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SELF-CONTROL CYCLES

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Abstract

Consumers often exhibit behavioral cycles with alternating abstinence and indulgence over time. In the framework of tempting good consumption under limited willpower, we develop a simple model of the self-control cycles. To do so, based on the empirically relevant property of self-control, we incorporate two countervailing effects that self-control behaviors have on willpower with different delays. First, exercising self-control as of restraining tempting consumption depletes willpower in the next instant, and thereby reduces mental capital available for self-control thereafter. Second, as the self-control experience is accumulated, the consumer's willpower is gradually enhanced. The resulting predator-prey type dynamics in consumers' cognitive mechanics lead to cycles in tempting good consumption. The self-control cycles occur when (i) the self-control cost reducing effect of willpower and (ii) the willpower enhancing effect of self-control are both sufficiently strong.
Keywords: Self-control, cycle, temptation, willpower, malleability, consumption.

1 Introduction

People often repeat cyclically self-regulatory and indulgent behaviors. For example, they repeatedly alternate between dieting and binge eating; hard working and excessive drinking; intensive studying and binge watching; and/or rehabilitation and reoffending. Although the cyclical nature of self-control behaviors is well-known, there has not been theoretical research on how it takes place.

By incorporating dynamic interactions between self-control and the underlying metal capital, willpower, this study aims at developing a model of self-control cycles, defined as alternating abstinence-indulgence behavioral phases in tempting good consumption, in which self-control of restraining tempting goods cyclically fluctuates over time as willpower repeats depleting and accumulating processes.

To do so, we develop a model of lifetime-utility maximizers who consume tempting and non-tempting goods, e.g., sweet and vegetable, in which self-control for restraining tempting good consumption depletes willpower, which in turn raises self-control costs in the next instant. With the willpower constraint, we describe consumption behavior in the context of intertemporal self-control allocation.

Our basic idea in describing consumers' dynamic self-regulatory behavior is to incorporate two empirically-relevant properties of willpower as self-control capacity. The first is depletability: as reported by, e.g., Kool et al. (2013), Dang (2018), Boat et al. (2020), and Baumeister et al. (2024), willpower is depletable owing to self-control execution. The second property is malleability: willpower can also be enhanced or trained through repeated self-control behaviors, as shown by Oaten and Cheng (2006a, b), Piquero et al. (2010), Kaplan and Berman (2010), Sultan et al. (2012), and Fries et al. (2017). By focusing on the two effects, we extend our previous model (Ikeda and Ojima, 2021), such that self-control behavior depletes willpower in next instant, as in the model of Ikeda and Ojma (2021), whereas the self-control experience contributes to enhancing willpower in later periods. By taking into account these countervailing effects that will occur with diffrent delays, consumers are assumed to allocate self-control of controlling tempting good consumption

In the resulting interactive dynamics of willpower and self-control experience capital, they have asymetric cross effects on each other: An increase in willpower reduces self-control costs required when restraining tempting good consumption, and hence reduces self-control experiences, on one hand. On the other hand, an increase in self-control experiences contributes to enhancing willpower. The dynamics with such asymmetic cross effects are known as the predator-prey type dynamics. Our idea is to describe self-control cycles by introducing the predator-prey structure in terms of willpower and self-control experience capital in consumers' cognitive mechanics.

As a natural consequence, we show that the self-control cycle takes place when the asymmetic cross effect is large enough, that is, when both of (i) the self-control cost reducing effect of willpower and (ii) the willpower enhancing effect of self-control are sufficiently strong.

2 Consumers with willpower

We start with modeling consumer behavior with depletable willpower. As the key element, we specify critical roles that are played by the willpower constraint both in the short run and in the long run. In the short run, a self-control plan should be scheduled within the historically given willpower capacity. In the long run, in which willpower varies owing to the depleting and enhancing processes, the stream of the self-control flow is determined jointly with the dynamics of the willpower capacity.

2.1 The model with tempting and non-tempting goods

Consider an infinitely-lived consumer. As in Ikeda and Ojima (2023), there supposed to be two distinct consumption goods, tempting good x (say, a sweet) and non-tempting good c (say, a vegetable). The good x is tempting, like a sweet, in the sense that there is a high temptation level x^T of quantity consumed, where he has to self-control so as to restrain himself from consuming such a great quantity. In contrast, good c, like a vegetable, is not tempting, in the sense that, how lower consumption level he may choose, he can realize it without bearing any self-control costs.

To restrain the consumption level of tempting goods, the consumer has to bear subjective self-control costs in the form of dis-utility. To describe dynamically interactive properties of the self-control behavior, we introduce willpower W, a mental resouce that eases self-control efforts. Formally, we specify the self-control costs as a function of x, x^T, and willpower stock W, $f\left(x, W; x^T\right)$, which is assumed to be twice continuously differentiable and satisfies

$$\begin{aligned} f &> 0, f_x < 0, f_W < 0, f_T > 0, \\ f_{xW} &\geq 0, \text{ and } f \text{ is convex in } (x, W), \end{aligned}$$

where $f_x = \partial f\left(x, W; x^T\right)/\partial x$, etc. and $f_T = \partial f\left(x, W; x^T\right)/\partial x^T$. As x^T is exogenous in the model, we will omit it from f to express $f(x, W)$ insofar as there is no need to make it explicit. From the above, a larger consumption of x and a stronger willpower imply smaller self-control costs, whereas a higher temptation level x^T entails larger self-control costs. With a greater willpower stock, the marginal cost of restraining x $(-f_x)$ is lower.

At each point in time t, the consumer's willpower is re-charged at the rate of ψ, whereas self-control activities at that time deplete willpower and thereby decrease the willpower stock available at the next instant. With the initial willpower stock being exogenously given by W_0, willpower is generated by the following equation:

$$\dot{W}(t) = \psi - \alpha \{f_0 + f(x(t), W(t))\}, \alpha > 0, \tag{1}$$

where a dot represents the time derivative, i.e., $\dot{W}(t) = \partial W(t)/\partial t$; parameter $\alpha (> 0)$ denotes the strength of the willpower-depleting effect of self-control; and f_0 represents exogenous subsistence mental costs, required for miminal self-control in daily life, such as keeping moral rules, surviving in a social competition, putting up with discrimination, etc.

We also incorporate the willpower-enhancing effect of exercising self-control. To do so, we introduce the discounted sum of the past self-control costs M as

$$M(t) = \int_{-\infty}^{t} k \{f_0 + f(x(\tau), W(\tau))\} \exp(-\sigma(\tau - t)) d\tau, \;\; k > 0, \sigma > 0. \tag{2}$$

It represents the cumulative experiences of self-control excution. We call it the self-control experience capital, or simply the self-control experience. By differentiating the above equation by t, the law of motion for the self-control experience is given by

$$\dot{M}(t) = k \{f_0 + f(x(t), W(t))\} - \sigma M(t), \tag{3}$$

where $M(0) = M_0$, a constant. The self-control experience, therefore, is accumulated by the difference of the self-control cost flow $k(f_0 + f)$ and the depreciation σM.

We specify the willpower-enhancing effect of self-control excution by assuming that the willpower re-charge rate ψ linearly and positively depends on the self-control experience M: $\psi(M) = \gamma M + \psi_0$, where $\gamma > 0$. Then, the willpower dynamics in (1) are rewritten as

$$\dot{W}(t) = \gamma M + \psi_0 - \alpha \{f_0 + f(x(t), W(t))\}. \tag{4}$$

As described by (3) and (4), exercising self-control $f_0 + f$ has two effects on willpower that differ both in timing and signs. To be intuitive, self-control behavior at time t $(f_0 + f(t))$, on the one hand, depletes next-instant willpower $(\dot{W}(t))$ by $\alpha(f_0 + f(t))$. On the other hand, it enlarges the next-instant self-control experience capital $(\dot{M}(t))$ if it is large enough to dominate the depreciation $\sigma M(t)$, which in turn contributes to enhancing the next-*next*-instant willpower $(\dot{W}(t + dt))$. Owing to these opposite effects of self-control on willpower that occur with delay, our model has a potential of generating cyclical self-control behavior through the predator-prey type dynamic mechanism.

We specify the consumer's utility as

$$U_0 = \int_0^\infty \{u(x(t)) + v(c(t)) - f_0 - f(x(t), W(t))\} \exp(-rt)\, dt \qquad (5)$$

where the instantaneous utility functions $u(x)$ and $v(c)$ are assumed to be twice continuously differentiable, strictly increasing, and strictly concave; r represents the subjective discount rate, which is constant.

Taking non-tempting good c as numeraire, let q denote the price of the tempting good. It is assumed to be constant. He is endowed with constant income y at each point in time. There is no asset and the consumer's flow budget constraint is given by

$$y = qx(t) + c(t). \qquad (6)$$

2.2 Optimal willpower-self-control dynamics

The consumer maximizes lifetime utility (5) by choosing the time profile of consumption $\{x(t), c(t), W(t)\}_{t=0}^{\infty}$ under the willpower constraint (4), the self-control experience dynamics (3), the flow budget constraint (6), and the two initial conditions, $W(0) = W_0$ and $M(0) = M_0$. Letting λ and π denote the current-value shadow prices of willpower charging $\dot{W}$ and self-control experience accumulation $\dot{M}$, respectively, the optimality conditions are given by

$$u'(x) - (1 + \alpha\lambda - k\pi) f_x(x, W) = qv'(y - qx), \qquad (7)$$

$$\dot{\lambda} = (r + \alpha f_W(x, W))\lambda - kf_W(x, W)\pi + f_W(x, W), \qquad (8)$$

$$\dot{\pi} = (r + \sigma)\pi - \gamma\lambda, \qquad (9)$$

together with the transversality conditions for W and M.

Equation (7) requires that the marginal utility of tempting good consumption x (the L.H.S.) be equal to that of non-tempting good consumption c (the R.H.S.). The marginal utility of x is composed of direct marginal utility $u'(x)$ and the marginal benefits of economizing self-control costs

$((1+\alpha\lambda-k\pi)\,f_x(x,W))$, where the marginal benefits are in turn composed of the direct benefit f_x and the indirect benefit and cost of economizing the present willpower, $-\alpha\lambda f_x$ and $k\pi f_x$. Equation (8) represents the Euler condition for the shadow price of willpower charging. The Euler condition (9) generates the shadow price dynamics for the self-control experience.

2.3 Local dynamics

Supposing that there exists a steady-state solution (x^*, c^*, W^*, M^*), we consider local dynamics around the steady state point. From (7), x can be expressed as a function $X(\lambda,\pi,W)$. By substituting it to (3), (4), (8), and (9), and linearizing the result around the steady state, the optimal dynamics can be reduced to the autonomous differential equation system:

$$\left(\begin{array}{cccc} \dot{W}_t & \dot{M}_t & \dot{\lambda}_t & \dot{\pi}_t \end{array}\right)^T = J\left(\begin{array}{cccc} W_t - W^* & M_t - M^* & \lambda_t - \lambda^* & \pi_t - \pi^* \end{array}\right)^T;$$

$$J = \left(\begin{array}{cccc} -\alpha\Sigma & \gamma & -\alpha^2 f_x^2/\Delta & \alpha k f_x^2/\Delta \\ k\Sigma & -\sigma & \alpha k f_x^2/\Delta & -k^2 f_x^2/\Delta \\ -\xi\Sigma_W & 0 & r+\alpha\Sigma & -k\Sigma \\ 0 & 0 & -\gamma & r+\sigma \end{array}\right), \tag{10}$$

where

$$\begin{aligned} \xi &= 1+\alpha\lambda-k\pi, \Delta = u'' + v'' - \xi f_{xx}, \\ \Sigma &= f_W + f_x f_{xW}\xi/\Delta, \Sigma_W = f_{WW} + f_{xW}^2\xi/\Delta, \end{aligned}$$

and the coefficients are all evaluated at the steady state point.

Let K denote the sum of the second order principal minors of J:

$$K = -\Sigma\left\{\alpha(r+\alpha\Sigma) + 2\gamma k\right\} + r\lambda\alpha^2 f_x \Sigma_W/\Delta - \sigma(r+\sigma).$$

Then, by adopting Lemma 2 of Dockner and Feichtinger (1991), we obtain the following stability condition.

Proposition 1: The optimal consumer dynamics under willpower are locally saddle-point stable if and only if the steady state point satisfies:

1. $\det(J) > 0$
2. $K < -r^2 + \sqrt{r^4 + 4\det(J)}$

Insert Figure 1 around here.

Proposition 2: Suppose that the conditions in Proposition 1 are met, so that the optimal consumer dynamics are locally saddle-point stable. Then, the stable roots are imaginary if and only if the steady state point satisfies

$$K > -2\sqrt{\det(J)}.$$

Propositions 1 and 2 provide the conditions for cyclical consumption behavior under costly self-control. However, it is impossibly hard to show precise conditions for the existence of such an optimal solution. Thus, we next focus on case of linear self-control costs and thereby derive cyclical self-control as a optimal behavior.

3 The linear self-control cost case

To analyze the optimal dynamics more explicitly, consider the case in which the self-control cost function $f(x, W)$ is linear: $f = -\delta x - \varepsilon W; \delta, \varepsilon > 0$. Then, the optimality conditions are rewritten as

$$u'(x) + \delta(1 + \alpha\lambda - k\pi) = qv'(y - qx), \tag{11}$$

$$\dot{\lambda} = (r - \alpha\varepsilon)\lambda + k\varepsilon\pi - \varepsilon, \tag{12}$$

$$\dot{\pi} = (r + \sigma)\pi - \gamma\lambda, \tag{13}$$

$$\dot{W} = \gamma M + \psi_0 - \alpha(f_0 - \delta x - \varepsilon W), \tag{14}$$

$$\dot{M} = k(f_0 - \delta x - \varepsilon W) - \sigma M. \tag{15}$$

where time denotation is omitted without the risk of confusion.

From (11), the optimal consumption of the tempting good x can be solved as a function of the two shadow prices:

$$x = X(\lambda, \pi); X_\lambda > 0, X_\pi < 0. \tag{16}$$

That is, in response to an increase in the shadow price of willpower, λ, the consumer increases tempting-good consumption to save willpower: $X_\lambda > 0$. Upon an increase in the price of the self-control experience capital, π, tempting-good consumption is reduced to accumulate the self-control experience: $X_\pi < 0$.

3.1 The steady-state solution

By setting $\begin{pmatrix} \dot{\lambda} & \dot{\pi} & \dot{W} & \dot{M} \end{pmatrix} = 0$ in (12) through (15), the steady state solution $(\lambda^*, \pi^*, W^*, M^*)$ is obtained as

$$\lambda^* = \frac{\varepsilon\,(r+\sigma)}{(r-\alpha\varepsilon)\,(r+\sigma)+k\gamma\varepsilon}, \pi^* = \frac{\gamma\varepsilon}{(r-\alpha\varepsilon)\,(r+\sigma)+k\gamma\varepsilon}, \tag{17}$$

$$W^* = \frac{f_0 - \delta X\,(\lambda^*, \pi^*)}{\varepsilon} + \frac{\sigma\psi_0}{\varepsilon\,(k\gamma - \alpha\sigma)}, M^* = -\frac{k\psi_0}{(k\gamma - \alpha\sigma)}. \tag{18}$$

For these values to be positive, the following condition should be met:

Assumption 1:

1. $(r-\alpha\varepsilon)\,(r+\sigma)+k\gamma\varepsilon > 0$,
2. $f_0 - \delta X\,(\lambda^*, \pi^*) - \frac{\sigma\psi_0}{\alpha\sigma - k\gamma} > 0$,
3. $\psi_0\,(\alpha\sigma - k\gamma) > 0$.

3.2 Linearized system

In this linear self-control cost model, the coefficient matrix of the linearized system with respect to $\begin{pmatrix} W_t - W^* & M_t - M^* & \lambda_t - \lambda^* & \pi_t - \pi^* \end{pmatrix}^T$, i.e., J in (10), is reduced to:

$$J = \begin{pmatrix} \alpha\varepsilon & \gamma & \alpha\delta X_\lambda & \alpha\delta X_\pi \\ -k\varepsilon & -\sigma & -k\delta X_\lambda & -k\delta X_\pi \\ 0 & 0 & r-\alpha\varepsilon & k\varepsilon \\ 0 & 0 & -\gamma & r+\sigma \end{pmatrix} \equiv \begin{pmatrix} J_{11} & J_{12} \\ 0 & J_{22} \end{pmatrix}. \tag{19}$$

As the matrix is block-recursive, the intrinsic property of the $(W_t - W^*, M_t - M^*)^T$ dynamics is determined by submatrix J_{11}, which has the predetor-prey type structure that antidiagonal elements have different signs: an increase in willpower stock W_t reduces self-control experience accumulation $\dot{M}_t$ by $-k\varepsilon$, whereas an increase in M_t in contrast enhances willpower accumulation $\dot{W}_t$ by γ. These couterveiling interaction of the willpower stock, playing a role as the number of predator, say wolves, and the self-control experience capital, playing a role of the number of prey, say dears, yields cyclical self-control behavior.

The characteristic roots are easily computed from second-order sub-matices J_{11} and J_{22} as

$$\chi_1 = \frac{\alpha\varepsilon - \sigma + \sqrt{(\alpha\varepsilon - \sigma)^2 - 4\varepsilon(k\gamma - \alpha\sigma)}}{2}, \tag{20}$$

$$\chi_2 = \frac{\alpha\varepsilon - \sigma - \sqrt{(\alpha\varepsilon - \sigma)^2 - 4\varepsilon(k\gamma - \alpha\sigma)}}{2}, \tag{21}$$

which are roots for J_{11}, and

$$\omega_1 = \frac{2r + \sigma - \alpha\varepsilon + \sqrt{(\alpha\varepsilon - \sigma)^2 - 4\varepsilon(k\gamma - \alpha\sigma)}}{2}, \tag{22}$$

$$\omega_2 = \frac{2r + \sigma - \alpha\varepsilon - \sqrt{(\alpha\varepsilon - \sigma)^2 - 4\varepsilon(k\gamma - \alpha\sigma)}}{2}, \tag{23}$$

which are roots for J_{12}.

As roots χ_i and ω_i have the same squared root parts, the four roots are either all real or all imaginary. Equations (20) through (23) imply the following:

Lemma 1: Four roots χ_i and ω_i are all real or all imaginary. They are imaginary if and only if

$$(\alpha\varepsilon - \sigma)^2 < 4\varepsilon(k\gamma - \alpha\sigma) \tag{24}$$

As seen from (24), for the roots to be imaginary, it is necessary that $k\gamma > \alpha\sigma$, i.e., the cross indirect effect ($k\gamma$), measured by the product of the antidiagonal elements of J_{11}, is greater than the own direct effect ($\alpha\sigma$), defined as the product of its diagonal elements, meaning that the predator-prey mechanism dominantly works. **Note that the antidiagonal effect $k\gamma$ captures the willpower-enhancing effect of self-control exertion, whereas the diagonal effect $\alpha\sigma$ depends on the willpower-depeleting effect α. Thus, the willpower-enhancing effect needs to be sufficiently large, compared to the willpower-depleting effect, for the cyclical behavior to occur.**

3.3 Saddle-point stability

The fourth-order differential equation system with coefficient matrix (19) describes interactive dynamics of two state variables W_t and M_t, which are historically determined, and two shadow prices λ_t and π_t, which are jumpable. For the system to be saddle-point stable, two of the four roots should be stable roots, and the other two be unstable roots. As shown by Lemma A1 in Appendix, we have $\text{sign}(\text{Re}(\omega_1)) = \text{sign}(\text{Re}(\omega_2))$ under the first condition of Assumption 1, where Re denotes the real part of a complex number. Thus, the system is saddle-point stable if and only if the following two conditions are met:

1. $\text{sign}(\text{Re}\,(\chi_1)) = \text{sign}(\text{Re}\,(\chi_2))$
2. $\text{sign}(\text{Re}\,(\chi_i)) \neq \text{sign}(\text{Re}\,(\omega_i))$

From (20) and (21), the first condition is met if and only if $k\gamma > \alpha\sigma$: the antidiagonal effect is greater than the diagonal effect. When $k\gamma > \alpha\sigma$, in turn, the third condition of Assumption 1 is equivalent to that $\psi_0 < 0$. In what follows, we assume that these inequalities hold valid:

Assumption 2: $k\gamma > \alpha\sigma$ and $\psi_0 < 0$.

Then, a close look at roots χ_i and ω_i $(i = 1, 2)$ yields the following result on saddle-point stability (see Appendix A1.6 for the derivation).

Lemma 2: Under Assumptions 1 and 2, the following two properties hold valid:

1. The steady-state point is locally saddle-point stable if and only if either (i) $\alpha\varepsilon < \sigma$ or (ii) $\alpha\varepsilon > 2r + \sigma$ is satisfied.
2. In case (i), two stable roots are given by χ_1 and χ_2, while in case (ii), they are given by ω_1 and ω_2.

3.4 Self-control cycles

From the recursive block structure of coefficient matrix J, (19), the dynamics of state variables (W, M) are generated by two stable roots (either (χ_1, χ_2) or (ω_1, ω_2)), whereas the dynamic properties of (λ, π) and hence of $x = X(\lambda, \pi)$ are determined solely by the corresponding roots ω_i.

More specifically, in case (i) of Lemma 2, the (W, M) dynamics are driven by stable roots χ_1 and χ_2, while, with ω_i being unstable rootes, (λ_0, π_0) and hence $x_0 = X(\lambda_0, \pi_0)$ jump to their steady-state values, i.e., the (λ, π, x) dynamics are degenerate, so that their time prifles are perfectly smoothed. In case (ii), after (λ_0, π_0) is determined on the saddle hyperplane by the initial condition, (W_0, M_0) =given, the dynamics of (W, M, λ, π) and, hence, of $X(\lambda, \pi)$ are generated with stable roots ω_1 and ω_2.

In sum, consumption (x, c) exhibits non-degenerate dynamics only when roots ω_1 and ω_2 work as stable roots, i.e., case (ii) in Lemma 2. In particular, when roots ω_1 and ω_2 are imaginary, consumption (x, c) exhibits a stable cyclical time-profile. We call it a self-control cycle.[1]

Defnition: Stable cyclical dynamics of consumption (x, c) are called *self-control cycles.*

Note that even when roots ω_i and χ_i are imaginary, self-control cycles do not take place if ω_i are unstable roots and χ_i are stable ones. In that case, cyclical dynamics occur only for mental variables (W, M), which are usually unobservable, whereas the shadow prices and hence consumption are perfectly smoothed and remain at the steady-state values. These behaviorally unobserveable mental cycles are out of interest for the present economic analysis.

Insert Table 1.

Based on Lemmas 1 and 2, Table 1 taxonomizes possible saddle-point stable dynamics into four cases (1)-(4), according to which roots are stable roots and which roots are real numbers, and characterizes behavioral dynamics in each case. Self-control cycles take place in case (4):

Proposition 3: Under Assumptions 1 and 2, self-control cycles occur if and only if case (4) holds valid: $\alpha\varepsilon > 2r + \sigma$ and $(\alpha\varepsilon - \sigma)^2 < 4\varepsilon(\gamma k - \alpha\sigma)$.

As seen from Table 1, four cases (1) - (4) can be characterized by relative magnitudes ε and γk in the parameter space. Figure 2 illustrates the regions of cases (1)-(4) in the parametric space of $(\varepsilon, \gamma k)$. To incorporate the restriction by Assumptions 1 and 2, the figure illustrates the regions $\gamma k > \alpha\sigma$ (Assumption 2) and $(r - \alpha\varepsilon)(r + \sigma) + k\gamma\varepsilon > 0$ (condition 1 of Assumption 2). As the latter inequality can be rewritten as $\varepsilon(\alpha(\sigma + r) - k\gamma) < r(r + \sigma)$, the restriction is depicted as the region $\{(\varepsilon, \gamma k) \,|k\gamma > \alpha(\sigma + r)\} \cup \left\{(\varepsilon, \gamma k) \,\middle|\, k\gamma < \alpha(\sigma + r) \text{ and } k\gamma - \alpha(\sigma + r) > -\frac{r(r+\sigma)}{\varepsilon}\right\}$.[2]

Insert Figure 2.

[1] Unstable consumption time-profiles cannot be a optimum solution, because they necessarily violate the transversality condition and non-negativity.

[2] Condition 2 of Assumption 1 is satisfied if free parameter f_0 is large enough. Condition 3 of Assumption 1 is met if $\psi_0 < 0$.

3.5 Self-control cycle solutions

We now focus on the self-control cycle case, i.e., case (4), where ω_i $(i = 1, 2)$ are stable imaginary roots and χ_i are unstable imaginary roots. As associated characteristic vectors can be complex, we decompose the characteristic vector ν associated with ω_2 into the real and imaginary parts as $\nu \equiv (\mu_W, \mu_M, \mu_\lambda, \mu_\pi)^T + (\phi_W, \phi_M, \phi_\lambda, \phi_\pi)^T i$. Then, expressing ω_2 as $\omega_2 = p - si$, where

$$p = \frac{2r + \sigma - \alpha\varepsilon}{2}, s = \frac{\sqrt{4\varepsilon(\gamma k - \alpha\sigma) - (\alpha\varepsilon - \sigma)^2}}{2},$$

the self-control cycle solution to (19) is obtained as:

$$\begin{pmatrix} W_t - W^* \\ M_t - M^* \\ \lambda_t - \lambda^* \\ \pi_t - \pi^* \end{pmatrix} = \exp(pt) \left\{ (C_1 \cos(st) - C_2 \sin(st)) \begin{pmatrix} \mu_W \\ \mu_M \\ \mu_\lambda \\ \mu_\pi \end{pmatrix} + (C_1 \cos(st) + C_2 \sin(st)) \begin{pmatrix} \phi_W \\ \phi_M \\ \phi_\lambda \\ \phi_\pi \end{pmatrix} \right\}, \quad (25)$$

with the x_t dynamics being generated by (16), where C_1 and C_2 are constants. Note that two similar terms associated with roots χ_i that should have appeared on the right hand side are not there, because χ_i are unstable roots, so that the associated constants are optimally set zero.

Set $t = 0$ in solution (25) and substitute the initial conditions for W_0 and M_0 into the result. For given steady-state values $(W^*, M^*, \lambda^*, \pi^*)$, the resulting fourth-order simultaneous equation system determines constants C_1 and C_2 and the intial values of the shadow prices λ_0 and π_0.[3] Once those values are given, (25) and (16) uniquely generates the cyclical time profile of $(W_t, M_t, \lambda_t, \pi_t, x_t)$ for $t \geq 0$.

3.6 Alternating inhibition and relaxation phases: Numerical examples

In the case of the self-control cycle, upon external shocks, consumer behavior exhibits alterantingly an inhibition phase, in which tempting good consumption x continues to decline, and a relaxation phase, in which x continues to go up. To show numerical examples of self-control cycles, we specify parameter

[3] For derivation, see Appendix A3.1.

values as in Table 2, where the values satisfy Assumptions 1 and 2 and the conditions in Proposition 3. Initial values W_0 and M_0 are set as equal to the steady state values of W and M under the parameter values.

Insert Table 2. Parameter values.

3.6.1 An increase in self-control needs f_0

Figure 3 illustrates cyclical adjustments to an permenent increase in external self-control needs f_0 from 50 to 55.

Upon the increased needs for self-control, W^* needs to be increased (see (18)), whereas M^* and x^* are unchanged. To attain the greater W^*, the consumer initially reduces x_t discretely from the initial steady-state level and, thereafter, further decreases it over time: an inhibition phase starts (I-1, in Figure 3). The inhibition leads to accumulation of self-control experience capital M_t (I-2) Although W_t initally gets depleted due to the enhanced self-contol behavior, it thereafter reverses to an upward process due to the accumulation of M_t (I-3). When M_t is accumulated enough that its scarcity reached a critial low level, the consumer stops inhibiting tempting good consumption and thereafter a relaxation phase starts, with x_t reverting into an increasing process (R-1). In this phase, M_t is decumulated due to relaxation with over-time increases in x_t (R-2), whereas W_t increases until M_t becomes low enough, and thereafter turns into a decreasing process (R-3). When M_t reaches a critial low level, the upward motion of x_t levels off, and thereafter the relaxtion phase reverts to the second inhibition phase (I-4).

In this way, through adjustment process with the two phases alternating continuously, tempting good consumption converges to the same steady-state level as the pre-shock level, where the greater post-shock self-control is attained with a greater willpower stock that was accumulated under enhanced self-control experiences in the interim run.

Figure 3. Cyclical adjustments to an increase in external self-control needs f_0.

3.6.2 An increase in income y

By considering a permenent increase in income y from 1,000 to 1,010, Figure 4 depicts another typical example of cyclical adjustments, in which the adjustment starts with the relaxation phase. Upon the positive income shock, the consumer increases steady-state tempting good consumption x^*. Owing to the resulting decrease in the self-control needs, willpower W^* is decumulated, whereas M^* is not affected (see (18)). To attain the new steady-state consumption plan, the consumer initially enjoys binge consumption by increasing x discretely and, thereafter continues to further increase it over time ((R-1) in Figure 4) until self-control experience capital M_t declines to a critical level (R-2). At the outset of the relaxation phase, W_t exhibits a short period of accumulation. But, sooner or later, it turns into a decumulation process owing to over-time decreases in M_t (R-3). When M_t reaches a critical low level, x_t reverts to a decreasing process, and the inhibition phase begins (I-1). In this way, through the opposite cyclical process that occurs in Figure 3, x_t converges to a higher steady-state level with a smaller willpower stock.

Figure 4. Cyclical adjustments to an increase in income y.

As a whole, as in the predetor-prey dynamics, in which the cycles of the number of the prey (deer) precede those of the number of the predator (wolf), and as shown in Figures 3 and 4, the cycles of self-control experience capital M_t (as prey) precede those of the willpower stock W_t (as predator). Those cycles are brought about by preceding, opposedly-directed cyclical behavior of tempting good consumption x_t that the consumer optimally chooses.

4 Conclusions

By incorporating dynamic interaction between depletable willpower and self-control as of restraining tempting consumption, we have examined the possibility of self-control cycles, or abstinence-indulgence cycles. The self-control cycle takes place when (i) the self-control cost reducing effect of willpower and (ii) the willpower enhancing effect of self-control are sufficiently strong.

Appendix

A1 Solutions in the linear self-control cost case

A1.1 Optimization

Consider the linear model with $f(x, W) = f_0 - (\delta x + \varepsilon W)$ and

$$\dot{W} = \gamma M + \psi_0 - \alpha f_0 + \alpha(\delta x + \varepsilon W) \quad (26)$$

$$\dot{M} = k(f_0 - (\delta x + \varepsilon W)) - \sigma M \quad (27)$$

Define the Hamiltonian function as

$$\begin{aligned} H &= u(x) + v(y - qx) - f_0 + (\delta x + \varepsilon W) + \lambda\{\gamma M + \psi_0 - \alpha f_0 + \alpha(\delta x + \varepsilon W)\} \\ &\quad + \pi\{k(f_0 - (\delta x + \varepsilon W)) - \sigma M\} \end{aligned}$$

where c is eliminated by using the budget equation $qx + c = y$. Then, the first-order conditions are given by

$$\begin{aligned} H_x &= 0; u'(x) - qv'(y - qx) + \delta + \lambda\alpha\delta - \pi k\delta = 0 \\ &\Rightarrow u'(x) + \delta(1 + \lambda\alpha - \pi k) = qv'(y - qx) \end{aligned}$$

$$\begin{aligned} \dot{\lambda} - r\lambda &= -H_W : \dot{\lambda} - r\lambda = -(\varepsilon + \alpha\varepsilon\lambda - k\varepsilon\pi) \\ &\Rightarrow \dot{\lambda} = (r - \alpha\varepsilon)\lambda + k\varepsilon\pi - \varepsilon \end{aligned}$$

$$\begin{aligned} \dot{\pi} - r\pi &= -H_M : \dot{\pi} - r\pi = -(\gamma\lambda - \sigma\pi) \\ &\Rightarrow \dot{\pi} = -\gamma\lambda + (r + \sigma)\pi \end{aligned}$$

which correspond to (11) though (13), respectively. From the first equation above, we have $u'(x) - qv'(y - qx) = -\delta(1 + \lambda\alpha - \pi k)$, which implies optimal x should depend positively on λ and negatively on π. We express this relation by $X(\lambda, \pi)$, where

$$X_\lambda = -\frac{\alpha\delta}{u'' + q^2 v''} > 0; X_\pi = \frac{k\delta}{u'' + q^2 v''} < 0.$$

A1.2 Steady-state solutions and Assimption 1

Condition $\left(\dot{\lambda}, \dot{\pi}, \dot{W}, \dot{M}\right) = 0$ implies

$$(r - \alpha\varepsilon)\lambda^* + k\varepsilon\pi^* - \varepsilon = 0$$

$$-\gamma\lambda^* + (r+\sigma)\,\pi^* = 0$$

$$\begin{aligned}\gamma M^* + \psi_0 - \alpha f_0 + \alpha\left(\delta x^* + \varepsilon W^*\right) &= 0 \\ &\Rightarrow \alpha\varepsilon W^* + \gamma M^* = -\psi_0 + \alpha\left(f_0 - \delta X\left(\lambda^*, \pi^*\right)\right)\end{aligned}$$

$$\begin{aligned}k\left(f_0 - \left(\delta x^* + \varepsilon W^*\right)\right) - \sigma M^* &= 0 \\ &\Rightarrow k\varepsilon W^* + \sigma M^* = k\left(f_0 - \delta X\left(\lambda^*, \pi^*\right)\right)\end{aligned}$$

The first two equations can be summarized as

$$\begin{pmatrix} r-\alpha\varepsilon & k\varepsilon \\ -\gamma & r+\sigma \end{pmatrix}\begin{pmatrix} \lambda^* \\ \pi^* \end{pmatrix} = \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}$$

which can be solved as

$$\begin{aligned}\begin{pmatrix} \lambda^* \\ \pi^* \end{pmatrix} &= \frac{1}{(r-\alpha\varepsilon)(r+\sigma)+\gamma k\varepsilon}\begin{pmatrix} r+\sigma & -k\varepsilon \\ \gamma & r-\alpha\varepsilon \end{pmatrix}\begin{pmatrix} \varepsilon \\ 0 \end{pmatrix} \\ &= \frac{1}{(r-\alpha\varepsilon)(r+\sigma)+\gamma k\varepsilon}\begin{pmatrix} \varepsilon(r+\sigma) \\ \varepsilon\gamma \end{pmatrix} \qquad (28)\end{aligned}$$

This solution corresponds to (17).

The other two equations can be summed up to

$$\begin{pmatrix} \alpha\varepsilon & \gamma \\ k\varepsilon & \sigma \end{pmatrix}\begin{pmatrix} W^* \\ M^* \end{pmatrix} = \begin{pmatrix} -\psi_0 + \alpha\left(f_0 - \delta X\left(\lambda^*, \pi^*\right)\right) \\ k\left(f_0 - \delta X\left(\lambda^*, \pi^*\right)\right) \end{pmatrix}$$

which can be solved as

$$\begin{aligned}\begin{pmatrix} W^* \\ M^* \end{pmatrix} &= \frac{1}{\varepsilon(\alpha\sigma - k\gamma)}\begin{pmatrix} \sigma & -\gamma \\ -k\varepsilon & \alpha\varepsilon \end{pmatrix}\begin{pmatrix} -\psi_0 + \alpha\left(f_0 - \delta X\left(\lambda^*, \pi^*\right)\right) \\ k\left(f_0 - \delta X\left(\lambda^*, \pi^*\right)\right) \end{pmatrix} \\ &= \frac{1}{\varepsilon(\alpha\sigma - k\gamma)}\begin{pmatrix} \sigma\left\{-\psi_0 + \alpha\left(f_0 - \delta X\left(\lambda^*, \pi^*\right)\right)\right\} - \gamma k\left(f_0 - \delta X\left(\lambda^*, \pi^*\right)\right) \\ -k\varepsilon\left\{-\psi_0 + \alpha\left(f_0 - \delta X\left(\lambda^*, \pi^*\right)\right)\right\} + \alpha\varepsilon k\left(f_0 - \delta X\left(\lambda^*, \pi^*\right)\right) \end{pmatrix} \\ &= \frac{1}{\varepsilon(\alpha\sigma - k\gamma)}\begin{pmatrix} (\alpha\sigma - \gamma k)\left(f_0 - \delta X\left(\lambda^*, \pi^*\right)\right) - \sigma\psi_0 \\ \varepsilon k\psi_0 \end{pmatrix} \qquad (29)\end{aligned}$$

corresponding to (18).

By considering the positivity conditions for $(\lambda^*, \pi^*, W^*, M^*)$, the conditions of Assumption 1 are derived. First, from (28), (λ^*, π^*) is positive if and only if

$$(r-\alpha\varepsilon)(r+\sigma)+\gamma k\varepsilon > 0,$$

which is equivalent to inequality 1 of Assumption 1. Second, from (29), $W^* > 0$ if and only if

$$f_0 - \delta X(\lambda^*, \pi^*) - \frac{\sigma\psi_0}{\alpha\sigma - k\gamma} > 0,$$

and $M^* > 0$ if and only if

$$\psi_0 (\alpha\sigma - k\gamma) > 0.$$

The last two inequalities are equivalent to inequalities 2 and 3 of Assumption 1.

A1.3 Linearized system with coefficient matrix (19)

The dynamic system is given by

$$\begin{aligned}\dot{W} &= \gamma M + \psi_0 - \alpha f_0 + \alpha(\delta X(\lambda, \pi) + \varepsilon W) \\ &\simeq \alpha\varepsilon(W - W^*) + \gamma(M - M^*) + \alpha\delta X_\lambda(\lambda - \lambda^*) + \alpha\delta X_\pi(\pi - \pi^*)\end{aligned}$$

$$\begin{aligned}\dot{M} &= k(f_0 - (\delta X(\lambda, \pi) + \varepsilon W)) - \sigma M \\ &\simeq -\varepsilon k(W - W^*) - \sigma(M - M^*) - k\delta X_\lambda(\lambda - \lambda^*) - k\delta X_\pi(\pi - \pi^*)\end{aligned}$$

$$\begin{aligned}\dot{\lambda} &= (r - \alpha\varepsilon)\lambda + k\varepsilon\pi - \varepsilon \\ &= (r - \alpha\varepsilon)(\lambda - \lambda^*) + k\varepsilon(\pi - \pi^*)\end{aligned}$$

$$\begin{aligned}\dot{\pi} &= -\gamma\lambda + (r + \sigma)\pi \\ &= -\gamma(\lambda - \lambda^*) + (r + \sigma)(\pi - \pi^*)\end{aligned}$$

It follows that the coefficient matrix J of the linearized system with respect to $\begin{pmatrix} W - W^* & M - M^* & \lambda - \lambda^* & \pi - \pi^* \end{pmatrix}$ is given by

$$J = \begin{pmatrix} \alpha\varepsilon & \gamma & \alpha\delta X_\lambda & \alpha\delta X_\pi \\ -k\varepsilon & -\sigma & -k\delta X_\lambda & -k\delta X_\pi \\ 0 & 0 & r - \alpha\varepsilon & k\varepsilon \\ 0 & 0 & -\gamma & r + \sigma \end{pmatrix} \left(\equiv \begin{pmatrix} J_{11} & J_{12} \\ 0 & J_{22} \end{pmatrix}\right)$$

as is shown by (19).

A1.4 Characteristic roots

Characteristic roots χ_i for J_{11} are obtained from

$$\begin{aligned}(|\chi I - J_{11}| &=)\begin{vmatrix} \chi - \alpha\varepsilon & -\gamma \\ k\varepsilon & \chi + \sigma \end{vmatrix} = 0 \\ &\Leftrightarrow (\chi - \alpha\varepsilon)(\chi + \sigma) + \varepsilon k\gamma = \chi^2 + (\sigma - \alpha\varepsilon)\chi + \varepsilon(k\gamma - \alpha\sigma) = 0\end{aligned}$$

Thus, we have

$$\chi_1 = \frac{\alpha\varepsilon - \sigma + \sqrt{(\alpha\varepsilon - \sigma)^2 - 4\varepsilon(k\gamma - \alpha\sigma)}}{2}; \tag{30}$$

$$\chi_2 = \frac{\alpha\varepsilon - \sigma - \sqrt{(\alpha\varepsilon - \sigma)^2 - 4\varepsilon(k\gamma - \alpha\sigma)}}{2} \tag{31}$$

which are (20) and (21) in the text. Roots ω_i for J_{22} are from

$$\begin{aligned}(|\omega I - J_{22}| &=)\begin{vmatrix} \omega - (r - \alpha\varepsilon) & -k\varepsilon \\ \gamma & \omega - (r + \sigma) \end{vmatrix} = 0 \\ &\Leftrightarrow \{\omega - (r - \alpha\varepsilon)\}\{\omega - (r + \sigma)\} + \varepsilon k\gamma \\ &= \omega^2 - (2r - \alpha\varepsilon + \sigma)\omega + (r - \alpha\varepsilon)(r + \sigma) + \varepsilon k\gamma = 0\end{aligned}$$

which can be solved as

$$\begin{aligned}\omega_1 &= \frac{2r - \alpha\varepsilon + \sigma + \sqrt{(2r - \alpha\varepsilon + \sigma)^2 - 4\{(r - \alpha\varepsilon)(r + \sigma) + \varepsilon k\gamma\}}}{2} \\ &= \frac{2r - \alpha\varepsilon + \sigma + \sqrt{(2r - \alpha\varepsilon + \sigma)^2 - 4r(r - \alpha\varepsilon + \sigma) - 4\varepsilon(k\gamma - \alpha\sigma)}}{2} \quad (32) \\ \omega_2 &= \frac{2r - \alpha\varepsilon + \sigma - \sqrt{(2r - \alpha\varepsilon + \sigma)^2 - 4r(r - \alpha\varepsilon + \sigma) - 4\varepsilon(k\gamma - \alpha\sigma)}}{2}\end{aligned}$$

As the first two terms in the root can be reduced to

$$\begin{aligned}& (2r - \alpha\varepsilon + \sigma)^2 - 4r(r - \alpha\varepsilon + \sigma) \\ =\ & 4r^2 + \alpha^2\varepsilon^2 + \sigma^2 - 4r\alpha\varepsilon + 4r\sigma - 2\alpha\varepsilon\sigma - 4r^2 + 4r\alpha\varepsilon - 4r\sigma \\ =\ & \alpha^2\varepsilon^2 - 2\alpha\varepsilon\sigma + \sigma^2 \\ =\ & (\alpha\varepsilon - \sigma)^2\end{aligned}$$

Thus we have

$$\begin{aligned}
\omega_1 &= \frac{2r - \alpha\varepsilon + \sigma + \sqrt{(\alpha\varepsilon - \sigma)^2 - 4\varepsilon(k\gamma - \alpha\sigma)}}{2} && (33) \\
&= \frac{\alpha\varepsilon - \sigma + \sqrt{(\alpha\varepsilon - \sigma)^2 - 4\varepsilon(k\gamma - \alpha\sigma)}}{2} + \frac{2r - \alpha\varepsilon + \sigma - \alpha\varepsilon + \sigma}{2} \\
&= \chi_1 + r - \alpha\varepsilon + \sigma && (34) \\
\omega_2 &= \frac{2r - \alpha\varepsilon + \sigma - \sqrt{(\alpha\varepsilon - \sigma)^2 - 4\varepsilon(k\gamma - \alpha\sigma)}}{2} && (35) \\
&= \frac{\alpha\varepsilon - \sigma - \sqrt{(\alpha\varepsilon - \sigma)^2 - 4\varepsilon(k\gamma - \alpha\sigma)}}{2} + \frac{2r - \alpha\varepsilon + \sigma - \alpha\varepsilon + \sigma}{2} \\
&= \chi_2 + r - \alpha\varepsilon + \sigma && (36)
\end{aligned}$$

(33) and (35) are (22) and (23) in the text, respectively.

A1.5 Imaginary roots or real roots

By comparing χ_i ((30) and (31)) and ω_i ((33) and (36)), we see that four roots are all real or all imaginary. It is easy to see from the equations that they are all imaginary if and only if $(\alpha\varepsilon - \sigma)^2 < 4\varepsilon(k\gamma - \alpha\sigma)$, as summarized in Lemma 1 in the text.

A1.6 Saddle-point stability and Lemma 2

We begin with proving the following property:

Lemma A1: Under inequality 1 in Assumption 1 ($(r - \alpha\varepsilon)(r + \sigma) + k\gamma\varepsilon > 0$), sign(Re (ω_1)) =sign(Re (ω_2)) .

Proof: When ω_1 and ω_2 are imaginary, it is trivially valid that sign(Re (ω_1)) = sign(Re (ω_2)). We thus focus on the case in which they are real. From (32), the inside of the square root satisfies

$$\begin{aligned}
& (2r - \alpha\varepsilon + \sigma)^2 - 4r(r - \alpha\varepsilon + \sigma) - 4\varepsilon(k\gamma - \alpha\sigma) \\
= \; & (2r - \alpha\varepsilon + \sigma)^2 - 4\left(r^2 + (\sigma - \alpha\varepsilon)r + \varepsilon\alpha\sigma\right) - 4\varepsilon k\gamma \\
= \; & (2r - \alpha\varepsilon + \sigma)^2 - 4\{(r + \sigma)(r - \alpha\varepsilon) + \varepsilon k\gamma\} \\
< \; & (2r - \alpha\varepsilon + \sigma)^2
\end{aligned}$$

under condition 1 in Assumption 1. Therefore, when ω_i are real roots, if $2r - \alpha\varepsilon + \sigma > 0$, $\omega_1 > \omega_2 > 0$. If $2r - \alpha\varepsilon + \sigma < 0$, $\omega_2 < \omega_1 < 0$.■

For the system to be saddle-point stable, two of the four roots should be stable roots, and the other two be unstable roots. As pointed out in the first paragraph of Section 3.3, From Lemma A1, the saddle point stability is the case if and only if

1. $\text{sign}(\text{Re}(\chi_1)) = \text{sign}(\text{Re}(\chi_2))$
2. $\text{sign}(\text{Re}(\chi_i)) \neq \text{sign}(\text{Re}(\omega_i))$

As for condition 1, (30) and (31) imply the following lemma:

Lemma A2: $\text{sign}(\text{Re}(\chi_1)) = \text{sign}(\text{Re}(\chi_2))$ if and only if $k\gamma - \alpha\sigma > 0$.
Proof: The inside of the square root in (30) and (31) is smaller than $\alpha\varepsilon - \sigma$ if and only if $k\gamma - \alpha\sigma > 0$.■

From condition 3 of Assumption 1 ($\psi_0 (\alpha\sigma - k\gamma) > 0$), inequality $k\gamma - \alpha\sigma > 0$ implies $\psi_0 < 0$. Assumption 2 in the text assumes that these two inequalities hold valid.

Suppose that Assumptions 1 and 2 hold. Then, from (30) and (31), $\text{sign}(\alpha\varepsilon - \sigma) = \text{sign}(\text{Re}(\chi_1)) = \text{sign}(\text{Re}(\chi_2))$. Similarly, from (33) and (35), $\text{sign}(2r - \alpha\varepsilon + \sigma) = \text{sign}(\text{Re}(\omega_1)) = \text{sign}(\text{Re}(\omega_2))$. Thus, the saddle-point stability condition $\text{sign}(\text{Re}(\chi_i)) \neq \text{sign}(\text{Re}(\omega_i))$ $(i = 1, 2)$ is satisfied if and only if $(\alpha\varepsilon - \sigma)(2r - \alpha\varepsilon + \sigma) < 0$. This is satisfied either when (i) $\alpha\varepsilon < \sigma$ (hence $\alpha\varepsilon < \sigma + 2r$) (i.e., case (i) in Lemma 2-1) or when (ii) $\alpha\varepsilon > \sigma + 2r$ (hence $\alpha\varepsilon > \sigma$) (case (ii) in Lemma 2-1). These conditions are summarized in item 1 of Lemma 2. Item 2 of Lemma 2 is implied from the expressions of χ_i and ω_i ((30), (31), (33), and (35)).

A2 Proposition 1 in the linear self-cotrol case

In this appendix, we show that, under Assumptions 1 and 2, the saddle-point stability condition obtained in the linear self-control case in Section 3, i.e., the item 1 of Lemma 2,

$$(\alpha\varepsilon - \sigma)(\alpha\varepsilon - \sigma - 2r) > 0, \tag{37}$$

is equivalenet to the saddle-point stability condition in Proposition 1. In the linear model, $\det(J)$ reduces to

$$\det(J) = \varepsilon(\gamma k - \alpha\varepsilon)\{(r - \alpha\varepsilon)(r + \sigma) + \varepsilon\gamma k\} > 0,$$

where the last inequality comes from Assumptions 1 and 2. As item 1 of Proposition 1 is thus satisfied in the linear case, what we should show here is that condition (37) is satisified if and only if item 2 of Proposition 1 is met, or equivalently, if and only if: either condition of the following two conditions 1 and 2 is satisified

1. $K > 0$ and $\det(J) > \left(\frac{K}{2}\right)^2 + r^2\frac{K}{2}$

2. $K < 0$

We shall prove that if either condition 1 or 2 is met, inequality (37) holds valid.

Lemma A3: Condition 1 implies inequality (37).
Proof: We have

$$\begin{aligned} & \det(J) - \left(\frac{K}{2}\right)^2 - r^2\frac{K}{2} \qquad (38) \\ = \ & \frac{1}{4}(\alpha\varepsilon - \sigma)(\alpha\varepsilon - \sigma - 2r)\{K + \varepsilon(\gamma k - \alpha\varepsilon) + (r - \alpha\varepsilon)(r + \sigma) + \varepsilon\gamma k\}. \end{aligned}$$

Under Assumptions 1 and 2, $\gamma k - \alpha\varepsilon > 0$ and $(r - \alpha\varepsilon)(r + \sigma) + \varepsilon\gamma k > 0$. Therefore, condition 1 implies (37). ■

Lemma A4: Condition 2 implies inequality (37).
Proof: We prove this by seperating two cases (i) $\alpha\varepsilon - \sigma \leq r$ and (ii) $\alpha\varepsilon - \sigma > r$.

Consider case (i): $\alpha\varepsilon - \sigma \leq r$. As K can be rewritten as $(\alpha\varepsilon - \sigma)\{r - (\alpha\varepsilon - \sigma)\} + 2\varepsilon(\gamma k - \alpha\varepsilon)$, condition 2 ($K < 0$) implies

$$\alpha\varepsilon - \sigma < -\frac{2\varepsilon(\gamma k - \alpha\varepsilon)}{r - (\alpha\varepsilon - \sigma)} < 0,$$

where the last inequality holds vaild in the present case (i) under Assumption 2. As $2r + \sigma - \alpha\varepsilon > r + \sigma - \alpha\varepsilon > 0$ in case (i), this implies (37).

Next, consider case (ii): $\alpha\varepsilon - \sigma > r$. Item 1 of Assumption 1 can be rewritten as

$$\varepsilon\gamma k - \varepsilon\alpha\sigma > -r^2 + (\varepsilon\alpha - \sigma)r. \qquad (39)$$

We have

$$\begin{aligned} K &= -\alpha^2\varepsilon^2 - \sigma r - \sigma^2 + \alpha\varepsilon r + 2\varepsilon\gamma k \\ &= (2\varepsilon\gamma k - 2\varepsilon\alpha\sigma) + 2\varepsilon\alpha\sigma - \alpha^2\varepsilon^2 - \sigma r - \sigma^2 + \alpha\varepsilon r \\ &> 2\left(-r^2 + (\varepsilon\alpha - \sigma)r\right) - \alpha^2\varepsilon^2 + (\varepsilon\alpha - \sigma)r - \sigma^2 + 2\varepsilon\alpha\sigma \quad \text{(from (39))} \\ &= -2r^2 + 3(\varepsilon\alpha - \sigma)r - (\varepsilon\alpha - \sigma)^2 \\ &= (2r - \alpha\varepsilon + \sigma)(\alpha\varepsilon - \sigma - r). \end{aligned}$$

Thus, condition 2 ($K < 0$) implies $2r - \alpha\varepsilon + \sigma < 0$,because $\alpha\varepsilon - \sigma - r > 0$ in case (ii). As $\alpha\varepsilon - \sigma > r > 0$ in this case, (37) holds valid under condition 2. ■

Lemmas A3 and A4 show that (37) holds valid if either condition 1 or 2 is met. In Lemma A5 below, we next show that the converse relationship is also true.

Lemma A5: Inequality (37) implies that condition 1 or 2 hold valid.
Proof: Suppose that (37) holds valid. Then, $K \geq 0$ or $K < 0$. Consider the case that $K \geq 0$. In this case, from equation (38), inequality (37) implies condition 1 under Assumptions 1 and 2. When $K < 0$ is the case, condition 2 trivially holds valid. ■

A3 Explicit solutions in the linear case

We here derive explicit cyclical solutions for optimal consumer behavior in the liear case by solving

$$\left(\begin{array}{cccc} \dot{W}_t & \dot{M}_t & \dot{\lambda}_t & \dot{\pi}_t \end{array} \right)^T = J \left(\begin{array}{cccc} W_t - W^* & M_t - M^* & \lambda_t - \lambda^* & \pi_t - \pi^* \end{array} \right)^T, \tag{40}$$

under the initial conditions that (W_0, M_0) is given, where J is given (19).

A3.1 General solutions in the case of self-control cycles

Consider the case that self-control cycles take place under Assumptions 1 and 2. From Proposition 3, it is valid here that $\alpha\varepsilon > 2r + \sigma$ and $(\alpha\varepsilon - \sigma)^2 < 4\varepsilon(\gamma k - \alpha\sigma)$, so that ω_i $(i = 1, 2)$ are stable imaginary roots ($\mathrm{Re}(\omega_i) = (2r + \sigma - \alpha\varepsilon)/2 < 0$), whereas χ_1 and χ_2 $(i = 1, 2)$ are unstable imaginary roots ($\mathrm{Re}(\chi_i) = \alpha\varepsilon - \sigma > 0$). As associated characteristic vectors can be complex, we denote the characteristic vector associated with ω_2, ν, as $\nu = \mu + \phi i$, where $\nu, \mu, \phi \in R^4$. More explicitly, we use the following notation:

$$\begin{aligned} v &\equiv (v_W, v_M, v_\lambda, v_\pi)^T \\ &\equiv \mu + \phi i \\ &\equiv (\mu_W, \mu_M, \mu_\lambda, \mu_\pi)^T + (\phi_W, \phi_M, \phi_\lambda, \phi_\pi)^T i. \end{aligned} \tag{41}$$

By definition, ν satisfies $(J - \omega_2 I)(\mu + \phi i) = 0$, with J being given by (19).

Expressing ω_1 and ω_2 as $\omega_1 = p + si$ and $\omega_2 = p - si$, where

$$\begin{aligned} p &= \frac{2r + \sigma - \alpha\varepsilon}{2}, \\ s &= \frac{\sqrt{4\varepsilon(\gamma k - \alpha\sigma) - (\alpha\varepsilon - \sigma)^2}}{2}, \end{aligned}$$

then, general solutions to (19) are obtained in the following form:

$$\begin{pmatrix} W_t - W^* \\ M_t - M^* \\ \lambda_t - \lambda^* \\ \pi_t - \pi^* \end{pmatrix} = \exp(pt) \left\{ (C_1 \cos(st) - C_2 \sin(st)) \begin{pmatrix} \mu_W \\ \mu_M \\ \mu_\lambda \\ \mu_\pi \end{pmatrix} + \right.$$

$$\left. (C_1 \cos(st) + C_2 \sin(st)) \begin{pmatrix} \phi_W \\ \phi_M \\ \phi_\lambda \\ \phi_\pi \end{pmatrix} \right\} \tag{42}$$

where C_1 and C_2 are constants. Note that two similar terms associated with roots χ_i that should have appeared on the right hand side are not there, because χ_i are unstable roots, so that the associated constants are optimally set zero.

We show explicitly characteristic vector $\nu = \mu + \phi i$. By definition, $v \equiv (v_W, v_M, v_\lambda, v_\pi)^T$ satisfies $(J - \omega_2 I)\, v = 0$:

$$\begin{pmatrix} \alpha\varepsilon - \omega_2 & \gamma & H_\lambda & H_\pi \\ -k\varepsilon & -\sigma - \omega_2 & I_\lambda & I_\pi \\ 0 & 0 & r - \alpha\varepsilon - \omega_2 & k\varepsilon \\ 0 & 0 & -\gamma & r + \sigma - \omega_2 \end{pmatrix} \begin{pmatrix} v_W \\ v_M \\ v_\lambda \\ v_\pi \end{pmatrix} = 0 \tag{43}$$

where $(H_\lambda, H_\pi, I_\lambda, I_\pi)$ denotes $(\alpha\delta X_\lambda, \alpha\delta X_\pi, -k\delta X_\lambda, -k\delta X_\pi)$, the elements of the 2×2 north-east submatrix J_{12} in matrix J in (19). As ω_2 is a characteristic root for the 2×2 south-east submatrix J_{22}, the third and fourth equations in (43) are not independent of each other. We thus set $v_\lambda = r + \sigma - \omega_2$ and $v_\pi = \gamma$, so that the real and imaginary parts of the vector elements are given as

$$\begin{aligned} \begin{pmatrix} v_\lambda \\ v_\pi \end{pmatrix} &\equiv \begin{pmatrix} \mu_\lambda \\ \mu_\pi \end{pmatrix} + \begin{pmatrix} \phi_\lambda \\ \phi_\pi \end{pmatrix} i \\ &= \begin{pmatrix} r + \sigma - p \\ \gamma \end{pmatrix} + \begin{pmatrix} s \\ 0 \end{pmatrix} i \end{aligned} \tag{44}$$

Substitute (44) for v_λ and v_π into (43). Then, the first and second equations in (43) can be solved for (v_W, v_M) to obtain

$$\begin{aligned} &\begin{pmatrix} v_W \\ v_M \end{pmatrix} \\ = &\frac{1}{D} \begin{pmatrix} (\omega_2 + \sigma)(\omega_2 - r - \sigma) H_\lambda - \gamma(\omega_2 + \sigma) H_\pi + \gamma(\omega_2 - r - \sigma) I_\lambda - \gamma^2 I_\pi \\ -k\varepsilon(\omega_2 - r - \sigma) H_\lambda + k\varepsilon\gamma H_\pi + (\omega_2 - \alpha\varepsilon)(\omega_2 - r - \sigma) I_\lambda - \gamma(\omega_2 - \alpha\varepsilon) I_\pi \end{pmatrix} \\ = &\frac{1}{D} \begin{pmatrix} R_W + C_W i \\ R_M + C_M i \end{pmatrix}, \end{aligned} \tag{45}$$

where D denotes the determinant of the 2×2 north-west submatrix J_{11} :

$$\begin{aligned} D &= (\omega_2 - \alpha\varepsilon)(\omega_2 + \sigma) + \gamma k \varepsilon \\ &= A - Bi; \end{aligned}$$

$$\begin{aligned} A &= p^2 - s^2 + p(\sigma - \alpha\varepsilon) - \alpha\varepsilon\sigma + k\varepsilon\gamma, \\ B &= 2p + \sigma - \alpha\varepsilon, \end{aligned}$$

and where

$$\begin{aligned} R_W &= \left(p^2 - s^2\right) H_\lambda + p\left(\gamma I_\lambda - r H_\lambda - \gamma H_\pi\right) - \sigma(r+\sigma) H_\lambda - \sigma\gamma H_\pi - \gamma(r+\sigma) I_\lambda - \gamma^2 I_\pi, \\ C_W &= -s\left(2pH_\lambda + \gamma I_\lambda - r H_\lambda - \gamma H_\pi\right), \\ R_M &= \left(p^2 - s^2\right) I_\lambda - p\left(k\varepsilon H_\lambda + (r + \sigma + \alpha\varepsilon) I_\lambda + \gamma I_\pi\right) + k\varepsilon(r+\sigma) H_\lambda + k\varepsilon\gamma H_\pi \\ & \quad + \alpha\varepsilon(r+\sigma) I_\lambda + \alpha\varepsilon\gamma I_\pi, \\ C_M &= s\left(2pI_\lambda + k\varepsilon H_\lambda + (r+\sigma+\alpha\varepsilon) I_\lambda + \gamma I_\pi\right). \end{aligned}$$

By multiplying the conjugate complex number of D to the denominator and the numerator of (45), we finally obtain

$$\begin{aligned} & \begin{pmatrix} v_W \\ v_M \end{pmatrix} \\ = & \frac{1}{A^2 + B^2} \begin{pmatrix} R_W A - C_W B + (C_W A + R_W B) i \\ R_M A - C_M B + (C_M A + R_M B) i \end{pmatrix}, \end{aligned}$$

which implies that the real part of $(v_W, v_M)^T$ is given by

$$\begin{pmatrix} \mu_W \\ \mu_M \end{pmatrix} = \begin{pmatrix} (R_W A - C_W B) / (A^2 + B^2) \\ (R_M A - C_M B) / (A^2 + B^2) \end{pmatrix}, \tag{46}$$

and the coefficients of the imaginary part are given by

$$\begin{pmatrix} \phi_W \\ \phi_M \end{pmatrix} = \begin{pmatrix} (C_W A + R_W B) / (A^2 + B^2) \\ (C_M A + R_M B) / (A^2 + B^2) \end{pmatrix}. \tag{47}$$

In sum, characteristic vector (41) is given by (45), (46) and (47). Given the vector, general solution (42) generates self-constrol cycles once constants (C_1,C_2) and the initial shadow prices (λ_0,π_0) are given, where they are determined by the initial conditions, as we shall show next.

A3.2 Detemining constants (C_1,C_2) and the initial shadow prices (λ_0,π_0)

Setting $t = 0$ in (42) yields

$$\begin{pmatrix} W_0 - W^* \\ M_0 - M^* \\ \lambda_0 - \lambda^* \\ \pi_0 - \pi^* \end{pmatrix} = \begin{pmatrix} \mu_W & \phi_W \\ \mu_M & \phi_M \\ \mu_\lambda & \phi_\lambda \\ \mu_\pi & \phi_\pi \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \tag{48}$$

where the elements of the coefficient matrix on the right hand side are given by (45), (46) and (47).

From the first and second equations in (48), constants C_1 and C_2 are determined by the initial values W_0 and M_0 as

$$\begin{aligned} C_1 &= \frac{\phi_M (W_0 - W^*) - \phi_W (M_0 - M^*)}{\mu_W \phi_M - \mu_M \phi_W}, \\ C_2 &= \frac{-\mu_M (W_0 - W^*) + \mu_W (M_0 - M^*)}{\mu_W \phi_M - \mu_M \phi_W}. \end{aligned} \tag{49}$$

Substituting these constants back to (48) yields the solution for the intial values of th shadow prices as

$$\begin{aligned} \lambda_0 - \lambda^* &= \frac{(\mu_\lambda \phi_M - \mu_M \phi_\lambda)(W_0 - W^*) + (-\mu_\lambda \phi_W + \mu_W \phi_\lambda)(M_0 - M^*)}{\mu_W \phi_M - \mu_M \phi_W}, \\ \pi_0 - \pi^* &= \frac{(\mu_\pi \phi_M - \mu_M \phi_\pi)(W_0 - W^*) + (-\mu_\pi \phi_W + \mu_W \phi_\pi)(M_0 - M^*)}{\mu_W \phi_M - \mu_M \phi_W} \end{aligned} \tag{50}$$

After all, for a give steady-state point and the intial values W_0 and M_0, the values for constants C_1 and C_2 are determined uniquely by (49), whereas the initial values of the two shadow prices are decided uniquely by (50). Given those values, the self-control cycle is generated uniquely by (42).

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Table 1. Taxonomy of consumer dynamics

	(i) $0 < \alpha\varepsilon < \sigma$ Saddle-point stable χ_1, χ_2 stable ω_1, ω_2 unstable	$\sigma < \alpha\varepsilon < 2r + \sigma$ Unstable	(ii) $2r + \sigma \le \alpha\varepsilon$ Saddle-point stable χ_1, χ_2 unstable ω_1, ω_2 stable
(a) All roots real $4\varepsilon(\gamma k - \alpha\sigma) < (\alpha\varepsilon - \sigma)^2$	**Case (1)** (λ_t, π_t, x_t) perfectly smoothed (W_t, M_t) non-cyclical	No solution	Case (3) All non-cyclical
(b) All roots imaginary $(\alpha\varepsilon - \sigma)^2 < 4\varepsilon(\gamma k - \alpha\sigma)$	**Case (2)**: willpower cycle (λ_t, π_t, x_t) perfectly smoothed (W_t, M_t) cyclical	No solution	**Case (4)**: **self-control cycle** All cyclical

Table 2. Parameter values

α	γ	δ	ε	σ	ψ_0	θ
0.5	0.6	0.1	0.7	0.05	-5	2

k	q	r	y	f_0	W_0	M_0
0.75	1	0.05	1,000	50	3.80	8.82

Note: These parameter values satisfy all the conditions in Proposition 3 for the self-control cycle to occur. The initial values W_0 and M_0 are set equal to the steady state values under the parameter values.

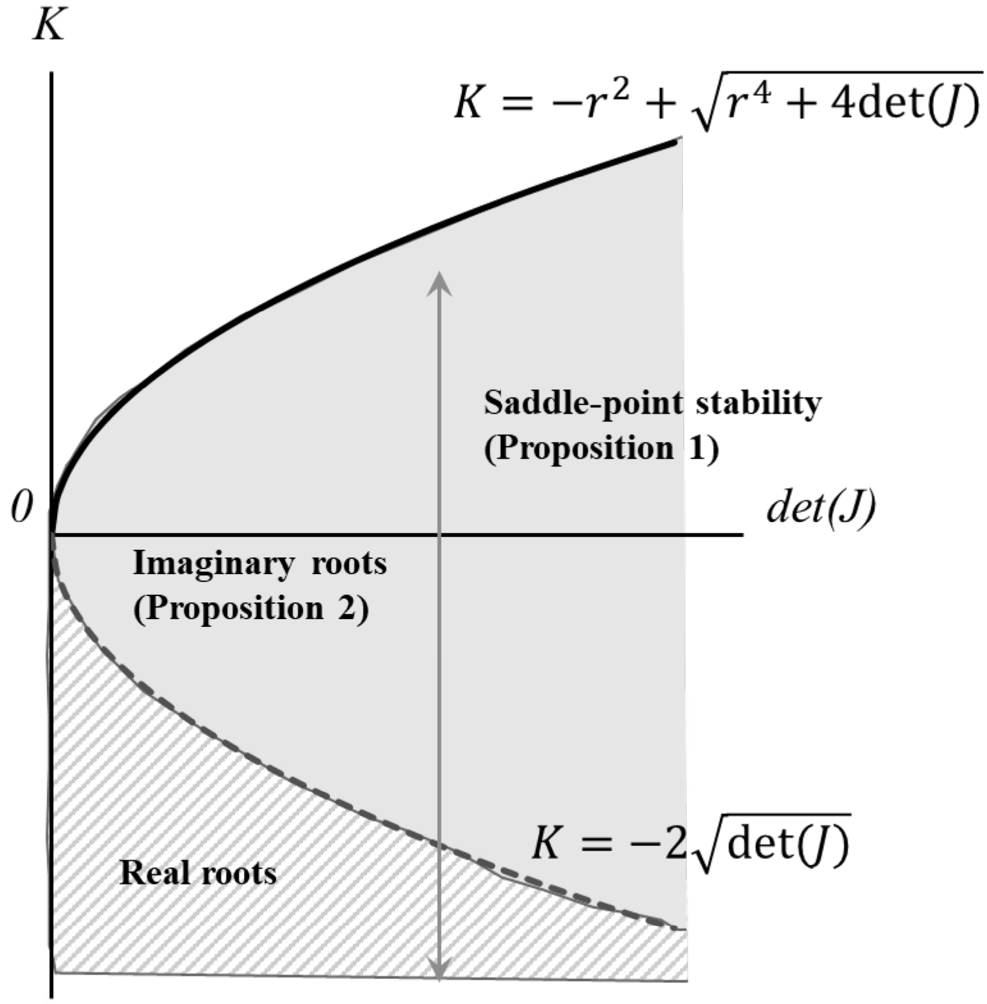

Figure 1. Saddle-point stability and roots. Note: The figure illustrates the region for the saddle-point stability, summarized by Proposition 1, and the regions for real and imaginary roots, summarized by Proposition 2.

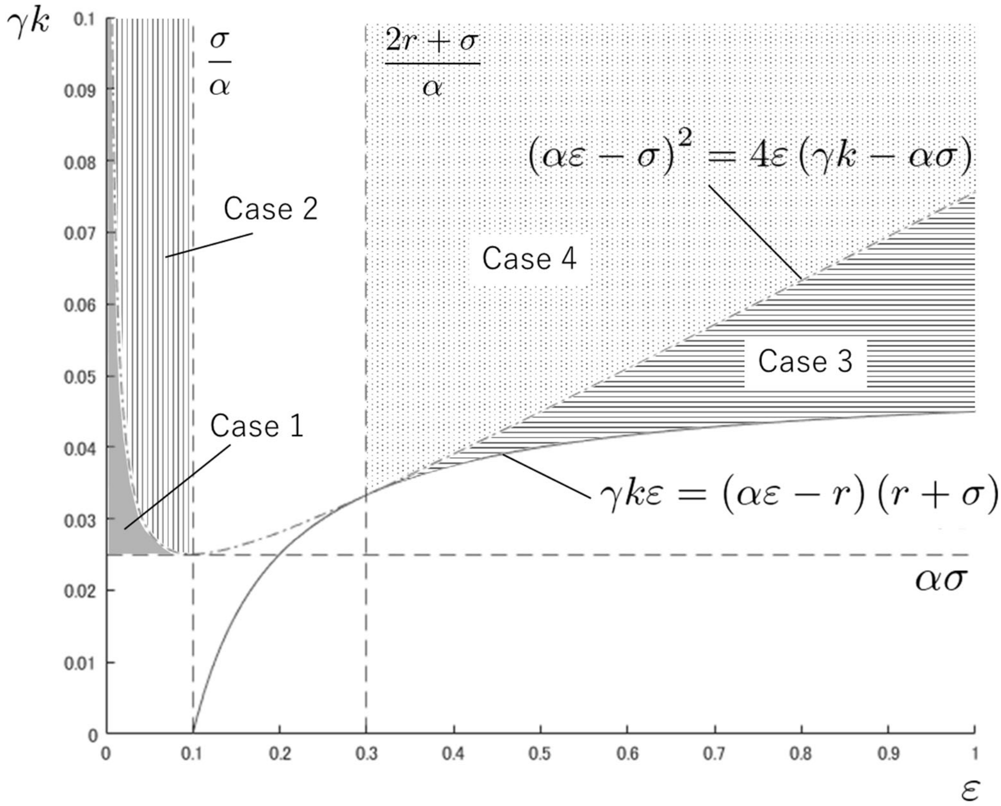

Figure 2. Dynamic properties of self-control behavior. Note: Case 1 through 4 are defined in Table 1. Parameter values for $\langle \alpha, \delta, \sigma, r \rangle$ are set as $(0.5, 0.1, 0.05, 0.05)$ as in Table 2 in Section 3.6.

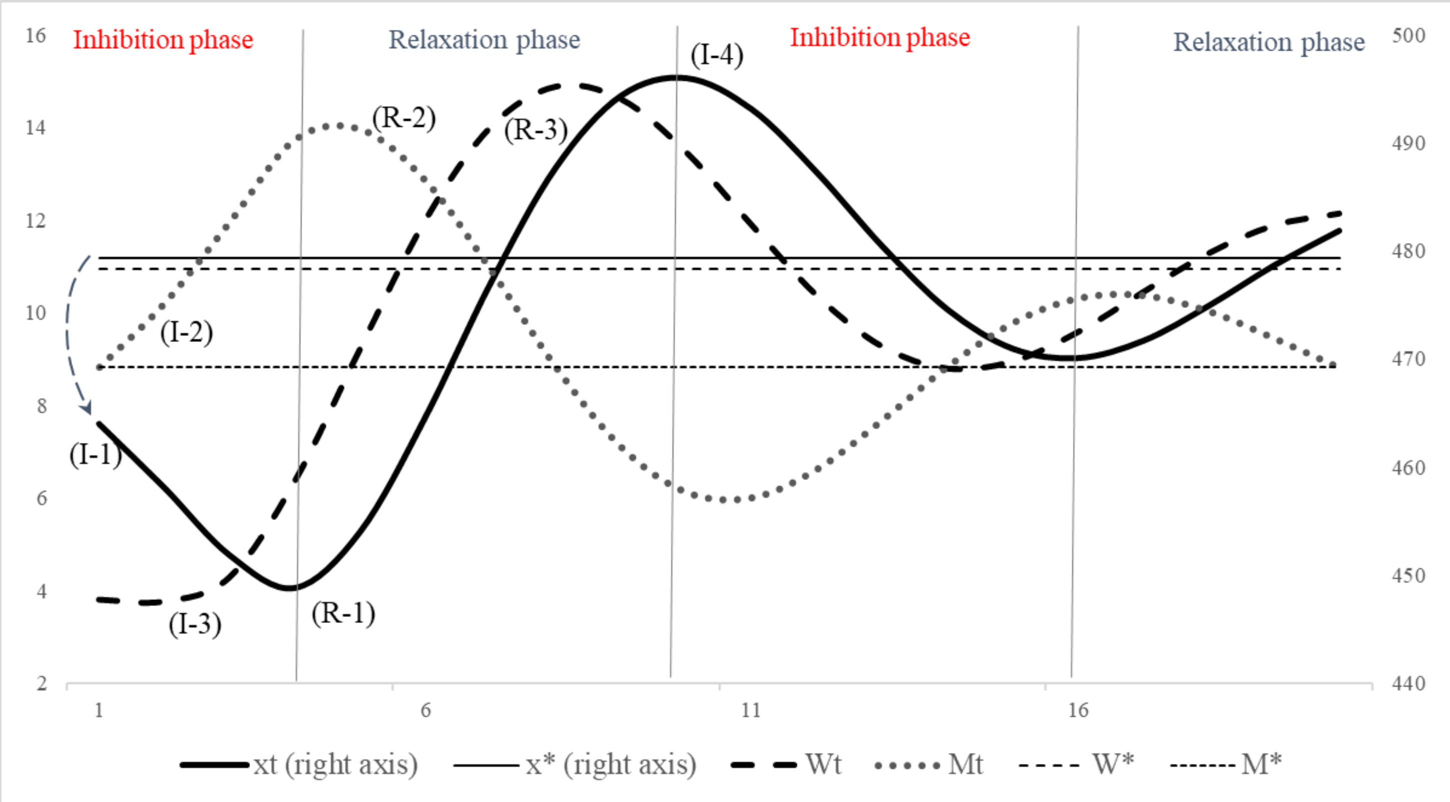

Figure 3. Cyclical adjustments to an increase in external self-control needs f_0. Note: The figure shows cyclical adjustments of tempting good consumption x, willpower stock W, and self-control experience capital M in response to a once-and-for-all increase in external self-control needs f_0 from 50 to 55, where the parameter values are specified as in Table 2. The variables with an asterisk denote post-shock steady-state levels, and where the initial point is assumed to be the steady-state point under the pre-shock environment. The arrow at the initial point in time indicates a discrete downward jump of x_0 to accumulate the self-control experience capital. The steady state values of x and M are unaffected by the increase in f_0, whereas the steady-state W needs to be increased to finance the increased external needs for self-control.

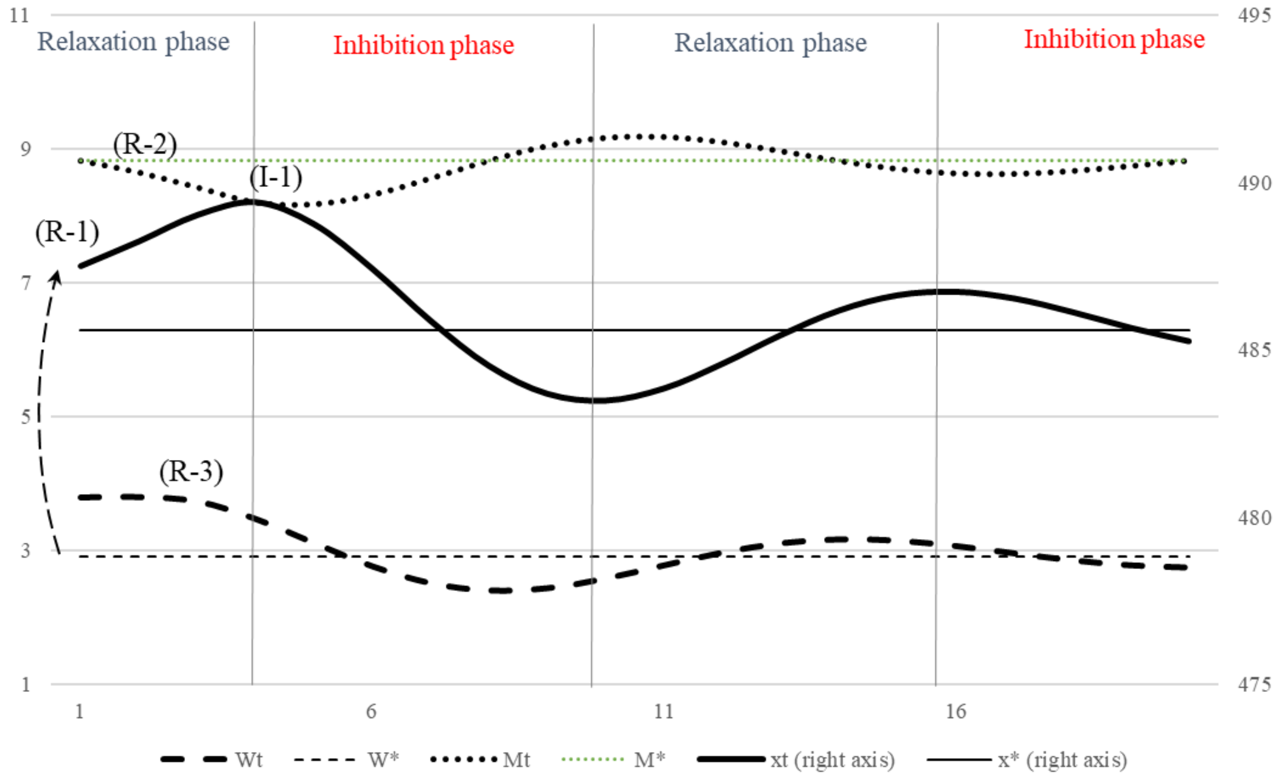

Figure 4. Cyclical adjustments to an increase in income *y*. Note: The figure shows cyclical adjustments of tempting good consumption *x*, willpower stock *W*, and self-control experience capital *M* in response to a once-and-for-all increase in income *y* from 1,000 to 1.010, where the parameter values are specified as in Table 2. The variables with an asterisk denote post-shock steady-state levels, and where the initial point is assumed to be the steady-state point under the pre-shock environment. The arrow at the initial point in time indicates a discrete upward jump of x_0 to decumulate the self-control experience capital. The steady-state value of *M* is unaffected by the increase in *y*, whereas the steady-state *W* decreases along with increase in x* and hence the reduced demand for steady-state self-control.