scale, σ, the Leontief function is placed at zero and the linear utility function at infinity,

and it disturbs the classical and important interpretation of α.

[13]In the literature, a censored normal distribution, i.e., the two-limit Tobit model, has been used for the distribution of the error term. However, it leads to an unnatural interpretation when the scale σ_ϵ is moderately large. For example, when $s^* = 0.8$ and $\sigma_\epsilon = 0.1$, the expenditure share is more likely to be in the right corner, i.e., $s = 1$, instead of a point closer to the theoretical share, i.e., $s = 0.8$. For this reason, we used the truncated normal distribution. Note that the truncation model is also standard in the relevant literature (Andreoni et al., 2013; Ahn et al., 2014; Echenique et al., 2020, Online Appendix).

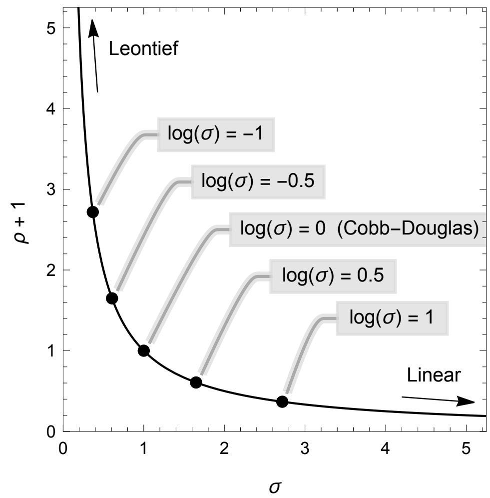

Figure 5: Symmetric reparameterization $\log(\sigma)$. *Notes:* Equation $\rho + 1 = 1/\sigma$ and points $(\sigma, \rho + 1)$ corresponding one-to-one to values $\log(\sigma) \in \{-1, -0.5, 0, 0.5, 1\}$ are depicted.

treating both asymmetrically. By contrast, this reparameterization allows us to treat them symmetrically by placing the Leontief case at negative infinity and the linear case at positive infinity, with the Cobb–Douglas case at the center.

The prior distribution for $\log(\sigma)$ was set to Student's t-distribution with degrees of freedom $\nu = 4$. Figure 6 shows the reason. Student's t-distribution prior on $\log(\sigma)$ (left panel) can treat each of the three limits more uniformly than can the *uniform* prior (right panel). The uniform distribution on $\log(\sigma)$ is not *non-informative* here. Instead, it unintentionally emphasizes the Leontief and linear functions, as seen in the right panel.[14]

Finally, for the scale parameter σ_ϵ, the exponential distribution with rate parameter $\lambda = 0.5$ was adopted. Because the expenditure share form (10) is always in the unit interval, the error term with $\sigma_\epsilon > 1.5$ is large enough to even mimic the uniform random selection model in Bronars (1987). Moreover, the 0.95-quantile and 0.99-quantile of the exponential distribution with $\lambda = 0.5$ are about 1.5 and 2.3, respectively. Hence, it assigns a positive probability value even to almost entirely random choice models and is not very restrictive. It also allows us to keep the model parsimonious (Gelman et al., 2017).

[14]Student's t-distribution with other ν, such as the Cauchy distribution ($\nu = 1$) or the normal distribution ($\nu = \infty$), can be a candidate for the prior distribution. Here, we adopted $\nu = 4$ to make the prior distribution weakly informative and, at the same time, to make the model parsimonious (Gelman et al., 2017).

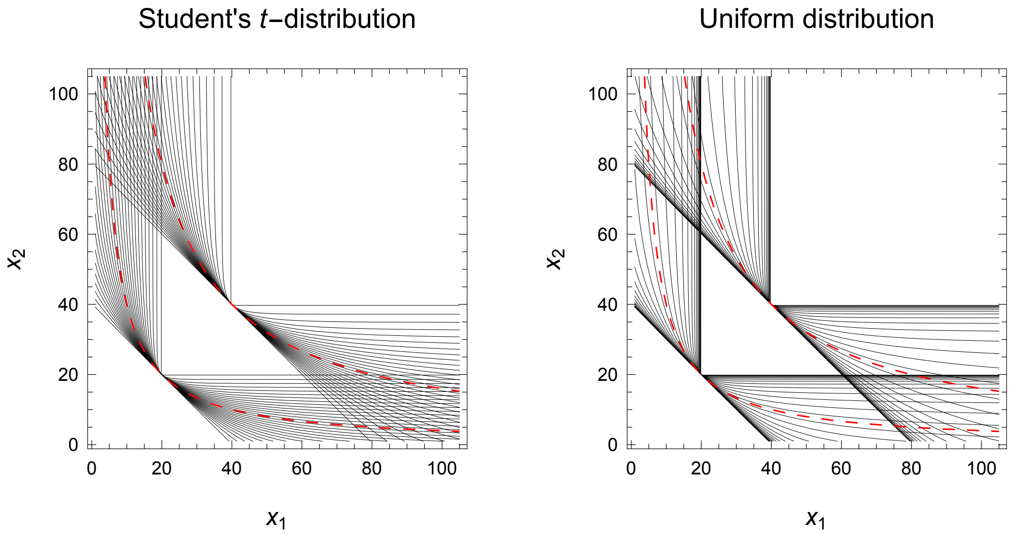

Figure 6: Prior distributions on $\log(\sigma)$. *Notes:* Indifference curves of the extended CES functions with $\alpha = 0.5$ and 30 different values of $\log(\sigma)$ (solid lines), and that of the Cobb–Douglas utility function (dotted lines) are depicted. **Left:** $\log(\sigma)$ are taken as r_i-quantile points ($i = 1, \ldots, 30$) of the t-distribution with $\nu = 4$, where r_i are points that divide interval $[0.005, 0.995]$ into 29 intervals of equal length. **Right:** For comparison, we visualized a uniform distribution of $\log(\sigma)$ between r_1 and r_{30} that we used in the left panel. Values of $\log(\sigma)$ are $k/29$-quantile points ($k = 0, \ldots, 29$) of this uniform distribution.

2.3 Rescaling $\log(\sigma)$ to τ

Although the statistical model was formulated on the parameters $\alpha, \log(\sigma)$, and σ_ϵ, we further transform $\log(\sigma)$ to τ by the cumulative distribution function of Student's t-distribution with $\nu = 4$. Formally, let τ be defined as:[15]

$$\tau = F_4\big(\log(\sigma)\big) = \frac{1}{2} + \frac{\log(\sigma)\big(\log(\sigma)^2 + 6\big)}{2\big(\log(\sigma)^2 + 4\big)^{\frac{3}{2}}} \tag{12}$$

where $\log(\sigma) \in \mathbb{R}$. We call τ the *rescaled substitution parameter* or, with some abuse of notation, the *substitution parameter*. The Leontief, Cobb–Douglas, and linear limits correspond to $\tau \to 0, 1/2$, and 1, respectively.

This rescaling has two methodological advantages. First, τ is in the unit interval $(0, 1)$, the same as α. Using the same scale helps us compare the variability of α and τ in the analysis. Second, τ is a more balanced measure of subjects' behavior than $\log(\sigma)$. Indeed, as shown in Figure 6, F_4 can map the three limits of the CES function on the unit interval

[15] See, for example, Shaw (2006).

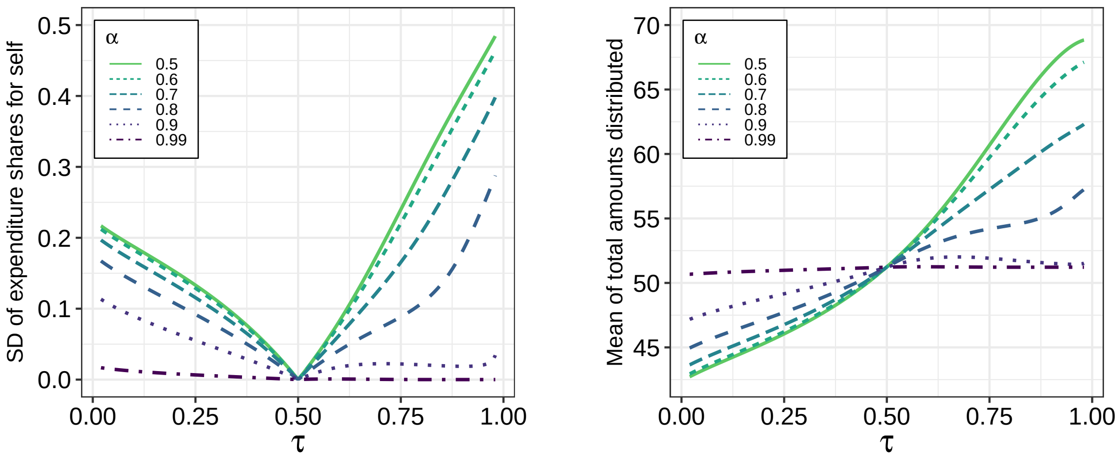

Figure 7: Relationship between τ and the described behavior. *Notes:* Each line corresponds to a fixed α and various τ. **Left:** The standard deviation (SD) of expenditure shares. For each pair of parameters, optimal expenditure shares are calculated by (10) with $f(\rho) = \max\{0, \rho\} + 1$ with a fixed set of prices, and we calculated the sample standard deviation of them. **Right:** The mean of total amounts distributed. We calculated the total amounts of optimal distribution, $x_1^* + x_2^*$, and took their sample mean.

in a balanced way.[16] Moreover, Figure 7 demonstrates this point in terms of the described behavior. The left panel shows the relationship between τ and the sample standard deviation of expenditure shares, which is a simple proxy for the sensitivity of behavioral change with respect to price changes.[17] Starting from the Cobb–Douglas case, $\tau = 1/2$, τ maps changes in behavior approximately proportionally toward two extreme cases: the Leontief and linear cases. Moreover, in the right panel, a clear relationship between τ and the total amount of the optimal distribution, $x_1^* + x_2^*$, is shown. It shows an approximately proportional relationship; hence, τ is a good measure of "efficiency-orientation."

By contrast, the original scales, such as ρ and σ, have much more complex relationships to the described behavior (Figure 8).[18] Problematically, σ has a wide range that describes little difference in behavior, and vice versa. It exaggerates some differences in behavior

[16]The prior distribution set to $\log(\sigma)$ can also be interpreted as a uniform prior distribution on the parameter τ. Moreover, our Bayesian method can be modified to a maximum likelihood method or a least squares method (Gelman et al., 2013, Section 4.5).

[17]At $\tau = 1/2$, the model corresponds to the Cobb–Douglas case, and hence the expenditure share is always α regardless of the price system, i.e., the standard deviation is 0 at $\tau = 1/2$ for any α. The increase in the standard deviation for $\tau < 1/2$ comes from the fact that behavior becomes inelastic in the Leontief case, i.e., one satisfying $x_1/x_2 = \alpha/(1-\alpha)$, so that the *expenditure share* varies under various price systems.

[18]Here, we show σ instead of ρ because ρ corresponds to τ in reverse order. ρ and $\log(\sigma)$ exhibit similar complex relationships as shown in Figure 23 in Appendix B.

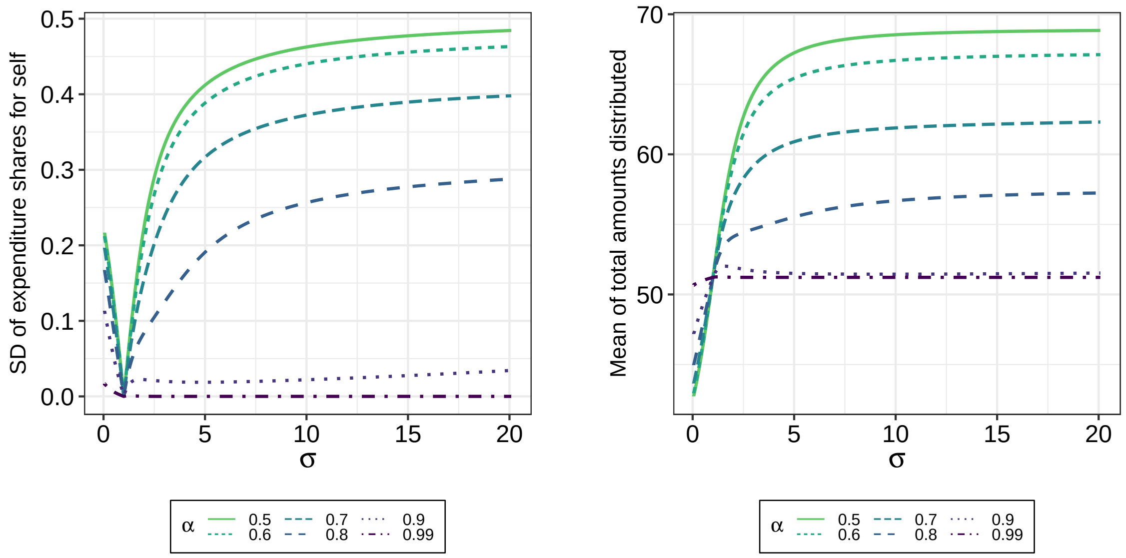

Figure 8: Relationship between σ and the described behavior. *Notes:* The graphs correspond to Figure 7, but the horizontal axes show values of σ. Parameter values used correspond one-to-one to those of Figure 7.

and ignores others: the same inappropriateness with α in the standard CES model.[19] Moreover, these figures clarify a reason behind the somewhat ad hoc treatment of ρ found in the literature: some studies focus on arbitrarily truncated ranges, such as $-1 < \rho < 1$ or $-1 < \rho < 2$.[20] By contrast, τ maps the behavior proportionally on the unit interval, so such arbitrariness is not necessary.

3 Empirical Illustration

To illustrate the empirical usefulness of our model, we applied it to the experimental data from Fisman et al. (2007) and Fisman et al. (2023). We also estimated the standard CES version of our model, that is, a model specified by the standard CES utility function (2) and the statistical assumptions described in Section 2.2, for comparison.[21] The two

[19]The original scales are rooted in a differential equation that describes indifference curves using an economic term (Arrow et al., 1961; Klump et al., 2012). Given this origin, it is natural that they are helpful in mathematical analysis and interpreting one behavior in absolute economic terms but not in coherent relative comparisons of various behaviors. They are not directly designed for such purposes.

[20]Furthermore, ρ and σ should increase the standard error as their values become high. Hence, the difference in the variance of estimated ρ between samples, such as the one reported in Müller (2019), can potentially be interpreted as the difference in the mean.

[21]Parameter estimation was conducted by simulating the posterior distribution of parameters by Markov chain Monte Carlo (MCMC) methods. The analysis was implemented in R (R Core Team, 2016) using MCMC methods in Stan (Carpenter et al., 2017; Stan Development Team, 2022).

experiments differ in the number of subjects, 76 and 687. In each experiment, a dataset consists of 50 allocation decisions. In Fisman et al. (2023), the subjects are asked to make 50 allocation decisions at two different time points. Because we analyzed the data in this demonstration without paying attention to time-invariant individual factors, there are 1,374 (= 687 × 2) datasets for Fisman et al. (2023).

3.1 Comparison of the distribution parameter α

First, we examine the results for α. Figure 9 shows the relationship between the estimated α and the observed mean expenditure shares for the self in Fisman et al. (2023). Note that it is an empirical counterpart of the demand structures depicted in Figures 3 and 4.

As we see in the top-left panel, the estimated α for the extended CES model, $\hat{\alpha}_e$, tracks the mean expenditure share reasonably well for each $\hat{\tau}$. By contrast, for the standard CES model (top-right panel), the behavior of $\hat{\alpha}_s$ is more complex and nonlinear when $\hat{\tau} \leqq 0.5$. If $\hat{\tau} \leqq 0.25$ and the mean expenditure share is around 0.5, then $\hat{\alpha}_s$ are highly variable. This is clearly implied by the nonlinear demand function depicted in Figure 4, i.e., if $\hat{\tau} \leqq 0.25$ and the mean expenditure share is around 0.5, then many α's are consistent with the observed behavior. Moreover, if the mean expenditure share is slightly different from 0.5, then $\hat{\alpha}_s$ are concentrated around the corresponding extreme value. This is also an empirical consequence of the demand structure.

The difference between the two models is directly shown in the bottom panel. As expected from the definition, the two models differ only when $\hat{\tau} \leqq 0.5$. The instability and inconsistency of $\hat{\alpha}_s$ needlessly disturb the consistent interpretation that is kept within the other half of the parameter space, $\hat{\tau} > 0.5$. The problem is noticeably mitigated in the extended CES model, thanks to the approximately proportional demand structure depicted in Figure 3.

Figure 10 shows essentially the same results for the data from Fisman et al. (2007). The visualized pattern is consistent with that of Figure 9, though the difference between the two models is less clear because the number of subjects is relatively small. Moreover, the variability of $\hat{\alpha}_s$ when $\hat{\tau} \leqq 0.25$ seems much smaller because no subjects distributed, on average, more to the other subject. Hence, in such a fortunate situation, the problem of the standard CES model can remain minor or even potential.

Finally, Figure 11 shows how this highly nonlinear variation of $\hat{\alpha}_s$ occurs in subjects with low $\hat{\tau}_s$. We handpicked three subjects in Fisman et al. (2007) to clarify the structure. Consistent with Figure 10, the estimate of the extended CES model, $\hat{\alpha}_e$, is proportionally associated with the observed behavior. By contrast, the association between $\hat{\alpha}_s$ and the observed behavior is complex. First, as the left panel (ID 16) shows, if the observed choices are around the diagonal line and some are slightly tilted toward the self, then $\hat{\alpha}_s$

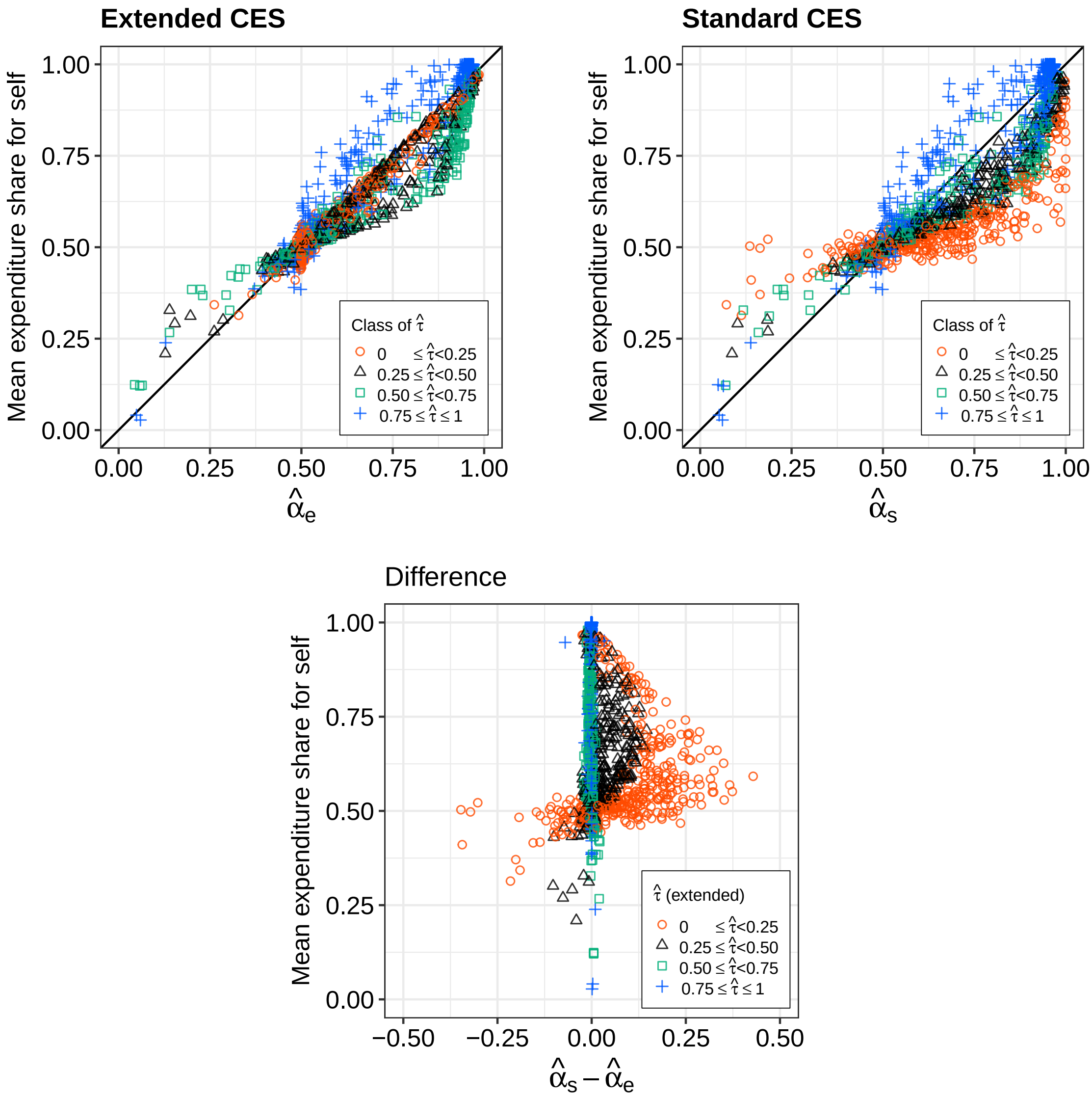

Figure 9: Estimated α for the data from Fisman et al. (2023). *Notes:* **Top:** The posterior mean estimates $\hat{\alpha}_e$ (for the extended CES model), $\hat{\alpha}_s$ (for the standard CES model), and the sample mean of the observed expenditure share for the self are presented. The solid lines are the diagonal lines. **Bottom:** The difference $\hat{\alpha}_e - \hat{\alpha}_s$ and the sample mean of the observed expenditure share for the self are depicted. $\hat{\alpha}_e$ and $\hat{\alpha}_s$ denote the estimates for the extended and the standard CES models, respectively. Class of $\hat{\tau}$ is defined by the estimated value of the extended CES model.

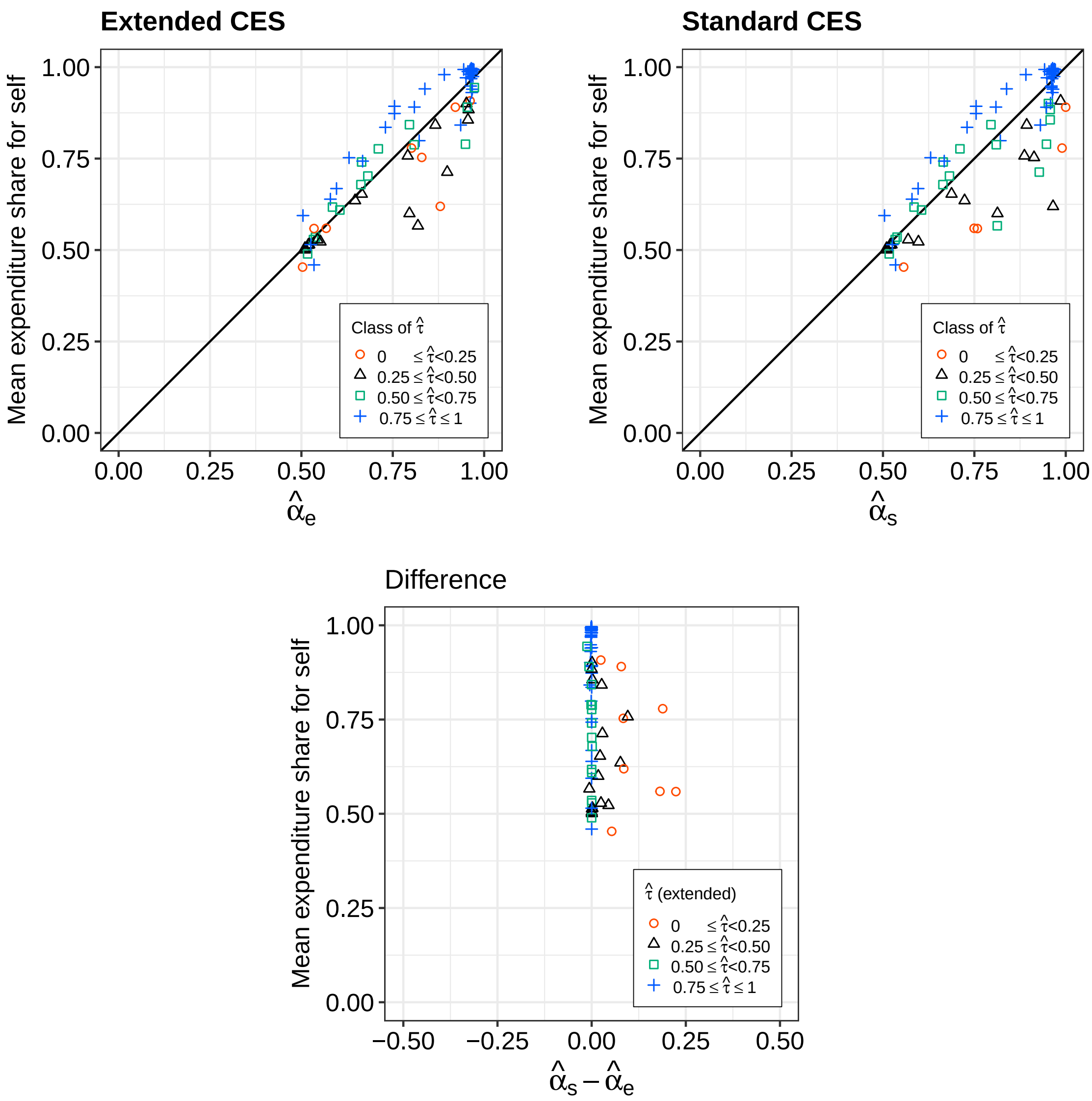

Figure 10: Estimated α for the data from Fisman et al. (2007). *Notes:* Each plot corresponds to that of Figure 9.

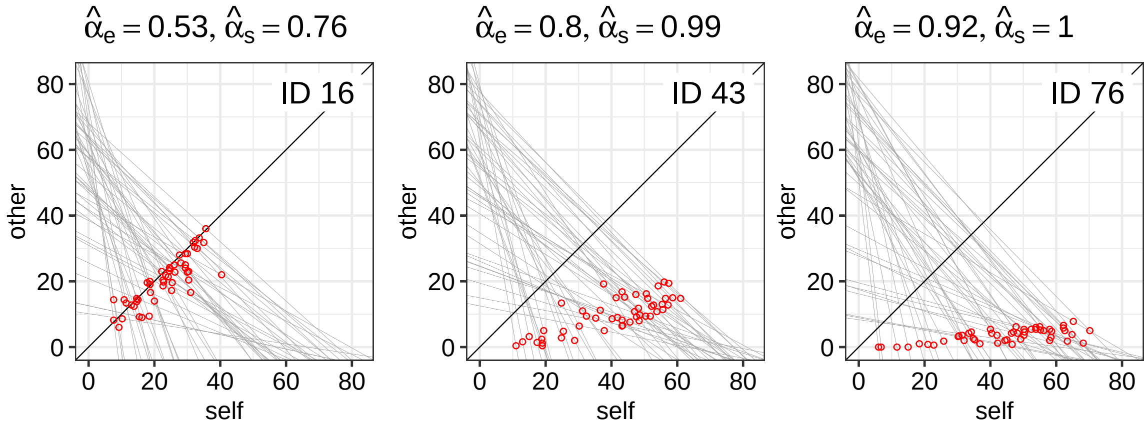

Figure 11: Comparison between estimates and the observed behavior. *Notes:* For each panel, 50 budget lines (gray line) and the observed choices of an individual subject (red circle) are depicted. The posterior mean estimates $\hat{\alpha}_e$ and $\hat{\alpha}_s$ are indicated at the top, and the individual's ID is indicated in the panel. The diagonal line is indicated by the solid black line.

is markedly larger than 0.5. Because the slight tilt can be toward either side, $\hat{\alpha}_s$ becomes unstable when the mean expenditure share is around 0.5 and $\hat{\tau}_s$ is low. Next, as the middle and right panels (ID 43 and 76) show, if the choices are not on the diagonal line, then $\hat{\alpha}_s$ shrinks around $\alpha = 1$ because, in the standard CES model, almost all α represent the behavior that distributes on the diagonal line. Thus, there is no room in $\hat{\alpha}_s$ to express the apparent difference in the choices of the two subjects. The extended CES model resolves such complex association by making α meaningful in the Leontief case, i.e., when τ is low.

3.2 Comparison of the transformed substitution parameter τ

First, we show the empirical usefulness of τ, i.e., the scale transformations described in Section 2.3. Figure 12 shows the relationship between the estimated τ and the sample mean of total amounts distributed, i.e., an empirical counterpart of Figure 7.[22] $\hat{\tau}$ is roughly proportionally associated with the total amount distributed, and hence it is a sound measure for "efficiency-orientation." Consistent with the theoretical demand in Figure 7, if $\hat{\alpha} \approx 1$, then $\hat{\tau}$ has little information on the difference in the observed behaviors. Hence, excluding subjects who have extreme values of $\hat{\alpha}$ can be beneficial when using it as a measure for "efficiency-orientation." By contrast, the original scales, such as ρ and σ, are less sound as a measure for "efficiency-orientation" (Figure 13).[23] Even in a truncated

[22]We examine the result for Fisman et al. (2023) here because similar observations hold for Fisman et al. (2007).

[23]We used $\hat{\sigma}$ instead of $\hat{\rho}$ here because $\hat{\tau}$ correspond to the latter in reverse order.

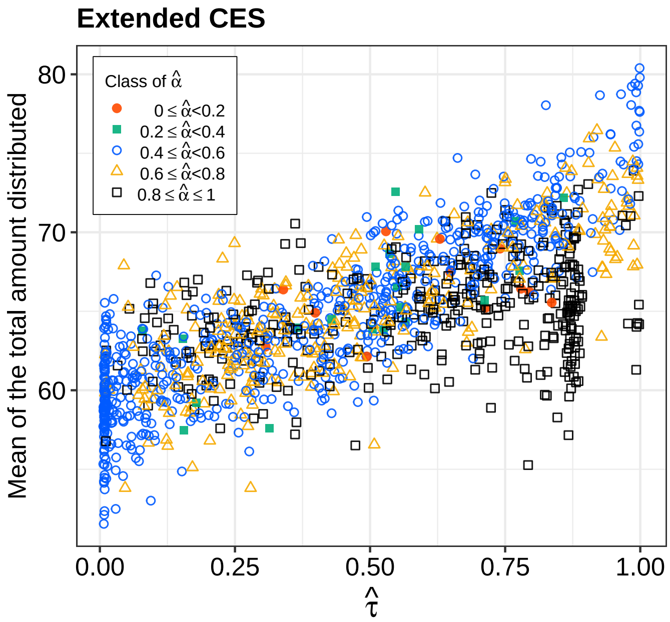

Figure 12: Estimated τ for the extended CES model. *Notes:* The posterior mean estimates $\hat{\tau}$ of the extended CES model and the sample mean of the total amount distributed are presented.

visualization (left panel), the association between $\hat{\sigma}$ and the total amount distributed is highly nonlinear and complex, and the situation is far worse in the non-truncated sample (right panel). Almost all variation in the total amount corresponds to a relatively small range $\hat{\sigma} \leqq 5$. $\hat{\tau}$ has no such nonlinearity and does not need any truncation for further analysis and visualization. Next, we examine the differences between the standard and extended CES models (Figure 14). The main difference lies in subjects with $\hat{\alpha} \geq 0.8$ or $\hat{\alpha} \leq 0.2$. As these points are in area $\hat{\tau}_s - \hat{\tau}_e > 0$, the estimates of the extended CES model are relatively smaller than those of the standard CES model. This difference is a by-product of the nonlinear demand with respect to α in the standard CES model (the left panel of Figure 4) and its absence in the extended CES model (Figure 3). Figure 15 visualizes this point with a subject we handpicked from Fisman et al. (2007). As can be seen in the right panel, in the standard CES model, the posterior distribution of τ is not in the area $\tau < 0.5$. This is because most of (α, τ) in this area correspond to the *symmetric* Leontief function, and hence α must be very close to 1 to be consistent with the observed "selfish" behavior presented in the left panel. By contrast, in the extended CES model, τ is consistent with the area $\tau < 0.5$ because α expresses the "selfish" tendency of the behavior regardless of the value of τ. Therefore, the posterior distribution of τ is broader, and hence its posterior mean is smaller.

Note that the wide posterior distribution of the extended CES model is not a drawback; it adequately represents uncertainty. Indeed, the observed "selfish" behavior should be consistent with a wide range of τ because τ is meaningless in the limit of α in any

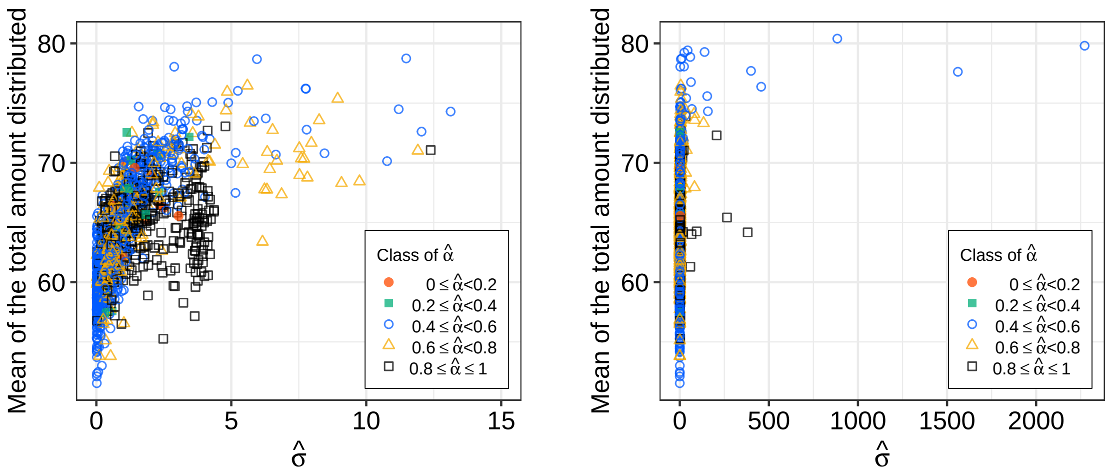

Figure 13: Estimated σ of the extended CES model. *Notes:* The posterior mean estimates $\hat{\sigma}$ of the extended CES model and the sample mean of the total amount distributed are presented. We present estimates with truncation (left panel) and without truncation (right panel).

CES-type parameterization (Section 2.1). The relatively narrow posterior distribution of the standard CES model is merely caused by the ineffectiveness of α instead of some clear tendency in subjects' behavior to respond to price changes.

3.3 On the interpretation of "the equality–efficiency trade-off"

Here, we point out that the substitution parameter, ρ, does not represent "the equality–efficiency trade-off." In the literature, ρ is often taken to represent "the equality–efficiency trade-off" that individual subjects face when distributing payoffs. This interpretation originates from the mathematical fact that in the standard CES model, the case $\rho \to \infty$ (or $\tau \to 0$) corresponds to the *symmetric* Leontief function. Because τ is a one-to-one transformation of ρ, if this interpretation were correct, it should carry over to τ.

However, this interpretation is empirically inconsistent. Indeed, our results show that subjects with $\hat{\alpha}_e \approx 0.5$ tend to distribute equally (Figure 9), while they are quite different in $\hat{\tau}$, which represents large differences in efficiency-orientation (Figure 12). Similarly, subjects with $\hat{\alpha}_e > 0.5$ tend to distribute "selfishly," although they differ in efficiency-orientation. Hence, efficiency-orientation and equality-orientation are not in a trade-off relationship. Indeed, this is easily seen from the theoretical demand function (Figures 3 and 7). As the extended CES model prescribes, there are instances of many patterns of behavioral orientation.

Therefore, "the equality–efficiency trade-off" interpretation does not come from subjects' behavior but is a mathematical artifact that stems from the nonlinear structure of the standard CES model. Crucially, this is not an inevitable interpretation; we can avoid

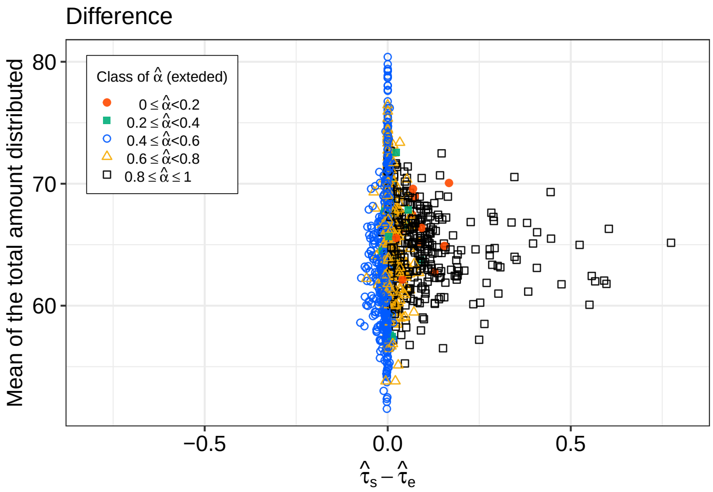

Figure 14: Difference in estimated τ between the standard and the extended CES models. *Notes:* The difference $\hat{\tau}_s - \hat{\tau}_e$ and the sample mean of the observed total amount distributed are depicted. $\hat{\tau}_s$ and $\hat{\tau}_e$ denote the estimates for the standard and extended CES models, respectively. Class of $\hat{\alpha}$ is defined by the estimated value of the extended CES model.

such empirically unfounded interpretations using another alternative, the extended CES model.[24]

3.4 Overall comparison of two models

Figures 16 and 17 show estimation differences between the extended and the standard CES models across all subjects. As expected from the definition of the extended CES utility function, the estimation differences for the distribution parameter α exist mainly when $\hat{\tau} < 0.5$. In most of these cases, the estimated α of our model is closer to 0.5 than that of the standard CES model. The reason behind this result has been shown in Figures 9, 10, and 11. If the mean expenditure share is around 0.5, then $\hat{\alpha}_s$ is closer to 1 or 0 than $\hat{\alpha}_e$, depending on whether the behavior is slightly "selfish" or "generous." Moreover, Figure 17 reconfirms that, for the three subjects (ID 16, 43, and 76 in Figure 11), $\hat{\alpha}_e$ represents the apparent differences in the observed behavior better than $\hat{\alpha}_s$. For the case $\hat{\tau} > 0.5$, the estimation differences in α are much less prevalent. However, the estimates have some significant differences, specifically around $\alpha = 1$ or 0. The reason behind this result is shown in Figure 15. As we argued above, in the extended CES model,

[24]A similar argument to that in Footnote 9 applies here. If we argue that, based on the standard CES model, $\rho = \infty$ means high equality-orientation, we can also argue that, based on Senhadji (1997)'s variant, $\rho = -1$ means high equality-orientation because it corresponds to *symmetric* linear utility function. Hence, which side of ρ corresponds to the high equality-orientation is a matter of modeling decision. Note that both of these interpretations are not based on subjects' behavior and are not inevitable ones.

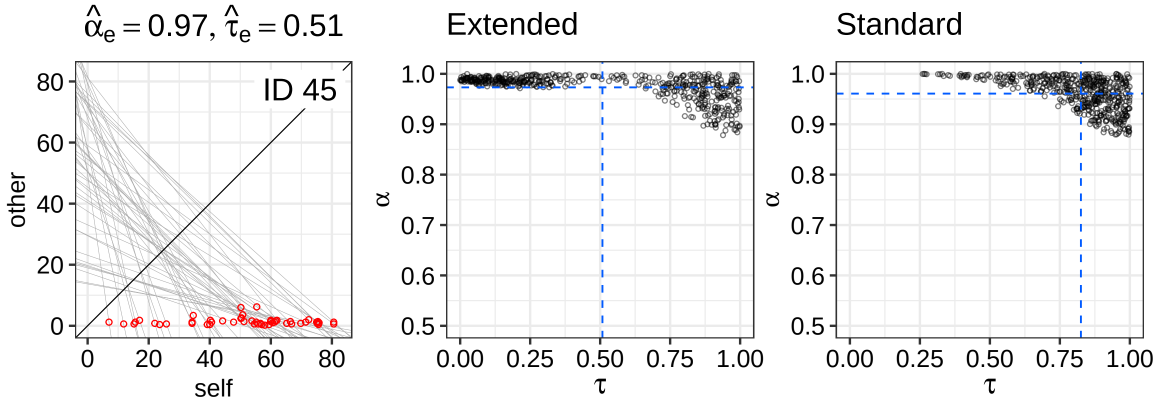

Figure 15: $\hat{\alpha} \approx 1$ causes smaller $\hat{\tau}_e$ than $\hat{\tau}_s$. *Notes:* **Left:** The subject's choices are depicted in the same way as in Figure 11. **Middle and Right:** The posterior samples of (α, τ) of the extended (middle panel) and the standard CES model (right panel) are depicted. The dashed lines indicate the estimated values.

the posterior distribution of τ becomes broader, and hence $\hat{\tau}_e$ becomes smaller than $\hat{\tau}_s$. The extended CES model adequately represents the uncertainty in estimating τ for those subjects.

Finally, we note that while the estimated parameters differ significantly, the extended CES model does not improve prediction accuracy over the standard CES model. This is theoretically reasonable. Indeed, for any ρ, the optimal solution of a model can be mapped one-to-one to that of another (Proposition 2 in Appendix A.2). Hence, a choice between two models amounts to a choice of a specific scale of α, as noted in Section 2.2.1.

To demonstrate this point empirically, we assessed the predictive performance of the two models using 10-fold cross-validation (Vehtari and Ojanen, 2012, Section 5.1.3). We estimated the root-mean-squared error (RMSE) of the expenditure share for the self.[25] Figure 18 shows the result. The difference in RMSE is modest, but the extended CES model has a slightly better predictive performance on average. The mean difference in RMSE for Fisman et al. (2007) is 5.135×10^{-4}, and the maximum and minimum are 1.395×10^{-2} and -9.250×10^{-4}, respectively. Similarly, for Fisman et al. (2023), the mean is 1.938×10^{-5}, and the maximum and minimum are 9.366×10^{-3} and -4.387×10^{-3},

[25] We choose RMSE for our metric because it is based on a very common utility function in point predictions (Vehtari and Ojanen, 2012, Section 3.1.1). For each subject, we divided the choice data into 10-folds, each of which consists of 5 data points. For each fold $k = 1, \ldots, 10$, the model was estimated on a dataset consisting of all but the k-th fold. Then, for each data point in the k-th fold, we evaluated the squared error between the predicted expenditure share for the self and the observed one. In this way, for each data point of the original dataset (50 allocation decisions), we obtained the squared error. Finally, an estimate of RMSE is obtained by taking the sample mean of the squared errors and then its square root.

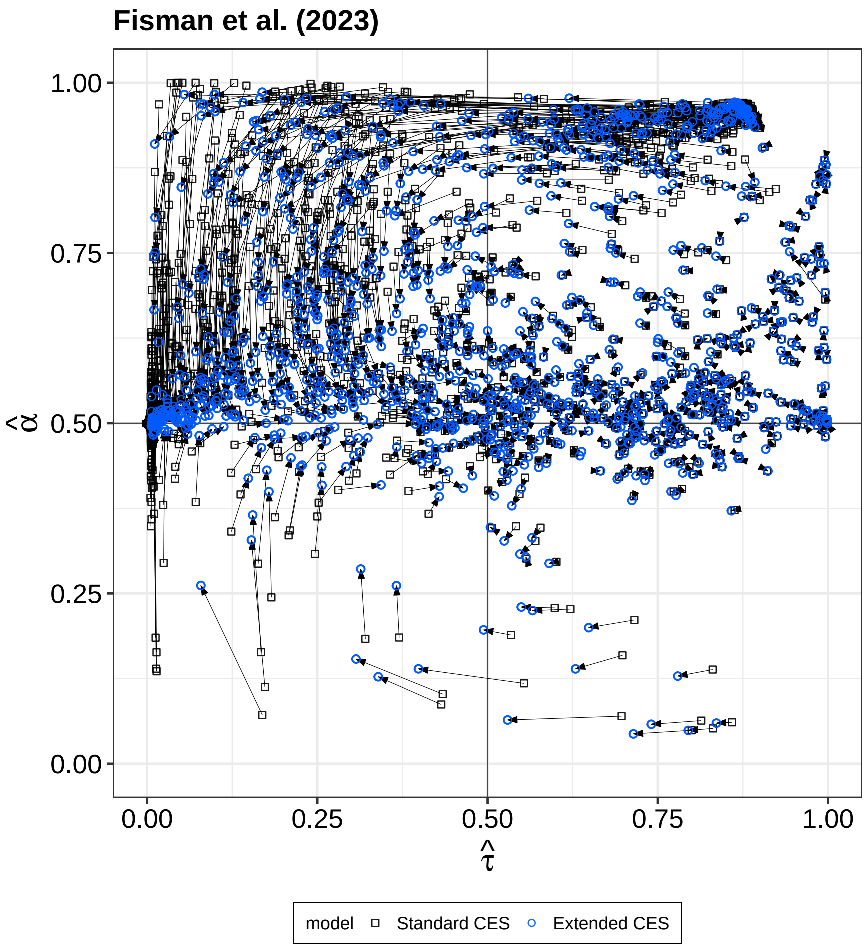

Figure 16: Estimation differences for data from Fisman et al. (2023). *Notes:* Posterior mean estimates of the extended CES model (blue circle) and the standard CES model (black square), and the differences between the two (black arrow).

respectively. These differences are insignificant because the expenditure share lies within $[0, 1]$. Hence, a choice between these two models amounts to a choice of a specific scale of α, also in practice. In conclusion, the extended CES model does not worsen predictive performance and improves measurement validity relative to the standard CES model.

4 Parameter Recovery Simulation

We conducted a parameter recovery simulation to assess the model's inferential uncertainty (Wilson and Collins, 2019). We simulated synthetic datasets fixing parameters α, τ, and σ_ϵ. In particular, α and τ were fixed so that the overall simulation covers a valid range

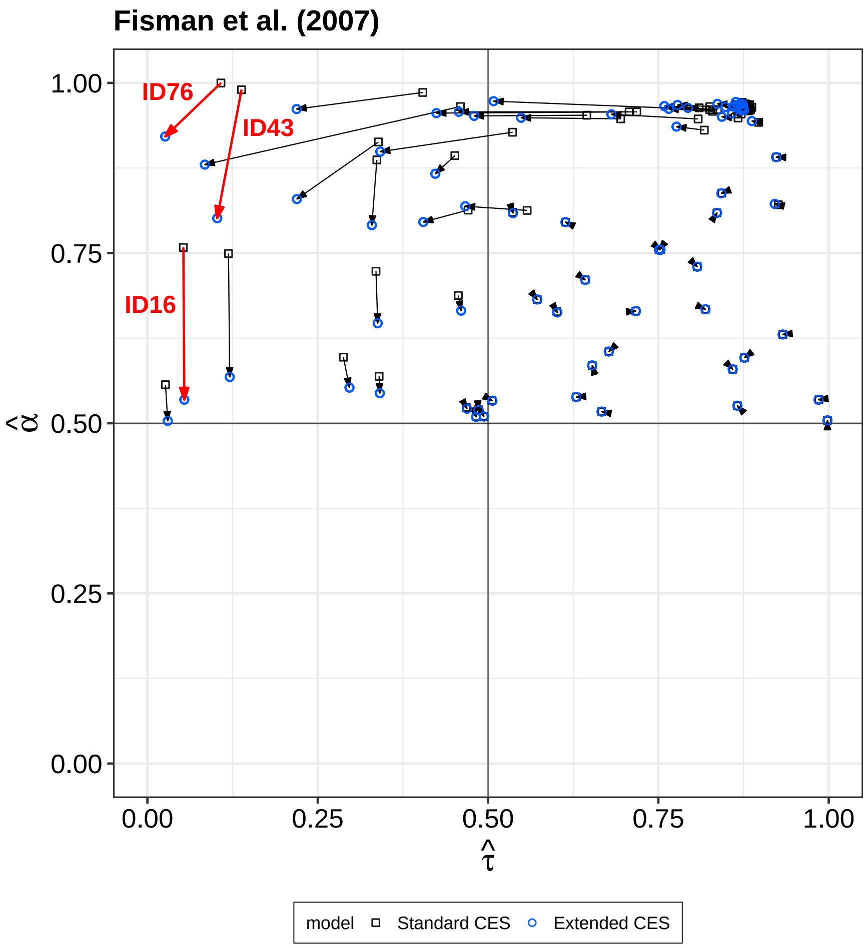

Figure 17: Estimation differences for data from Fisman et al. (2007). *Notes:* As in Figure 16, the estimates of the two models and the difference between them are depicted. The three subjects in Figure 11 are highlighted here with ID labels and thicker arrows.

such that $\alpha \geqq 0.5$ and $\tau \in [0, 1]$.[26] σ_ϵ was fixed at $\sigma_\epsilon = 0.01, 0.05, 0.1, 0.15$. We generated 200 datasets for each combination of parameters, each with 50 allocation decisions. In total, we generated $6 \times 11 \times 4 \times 200 = 52,800$ datasets in the entire simulation. Note that, for comparison, we used an identical set of 50 price systems throughout the entire simulation. In the following, we focus on the cases $\sigma_\epsilon = 0.01$ and $\sigma_\epsilon = 0.1$. Results for other cases are presented in Appendix B.

[26]More precisely, α was set at $\alpha = 0.5, 0.6, 0.7, 0.8, 0.9, 0.99$, and τ was set at $\tau = 0.01, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.99$. We ignored $\alpha < 0.5$ because of the symmetry.

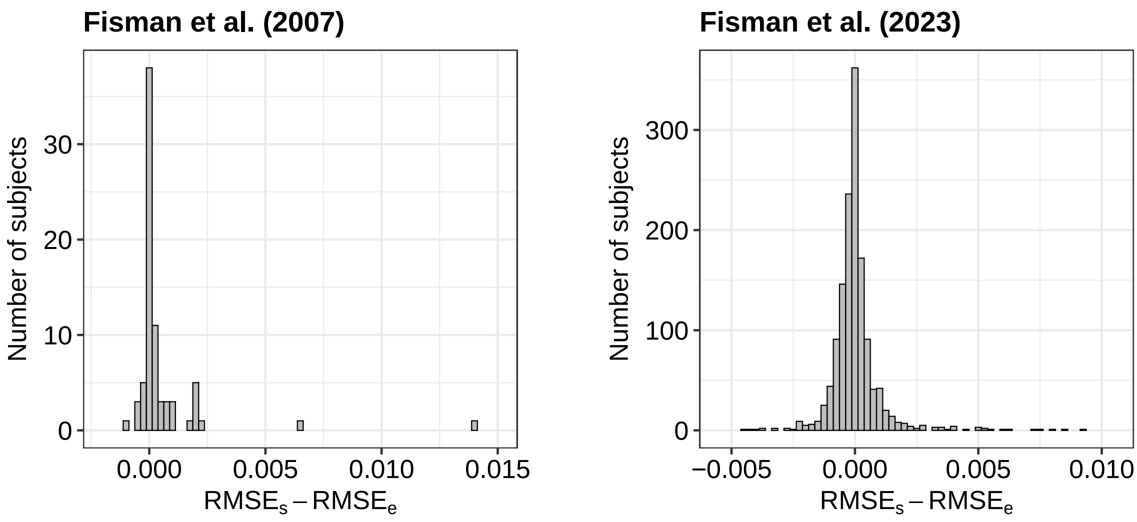

Figure 18: Difference in predictive performance. *Notes:* The estimated differences in RMSEs between the extended and standard CES models for each subject are depicted.

4.1 Parameter identification

First, by focusing on the case $\sigma_\epsilon = 0.01$, we observe the impact of the parameter identification problem. Figure 19 shows the result. We can recover the values of most of the true parameters using our procedure. In particular, we can recover α at a sufficiently reliable level. This is good news because α constitutes an essential interest in the context of distributional preferences. Moreover, τ can be recovered from most of the entire combinations of true parameter values. Furthermore, $\hat{\tau}$ is relatively more variable than $\hat{\alpha}$ in each combination.

As expected from the definition (5), the parameter identification problem for τ is severe when $\alpha = 0.99$. This is a common feature for any CES-type specification and hence inevitable. Even when $\alpha = 0.9$, identifying τ is difficult for cases such that $\tau \geqq 0.7$. It should also be noted that this latter difficulty stems from not only the CES specification but also the finite nature of the experimental method. Because the number of price systems is finite, there is limited information to identify subtle differences in τ.

4.2 Inferential uncertainty

Next, we consider the case $\sigma_\epsilon = 0.1$. Figure 20 shows the result. The inferential uncertainty is more significant than in the previous case because σ_ϵ is larger. As in the previous case, α can be recovered more precisely than τ. Indeed, $\hat{\alpha}$ is less likely to overlap with its neighbor 0.1 away when compared with $\hat{\tau}$. While the estimates for τ are not so precise, they do not overlap significantly with parameters that are 0.2 apart, which suggests that they hold valuable information. However, for $\alpha = 0.9$ and 0.99, the bias and variability of the estimates become severe.

To examine this uncertainty more quantitatively, Figure 21 summarizes the same results regarding inferential error. For both parameters, the magnitude of estimation error mainly depends on the value of α rather than τ. The estimation error is relatively minor for α than for τ, but even for α, distinguishing subtle differences seems difficult when $\alpha \approx 1$. In general, $\hat{\alpha}$ is reasonably reliable for distinguishing significant differences. Moreover, $\hat{\tau}$ is reliable when we focus on the cases where the value of α is not extreme, such as $\alpha \leqq 0.8$. Hence, even though these estimates are not perfect, this method can provide valid and meaningful information about behavioral differences.

Overall, these results suggest that special care should be taken when $\alpha \approx 1$, which typically arises in the context of distributional preferences. For example, $\hat{\tau}$ can be dropped in the subsequent analysis if $\hat{\alpha}$ is above a certain threshold. However, the threshold will depend on the purpose of the analysis, the set of price systems, and the statistical model setting, so no generic threshold value is likely to exist. Therefore, setting the threshold for each study would be desirable.

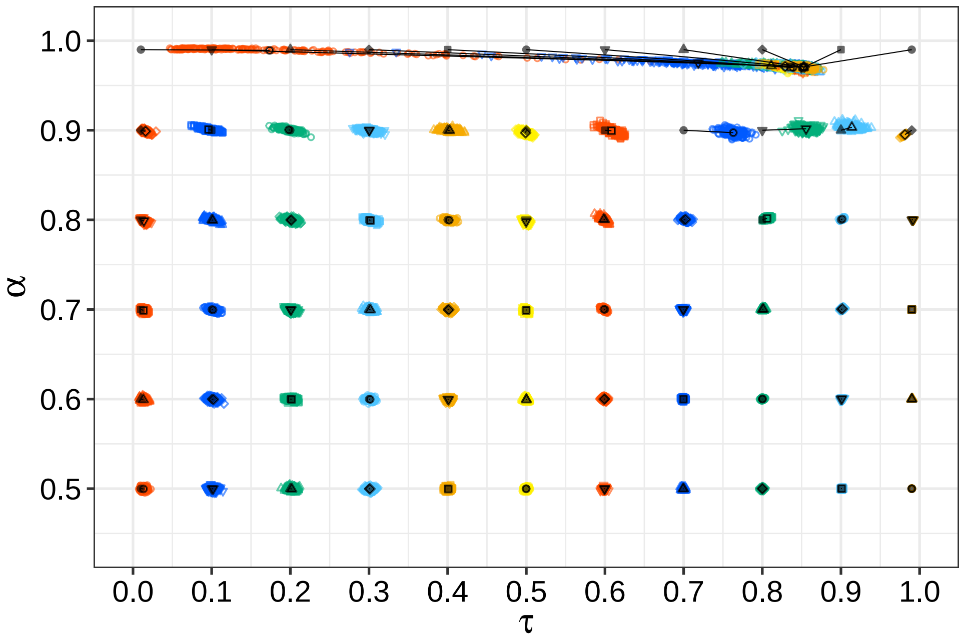

Figure 19: Estimation results of the simulation with $\sigma_\epsilon = 0.01$. *Notes:* For each true parameter (τ, α), the true parameter value (black filled marker), 200 corresponding simulation results (unfilled marker), and the mean of the results (black unfilled marker) are depicted. Black lines connect true values and corresponding means.

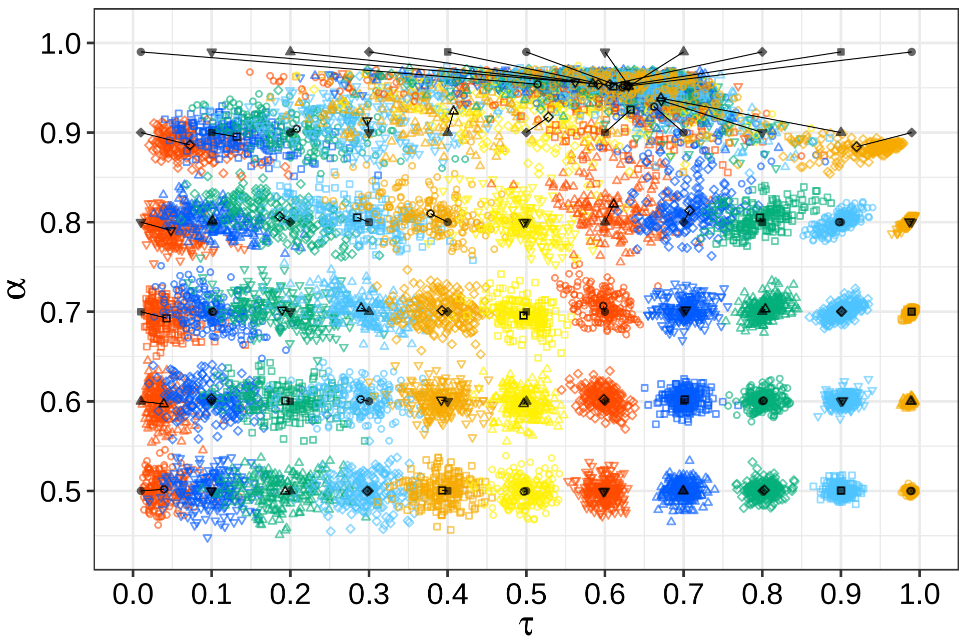

Figure 20: Estimation results of the simulation with $\sigma_\epsilon = 0.1$. *Notes:* As for Figure 19, the true parameter value, simulation results, and the mean of the results are depicted.

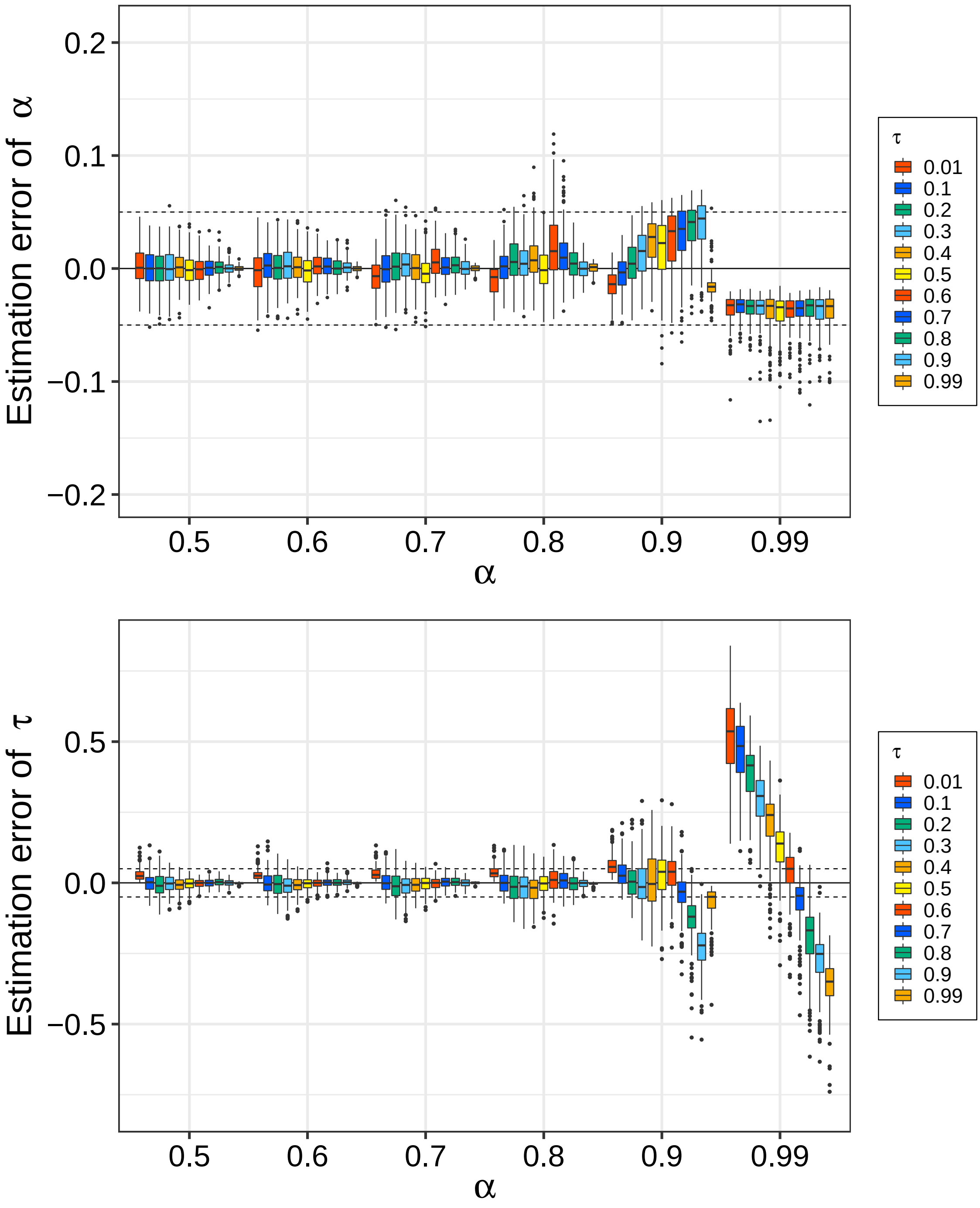

Figure 21: Estimation error of the simulation with $\sigma_\epsilon = 0.1$. *Notes:* Each box corresponds to 200 estimation errors defined by the estimated values minus the true parameter value α or τ. Dashed horizontal lines indicate ± 0.05. **Top:** Estimation error of α. **Bottom:** Estimation error of τ. Note that the vertical axis scale is different from the top panel.

5 Conclusion

In this study, we proposed the extended CES utility function and demonstrated its usefulness. We have shown that α describes subjects' behavior better than the standard CES utility function. Moreover, we proposed a rescaled parameter τ that describes subjects' efficiency-orientation better than the original scale, ρ. As we explored theoretically and demonstrated empirically, the conventional usage of previously proposed CES functions has undesirable features in measuring the heterogeneity of preferences. Furthermore, the conventional interpretation of ρ, "the equality–efficiency trade-off," is not based on subjects' behavior but on the standard CES function's limitation. Therefore, using the extended CES model instead of other CES specifications should help strengthen the conclusions of applied studies by reducing estimate instability and enhance our understanding of distributional preferences by avoiding inconsistent interpretations.

In our simulation study, we found that $\hat{\alpha}$ is more reliable for further analyses than $\hat{\tau}$. In summary, these estimates provide valuable information if we use them with special care. We emphasize that this type of simulation study is a simple and important practice for any model-based behavioral analysis (Wilson and Collins, 2019).

Finally, we mention further cautions regarding this experimental method. First, our conclusion does not suggest that individuals never face such trade-offs. Instead, to examine whether people face the trade-off, we should address it more elaborately in the research design. Engelmann and Strobel (2004, 2006) are examples of such experiments. Moreover, note that the existence of "the equality–efficiency trade-off" itself is context-dependent and debatable (Okun, 2015, Sections 3 and 4; Herzog, 2021, Section 4.2; Andersen and Maibom, 2020). Second, we should be cautious in interpreting α beyond the behavioral interpretation. For example, the equality-oriented behavior represented by $\alpha = 1/2$ constitutes a subset of the broad notion of "proportional equality," which includes egalitarian and non-egalitarian ideas of justice (Gosepath, 2021, p. 8). Hence, in particular, we should be cautious in making a priori judgments about whether a value of α is desirable or just. Moreover, this experiment focuses on a local and abstract situation including too few people. The concepts discussed in political philosophy focus on more global and institutional problems in more specific contexts and, hence, may not directly apply to this experimental setting. Third, while this study focuses on the CES utility function, some other aspects can be reexamined. Future work will consider other aspects of the model, such as the residual error term or the class of utility functions in the first place, to make the model more general and flexible, better representing subjects' preferences.

Appendix A.

A.1 Limits of CES utility functions

We focus on the two specifications used in the main text, Arrow et al. (1961) and Senhadji (1997), but we also consider two other specifications discussed in Thöni (2015): specifications of Thöni (2015) and David and Van de Klundert (1965).

Let $\alpha \in (0,1)$ be the distribution parameter and $\rho \in (-1,\infty) \setminus \{0\}$ be the substitution parameter. Let $x_1 > 0$ and $x_2 > 0$. We consider the following four specifications.

1) The standard CES of Arrow et al. (1961):

$$\left(\alpha x_1^{-\rho} + (1-\alpha) x_2^{-\rho}\right)^{-\frac{1}{\rho}} \tag{13}$$

2) The CES variant used by Senhadji (1997):

$$\left(\alpha^{\rho+1} x_1^{-\rho} + (1-\alpha)^{\rho+1} x_2^{-\rho}\right)^{-\frac{1}{\rho}} \tag{14}$$

3) The CES variant proposed by Thöni (2015):

$$\left(\alpha^{\rho+2} x_1^{-\rho} + (1-\alpha)^{\rho+2} x_2^{-\rho}\right)^{-\frac{1}{\rho}} \tag{15}$$

4) The CES variant of David and Van de Klundert (1965):

$$\left(\alpha^{-\rho} x_1^{-\rho} + (1-\alpha)^{-\rho} x_2^{-\rho}\right)^{-\frac{1}{\rho}} \tag{16}$$

The only difference between these four specifications is the power exponent of α. Hence, considering a given function $f(\rho)$ that maps ρ to a real number, these specifications have the following general form:

$$u(x_1, x_2) = \left(\alpha^{f(\rho)} x_1^{-\rho} + (1-\alpha)^{f(\rho)} x_2^{-\rho}\right)^{-\frac{1}{\rho}}. \tag{17}$$

Note that $f_1(\rho) = 1$ for Arrow et al. (1961), $f_2(\rho) = \rho+1$ for Senhadji (1997), $f_3(\rho) = \rho+2$ for Thöni (2015), and $f_4(\rho) = -\rho$ for David and Van de Klundert (1965). In the following, we derive the limit of (17) as ρ tends to $-1, 0$, and ∞ in turn.[27]

[27] The proof is a straightforward extension of a well-known argument but is provided here for reference.

Lemma 1. *Suppose that $f(\rho)$ is continuous at $\rho = -1$. Then, as ρ tends to -1, $u(x_1, x_2)$ tends to*

$$\alpha^{f(-1)}x_1 + (1-\alpha)^{f(-1)}x_2. \tag{18}$$

Proof. Consider the logarithmic form of the utility function. Then, because the function $f(\rho)$ is continuous at $\rho = -1$, we have

$$\begin{aligned}\lim_{\rho \to -1} \log(u(x_1, x_2)) &= -\frac{1}{-1} \lim_{\rho \to -1} \log\left(\alpha^{f(\rho)}x_1^{-\rho} + (1-\alpha)^{f(\rho)}x_2^{-\rho}\right) \\ &= \log\left(\alpha^{f(-1)}x_1 + (1-\alpha)^{f(-1)}x_2\right).\end{aligned}$$

□

From this lemma, it follows that (13), (15), and (16) converge to linear function $\alpha x_1 + (1-\alpha)x_2$ because $f_1(-1) = f_3(-1) = f_4(-1) = 1$. By contrast, as Thöni (2015) mentioned, the variant of Senhadji (1997) converges to the symmetric one, $x + y$, because $f_2(-1) = 0$.

Lemma 2. *Suppose that $f(\rho)$ is continuously differentiable on an open interval that contains* 0. *Then, as ρ tends to* 0, *$u(x_1, x_2)$ tends to*

$$\left(\alpha^{-f'(0)}x_1\right)^D \left((1-\alpha)^{-f'(0)}x_2\right)^{1-D} \tag{19}$$

where

$$D = \frac{\alpha^{f(0)}}{\alpha^{f(0)} + (1-\alpha)^{f(0)}}. \tag{20}$$

Proof. Again, we calculate the limit with logarithmic form $\log(u(x_1, x_2))$. First, by L'Hôpital's rule, we have

$$\begin{aligned}\lim_{\rho \to 0} \log(u(x_1, x_2)) &= \lim_{\rho \to 0} \left\{-\frac{1}{\rho'}\left(\log\left(\alpha^{f(\rho)}x_1^{-\rho} + (1-\alpha)^{f(\rho)}x_2^{-\rho}\right)\right)'\right\} \\ &= -\lim_{\rho \to 0} \frac{\left(\alpha^{f(\rho)}x_1^{-\rho} + (1-\alpha)^{f(\rho)}x_2^{-\rho}\right)'}{\alpha^{f(\rho)}x_1^{-\rho} + (1-\alpha)^{f(\rho)}x_2^{-\rho}}.\end{aligned} \tag{21}$$

Because $f(\rho)$ is continuous at $\rho = 0$, the limit of the denominator as $\rho \to 0$ is $\alpha^{f(0)} + (1-\alpha)^{f(0)}$. Moreover, the limit of the numerator is

$$\begin{aligned}&\lim_{\rho \to 0}\left(\alpha^{f(\rho)}x_1^{-\rho}\log\left(\frac{\alpha^{f'(\rho)}}{x_1}\right) + (1-\alpha)^{f(\rho)}x_2^{-\rho}\log\left(\frac{(1-\alpha)^{f'(\rho)}}{x_2}\right)\right) \\ &= \alpha^{f(0)}\log\left(\frac{\alpha^{f'(0)}}{x_1}\right) + (1-\alpha)^{f(0)}\log\left(\frac{(1-\alpha)^{f'(0)}}{x_2}\right).\end{aligned}$$

Therefore, by defining constant D as $D = \alpha^{f(0)}/(\alpha^{f(0)} + (1-\alpha)^{f(0)})$, the right-hand side of (21) is

$$\begin{aligned}
&-\left\{D \log\left(\frac{\alpha^{f'(0)}}{x_1}\right) + (1-D)\log\left(\frac{(1-\alpha)^{f'(0)}}{x_2}\right)\right\}\\
&= \log\left\{\left(\frac{x_1}{\alpha^{f'(0)}}\right)^D \left(\frac{x_2}{(1-\alpha)^{f'(0)}}\right)^{1-D}\right\}\\
&= \log\left\{\left(\alpha^{-f'(0)}x_1\right)^D \left((1-\alpha)^{-f'(0)}x_2\right)^{1-D}\right\}.
\end{aligned}$$

Hence, the conclusion follows. □

This lemma implies that the limits of (13) and (14) are behaviorally equivalent to the Cobb–Douglas function $x_1^{\alpha}x_2^{1-\alpha}$, (15) is equivalent to the Cobb–Douglas function $x_1^D x_2^{1-D}$ where $D = \alpha^2/(\alpha^2 + (1-\alpha)^2)$, and (16) is equivalent to the symmetric Cobb–Douglas function $x_1^{0.5}x_2^{0.5}$.[28]

Notably, this provides a good reason to believe that the solution proposed by Thöni (2015) is not very successful in providing a straightforward interpretation of α. Indeed, the parameter α is not equal to the expenditure share of the first good if $\alpha \neq 0, 0.5$, or 1 (Figure 22 in Appendix B). Instead, the expenditure share is equal to $D = \alpha^2/(\alpha^2 + (1-\alpha)^2)$. For example, if $\alpha = 0.75$, then the expenditure share of the first good is equal to $D = 0.9$. This discrepancy between α and the behavior (characterized by D) makes it hard to interpret α in this limit. Furthermore, the name *distribution parameter* itself comes from the fact that the limit of the standard CES function (as $\rho \to 0$) is the Cobb–Douglas function with parameter α (Arrow et al., 1961, Footnote 10). We believe adhering to this classical notion is essential to provide a straightforward interpretation of α.

Lemma 3. *For $\rho > 0$, let $f(\rho) = a\rho + b$ for some real numbers a and b. Then, as ρ tends to ∞, $u(x_1, x_2)$ tends to*

$$\min\left\{\frac{x_1}{\alpha^a}, \frac{x_2}{(1-\alpha)^a}\right\}. \tag{22}$$

[28] First, for Arrow et al. (1961), we have $f_1(0) = 1$ and $f_1'(0) = 0$, and hence $u(x_1, x_2) = x_1^{\alpha}x_2^{1-\alpha}$, which is the Cobb–Douglas function with the distribution parameter α. Second, for Senhadji (1997), it holds that $f_2(0) = 1$ and $f_2'(0) = 1$. This implies that $u(x_1, x_2) = \left(\alpha^{\alpha}(1-\alpha)^{1-\alpha}\right)^{-1} x_1^{\alpha}x_2^{1-\alpha}$, which is a positive transformation of the Cobb–Douglas function in the previous case. Third, for Thöni (2015), we have $f_3(0) = 2$ and $f_3'(0) = 1$. Hence, it follows that $D = \alpha^2/(\alpha^2 + (1-\alpha)^2)$ and $u(x_1, x_2) = \left(\alpha^D(1-\alpha)^{1-D}\right)^{-1} x_1^D x_2^{1-D}$, which is a positive transformation of the Cobb–Douglas function with the distribution parameter $D = \alpha^2/(\alpha^2 + (1-\alpha)^2)$. Finally, for David and Van de Klundert (1965), we have $f_4(0) = 0$ and $f_4'(0) = -1$. Hence, $D = 0.5$ and $u(x_1, x_2) = \sqrt{\alpha(1-\alpha)}\, x_1^{0.5}x_2^{0.5}$, which is a positive transformation of the symmetric Cobb–Douglas function.

Proof. As in the previous proof, we have the following by L'Hôpital's rule:

$$\lim_{\rho\to\infty} \log(u(x_1,x_2)) = -\lim_{\rho\to\infty} \frac{\left(\alpha^{f(\rho)}x_1^{-\rho} + (1-\alpha)^{f(\rho)}x_2^{-\rho}\right)'}{\alpha^{f(\rho)}x_1^{-\rho} + (1-\alpha)^{f(\rho)}x_2^{-\rho}}. \tag{23}$$

Because we assume $f(\rho) = a\rho + b$, the right-hand side of this equation is equal to

$$\begin{aligned} &-\lim_{\rho\to\infty} \frac{\alpha^{a\rho+b}x_1^{-\rho}\log\left(\alpha^a x_1^{-1}\right) + (1-\alpha)^{a\rho+b}x_2^{-\rho}\log\left((1-\alpha)^a x_2^{-1}\right)}{\alpha^{a\rho+b}x_1^{-\rho} + (1-\alpha)^{a\rho+b}x_2^{-\rho}} \\ =&-\lim_{\rho\to\infty} \frac{\alpha^b(\alpha^a x_1^{-1})^\rho\log\left(\alpha^a x_1^{-1}\right) + (1-\alpha)^b((1-\alpha)^a x_2^{-1})^\rho\log\left((1-\alpha)^a x_2^{-1}\right)}{\alpha^b(\alpha^a x_1^{-1})^\rho + (1-\alpha)^b((1-\alpha)^a x_2^{-1})^\rho} \end{aligned} \tag{24}$$

Now, consider the case $\alpha^a x_1^{-1} > (1-\alpha)^a x_2^{-1}$. Then, we can compute (24) as

$$\begin{aligned} &-\lim_{\rho\to\infty} \frac{\alpha^b\log\left(\alpha^a x_1^{-1}\right) + (1-\alpha)^b\left(\frac{(1-\alpha)^a x_2^{-1}}{\alpha^a x_1^{-1}}\right)^\rho\log\left((1-\alpha)^a x_2^{-1}\right)}{\alpha^b + (1-\alpha)^b\left(\frac{(1-\alpha)^a x_2^{-1}}{\alpha^a x_1^{-1}}\right)^\rho} \\ =&-\frac{\alpha^b\log\left(\alpha^a x_1^{-1}\right) + (1-\alpha)^b\cdot 0\cdot\log\left((1-\alpha)^a x_2^{-1}\right)}{\alpha^b + (1-\alpha)^b\cdot 0} = \log\left(\alpha^{-a}x_1\right). \end{aligned} \tag{25}$$

Because $\alpha^a x_1^{-1} > (1-\alpha)^a x_2^{-1}$ implies $\alpha^{-a}x_1 < (1-\alpha)^{-a}x_2$, we have the desired conclusion in this case. It is clear that in the opposite case $\alpha^a x_1^{-1} < (1-\alpha)^a x_2^{-1}$, the same argument follows. Finally, if we have $\alpha^a x_1^{-1} = (1-\alpha)^a x_2^{-1}$, then we can compute (24) as

$$\begin{aligned} &-\frac{\alpha^b\log\left(\alpha^a x_1^{-1}\right) + (1-\alpha)^b\log\left((1-\alpha)^a x_2^{-1}\right)}{\alpha^b + (1-\alpha)^b} \\ =&-\frac{(\alpha^b + (1-\alpha)^b)\log\left(\alpha^a x_1^{-1}\right)}{\alpha^b + (1-\alpha)^b} = \log\left(\alpha^{-a}x_1\right). \end{aligned} \tag{26}$$

Therefore, the desired conclusion is also valid in this case. □

From this lemma, we can see that (14) and (15) converge to the Leontief function $\min\{x_1/\alpha, x_2/(1-\alpha)\}$ because $a = 1$ for $f_2(\rho) = \rho + 1$ and $f_3(\rho) = \rho + 2$. Moreover, the well-known property of the standard CES function (13) follows from this lemma, i.e., it converges to the *symmetric* Leontief function $\min\{x_1, x_2\}$ regardless of the value of α because $a = 0$ for $f_1(\rho) = 1$. Finally, the limit of (16) is $\min\{\alpha x_1, (1-\alpha)x_2\}$ because $a = -1$ for $f_4(\rho) = -\rho$.[29]

[29]Note that the CES variant of David and Van de Klundert (1965) has more clear shortcomings in this limit, as pointed out in Thöni (2015). For example, consider the case $\alpha > 1-\alpha$. Then, the form $\min\{\alpha x_1, (1-\alpha)x_2\}$ has a higher utility value if $x_2 > x_1$, and hence the interpretation of α is reversed

Here, we provide an essential result concerning the flexibility of CES functions in the limits. As we have seen, the previously proposed specifications do not have one of the three desirable limits: $\alpha x_1 + (1-\alpha)x_2$, (a positive multiple of) $x_1^{\alpha}x_2^{1-\alpha}$, and $\min\{x_1/\alpha, x_2/(1-\alpha)\}$ as $\rho \to -1, 0$, and ∞, respectively. Indeed, Lemma 1, 2, and 3 imply that this generally holds in the class (17) with the affine function form of $f(\rho)$, i.e., $f(\rho) = a\rho + b$.

Theorem 1. *Let $f(\rho) = a\rho + b$ for some real numbers a and b, and $\alpha \in (0,1)$. Then, the utility function* (17) *cannot have all of the three limits $\alpha x_1 + (1-\alpha)x_2$, $\alpha^{-a\alpha}(1-\alpha)^{-a(1-\alpha)}x_1^{\alpha}x_2^{1-\alpha}$, and* $\min\{x_1/\alpha, x_2/(1-\alpha)\}$ *as $\rho \to -1, 0$, and ∞, respectively.*

Proof. If we have the limit $\alpha x_1 + (1-\alpha)x_2$ as $\rho \to -1$, then from Lemma 1, we must have $f(-1) = 1$ and hence $b = 1 + a$. Moreover, if we have the limit $\min\{x_1/\alpha, x_2/(1-\alpha)\}$ as $\rho \to \infty$, then from Lemma 3 we must have $a = 1$. Therefore, we must also have $b = 1 + 1 = 2$, and $f(\rho) = \rho + 2$, i.e., we must have Thöni (2015)'s specification. However, as we see above, the limiting form of this specification, as $\rho \to 0$, is equivalent to the Cobb–Douglas function $x_1^{D}x_2^{1-D}$ where $D = \alpha^2/(\alpha^2 + (1-\alpha)^2)$. Hence, in particular, it is not the stated form that is equivalent to the standard Cobb–Douglas function, $x_1^{\alpha}x_2^{1-\alpha}$. $\square$

This proof shows that requiring the two limits implies the CES variant of Thöni (2015). Similarly, requiring $\alpha x_1 + (1-\alpha)x_2$ and $\alpha^{-a\alpha}(1-\alpha)^{-a(1-\alpha)}x_1^{\alpha}x_2^{1-\alpha}$ implies Arrow et al. (1961)'s one. Moreover, requiring $\alpha^{-a\alpha}(1-\alpha)^{-a(1-\alpha)}x_1^{\alpha}x_2^{1-\alpha}$ and $\min\{x_1/\alpha, x_2/(1-\alpha)\}$ implies Senhadji (1997)'s variant. Furthermore, requiring all three limits implies $b = 1 + a$, $b = 1$, and $a = 1$, an apparent contradiction. Therefore, these three specifications are the most flexible instances in this class.

Now, we define the *extended CES utility function* by (17) with $f(\rho) = \max\{0, \rho\} + 1$, where $\rho \in (-1, \infty) \setminus \{0\}$. This does not belong to the class studied in Theorem 1. Furthermore, it has the following desirable limits.

Theorem 2. *Let $\alpha \in (0,1)$ and $f(\rho) = \max\{0, \rho\} + 1$ for any $\rho \in (-1, \infty) \setminus \{0\}$. Then,*

(Figure 22 in Appendix B). Moreover, this specification loses the classical interpretation of α in the Cobb–Douglas case. These properties could have problematic consequences beyond our specific context if one needs to interpret α consistently with behavior.

the utility function (17) *has the following limits:*

$$\lim_{\rho \to -1} u(x_1, x_2) = \alpha x_1 + (1-\alpha) x_2, \tag{27}$$

$$\lim_{\rho \to 0^-} u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}, \tag{28}$$

$$\lim_{\rho \to 0^+} u(x_1, x_2) = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} x_1^{\alpha} x_2^{1-\alpha}, \tag{29}$$

$$\lim_{\rho \to \infty} u(x_1, x_2) = \min\left\{ \frac{x_1}{\alpha}, \frac{x_2}{1-\alpha} \right\}. \tag{30}$$

Proof. For $\rho < 0$, we have $f(\rho) = 1$, and hence the first and second limits follow from Lemma 1 and 2, respectively. Similarly, for $\rho > 0$, we have $f(\rho) = \rho + 1$, and hence the third and fourth limits follow from Lemma 2 and 3, respectively. $\square$

While the two one-sided limits differ, their optimal solutions coincide. This is because the optimal solution of the utility maximization problem does not change with a positive constant multiplication. See Proposition 1.

A.2 Optimal solution for CES utility functions

We first describe the optimal solution of the utility maximization problem with the general form of the CES utility function (17). Because the mathematical behavior of this function at the boundary of R_+^2 is subtle, we restrict the problem for simplicity.[30] Let $c > 0$ be an arbitrarily small number. We consider the following utility maximization problem:

$$\begin{aligned} \max_{x_1, x_2} \quad & u(x_1, x_2) \\ \text{s.t.} \quad & p_1 x_1 + p_2 x_2 \leqq 1 \\ & x_1, x_2 \geqq c \end{aligned} \tag{31}$$

where $\alpha \in (0,1)$ and $\rho \in (-1, \infty) \setminus \{0\}$. Because c is arbitrarily small, this modification should not affect the study's conclusion in the current context.

Because $u(x_1, x_2)$ is a concave function and the constraints are defined by continuously differentiable convex functions, the Karush–Kuhn–Tucker condition gives us the optimal

[30] For example, let $f(\rho) = 1$. If $\rho = -1/2$, then $u(x_1, x_2) = (\alpha\sqrt{x_1} + (1-\alpha)\sqrt{x_2})^2$. This instance does not have $\partial u / \partial x_1 (x_1, x_2)$ at $(0, x_2)$ or $\partial u / \partial x_2 (x_1, x_2)$ at $(x_1, 0)$.

solution. For any $\alpha \in (0,1)$ and $\rho \in (-1,\infty) \setminus \{0\}$, the optimal expenditure share is

$$p_1 x_1 = \begin{cases} 1 - p_2 c & \text{if } \left(\frac{\alpha}{1-\alpha}\right)^{f(\rho)} \geqq \frac{p_1}{p_2}\left(\frac{1-p_2 c}{p_1 c}\right)^{\rho+1}, \\ p_1 c & \text{if } \left(\frac{\alpha}{1-\alpha}\right)^{f(\rho)} \leqq \frac{p_1}{p_2}\left(\frac{p_2 c}{1-p_1 c}\right)^{\rho+1}, \\ \frac{\left(\frac{\alpha}{1-\alpha}\right)^{\frac{f(\rho)}{\rho+1}}}{\left(\frac{\alpha}{1-\alpha}\right)^{\frac{f(\rho)}{\rho+1}} + \left(\frac{p_2}{p_1}\right)^{\frac{\rho}{\rho+1}}} & \text{otherwise.} \end{cases} \tag{32}$$

Note that this optimal solution is also consistent with (17) at $\rho = 0$. Indeed, if $f(\rho)$ is continuous at $\rho = 0$, then the limit of (32) as $\rho \to 0$ is equal to the optimal solution for the limiting form (19). Moreover, substituting $\rho = 0$ into (32) gives the optimal solution for the limit form (19). Hence, the optimal expenditure (32) is continuous at $\rho = 0$, and its value is the optimal expenditure of the limit (19). We summarize this fact as follows.

Proposition 1. *Suppose that $f(\rho)$ is continuously differentiable on an open interval that contains* 0. *If we extend the definition* (17) *at $\rho = 0$ by equation* (19)*, then equation* (32) *is the optimal solution of* (31) *and continuous for all $\rho \in (-1,\infty)$.*

Given this fact, we can also extend the optimal expenditure for the extended CES utility function. If we extend definition (17) at $\rho = 0$ by the right-hand side of (28), then the same result follows. Hence, we can use equation (32) for all ρ, including the case $\rho = 0$.

Based on the optimal solution expression, we clarify that our focus in this paper is not on the predictive accuracy or goodness of fit of various CES utility functions. That is because, for two CES function specifications, $f(\rho)$ and $\bar{f}(\rho)$, the optimal solution for one can be obtained by rescaling the distribution parameter of the other.

Proposition 2. *For any $\rho \in (-1,\infty)$, let α and $\bar{\alpha}$ be distribution parameters of two CES functions,* (17) *with $f(\rho)$ and $\bar{f}(\rho)$, respectively. If $f(\rho) \neq 0$, then the optimal solutions of the two specifications are equal if and only if*

$$\alpha = \frac{\bar{\alpha}^{\frac{\bar{f}(\rho)}{f(\rho)}}}{\bar{\alpha}^{\frac{\bar{f}(\rho)}{f(\rho)}} + (1-\bar{\alpha})^{\frac{\bar{f}(\rho)}{f(\rho)}}}. \tag{33}$$

Proof. From expression (32), it is clear that the optimal solutions are equal if and only if $\left(\frac{\alpha}{1-\alpha}\right)^{f(\rho)} = \left(\frac{\bar{\alpha}}{1-\bar{\alpha}}\right)^{\bar{f}(\rho)}$. Hence, the conclusion can be checked by a direct calculation. □

For the CES functions of Arrow et al. (1961), Senhadji (1997), and Thöni (2015), and the extended CES function, this rescaling is always mutually possible because $f(\rho) \neq 0$ for any $\rho \in (-1,\infty)$ for all of these specifications. Therefore, a dataset generated from one of these specifications can be generated consistently from the others. In this sense, the choice of $f(\rho)$ is not expected to change the predictive accuracy significantly. Thus, it is clear that our focus is not on improving these aspects. Instead, we focus on the scales of the distribution parameter, which can be mutually translated by equation (33), especially on scales appropriate for measuring the heterogeneity of preferences.

Appendix B.

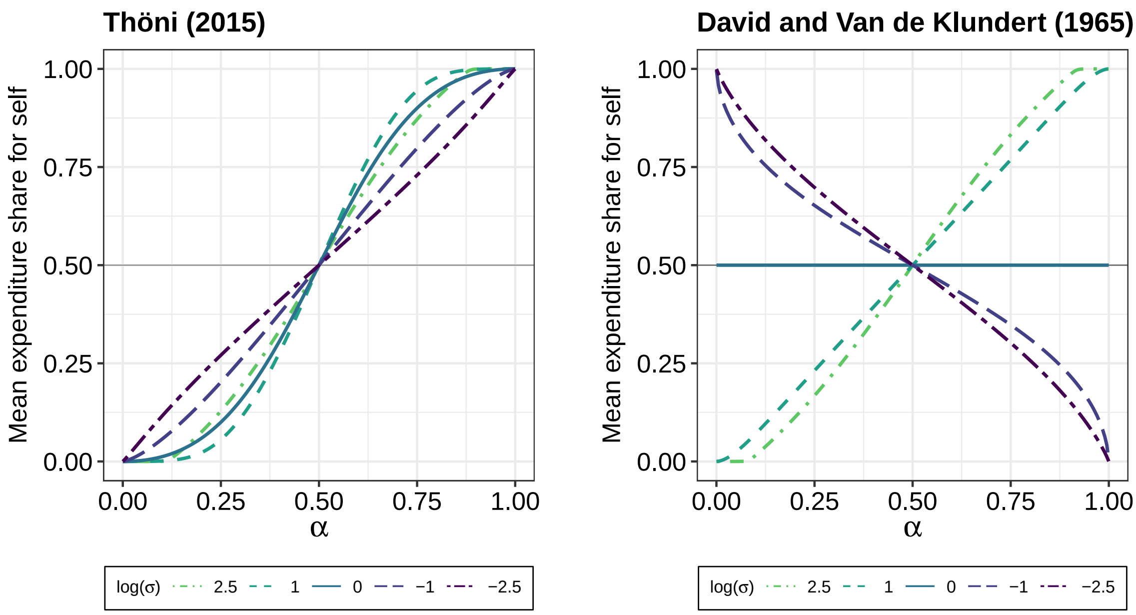

Figure 22: The expenditure share of the CES functions. *Notes:* Each line corresponds to the mean expenditure shares obtained by the same calculation method as in Figure 3. **Left**: The CES variant of Thöni (2015). Expenditure shares are calculated based on equation (10) with $f(\rho) = \rho + 2$. **Right**: The CES variant of David and Van de Klundert (1965). Corresponding graphs based on equation (10) with $f(\rho) = -\rho$.

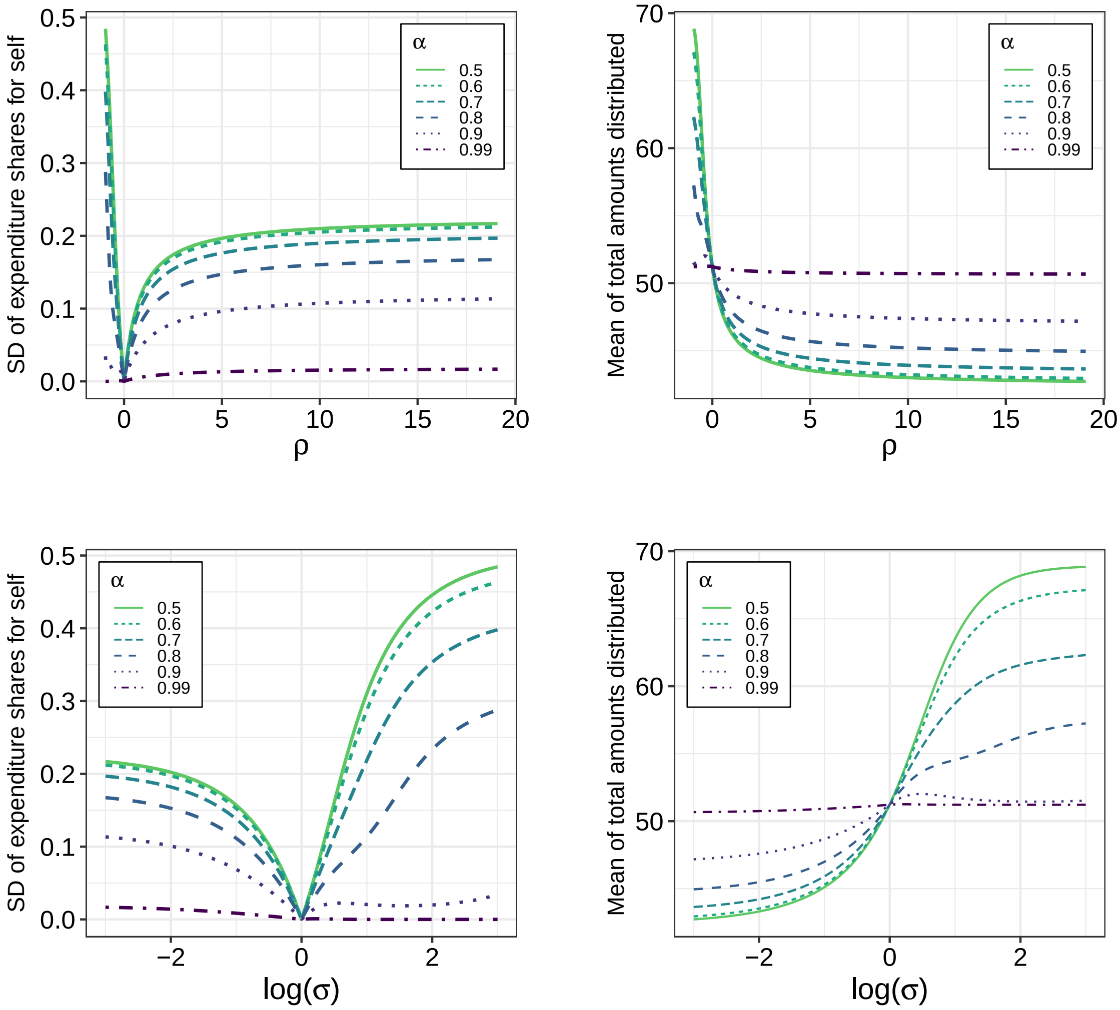

Figure 23: Relationship between other parameterizations and the response to price changes. *Notes:* The graphs correspond to Figure 7, but the horizontal axes are drawn on the scale of ρ (the top panels) and $\log(\sigma)$ (the bottom panels). The parameter values used in these graphs correspond one-to-one to those in Figure 7.

Appendix C.

In this appendix, we present the simulation results for the cases $\sigma_\epsilon = 0.05$ (Figure 24) and $\sigma_\epsilon = 0.15$ (Figure 25). The general patterns are similar to $\sigma_\epsilon = 0.01$ and $\sigma_\epsilon = 0.1$. In particular, the estimation error is relatively minor for α than for τ. The estimation error becomes more severe when noise is large, such as $\sigma_\epsilon = 0.15$. That is, even for α, distinguishing subtle differences seems difficult when $\alpha \approx 1$. Therefore, as discussed in the main text, they should be used with special care to extract valid and meaningful information about behavioral differences and their implications.

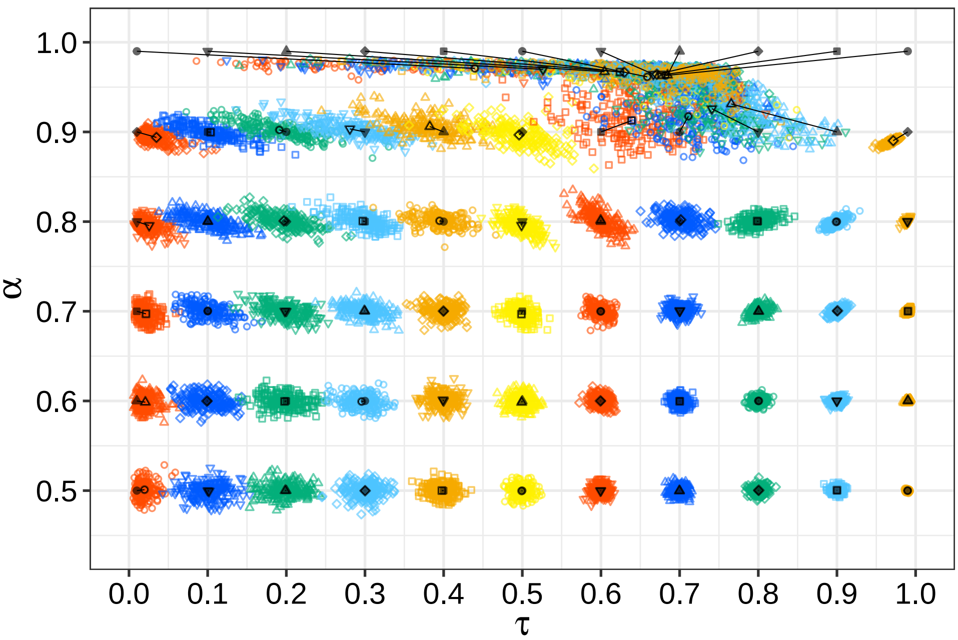

Figure 24: Estimation error of the simulation with $\sigma_\epsilon = 0.05$. *Notes:* As for Figure 19, the true parameter value, simulation results, and the mean of the results are depicted.

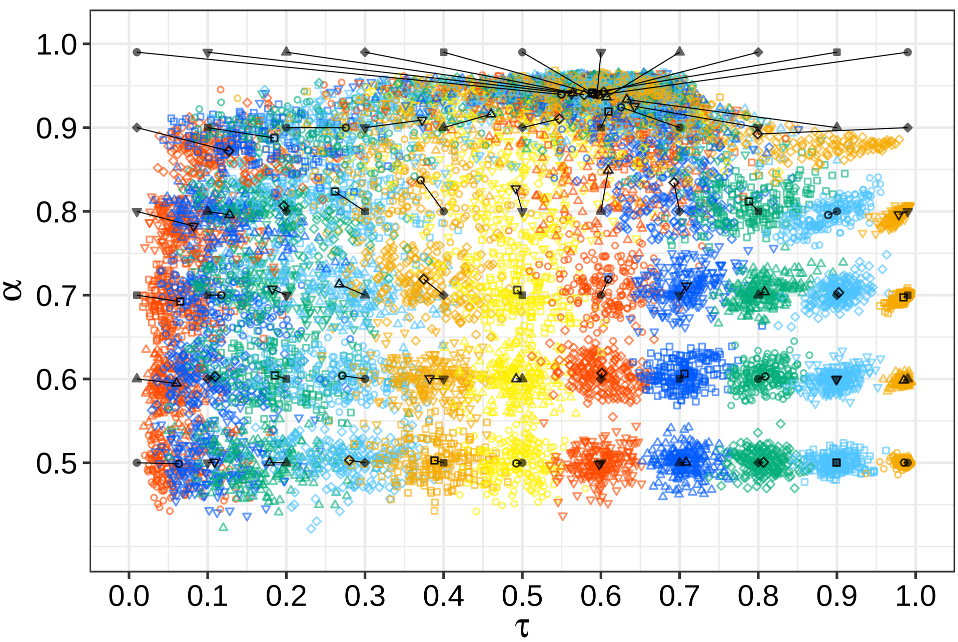

Figure 25: Estimation error of the simulation with $\sigma_\epsilon = 0.15$. *Notes:* As for Figure 19, the true parameter value, simulation results, and the mean of the results are depicted.

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