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BAYESIAN LEARNING WHEN PLAYERS ARE MISSPECIFIED ABOUT OTHERS

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Abstract

This paper considers Bayesian learning when players are biased about the data-generating process, and are biased about the opponent's bias about the data-generating process. Specifically, we assume that each player's bias about others takes the form of interpersonal projection, which is a tendency to overestimate the extent to which others share the player's own view. We show that even an arbitrarily small amount of bias can destroy correct learning of an unknown state, i.e., there is zero probability of the posterior belief staying in a neighborhood of the true state.

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1 Introduction

Economic agents often have a misspecified view of the world: workers may be overconfident about their own capability, political actors may believe that their opinion is more common than in reality, players may erroneously think that they are unfairly treated in competition, and so on.[1] Recent literature on misspecified learning studies how such misspecifications influence the agents' behavior and payoffs, assuming either a single-agent setup or a multi-agent setup in which the agents' misspecifications are common knowledge (e.g., Esponda and Pouzo, 2016; Heidhues, Kőszegi, and Strack, 2018; Ba and Gindin, 2023). However, this common knowledge assumption leaves out many potential applications, as it does not allow players to have a bias about the opponent's bias. For example, a worker may not be aware of her colleague's overconfidence, in which case she has a misspecification about the colleague's view of the world.[2] We study how such misperception about others influences learning. Our main finding is that even a vanishingly small amount of bias can have a substantial impact on the learning outcome.

Section 2 introduces our model. We consider the case in which players' bias on others takes the form of *interpersonal projection bias*, which has been studied in economics as well as other fields such as psychology, marketing, and political science. Interpersonal projection (also known as *false-consensus effect*) is a tendency for individuals to believe that others' views are more similar to their own views than in the reality. For example, a person who prefers a particular political candidate tends to think that other voters also prefer the same candidate. Past work provides empirical evidence of this bias in various economic situations; we will review this literature in Section 2.2.

Section 3.1 illustrates our idea via a team-production example. Suppose that two myopic players work on a joint project. There are infinitely many periods. The project output each period depends on the total effort in that period, as well as players' capability a and an unknown state θ (which can be interpreted as the profitability of their business). Efforts are private information. Players are Bayesian, and update their beliefs about the state θ every period.

[1] See Daniel and Hirshleifer (2015), Malmendier and Tate (2015), and Grubb (2015) for reviews of the literature on overconfidence.

[2] See Bursztyn and Yang (2022) for a review of the literate on misspecifications about others.

Suppose that one of the players (say, player 2) is misspecified and has overconfidence about the capability. Formally, let A_i denote player i's perception of the total capability of the team, and assume that $A_2 > A_1 = a$. On top of that, suppose that these perceptions are not common knowledge, and each player projects her perception on the opponent. That is, player i thinks that the opponent j thinks that the capability is $\hat{A}_j = \gamma A_i + (1-\gamma)A_j$, where $\gamma \in [0,1]$ is a parameter which measures the degree of interpersonal projection. This framework (of interpersonal projection) is borrowed from Gagnon-Bartsch, Pagnozzi, and Rosato (2021), who consider an auction model in which bidders project their tastes on others. They characterize how this projection influences the equilibrium in the one-shot game. In contrast, we consider the infinite-horizon model to study how misperception about others influences learning.

For the special case in which $A_2 = a$, there is no bias at all. (Indeed, $\hat{A}_i = A_i = a$ regardless of the parameter γ.) Hence the standard argument shows that correct learning occurs. That is, the degenerate belief on the true state θ^* is a steady-state belief of the learning process, and players' posterior beliefs almost surely converge to this steady state, regardless of the initial common prior.

Suppose now that player 2 has small overconfidence, so that A_2 is slightly higher than a. Not surprisingly, the steady state in this model is continuous in A_2. In particular, when A_2 is close to a, there is a steady state in which "almost correct learning" occurs, i.e., each player's belief puts a probability mass on a state close to the true state θ^*. However, it turns out that this steady state is *not* the long-run outcome of the learning. Indeed, we find that in the presence of interpersonal projection (i.e., $\gamma > 0$), players' beliefs converge to this steady state with *zero probability*. Note that this result holds for any $A_2 > a$, so even a vanishingly small amount of overconfidence destroys correct learning. Similarly, this result holds for any $\gamma > 0$, which means that an arbitrarily small amount of interpersonal projection is enough to destroy correct learning.

Why is correct learning vulnerable to such a small bias? A key is that projecting players have *inferential naivety* and make incorrect prediction about the opponent's play. Initially, this inferential naivety is small. Indeed, because players have only a small amount of misspecification, they make "almost correct" predictions about the opponent's action in earlier periods. However, there is a snowball effect on the inferential naivety, and players may make larger prediction errors in later periods. To see this, suppose that players (slightly) mispredict the opponent's action today. As we will explain in Section 3.1, this misprediction influences players' belief updating and causes divergence of the posterior beliefs. That is, a player who has a relatively optimistic belief (about the unknown state) tends to be more optimistic tomorrow, and a player who has a relatively pessimistic belief tends to be more pessimistic tomorrow. Then due to this expanding difference between the posteriors, players make more serious misprediction of the opponent's action tomorrow; for example, the optimistic player expects that the opponent is similarly optimistic and maximizes payoffs, while in reality the opponent is pessimistic. Such misprediction leads to further belief divergence, which in turn causes more serious prediction errors in later periods, and so on. Due to this process, players eventually have a huge amount of misspecification about the opponent's action, which destroys correct learning. In short, misspecification about the opponent's action — which endogenously grows in our model — is the main source of learning failure.

In Section 3.3, we extend the analysis to a general Bayesian learning model, and provide a sufficient condition under which small misspecification leads to a failure of learning just as in the team-production example. This result applies to a wide range of economic applications, such as air pollution, lobbying, games with conflicting interests, and Cournot duopoly.

We also study what happens when players' bias on others take different forms. Section 4 considers team production where different players have different initial priors about the unknown state, and are unaware of the opponent having a different prior. We assume that players correctly understand their capability, so they are misspecified only about the opponent's initial belief (and the opponent's belief about her own initial belief, and so on). In Section 5, we consider the case of one-sided misspecification where only player 2 has international projection bias. In both cases, we find that our discontinuity result still holds, i.e., a small amount of misspecification can still destroy correct learning. Again, the main source of learning failure is misspecification about the opponent's action, which endogenously grows through learning.

As we will discuss in Section 6, the literature on misspecified learning is rapidly growing, and among these papers, our work is most closely related to Frick, Iijima, and Ishii (2020). They look at a social learning problem where agents observe the opponents' actions every period and learn a payoff-relevant unknown state from them. The agents are misspecified in that they have incorrect views about how the opponents interpret information (and hence they have incorrect views about the opponents' behavior). They show that a steady state is discontinuous in the amount of misspecification, and in particular, even with a vanishingly small amount of misspecification, in the unique steady state, the agents have a point-mass belief on a state which is far away from the true state. So a small misspecification leads to a compete breakdown of correct learning.

Frick, Iijima, and Ishii (2020) also argue that their discontinuity result relies on the assumption that the agents have only a limited amount of information about the state; in their model, the agents observe a noisy signal about the state only once, so the agents learn mostly from the opponents' actions.[3] Indeed, they show that if the agents observe feedback (signals) every period, then the result is overturned and steady states are continuous in the amount of misspecification.

We complement their work by showing that even in the case of repeated feedback, correct learning can be vulnerable to small misspecification.[4] In our model, small misspecification has only a small impact on the steady state. However, it also influences the learning dynamics, and the probability of the belief converging to the steady state is pushed down to zero. One of the contributions of this paper is to identify this new mechanism which causes learning failure. We also believe that the analysis of the case of repeated feedback is important, because repeated feedback is common in many economic applications. For example, if agents observe their own payoffs every period, then it should be regarded as a case of repeated feedback, as payoffs are informative about the state in general.[5]

As a technical contribution, we extend Pemantle (1990) and provide a condition under which there is zero probability of a stochastic process converging to a steady state which satisfies a property called linear instability. A notable difference from Pemantle (1990) is that our result applies to

[3] Gagnon-Bartsch and Rosato (2024) study finite-period dynamic pricing when consumers have an interpersonal projection bias. They assume that each consumer observes a noisy signal about the state only once, as in Frick, Iijima, and Ishii (2020).

[4] Recent work by Frick, Iijima, and Ishii (2023) also show that small misspecification can lead to learning failure in the case of repeated feedback, but they assume *slow learning*, in that signals can be arbitrarily uninformative depending on the agent's action. Our model does not have such a feature, and learning fails even when players observe informative signals every period.

[5] Frick, Iijima, and Ishii (2020) assume that the agents do not observe payoffs.

the case of a Gaussian noise.[6] We believe that this extension is useful for future research, as many applied papers consider a Gaussian noise.

2 Model

2.1 Setup

There are two players $i = 1,2$ and infinitely many periods $t = 1,2,\cdots$. At the beginning of the game, an unobservable economic state θ^* is drawn from a closed interval $\Theta = [\underline{\theta},\overline{\theta}]$, according to a common prior distribution $\mu \in \triangle\Theta$. We assume that μ has a continuous density μ' with full support. In each period t, each player i has a belief $\mu_i^t \in \triangle\Theta$ about the state θ, and chooses an action x_i from a closed interval $X_i = [0,\bar{x}_i]$. Player i's action x_i is not observable by the opponent $j \neq i$. Given an action profile $x = (x_1,x_2)$, players observe a noisy public signal

$$y = Q(x_1,x_2,a,\theta^*) + \varepsilon,$$

where $a \in \boldsymbol{R}$ is a fixed parameter and ε is a random noise which follows the standard normal distribution $N(0,1)$. Player i's stage-game payoff is $u_i(x_i,y)$. We assume that both Q and u_i are twice continuously differentiable.

We consider a situation in which players have incorrect views about the data-generating process above. Formally, we assume that each player i believes that the true parameter is A_i, rather than a. One of the examples we have in mind is that the parameter a denotes the capability of players and they are overconfident about the capability, in which case we have $A_i > a$ (Heidhues, Kőszegi, and Strack, 2018). We allow $A_1 \neq A_2$, so different players may have different biases about the parameter a.

We assume that players' biases, (A_1,A_2), need not be common knowledge, and they may have a biased view about their opponent's bias. Specifically, we assume that each player has *interpersonal*

[6] Benaïm (1999) also extends Pemantle (1990), but he does not allow a Gaussian noise. Benaïm and Faure (2012) allow a Gaussian noise, but they focus on the case in which the process is cooperative. Also, they make various technical assumptions on the noise term, which are not satisfied in our model (e.g., i.i.d. noise, positive-definite assumption which rules out perfect correlation of a noise).

projection bias, in that she overestimates the extent to which others share her opinion. Following Gagnon-Bartsch, Pagnozzi, and Rosato (2021) and Gagnon-Bartsch and Rosato (2024), we model this bias by assuming that each player i thinks that

(a) The true parameter is A_i.

(b) The opponent $-i$ thinks that the true parameter is $\hat{A}_{-i} = \gamma A_i + (1-\gamma)A_{-i}$.

(c) The information above is common knowledge.

Part (a) states that player i may have a biased view about the data-generating process. Part (b) describes player i's interpersonal projection bias, and the parameter $\gamma \in [0,1]$ measures the degree of this bias. When $\gamma = 0$, each player i correctly understands the opponent's view about the data-generating process. When $\gamma = 1$, each player i is completely unaware of the opponent's view being different from her own view, and naively thinks that the opponent $-i$ also thinks that the true parameter is A_i (in reality, the opponent thinks that the true parameter is A_{-i}). When γ takes intermediate values, each player i recognizes that the opponent has a different opinion, but underestimates this difference; she incorrectly thinks that the opponent's opinion is closer to her own opinion than in the reality. To make our exposition as simple as possible, we follow earlier work (Gagnon-Bartsch, Pagnozzi, and Rosato, 2021; Gagnon-Bartsch and Rosato, 2024) and assume that this parameter γ is common for all players. However, this assumption is not essential; indeed, it is not difficult to show that all our results hold even when different players have different γ.[7]

Part (c) asserts that player i has naive higher-order beliefs, in that she neglects the possibility that the opponent misunderstands player i's view about the world. Note that Gagnon-Bartsch, Pagnozzi, and Rosato (2021) and Gagnon-Bartsch and Rosato (2024) impose the same assumption. As they argue, this assumption is motivated by the idea that people who are unaware of their own interpersonal projection bias are likely not attentive to others' interpersonal projection bias.

The following examples highlight that our model is flexible and covers a number of economic examples.[8]

[7] Similarly, the linearity of $\hat{A}_{-i}$ is not essential for our analysis. Our Proposition 3 holds even when $\hat{A}_{-i} = f_{-i}(A_i, A_{-i}, \gamma)$, as long as f_{-i} is continuous and $f(a,a,\gamma) = a$ for all γ.

[8] Here the interpersonal projection bias is modeled as a cognitive bias which underestimates the difference between

Example 1. *Underestimating Heterogeneity of Confidence*. Suppose that two players work on a joint project. Let a_i denote player i's capability, and let $a = a_1 + a_2$ denote the total capability. Each period, each player i chooses an effort level x_i. The output of the joint project is $y = Q(x_1, x_2, \theta) + a + \varepsilon$, and each player i's payoff is $y - c(x_i)$, where $c(x_i)$ is the effort cost and θ is an unknown project quality. Suppose that player 1 correctly understands each player's capability (so $A_1 = a$), while player 2 incorrectly believes that the total capability is $A_2 \neq a$. Intuitively, when $A_2 > a$, player 2 has *overconfidence* about their capabilities. When $A_2 < a$, player 2 has *underconfidence* about her own capability or *prejudice* about the opponent's capability. These perceptions (A_1, A_2) are common knowledge when we set $\gamma = 0$. On the other hand, when $\gamma > 0$, it describes the situation in which (A_1, A_2) are not common knowledge, and player 1 underestimates the opponent's misperception (and similarly, player 2 underestimates how much player 1's view differs from her own view).

Example 2. *Underestimating Heterogeneity of Capabilities*. Still consider the joint work problem, but suppose now that $a_1 = \overline{a} - \Delta$ and $a_2 = \overline{a} + \Delta$. That is, we consider the case in which player 2 is more capable than player 1. Suppose that each player underestimates the difference in capabilities; each player i thinks that the opponent's capability is $\hat{a}_{-i} = \beta_i a_i + (1 - \beta_i) a_{-i}$ for some parameter $\beta_i \in (0, 1]$. Suppose that her higher-order belief is naive in that she thinks that it is common knowledge that players' capabilities are $(a_i, \hat{a}_{-i})$. This situation is a special case of our framework where $a = 2\overline{a}$, $A_i = a_i + \hat{a}_{-i}$, and $\gamma = 1$.

Example 3. *Underestimating Impacts of Actions by Others*. Consider a two-player game in which the output is $y = Q(ax_1, x_2, \theta) + \varepsilon$, where a represents the degree of (marginal) influence of player 1's action. Suppose that player 1 correctly understands his own influence (so $A_1 = a$), while player

an agent's own perception and the opponent's perception. Some papers in the literature consider a slightly different model of the bias; they look at a model with infinitely many agents, where each agent is endowed with a type and overestimates the share of her own type (e.g., Gagnon-Bartsch and Rosato (2024)). The tools developed in this paper are also useful to study such a model. Specifically, in Appendix D, we consider a continuous-agent model where a half of the population is type 1 who thinks that the true parameter is A_1 (just as player 1 does in the model above), while the other half of the population is type 2 who thinks that the true parameter is A_2. Each type i overestimates the share of her own type. It turns out that the analysis of this model is analogous to that of our main model. Indeed, our result (Proposition 3) remains true as is even in this continuous-agent model.

2 incorrectly believes that it is $A_2 < a$. Intuitively, player 2 is inattentive to the effect of player 1's action on the output.

An important feature of our setup above is that if players' misperception is small in the sense that both A_1 and A_2 are ε-close to the parameter a, then it is common knowledge that the true parameter is in the ε-neighborhood of a, regardless of the parameter γ. In this sense, small misperception has a small impact on the whole information structure; it induces an information structure close to that for the case in which the parameter a is common knowledge.[9] We will show that such a small change in the information structure can cause a huge difference in the equilibrium outcome.

When players project their own view about the world to the opponent's view, they have *inferential naivety* and make incorrect predictions about the opponent's play (Eyster, 2019). Indeed, while player i believes that the opponent $-i$ maximizes the payoff and updates the belief conditional on the parameter $\hat{A}_{-i} = \gamma A_i + (1-\gamma)A_{-i}$, in reality the opponent does so conditional on the parameter A_{-i}. To analyze players' behavior in the presence of such inferential naivety, it is useful to consider two *hypothetical players* $-i = 1,2$. Hypothetical player $-i$ is player $-i$ who thinks that (a) the true parameter is $\hat{A}_{-i} = \gamma A_i + (1-\gamma)A_{-i}$, (b) the opponent (i.e., player i) thinks that the true parameter is A_i, and (c) the information above is common knowledge. Intuitively, player i thinks that her opponent is hypothetical player $-i$, and hence each period, she best-responds to this hypothetical player's action. We assume that players are myopic, so that each player i chooses a static Nash equilibrium action against hypothetical player $-i$ every period.[10,11] This assumption shuts down the repeated-game effect, so that a result similar to the folk theorem (which is not of

[9] Indeed, it is not difficult to show that player i's belief hierarchy is continuous with respect to the parameter A_i in the uniform weak topology of Chen, Di Tillio, Faingold, and Xiong (2017). So small misperception has a negligible impact on the whole information structure in the uniform weak topology.

[10] In the literature on Bayesian games, rationalizability is often used as a solution concept when players do not have a common prior. It turns out that in all the example studied in this paper, the game is dominance solvable, which means that the set of Nash equilibrium coincides with the set of rationalizable actions. Hence all our results still apply to these examples, even if we use rationalizability as a solution concept.

[11] Here we consider players who recognize that the opponent also learns the state and changes the action as time goes. This setup is different from the one in the literature on learning in games (e.g., Fudenberg and Kreps, 1993; Esponda and Pouzo, 2016), which asks when players play equilibria and how they learn the opponent's strategy from signals.

our interest) does not arise.[12]

Formally, players' behavior each period is described as follows. Let $\hat{x}_i$ and $\hat{\mu}_i$ denote hypothetical player i's action and belief, and let $x = (x_1, \hat{x}_2, x_2, \hat{x}_1)$ denote an action profile. (Here we have $(x_1, \hat{x}_2)$ in the first two components, in order to emphasize that they best-respond to each other.) Player i's expected stage-game payoff is defined as

$$U_i(x, A_i, \theta) = E[u_i(x_i, Q(x_i, \hat{x}_{-i}, A_i, \theta) + \varepsilon)],$$

because she thinks that the parameter is A_i and the opponent is a hypothetical player. Similarly, hypothetical player i's expected stage-game payoff given θ is

$$\hat{U}_i(x, \hat{A}_i, \theta) = E[u_i(\hat{x}_i, Q(\hat{x}_i, x_{-i}, \hat{A}_i, \theta) + \varepsilon)].$$

In period one, all players have the same belief $\mu_i^1 = \hat{\mu}_i^1 = \mu$. So they play a Nash equilibrium $(x_1^1, \hat{x}_2^1, x_2^1, \hat{x}_1^1)$, which (assuming interior solutions) satisfies the first-order condition $\frac{\partial E[U_i(x, A_i, \theta)|\mu]}{\partial x_i} = \frac{\partial E[\hat{U}_i(x, \hat{A}_i, \theta)|\mu]}{\partial \hat{x}_i} = 0$. At the end of period one, players observe a public signal $y^1 = Q(x_1^1, x_2^1, a, \theta^*) + \varepsilon$, and update the posterior beliefs using Bayes' rule. Their beliefs in period two are given by

$$\mu_i^2(\theta) = \frac{\mu_i^1(\theta) f(y - Q(x_i^1, \hat{x}_{-i}^1, A_i, \theta))}{\int_\Theta \mu_i^1(\tilde{\theta}) f(y - Q(x_i^1, \hat{x}_{-i}^1, A_i, \tilde{\theta})) d\tilde{\theta}},$$
$$\hat{\mu}_i^2(\theta) = \frac{\hat{\mu}_i^1(\theta) f(y - Q(\hat{x}_i^1, x_{-i}^1, \hat{A}_i, \theta))}{\int_\Theta \hat{\mu}_i^1(\tilde{\theta}) f(y - Q(\hat{x}_i^1, x_{-i}^1, \hat{A}_i, \tilde{\theta})) d\tilde{\theta}}.$$

As is clear from this formula, player i's posterior belief is biased in two ways: She updates the belief conditional on the incorrect parameter A_i, and on the incorrect prediction $\hat{x}_{-i}^1$ about the opponent's play. Then in period two, players play a Nash equilibrium given this belief profile $\mu^2 = (\mu_1^2, \hat{\mu}_2^2, \mu_2^2, \hat{\mu}_1^2)$.[13] Likewise, in any subsequent period t, players play a Nash equilibrium given the posterior beliefs computed by Bayes' rule.

[12] Another way to avoid the repeated-game effect is to use a Markov-perfect equilibrium (where the state is players' beliefs about θ) as a solution concept. If players play a Markovian equilibrium, (with additional technical assumptions) we can show that their long-run behavior is asymptotically the same as that of myopic players studied in this section. In this sense, our result remains true even for forward-looking players.

[13] Because y is public, player 1 correctly predicts hypothetical player 2's posterior belief $\hat{\mu}_2^2$, and similarly, hypothetical player 2 correctly predicts player 1's posterior belief μ_1^2. So they will indeed play a Nash equilibrium given these beliefs.

It is well-known in the literature on misspecified learning that if players' actions and beliefs converge, the limit must be a *steady state* (which is also known as *Berk-Nash equilibrium*, see Esponda and Pouzo (2016), Esponda, Pouzo, and Yamamoto (2021), Murooka and Yamamoto (2023)). In our setup, a *steady state* is defined as $(x_1^*, \hat{x}_2^*, x_2^*, \hat{x}_1^*, \mu_1^*, \hat{\mu}_2^*, \mu_2^*, \hat{\mu}_1^*)$ which satisfies

$$x_i^* \in \arg\max_{x_i} E[U_i(x_i, \hat{x}_{-i}^*, A_i, \theta) | \mu_i^*] \quad \forall i, \tag{1}$$

$$\hat{x}_i^* \in \arg\max_{\hat{x}_i} E[\hat{U}_i(\hat{x}_i, x_{-i}^*, \hat{A}_i, \theta) | \hat{\mu}_i^*] \quad \forall i, \tag{2}$$

$$\mu_i^* = 1_{\theta_i} \quad \text{where } \theta_i \in \arg\min_{\theta' \in \Theta} |Q(x_i^*, \hat{x}_{-i}^*, A_i, \theta') - Q(x_1^*, x_2^*, a, \theta^*)| \quad \forall i, \tag{3}$$

$$\hat{\mu}_i^* = 1_{\hat{\theta}_i} \quad \text{where } \hat{\theta}_i \in \arg\min_{\theta' \in \Theta} |Q(\hat{x}_i^*, x_{-i}^*, \hat{A}_i, \theta') - Q(x_1^*, x_2^*, a, \theta^*)| \quad \forall i. \tag{4}$$

The first two conditions are the incentive-compatibility conditions, which require that each player maximize her own payoff given some beliefs. The next two conditions require that these beliefs satisfy consistency, in that each (actual and hypothetical) player's belief is concentrated on a state θ which best explains the data given the equilibrium action x^*. For example, $|Q(x_i^*, \hat{x}_{-i}^*, A_i, \theta') - Q(x_1^*, x_2^*, a, \theta^*)|$ measures the gap between player i's expectation and the actual observation, and (3) asserts that her belief must be concentrated on the state which minimize this gap. For simplicity, in what follows, we assume that for each x and A_i, the minimizer of $|Q(x_i, \hat{x}_{-i}, A_i, \theta') - Q(x_1, x_2, a, \theta^*)|$ is unique.

2.2 Evidence on Interpersonal Projection

The empirical and experimental literatures report that people often have systematic biases when predicting others' view. Bursztyn and Yang (2022) conduct a meta-analysis and review how misperceptions about others are widespread. The interpersonal projection studied in this paper is one of such biases, also known as "social/taste projection" or "false-consensus effect/bias" in the literature.[14] Seminal papers of the interpersonal projection bias are Van Boven, Dunning, and Loewenstein (2000) and Van Boven and Loewenstein (2003), who show in their experiments that people overestimate the similarity between their own preferences/actions and other subjects' ones.

[14] False-consensus effect is originated in social psychology (e.g., Ross, Greene, and House, 1977; Krueger and Clement, 1994).

Subsequent studies find further evidence of interpersonal projection in various situations, such as election outcomes (Delavande and Manski, 2012), evaluation of political candidates and consumer products (Orhun and Urminsky, 2013), and public opinions (Furnas and LaPira, 2024) in political science; lottery choices (Engelmann and Strobel, 2012), investment decisions (Egan, Merkle, and Weber, 2014), and worker effort (Bushong and Gagnon-Bartsch, 2023) in economics.[15]

Recently, Gagnon-Bartsch and Bushong (2024) conduct belief-updating experiments on consumer choice for others. They find that subjects project their own tastes to others when evaluating products, and this inferential bias remains even after observing an informative signal about other subjects' preferences. This result suggests that interpersonal projection persists even after learning, as we assume in this paper.

3 Discontinuity of the Limiting Beliefs

3.1 Example of Non-Convergence: Team Production

To illustrate our idea, we will consider a simple model of team production. Suppose that two players work on a joint project. Each period, each player $i = 1,2$ chooses how much she shirks in that period. Let $x_i \in [0,1]$ denote player i's action. Player i's payoff is $y + x_i - \frac{1}{2}x_i^2$, where $x_i - \frac{1}{2}x_i^2$ is her private benefit from shirking. Note that this benefit is increasing in x_i, while the marginal benefit is decreasing in x_i. The output of the joint project each period is

$$y = Q(x,a,\theta) + \varepsilon = a - \theta(x_1 + x_2) + \varepsilon. \tag{5}$$

Here, $a \in \boldsymbol{R}$ is the capability of the team, $\theta \in \Theta = [0.7, 0.9]$ is an unknown state, and ε is a noise term which follows the standard normal distribution $N(0,1)$.

Because we assume $\theta \in [0.7, 0.9]$, regardless of players' beliefs, the Nash equilibrium action in the one-shot game is unique and given by $x_i = 1 - E[\theta|\mu_i^t]$. We assume that the initial prior is uniform on $\Theta = [0.7, 0.9]$ and that the true state is $\theta^* = 0.8$.[16]

[15] Gagnon-Bartsch, Pagnozzi, and Rosato (2021) and Gagnon-Bartsch and Rosato (2024) theoretically analyze the implications of interpersonal projection in auction and pricing, respectively.

[16] Our result does not rely on this specification of the state space $\Theta = [0.7, 0.9]$. Indeed, for any state space Θ

Suppose that each player i has a bias on their capability a so that player i thinks that the true capability is A_i. Also, there is interpersonal projection, so that each player i thinks that the opponent's perception is $\hat{A}_{-i} = \gamma A_i + (1-\gamma)A_{-i}$.[17]

Remark 1. As discussed in Chapter 24 of Varian (1992), the model above can be interpreted as a model of negative externalities in general. One of the examples which fit such an interpretation is an environmental problem, where y is the quality of the environment (e.g., air pollution, deforestation, and fishery), x_i is player i's production level which has a negative impact on the environmental quality, and $A_i > a$ is player i's optimism about the environmental quality. Such optimism is observed in various environmental problems; see Dechezleprêtre et al. (forthcoming) and references therein. Another example is political lobbying, where y is the quality of the public service, x_i is the expenditure on lobbying activity, and a is the public opinion (on the quality of the public service). Indeed, a recent paper by Furnas and LaPira (2024) reports interpersonal projection in such a context that many unelected political actors, including lobbyists, civil servants, and journalists, tend to believe that others' views about public opinions systematically and erroneously correspond to their own one.

For ease of exposition, throughout this example, we will focus on the full-projection case (i.e., $\gamma = 1$). Intuitively, this is the situation in which each player is completely unaware of the opponent having a different view about the capability. Under this assumption, player i's posterior μ_i^t is the same as hypothetical player $-i$'s posterior $\hat{\mu}_{-i}^t$ after every history. Hence we only need to check how two beliefs (μ_1, μ_2) evolve over time (rather than the evolution of four beliefs $(\mu_1, \hat{\mu}_2, \mu_2, \hat{\mu}_1)$), which greatly simplifies our analysis. In Section 3.3, we will show that this assumption is not essential, in that a similar result still holds even in the case of partial projection (i.e., $\gamma < 1$).

The steady state in this setup is characterized by the conditions (1) through (4). As we assume

which contains the true state $\theta^* = 0.8$ in its interior, we can show that when players' biases are small, there is zero probability of the beliefs converging to the interior steady state. This result directly follows from Proposition 3.

[17] As discussed in Examples 1 and 2 in the previous section, such interpersonal projection can arise when players underestimate heterogeneity of perceptions or heterogeneity of capability itself. For example, when $A_1 = a$ and $A_2 > a$, player 1 correctly understands the capability but player 2 has overconfidence, and each player underestimates how different the opponent's perception is. Such underestimation about the opponent's overconfidence seems relevant in practice. Indeed, Ludwig and Nafziger (2011) report that most subjects in their experiments are not aware of or underestimate overconfidence of other subjects.

$\gamma = 1$, we have $\theta_i = \hat{\theta}_{-i}$ in the steady state, so the steady state belief is described by two parameters, (θ_1, θ_2).

Consider the case with $A_1 = A_2 = a$; this is the case in which players have no bias on the parameter a, but are unaware of the difference of the posterior beliefs (if any) due to interpersonal projection. In this special case, there are three steady states: One of the steady states is an interior point, in which both players learn the true state ($\theta_1 = \theta_2 = \theta^*$), and choose the Nash equilibrium for this state θ^*. There are also two boundary steady states: In these steady states, players' beliefs are $(\theta_1, \theta_2) = (\underline{\theta}, \overline{\theta})$ or $(\theta_1, \theta_2) = (\overline{\theta}, \underline{\theta})$, and they choose a Nash equilibrium given these beliefs.[18] These boundary steady states do not arise as a long-run outcome, however. Indeed, because there is no misspecification about the parameter a, starting from a common prior μ, players learn the true state with probability one, i.e., the beliefs converge to the interior steady state almost surely when $A_1 = A_2 = a$.

Now, consider the case in which players are slightly biased (i.e., (A_1, A_2) is perturbed from (a, a) a small amount). By continuity, there is an interior steady state in which players' beliefs are close to (θ^*, θ^*). Let $m^*(A_1, A_2) = (m_1^*(A_1, A_2), m_2^*(A_1, A_2))$ denote this steady-state belief. Also, the boundary points $(\theta_1, \theta_2) = (\underline{\theta}, \overline{\theta})$ and $(\theta_1, \theta_2) = (\overline{\theta}, \underline{\theta})$ are still steady states in this case.

One may expect that the boundary states are still immaterial, and that the the beliefs converge to the interior steady state $m^*(A_1, A_2)$, just as in the case of $A_1 = A_2 = a$. Proposition 1 shows that such a conjecture is incorrect, and the beliefs converge to the boundary points when (A_1, A_2) is perturbed.

Proposition 1. *Suppose that $\gamma = 1$. Then there are $\underline{A} < a$ and $\overline{A} > a$ such that the following results hold.*

(i) For any $(A_1, A_2) \in (\underline{A}, \overline{A})^2$ with $A_1 = A_2$, players' posterior beliefs $\mu^t = (\mu_1^t, \mu_2^t)$ converge to

[18] To see that $(\theta_1, \theta_2) = (\underline{\theta}, \overline{\theta})$ is a steady-state belief, note that $\frac{\partial^2 Q}{\partial x_i \partial \theta} < 0$, so we have $x_1 > \hat{x}_1$ in this steady state. This means that player 2 underestimates how much the opponent shirks, and thus finds that the output is worse than the anticipation. This makes player 2 more pessimistic, but her current belief $\overline{\theta}$ already hits the upper bound of the set Θ, so her belief stays there. Similarly, player 1's belief stays at $\underline{\theta}$, which imply that $(\theta_1, \theta_2) = (\underline{\theta}, \overline{\theta})$ is indeed a steady-state belief. For the same reason, $(\theta_1, \theta_2) = (\overline{\theta}, \underline{\theta})$ is a steady-state belief.

the interior steady state almost surely, i.e.,

$$\Pr\left(\lim_{t\to\infty}\mu^t = (1_{m_1^*(A_1,A_2)}, 1_{m_2^*(A_1,A_2)})\right) = 1.$$

In particular, when $A_1 = A_2 = a$, correct learning occurs in that $\lim_{t\to\infty}(\mu_1^t, \mu_2^t) = (1_{\theta^*}, 1_{\theta^*})$ almost surely.

(ii) For any $(A_1,A_2) \in (\underline{A},\overline{A})^2$ with $A_1 \neq A_2$, players' posterior beliefs converge to the interior steady state $(1_{m_1^*}, 1_{m_2^*})$ with zero probability. The beliefs converge to the boundary points almost surely, i.e.,

$$\Pr\left(\lim_{t\to\infty}\mu^t \in \{(1_{\underline{\theta}}, 1_{\overline{\theta}}), (1_{\overline{\theta}}, 1_{\underline{\theta}})\}\right) = 1.$$

This proposition shows that the learning outcome is discontinuous in players' perceptions (A_1,A_2), and in particular correct learning is vulnerable to a small amount of misspecification. For example, assuming that player 1 has a correct perception $A_1 = a$, an arbitrarily small amount of player 2's overconfidence $A_2 > a$ leads to zero probability of the beliefs converging to the interior steady state.

In the literature of incomplete-information games, it is well-known that an equilibrium in a normal-form game is continuous with respect to the information structure; Chen, Di Tillio, Faingold, and Xiong (2017) show that a small perturbation of one's belief hierarchy (a belief about an economic state, a belief about the opponent's belief about the state, and so on) has only a marginal impact on the equilibrium. The proposition above does not contradict this result. Indeed, in our model, the *equilibrium strategy* in the infinite-horizon game, which maps one's belief μ_i to an action, is continuous in the parameter (A_1,A_2), so a small perturbation of one's belief hierarchy has a negligible impact on the equilibrium strategy.[19] In this sense, the main result of Chen, Di Tillio, Faingold, and Xiong (2017) still holds in our model. However, this need not imply that the resulting beliefs are continuous in the parameter (A_1,A_2), and Proposition 1 shows that our model is one of the cases in which such discontinuity arises.

Our result also shows that misperception about the opponent's bias can have a huge impact on the equilibrium outcome. Consider a model in which each player correctly understands the

[19] In our model, the belief hierarchies induced by (A_1,A_2) and $(A_1',A_2') \neq (A_1,A_2)$ are close in the uniform-weak topology of Chen, Di Tillio, Faingold, and Xiong (2017) if (A_1,A_2) and (A_1',A_2') are close.

opponent's perception, i.e., suppose that the perceptions (A_1, A_2) are common knowledge. In such a model, if players' biases are small (i.e., (A_1, A_2) is close to (a, a)), then players' beliefs almost surely converge to the unique interior steady state — see our companion paper (Murooka and Yamamoto, 2023) for a proof. This steady state is continuous in (A_1, A_2), so approximately correct learning occurs even if there is small misperception. In contrast, our proposition above shows that once we introduce unawareness about the opponent's bias, even vanishingly small misperception (e.g., overconfidence) completely destroys correct learning.

A rough intuition behind this proposition is as follows. In our model, when each player's misperception is small (i.e., A_i is close to a), the evolution of players' beliefs is governed by the following two forces:

- *Regular learning effect.* If the inferential naivety does not exist and each player correctly predicts the opponent's action every period, their beliefs move toward a neighborhood of the true state $\theta^* = 0.8$.
- *Polarization effect.* When $\mu_1^t \neq \mu_2^t$ so that players have different beliefs about the state, they have inferential naivety. That is, each player i believes that the opponent's belief is also μ_i^t, but in reality the opponent's belief is μ_j^t. It turns out that this inferential naivety is *amplified* through learning, in that on average, the gap between players' beliefs μ_1^t and μ_2^t increases over time, and the beliefs eventually move toward the boundary steady states.[20] To see this, suppose that player 1's belief μ_1^t is more optimistic than player 2's belief μ_2^t. In this case, player 1 overestimates the opponent's optimism about the state θ, and hence underestimates the opponent's effort. Accordingly, the realized outcome y is on average better than player 1's expectation, which makes her even more optimistic about the state. A similar argument shows that pessimistic player 2 becomes even more pessimistic, so the beliefs are indeed polarized and move toward the boundary steady state $(1_{\overline{\theta}}, 1_{\underline{\theta}})$.

When $A_1 = A_2$, players are symmetric, and hence we have $\mu_1^t = \mu_2^t$ every period. This means

[20] The argument here is informal in that we have not defined how to measure the difference of the two beliefs. Using the notation introduced in the proof sketch in Section 3.3, it can be measured by the difference of the mean beliefs, i.e., $|m_1^t - m_2^t|$.

that the inferential naivety is never in effect (in other words, players do not misspecify the opponent's action), and the evolution of the posterior belief is solely governed by the regular learning effect. This leads to appropximate correct learning, as stated in part (i) of the proposition.

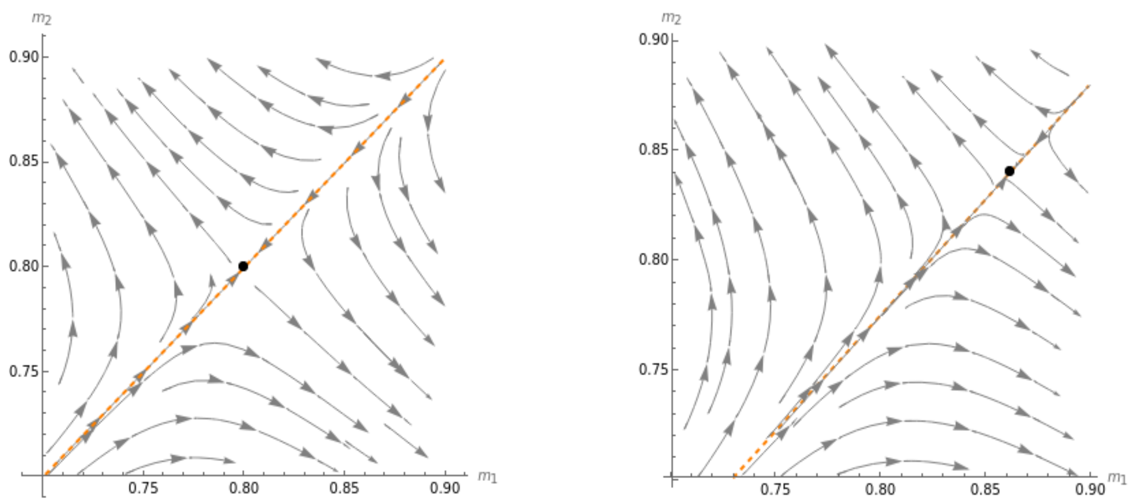
On the other hand, when $A_1 \neq A_2$, players are not symmetric, and hence even if they start with a common prior about the state, they have different posteriors in later periods. Then the inferential naivety starts to influence the belief evolution, and in particular, the polarization effect described above forces the mean belief to move toward the boundary steady states. This leads to part (ii) of the proposition.

Remark 2. In our view, the main point of Proposition 1 is to show vulnerability of correct learning to a small amount of misspecification, rather than convergence to boundary steady states. Indeed, these boundary steady states are not self-confirming, in that in these steady states, players keep being surprised by an actual output being different from their expectations on average. Hence, if players' beliefs stay at these steady states for a while, they might realize that their models are incorrect and revise the models. So when there is small misspecification, we may expect that the beliefs cannot stay in a neighborhood of the interior steady state, but it is less clear if the beliefs indeed stay at the boundary points forever.

3.2 Proof Idea of Proposition 1

We will describe a more detailed proof sketch. To simplify the discussion, for now we assume that the state space is $\Theta = \boldsymbol{R}$, rather than the closed interval $[0.7, 0.9]$. This assumption, together with the normality of the distribution of the noise term ε, implies that each player's posterior belief μ_i^t in period t is normal. Hence the posterior can be written as $\mu_i^t = N(m_i^t, (\sigma_i^t)^2)$, where m_i^t is the mean and $(\sigma_i^t)^2$ is the variance of the normal distribution. It is not difficult to show that the variance $(\sigma_i^t)^2$ converges to zero as time goes, so the posterior μ_i^t in a later period t is approximately a degenerate belief $1_{m_i^t}$ on the mean m_i^t. In what follows, we will describe how this mean m_i^t changes over time, and explain that it cannot converge to the interior steady state when $A_1 \neq A_2$.

Step 1: Linear Instability of the Interior Steady State In our model, the information y^t each period is influenced by the stochastic noise ε, and hence the mean belief m_i^t evolves stochastically. However, using the theory of stochastic approximation (see, for example, Kushner and Yin (2003)), we can show that after a long time, the motion of the mean belief m_i^t can be approximated by ordinal differential equations (ODEs) which do not involve a stochastic noise. That is, the motion of m_i^t is almost deterministic after a long time.



(a) $A_1 = A_2 = a$ **(b)** $A_2 > A_1 = a$

Figure 1: Motion of the mean belief m^t. The black dot is the interior steady state. The dashed orange line is the basin of attraction of the interior steady state.

Figure 1(a) is the phase portrait which describes the solution to the ODE for the case of $A_1 = A_2 = a$ (i.e., players have no misspecification about a, but are unaware of the difference of the posterior beliefs). In the figure, the horizontal axis is player 1's mean belief m_1^t, and the vertical axis is player 2's belief m_2^t. The black dot in the middle is the interior steady state m^* where $m_1^* = m_2^* = 0.8$. As can be seen, there are only two paths converging to this steady state; one from the top-right corner and the one from the bottom-left corner. So the basin of attraction of this steady state is simply the 45-degree line (described by the dashed orange line in the figure), which has *measure zero*. In particular, if the initial value is perturbed and leaves this basin, then the mean belief m^t eventually moves toward the boundary points, the top-left corner or the bottom-right corner. In this sense, the interior steady state is *linearly unstable*.

This phase portrait can be seen as a consequence of the regular learning effect and the polarization effect discussed in the previous subsection. For example, suppose that the current belief is on the 45-degree line, i.e., $m_1^t = m_2^t$ today. In this case, each player has a correct belief about the opponent's belief, so that inferential naivety does not exist. Accordingly, only the regular learning effect matters, and hence the mean belief moves toward the interior steady state. This means that the 45-degree line is indeed the basin of the interior steady state.

Suppose now that the current mean belief m^t is slightly perturbed and is not on the 45-degree line; for concreteness, suppose that the current belief is $m^t = (0.9-\varepsilon, 0.9)$. Initially, the gap $|m_1^t - m_2^t|$ of the beliefs is small, and hence players make almost correct prediction of the opponent's action. Hence the impact of the polarization effect is almost negligible, and learning is mostly governed by the regular learning effect. Accordingly, the belief m^t moves toward the interior steady state. However, as the belief m^t approaches the interior steady state, the regular learning effect slows down, and eventually it becomes smaller than the polarization effect. Then the belief m^t starts to move toward the top-left corner, as described in the phase portrait.

So far we have focused on the case with $A_1 = A_2 = a$, but even if players have small misperception, the phase portrait does not change qualitatively. Figure 1(b) is the phase portrait which approximates the belief dynamic when $A_1 = a$ and $A_2 = a + 0.03$. Due to the misperception, now the steady state m^* moves toward the top-right corner. Other than that, the belief dynamic is very similar to that for the case with $A_1 = A_2 = a$. In particular, the steady state m^* is still unstable, in that its basin of attraction (the dashed orange line) has measure zero.

Step 2: Non-Convergence to Unstable Steady State Because the public information y is influenced by the noise term ε every period, the evolution of the mean belief m^t is stochastic, and its actual path can occasionally deviate from the one described in the phase portrait. (Recall that the phase portrait is just an *approximation* of the actual path. For the precise meaning of "approximation" here, see Lemma 1 in the appendix.) It turns out that this stochastic deviation from the phase portrait is critical for our discontinuity result.

The blue thick lines in Figure 2 describe how the mean belief m^t is perturbed due to the noise. More precisely, it is the set of all possible mean beliefs m^{t+1} in the next period, when the current

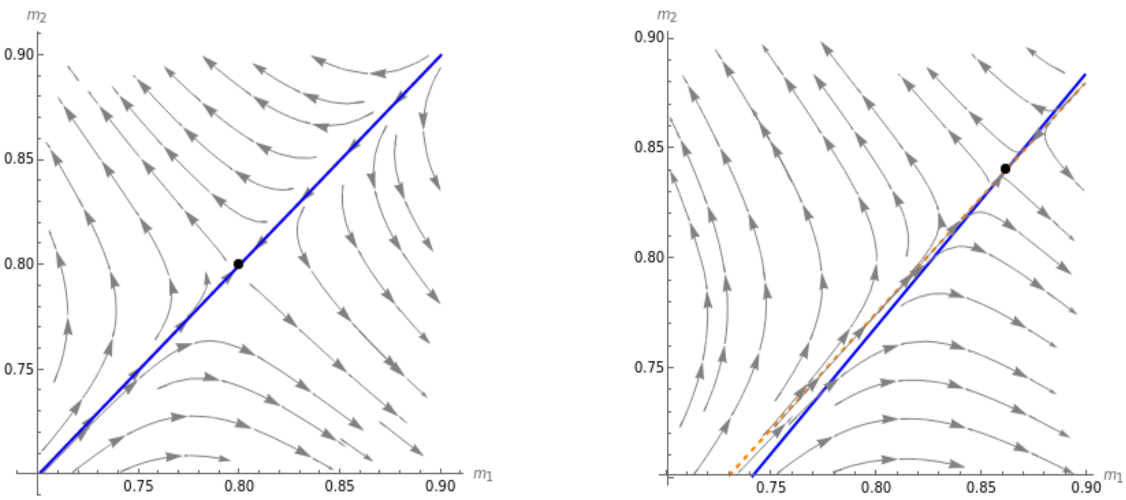


Figure 2: The mean belief m^t is perturbed along with the blue thick line, due to the noise ε.

mean belief m^t is exactly at the interior steady state. The exact value of m^{t+1} depends on the realization of the noise term ε today. For example, if $\varepsilon = 0$ today, then the mean belief m^{t+1} is unchanged and stays at the steady state.

Figure 2(a) considers the case of $A_1 = A_2 = a$. Here, the blue thick line coincides with the basin of the interior steady state (the dashed orange line in Figure 1(a)). This implies that even if the mean belief leaves the steady state due to the noise, it remains on the basin, and hence it eventually returns to the steady state. This is why the belief converges to the interior steady state as stated in part (i) of Proposition 1.

In contrast, when $A_1 \neq A_2$ so that players have different perceptions, the posteriors of the two players react to the noise ε differently. In particular, as described in Figure 2(b), the blue thick line does not coincide with the basin of the interior steady state; this means that the mean belief m^t is going to be "kicked out" from the basin, unless we have $\varepsilon = 0$ in all periods. Using this property, we show in the proof that the mean belief leaves a neighborhood of the basin infinitely often, after which the belief tends to move toward one of the boundary steady states as described in the phase portrait. This leads to the non-convergence result stated in part (ii) of Proposition 1.

3.3 Discontinuity in a General Setup

We have seen that in the above team-production example, players' beliefs do not converge to the interior steady state. Now we will consider a general model and provide a condition under which a similar non-convergence result holds.

Consider a general model introduced in Section 2.1. To simplify our exposition, here we assume that the initial prior μ is uniform; but all our results extend to any initial prior μ with a continuous density.[21]

Assume that the output function Q is *linear* in θ:

$$Q(x_1,x_2,a,\theta) = R(x_1,x_2,a)\theta + S(x_1,x_2,a).$$

We assume that $R(x_1,x_2,a)$ is uniformly bounded away from zero for all on-path action profiles. This ensures that each player's state learning never stops. Let $\underline{R} > 0$ denote the minimum of $|R(x_1,x_2,a)|$.

Given an action profile $x = (x_1,\hat{x}_2,x_2,\hat{x}_1)$, let $\theta_i(x)$ denote a solution to

$$Q(x_i,\hat{x}_{-i},A_i,\theta) = Q(x_1,x_2,a,\theta^*),$$

and let

$$I_i(x_i,\hat{x}_{-i}) = (R(x_i,\hat{x}_{-i},A_i))^2.$$

Intuitively, $\theta_i(x)$ is player i's average estimate of the state θ; if the current action profile is x and the realized noise ε is zero today, then (ignoring the impact of the prior) player i's posterior mean is $\theta_i(x)$. $I_i(x)$ measures the informativeness of the signal for player i, when she thinks that the current action is $(x_i,\hat{x}_{-i})$.

The linearlity assumption above implies that player i's posterior belief at the beginning of period $t+1$ is the truncated normal distribution induced by the normal distribution $N(m_i^{t+1}, \frac{1}{t\xi_i^{t+1}})$,

[21] When the initial prior is not uniform, one's posterior belief need not be a truncated normal distribution. However, for large t, the posterior belief is still approximated by a truncated normal distribution, which is enough for our result.

where

$$m_i^{t+1} = \frac{\sum_{\tau=1}^{t} I_i(x_i^\tau, \hat{x}_{-i}^\tau)\left(\theta_i(x^\tau) - \frac{\varepsilon^\tau}{\sqrt{I_i(x_i^\tau, \hat{x}_{-i}^\tau)}}\right)}{\sum_{\tau=1}^{t} I_i(x_i^\tau, \hat{x}_{-i}^\tau)}, \tag{6}$$

$$\xi_i^{t+1} = \frac{1}{t}\sum_{\tau=1}^{t} I_i(x_i^\tau, \hat{x}_{-i}^\tau). \tag{7}$$

In words, the posterior mean m_i^{t+1} is the weighted average of player i's estimate $\theta_i(x^\tau) - \frac{\varepsilon^\tau}{\sqrt{I_i(x_i^\tau, \hat{x}_{-i}^\tau)}}$ in the past periods, where the weight is the informativeness I_i. Note that the estimate here involves the term $\frac{\varepsilon^\tau}{\sqrt{I_i(x_i^\tau, \hat{x}_{-i}^\tau)}}$, which measures how the noise ε^τ on the signal y^τ influences the estimate. The parameter ξ_i^{t+1} is the average of the informativeness I_i of the past signals. By the assumption that $R(x_1, x_2, a)$ is bounded away from zero, $\frac{1}{t\xi_i^{t+1}}$ must goes to zero as $t \to \infty$. This means that after a long time, each player i's belief μ_i^t is approximately a degenerate belief on m_i^t.

Similarly, hypothetical player $-i$'s posterior belief at the beginning of period $t+1$ is the truncated normal distribution induced by the normal distribution $N(\hat{m}_{-i}^{t+1}, \frac{1}{t\hat{\xi}_{-i}^{t+1}})$ where

$$\hat{m}_{-i}^{t+1} = \frac{\sum_{\tau=1}^{t} \hat{I}_{-i}(x_i^\tau, \hat{x}_{-i}^\tau)\left(\hat{\theta}_{-i}(x^\tau) - \frac{\varepsilon^\tau}{\sqrt{\hat{I}_{-i}(x_i^\tau, \hat{x}_{-i}^\tau)}}\right)}{\sum_{\tau=1}^{t} \hat{I}_{-i}(x_i^\tau, \hat{x}_{-i}^\tau)}, \tag{8}$$

$$\xi_i^{t+1} = \frac{1}{t}\sum_{\tau=1}^{t} \hat{I}_{-i}(x_i^\tau, \hat{x}_{-i}^\tau). \tag{9}$$

Here $\hat{\theta}_{-i}(x)$ is defined as a solution to

$$Q(x_i, \hat{x}_{-i}, \hat{A}_{-i}, \theta) = Q(x_1, x_2, a, \theta^*),$$

and $\hat{I}_{-i}(x_i, \hat{x}_{-i}) = I_i(x_i, \hat{x}_{-i})$. In what follows, we assume that $\theta_i(x)$ and $\hat{\theta}_i(x)$ are Lipschitz-continuous in x, and that $I_i(x_i, \hat{x}_{-i}) = \hat{I}_{-i}(x_i, \hat{x}_{-i})$ is Lipschitz-continuous in $(x_i, \hat{x}_{-i})$.

In each period $t+1$, player i computes a one-shot Nash equilibrium $(x_i, \hat{x}_{-i})$ given the posterior belief profile $(\mu_i^{t+1}, \hat{\mu}_{-i}^{t+1})$ in her mind, and chooses the equilibrium action x_i. In what follows, we will assume that this Nash equilibrium is unique for each belief, and denote it by $\left(x_i(m_i^{t+1}, \frac{1}{t\xi_i^{t+1}}, \hat{m}_{-i}^{t+1}, \frac{1}{t\hat{\xi}_{-i}^{t+1}}), \hat{x}_{-i}(m_i^{t+1}, \frac{1}{t\xi_i^{t+1}}, \hat{m}_{-i}^{t+1}, \frac{1}{t\hat{\xi}_{-i}^{t+1}})\right)$, in order to emphasize the dependence on the posterior belief.

Now, recall that when t is large, each player i's posterior belief μ_i^t is approximately a degenerate belief on m_i^t. In other words, the parameter ξ_i^t has almost no impact on the posterior belief. Then it is natural to expect that the same is true for the equilibrium action, and the parameter ξ_i^t has almost no impact on it. The following assumption captures this idea. For each i, m_i, $\hat{m}_{-i}$, let $(x_i(m_i,\hat{m}_{-i}),\hat{x}_{-i}(m_i,\hat{m}_{-i}))$ denote a Nash equilibrium when player i's posterior belief is degenerate in that $(\mu_i,\hat{\mu}_{-i}) = (1_{\theta_i}, 1_{\hat{\theta}_{-i}})$, where $\theta_i \in \arg\min_{\tilde{\theta}\in\Theta} |m_i - \tilde{\theta}|$ and $\hat{\theta}_{-i} \in \arg\min_{\tilde{\theta}\in\Theta} |\hat{m}_{-i} - \tilde{\theta}|$.

Assumption 1. There are $\underline{A} < a$ and $\overline{A} > a$ such that for any $(A_1,A_2) \in (\underline{A},\overline{A})^2$, the following properties hold.

(i) There are $K > 0$ and $\alpha > 0$ such that

- For all i, m_i, $\hat{m}_{-i}$, $\xi_i \geq \underline{R}^2$, $\hat{\xi}_{-i} \geq \underline{R}^2$, and for sufficiently large t,

$$\left| x_i\left(m_i, \frac{1}{t\xi_i}, \hat{m}_{-i}, \frac{1}{t\hat{\xi}_{-i}}\right) - x_i(m_i,\hat{m}_{-i}) \right| < \frac{K}{t^\alpha},$$
$$\left| \hat{x}_{-i}\left(m_i, \frac{1}{t\xi_i}, \hat{m}_{-i}, \frac{1}{t\hat{\xi}_{-i}}\right) - \hat{x}_{-i}(m_i,\hat{m}_{-i}) \right| < \frac{K}{t^\alpha}.$$

- When $m_i \in \mathrm{int}\Theta$, the inequalities above hold for $\alpha = 1$.

(ii) The limit equilibrium actions $(x_i(m_i,\hat{m}_{-i}),\hat{x}_{-i}(m_i,\hat{m}_{-i}))$ are Lipschitz-continuous in $(m_i,\hat{m}_{-i})$.

The above assumption asserts that when t is large so that the posterior belief is approximately degenerate, the Nash equilibrium action is approximated by the equilibrium $(x_i(m_i,\hat{m}_{-i}),\hat{x}_{-i}(m_i,\hat{m}_{-i}))$ for the corresponding degenerate belief. It also requires that the approximation error is at most of order $O(\frac{1}{t})$ if m_i is in the interior of Θ, and at most of order $O(\frac{1}{t^\alpha})$ otherwise.

In the team-production example, small misspecification has a small impact on the steady state, that is, when players' misperceptions about the capability is small, there is an interior steady state in which players learn the state almost correctly. The following proposition shows that the same result holds for generic games.

Proposition 2. *Suppose that when $A_1 = A_2 = a$, we have*

$$\frac{\partial Q}{\partial \theta} + \frac{\partial Q}{\partial x_1}\frac{\partial x_1}{\partial m_1} + \frac{\partial Q}{\partial x_1}\frac{\partial x_1}{\partial \hat{m}_2} + \frac{\partial Q}{\partial x_2}\frac{\partial \hat{x}_2}{\partial m_1} + \frac{\partial Q}{\partial x_2}\frac{\partial \hat{x}_2}{\partial \hat{m}_2} \neq 0 \tag{10}$$

at the steady state belief (i.e., $\theta = m_i = \hat{m}_i = \theta^*$ and $x_i = \hat{x}_i = x_i(\theta^*, \theta^*)$ for each i). Then there is an open neighborhood $U \subset \boldsymbol{R}^4$ of $(m_1, \hat{m}_2, m_2, \hat{m}_1) = (\theta^*, \theta^*, \theta^*, \theta^*)$ such that

(i) When $A_1 = A_2 = a$, the steady state belief in the neighborhood U is unique and it is $m_1 = m_2 = \hat{m}_1 = \hat{m}_2 = \theta^*$.

(ii) There are $\underline{A} < a$ and $\overline{A} > a$ such that for any $\gamma \in [0,1]$, there is a unique continuous function $m^* : [\underline{A}, \overline{A}]^2 \to U$ such that $m^*(a,a) = (\theta^*, \theta^*, \theta^*, \theta^*)$ and such that for each $(A_1, A_2) \in (\underline{A}, \overline{A})^2$, $m^*(A_1, A_2)$ is a steady state belief given (A_1, A_2).

To interpret assumption (10), suppose that player 1's belief m_1 and her belief $\hat{m}_2$ about the opponent's belief increase a bit, by the same amount. Then her subjective expectation $Q(x_1, \hat{x}_2, A_1, m_1)$ changes by $\frac{\partial Q}{\partial \theta} + \frac{\partial Q}{\partial x_1}\frac{\partial x_1}{\partial m_1} + \frac{\partial Q}{\partial x_1}\frac{\partial x_1}{\partial m_2} + \frac{\partial Q}{\partial x_2}\frac{\partial x_2}{\partial m_1} + \frac{\partial Q}{\partial x_2}\frac{\partial x_2}{\partial m_2}$, and our assumption (10) asserts that this impact is non-zero. This assumption is satisfied for generic games, and the proposition above shows that in such games, the interior steady state is continuous in (A_1, A_2). In other words, when players' misspecification is small, there is an interior steady state m^* in which they approximately learn the true state.

Now, we will show that for a class of games, there is zero probability of the beliefs converging to this steady state m^*. To state the result, the following terminology is useful. Given a mean belief profile $m = (m_1, \hat{m}_2, m_2, \hat{m}_1)$, let $x(m) = (x_1(m_1, \hat{m}_2), \hat{x}_2(m_1, \hat{m}_2), x_2(m_2, \hat{m}_1), \hat{x}_1(m_2, \hat{m}_1))$ denote the equilibrium action for the corresponding degenerate beliefs. Given perceptions $(A_1, A_2) \in (\underline{A}, \overline{A})^2$, let

$$b' = -\left(\frac{1}{R(x_1, \hat{x}_2, A_1)}, \frac{1}{R(x_1, \hat{x}_2, \hat{A}_2)}, \frac{1}{R(x_2, \hat{x}_1, A_2)}, \frac{1}{R(x_2, \hat{x}_1, \hat{A}_1)}\right)$$

where the actions are the steady state actions, i.e., $(x_1, \hat{x}_2, x_2, \hat{x}_1) = x(m^*(A_1, A_2))$. Intuitively, this vector b' is the direction toward which the mean belief $m = (m_1, \hat{m}_2, m_2, \hat{m}_1)$ moves due to the noise ε. In the team-production example, it corresponds to the blue thick line in Figure 2.

Also, given perceptions $(A_1, A_2) \in (\underline{A}, \overline{A})^2$, let $H' \subseteq \boldsymbol{R}^4$ denote the affine space spanned by the generalized eigenvectors associated with negative eigenvalues of the matrix

$$J' = \begin{pmatrix} \frac{\partial\theta_1(x(m))}{\partial m_1}-1 & \frac{\partial\theta_1(x(m))}{\partial\hat{m}_2} & \frac{\partial\theta_1(x(m))}{\partial m_2} & \frac{\partial\theta_1(x(m))}{\partial\hat{m}_1} \\ \frac{\partial\hat{\theta}_2(x(m))}{\partial m_1} & \frac{\partial\hat{\theta}_2(x(m))}{\partial\hat{m}_2}-1 & \frac{\partial\hat{\theta}_2(x(m))}{\partial m_2} & \frac{\partial\hat{\theta}_2(x(m))}{\partial\hat{m}_1} \\ \frac{\partial\theta_2(x(m))}{\partial m_1} & \frac{\partial\theta_2(x(m))}{\partial\hat{m}_2} & \frac{\partial\theta_2(x(m))}{\partial m_2}-1 & \frac{\partial\theta_2(x(m))}{\partial\hat{m}_1} \\ \frac{\partial\hat{\theta}_1(x(m))}{\partial m_1} & \frac{\partial\hat{\theta}_1(x(m))}{\partial\hat{m}_2} & \frac{\partial\hat{\theta}_1(x(m))}{\partial m_2} & \frac{\partial\hat{\theta}_1(x(m))}{\partial\hat{m}_1}-1 \end{pmatrix}$$

where the derivatives are evaluated at the steady state belief $m = m^*(A_1,A_2)$. Intuitively, this set H' is (local approximation of) the basin of the interior steady state $m^*(A_1,A_2)$. In the team-production example, it is represented by the dashed orange line in Figure 1.

We say that perceptions (A_1,A_2) are *regular* if $b' \notin H'$. Under this regularity condition, the mean belief is kicked out from the basin, due to the noise ε. This is similar to the team-production example with $A_1 \neq A_2$. In contrast, when this regularity condition does not hold, the mean belief m^t stays on the basin of the interior steady state even when it is perturbed by the noise ε. This is similar to the team-production example with $A_1 = A_2$.

Proposition 3. *Suppose that Assumption 1 holds. Suppose also that the assumption stated in Proposition 2 holds, so that there is a function m^*. Suppose that when $A_1 = A_2 = a$, we have*

$$\frac{\partial\theta_1(x(m))}{\partial m_1} + \frac{\partial\theta_1(x(m))}{\partial\hat{m}_2} - \frac{\partial\hat{\theta}_1(x(m))}{\partial m_1} - \frac{\partial\hat{\theta}_1(x(m))}{\partial\hat{m}_2} > 1 \tag{11}$$

at the steady state with correct learning (i.e., $m_1 = m_2 = \hat{m}_1 = \hat{m}_2 = \theta^*$). Then there are $\underline{A} < a$ and $\overline{A} > a$ such that the following results hold:

(i) For any $\gamma \in (0,1]$ and for any regular $(A_1,A_2) \in (\underline{A},\overline{A})$, we have

$$\Pr\left(\lim_{t\to\infty}(\mu_1^t,\mu_2^t) = (1_{m_1^*(A_1,A_2)}, 1_{m_2^*(A_1,A_2)})\right) = 0.$$

(ii) For any $\gamma \in (0,1]$ and for any $(A_1,A_2) \in (\underline{A},\overline{A})$, the matrix J' has three negative eigenvalues and one positive eigenvalue; hence $\dim H' = 3$ (and hence the regularity condition $b' \notin H'$ holds for almost all cases).

(iii) For the special case of $\gamma = 1$, (A_1,A_2) satisfies the regularity condition $b' \notin H'$ if

$$g_1(A_1,A_2) \neq g_2(A_1,A_2) \tag{12}$$

where

$$g_i(A_1,A_2) = \frac{\partial \theta_i(x(m))}{\partial m_i} + \frac{\partial \theta_i(x(m))}{\partial \hat{m}_{-i}} + \frac{R(x_i(m),\hat{x}_{-i}(m),A_i)}{R(x_{-i}(m),\hat{x}_i(m),A_{-i})}\left(\frac{\partial \theta_i(x(m))}{\partial m_{-i}} + \frac{\partial \theta_i(x(m))}{\partial \hat{m}_i}\right)$$

for $m = m^*(A_1,A_2)$.

Part (i) of the proposition shows that if there is interpersonal projection (i.e., $\gamma > 0$) and the payoff function satisfies (11),[22] then the long-run outcome is discontinuous in the parameter (A_1,A_2): It shows that the probability of the beliefs converging to the steady state $m^*(A_1,A_2)$ suddenly drops to zero, once players have a small amount of misperception which satisfies the regularity condition. Parts (ii) and (iii) of the proposition show that this regularity condition is satisfied for almost all parameters (A_1,A_2). Indeed, in the team-production example, any (A_1,A_2) with $A_1 \neq A_2$ satisfies (12), and hence is regular. (See the proof of Proposition 1 for details.)

The proposition also shows that an arbitrarily small $\gamma > 0$ is enough to destroy correct learning. Note that when $\gamma = 0$ (i.e., if there is no inferential naivety), the beliefs converge almost surely to the steady state, even for perturbed (A_1,A_2). (See our companion paper Murooka and Yamamoto (2023) for the proof.) This means that for a fixed (A_1,A_2), the long-run belief is discontinuous in γ as well; even a negligible amount of interpersonal projection bias can have a significant impact on the limit outcome.

A key assumption in Proposition 3 is (11), which ensures that the polarization effect is strong enough. Under (11), once the belief is perturbed from the steady state for some direction, the belief leaves a neighborhood of the steady state, just as in the team-production example. To see this, suppose that $A_1 = A_2 = a$ and that the players' current beliefs are at the steady state $m_1 = m_2 = \hat{m}_1 = \hat{m}_2 = \theta^*$. Suppose now that the beliefs are slightly perturbed so that $m_1 = \hat{m}_2 = \theta^* + \Delta$, while m_2 and $\hat{m}_1$ are unchanged. That is, we perturb the belief in such a way that the inferential naivety about each player i's belief is $|m_i - \hat{m}_i| = \Delta$. Our assumption (11) implies that we have $\theta_1(x(m)) - \hat{\theta}_1(x(m)) > \Delta$ with this new belief profile m, which means that the inferential naivety $m_1 - \hat{m}_1$ regarding player 1's belief is *amplified* through learning over time. Similarly, because we

[22] Note that this assumption (11) does not depend on the parameter γ, as it considers the case with $A_1 = A_2 = a$, where the parameter γ has no impact on θ_i or $\hat{\theta}_i$.

assume $A_1 = A_2 = a$, we have $\theta_1(x) = \hat{\theta}_2(x)$ and $\theta_2(x) = \hat{\theta}_1(x)$ for all x, and hence (11) implies

$$\frac{\partial \hat{\theta}_2(x(m))}{\partial m_1} + \frac{\partial \hat{\theta}_2(x(m))}{\partial \hat{m}_2} - \frac{\partial \theta_2(x(m))}{\partial m_1} - \frac{\partial \theta_2(x(m))}{\partial \hat{m}_2} > 1.$$

Accordingly we have $\hat{\theta}_2(x(m)) - \theta_2(x(m)) > \Delta$ with the new belief profile m, which means that the inferential naivety $\hat{m}_2 - m_2$ regarding player 2's belief is also *amplified* over time. In sum, if the belief is perturbed in the above fashion, then each player's inferential naivety is amplified through learning, and the belief does not return to the steady state $m_1 = m_2 = \hat{m}_1 = \hat{m}_2 = \theta^*$.

Simple algebra shows that[23] our key assumption (11) can be rewritten as

$$-\frac{\frac{\partial Q}{\partial x_2}\frac{\partial \hat{x}_2(m_1,\hat{m}_2)}{\partial m_1} + \frac{\partial Q}{\partial x_2}\frac{\partial \hat{x}_2(m_1,\hat{m}_2)}{\partial \hat{m}_2} + \frac{\partial Q}{\partial x_1}\frac{\partial x_1(m_1,\hat{m}_2)}{\partial m_1} + \frac{\partial Q}{\partial x_1}\frac{\partial x_1(m_1,\hat{m}_2)}{\partial \hat{m}_2}}{\frac{\partial Q}{\partial \theta}} > 1. \tag{13}$$

Note that the denominator of the left-hand side of (13) measures how one's belief directly influences the expected output Q, while the numerator measures the indirect impact on the output Q through the equilibrium action. Condition (13) requires that these two effects have opposite signs, and that the magnitude of the former effect is smaller than that of the latter effect.

It turns out that Condition (13) is satisfied in many economic applications:

Example 4. *Team production when efforts are substitutes.* In Section 3.1, we have considered a model of team production where each player's optimal action is independent of the opponent's action. This independence assumption is not critical for our result, as illustrated by the following example. Suppose that each period, each player i chooses an effort level $x_i \in [0,1]$ (instead of the amount of shirking). The output is given by

$$y = a - \theta\left(\frac{1}{x_1 + x_2} - \frac{1}{2}\right) + \varepsilon, \tag{14}$$

where a is the total capability of the team, θ is an unknown state, and ε is a noise term which follows the standard normal distribution. The difference from the model presented in Section 3.1 is that efforts are substitutes and hence one's optimal effort depends on the opponent's effort. Player i's payoff is $y - c(x_i)$, where $c(x_i) = \frac{1}{8}x^2$ is i's effort cost. Assume that the true state is $\theta^* = 0.5$. When $A_1 = A_2 = a$, simple algebra shows that the left-hand side of (13) is approximately 1.62 at

[23] This follows from (46) in the appendix with $X_i = \hat{x}_i$ at $A_1 = A_2 = a$.

the interior steady state $\theta_i = \hat{\theta}_i = 0.5$. So Proposition 3 implies that for any $\gamma > 0$ and for any regular (A_1, A_2) close to (a, a), there is zero probability of the beliefs converge to the steady state $m^*(A_1, A_2)$, i.e., correct learning is destroyed if players (slightly) underestimate the heterogeneity of their perceptions.

Example 5. *Games with conflicting interests.* Consider a situation in which two players compete with each other. Specifically, in each period, each player i chooses effort $x_i \geq 0$ and observes a public outcome

$$y = Q(x_1, x_2, a, \theta) + \varepsilon = \theta\,(x_2 - x_1 + a) + \varepsilon,$$

where $\theta > 0$ is the state, and $a \in \mathbb{R}$ is player 2's relative capability compared to player 1 (or any feature which creates asymmetry in the competition). Intuitively, y represents player 2's relative performance in the competition, and indeed, y is increasing in x_2 while it is decreasing in x_1. Player 2' payoff is $y - \frac{1}{2}x_2^2$, while player 1's payoff is $-y - \frac{1}{8}x_1^2$, where the second term in each player's payoff is the effort cost. This framework can be used to analyze various tournament-like situations such as advertisement (where x_i is the amount of advertisement and y is a variable which influences the market share). Suppose that the true state is $\theta^* = 0.5$. When $A_1 = A_2 = a = 2$, simple algebra shows that the left-hand side of (13) equals 3 at the interior steady state $\theta_i = \hat{\theta}_i = 0.5$. Hence by Proposition 3, correct learning is destroyed for any $\gamma > 0$ and for any regular (A_1, A_2) close to (a, a).

Example 6. *Cournot duopoly.* Suppose that each period, each firm $i = 1, 2$ chooses its quantity $x_i \in [0, \bar{x}]$, and a publicly observable market price is given by

$$y = a - \theta\,(x_1 + x_2) + \varepsilon,$$

where θ is an unknown state and ε is a noise term which follows the standard normal distribution. Firm i's payoff is $yx_i - c(x_i)$, where yx_i is firm i's revenue and $c(x_i)$ is firm i's production cost. Suppose that the true state is $\theta^* > 0$. When $A_1 = A_2 = a$, the Nash equilibrium actions at the steady state belief $\theta_i = \hat{\theta}_i = \theta^*$ are $x_i = \hat{x}_i$ for $i = 1, 2$. Simple algebra shows that the left-hand side of (13) is larger than 1 if and only if the cost function is concave at this equilibrium production level, i.e., $c''(x_i) < 0$. Hence, Proposition 3 applies to this example in such a case, and for any $\gamma > 0$ and for any regular (A_1, A_2) close to (a, a), there is zero probability of the beliefs converging to the steady state $m^*(A_1, A_2)$.

4 Different Initial Priors

As shown in Proposition 1, in the team-production example with correct perception (i.e., $A_1 = A_2 = a$), correct learning occurs almost surely. A key is that the initial belief is on the basin of attraction of the interior steady state (the dashed orange line in Figure 1(a)), and it never leaves the basin. Hence it eventually converges to the interior steady state.

This section studies what happens when players have different initial priors. In this case, the initial belief is not on the basin of the interior steady state, and Figure 1(a) suggests that the beliefs move toward the boundary points, rather than the interior steady state. We will show that this is indeed the case, and the belief converges to the boundary points with positive probability (but not with probability one, we will discuss this later).

Formally, consider the team-production problem with $A_1 = A_2 = a$. For simplicity, we assume that each player i's initial prior is the truncated normal distribution induced by $N(m_i^1, \frac{1}{\xi_i^1})$, where $(m_1^1, \xi_1^1) \neq (m_2^1, \xi_2^1)$. Assume also that each player i is unaware of the opponent having a different belief; she incorrectly believes that the opponent's initial prior is induced by $N(m_i^1, \frac{1}{\xi_i^1})$; in reality, the opponent's belief is induced by $N(m_{-i}^1, \frac{1}{\xi_{-i}^1})$. Then we have the following result:

Proposition 4. *For any initial beliefs with* $(m_1^1, \xi_1^1) \neq (m_2^1, \xi_2^1)$,

$$\Pr\left(\lim_{t\to\infty} \mu^t \in \{(1_{\overline{\theta}}, 1_{\underline{\theta}}), (1_{\underline{\theta}}, 1_{\overline{\theta}})\}\right) > 0.$$

This result is weaker than Proposition 1(ii), in that it does not rule out the possibility that correct learning occurs with positive probability. Intuitively, even though the initial belief is not on the basin of the interior steady state, depending on the realization of the noise ε, the posterior may move to (a neighborhood of) the basin. And once it happens, the belief may stay at the basin forever; this is so because we assume $A_1 = A_2 = a$, in which case the noise does not kick out the mean belief from the basin, as described by Figure 2(a). Accordingly, we cannot rule out the possibility of the beliefs converging to the interior steady state.

In contrast, Proposition 1(ii) assumes $A_1 \neq A_2$, in which case the blue line in Figure 2 does not coincide with the basin of the interior steady state. This means that even if the posterior belief returns to the basin in some period t, it cannot stay there and will be kicked out again due to the noise. This property leads to the zero-probability convergence.

5 One-Sided Misspecification

We have focused on the situation in which both players are misspecified and have inferential naivety about the opponent's action and belief. This section studies how our result is affected when the misspecification is only on one side. We consider the model in which only one of the players is misspecified and has inferential naivety; the other player is perfectly rational, and in particular knows that the opponent is misspecified. It turns out that a result similar to Proposition 3 still holds. That is, when one of the players is slightly misspecified, the beliefs converge to the interior steady state with zero probability. So correct learning is vulnerable to small misspecification even in this one-sided misspecification case.

As in Section 3.3, suppose that players observe a public signal

$$\begin{aligned} y &= Q(x_1, x_2, a, \theta) + \varepsilon \\ &= R(x_1, x_2, a)\theta + S(x_1, x_2, a) + \varepsilon \end{aligned}$$

every period. We assume that player 2's information structure is the same as before. That is, she thinks that the true parameter is $A_2 \neq a$, and projects her view on the opponent. To simplify our exposition, we will focus on the full-projection case (i.e., $\gamma = 1$), so player 2 thinks that player 1 also thinks that the true parameter is $\hat{A}_1 = A_2$.[24] On the other hand, player 1 is correctly specified, so she knows the true parameter a (so $A_1 = a$) and knows player 2's information structure above.

Let $x = (x_1, x_2, \hat{x}_1)$ denote the action profile in this setup; here $\hat{x}_2$ is dropped, as player 1 correctly predicts the opponent's action every period. Define $\theta_2(x)$ and $I_2(x_2, \hat{x}_1)$ as in Section 3.3.

[24]This assumption is not essential, and the result does not change much even in the case of partial projection ($\gamma < 1$).

Then player 2's posterior is the truncated normal distribution induced by the normal distribution $N(m_2^t, \frac{1}{(t-1)\xi_2^t})$, where the parameters (m_2^t, ξ_2^t) are given by (6) and (7). Each period, she computes a Nash equilibrium $(\hat{x}_1, x_2)$ given her perception A_2 and this posterior, and chooses this equilibrium action x_2. As in section 3.3, we denote this equilibrium action by $\left(x_2\left(m_2^t, \frac{1}{(t-1)\xi_2^t}\right), \hat{x}_1\left(m_2^t, \frac{1}{(t-1)\xi_2^t}\right)\right)$.

Player 1 correctly predicts this action x_2 and best-responds to it every period. Specifically, let $\theta_1(x) = \theta^*$ and let $I_1(x_1, x_2) = (R(x_1, x_2, a))^2$. Then player 1's posterior in period t is the truncated normal distribution induced by $N(m_1^t, \frac{1}{(t-1)\xi_1^t})$ where the parameters (m_1^t, ξ_1^t) are given by (6) and (7) with $I_1(x_1, \hat{x}_2)$ being replaced by $I_1(x_1, x_2)$. In each period t, player 1 best-responds to x_2^t given this posterior belief. In what follows, let $x_1\left(m_1^t, \frac{1}{(t-1)\xi_1^t}, m_2^t, \frac{1}{(t-1)\xi_2^t}\right)$ denote this action. Note that this action depends on m_2^t and ξ_2^t, as they influence x_2^t.

As in Section 3.3, we assume that these equilibrium actions satisfy a certain continuity property. Let $(x_2(m_2), \hat{x}_1(m_2))$ denote a Nash equilibrium when everyone thinks that the true parameter is A_2 and the true state is $\theta \in \arg\min_{\theta' \in \Theta} |m_2 - \theta'|$. Also, let $x_1(m_1, m_2)$ denote player 1's best response to $x_2(m_2)$ when she thinks that the true state is $\theta \in \arg\min_{\theta' \in \Theta} |m_1 - \theta'|$.

Assumption 2. There are $\underline{A} < a$ and $\overline{A} > a$ such that for any $A_2 \in (\underline{A}, \overline{A})$, the following properties hold.

(i) There are $K > 0$ and $\alpha > 0$ such that

- For any (m_1, m_2), for any sufficiently large t, for any $\xi_1 \in \left(0, \frac{1}{t\underline{R}^2}\right)$, and for any $\xi_2 \in \left(0, \frac{1}{t\underline{R}^2}\right)$, we have $|x_1(m_1, \frac{1}{\xi_1}, m_2, \frac{1}{\xi_2}) - x_1(m_1, m_2)| < \frac{K}{t^\alpha}$, $|x_2(m_2, \frac{1}{\xi_2}) - x_2(m_2)| < \frac{K}{t^\alpha}$, and $|\hat{x}_1(m_2, \frac{1}{\xi_2}) - \hat{x}_1(m_2)| < \frac{K}{t^\alpha}$, and
- For any $(m_1, m_2) \in (\text{int}\Theta)^2$, for any sufficiently large t, for any $\xi_1 \in (0, \frac{1}{t\underline{R}^2})$, and for any $\xi_2 \in \left(0, \frac{1}{t\underline{R}^2}\right)$, the above inequalities hold for $\alpha = 1$.

(ii) The limit equilibrium actions $(x_1(m_1, m_2), x_2(m_2), \hat{x}_1(m_2))$ are Lipschitz-continuous in (m_1, m_2).

As we assume that player 1 is perfectly rational, she "appropriately" updates her posterior belief each period, in that she uses Bayes' rule with the correct parameter a and the correct prediction about the opponent's action. Accordingly, the standard argument shows that she eventually learns the true state θ^* almost surely. Hence the *steady state* in this environment is defined as $(x_1^*, x_2^*, \hat{x}_1^*, \mu_1^*, \mu_2^*, \hat{\mu}_1^*)$ which solves

$$
\begin{aligned}
& x_1^* \in \arg\max_{x_1} U_1(x_1, x_2^*, a, \theta^*) \\
& x_2^* \in \arg\max_{x_2} U_2(\hat{x}_1^*, x_2, A_2, \theta_2) \\
& \hat{x}_1^* \in \arg\max_{\hat{x}_1} U_2(\hat{x}_1, x_2^*, A_2, \theta_2) \\
& \mu_1^* = 1_{\theta^*} \\
& \mu_2^* = \hat{\mu}_1^* = 1_{\theta} \quad \text{where } \ \theta \in \arg\min_{\theta' \in \Theta} |Q(\hat{x}_1, x_2, A_2, \theta') - Q(x_1, x_2, a, \theta^*)|.
\end{aligned}
$$

Intuitively, the last constraint implies that player 2's belief is concentrated on the state which best explains the data.

The following proposition is a counterpart to Proposition 2, and shows that under the assumption (15), the steady state defined above is continuous with respect to player 2's perception A_2. Note that the assumption (15) is satisfied for generic games, so this proposition implies that small misspecification has a small impact on steady states for "almost all" games. The proof of the proposition is very similar to that of Proposition 2, and hence omitted.

Proposition 5. *Suppose that when $A_1 = A_2 = a$, we have*

$$\frac{\partial Q}{\partial x_1} \frac{\partial x_1(\theta_1, \theta_2)}{\partial \theta_1} + \frac{\partial Q}{\partial \theta} \neq 0 \tag{15}$$

at the steady state with correct learning (i.e., $\theta_i = \theta^*$ and $x_i = x_i(\theta^*)$ for each i). Then there is an open neighborhood $U \subset \boldsymbol{R}$ of $\theta_2 = \theta^*$ such that

(i) When $A_1 = A_2 = a$, the steady state belief in the neighborhood U is unique and it is $\theta_2 = \theta^*$.

(ii) There are $\underline{A} < a$, $\overline{A} > a$, and a unique continuous function $m_2^* : [\underline{A}, \overline{A}] \to U$ such that $m_2^*(a) = \theta^*$ and such that for each $A_2 \in [\underline{A}, \overline{A}]$, $m_2^*(A_2)$ is a steady state belief given A_2.

The next proposition is the main result of this section, which shows that the limit outcome is discontinuous in A_2 even in this one-sided misspecification case. Let $m = (m_1, m_2)$, and let $x(m)$ denote the action profile $(x_1(m), x_2(m_2), \hat{x}_1(m_2))$.

Proposition 6. *Suppose that Assumption 2 holds. Suppose that the assumption stated in Proposition 5 holds, so that there is a function m_2^*. Suppose also that when $A_1 = A_2 = a$, we have*

$$\frac{\partial \theta_2(x(m))}{\partial m_2} > 1 \tag{16}$$

at the steady state with correct learning (i.e., $m_1 = m_2 = \theta^*$). Then, there are $\underline{A} < a$ and $\overline{A} > a$ such that for any $A_2 \in (\underline{A}, \overline{A})$ such that

$$\frac{\partial \theta_2(x(m))}{\partial m_2} + \frac{R(x_2(m_2), \hat{x}_1(m_2), A_2)}{R(x_1(m_1), \hat{x}_2(m_1), a)} \frac{\partial \theta_2(x(m))}{\partial m_1} \neq 0, \tag{17}$$

we have

$$\Pr\left(\lim_{t\to\infty}(\mu_1^t, \mu_2^t) = (1_{\theta^*}, 1_{m_2^*(A_2)})\right) = 0.$$

Note that (17) is similar to the regularity condition imposed in Proposition 3, and is satisfied in generic games. So the critical assumption in the proposition above is (16). Simple algebra shows that the team-production example discussed in Section 3.1 satisfies this condition (16). Hence small misspecification destroys correct learning even in the case of one-sided misspecification. Similarly, the above proposition applies to the games with conflicting interests (Example 5) when $\theta^* = 0.8$ and $a = 2$.

6 Related Literature and Concluding Remarks

There is a rapidly growing literature on Bayesian learning with model misspecification. Nyarko (1991) presents a model in which the agent's action does not converge. Fudenberg, Romanyuk, and Strack (2017) consider a general two-state model and characterize the agent's asymptotic actions and behavior. Ba and Gindin (2023), He (2022), and Heidhues, Kőszegi, and Strack (2018, 2021) study a continuous-state setup, and they show that the agent's action and belief converge to a Berk-Nash equilibrium of Esponda and Pouzo (2016), under some assumptions on payoffs and information structure. Esponda, Pouzo, and Yamamoto (2021) characterize the agent's asymptotic behavior in a general single-agent model. Fudenberg, Lanzani, and Strack (2021) and Frick, Iijima, and Ishii (2023) discuss stability of steady states. All these papers look at a single-agent problem or a multi-agent setup in which each player's bias is common knowledge.

Misspecification of others has been studied in the literature on social learning (e.g., DeMarzo, Vayanos, and Zwiebel, 2003; Eyster and Rabin, 2010; Gagnon-Bartsch, 2016; Gagnon-Bartsch and Rabin, 2016; Bohren and Hauser, 2021). Most of these papers do not discuss discontinuity of the equilibrium outcome, and indeed, one of the main result of Bohren and Hauser (2021) is that the long-run outcome is robust to a small perturbation of the information structure. An exception is Frick, Iijima, and Ishii (2020), who show that the equilibrium outcome is discontinuous in the information structure in a model of information aggregation. As explained in Introduction, a key assumption is that the agents observe a noise signal about the state only once, which is critical for the discontinuity of the steady state. In contrast, in our model, the agents have repeated feedback about the state, and accordingly the steady states are continuous in the information structure. Nonetheless the equilibrium outcome is discontinuous, because small misspecification influences the entire learning dynamics and the convergence probability suddenly drops to zero.

In this paper, we have assumed that players' bias about the opponent's bias takes a form of interpersonal projection. We conjecture that the main finding of this paper goes through even if we consider other forms of misspecification: Roughly, whenever each player has a bias about the opponent's bias, they misspecify the opponent's action. So for games in which there is a snowball effect discussed in this paper, we may expect that small misperception leads to learning failure.

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Online Appendix

A Non-Convergence Theorem for a General Stochastic Process

In this appendix, we will extend the non-convergence theorem of Pemantle (1990) and show that the same non-convergence result holds in a more general environment which includes our model as a special case. This result is used in the proofs of the various non-convergence results in the main text.

Consider a stochastic difference equation

$$v(t+1)-v(t)=\frac{1}{t+1}\left(F(v(t))+b(t,v(t))\varepsilon\right) \tag{18}$$

where $v(t)\in\boldsymbol{R}^n$, $F:\boldsymbol{R}^n\to\boldsymbol{R}^n$, $b(t,v(t))\in\boldsymbol{R}^n$, and $\varepsilon\sim N(0,1)$. We assume that F is Lipschitz-continuous, and that there is $\overline{b}$ such that $|b_i(t,v)|<\overline{b}$ for all i, t, and $v\in\boldsymbol{R}^n$, where $b_i(t,v)$ is the ith component of the vector $b(t,v)$. This second assumption essentially means that the variance of the noise is bounded.

A stochastic process $\{v(t)\}_{t=1}^{\infty}$ is a *perturbed solution to* (18) if it solves

$$v(t+1)-v(t)=\frac{1}{t+1}\left(\tilde{F}(t,v(t))+b(t,v(t))\varepsilon\right)$$

for some $\tilde{F}$ such that there is $K>0$ and $\alpha>0$ such that for all t and v,

$$|F(v)-\tilde{F}(t,v)|<\frac{K}{t^{\alpha}}.$$

The following is a famous result in the theory of stochastic approximation (e.g, Theorem 2.1 of Kushner and Yin (2003)), which shows that if a stochastic process $\{v(t)\}_{t=1}^{\infty}$ is a perturbed solution to (18), and if this process $\{v(t)\}_{t=1}^{\infty}$ is bounded with probability one, then the asymptotic motion $\{v(t)\}$ is approximated by the associated ODE

$$\frac{dw(t)}{dt}=F(w(t)). \tag{19}$$

To state the result formally, we use the following terminologies. Given a realized infinite-horizon outcome $\{v(t)\}_{t=1}^{\infty}$, define the *continuous-time interpolation* as a mapping $\boldsymbol{w}:[0,\infty)\to\boldsymbol{R}^n$ such that

$$\boldsymbol{w}[\tau_t+s]=v(t)+\frac{\tau}{\tau_{t+1}-\tau_t}(v(t)-v(t))$$

for all $t = 0, 1, \cdots$ and $\tau \in [0, \frac{1}{t+1})$. This $\boldsymbol{w}$ is an *asymptotic pseudotrajectory* of the ODE (19) if for any $T > 0$,

$$\lim_{t \to \infty} \sup_{\tau \in [0,T]} |\boldsymbol{w}(t+\tau) - s(\boldsymbol{w}(t))[\tau]| = 0 \tag{20}$$

where $s(v(0)) : \boldsymbol{R}_+ \to \boldsymbol{R}^n$ is a solution to the ODE (19) given the initial value $v(0)$.

Lemma 1. *Suppose that a stochastic process $\{v(t)\}_{t=1}^{\infty}$ is a perturbed solution to (18), and this process $\{v(t)\}_{t=1}^{\infty}$ is bounded almost surely. Then with probability one, $\boldsymbol{w}$ is an asymptotic pseudo-trajectory of the ODE (19).*

A point $p \in \boldsymbol{R}^n$ is a *steady state* of the ODE if $F(p) = 0$. A steady state p is *linearly unstable* if the Jacobian of F at p has at least one eigenvalue with a positive real part. Pemantle (1990) shows that there is zero probability of the stochastic process $\{v(t)\}$ converging to linearly unstable steady states, assuming that (i) $\{v(t)\}$ is an exact solution to (18), and (ii) the noise term ε has a bounded support. An important consequence of (ii) is that the step size $v(t+1) - v(t)$ is bounded by $\frac{\tilde{c}}{t+1}$, which is frequently used in Pemantle's proof.

The following proposition shows that the same result holds even if $\{v(t)\}$ is an perturbed solution (rather than an exact solution) to (18) and the noise term ε is normally distributed (and hence has an unbounded support). This extension is critical in order to obtain our discontinuity result (Proposition 3).

Proposition 7. *Let p be a linearly unstable steady state of the ODE (19). Let H be the affine space spanned by the generalized eigenvectors associated with the eigenvalues with negative real parts, and let H^* be the set of all unit vectors orthogonal to H. Assume that there is $\kappa > 0$, $t^* > 0$, and a neighborhood U of p such that $|b(t,v) \cdot h| \geq \kappa$ for all $h \in H^*$, $t \geq t^*$, and $v \in U$. If there is K and a neighborhood U' of p such that $|F(v) - \tilde{F}(t,v)| < \frac{K}{t}$ for all $v \in U'$ and t, then* $\Pr(\lim_{t\to\infty} v(t) = p) = 0$.

B Proofs

B.1 Proof of Proposition 1(i)

We will first show that Assumption 1 in the main text is satisfied in this team-production example. Recall that each player's postrior belief $\mu_i^{t+1} = \hat{\mu}_{-i}^{t+1}$ is the truncated normal induced by $N(m_i^{t+1}, \frac{1}{t\xi_i^{t+1}})$. Thus the Nash equilibrium action today is $x_i(m_i^{t+1}, \frac{1}{t\xi_i^{t+1}}) = \hat{x}_{-i}(m_i^{t+1}, \frac{1}{t\xi_i^{t+1}}) = 1 - \tilde{m}(m_i^{t+1}, t\xi_i^{t+1})$, where $\tilde{m}(m_i^{t+1}, t\xi_i^{t+1})$ denote the mean of the truncated normal distribution μ_i^{t+1}. Let $x_i(m_i) = \hat{x}_{-i}(m_i) = 1 - \theta$ denote the Nash equilibrium action when player i has a degenerate belief $\mu_i = 1_\theta$ where θ solves $\min_{\tilde{\theta}\in\Theta} |m_i - \tilde{\theta}|$. Part (iv) of the following lemma implies that these equilibrium actions satisfy Assumption 1. (Parts (i)-(iii) of this lemma are not used here, but we will use them in the proof of Proposition 1(ii).)

Lemma 2. *There is $k > 0$ and $\bar{t} > 0$ such that for all $t > \bar{t}$ and $\xi \geq \underline{R}^2$,*

(i) $|\tilde{m}(m, t\xi) - m| < \frac{k}{\sqrt{t}}$ *for all* $m \in \Theta$,

(ii) $|\tilde{m}(m, t\xi) - \underline{\theta}| < \frac{k}{\sqrt{t}}$ *for all* $m < \underline{\theta}$,

(iii) $|\tilde{m}(m, t\xi) - \overline{\theta}| < \frac{k}{\sqrt{t}}$ *for all* $m > \overline{\theta}$.

Also, for any interior point $\theta^* \in \Theta$, there is a neighborhood U of θ^*, $k > 0$, and $\bar{t}' > 0$ such that for all $t > \bar{t}'$ and $m \in U$,

(iv) $|\tilde{m}(m, t\xi) - m| < \frac{k}{t}$.

Proof. Let $\xi = \underline{R}^2$. We will show that (i)-(iv) hold for this particular ξ. Then it is straightforward to see that (i)-(iv) holds for all other $\xi > \underline{R}^2$.

Let ϕ denote the pdf of the standard normal $N(0,1)$, and let Φ denote its cdf. Pick a truncated normal distribution induced by some normal distribution $N(m, \frac{1}{t\xi})$. It is well-known that that the mean of this truncated normal distribution is

$$\tilde{m}(m, t\xi) = m + \frac{1}{\sqrt{t\xi}} \cdot \frac{\phi(\sqrt{t\xi}(\overline{\theta} - m)) - \phi(\sqrt{t\xi}(\underline{\theta} - m))}{\Phi(\sqrt{t\xi}(\overline{\theta} - m)) - \Phi(\sqrt{t\xi}(\underline{\theta} - m))} \tag{21}$$

Since $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-0.5x^2) \leq \frac{1}{\sqrt{2\pi}}$,

$$|\phi(\sqrt{t\xi}(\overline{\theta}-m)) - \phi(\sqrt{t\xi}(\underline{\theta}-m))| < \frac{1}{\sqrt{2\pi}}.$$

Also there is $\bar{t} > 0$ such that for all $m \in \Theta$ and $t > \bar{t}$,

$$\Phi(\sqrt{t\xi}(\overline{\theta}-m)) - \Phi(\sqrt{t\xi}(\underline{\theta}-m)) > \frac{1}{3}. \tag{22}$$

Plugging these into (21), we have

$$|\tilde{m}(m,t\xi) - m| < \frac{1}{\sqrt{t\xi}} \cdot \frac{3}{\sqrt{2\pi}}$$

for all $m \in \Theta$ and $t > \bar{t}$, which implies (i).

Next, we will prove (iv). Note that

$$|\phi(\sqrt{t\xi}(\overline{\theta}-m)) - \phi(\sqrt{t\xi}(\underline{\theta}-m))| = \frac{1}{\sqrt{2\pi}} \left| \left(\frac{1}{(\sqrt{e})^{(\overline{\theta}-m)^2}} \right)^{t\xi} - \left(\frac{1}{(\sqrt{e})^{(\underline{\theta}-m)^2}} \right)^{t\xi} \right|.$$

Pick $\theta^* \in (\underline{\theta}, \overline{\theta})$. Then there is a neighborhood U of θ^* such that we have $\frac{1}{(\sqrt{e})^{(\overline{\theta}-m)^2}} < 1$ and $\frac{1}{(\sqrt{e})^{(\underline{\theta}-m)^2}} < 1$ for all $m \in U$. Then there is $\bar{t}'$ such that

$$|\phi(\sqrt{t\xi}(\overline{\theta}-m)) - \phi(\sqrt{t\xi}(\underline{\theta}-m))| < \frac{1}{t}$$

for all $m \in U$ and $t > \bar{t}'$. Plugging this and (22) into (21), we have (iv).

Finally, we will prove (ii) and (iii). Let $\tilde{\phi}(m, \frac{1}{\xi})$ denote the pdf of the truncated normal induced by $N(m, \frac{1}{\xi})$. Then for any $x > 0$ and $m < \underline{\theta}$,

$$\frac{\tilde{\phi}\left(\underline{\theta}, \frac{1}{\xi}\right)[\underline{\theta}+x]}{\tilde{\phi}\left(\underline{\theta}, \frac{1}{\xi}\right)[\underline{\theta}]} = \frac{\phi(\sqrt{\xi}x)}{\phi(0)} > \frac{\phi(\sqrt{\xi}(\underline{\theta}-m+x))}{\phi(\sqrt{\xi}(\underline{\theta}-m))} = \frac{\tilde{\phi}\left(m, \frac{1}{\xi}\right)[\underline{\theta}+x]}{\tilde{\phi}\left(m, \frac{1}{\xi}\right)[\underline{\theta}]}.$$

This means that the truncated normal induced by $N(\underline{\theta}, \frac{1}{\xi})$ first-order stochastically dominates that induced by $N(m, \frac{1}{\xi})$ for all $m < \underline{\theta}$. Hence

$$\underline{\theta} < \tilde{m}(m,\xi) < \tilde{m}(\underline{\theta},\xi)$$

for all $m < \underline{\theta}$. Together with part (i) of the lemma, we obtain (ii). The proof of (iii) is similar and hence omitted. *Q.E.D.*

When $A_1 = A_2$, players have the same view about the world and hence have the same posterior belief every period, i.e., we have $m_1^t = m_2^t$ and $\xi_1^t = \xi_2^t$ after every history. Hence we only need to consider how player 1's belief (m_1^t, ξ_1^t) evolves over time.

Since Assumption 1 holds in this example, just as we show in the proof of Proposition 3, the stochastic process (m_1^t, ξ_1^t) is a perturbed solution to the difference equations (36) and (38) for $i = 1$. (Here we do not need to consider (37) or (39), as we know that $(m_i^t, \xi_i^t) = (\hat{m}_{-i}^t, \hat{\xi}_{-i}^t)$ in the case of full projection.) Also, as shown in Lemma 3 in the proof of Proposition 1(ii), the process (m_1^t, ξ_1^t) is bounded almost surely. Hence Lemma 1 applies, so that the continuous-time interpolation $\boldsymbol{w}$: $[0, \infty) \to \boldsymbol{R}^2$ of the stochastic process $(m_1^t, \xi_1^t)_{t=1}^{\infty}$ is an *asymptotic pseudotrajectory* of the ODE

$$\frac{dm_1(t)}{dt} = \frac{I_1(m_1(t))(\theta_1(m_1(t)) - m_1(t))}{\xi_i(t)}, \tag{23}$$

$$\frac{d\xi_1(t)}{dt} = I_1(m_1(t)) - \xi_1(t), \tag{24}$$

where $\theta_1(m_1)$ and $I_1(m_1)$ are defined as $\theta_1(m_1, m_2, \hat{m}_1, \hat{m}_2)$ and $I_1(m_1, \hat{m}_2)$ for $m_2 = \hat{m}_1 = \hat{m}_2 = m_1$.

For the special case of $A_1 = A_2 = a$, simple algebra shows that $\theta_1(m_1) - m_1 < 0$ for any $m_1 > m_1^*(A_1, A_2) = \theta^*$, while $\theta_1(m_1) - m_1 > 0$ for any $m_1 < m_1^*(A_1, A_2) = \theta^*$. So regardless of the initial value $(m_1(0), \xi_1(0))$, the solution to the above ODE converges to the interior steady state $(m_1^*(A_1, A_2), I_1(m_1^*(A_1, A_2)))$.

A standard argument shows that the same result holds even when $A_1 = A_2$ is slightly perturbed from a; we have $\theta_1(m_1) - m_1 < 0$ for any $m_1 > m_1^*(A_1, A_2)$, while $\theta_1(m_1) - m_1 > 0$ for any $m_1 < m_1^*(A_1, A_2)$. So the solution to the ODE always converges to the interior steady state.

This means that when $A_1 = A_2$ is close to a, the interior steady state $(m_1^*(A_1, A_2), I_1(m_1^*(A_1, A_2)))$ is *globally attracting* in the sense of Benaïm (1999). Then his theorem shows that the stochastic process (m_1^t, ξ_1^t) almost surely converges to the interior steady state, as desired.

B.2 Proof of Proposition 1(ii)

We will first show that the process converges to the interior steady state with zero probability. Then we will show that the process converges to the boundary steady states.

Part 1: Non-convergence to the interior steady state First, we will show that there is zero probability of the process converging to the interior steady state. As we have seen in the proof of Proposition 1(i), Assumption 1 is satisfied in this team-production example. Also, simple algebra shows that when $A_1 = A_2 = a$, we have $\frac{\partial\theta_1(x(m))}{\partial m_1} + \frac{\partial\theta_1(x(m))}{\partial\hat{m}_2} = 2$ and $\frac{\partial\hat{\theta}_1(x(m))}{\partial m_1} + \frac{\partial\hat{\theta}_1(x(m))}{\partial\hat{m}_2} = -2$. This immediately implies that (11) holds in this example. Hence from Proposition 3(i), the beliefs converge to the interior steady state with zero probability, whenever (A_1, A_2) is regular.

So it suffices to show that any (A_1, A_2) with $A_1 \neq A_2$ is regular. From Proposition 3(iii), we only need to show that any (A_1, A_2) with $A_1 \neq A_2$ satisfies (12).

Recall that given m, $\theta_i(m)$ solves $Q(x_i(m_i), x_{-i}(m_{-i}), a, \theta^*) - Q(x_i(m_i), \hat{x}_{-i}(\hat{m}_{-i}), A_i, \theta_i) = 0$. By the implicit function theorem, we have

$$\frac{\partial\theta_i}{\partial m_i} = \frac{\frac{\partial Q^*}{\partial x_i}\frac{\partial x_i}{\partial m_i} - \frac{\partial Q_i}{\partial x_i}\frac{\partial x_i}{\partial m_i}}{\frac{\partial Q_i}{\partial\theta_i}} = \frac{\frac{\partial Q^*}{\partial x_i}\frac{\partial x_i}{\partial m_i} - \frac{\partial Q_i}{\partial x_i}\frac{\partial x_i}{\partial m_i}}{R(x_i(m_i), \hat{x}_{-i}(\hat{m}_{-i}), A_i)},$$

$$\frac{\partial\theta_i}{\partial m_{-i}} = \frac{\frac{\partial Q^*}{\partial x_{-i}}\frac{\partial x_{-i}}{\partial m_{-i}}}{\frac{\partial Q_i}{\partial\theta_i}} = \frac{\frac{\partial Q^*}{\partial x_{-i}}\frac{\partial x_{-i}}{\partial m_{-i}}}{R(x_i(m_i), \hat{x}_{-i}(\hat{m}_{-i}), A_i)},$$

$$\frac{\partial\theta_i}{\partial\hat{m}_i} = 0,$$

$$\frac{\partial\theta_i}{\partial\hat{m}_{-i}} = -\frac{\frac{\partial Q_i}{\partial\hat{x}_{-i}}\frac{\partial\hat{x}_{-i}}{\partial\hat{m}_{-i}}}{\frac{\partial Q_i}{\partial\theta_i}} = -\frac{\frac{\partial Q_i}{\partial\hat{x}_{-i}}\frac{\partial\hat{x}_{-i}}{\partial\hat{m}_{-i}}}{R(x_i(m_i), \hat{x}_{-i}(\hat{m}_{-i}), A_i)}$$

where $Q_i = Q(x_i(m_i), \hat{x}_{-i}(\hat{m}_{-i}), A_i, \theta_i)$ denotes player i's subjective expectation and $Q^* = Q(x_i(m_i), x_{-i}(m_{-i}), a, \theta^*)$ denotes the true mean. Using these equations, the left-hand side of (12) can be rewritten as

$$\frac{\frac{\partial Q^*}{\partial x_1}\frac{\partial x_1}{\partial m_1} - \frac{\partial Q_1}{\partial x_1}\frac{\partial x_1}{\partial m_1}}{R(x_1(m_1), \hat{x}_2(\hat{m}_2), A_1)} - \frac{\frac{\partial Q_1}{\partial\hat{x}_2}\frac{\partial\hat{x}_2}{\partial\hat{m}_2}}{R(x_1(m_1), \hat{x}_2(\hat{m}_2), A_1)} + \frac{\frac{\partial Q^*}{\partial x_2}\frac{\partial x_2}{\partial m_2}}{R(x_2(m_2), \hat{x}_1(\hat{m}_1), A_2)}$$
$$= \frac{\frac{\partial Q^*}{\partial x_1}\frac{\partial x_1}{\partial m_1} - 2\frac{\partial Q_1}{\partial x_1}\frac{\partial x_1}{\partial m_1}}{R(x_1(m_1), \hat{x}_2(\hat{m}_2), A_1)} + \frac{\frac{\partial Q^*}{\partial x_2}\frac{\partial x_2}{\partial m_2}}{R(x_2(m_2), \hat{x}_1(\hat{m}_1), A_2)}.$$

Here the equality follows from the fact that we have $\frac{\partial Q_1}{\partial x_1}\frac{\partial x_1}{\partial m_1} = \frac{\partial Q_1}{\partial\hat{x}_2}\frac{\partial\hat{x}_2}{\partial\hat{m}_2}$ at any steady state where $m_1 = \hat{m}_2$. Similarly, the right-hand side of (12) is written as

$$\frac{\frac{\partial Q^*}{\partial x_2}\frac{\partial x_2}{\partial m_2} - 2\frac{\partial Q_2}{\partial x_2}\frac{\partial x_2}{\partial m_2}}{R(x_2(m_2), \hat{x}_1(\hat{m}_1), A_2)} + \frac{\frac{\partial Q^*}{\partial x_1}\frac{\partial x_1}{\partial m_1}}{R(x_1(m_1), \hat{x}_2(\hat{m}_2), A_1)}.$$

Hence (12) reduces to

$$\frac{-2\frac{\partial Q_1}{\partial x_1}\frac{\partial x_1}{\partial m_1}}{R(x_1(m_1),\hat{x}_2(\hat{m}_2),a)} \neq \frac{-2\frac{\partial Q_2}{\partial x_2}\frac{\partial x_2}{\partial m_2}}{R(x_2(m_2),\hat{x}_1(\hat{m}_1),A_2)}$$

which is further simplified to

$$-\frac{m_1}{1-m_1} \neq -\frac{m_2}{1-m_2}$$

because at the steady state, we have $m_i = \hat{m}_{-i}$ so that $x_i(m_i) = \hat{x}_i(\hat{m}_{-i}) = 1-m_i$, $\frac{\partial Q_i}{\partial x_i} = -\theta_i(m) = -m_i$, and $R(x_i(m_i),\hat{x}_{-i}(\hat{m}_{-i}),A_i) = x_i(m_i)+\hat{x}_{-i}(\hat{m}_{-i}) = 2(1-m_i)$. This inequality indeed holds (and hence (12) is satisfied) whenever $A_1 \neq A_2$, because $\frac{m_i}{1-m_i}$ is increasing in m_i on the set Θ, and the consistency condition implies that $m_1^* \neq m_2^*$ in any interior steady state with $A_1 \neq A_2$.

Part 2: Convergence to boundary beliefs With an abuse of notation, we will write $(m^t,\xi^t) = (m_i^t,\xi_i^t)_{i=1}^t$, because we do not need to compute $(\hat{m}_i^t,\hat{\xi}_i^t)$ in the case of full projection.

We will first show that the stochastic process (m^t,ξ^t) is bounded with probability one. Recall that regardless of the parameter A_i, a Nash equilibrium given a state θ_i is $x_i = \hat{x}_{-i} = 1-\theta_i$. Hence on the equilibrium path, each player's production is at least $\underline{x} = 1-\overline{\theta}$ and does not exceed $\overline{x} = 1-\underline{\theta}$.

Let $\underline{m}_i$ be such that

$$A_i - \underline{m}_i(\overline{x}+\overline{x}) = a - \theta^*(\overline{x}+\underline{x}).$$

In words, $\underline{m}_i \in \boldsymbol{R}$ denotes a state with which player i's subjective expectation about the output matches the true mean, when player i thinks that the opponent chooses the maximal effort $\overline{x}$ but in reality she chooses the minimal effort $\underline{x}$. Note that this $\underline{m}_i$ is the minimum of $\theta_i(m)$ over all m, and that $\underline{m}_i$ need not be in the state space Θ. Similarly, let $\overline{m}_i$ be such that

$$A_i - \overline{m}_i(\underline{x}+\underline{x}) = a - \theta^*(\underline{x}+\overline{x}).$$

This is a state with which player i's subjective expectation about the output matches the true mean, when player i thinks that the opponent chooses the minimal effort $\underline{x}$ but in reality she chooses the maximal effort $\overline{x}$. Note that this $\overline{m}_i$ is the maximum of $\theta_i(m)$ over all m.

The following lemma shows that almost surely, m_i^t is in a neighborhood of $[\underline{m}_i,\overline{m}_i]$ after a long time. This immediately implies that the process (m^t,ξ^t) is bounded almost surely; indeed, since $x_i \in [\underline{x},\overline{x}]$ for each i, $I_i(x^\tau)$ has the minimal value $\underline{I} = I_i(\overline{x},\overline{x})$ and the maximal value $\overline{I} = I_i(\underline{x},\underline{x})$, and hence it is obvious that ξ_i^t is always in the bounded interval $[\underline{I}, \overline{I}]$. We omit the proof of the lemma, as it is very similar to that of Lemma 5 of Heidhues, Kőszegi, and Strack (2021).

Lemma 3. *Given any* (A_1, A_2)*, almost surely,* $\underline{m}_i \leq \liminf_{t\to\infty} m_i^t \leq \limsup_{t\to\infty} m_i^t \leq \overline{m}_i$ *for each i.*

As we have seen in the proof of Proposition 1(i), Assumption 1 holds in this team-production game. Then just as we show in the proof of Proposition 3, the stochastic process (m^t, ξ^t) is a perturbed solution to the difference equations (36) and (38). (Here we do not need to consider (37) or (39), as we know that $(m_i^t, \xi_i^t) = (\hat{m}_{-i}^t, \hat{\xi}_{-i}^t)$ in the case of full projection.) This, together with the boundedness of (m^t, ξ^t), implies that Lemma 1 applies, so that the continuous-time interpolation $\boldsymbol{w} : [0, \infty) \to \boldsymbol{R}^4$ of the stochastic process $(m^t, \xi^t)_{t=1}^{\infty}$ is an *asymptotic pseudotrajectory* of the ODE

$$\frac{dm_i(t)}{dt} = \frac{I_i(m_i(t))(\theta_i(m(t)) - m_i(t))}{\xi_i(t)}, \tag{25}$$

$$\frac{d\xi_i(t)}{dt} = I_i(m_i(t)) - \xi_i(t), \tag{26}$$

where $\theta_i(m_i, m_{-i})$ and $I_i(m_i)$ are defined as $\theta_i(m_i, m_{-i}, \hat{m}_i, \hat{m}_{-i})$ and $I_i(m_i, \hat{m}_{-i})$ for $\hat{m}_i = m_{-i}$ and $\hat{m}_{-i} = m_i$.

Intuitively, this means that the asymptotic motion of the stochastic process (m^t, ξ^t) is approximated by the ODE (25) and (26). So in order to know the long-run outcome of the stochastic process, it suffices to investigate the ODE.

The next lemma characterizes the behavior of the solution to the ODE when players' misperception is small.

Lemma 4. *There are* $\underline{A} < a$ *and* $\overline{A} > a$ *such that for any* $(A_1, A_2) \in (\underline{A}, \overline{A})^2$ *and for any i, there are values* θ'_{-i} *and* θ''_{-i} *with* $\underline{\theta} < \theta'_{-i} < \theta''_{-i} < \overline{\theta}$ *and differentiable functions* $f_i : [\theta'_{-i}, \theta''_{-i}] \to \Theta$, $\tilde{f}_i : [\underline{\theta}, \theta''_{-i}] \to [\overline{\theta}, \overline{m}_i]$*, and* $\hat{f}_i : [\theta'_{-i}, \overline{\theta}] \to [\underline{m}_i, \underline{\theta}]$ *such that the following properties hold:*

(i) $f_i'(m_{-i}) > 1$ *for all* m_{-i}*,* $f_i(\theta'_{-i}) = \underline{\theta}$*,* $f_i(\theta''_{-i}) = \overline{\theta}$*,* $\tilde{f}_i'(m_{-i}) < 0$ *for all* m_{-i}*,* $\tilde{f}_i(\underline{\theta}) = \overline{m}_i$*,* $\tilde{f}_i(\theta''_{-i}) = \overline{\theta}$*,* $\hat{f}_i'(m_{-i}) < 0$ *for all* m_{-i}*,* $\hat{f}_i(\theta'_{-i}) = \underline{\theta}$*,* $\hat{f}_i(\overline{\theta}) = \underline{m}_i$*,*

(ii) For any $m_{-i} < \underline{\theta}$*,* $\theta_i(m) - m_i$ *is positive if* $m_i < \overline{m}_i$*, is zero if* $m_i = \overline{m}_i$*, and is negative if* $m_i > \overline{m}_i$*.*

(iii) For any $m_{-i} \in [\underline{\theta}, \theta'_{-i})$, $\theta_i(m) - m_i$ is positive if $m_i < \tilde{f}_i(m_{-i})$, is zero if $m_i = \tilde{f}_i(m_{-i})$, and is negative if $m_i > \tilde{f}_i(m_{-i})$,

(iv) For any $m_{-i} \in [\theta'_{-i}, \theta''_{-i}]$, $\theta_i(m) - m_i$ is positive if $m_i < \hat{f}_i(m_{-i})$, is zero if $m_i = \hat{f}_i(m_{-i})$, is negative if $m_i \in (\hat{f}_i(m_{-i}), f_i(m_{-i}))$, is zero if $m_i = f_i(m_{-i})$, is positive if $m_i \in (f_i(m_{-i}), \tilde{f}_i(m_{-i}))$, is zero if $m_i = \tilde{f}_i(m_{-i})$, and is negative if $m_i > \tilde{f}_i(m_{-i})$.

v) For any $m_{-i} \in (\theta''_{-i}, \overline{\theta}]$, $\theta_i(m) - m_i$ is positive if $m_i < \hat{f}_i(m_{-i})$, is zero if $m_i = \hat{f}_i(m_{-i})$, and and is negative if $m_i > \hat{f}_i(m_{-i})$.

(vi) For any $m_{-i} > \overline{\theta}$, $\theta_i(m) - m_i$ is positive if $m_i < \underline{m}_i$, is zero if $m_i = \underline{m}_i$, and is negative if $m_i > \underline{m}_i$.

Proof. We will first explain how to choose θ'_{-i}, θ''_{-i}, f_i, $\tilde{f}_i$, and $\hat{f}_i$. Let θ'_{-i} be a state θ which solves

$$A_i - \underline{\theta}(x_i(\underline{\theta}) + x_{-i}(\underline{\theta})) = a - \theta^*(x_i(\underline{\theta}) + x_{-i}(\theta)).$$

When $A_i = a$, the right-hand side $(a - \theta^*(2 - \underline{\theta} - \theta))$ is less than the left-hand side $(a - \underline{\theta}(2 - 2\underline{\theta}))$ at $\theta = \underline{\theta}$, and is greater than that at $\theta = \theta^*$. Also the right-hand side is increasing in θ. Hence θ'_{-i} which solves the equality above is unique and $\underline{\theta} < \theta'_{-i} < \theta^*$. Then by the continuity, the same result holds as long as (A_1, A_2) is close to (a, a).

Similarly, let θ''_{-i} be a state θ which solves

$$A_i - \overline{\theta}(x_i(\overline{\theta}) + x_{-i}(\overline{\theta})) = a - \theta^*(x_i(\overline{\theta}) + x_{-i}(\theta)).$$

Then again, for A_i close to a, θ''_{-i} is uniquely determined and $\theta^* < \theta''_{-i} < \overline{\theta}$. Hence we have $\underline{\theta} < \theta'_{-i} < \theta''_{-i} < \overline{\theta}$ as stated in the lemma.

Then for each $m_{-i} \in [\underline{\theta}, \theta''_{-i}]$, define $\tilde{f}_i(m_{-i})$ as a value m_i which solves

$$A_i - m_i(x_i(\overline{\theta}) + x_{-i}(\overline{\theta})) = a - \theta^*(x_i(\overline{\theta}) + x_{-i}(m_{-i})),$$

i.e., with this belief m_i, player i's subjective expectation about the output matches the true mean when she believes that the Nash equilibrium for $\overline{\theta}$ will be chosen but in reality the opponent chooses the Nash equilibrium action for m_{-i}. Note that the above equation is linear in m_i, and hence indeed has a unique solution. By the definition, $\tilde{f}_i(\underline{\theta}) = \overline{m}_i$ and $\tilde{f}_i(\theta''_{-i}) = \overline{\theta}$. Also by the implicit function theorem, $\tilde{f}_i(m_{-i})$ is decreasing in m_{-i}, as stated in the lemma.

Similarly, for each $m_{-i} \in [\theta'_{-i}, \overline{\theta}]$, define $\hat{f}_i(m_{-i})$ as a value m_i which solves

$$A_i - m_i(x_i(\underline{\theta}) + x_{-i}(\underline{\theta})) = a - \theta^*(x_i(\underline{\theta}) + x_{-i}(m_{-i})).$$

Again this equation is linear in m_i, and hence has a unique solution. Also it is easy to check that $\hat{f}_i(\theta'_{-i}) = \underline{\theta}$, $\hat{f}_i(\overline{\theta}) = \underline{m}_i$, and $\hat{f}_i(m_{-i})$ is decreasing in m_{-i}.

Also for each $m_{-i} \in [\theta'_{-i}, \theta'']$, define $f_i(m_{-i})$ as a value $m_i \in \Theta$ which solves

$$A_i - m_i(x_i(m_i) + x_{-i}(m_i)) = a - \theta^*(x_i(m_i) + x_{-i}(m_{-i})).$$

To see that this equation has a solution, let

$$g(m_i, m_{-i}) = A_i - m_i(x_i(m_i) + x_{-i}(m_i)) - a + \theta^*(x_i(m_i) + x_{-i}(m_{-i})).$$

By the definition of θ''_{-i}, $g(\overline{\theta}, \theta''_{-i}) = 0$. Then since g is decreasing in m_{-i}, we have $g(\overline{\theta}, m_{-i}) \geq 0$ for all $m_{-i} \in [\theta'_{-i}, \theta'']$. Likewise, since $g(\underline{\theta}, \theta'_{-i}) = 0$. we have $g(\underline{\theta}, m_{-i}) \leq 0$ for all $m_{-i} \in [\theta'_{-i}, \theta'']$. Taken together, given any $m_{-i} \in [\theta'_{-i}, \theta'']$, we have $g(\underline{\theta}, m_{-i}) \leq 0 \leq g(\overline{\theta}, m_{-i})$, so there is at least one $m_i \in \Theta$ which solves $g(m_i, m_{-i}) = 0$. Also this solution is unique, because given any $m_{-i} \in [\theta'_{-i}, \theta'']$, g is strictly increasing in m_i when $m_i \in \Theta$. (Note that g is a quadratic function of m_i.)

By the definition of θ'_{-i} and θ''_{-i}, we have $f_i(\theta'_{-i}) = \underline{\theta}$ and $f_i(\theta''_{-i}) = \overline{\theta}$. Also, by the implicit function theorem,

$$f'_i(m_{-i}) = -\frac{\frac{\partial g}{\partial m_{-i}}}{\frac{\partial g}{\partial m_i}} = \frac{\theta^*}{-2 + 4m_i - \theta^*}.$$

We have $f'_i(m_{-i}) = 2$ at $m_i = m_{-i} = \theta^* = 0.8$ and $f'_i(m_{-i}) > 1$ for any $m_i, m_{-i} \in \Theta$. So all the properties stated in part (i) holds.

Next, we will prove part (iv). Pick $m_{-i} \in (\theta'_{-i}, \theta''_{-i})$ arbitrarily. By the definition of $\hat{f}_i$, we have $\theta_i(m) = \hat{f}_i(m_{-i})$ for any $m_i \leq \underline{\theta}$. Hence $\theta_i(m) - m_i$ is positive for $m_i < \hat{f}_i(m_{-i})$, is zero for $m_i = \hat{f}_i(m_{-i})$, and is negative for $m_i \in (\hat{f}_i(m_{-i}), \underline{\theta}]$, as stated in the lemma.

For $m_i \in (\underline{\theta}, f_i(m_{-i})$, we claim that $\theta_i(m) - m_i$ is negative. Suppose not so that $\theta_i(m) - m_i \geq 0$. If $\theta_i(m) - m_i = 0$, then by the definition of f_i, we must have $m_i = f_i(m_{-i})$, which contradicts with

$m_i < f_i(m_{-i})$. If $\theta_i(m) - m_i > 0$, then there must be $m_i' \in (\underline{\theta}, m_i)$ such that $\theta_i(m_i', m_{-i}) - m_i' = 0$. (This is so because $\theta_i(\underline{\theta}, m_{-i}) - \underline{\theta} < 0$.) But then we must have $m_i' = f_i(m_{-i})$, which is a contradiction. Hence $\theta_i(m) - m_i$ is negative in this case.

By the symmetry, for $m_i > f_i(m_{-i})$, all the properties stated in part (iv) of the lemma are satisfied. Also, by the definition of f_i, we have $\theta_i(m) - m_i = 0$ for $m_i = f_i(m_{-i})$. Hence part (iv) follows.

The proofs of the other parts of the lemma are very similar, and hence omitted. *Q.E.D.*

Figure 3 highlights what is shown in the lemma above. Here the the horizontal axis represents m_{-i} and the vertical axis represents m_i. The origin is the interior steady-state belief. The large dotted square is $\times_{i=1,2}[\underline{m}_i, \overline{m}_i]$, and recall that after a long time, (m_1^t, m_2^t) is in a neighborhood of this square almost surely. The small dotted square is the state space $\times_{i=1,2}\Theta$. The thick polygonal line is the set of points at which $\frac{d\theta_i(t)}{dt} = \theta_i(m(t)) - m_i(t) = 0$; the downward-sloping line at the top is the graph of the function $\tilde{f}_i(m_{-i})$ defined in the lemma above, the upward-sloping line in the middle is the graph of f_i, and the downward-sloping line at the bottom is the graph of $\hat{f}_i$. On the left side of this thick line, $\frac{d\theta_i(t)}{dt} = \theta_i(m(t)) - m_i(t) > 0$, which means that the solution $\theta_i(t)$ to the ODE increases over time. In contrast, on the right side of the line, $\frac{d\theta_i(t)}{dt} = \theta_i(m(t)) - m_i(t) < 0$, and hence $\theta_i(t)$ decreases over time. See the thick arrows in the figure.

Figure 4 describes how the solution to the ODE behaves when both $m_1(t)$ and $m_2(t)$ change over time. The horizontal axis represents m_1 and the vertical axis represents m_2. The two thick polygonal lines are the set of points at which $\frac{dm_i(t)}{dt} = 0$. If the current value $m(t)$ is on the polygonal line with $\frac{dm_1(t)}{dt} = 0$, only $m_2(t)$ changes at the next instant, so $m(t)$ moves vertically, as shown by the arrows in the figure. Similarly, If the current value is on the polygonal line with $\frac{dm_2(t)}{dt} = 0$, only $m_1(t)$ changes at the next instant, so $m(t)$ moves horizontally. For all other points, both m_1 and m_2 move simultaneously. We cannot pin down the exact motion of $m(t)$ in this case (hence we have fork arrows in the picture) because it depends on the current value of $\xi(t)$, which is not specified here; in general, when ξ_1 is relatively larger than ξ_2, m_1 moves faster than m_2, and hence the arrow becomes flatter.

As can be seen from the figure, the polygonal lines intersect three times, and these are the steady states of the ODE. That is, the ODE have one interior steady state (the origin) and two boundary

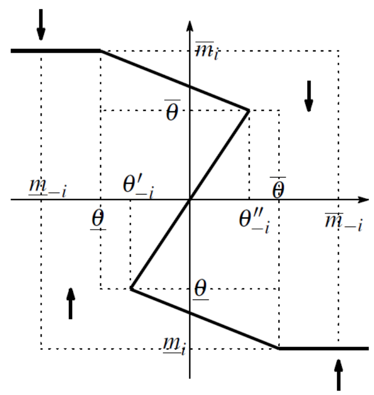

Figure 3: Motion of $m_i(t)$ for Fixed m_{-i}

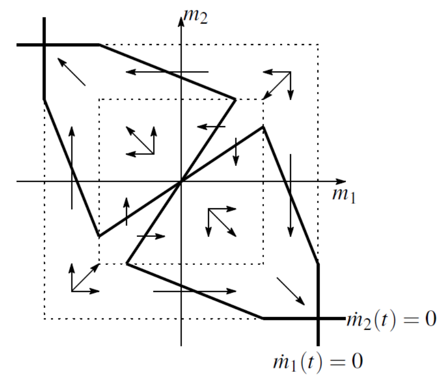

Figure 4: Motion of $m(t)$

steady states ($(\underline{m}_1, \overline{m}_2)$ and $(\overline{m}_1, \underline{m}_2)$). From the figure, it is easy to check that given any initial value (m, ξ), the solution to the ODE eventually converges to one of these steady states. However, this does not imply that the set of steady states is globally attracting in the sense of Benaïm (1999); a problem is that in a neighborhood of the origin (the interior steady state), $(\frac{dm_1(t)}{dt}, \frac{dm_2(t)}{dt})$ is approximately $(0,0)$, meaning that the motion of $m(t)$ can be very slow. Accordingly, for some initial value, it takes arbitrarily long time for the solution to reach a neighborhood of the boundary steady state, so we cannot find a uniform bound T appearing in the definition of attracting sets.

Nonetheless, we can show that m^t converge to the boundary steady states. This implies the result we want, as in such a case the actual belief μ_i^t converges to $1_{\underline{\theta}}$ or $1_{\overline{\theta}}$.

Formally, our goal is to prove the following lemma. Let $B = \{(\underline{m}_1, \overline{m}_2, \underline{I}, \overline{I}), (\overline{m}_1, \underline{m}_2, \overline{I}, \underline{I})\}$ denote the set of the boundary steady states. Also, let $M = (\times_{i=1,2}[\underline{m}_i, \overline{m}_i]) \times [\underline{I}, \overline{I}]^2$. For a point $v \in \boldsymbol{R}^n$ and a set $B \subset \boldsymbol{R}^n$, let $d(v, B) = \min_{v' \in B} |v - v'|$ denote the distance from v to B.

Lemma 5. *Pick a particular path* $\boldsymbol{w} : \boldsymbol{R} \to \boldsymbol{R}^4$ *such that (i)* $\boldsymbol{w}$ *is an asymptotic pseudotrajectory of the ODE, (ii)* $\lim_{t\to\infty} d(\boldsymbol{w}(t), M) = 0$*, and (iii)* $\lim_{t\to\infty} \boldsymbol{w}(t) \neq p$*. (Note that these properties hold with probability one, as shown by the earlier lemmas.) Then* $\lim_{t\to\infty} d(\boldsymbol{w}, B) = 0$.

Proof. Pick $\boldsymbol{w}$ as stated. Since $\lim_{t\to\infty} \boldsymbol{w}(t) \neq p$, there is $\varepsilon > 0$ such that for any $T > 0$, there is $t > T$ such that $\boldsymbol{w}(t) \notin (\times_{i=1,2}[m_i^* - \varepsilon, m_i^* + \varepsilon]) \times [\underline{I}, \overline{I}]^2$. Pick such ε.

Now, note that the inverse function f_i^{-1} is increasing and $f_i^{-1}(m_i^*) = m_{-i}^*$. Hence we have

$f_i^{-1}(m_i^* - \varepsilon) < f_i^{-1}(m_i^*) < f_i^{-1}(m_i^* + \varepsilon)$. Then there is $\eta > 0$ such that

$$f_i^{-1}(m_i^* - \varepsilon) + 2\eta < f_i^{-1}(m_i^*) < f_i^{-1}(m_i^* + \varepsilon) - 2\eta \tag{27}$$

for all i. Pick such $\eta > 0$. Then let $A \subset \boldsymbol{R}^4$ be such that

$$A = \{(m,\xi) \in M | \min\{m_1 - m_1^*, m_2 - m_2^*\} \leq \eta\} \cap \{(m,\xi) | \max\{m_1 - m_1^*, m_2 - m_2^*\} \geq -\eta\}.$$

See Figure 5.

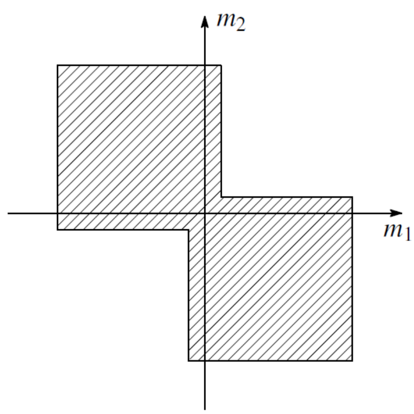

Figure 5: The projection of the set A.

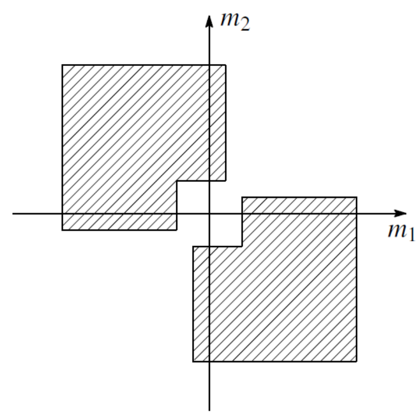

Figure 6: The projection of the set A'.

From Figure 4, given any initial value chosen from the ε-neighborhood of M, the solution to the ODE converges to this set A. Also, the solution does not enter a neighborhood of the origin on the way to a neighborhood of A; this means that the solution reaches a neighborhood of A by some time T, which is independent of the initial value. Thus the set A is attracting in the sense of Benaïm (1999), and its basin is the ε-neighborhood of M.

Theorem 6.10 of Benaïm (1999) asserts that if a path $\boldsymbol{w}$ visits the basin W of an attracting set A infinitely often and if W is compact, then $\boldsymbol{w}$ converges to the set A. Since we assume that $\lim_{t\to\infty} d(\boldsymbol{w}(t), M) = 0$, our path $\boldsymbol{w}$ indeed visits the ε-neighborhood of M infinitely often (actually $\boldsymbol{w}$ stays there forever, after a long time). Also ε-neighborhood of M is compact. Hence $\boldsymbol{w}$ converges to the set A, i.e., $\lim_{t\to\infty} d(\boldsymbol{w}(t), A) = 0$. This in particular implies that there is $T > 0$ such that for any $t > T$, $\boldsymbol{w}(t)$ stays in the η-neighborhood of the set A.

At the same time, by the assumption $\boldsymbol{w}$ leaves the set $(\times_{i=1,2}[m_i^* - \varepsilon, m_i^* + \varepsilon]) \times [\underline{I}, \bar{I}]^2$ infinitely often. This means that $\boldsymbol{w}$ visits the set

$$A' = \{(m,\xi) | d((m,\xi),A) \leq \eta \text{ and } m \notin \times_{i=1,2}(m_i^* - \varepsilon, m_i^* + \varepsilon)\}$$

infinitely often. See Figure 6.

Note that this set A' is compact and is a basin of the set B of the boundary steady states.[25] Hence again from Theorem 6.10 of Benaïm (1999), $\boldsymbol{w}$ converges to B, as desired. *Q.E.D.*

B.3 Proof of Proposition 2

Part (i). Suppose that $A_1 = A_2 = a$. Then the interior steady state belief $(m_1, m_2, \hat{m}_1, \hat{m}_2)$ solves the system of equations

$$Q(x_1(m_1,\hat{m}_2), x_2(\hat{m}_1,m_2), \theta^*, a) - Q(x_1(m_1,\hat{m}_2), \hat{x}_2(m_1,\hat{m}_2), m_1, a) = 0, \quad (28)$$

$$Q(x_1(m_1,\hat{m}_2), x_2(\hat{m}_1,m_2), \theta^*, a) - Q(x_1(m_1,\hat{m}_2), \hat{x}_2(m_1,\hat{m}_2), \hat{m}_2, a) = 0, \quad (29)$$

$$Q(x_1(m_1,\hat{m}_2), x_2(\hat{m}_1,m_2), \theta^*, a) - Q(\hat{x}_1(\hat{m}_1,m_2), x_2(\hat{m}_1,m_2), m_2, a) = 0, \quad (30)$$

$$Q(x_1(m_1,\hat{m}_2), x_2(\hat{m}_1,m_2), \theta^*, a) - Q(\hat{x}_1(\hat{m}_1,m_2), x_2(\hat{m}_1,m_2), \hat{m}_1, a) = 0. \quad (31)$$

At the steady state belief $m_i = \hat{m}_i = \theta^*$, the Jacobian of the above system is

$$\begin{pmatrix} -\frac{\partial Q}{\partial x_2}\frac{\partial \hat{x}_2}{\partial m_1} - \frac{\partial Q}{\partial \theta} & -\frac{\partial Q}{\partial x_2}\frac{\partial \hat{x}_2}{\partial \hat{m}_2} & \frac{\partial Q}{\partial x_2}\frac{\partial x_2}{\partial m_2} & \frac{\partial Q}{\partial x_2}\frac{\partial x_2}{\partial \hat{m}_1} \\ -\frac{\partial Q}{\partial x_2}\frac{\partial \hat{x}_2}{\partial m_1} & -\frac{\partial Q}{\partial x_2}\frac{\partial \hat{x}_2}{\partial \hat{m}_2} - \frac{\partial Q}{\partial \theta} & \frac{\partial Q}{\partial x_2}\frac{\partial x_2}{\partial m_2} & \frac{\partial Q}{\partial x_2}\frac{\partial x_2}{\partial \hat{m}_1} \\ \frac{\partial Q}{\partial x_1}\frac{\partial x_1}{\partial m_1} & \frac{\partial Q}{\partial x_1}\frac{\partial x_1}{\partial \hat{m}_2} & -\frac{\partial Q}{\partial x_1}\frac{\partial \hat{x}_1}{\partial m_2} - \frac{\partial Q}{\partial \theta} & -\frac{\partial Q}{\partial x_1}\frac{\partial \hat{x}_1}{\partial \hat{m}_1} \\ \frac{\partial Q}{\partial x_1}\frac{\partial x_1}{\partial m_1} & \frac{\partial Q}{\partial x_1}\frac{\partial x_1}{\partial \hat{m}_2} & -\frac{\partial Q}{\partial x_1}\frac{\partial \hat{x}_1}{\partial m_2} & -\frac{\partial Q}{\partial x_1}\frac{\partial \hat{x}_1}{\partial \hat{m}_1} - \frac{\partial Q}{\partial \theta} \end{pmatrix}.$$

[25] To see that A' is a basin of B, pick any point $(m,\xi) \in A'$. If (m,ξ) is in the fourth quadrant, we have $\frac{dm_1(0)}{dt} > 0$ and $\frac{dm_2(0)}{dt} < 0$, i.e., the solution $m(t)$ to the ODE move toward the south-east direction, and eventually converge to the boundary point $(\overline{m}_1, \underline{m}_2)$. See Figure 4. Also the solution does not enter the ε-neighborhood of the origin, so it reaches a neighborhood of the boundary point by some time T which is independent of the initial value. Next, consider the case in which (m,ξ) is in the first quadrant. In this case we have either $m_1 < m_1^* + 2\eta$ or $m_2 < m_2^* + 2\eta$, and without loss of generality, we will focus on the case with $m_2 < m_2^* + 2\eta$. Then from (27), the point (m,ξ) is below the graph of f_1 (the flatter upward-sloping line in Figure 4). Then again we have $\frac{dm_1(0)}{dt} > 0$ and $\frac{dm_2(0)}{dt} < 0$, so that the solution $m(t)$ moves toward the south-east direction and eventually converges to the boundary point $(\overline{m}_1, \underline{m}_2)$. A similar argument applies when (m,ξ) is in the second or the third quadrant. Hence A' is indeed a basin of B.

This matrix is regular, as its determinant is

$$D = \begin{vmatrix} -\frac{\partial Q}{\partial \theta} & \frac{\partial Q}{\partial \theta} & 0 & 0 \\ -\frac{\partial Q}{\partial x_2}\frac{\partial \hat{x}_2}{\partial m_1} & -\frac{\partial Q}{\partial x_2}\frac{\partial \hat{x}_2}{\partial \hat{m}_2} - \frac{\partial Q}{\partial \theta} & \frac{\partial Q}{\partial x_2}\frac{\partial x_2}{\partial m_2} & \frac{\partial Q}{\partial x_2}\frac{\partial x_2}{\partial \hat{m}_1} \\ 0 & 0 & -\frac{\partial Q}{\partial \theta} & \frac{\partial Q}{\partial \theta} \\ \frac{\partial Q}{\partial x_1}\frac{\partial x_1}{\partial m_1} & \frac{\partial Q}{\partial x_1}\frac{\partial x_1}{\partial \hat{m}_2} & -\frac{\partial Q}{\partial x_1}\frac{\partial \hat{x}_1}{\partial m_2} & -\frac{\partial Q}{\partial x_1}\frac{\partial \hat{x}_1}{\partial \hat{m}_1} - \frac{\partial Q}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{\partial Q}{\partial \theta} & 0 & 0 & 0 \\ -\frac{\partial Q}{\partial x_2}\frac{\partial \hat{x}_2}{\partial m_1} & -\frac{\partial Q}{\partial x_2}\frac{\partial \hat{x}_2}{\partial m_1} - \frac{\partial Q}{\partial x_2}\frac{\partial \hat{x}_2}{\partial \hat{m}_2} - \frac{\partial Q}{\partial \theta} & \frac{\partial Q}{\partial x_2}\frac{\partial x_2}{\partial m_2} & \frac{\partial Q}{\partial x_2}\frac{\partial x_2}{\partial \hat{m}_1} + \frac{\partial Q}{\partial x_2}\frac{\partial x_2}{\partial m_2} \\ 0 & 0 & -\frac{\partial Q}{\partial \theta} & 0 \\ \frac{\partial Q}{\partial x_1}\frac{\partial x_1}{\partial m_1} & \frac{\partial Q}{\partial x_1}\frac{\partial x_1}{\partial m_1} + \frac{\partial Q}{\partial x_1}\frac{\partial x_1}{\partial \hat{m}_2} & -\frac{\partial Q}{\partial x_1}\frac{\partial \hat{x}_1}{\partial m_2} & -\frac{\partial Q}{\partial x_1}\frac{\partial \hat{x}_1}{\partial \hat{m}_1} - \frac{\partial Q}{\partial x_1}\frac{\partial \hat{x}_1}{\partial m_2} - \frac{\partial Q}{\partial \theta} \end{vmatrix}$$

$$= \left(\frac{\partial Q}{\partial \theta}\right)^2 \begin{vmatrix} -\frac{\partial Q}{\partial x_2}\frac{\partial \hat{x}_2}{\partial m_1} - \frac{\partial Q}{\partial x_2}\frac{\partial \hat{x}_2}{\partial \hat{m}_2} - \frac{\partial Q}{\partial \theta} & \frac{\partial Q}{\partial x_2}\frac{\partial x_2}{\partial \hat{m}_1} + \frac{\partial Q}{\partial x_2}\frac{\partial x_2}{\partial m_2} \\ \frac{\partial Q}{\partial x_1}\frac{\partial x_1}{\partial m_1} + \frac{\partial Q}{\partial x_1}\frac{\partial x_1}{\partial \hat{m}_2} & -\frac{\partial Q}{\partial x_1}\frac{\partial \hat{x}_1}{\partial \hat{m}_1} - \frac{\partial Q}{\partial x_1}\frac{\partial \hat{x}_1}{\partial m_2} - \frac{\partial Q}{\partial \theta} \end{vmatrix}$$

$$= \left(\frac{\partial Q}{\partial \theta}\right)^2 \left\{ \left(\frac{\partial Q}{\partial \theta}\right)^2 + \frac{\partial Q}{\partial \theta}\left(\frac{\partial Q}{\partial x_1}\frac{\partial x_1}{\partial m_1} + \frac{\partial Q}{\partial x_1}\frac{\partial x_1}{\partial \hat{m}_2} + \frac{\partial Q}{\partial x_2}\frac{\partial \hat{x}_2}{\partial m_1} + \frac{\partial Q}{\partial x_2}\frac{\partial \hat{x}_2}{\partial \hat{m}_2}\right)\right\}$$

$$= \left(\frac{\partial Q}{\partial \theta}\right)^3 \left(\frac{\partial Q}{\partial \theta} + \frac{\partial Q}{\partial x_1}\frac{\partial x_1}{\partial m_1} + \frac{\partial Q}{\partial x_1}\frac{\partial x_1}{\partial \hat{m}_2} + \frac{\partial Q}{\partial x_2}\frac{\partial \hat{x}_2}{\partial m_1} + \frac{\partial Q}{\partial x_2}\frac{\partial \hat{x}_2}{\partial \hat{m}_2}\right) \neq 0.$$

Here the second to the last inequality uses the fact that $x_i = \hat{x}_i$ when $A_1 = A_2 = a$, and the last inequality follows from (10) and $\frac{\partial Q}{\partial \theta} = R(x_1(\theta^*), x_2(\theta^*), a) \neq 0$.

The regularity of the Jacobian implies that for any direction $b = (b_1, b_2, b_3, b_4)$, if we slightly perturb $(m_1, m_2, \hat{m}_1, \hat{m}_2)$ toward the direction b from the steady state belief $m_i = \hat{m}_i = \theta^*$, then the resulting belief $(m_1', m_2', \hat{m}_1', \hat{m}_2')$ does not solve the system of equations (28) through (31). This implies the result we want.

Part (ii). Pick $\lambda \in [0,1]$ arbitrarily. Given any (A_1, A_2), the interior steady state belief $(m_1, m_2, \hat{m}_1, \hat{m}_2)$ solves the system of equations

$$Q(x_1(m_1,\hat{m}_2), x_2(\hat{m}_1,m_2), \theta^*, a) - Q(x_1(m_1,\hat{m}_2), \hat{x}_2(m_1,\hat{m}_2), m_1, A_1) = 0,$$

$$Q(x_1(m_1,\hat{m}_2), x_2(\hat{m}_1,m_2), \theta^*, a) - Q(x_1(m_1,\hat{m}_2), \hat{x}_2(m_1,\hat{m}_2), \hat{m}_2, \hat{A}_2) = 0,$$

$$Q(x_1(m_1,\hat{m}_2), x_2(\hat{m}_1,m_2), \theta^*, a) - Q(\hat{x}_1(\hat{m}_1,m_2), x_2(\hat{m}_1,m_2), m_2, A_2) = 0,$$

$$Q(x_1(m_1,\hat{m}_2), x_2(\hat{m}_1,m_2), \theta^*, a) - Q(\hat{x}_1(\hat{m}_1,m_2), x_2(\hat{m}_1,m_2), \hat{m}_1, \hat{A}_1) = 0.$$

When $A_1 = A_2 = a$, $m_i = \hat{m}_i = \theta^*$ solves this system of equations. At this steady state, the Jacobian of this system is exactly the same as that appearing in the proof of part (i), which is regular. Hence it follows from the implicit function theorem that there are $\underline{A}_\lambda < a, \overline{A}_\lambda > a$, a neighborhood $U_\lambda \subset \boldsymbol{R}^4$ of $(\theta^*,\theta^*,\theta^*,\theta^*)$, and a continuous function $m^*_\lambda : [\underline{A}_\lambda, \overline{A}_\lambda]^2 \to U_\lambda$ such that given any $(A_1,A_2) \in [\underline{A}_\lambda, \overline{A}_\lambda]^2$, $m^*_\lambda(A_1,A_2)$ is a unique steady state in U_λ. Since λ is chosen from the compact interval $[0,1]$, a standard argument shows that we can find $\underline{A}$, $\overline{A}$, U, and m^* which work for all $\lambda \in [0,1]$.

B.4 Proof of Proposition 3

We will first prove part (i). As explained in the main text, each player i's posterior belief at the beginning of period $t+1$ is the truncated normal distribution induced by $N(m_i^{t+1}, \frac{1}{t\xi_i^{t+1}})$, where the parameters m_i^{t+1} and ξ_i^{t+1} are given by (6) and (7). Similarly, each hypothetical player i's belief is the truncated normal distribution induced by $N(\hat{m}_i^{t+1}, \frac{1}{t\hat{\xi}_i^{t+1}})$, where the parameters m_i^{t+1} and ξ_i^{t+1} are given by (8) and (9).

Arranging (6) through (9), we obtain the following recursive equations which completely describe the evolution of $(m^t,\xi^t) = (m_i^t, \xi_i^t, \hat{m}_i^t, \hat{\xi}_i^t)_{i=1}^2$:

$$m_i^{t+1} - m_i^t = \frac{1}{t}\left\{ \frac{I_i(x_i^t, \hat{x}_{-i}^t)\left(\theta_i(x^t) - m_i^t - \frac{\varepsilon^t}{R(x_i^t,\hat{x}_{-i}^t,A_i)}\right)}{\frac{t-1}{t}\xi_i^t + \frac{1}{t}I_i(x_i^t,\hat{x}_{-i}^t)} \right\}, \tag{32}$$

$$\hat{m}_{-i}^{t+1} - \hat{m}_{-i}^t = \frac{1}{t}\left\{ \frac{\hat{I}_{-i}(x_i^t, \hat{x}_{-i}^t)\left(\hat{\theta}_{-i}(x^t) - \hat{m}_{-i}^t - \frac{\varepsilon^t}{R(x_i^t,\hat{x}_{-i}^t,\hat{A}_{-i})}\right)}{\frac{t-1}{t}\hat{\xi}_{-i}^t + \frac{1}{t}\hat{I}_{-i}(x_i^t,\hat{x}_{-i}^t)} \right\}, \tag{33}$$

$$\xi_i^{t+1} - \xi_i^t = \frac{1}{t}\left(I_i(x_i^t,\hat{x}_{-i}^t) - \xi_i^t\right), \tag{34}$$

$$\hat{\xi}_{-i}^{t+1} - \hat{\xi}_{-i}^t = \frac{1}{t}\left(\hat{I}_{-i}(x_i^t,\hat{x}_{-i}^t) - \hat{\xi}_{-i}^t\right). \tag{35}$$

The first equation (32) implies that player i updates the mean belief m_i^t depending on how her estimate $\theta_i(x^t) - \frac{\varepsilon^\tau}{R(x_i^t,\hat{x}_{-i}^t,A_i)}$ based on the new information today differs from her current mean belief m_i^t. If the new estimate coincides with the current mean belief, she does not update it. Otherwise, the mean belief moves toward the new estimate, and its magnitude is amplified by the (relative) informativeness of the signal today. The second equation (33) asserts that hypothetical player $-i$ updates the mean belief $\hat{m}_{-i}^t$ in a similar way.

The third and fourth equations also have a similar interpretation; ξ_i^t and $\hat{\xi}_{-i}^t$ are updated depending on how the informativeness $I_i(x_i^t, \hat{x}_{-i}^t)$ of the signal today differs from the average informativeness of the past signals.

Since we assume that the equilibrium action $(x_i^t, \hat{x}_{-i}^t)$ is approximated by $(x_i(m_i^t, \hat{m}_{-i}^t), \hat{x}_{-i}(m_i^t, \hat{m}_{-i}^t))$ with approximation error $O(\frac{1}{t^\alpha})$ and that $\theta_i(x)$ and $I_i(x)$ are Lipschitz-continuous, the parameters (m^t, ξ^t), which solves the difference equations above, is a perturbed solution to

$$m_i^{t+1} - m_i^t = \frac{1}{t}\left\{\frac{I_i(m_i^t, \hat{m}_{-i}^t)\left(\theta_i(m^t) - m_i^t - \frac{\varepsilon^t}{R(x_i(m_i^t, \hat{m}_{-i}^t), \hat{x}_{-i}(m_i^t, \hat{m}_{-i}^t), A_i)}\right)}{\xi_i^t}\right\}, \tag{36}$$

$$\hat{m}_{-i}^{t+1} - \hat{m}_{-i}^t = \frac{1}{t}\left\{\frac{\hat{I}_{-i}(m_i^t, \hat{m}_{-i}^t)\left(\hat{\theta}_{-i}(m^t) - \hat{m}_{-i}^t - \frac{\varepsilon^t}{R(x_i(m_i^t, \hat{m}_{-i}^t), \hat{x}_{-i}(m_i^t, \hat{m}_{-i}^t), \hat{A}_{-i})}\right)}{\hat{\xi}_{-i}^t}\right\}, \tag{37}$$

$$\xi_i^{t+1} - \xi_i^t = \frac{1}{t}\left(I_i(m_i^t, \hat{m}_{-i}^t) - \xi_i^t\right), \tag{38}$$

$$\hat{\xi}_{-i}^{t+1} - \hat{\xi}_{-i}^t = \frac{1}{t}\left(\hat{I}_{-i}(m_i^t, \hat{m}_{-i}^t) - \xi_i^t\right), \tag{39}$$

where $\theta_i(m) = \theta_i(x(m))$, $\hat{\theta}_{-i}(m) = \hat{\theta}_{-i}(x(m))$, $I_i(m_i, \hat{m}_{-i}) = I_i(x_i(m_i), \hat{x}_i(m_i))$, and $\hat{I}_{-i}(m_i, \hat{m}_{-i}) = \hat{I}_{-i}(x_i(m_i), \hat{x}_i(m_i))$. Note that this system of equations is a special case of the difference equation (18) considered in Appendix A. Hence the asymptotic behavior of (m^t, ξ^t) is approximated by the ODE

$$\frac{dm_i(t)}{dt} = \frac{I_i(m_i(t), \hat{m}_{-i}(t))(\theta_i(m(t)) - m_i(t))}{\xi_i(t)}, \tag{40}$$

$$\frac{d\hat{m}_{-i}(t)}{dt} = \frac{\hat{I}_{-i}(m_i(t), \hat{m}_{-i}(t))(\hat{\theta}_{-i}(m(t)) - \hat{m}_{-i}(t))}{\hat{\xi}_{-i}(t)}, \tag{41}$$

$$\frac{d\xi_i(t)}{dt} = I_i(m_i(t), \hat{m}_{-i}(t)) - \xi_i(t), \tag{42}$$

$$\frac{d\hat{\xi}_{-i}(t)}{dt} = \hat{I}_{-i}(m_i(t), \hat{m}_{-i}(t)) - \hat{\xi}_{-i}(t). \tag{43}$$

Suppose that $A_1 = A_2 = a$, and let $\xi^* = I_1(\theta^*, \theta^*) = I_2(\theta^*, \theta^*)$. Then obviously $m_i = \hat{m}_i = \theta^*$ and $\xi_i = \hat{\xi}_i = \xi^*$ constitute a steady state of the ODE above. The following lemma shows that this steady state is linearly unstable if (and only if) the assumption (11) holds.

Lemma 6. *Suppose that $A_1 = A_2 = a$. The following two conditions are equivalent.*

(i) At the steady state $m_i = \hat{m}_i = \theta^*$ and $\xi_i = \hat{\xi}_i = \xi^*$, the Jacobian of the system (40) through (43) has at least one eigenvalue whose real part is positive.

(ii) (11) holds.

Also whenever (i) holds, the Jacobian has only one positive eigenvalue; the other eigenvalues are negative.

Proof. Note that $\theta_i(m(t)) - m_i(t) = \hat{\theta}_{-i}(m(t)) - \hat{m}_{-i}(t) = 0$ in any steady state of the ODE. Hence the Jacobian J of the ODE at the steady state is

$$J = \begin{pmatrix} \frac{\partial\theta_1}{\partial m_1} - 1 & \frac{\partial\theta_1}{\partial \hat{m}_2} & \frac{\partial\theta_1}{\partial m_2} & \frac{\partial\theta_1}{\partial \hat{m}_1} & 0 & 0 & 0 & 0 \\ \frac{\partial\hat{\theta}_2}{\partial m_1} & \frac{\partial\hat{\theta}_2}{\partial \hat{m}_2} - 1 & \frac{\partial\hat{\theta}_2}{\partial m_2} & \frac{\partial\hat{\theta}_2}{\partial \hat{m}_1} & 0 & 0 & 0 & 0 \\ \frac{\partial\theta_2}{\partial m_1} & \frac{\partial\theta_2}{\partial \hat{m}_2} & \frac{\partial\theta_2}{\partial m_2} - 1 & \frac{\partial\theta_2}{\partial \hat{m}_1} & 0 & 0 & 0 & 0 \\ \frac{\partial\hat{\theta}_1}{\partial m_1} & \frac{\partial\hat{\theta}_1}{\partial \hat{m}_2} & \frac{\partial\hat{\theta}_1}{\partial m_2} & \frac{\partial\hat{\theta}_1}{\partial \hat{m}_1} - 1 & 0 & 0 & 0 & 0 \\ \frac{\partial I_1}{\partial m_1} & \frac{\partial I_1}{\partial \hat{m}_2} & 0 & 0 & -1 & 0 & 0 & 0 \\ \frac{\partial \hat{I}_2}{\partial m_1} & \frac{\partial \hat{I}_2}{\partial \hat{m}_2} & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{\partial I_2}{\partial m_2} & \frac{\partial I_2}{\partial \hat{m}_1} & 0 & 0 & -1 & 0 \\ 0 & 0 & \frac{\partial \hat{I}_1}{\partial m_2} & \frac{\partial \hat{I}_1}{\partial \hat{m}_1} & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{44}$$

Obviously the above matrix J has an eigenvalue $\lambda = -1$ (multiplicity 4). The remaining eigenvalues of J are the ones for the submatrix

$$J' = \begin{pmatrix} \frac{\partial\theta_1}{\partial m_1} - 1 & \frac{\partial\theta_1}{\partial \hat{m}_2} & \frac{\partial\theta_1}{\partial m_2} & \frac{\partial\theta_1}{\partial \hat{m}_1} \\ \frac{\partial\hat{\theta}_2}{\partial m_1} & \frac{\partial\hat{\theta}_2}{\partial \hat{m}_2} - 1 & \frac{\partial\hat{\theta}_2}{\partial m_2} & \frac{\partial\hat{\theta}_2}{\partial \hat{m}_1} \\ \frac{\partial\theta_2}{\partial m_1} & \frac{\partial\theta_2}{\partial \hat{m}_2} & \frac{\partial\theta_2}{\partial m_2} - 1 & \frac{\partial\theta_2}{\partial \hat{m}_1} \\ \frac{\partial\hat{\theta}_1}{\partial m_1} & \frac{\partial\hat{\theta}_1}{\partial \hat{m}_2} & \frac{\partial\hat{\theta}_1}{\partial m_2} & \frac{\partial\hat{\theta}_1}{\partial \hat{m}_1} - 1 \end{pmatrix}.$$

Let

$$J'_\lambda = \begin{pmatrix} \frac{\partial\theta_1}{\partial m_1} - 1 - \lambda & \frac{\partial\theta_1}{\partial \hat{m}_2} & \frac{\partial\theta_1}{\partial m_2} & \frac{\partial\theta_1}{\partial \hat{m}_1} \\ \frac{\partial\hat{\theta}_2}{\partial m_1} & \frac{\partial\hat{\theta}_2}{\partial \hat{m}_2} - 1 - \lambda & \frac{\partial\hat{\theta}_2}{\partial m_2} & \frac{\partial\hat{\theta}_2}{\partial \hat{m}_1} \\ \frac{\partial\theta_2}{\partial m_1} & \frac{\partial\theta_2}{\partial \hat{m}_2} & \frac{\partial\theta_2}{\partial m_2} - 1 - \lambda & \frac{\partial\theta_2}{\partial \hat{m}_1} \\ \frac{\partial\hat{\theta}_1}{\partial m_1} & \frac{\partial\hat{\theta}_1}{\partial \hat{m}_2} & \frac{\partial\hat{\theta}_1}{\partial m_2} & \frac{\partial\hat{\theta}_1}{\partial \hat{m}_1} - 1 - \lambda \end{pmatrix}.$$

Since we look at the case with $A_1 = A_2 = a$, we have $\theta_i(m) = \hat{\theta}_{-i}(m)$ for all m, and hence $\frac{\partial \theta_i}{\partial m_i} = \frac{\partial \hat{\theta}_{-i}}{\partial m_i}$, $\frac{\partial \theta_i}{\partial m_{-i}} = \frac{\partial \hat{\theta}_{-i}}{\partial m_{-i}}$, $\frac{\partial \theta_i}{\partial \hat{m}_i} = \frac{\partial \hat{\theta}_{-i}}{\partial \hat{m}_i}$, and $\frac{\partial \theta_i}{\partial \hat{m}_{-i}} = \frac{\partial \hat{\theta}_{-i}}{\partial \hat{m}_{-i}}$. Hence

$$
\begin{aligned}
|J'_\lambda| &= \begin{vmatrix} -1-\lambda & 1+\lambda & 0 & 0 \\ \frac{\partial \hat{\theta}_2}{\partial m_1} & \frac{\partial \hat{\theta}_2}{\partial \hat{m}_2} - 1 - \lambda & \frac{\partial \hat{\theta}_2}{\partial m_2} & \frac{\partial \hat{\theta}_2}{\partial \hat{m}_1} \\ 0 & 0 & -1-\lambda & 1+\lambda \\ \frac{\partial \hat{\theta}_1}{\partial m_1} & \frac{\partial \hat{\theta}_1}{\partial \hat{m}_2} & \frac{\partial \hat{\theta}_1}{\partial m_2} & \frac{\partial \hat{\theta}_1}{\partial \hat{m}_1} - 1 - \lambda \end{vmatrix} \\
&= \begin{vmatrix} -1-\lambda & 0 & 0 & 0 \\ \frac{\partial \hat{\theta}_2}{\partial m_1} & \frac{\partial \hat{\theta}_2}{\partial m_1} + \frac{\partial \hat{\theta}_2}{\partial \hat{m}_2} - 1 - \lambda & \frac{\partial \hat{\theta}_2}{\partial m_2} & \frac{\partial \hat{\theta}_2}{\partial m_2} + \frac{\partial \hat{\theta}_2}{\partial \hat{m}_1} \\ 0 & 0 & -1-\lambda & 0 \\ \frac{\partial \hat{\theta}_1}{\partial m_1} & \frac{\partial \hat{\theta}_1}{\partial m_1} + \frac{\partial \hat{\theta}_1}{\partial \hat{m}_2} & \frac{\partial \hat{\theta}_1}{\partial m_2} & \frac{\partial \hat{\theta}_1}{\partial m_2} + \frac{\partial \hat{\theta}_1}{\partial \hat{m}_1} - 1 - \lambda \end{vmatrix} \\
&= (1+\lambda)^2 \begin{vmatrix} \frac{\partial \hat{\theta}_2}{\partial m_1} + \frac{\partial \hat{\theta}_2}{\partial \hat{m}_2} - 1 - \lambda & \frac{\partial \hat{\theta}_2}{\partial m_2} + \frac{\partial \hat{\theta}_2}{\partial \hat{m}_1} \\ \frac{\partial \hat{\theta}_1}{\partial m_1} + \frac{\partial \hat{\theta}_1}{\partial \hat{m}_2} & \frac{\partial \hat{\theta}_1}{\partial m_2} + \frac{\partial \hat{\theta}_1}{\partial \hat{m}_1} - 1 - \lambda \end{vmatrix}.
\end{aligned}
$$

Hence the matrix J' has an eigenvalue $\lambda = -1$ (multiplicity 2). The remaining eigenvalues solve the quadratic equation

$$
\begin{aligned}
&\lambda^2 + \left(2 - \frac{\partial \hat{\theta}_2}{\partial m_1} - \frac{\partial \hat{\theta}_2}{\partial \hat{m}_2} - \frac{\partial \hat{\theta}_1}{\partial m_2} - \frac{\partial \hat{\theta}_1}{\partial \hat{m}_1}\right)\lambda \\
&\quad + \left\{ \left(\frac{\partial \hat{\theta}_2}{\partial m_1} + \frac{\partial \hat{\theta}_2}{\partial \hat{m}_2} - 1\right)\left(\frac{\partial \hat{\theta}_1}{\partial m_2} + \frac{\partial \hat{\theta}_1}{\partial \hat{m}_1} - 1\right) - \left(\frac{\partial \hat{\theta}_2}{\partial m_2} + \frac{\partial \hat{\theta}_2}{\partial \hat{m}_1}\right)\left(\frac{\partial \hat{\theta}_1}{\partial m_1} + \frac{\partial \hat{\theta}_1}{\partial \hat{m}_2}\right) \right\} = 0 \quad (45)
\end{aligned}
$$

Recall that when $A_1 = A_2 = a$, $\hat{\theta}_{-i}$ solves

$$Q(x_i(m_i, \hat{m}_{-i}), x_{-i}(m_{-i}, \hat{m}_i), \theta^*, a) - Q(x_i(m_i, \hat{m}_{-i}), \hat{x}_{-i}(m_i, \hat{m}_{-i}), \hat{\theta}_{-i}, a) = 0.$$

So by the implicit function theorem, at the steady state, we have

$$\frac{\partial \hat{\theta}_{-i}}{\partial m_i} = -\frac{\frac{\partial Q}{\partial x_{-i}} \frac{\partial x_{-i}}{\partial m_i}}{\frac{\partial Q}{\partial \theta}} = -\frac{\partial \hat{\theta}_{-i}}{\partial \hat{m}_i} \quad \text{and} \quad \frac{\partial \hat{\theta}_{-i}}{\partial \hat{m}_{-i}} = -\frac{\frac{\partial Q}{\partial x_{-i}} \frac{\partial x_{-i}}{\partial m_{-i}}}{\frac{\partial Q}{\partial \theta}} = -\frac{\partial \hat{\theta}_{-i}}{\partial m_{-i}}. \quad (46)$$

Plugging this, the quadratic equation (45) reduces to

$$\lambda^2 + \left(2 - \frac{\partial \hat{\theta}_2}{\partial m_1} - \frac{\partial \hat{\theta}_2}{\partial \hat{m}_2} - \frac{\partial \hat{\theta}_1}{\partial m_2} - \frac{\partial \hat{\theta}_1}{\partial \hat{m}_1}\right)\lambda + \left(1 - \frac{\partial \hat{\theta}_2}{\partial m_1} - \frac{\partial \hat{\theta}_2}{\partial \hat{m}_2} - \frac{\partial \hat{\theta}_1}{\partial m_2} - \frac{\partial \hat{\theta}_1}{\partial \hat{m}_1}\right) = 0 \quad (47)$$

The solution to this equation is

$$
\begin{aligned}
\lambda &= \frac{\left(\frac{\partial\hat{\theta}_2}{\partial m_1}+\frac{\partial\hat{\theta}_2}{\partial\hat{m}_2}+\frac{\partial\hat{\theta}_1}{\partial m_2}+\frac{\partial\hat{\theta}_1}{\partial\hat{m}_1}-2\right)\pm\sqrt{\left(\frac{\partial\hat{\theta}_2}{\partial m_1}+\frac{\partial\hat{\theta}_2}{\partial\hat{m}_2}+\frac{\partial\hat{\theta}_1}{\partial m_2}+\frac{\partial\hat{\theta}_1}{\partial\hat{m}_1}\right)^2}}{2} \\
&= -1, \quad \frac{\partial\hat{\theta}_2}{\partial m_1}+\frac{\partial\hat{\theta}_2}{\partial\hat{m}_2}+\frac{\partial\hat{\theta}_1}{\partial m_2}+\frac{\partial\hat{\theta}_1}{\partial\hat{m}_1}-1
\end{aligned}
$$

This means that the eigenvalues of the matrix J' are $\lambda=-1$ (multiplicity 3) and $\lambda=\frac{\partial\hat{\theta}_2}{\partial m_1}+\frac{\partial\hat{\theta}_2}{\partial\hat{m}_2}+\frac{\partial\hat{\theta}_1}{\partial m_2}+\frac{\partial\hat{\theta}_1}{\partial\hat{m}_1}-1$. This proves the result we want, as (11) implies that the latter eigenvalue is positive.

Q.E.D.

Suppose that (11) holds. Then from the lemma above, when $A_1=A_2=a$, the Jacobian of the ODE has one positive eigenvalue and seven negative eigenvalues (actually all of them are $\lambda=-1$) at the steady state. Now, by the continuity, the same result holds even when (A_1,A_2) is perturbed; that is, there are $\underline{A}<a$ and $\overline{A}>a$ such that for any $\gamma\in[0,1]$ and for any $(A_1,A_2)\in(\underline{A},\overline{A})^2$, the Jacobian of the ODE has one positive eigenvalue and seven negative eigenvalues at the steady state $m^*(A_1,A_2)$ defined in Proposition 2.[26]

Pick $(A_1,A_2)\in(\underline{A},\overline{A})^2$ and $\gamma\in(0,1]$ arbitrarily, and let J denote the Jacobian of the ODE at the steady state. Let H be the affine space spanned by the generalized eigenvectors associated with these seven negative eigenvalues. Also let

$$
b=\left(-\frac{1}{R(x_1,\hat{x}_2,A_1)},-\frac{1}{R(x_1,\hat{x}_2,\hat{A}_2)},-\frac{1}{R(x_2,\hat{x}_1,A_2)},-\frac{1}{R(x_2,\hat{x}_1,\hat{A}_1)},0,0,0,0\right)
$$

be the coefficient on the noise term ε in the difference equations (36) through (39). (Note that both H and b are evaluated at the steady state $m^*(A_1,A_2)$. So for example, x_i in the above display is $x_i(m_i^*,\hat{m}_{-i}^*)$, and $\hat{x}_{-i}$ is $\hat{x}_{-i}(m_i^*,\hat{m}_{-i}^*)$.) Then it directly follows from Proposition 7 that if $b\notin H$, then players' beliefs converge to $m^*(A_1,A_2)$ with zero probability.

Now, since the matrix J has the form (44), it is obvious that the space H is represented as

$$
H=\{(h_1,\hat{h}_2,h_2,\hat{h}_1,\xi_1,\hat{\xi}_2,\xi_2,\hat{\xi}_1)|\forall(h_1,\hat{h}_2,h_2,\hat{h}_1)\in H'\forall\xi_1,\xi_2,\hat{\xi}_1,\hat{\xi}_2\in\boldsymbol{R}\}
$$

where $H'\subset\boldsymbol{R}^4$ is the space spanned by the generalized eigenvectors associated with the negative eigenvalues of the matrix J'. Hence the condition $b\notin H$ is equivalent to $b'\notin H'$, which establishes part (i).

[26] Since γ is chosen from the compact space $[0,1]$, a standard argument shows that there are $\underline{A}$ and $\overline{A}$ which work for all γ.

For part (ii), note that when $A_1 = A_2 = a$, the eigenvalues of the matrix J' are $\lambda = -1$ (multiplicity 3) and $\lambda = \frac{\partial \hat{\theta}_2}{\partial m_1} + \frac{\partial \hat{\theta}_2}{\partial \hat{m}_2} + \frac{\partial \hat{\theta}_1}{\partial m_2} + \frac{\partial \hat{\theta}_1}{\partial \hat{m}_1} - 1 > 0$, as shown in the proof of Lemma 6. This immediately implies part (ii), as a small perturbation of (A_1, A_2) does not change the signs of the eigenvalues.

Now we will prove part (iii). So let $\gamma = 1$ and pick $(A_1, A_2) \in (\underline{A}, \overline{A})^2$. Let J denote the Jacobian of the ODE at the steady state for this (A_1, A_2). Since J has the form (44), it has an eigenvalue $\lambda = -1$ (multiplicity 4), and the remaining eigenvalues are the ones for the submatrix J'.

Since $\gamma = 1$, we have $\theta_i(m) = \hat{\theta}_{-i}(m)$ for all m. Hence the argument similar to the one in the proof of Lemma 6 shows that the submatrix J' has an eigenvalue $\lambda = -1$ (multiplicity 2), and the remaining two eigenvalues are the ones for the matrix

$$J'' = \begin{pmatrix} \frac{\partial \hat{\theta}_2}{\partial m_1} + \frac{\partial \hat{\theta}_2}{\partial \hat{m}_2} - 1 & \frac{\partial \hat{\theta}_2}{\partial m_2} + \frac{\partial \hat{\theta}_2}{\partial \hat{m}_1} \\ \frac{\partial \hat{\theta}_1}{\partial m_1} + \frac{\partial \hat{\theta}_1}{\partial \hat{m}_2} & \frac{\partial \hat{\theta}_1}{\partial m_2} + \frac{\partial \hat{\theta}_1}{\partial \hat{m}_1} - 1 \end{pmatrix}.$$

Note that the eigenvalues of this matrix are $\lambda = -1$ and $\lambda = \frac{\partial \hat{\theta}_2}{\partial m_1} + \frac{\partial \hat{\theta}_2}{\partial \hat{m}_2} + \frac{\partial \hat{\theta}_1}{\partial m_2} + \frac{\partial \hat{\theta}_1}{\partial \hat{m}_1} - 1$ at $A_1 = A_2 = a$. When (A_1, A_2) is perturbed, it still has one negative eigenvalue and one positive eigenvalue. Let λ_1 denote the negative one and λ_2 denote the positive one.

Let

$$e_i = \begin{pmatrix} e_{i,1} \\ e_{i,2} \end{pmatrix}$$

denote the eigenvector of the matrix J'' associated with the eigenvalue λ_i. Then by the definition, this vector e_i solves

$$J'' e_i = \lambda_i e_i.$$

This, together with the fact that $\theta_i(m) = \hat{\theta}_{-i}(m)$, implies that

$$J' e_i^* = \lambda_i e_i^*$$

where

$$e_i^* = \begin{pmatrix} e_{i,1} \\ e_{i,1} \\ e_{i,2} \\ e_{i,2} \end{pmatrix}.$$

So e_1^* and e_2^* are the eigenvectors of J'.

When $\gamma = 1$, the first two components of b' are the same, and so are the remaining two components. Accordingly, b' is represented as a linear combination of e_1^* and e_2^*. Then since the generalized eigenvalues of J' are linearly independent, the condition $b' \notin H'$ holds if and only if $b' \neq \alpha e_1^*$ for all $\alpha \in \boldsymbol{R}$. In other words, we have $b' \notin H'$ if and only if b' is not an eigenvector of H' for the eigenvalue $\lambda = \lambda_1$. This condition is equivalent to

$$H'b' \neq \lambda_1 b' \Leftrightarrow \begin{pmatrix} \frac{\frac{\partial \hat{\theta}_2}{\partial m_1} + \frac{\partial \hat{\theta}_2}{\partial \hat{m}_2} - 1}{R(x_1,\hat{x}_2,A_1)} + \frac{\frac{\partial \hat{\theta}_2}{\partial m_2} + \frac{\partial \hat{\theta}_2}{\partial \hat{m}_1}}{R(x_2,\hat{x}_1,A_2)} \\ \frac{\frac{\partial \hat{\theta}_1}{\partial m_1} + \frac{\partial \hat{\theta}_1}{\partial \hat{m}_2}}{R(x_1,\hat{x}_2,A_1)} + \frac{\frac{\partial \hat{\theta}_1}{\partial m_2} + \frac{\partial \hat{\theta}_1}{\partial \hat{m}_1} - 1}{R(x_2,\hat{x}_1,A_2)} \end{pmatrix} \neq \lambda_1 \begin{pmatrix} \frac{1}{R(x_1,\hat{x}_2,A_1)} \\ \frac{1}{R(x_2,\hat{x}_1,A_2)} \end{pmatrix}$$

This condition is satisfied whenever

$$\frac{\partial \hat{\theta}_2}{\partial m_1} + \frac{\partial \hat{\theta}_2}{\partial \hat{m}_2} - 1 + \frac{R(x_1,\hat{x}_2,A_1)}{R(x_2,\hat{x}_1,A_2)} \left(\frac{\partial \hat{\theta}_2}{\partial m_2} + \frac{\partial \hat{\theta}_2}{\partial \hat{m}_1} \right) \neq \frac{\partial \hat{\theta}_1}{\partial m_2} + \frac{\partial \hat{\theta}_1}{\partial \hat{m}_1} - 1 + \frac{R(x_2,\hat{x}_1,A_2)}{R(x_1,\hat{x}_2,A_1)} \left(\frac{\partial \hat{\theta}_1}{\partial m_1} + \frac{\partial \hat{\theta}_1}{\partial \hat{m}_2} \right),$$

which implies part (iii).

B.5 Proof of Proposition 4

We show that if different players have different initial priors in the team-production example, then the posteriors eventually converge to the boundary steady states with positive probability. The proof consists of two steps. In the first step, we show that for some initial priors, the posterior will eventually converge to the boundary steady state with positive probability. Then in the second step, we show that whenever different players have different initial priors, with positive probability, the posterior will move to the belief considered in the first step (which means that the posterior will converge to the boundary steady state afterwards).

Before we start the proof, note that in this setup, each player i's posterior in period t is a truncated normal distribution induced by $N(m_i^t, \frac{1}{t\xi_i^t})$, where

$$m_i^t = \frac{\xi_i^1 m_i^1 + \sum_{\tau=1}^{t-1} I_i(x_i^\tau, \hat{x}_{-i}^\tau) \left(\theta_i(x^\tau) - \frac{\varepsilon^\tau}{\sqrt{I_i(x_i^\tau, \hat{x}_{-i}^\tau)}} \right)}{\xi_i^1 + \sum_{\tau=1}^{t-1} I_i(x_i^\tau, \hat{x}_{-i}^\tau)},$$

$$\xi_i^t = \frac{1}{t} \left(\xi_i^1 + \sum_{\tau=1}^{t-1} I_i(x_i^\tau, \hat{x}_{-i}^\tau) \right).$$

Note that these formulas are a bit different from those for the case with the uniform initial prior.

Step 1: Convergence for some initial priors. We show that if the initial belief is close to the one for the boundary steady state, then the belief will converge to that steady state with positive probability. Without loss of generality, we will focus on one of the boundary steady state, say, $(1_{\overline{\theta}}, 1_{\underline{\theta}})$.

Note that in this steady state, player 1's belief $\theta_1 = \overline{\theta}$ solves the consistency condition (1), but does not explain the actual observation perfectly. Indeed, for player 1's expectation (about the output y) to match the true mean, her belief must be concentrated on $\overline{m} > \overline{\theta}$ which solves

$$\theta^*(x_1(\overline{\theta}) + x_2(\underline{\theta})) = \overline{m}(x_1(\overline{\theta}) + x_2(\overline{\theta})).$$

Let f_m denote the density of the normal distribution $N(m,1)$ with mean m and variance 1, and $\text{KL}(f|g)$ denote the KL divergence between densities f and g. Then since the true output distribution at the steady state is $N(\theta^*(x_1(\overline{\theta}) + x_2(\underline{\theta})), 1)$ and player 1's subjective distribution is $N(\overline{\theta}(x_1(\overline{\theta}) + x_2(\overline{\theta})), 1)$, we have

$$\begin{aligned}
&-\text{KL}(f_{\overline{\theta}(x_1(\overline{\theta})+x_2(\overline{\theta}))}|f_{\theta^*(x_1(\overline{\theta})+x_2(\underline{\theta}))}) < 0 \\
&\Leftrightarrow \int f_{\theta^*(x_1(\overline{\theta})+x_2(\underline{\theta}))}(y) \log \frac{f_{\overline{\theta}(x_1(\overline{\theta})+x_2(\overline{\theta}))}(y)}{f_{\theta^*(x_1(\overline{\theta})+x_2(\underline{\theta}))}(y)} dy < 0 \\
&\Leftrightarrow \int f_{\theta^*(x_1(\overline{\theta})+x_2(\underline{\theta}))}(y) \log \frac{f_{\overline{\theta}(x_1(\overline{\theta})+x_2(\overline{\theta}))}(y)}{f_{\overline{m}(x_1(\overline{\theta})+x_2(\overline{\theta}))}(y)} dy < 0.
\end{aligned}$$

Then from Lemma 2 of Frick, Iijima, and Ishii (2023), there is $l_1 \in (0,1)$ such that

$$\int f_{\theta^*(x_1(\overline{\theta})+x_2(\underline{\theta}))}(y) \left(\frac{f_{\overline{\theta}(x_1(\overline{\theta})+x_2(\overline{\theta}))}(y)}{f_{\overline{m}(x_1(\overline{\theta})+x_2(\overline{\theta}))}(y)} \right)^{l_1} dy < 1.$$

Similarly, in this steady state, player 2's belief $\theta_2 = \underline{\theta}$ does not explain the actual observation perfectly; for player 2's subjective expectation to match the true mean, her belief must be concentrated on $\underline{m} < \underline{\theta}$ which solves

$$\theta^*(x_1(\overline{\theta}) + x_2(\underline{\theta})) = \underline{m}(x_1(\underline{\theta}) + x_2(\underline{\theta})).$$

Then the argument similar to the above one shows that there is $l_2 \in (0,1)$ such that

$$\int f_{\theta^*(x_1(\overline{\theta})+x_2(\underline{\theta}))}(y) \left(\frac{f_{\underline{\theta}(x_1(\underline{\theta})+x_2(\underline{\theta}))}(y)}{f_{\underline{m}(x_1(\underline{\theta})+x_2(\underline{\theta}))}(y)} \right)^{l_2} dy < 1.$$

By the continuity, the above inequalities still hold even if the actions are perturbed a bit. Formally, we have the following result:

Lemma 7. *There is $\xi^* > 0$ such that for any $m_1 \geq \overline{\theta}$, $m_2 \leq \underline{\theta}$, and $\xi_1, \xi_2 > \xi^*$, we have*

$$\int f_{\theta^*(x_1(m_1,\frac{1}{\xi_1})+x_2(m_2,\frac{1}{\xi_2}))}(y)\left(\frac{f_{\overline{\theta}(x_1(m_1,\frac{1}{\xi_1})+x_2(m_1,\frac{1}{\xi_1}))}(y)}{f_{\overline{m}(x_1(m_1,\frac{1}{\xi_1})+x_2(m_1,\frac{1}{\xi_1}))}(y)}\right)^{l_1} dy < 1$$

and

$$\int f_{\theta^*(x_1(m_1,\frac{1}{\xi_1})+x_2(m_2,\frac{1}{\xi_2}))}(y)\left(\frac{f_{\underline{\theta}(x_1(m_2,\frac{1}{\xi_2})+x_2(m_2,\frac{1}{\xi_2}))}(y)}{f_{\underline{m}(x_1(m_2,\frac{1}{\xi_2})+x_2(m_2,\frac{1}{\xi_2}))}(y)}\right)^{l_2} dy < 1.$$

In what follows, pick $\xi^* > 0$ as stated in the lemma above. Without loss of generality, we assume that

$$\exp(\frac{\xi^*(\overline{m}-\overline{\theta})^2}{2}) > 4, \tag{48}$$

$$\exp(\frac{\xi^*(\underline{\theta}-\underline{m})^2}{2}) > 4. \tag{49}$$

Note that these inequalities indeed hold as long as ξ^* is large enough.

Our goal in this step is to prove the following lemma, which shows that for some initial prior, the posterior converges to the boundary steady state with positive probability.

Lemma 8. *Suppose that $m_1^1 > \frac{\overline{\theta}+\overline{m}}{2}$, $m_2^1 < \frac{\overline{\theta}+\overline{m}}{2}$, and $\xi_i^1 > \xi^*$ for each i. Then*

$$\Pr\left(\lim_{t\to\infty} \mu^t = (1_{\overline{\theta}}, 1_{\underline{\theta}})\right) > 0.$$

In the rest of this step, we will prove this lemma. Pick the initial prior as stated in the lemma. Let

$$L_1^1 = \left(\frac{\exp\left(-\frac{\xi_1^1(\overline{\theta}-m_1^1)^2}{2}\right)}{\exp\left(-\frac{\xi_1^1(\overline{m}-m_1^1)^2}{2}\right)}\right)^{l_1}.$$

In words, L_1^1 is the l_1th-power of the likelihood of the parameters $\overline{\theta}$ and $\overline{m}$ induced by the normal distribution $N(m_1, \frac{1}{\xi_1})$. (Intuitively, this distribution is player 1's initial prior when the state space is not restricted to Θ.) Similarly define L_1^t for each later period $t = 2, 3, \cdots$, using the posterior $N(m_1^t, \frac{1}{t\xi_1^t})$ in period t. By the definition, we have

$$L_1^{t+1} = L_1^t \left(\frac{f_{\overline{\theta}(x_1(m_1^t, \frac{1}{t\xi_1^t}) + x_2(m_1^t, \frac{1}{t\xi_1^t})}(y^t)}{f_{\overline{m}(x_1(m_1^t, \frac{1}{t\xi_1^t}) + x_2(m_1^t, \frac{1}{t\xi_1^t})}(y^t)} \right)^{l_1}.$$

Likewise, for each t, let L_2^t be the l_2th-power of the likelihood between the parameters $\underline{m}$ and $\underline{\theta}$ induced by the normal distribution $N(m_2^t, \frac{1}{t\xi_2^t})$. Then we have

$$L_2^{t+1} = L_2^t \left(\frac{f_{\underline{\theta}(x_1(m_2^t, \frac{1}{t\xi_2^t}) + x_2(m_2^t, \frac{1}{t\xi_2^t})}(y^t)}{f_{\underline{m}(x_1(m_2^t, \frac{1}{t\xi_2^t}) + x_2(m_2^t, \frac{1}{t\xi_2^t})}(y^t)} \right)^{l_2}.$$

Consider a hypothetical situation such that starting from the initial prior considered above, the actions in each period t is given by

$$(x_1^t, \hat{x}_2^t) = \begin{cases} (x_1(m_1^t, \frac{1}{t\xi_1^t}), \hat{x}_2(m_1^t, \frac{1}{t\xi_1^t})) & \text{if } m_1^t \geq \overline{\theta} \\ (x_1(\overline{\theta}, \frac{1}{t\xi_1^t}), \hat{x}_2(\overline{\theta}, \frac{1}{t\xi_1^t})) & \text{if } m_1^t < \overline{\theta} \end{cases},$$

$$(x_2^t, \hat{x}_1^t) = \begin{cases} (x_2(m_2^t, \frac{1}{t\xi_2^t}), \hat{x}_1(m_2^t, \frac{1}{t\xi_2^t})) & \text{if } m_2^t \leq \underline{\theta} \\ (x_2(\underline{\theta}, \frac{1}{t\xi_2^t}), \hat{x}_1(\underline{\theta}, \frac{1}{t\xi_2^t})) & \text{if } m_2^t > \underline{\theta} \end{cases}$$

In words, this is the situation in which players choose actions as if their mean beliefs are always in a neighborhood of the initial value, in that player 1's mean belief m_1^t is never below $\overline{\theta}$ and player 2's mean belief is never above $\underline{\theta}$.

From Lemma 7, in this hypothetical situation, both L_1^t and L_2^t are supermartingales. Hence from Dubin's uncrossing inequality,

$$\Pr\left(L_1^t > \exp\left(\frac{\xi^*(\overline{m} - \overline{\theta})^2}{2} \right) \text{ for some } t \right) \leq \frac{L_1^1}{\exp\left(\frac{\xi^*(\overline{m} - \overline{\theta})^2}{2} \right)} < \frac{1}{4} \tag{50}$$

$$\Pr\left(L_2^t > \exp\left(\frac{\xi^*(\underline{\theta} - \underline{m})^2}{2} \right) \text{ for some } t \right) \leq \frac{L_2^1}{\exp\left(\frac{(\xi^*\underline{\theta} - \underline{m})^2}{2} \right)} < \frac{1}{4} \tag{51}$$

Here, the last inequality in each line follows from (48), (49), and $L_i^1 \leq 1$ (which follows from $m_1^1 > \frac{\overline{\theta} + \overline{m}}{2}$ and $m_2^1 < \frac{\overline{\theta} + \overline{m}}{2}$), Note that if $m_1^t < \overline{\theta}$ in some t, then

$$
\begin{aligned}
L_1^t &= \left(\frac{\exp\left(-\frac{t\xi_1^t(\overline{\theta}-m_1^t)^2}{2}\right)}{\exp\left(-\frac{t\xi_1^t(\overline{m}-m_1^t)^2}{2}\right)} \right)^{l_1} \\
&= \exp\left(\frac{t\xi_1^t\{(\overline{m}-m_1^t)^2-(\overline{\theta}-m_1^t)^2\}}{2} \right)^{l_1} \\
&> \exp\left(\frac{\xi^*(\overline{m}-\overline{\theta})^2}{2} \right)^{l_1}
\end{aligned}
$$

where the last inequality follows from the fact that $t\xi_i^t$ is increasing in t and $\xi_1^1 > \xi^*$. Hence $L_1^t \leq \exp\left(\frac{\xi^*(\overline{m}-\overline{\theta})^2}{2}\right)$ implies $m_1^t \geq \overline{\theta}$. Likewise, $L_2^t \leq \exp\left(\frac{\xi^*(\underline{\theta}-\underline{m})^2}{2}\right)$ implies $m_2^t \leq \underline{\theta}$. Accordingly, in the hypothetical situation,

$$
\begin{aligned}
&\Pr\left(m_1^t \geq \overline{\theta} \text{ and } m_2^t \leq \underline{\theta} \text{ for all } t = 1,2,\cdots\right) \\
&\geq \Pr\left(L_1^t \leq \exp\left(\frac{\xi^*(\overline{m}-\overline{\theta})^2}{2}\right) \text{ and } L_2^t \leq \exp\left(\frac{\xi^*(\underline{\theta}-\underline{m})^2}{2}\right) \text{ for all } t = 1,2,\cdots\right) \\
&> 1 - \Pr\left(L_1^t > \exp\left(\frac{\xi^*(\overline{m}-\overline{\theta})^2}{2}\right) \text{ for some } t\right) - \Pr\left(L_2^t > \exp\left(\frac{\xi^*(\underline{\theta}-\underline{m})^2}{2}\right) \text{ for some } t\right) \\
&> \frac{1}{2}.
\end{aligned}
$$

where the last inequality follows from (50) and (51). This in turn implies that

$$
\Pr\left(m_1^t \geq \overline{\theta} \text{ and } m_2^t \leq \underline{\theta} \text{ for all } t = 1,2,\cdots\right) > \frac{1}{2}
$$

for the original situation, and since $m_1^t \geq \overline{\theta}$ for all t implies $\mu_1^t \to 1_{\overline{\theta}}$ and $m_2^t \leq \underline{\theta}$ for all t implies $\mu_2^t \to 1_{\underline{\theta}}$, we obtain the result we want.

Step 2: Convergence for all priors with $(m_1^1, \xi_1^1) \neq (m_2^1, \xi_2^1)$. Now we will show that whenever different players have different initial beliefs, with positive probability, the posterior beliefs converge to the boundary steady state. The following lemma proves a result slightly weaker than that; it shows the convergence to the boundary steady state when $E[\theta|\mu_1] \neq E[\theta|\mu_2]$.

Lemma 9. *Pick the initial values (m_1^1, ξ_1^1) and (m_2^1, ξ_2^1) such that $E[\theta|\mu_1] \neq E[\theta|\mu_2]$. Then*

$$
\Pr\left(\lim_{t\to\infty} \mu^t = (1_{\overline{\theta}}, 1_{\underline{\theta}})\right) > 0.
$$

Proof. For shorthand notation, let I_i^t denote the informativeness $I_i(x_i^t, \hat{x}_{-i}^t)$ of the signal in period t. Pick initial beliefs as stated in the lemma. Since $E[\theta|\mu_1] \neq E[\theta|\mu_2]$, we have $x_1^1 = \hat{x}_2^1 \neq x_2^1 = \hat{x}_1^t$, which implies $I_1^1 \neq I_2^1$. For now we assume $I_1^1 > I_2^1$ and prove the lemma. Later on we will explain how to fix the proof when $I_1^1 < I_2^1$.

Since we assume $I_1^1 > I_2^1$, there is $\lambda > 0$ such that

$$\sqrt{I_2^1} < \lambda(2 - 2\overline{\theta}) < \sqrt{I_1^1}. \tag{52}$$

Pick such λ. Also, pick a natural number T^* such that $T^*\underline{R}^2 > \xi^*$, where $\xi^* > 0$ is chosen as in the first step.

Suppose that the noise in the first T^* periods is

$$(\varepsilon^1, \cdots, \varepsilon^{T^*}) = (-K, \lambda K, 0, 0, \cdots, 0) \tag{53}$$

for some $K > 0$. That is, assume that the noise takes a negative value, then a positive value, and then zero for a while. Then the parameter $m_i^{T^*+1}$ for the posterior belief is

$$\begin{aligned} m_i^{T^*+1} &= \frac{\xi_i^1 m_i^1 + I_i^1\left(m_i(x^1) + \frac{K}{\sqrt{I_i^1}}\right) + I_i^2\left(m_i(x^2) - \frac{\lambda K}{\sqrt{I_i^2}}\right) + \sum_{t=3}^{T^*} I_i^t m_i(x^t)}{\xi_i^1 + \sum_{t=1}^{T^*} I_i^t} \\ &= \frac{\xi_i^1 m_i^1 + \sum_{t=1}^{T^*} I_i^t m_i(x^t) + K\left(\sqrt{I_i^1} - \lambda\sqrt{I_i^2}\right)}{\xi_i^1 + \sum_{t=1}^{T^*} I_i^t}. \end{aligned}$$

When $K \to \infty$, we have

$$m_i^2 = \frac{\xi_i^1 m_i^1 + I_i^1\left(m_i(x^1) + \frac{K}{\sqrt{I_i^1}}\right)}{\xi_i^1 + I_i^1} \to \infty$$

for each i, and hence $x_i^2 = \hat{x}_{-i}^2 \to 1 - \overline{\theta}$, implying $\sqrt{I_i^2} \to 2 - 2\overline{\theta}$. Plugging this into the above equation for $m_i^{t^*+1}$, it follows from (52) and the boundedness of $m_i(x^t)$ and I_i^t that[27]

$$\lim_{K\to\infty} m_1^{T^*+1} = \infty \quad \text{and} \quad \lim_{K\to\infty} m_2^{T^*+1} = -\infty.$$

So if we take a large $K > 0$,

$$m_1^{T^*+1} > \overline{m} \quad \text{and} \quad m_2^{T^*+1} < \underline{m}. \tag{54}$$

[27] Indeed, $\underline{m} \leq m_i(x^t) \leq \overline{m}$ and $(2-2\overline{\theta})^2 \leq I_i^t \leq (2-2\underline{\theta})^2$.

Pick such K. The inequalities (54) imply that, after the particular noise realization (53), the posterior belief μ^{T^*+1} satisfies the condition stated in Lemma 8 in the first step. (By the definition of T^*, we have $(T^*+1)\xi_i^{T^*+1} > \xi^*$ for any noise realization.)

Also, since the inequalities (54) are strict, they are still satisfied even if the noise sequence is perturbed; this means that as long as the noise sequence $(\varepsilon^1, \cdots, \varepsilon^{T^*})$ is in a neighborhood of $(-K, \lambda K, 0, 0, \cdots, 0)$, the posterior belief μ^{T^*+1} satisfies the condition stated in Lemma 8. Since this event occurs with positive probability, the result follows.

Next, consider the case in which $I_1^1 < I_2^1$. In this case, there is $\lambda > 0$ such that

$$\sqrt{I_1^1} < \lambda(2 - 2\underline{\theta}) < \sqrt{I_2^1}.$$

Pick such λ, and consider the noise realization

$$(\varepsilon^1, \cdots, \varepsilon^{T^*}) = (K, -\lambda K, 0, 0, \cdots, 0).$$

Then the argument similar to the above one shows the result we want. *Q.E.D.*

To complete the proof, it suffices to show that starting from any initial beliefs with $E[\theta|\mu_1] = E[\theta|\mu_2]$, the posterior in some period $t > 1$ satisfies $E[\theta|\mu_1^t] \neq E[\theta|\mu_2^t]$ with positive probability. The following lemma shows that such a result indeed holds for $t = 2$.

Lemma 10. *Suppose that* $(m_1^1, \xi_1^1) \neq (m_2^1, \xi_2^1)$ *and that* $E[\theta|\mu_1] = E[\theta|\mu_2]$. *Then*

$$\Pr\left(E[\theta|\mu_1^2] \neq E[\theta|\mu_2^2]\right) > 0.$$

Proof. Pick initial beliefs as stated, and let $\theta' = E[\theta|\mu_1]$.

We first prove that $\xi_1^1 \neq \xi_2^1$. Suppose not so that $\xi_1^1 = \xi_2^1$. Since we assume $(m_1^1, \xi_1^1) \neq (m_2^1, \xi_2^1)$, we have $m_1^1 \neq m_2^1$. But then we have $E[\theta|\mu_1] \neq E[\theta|\mu_2]$, a contradiction. So we must have $\xi_1^1 \neq \xi_2^1$.

Given the noise ε^1 in period one, the parameter m_i^2 of the posterior belief is

$$m_i^2 = \frac{\xi_i^1 m_i^1 + (2-2\theta')^2 m_i(x^1) - (2-2\theta')\varepsilon^1}{\xi_i^1 + (2-2\theta')^2}.$$

Then simple algebra shows that the parameters m_1^2 and m_2^2 coincides (i.e., $m_1^2 = m_2^2$) if

$$\varepsilon^1 = \left(\frac{\xi_1^1 m_1^1 + (2-2\theta')^2 m_1(x^1)}{\xi_1^1 + (2-2\theta')^2} - \frac{\xi_2^1 m_2^1 + (2-2\theta')^2 m_2(x^1)}{\xi_2^1 + (2-2\theta')^2}\right)\left(\frac{2-2\theta'}{\xi_1^1 + (2-2\theta')^2} - \frac{2-2\theta'}{\xi_2^1 + (2-2\theta')^2}\right)^{-1}.$$

Let ε^* denote this right-hand side. Note that this value ε^* is well-defined since $\xi_1^1 \neq \xi_2^1$.

Suppose that the noise in period one is exactly $\varepsilon^1 = \varepsilon^*$. Then from the argument above, we have $m_1^2 = m_2^2$. At the same time, we have $\xi_1^2 \neq \xi_2^2$, since $\xi_i^2 = \frac{\xi_1^1 + I_i^1}{2}$, $\xi_1^1 \neq \xi_2^1$, and $I_1^1 = I_2^1 = (2-2\theta')^2$. Accordingly, we have $E[\theta|\mu_1^2] \neq E[\theta|\mu_2^2]$ given the noise $\varepsilon^1 = \varepsilon^*$.

Also, by the continuity, the same inequality holds even if the noise is perturbed. That is, we have $E[\theta|\mu_1^2] \neq E[\theta|\mu_2^2]$ as long as ε^1 is in a neighborhood of ε^*. This completes the proof, as such an event occurs with positive probability. *Q.E.D.*

B.6 Proof of Proposition 6

The proof is similar to that of Proposition 3, so we will give only the outline of the proof.

Note first that the evolution of the posterior belief in our one-sided misspecification model is very similar to that for the two-sided misspecification case. Indeed, each player i's posterior belief at the beginning of period $t+1$ is the truncated normal distribution induced by $N(m_i^{t+1}, \frac{1}{t\xi_i^{t+1}})$, where the evolution of the parameters m_i^{t+1} and ξ_i^{t+1} is determined by the difference equations (32) and (34) with $I_1(x_1, \hat{x}_2)$ being replaced with $I_1(x_1, x_2)$. Then the argument similar to that in the proof of Proposition 3 shows that this evolution is approximated by the following ODEs

$$\frac{dm_i(t)}{dt} = \frac{I_i(m(t))(\theta_i(m(t)) - m_i(t))}{\xi_i(t)}, \tag{55}$$

$$\frac{d\xi_i(t)}{dt} = I_i(m(t)) - \xi_i(t). \tag{56}$$

where $\theta_i(m) = \theta_i(x(m))$, $I_1(m) = I_1(x_1(m), x_2(m_2))$, and $I_2(m) = I_2(x_2(m_2), \hat{x}_1(m_2))$.

Suppose that $A_1 = A_2 = a$. Obviously $(m_1, m_2, \xi_1, \xi_2) = (\theta^*, \theta^*, I_1(\theta^*, \theta^*), I_2(\theta^*, \theta^*))$ is a steady state of the ODE. The following lemma shows that this steady state is linearly unstable if (and only if) the assumption (16) holds.

Lemma 11. *Suppose that $A_1 = A_2 = a$. The following two conditions are equivalent.*

(i) At the steady state $(m_1, m_2, \xi_1, \xi_2) = (\theta^*, \theta^*, I_1(\theta^*, \theta^*), I_2(\theta^*, \theta^*))$, the Jacobian of the ODE (55) and (56) has at least one eigenvalue whose real part is positive.

(ii) (16) holds.

Proof. Note that $\theta_i(m) - m_i = 0$ and $\xi_i = I_i$ in any steady state. Note also that $\frac{\partial I_2}{\partial m_1} = 0$, as player 2 does not know player 1's actual belief. Hence the Jacobian J of the ODE at the steady state can be written as

$$J = \begin{pmatrix} \frac{\partial \theta_1}{\partial m_1} - 1 & \frac{\partial \theta_1}{\partial m_2} & 0 & 0 \\ \frac{\partial \theta_2}{\partial m_1} & \frac{\partial \theta_2}{\partial m_2} - 1 & 0 & 0 \\ \frac{\partial I_1}{\partial m_1} & \frac{\partial I_1}{\partial m_2} & -1 & 0 \\ 0 & \frac{\partial I_2}{\partial m_2} & 0 & -1 \end{pmatrix}.$$

Obviously this Jacobian J has an eigenvalue $\lambda = -1$ (multiplicity 2). The remaining two eigenvalues of J are the ones for the submatrix

$$J' = \begin{pmatrix} \frac{\partial \theta_1}{\partial m_1} - 1 & \frac{\partial \theta_1}{\partial m_2} \\ \frac{\partial \theta_2}{\partial m_1} & \frac{\partial \theta_2}{\partial m_2} - 1 \end{pmatrix}.$$

The eigenvalues of this matrix J' solve

$$\lambda^2 - \left(\frac{\partial \theta_1}{\partial m_1} + \frac{\partial \theta_2}{\partial m_2} - 2 \right) \lambda + \left(\frac{\partial \theta_1}{\partial m_1} - 1 \right) \left(\frac{\partial \theta_2}{\partial m_2} - 1 \right) - \frac{\partial \theta_1}{\partial m_2} \frac{\partial \theta_2}{\partial m_1} = 0.$$

Since player 1 is perfectly rational, we have $\theta_1(m) = \theta^*$ for all m, which implies $\frac{\partial \theta_1}{\partial m_1} = \frac{\partial \theta_1}{\partial m_2} = 0$. Hence the quadratic equation above is simplified to

$$\lambda^2 + \left(2 - \frac{\partial \theta_2}{\partial m_2} \right) \lambda + \left(1 - \frac{\partial \theta_2}{\partial m_2} \right) = 0,$$

whose solution is $\lambda = -1, \frac{\partial \theta_2}{\partial m_2} - 1$. This proves the result we want. *Q.E.D.*

By the continuity, the result of the lemma above holds even when A_2 is perturbed. That is, there are $\underline{A} < a$ and $\overline{A} > a$ such that for any $A_2 \in (\underline{A}, \overline{A})$, the interior steady state $(\theta^*, m_2^*(A_2))$ is linearly unstable. Hence it follows from Proposition 7 that for any $A_2 \in (\underline{A}, \overline{A})$ such that $b \notin H$, players' beliefs converge to $m^*(A_2)$ with zero probability. Here, H is the affine space spanned by the eigenvectors of J associated with the eigenvalues with negative real parts, and

$$b = \left(-\frac{1}{R(x_1(m_1^*), x_2(m_2^*), A_2)}, -\frac{1}{R(x_2(m_2^*), \hat{x}_1(m_2^*), A_2)}, 0, 0 \right).$$

The argument similar to that in the proof of Proposition 3 shows that (17) implies $b \notin H$. Hence the result follows.

B.7 Proof of Proposition 7

Let $\mathscr{N}$ be a neighborhood of p, and choose a function $\eta : \mathscr{N} \to \boldsymbol{R}_+$ as in Section 3 of Pemantle (1990), given the ODE (19). Roughly, $\eta(v)$ measures the distance between a point v and (the set of) the paths pointing to the steady state p. For example, any point v with $\eta(v) = 0$ is on such a path, so starting from this point v, a solution to the ODE (19) converges to p.

On the other hand, any point v with $\eta(v) > 0$ is not on such a path. So the solution to the ODE does not converge to p. Indeed, as shown by Proposition 3(v) of Pemantle (1990), we have $D_v(\eta)(F(v)) > 0$ for any v with $\eta(v) > 0$. So a solution to the ODE moves away from the paths converging to p. (Here, the notation for multidimensional derivatives uses $D_v(\eta)$ for the differential of η at a point v.)

Let $S_t = \eta(v(t))$ and $X_t = S_t - S_{t-1}$. Lemma 1 of Pemantle (1990) shows that after every history $\mathscr{F}_t$, the stochastic process $\{S_k\}$ can exceed $\frac{c^*}{\sqrt{t}}$ (i.e., $v(t)$ leaves a neighborhood of the paths converging to p) at some point in the future with probability at least 0.5. The following lemma shows that the same result holds in our setup. The proof can be found in Appendix B.7.1

Lemma 12. *There is a constant $c^* > 0$ and t^* such that for any $t > t^*$ and $\mathscr{F}_t$,*

$$\Pr\left(\sup_{k>t} S_k > \frac{c^*}{\sqrt{t}} \text{ or } v(k) \notin \mathscr{N} \text{ for some } k > t \,\middle|\, \mathscr{F}_t \right) > 0.5.$$

Lemma 2 of Pemantle (1990) shows that once the process $\{v(t)\}$ leaves a $\frac{c^*}{\sqrt{t}}$-neighborhood of p as stated in the lemma above, then it fails to return to p with positive probability. The proof can be found in Appendix B.7.2

Lemma 13. *Let $c^* > 0$ be as in Lemma 12. Then there is $a > 0$ such that*

$$\Pr\left(\inf_{k>t} S_k > \frac{c^*}{2\sqrt{t}} \text{ or } v(k) \notin \mathscr{N} \text{ for some } k \geq t \,\middle|\, \mathscr{F}_t, S_t \geq \frac{c^*}{\sqrt{t}} \right) \geq a.$$

The rest of the proof is exactly the same as the argument in the full paragraph on page 711 of Pemantle (1990): Suppose that $\Pr(v(t) \to p) > 0$. Then there is some history $\mathscr{F}_t$ after which the probability that $v(M)$ converges to p and never leaves the neighborhood $\mathscr{N}$ is at least $1 - \frac{a}{2}$. However, Lemmas 12 and 13 imply that the probability that $v(M)$ fails to converge to p or leaves $\mathscr{N}$ is at least $\frac{a}{2}$ conditional on any history $\mathscr{F}_t$. This is a contradiction.

B.7.1 Proof of Lemma 12

Without loss of generality, assume that $\mathscr{N}$ (the domain of the "distance function" η) is a closed ball surrounding p. (This is so because given a neighborhood U of the point p, we can always find a closed ball $\mathscr{N} \subseteq U$ containing p.) Then enlarge the domain of η by letting $\eta(v) = \eta(\arg\max_{\tilde{v}\in\mathscr{N}} d(\tilde{v},\mathscr{N}))$ for each $v \notin \mathscr{N}$. Here $d(v,\mathscr{N})$ measures the Euclidean distance between v and the ball $\mathscr{N}$. This function η is well-defined because $\mathscr{N}$ is a closed ball. Since η is Lipschitz in $\mathscr{N}$, it is so in the entire space $\boldsymbol{R}^n$.

Pick a sufficiently large t, and define a stopping time $\tau = \{M \geq t | S_M > \frac{c^*}{\sqrt{t}}\}$. We will show that $\Pr(\tau = \infty | \mathscr{F}_t) < 0.5$.

Step 1: Inequalities (12) and (14) of Pemantle (1990).

In the proof Pemantle (1990), he shows that there is $k_2 > 0$ such that for any $M > t$ with $S_M \leq \frac{c^*}{\sqrt{t}}$,

$$E[2X_{M+1}S_M|\mathscr{F}_M] \geq \frac{k_2 S_M^2}{M+1} + O\left(\frac{S_M}{M^2}\right), \tag{57}$$

$$E[X_{M+1}^2|\mathscr{F}_M] \text{ is at least } \frac{\text{const.}}{M^2}. \tag{58}$$

See (12) and (14) of Pemantle (1990). His proof relies on the assumption that the noise term has a bounded support (and hence the step size is of order $\frac{1}{t+1}$). We will show that the same result holds in our setup where the noise is Gaussian.

Note that for any $v, \tilde{v} \in \boldsymbol{R}^n$ and sufficiently large M,

$$\begin{aligned}
&E\left[\eta\left(v + \frac{b(M,\tilde{v})\varepsilon}{M+1}\right)\middle|\mathscr{F}_M\right] \\
=&E[\eta(v + zb(M,\tilde{v})\varepsilon)], \quad \text{where } z = \tfrac{1}{M+1} \\
=&\eta(v) + \left.\frac{\partial E[\eta(v + zb(M,\tilde{v})\varepsilon)]}{\partial z}\right|_{z=0} z + O(z^2) \\
=&\eta(v) + \sum_{i=1}^{n} \frac{\partial \eta(v)}{\partial v_i} b_i(M,\tilde{v}) E[\varepsilon] z + O(z^2) \\
=&\eta(v) + O(z^2) = \eta(v) + O\left(\frac{1}{M^2}\right).
\end{aligned}$$

To obtain the second equation, we regard the whole term as a function of z and apply Taylor expansion at $z = 0$. Intuitively, this shows that the impact of the noise ε in period M on the expected value of $\eta(v(M+1))$ is of order $O(\frac{1}{M^2})$. Then we have

$$\begin{aligned}
&E[S_{M+1}|\mathscr{F}_M] \\
=&E\left[\eta\left(v(M)+\frac{1}{M+1}\left(\tilde{F}(t,v(M))+b(M,v(M))\varepsilon\right)\right)\middle|\mathscr{F}_M\right] \\
=&\eta\left(v(M)+\frac{\tilde{F}(t,v(M))}{M+1}\right)+O\left(\frac{1}{M^2}\right)=\eta\left(v(M)+\frac{F(v(M))}{M+1}\right)+O\left(\frac{1}{M^2}\right) \\
\geq&\frac{k_2 S_M}{M+1}+O\left(\frac{1}{M^2}\right),
\end{aligned}$$

which immediately implies (57). Here the third equation follows from the Lipschitz continuity of η, and $|F(v)-\tilde{F}(M,v)| < \frac{K}{M}$. The last inequality follows from Proposition 3(iv) of Pemantle (1990).

To obtain (58), note that

$$\begin{aligned}
E[X_{M+1}^2|\mathscr{F}_M] =&(E[X_{M+1}^+|\mathscr{F}_M])^2 \\
&\geq(\Pr(|\varepsilon(M)|<1|\mathscr{F}_M)E[X_{M+1}^+|\mathscr{F}_M,|\varepsilon(M)|<1])^2.
\end{aligned}$$

Conditional on $|\varepsilon(M)| < 1$, the step size $v(M+1)-v(M)$ is of order $\frac{1}{M+1}$. Hence as in the first display on page 709 of Pemantle (1990), we have

$$\begin{aligned}
&E[X_{M+1}^+|\mathscr{F}_M,|\varepsilon(M)|<1] \\
\geq&E\left[\left(D_{v(M)}(\eta)\left(\frac{\tilde{F}(M,v(M))+b(M,v(M))\varepsilon}{M+1}\right)+O(|v(M+1)-v(M)|^2)\right)^+\middle|\mathscr{F}_M,|\varepsilon(M)|<1\right] \\
=&E\left[\left(D_{v(M)}(\eta)\left(\frac{F(v(M))+b(M,v(M))\varepsilon}{M+1}\right)+O\left(\frac{1}{M^2}\right)\right)^+\middle|\mathscr{F}_M,|\varepsilon(M)|<1\right] \\
\geq&E\left[\left(D_{v(M)}(\eta)\left(\frac{b(M,v(M))\varepsilon}{M+1}\right)+O\left(\frac{1}{M^2}\right)\right)^+\middle|\mathscr{F}_M,|\varepsilon(M)|<1\right] \\
\geq&\frac{\text{const.}}{M+1}+O\left(\frac{1}{M^2}\right).
\end{aligned}$$

Here the equality follows from linearity of $D_v(\eta)$, $|F(v)-\tilde{F}(M,v)| < \frac{K}{M}$, and the fact that the step size $v(M+1)-v(M)$ is of order $\frac{1}{M+1}$. The second to the last inequality follows from Proposition

3(v) of Pemantle (1990). The last inequality uses the fact that the gradient of η at p is $c'h$ for some $c' > 0$ and $h \in H^*$, which implies $D_v(\eta)(b(M, v(M))) \geq c'\kappa$ for any $v(M)$ in a neighborhood of p.

Substituting this inequality to the previous one, we obtain (58).

Step 2: Main Proof.

As argued in the full paragraph on page 709 of Pemantle (1990), combining (57) and (58) yields

$$E[2X_{M+1}S_M + X_{M+1}^2 | \mathscr{F}_M] \geq \frac{\text{const.}}{M^2},$$

which in turn implies

$$\begin{aligned}
E[S^2_{\tau\wedge(M+1)}|\mathscr{F}_t] - E[S^2_{\tau\wedge M}|\mathscr{F}_t] &= E[1_{\tau>M}(2X_{M+1}S_M + X^2_{M+1})|\mathscr{F}_t] \\
&= E[E[1_{\tau>M}(2X_{M+1}S_M + X^2_{M+1})|\mathscr{F}_M]|\mathscr{F}_t] \\
&\geq \frac{\text{const.}}{M^2} E[1_{\tau>M}|\mathscr{F}_t] \\
&\geq \frac{\text{const.}}{M^2} \Pr(\tau = \infty|\mathscr{F}_t)
\end{aligned}$$

for $M > t$. Pemantle (1990) applies this inequality iteratively and obtains

$$\begin{aligned}
E[S^2_{\tau\wedge M}|\mathscr{F}_t] &\geq S_t^2 + \text{const.} \cdot \Pr(\tau = \infty|\mathscr{F}_t) \sum_{i=t}^{M-1} \frac{1}{i^2} \\
&\geq \text{const.} \cdot \Pr(\tau = \infty|\mathscr{F}_t) \left(\frac{1}{t} - \frac{1}{M} \right). \qquad (59)
\end{aligned}$$

Then in the first paragraph on page 710, Pemantle (1990) shows that

$$\frac{4(c^*)^2}{t} \geq E(S^2_{M\wedge\tau}|\mathscr{F}_t), \qquad (60)$$

using the assumption that the noise has a bounded support (which ensures that the step size is of order $\frac{1}{M+1}$). We can show that the same result holds in our setup, the proof can be found at the end.

Then the rest of the proof is the same as Pemantle (1990): Combining (59) and (60),

$$\frac{4(c^*)^2}{t} \geq \text{const.} \cdot \Pr(\tau = \infty|\mathscr{F}_t) \left(\frac{1}{t} - \frac{1}{M} \right).$$

This inequality holds for all M, and when $M \to \infty$, it reduces to

$$4(c^*)^2 \geq \text{const.} \cdot \Pr(\tau = \infty|\mathscr{F}_t).$$

By taking c^* small enough, we have $\Pr(\tau = \infty|\mathscr{F}_t) \leq 0.5$, as desired.

Step 3: Proof of (60).

We will show that (60) holds in our setup: Let $L > 0$ be the Lipschitz constant of η, and $\hat{c} > 0$ be such that $|\tilde{F}(M,v) - v| < \hat{c}$ for all $M > t$ and for all v in a neighborhood of p. Then

$$\begin{aligned} |S_{M\wedge\tau} - S_{(M\wedge\tau)-1})| &< L|v(M\wedge\tau) - v((M\wedge\tau)-1)| \\ &\leq \frac{\hat{c}L + n\bar{b}|\varepsilon|L}{M\wedge\tau} \\ &\leq \frac{\hat{c}L + n\bar{b}|\varepsilon|L}{t}. \end{aligned} \tag{61}$$

whenever $v(M)$ is in the neighborhood of p. Since the mean of the half-normal distribution is $\frac{\sqrt{2}}{\sqrt{\pi}}$ and its variance is $1 - \frac{2}{\pi}$, we have

$$\begin{aligned} E[|S_{M\wedge\tau} - S_{(M\wedge\tau)-1}||\mathscr{F}_t] &< \frac{\text{const.}}{t}, \\ E[(S_{M\wedge\tau} - S_{(M\wedge\tau)-1})^2|\mathscr{F}_t] &< \frac{\text{const.}}{t^2}. \end{aligned}$$

Then we have

$$\begin{aligned} E[S^2_{M\wedge\tau}|\mathscr{F}_t] &= E[\{S_{(M\wedge\tau)-1} + (S_{M\wedge\tau} - S_{(M\wedge\tau)-1})\}^2|\mathscr{F}_t] \\ &\leq \left(\frac{c^*}{\sqrt{t}}\right)^2 + 2\frac{c^*}{\sqrt{t}}\frac{\text{const.}}{t} + \frac{\text{const.}}{t^2} \end{aligned}$$

where the inequality uses $S_{(M\wedge\tau)-1} \leq \frac{c^*}{\sqrt{t}}$ (which follows from the definition of τ), the Lipschitz-continuity of η, and the previous inequalities. When t is large, the last line is less than $\frac{4(c^*)^2}{t}$, and hence (60) follows.

B.7.2 Proof of Lemma 13

The proof is almost the same as that of Pemantle (1990). However, at some places, his proof uses the assumption that the noise has a bounded support (which ensures that the step size of the process is of order $\frac{1}{t+1}$). In what follows, we will explain how to extend his argument to our setup with Gaussian noise.

Enlarge the domain of η as in the proof of Lemma 12. Pick t large enough, and assume that $S_t \geq \frac{c^*}{\sqrt{t}}$. Let $\tau = \inf\{k \geq t | S_k \leq \frac{c^*}{2\sqrt{t}}\}$. Recall that $X_k = S_k - S_{k-1}$ is a difference sequence. Let

$\mu_k = E[X_k|\mathscr{F}_{k-1}]$. Consider a martingale $\{Z_k\}_{k=t}^{\infty}$ defined as $Z_k = S_t + \sum_{j=t+1}^{k} Y_j$, where $Y_k = 0$ for $\tau > k$ and $Y_k = X_k - \mu_k$ for $\tau \leq k$.

In the seventh to the last line on page 710, Pemantle (1990) argues that if the step size is of order $\frac{1}{k}$, then $\{Z_k\}$ is L^2-bounded (and hence the martingale convergence theorem applies).

In our setup, we can still prove that $\{Z_k\}$ is L^2-bounded. It is well-known that a martingale $\{Z_k\}$ is L^2-bounded if and only if

$$\sum_k E[(Z_k - Z_{k-1})^2] < \infty.$$

We will show that this inequality holds in our model. Using the argument similar to (61), we have

$$|S_k - S_{k-1}| < \frac{\hat{c}L + n\bar{b}|\varepsilon|L}{k}, \tag{62}$$

and hence

$$E[(S_k - S_{k-1})^2|\mathscr{F}_{k-1}] < \frac{\text{const.}}{k^2}, \qquad E[|S_k - S_{k-1}||\mathscr{F}_{k-1}] < \frac{\text{const.}}{k}.$$

These inequalities imply

$$\begin{aligned}
E[(Z_k - Z_{k-1})^2|\mathscr{F}_{k-1}] &\leq E[(X_k - \mu_k)^2|\mathscr{F}_{k-1}] \\
&= E[(S_k - S_{k-1})^2 - 2\mu_k(S_k - S_{k-1}) + \mu_k^2|\mathscr{F}_{k-1}] \\
&= E[(S_k - S_{k-1})^2 - 2E[S_k - S_{k-1}|\mathscr{F}_{k-1}](S_k - S_{k-1}) + (E[S_k - S_{k-1}|\mathscr{F}_{k-1}])^2|\mathscr{F}_{k-1}] \\
&< \frac{\text{const.}}{k^2}.
\end{aligned}$$

Hence we have $E[(Z_k - Z_{k-1})^2] < \frac{\text{const.}}{k^2}$ for every k. Then obviously $\sum_k E[(Z_k - Z_{k-1})^2] < \infty$, as desired.

Also in the last line on page 710, Pemantle (1990) shows that if the step size is of order $\frac{1}{k}$, then

$$\text{Var}\left(\sum_{k=t+1}^{\tau} Y_k\right) \leq \sum_{k=t+1}^{\infty} \frac{\text{const.}}{k^2},$$

In our model, we can still prove the same inequality. It is well-known that the covariance of the martingale difference (Y_i, Y_j) is zero, and hence

$$\text{Var}\left(\sum_{k=t+1}^{\tau} Y_k\right) = \text{Var}\left(\sum_{k=t+1}^{\infty} Y_k\right) = \sum_{k=t+1}^{\infty} \text{Var}(Y_k).$$

Note that $E[(Z_k - Z_{k-1})^2] < \frac{\text{const.}}{k^2}$. Because $\text{Var}(Y_k) = E[(Z_k - Z_{k-1})^2]$, the desired inequality holds.

C Examples

Example of Non-Convergence: Team Production

Suppose that the output function is given by

$$Q(x_1+x_2, a, \theta) = a - \theta\,(x_1+x_2)$$

and player i's payoff is $y + x_i - \frac{1}{2}x_i^2$, where $y = Q + \varepsilon$ and $a \in \mathbb{R}$. Suppose also that $\Theta = [\underline{\theta}, \overline{\theta}]$ where $0 < \underline{\theta} < \overline{\theta} < 1$.

Given misspecified parameters A_1, A_2, the incentive-compatibility conditions are:

$$x_1 = 1 - \theta_1,$$

$$\hat{x}_2 = 1 - \hat{\theta}_2,$$

$$x_2 = 1 - \theta_2,$$

$$\hat{x}_1 = 1 - \hat{\theta}_1.$$

From these equations, we obtain $\frac{\partial x_i(m_i, \hat{m}_{-i})}{\partial m_i} = -1$ and $\frac{\partial x_i(m_i, \hat{m}_{-i})}{\partial \hat{m}_{-i}} = 0$. Note also that $\frac{\partial Q}{\partial \theta} = -(x_1+x_2)$ and $\frac{\partial Q}{\partial x_i} = -\theta$.

At $m_1 = m_2 = \hat{m}_1 = \hat{m}_2 = \theta^*$, we have $x_i = \hat{x}_i = 1 - \theta^*$. Hence, the left hand side of Condition (13) equals

$$-\frac{-\theta^* \cdot (0-1-1+0)}{-2(1-\theta^*)} = \frac{\theta^*}{1-\theta^*}.$$

At $\theta^* = 0.8$, the above value equals 4.

We next consider one-sided misspecification where only player 2 has the above bias. Given the misspecified parameter A_2, the incentive-compatibility conditions at the interior steady state are:

$$x_1 = 1 - \theta^*,$$

$$x_2 = 1 - \theta_2,$$

$$\hat{x}_1 = 1 - \hat{\theta}_1.$$

By the similar derivation as above, the left hand side of Condition (16) equals

$$-\frac{-\theta^* \cdot (-1)}{-2(1-\theta^*)} = \frac{\theta^*}{2(1-\theta^*)}.$$

Note that the above value is larger than 1 if and only if $\theta^* > \frac{2}{3}$.

Example of Non-Convergence: Team Production When Efforts Are Substitutes

Suppose that the output function is given by

$$Q(x_1+x_2,a,\theta)=a-\theta\left(\frac{1}{x_1+x_2}-s\right)$$

where $a,s\in\mathbb{R}$, and player i's payoff is $y-c(x_i)$ where the cost function is $c(x_i)=\frac{1}{8}x_i^2$. Suppose also that $\Theta=[\underline{\theta},\overline{\theta}]$ where $0<\underline{\theta}<\overline{\theta}$. Note that $Q_{x_i},Q_a>0$, $Q_{x_iA}=0$, and $Q_{x_i\theta}>0$. In what follows, we assume $s<\frac{1}{2(\overline{\theta})^{\frac{1}{3}}}$; it implies that Q_θ is negative and bounded away from zero given players' incentive-compatibility conditions.

For example, suppose a is large so that $Q>0$. The output function Q represents the outcome of joint production, where the marginal benefit of effort is decreasing in θ.[28]

Given misspecified parameters A_1,A_2, the incentive-compatibility conditions are:

$$\frac{\theta_1}{(x_1+\hat{x}_2)^2}=\frac{1}{4}x_1,$$
$$\frac{\hat{\theta}_2}{(x_1+\hat{x}_2)^2}=\frac{1}{4}\hat{x}_2,$$
$$\frac{\theta_2}{(\hat{x}_1+x_2)^2}=\frac{1}{4}x_2,$$
$$\frac{\hat{\theta}_1}{(\hat{x}_1+x_2)^2}=\frac{1}{4}\hat{x}_1.$$

These four equations imply $x_1=\frac{2^{\frac{2}{3}}\theta_1}{\left(\theta_1+\hat{\theta}_2\right)^{\frac{2}{3}}}$, $\hat{x}_2=\frac{2^{\frac{2}{3}}\hat{\theta}_2}{\left(\theta_1+\hat{\theta}_2\right)^{\frac{2}{3}}}$, $x_2=\frac{2^{\frac{2}{3}}\theta_2}{\left(\theta_2+\hat{\theta}_1\right)^{\frac{2}{3}}}$, $\hat{x}_1=\frac{2^{\frac{2}{3}}\hat{\theta}_1}{\left(\theta_2+\hat{\theta}_1\right)^{\frac{2}{3}}}$. From this, we obtain $\frac{\partial x_i(m_i,\hat{m}_{-i})}{\partial m_i}=\frac{2^{\frac{2}{3}}}{(m_i+\hat{m}_{-i})^{\frac{2}{3}}}-\frac{2}{3}\frac{2^{\frac{2}{3}}m_i}{(m_i+\hat{m}_{-i})^{\frac{5}{3}}}$ and $\frac{\partial x_i(m_i,\hat{m}_{-i})}{\partial \hat{m}_{-i}}=-\frac{2}{3}\frac{2^{\frac{2}{3}}m_i}{(m_i+\hat{m}_{-i})^{\frac{5}{3}}}$. Note also that $\frac{\partial Q}{\partial\theta}=-\frac{1}{x_1+x_2}+s$ and $\frac{\partial Q}{\partial x_i}=\theta\frac{1}{(x_1+x_2)^2}$.

At $m_1=m_2=\hat{m}_1=\hat{m}_2=\theta^*$, we have $x_i=\hat{x}_i=(\theta^*)^{\frac{1}{3}}$, $\frac{\partial x_i(m_i,\hat{m}_{-i})}{\partial m_i}=\frac{2}{3(\theta^*)^{\frac{2}{3}}}$, and $\frac{\partial x_i(m_i,\hat{m}_{-i})}{\partial \hat{m}_{-i}}=$

[28] Note that $a\leq 0$ (and hence $Q<0$) also fits other economic situations. For example, the output function Q represents the degree of pollution or damage (e.g., CO2 emissions or agricultural drought), where agents can mitigate it by exerting effort (e.g., using eco-friendly products or investing to irrigation) and they have different views on Q.

$-\frac{1}{3(\theta^*)^{\frac{2}{3}}}$. Hence, the left hand side of Condition (13) equals

$$-\frac{\frac{(\theta^*)^{\frac{1}{3}}}{4}\cdot\left[-\frac{1}{3(\theta^*)^{\frac{2}{3}}}+\frac{2}{3(\theta^*)^{\frac{2}{3}}}+\frac{2}{3(\theta^*)^{\frac{2}{3}}}-\frac{1}{3(\theta^*)^{\frac{2}{3}}}\right]}{-\frac{1}{2(\theta^*)^{\frac{1}{3}}}+s}=\frac{\frac{1}{6(\theta^*)^{\frac{1}{3}}}}{\frac{1}{2(\theta^*)^{\frac{1}{3}}}-s}.$$

At $s=\theta^*=0.5$, the above value is about 1.62, thus Condition (13) is satisfied.

Example of Non-Convergence: Games with Conflicting Interests

Suppose that the output function is given by

$$Q(x_1,x_2,a,\theta)=\theta\left(x_2-x_1+a\right),$$

player 2' payoff is $y-\frac{1}{2}x_2^2$, player 1's payoff is $-y-\frac{1}{2k}x_1^2$, and $y=Q+\varepsilon$, where $x_i\geq 0$, $k>0$, and $a\in\mathbb{R}$. Suppose also that $\Theta=[\underline{\theta},\overline{\theta}]$ where $0<\underline{\theta}<\overline{\theta}$.

Given misspecified parameters A_1,A_2, the incentive-compatibility conditions are:

$$\begin{aligned}x_1&=k\theta_1,\\ \hat{x}_2&=\hat{\theta}_2,\\ x_2&=\theta_2,\\ \hat{x}_1&=k\hat{\theta}_1.\end{aligned}$$

From these equations, we obtain $\frac{\partial x_1(m_1,\hat{m}_2)}{\partial m_1}=k$, $\frac{\partial x_2(m_2,\hat{m}_1)}{\partial m_2}=1$, and $\frac{\partial x_i(m_i,\hat{m}_{-i})}{\partial \hat{m}_{-i}}=0$. Note also that $\frac{\partial Q}{\partial x_1}=-\theta$, $\frac{\partial Q}{\partial x_2}=\theta$, and $\frac{\partial Q}{\partial \theta}=(x_2-x_1+a)$.

At $m_1=m_2=\hat{m}_1=\hat{m}_2=\theta^*$, we have $x_1=\hat{x}_1=k\theta^*$ and $x_2=\hat{x}_2=\theta^*$. Hence, the left hand side of Condition (13) equals

$$-\frac{0+\theta^*-k\theta^*+0}{\theta^*-k\theta^*+a}=\frac{(k-1)\theta^*}{a-(k-1)\theta^*}.$$

So Condition (13) is satisfied if and only if $(k-1)\theta^*<a<2(k-1)\theta^*$.

We next consider one-sided misspecification where only player 2 has the above bias. Given the misspecified parameter A_2, the incentive-compatibility conditions at the interior steady state are:

$$x_1 = \hat{x}_2 = k\theta^*,$$
$$x_2 = \theta_2,$$
$$\hat{x}_1 = \hat{\theta}_1.$$

By the similar derivation as above, the left hand side of Condition (16) equals

$$-\frac{\theta^*}{\theta^* - k\theta^* + a} = \frac{\theta^*}{(k-1)\theta^* - a}.$$

Note that the above value is larger than 1 if and only if $(k-2)\theta^* < a < (k-1)\theta^*$.

Example of Non-Convergence: Cournot Duopoly

Suppose that the output function is given by

$$Q(x_1 + x_2, a, \theta) = a - \theta(x_1 + x_2)$$

where $a > 0$. Player i's payoff is $yx_i - c(x_i)$, where yx_i is firm i's revenue and $c(x_i)$ is firm i's production cost. Suppose also that $\Theta = [\underline{\theta}, \overline{\theta}]$ where $0 < \underline{\theta} < \overline{\theta}$.

Given misspecified parameters A_1, A_2, the incentive-compatibility conditions are:

$$A_1 - \theta_1(2x_1 + \hat{x}_2) - c'(x_1) = 0,$$
$$\hat{A}_2 - \hat{\theta}_2(x_1 + 2\hat{x}_2) - c'(\hat{x}_2) = 0,$$
$$A_2 - \theta_2(2x_2 + \hat{x}_1) - c'(x_2) = 0,$$
$$\hat{A}_1 - \hat{\theta}_1(x_2 + 2\hat{x}_1) - c'(\hat{x}_1) = 0.$$

By applying the implicit function theorem to these equations, we have

$$\begin{pmatrix} -2m_1 - c''(x_1) & -m_1 \\ -\hat{m}_2 & -2\hat{m}_2 - c''(\hat{x}_2) \end{pmatrix} \begin{pmatrix} \frac{\partial x_1(m_1,\hat{m}_2)}{\partial m_1} \\ \frac{\partial \hat{x}_2(m_1,\hat{m}_2)}{\partial m_1} \end{pmatrix} = \begin{pmatrix} 2x_1 + \hat{x}_2 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} -2m_1 - c''(x_1) & -m_1 \\ -\hat{m}_2 & -2\hat{m}_2 - c''(\hat{x}_2) \end{pmatrix} \begin{pmatrix} \frac{\partial x_1(m_1,\hat{m}_2)}{\partial \hat{m}_2} \\ \frac{\partial \hat{x}_2(m_1,\hat{m}_2)}{\partial \hat{m}_2} \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 + 2\hat{x}_2 \end{pmatrix}.$$

From this, we obtain $\frac{\partial x_1(m_1,\hat{m}_2)}{\partial m_1} = -\frac{(2x_1+\hat{x}_2)(2\hat{m}_2+c''(\hat{x}_2))}{(2m_1+c''(x_1))(2\hat{m}_2+c''(\hat{x}_2))-m_1\hat{m}_2}$ and $\frac{\partial x_1(m_1,\hat{m}_2)}{\partial \hat{m}_2} = \frac{(x_1+2\hat{x}_2)m_1}{(2m_1+c''(x_1))(2\hat{m}_2+c''(\hat{x}_2))-m_1\hat{m}_2}$ (as well as the above, $\frac{\partial x_2(m_2,\hat{m}_1)}{\partial m_2}$, $\frac{\partial x_2(m_2,\hat{m}_1)}{\partial \hat{m}_1}$ are obtained in the same manner). Note also that $\frac{\partial Q}{\partial \theta} = -(x_1+x_2)$ and $\frac{\partial Q}{\partial x_i} = -\theta$.

At $A_i = a$ and hence $m_1 = m_2 = \hat{m}_1 = \hat{m}_2 = \theta^*$, we have $\frac{\partial x_i(m_i,\hat{m}_{-i})}{\partial m_i} = -\frac{3x_i(2\theta^*+c''(x_i))}{(2\theta^*+c''(x_i))^2-\theta^{*2}}$ and $\frac{\partial x_i(m_i,\hat{m}_{-i})}{\partial \hat{m}_{-i}} = \frac{3x_i\theta^*}{(2\theta^*+c''(x_i))^2-\theta^{*2}}$. Hence, Condition (13) is

$$\theta^* \frac{-2\frac{3x_i(2\theta^*+c''(x_i))}{(2\theta^*+c''(x_i))^2-\theta^{*2}} + 2\frac{3x_i\theta^*}{(2\theta^*+c''(x_i))^2-\theta^{*2}}}{-2x_i} = \theta^* \frac{3(\theta^*+c''(x_i))}{(2\theta^*+c''(x_i))^2-\theta^{*2}} > 1. \tag{63}$$

By the assumptions of the interior and unique Nash equilibrium for any $\theta \in \Theta$, the second-order conditions must hold at the Nash-equilibrium action x_i: $\theta^* + c''(x_i) > 0$ and $(2\theta^* + c''(x_i))^2 - \theta^{*2} > 0$. By rearranging (63) with using these second-order conditions, Condition (13) can be simplified to $c''(x_i) < 0$.

D A Model with Misspecified Type Distribution

Here we will present a model with a continuum of agents where they overestimste how common their opinion is. We will show that the result in the main text remains true even in this setup.

Suppose that there is a unit mass of players. Each period, each player chooses an action and observes a public signal y, which is given by

$$y = Q(\bar{x}, a, \theta) + \varepsilon.$$

Here, $\bar{x}$ is the average of the chosen action in that period, θ is an unknown state, a is a fixed parameter, and ε is a noise which follows the standard normal distribution. Each player's payoff is $u(x,y)$, which depends on her own action x and the realized signal y.

We assume that there are two types of the players. A half of the population is type 1, who (incorrectly) thinks that the true parameter is A_1. The other half of the population is type 2, who thinks that the true parameter is A_2.

Crucially, we assume that each type overestimates how common her own type is. That is, each type i (incorrectly) believes that the share of type i is $0.5+\gamma$, where $\gamma > 0$. We also assume that higher-order beliefs are naive, in that each type i thinks that it is common knowledge that the share of type i is $0.5+\gamma$.

In this model, each type i has inferential naivety and mispredicts the other type's action. To analyze such a situation, it is useful to consider two hypothetical types. Specifically, hypothetical type $-i$ is a player who thinks that (a) the true parameter is A_{-i} and that (b) the fraction of type i is $0.5+\gamma$ while the fraction of hypothetical type $-i$ is $0.5-\gamma$. Intuitively, hypothetical type $-i$ is the type which appears in type i's strategic thinking, i.e., type i thinks that there are only type i and hypothetical type $-i$, and optimizes the behavior.

Let x_i^t and μ_i^t denote type i's action and belief in period t. Likewise, let $\hat{x}_i^t$ and $\hat{\mu}_i^t$ denote hypothetical type i's action and belief. Suppose that players are myopic and Bayesian, so that the posteriors are updated by Bayes' rule and players choose a Nash equilibrium each period, just as in our main model. In this setup, a steady state is defined as $(x_1^*, x_2^*, \hat{x}_1^*, \hat{x}_2^*, \theta_1^*, \theta_2^*, \hat{\theta}_1^*, \hat{\theta}_2^*)$ which satisfies

$$
\begin{aligned}
&x_i^* \in \arg\max_{x_i} E[u(x_i, Q((0.5+\gamma)x_i^* + (0.5-\gamma)\hat{x}_{-i}^*, A_i, \theta_i) + \varepsilon)] \quad \forall i, \\
&\hat{x}_{-i}^* \in \arg\max_{\hat{x}_{-i}} E[u(\hat{x}_{-i}, Q((0.5+\gamma)x_i^* + (0.5-\gamma)\hat{x}_{-i}^*, A_{-i}, \hat{\theta}_{-i}) + \varepsilon)] \quad \forall i, \\
&\theta_i \in \arg\min_{\theta' \in \Theta} |Q((0.5+\gamma)x_i^* + (0.5-\gamma)\hat{x}_{-i}^*, A_i, \theta') - Q(0.5(x_1^* + x_2^*), a, \theta^*)| \quad \forall i, \\
&\hat{\theta}_{-i} \in \arg\min_{\theta' \in \Theta} |Q((0.5+\gamma)x_i^* + (0.5-\gamma)\hat{x}_{-i}^*, A_{-i}, \theta') - Q(0.5(x_1^* + x_2^*), a, \theta^*)| \quad \forall i.
\end{aligned}
$$

In what follows, we assume that the function Q is linear in θ as in the main model. Then each type i's posterior belief is truncated-normal. Formally, given an action profile $x = (x_1, x_2, \hat{x}_1, \hat{x}_2)$, let $\theta_i(x)$ denote a solution to

$$Q((0.5+\gamma)x_i + (0.5-\gamma)\hat{x}_{-i}, A_i, \theta) = Q(0.5(x_1+x_2), a, \theta^*),$$

and let

$$I_i(x_i, \hat{x}_{-i}) = (R((0.5+\gamma)x_i + (0.5-\gamma)\hat{x}_{-i}, A_i))^2.$$

Then type i's posterior belief μ_i^{t+1} is the truncated normal distribution induced by $N(m_i^{t+1}, \frac{1}{t\xi_i^{t+1}})$, where m_i^{t+1} and ξ_i^{t+1} are given by (6) and (7).

Similarly, let $\hat{\theta}_i(x)$ denote a solution to

$$Q((0.5+\gamma)x_{-i} + (0.5-\gamma)\hat{x}_i, A_i, \theta) = Q(0.5(x_1+x_2), a, \theta^*),$$

and let $\hat{I}_i(x_{-i},\hat{x}_i) = I_i(x_{-i},\hat{x}_i)$. Then hypothetical player i's posterior is the truncated normal distribution induced by $N(\hat{m}_i^{t+1}, \frac{1}{t\hat{\xi}_i^{t+1}})$, where $\hat{m}_i^{t+1}$ and $\hat{\xi}_i^{t+1}$ are given by (8) and (9).

Suppose that Assumption 1 holds, so that the action in period $t+1$ is approximated by the limit action $(x_i(m_i^t,\hat{m}_{-i}^t),\hat{x}_{-i}(m_i^t,\hat{m}_{-i}^t))$ for large t.

The following is a counterpart to Proposition 2, which shows that small misspecification has a small impact on the steady state for generic games. The proof is very similar to that of Proposition 2 and can be found at the end of this appendix.

Proposition 8. *Suppose that when $A_1 = A_2 = a$, we have*

$$\frac{\partial Q}{\partial \theta} + 0.5\frac{\partial Q}{\partial \bar{x}}\left(\frac{\partial x_1}{\partial m_1} + \frac{\partial x_1}{\partial \hat{m}_2} + \frac{\partial \hat{x}_2}{\partial m_1} + \frac{\partial \hat{x}_2}{\partial \hat{m}_2}\right) \neq 0 \tag{64}$$

at the steady state belief (i.e., $\theta = m_i = \hat{m}_i = \theta^*$ and $x_i = \hat{x}_i = x_i(\theta^*,\theta^*)$ for each i). Then there is an open neighborhood $U \subset \boldsymbol{R}^4$ of $(m_1,m_2,\hat{m}_1,\hat{m}_2) = (\theta^*,\theta^*,\theta^*,\theta^*)$ such that

(i) When $A_1 = A_2 = a$, the steady state belief in the neighborhood U is unique and it is $(m_1,m_2,\hat{m}_1,\hat{m}_2) = (\theta^*,\theta^*,\theta^*,\theta^*)$.

(ii) There are $\underline{A} < a$ and $\overline{A} > a$ such that for any $\gamma \geq 0$, there is a unique continuous function m^* : $[\underline{A},\overline{A}]^2 \to U$ such that $m^*(a,a) = (\theta^*,\theta^*,\theta^*,\theta^*)$ and such that for each $(A_1,A_2) \in (\underline{A},\overline{A})^2$, $m^*(A_1,A_2)$ is a steady state belief given (A_1,A_2).

Define b' and H' as in Section 3.3, using $\theta_i(x)$ and $\hat{\theta}_i(x)$ specified in this appendix. A perception profile (A_1,A_2) is *regular* if $b' \notin H'$.

The next proposition shows that our main result still holds even with a false-consensus effect: It shows that the limit outcome is discontinuous in (A_1,A_2), and the probability of the beliefs converging to the interior steady state suddenly drops from one to zero when (A_1,A_2) is perturbed from (a,a). It also shows that an arbitrarily small γ (which measures the degree of the false-consensus effect) is enough for this discontinuity result. Again the proof can be found at the end of this appendix.

Proposition 9. *Suppose that the assumption stated in Proposition 8 holds, so that there is a function m^*. Suppose that when $A_1 = A_2 = a$, $\theta_i(x)$ and $\hat{\theta}_i(x)$ defined in this appendix satisfies (11) at* *the steady state with correct learning (i.e., $m_1 = m_2 = \hat{m}_1 = \hat{m}_2 = \theta^*$). Then there are $\underline{A} < a$ and $\overline{A} > a$ such that for any $\gamma > 0$ and for any regular $(A_1, A_2) \in (\underline{A}, \overline{A})$, we have*

$$\Pr(\lim_{t\to\infty}(\mu_1^t, \mu_2^t) = (1_{m_1^*(A_1,A_2)}, 1_{m_2^*(A_1,A_2)})) = 0.$$

D.1 Proof of Proposition 8

Then the interior steady state belief $(m_1, m_2, \hat{m}_1, \hat{m}_2)$ solves the system of equations

$$Q(0.5(x_1(m_1,\hat{m}_2)+x_2(\hat{m}_1,m_2)),\theta^*,a) - Q((0.5+\gamma)x_1(m_1,\hat{m}_2)+(0.5-\gamma)\hat{x}_2(m_1,\hat{m}_2),m_1,a) = 0,$$
$$Q(0.5(x_1(m_1,\hat{m}_2)+x_2(\hat{m}_1,m_2)),\theta^*,a) - Q((0.5+\gamma)x_1(m_1,\hat{m}_2)+(0.5-\gamma)\hat{x}_2(m_1,\hat{m}_2),\hat{m}_2,a) = 0,$$
$$Q(0.5(x_1(m_1,\hat{m}_2)+x_2(\hat{m}_1,m_2)),\theta^*,a) - Q((0.5-\gamma)\hat{x}_1(\hat{m}_1,m_2)+(0.5+\gamma)x_2(\hat{m}_1,m_2),m_2,a) = 0,$$
$$Q(0.5(x_1(m_1,\hat{m}_2)+x_2(\hat{m}_1,m_2)),\theta^*,a) - Q((0.5-\gamma)\hat{x}_1(\hat{m}_1,m_2)+(0.5+\gamma)x_2(\hat{m}_1,m_2),\hat{m}_1,a) = 0.$$

At the steady state belief $m_i = \hat{m}_i = \theta^*$, the Jacobian of the above system is

$$\begin{pmatrix} -E_1 - \frac{\partial Q}{\partial \theta} & -E_2 & 0.5\frac{\partial Q}{\partial \bar{x}}\frac{\partial x_2}{\partial m_2} & 0.5\frac{\partial Q}{\partial \bar{x}}\frac{\partial x_2}{\partial \hat{m}_1} \\ -E_1 & -E_2 - \frac{\partial Q}{\partial \theta} & 0.5\frac{\partial Q}{\partial \bar{x}}\frac{\partial x_2}{\partial m_2} & 0.5\frac{\partial Q}{\partial \bar{x}}\frac{\partial x_2}{\partial \hat{m}_1} \\ 0.5\frac{\partial Q}{\partial \bar{x}}\frac{\partial x_1}{\partial m_1} & 0.5\frac{\partial Q}{\partial \bar{x}}\frac{\partial x_1}{\partial \hat{m}_2} & -E_3 - \frac{\partial Q}{\partial \theta} & -E_4 \\ 0.5\frac{\partial Q}{\partial \bar{x}}\frac{\partial x_1}{\partial m_1} & 0.5\frac{\partial Q}{\partial \bar{x}}\frac{\partial x_1}{\partial \hat{m}_2} & -E_3 & -E_4 - \frac{\partial Q}{\partial \theta} \end{pmatrix}$$

where

$$E_1 = \frac{\partial Q}{\partial \bar{x}}\left\{\gamma\frac{\partial x_1}{\partial m_1} + (0.5-\gamma)\frac{\partial \hat{x}_2}{\partial m_1}\right\},$$
$$E_2 = \frac{\partial Q}{\partial \bar{x}}\left\{\gamma\frac{\partial x_1}{\partial \hat{m}_2} + (0.5-\gamma)\frac{\partial \hat{x}_2}{\partial \hat{m}_2}\right\},$$
$$E_3 = \frac{\partial Q}{\partial \bar{x}}\left\{\gamma\frac{\partial x_2}{\partial m_2} + (0.5-\gamma)\frac{\partial \hat{x}_1}{\partial m_2}\right\},$$
$$E_4 = \frac{\partial Q}{\partial \bar{x}}\left\{\gamma\frac{\partial x_2}{\partial \hat{m}_1} + (0.5-\gamma)\frac{\partial \hat{x}_1}{\partial \hat{m}_1}\right\}.$$

This matrix is regular, as its determinant is

$$
\begin{aligned}
D &= \begin{vmatrix}
-\frac{\partial Q}{\partial \theta} & \frac{\partial Q}{\partial \theta} & 0 & 0 \\
-E_1 & -E_2-\frac{\partial Q}{\partial \theta} & 0.5\frac{\partial Q}{\partial \bar{x}}\frac{\partial x_2}{\partial m_2} & 0.5\frac{\partial Q}{\partial \bar{x}}\frac{\partial x_2}{\partial \hat{m}_1} \\
0 & 0 & -\frac{\partial Q}{\partial \theta} & \frac{\partial Q}{\partial \theta} \\
0.5\frac{\partial Q}{\partial \bar{x}}\frac{\partial x_1}{\partial m_1} & 0.5\frac{\partial Q}{\partial \bar{x}}\frac{\partial x_1}{\partial \hat{m}_2} & -E_3 & -E_4-\frac{\partial Q}{\partial \theta}
\end{vmatrix} \\
&= \begin{vmatrix}
-\frac{\partial Q}{\partial \theta} & 0 & 0 & 0 \\
-E_1 & -E_1-E_2-\frac{\partial Q}{\partial \theta} & 0.5\frac{\partial Q}{\partial \bar{x}}\frac{\partial x_2}{\partial m_2} & 0.5\frac{\partial Q}{\partial \bar{x}}\left(\frac{\partial x_2}{\partial m_2}+\frac{\partial x_2}{\partial \hat{m}_1}\right) \\
0 & 0 & -\frac{\partial Q}{\partial \theta} & 0 \\
0.5\frac{\partial Q}{\partial \bar{x}}\frac{\partial x_1}{\partial m_1} & 0.5\frac{\partial Q}{\partial \bar{x}}\left(\frac{\partial x_1}{\partial m_1}+\frac{\partial x_1}{\partial \hat{m}_2}\right) & -E_3 & -E_3-E_4-\frac{\partial Q}{\partial \theta}
\end{vmatrix} \\
&= \left(\frac{\partial Q}{\partial \theta}\right)^2 \begin{vmatrix}
-E_1-E_2-\frac{\partial Q}{\partial \theta} & 0.5\frac{\partial Q}{\partial \bar{x}}\left(\frac{\partial x_2}{\partial m_2}+\frac{\partial x_2}{\partial \hat{m}_1}\right) \\
0.5\frac{\partial Q}{\partial \bar{x}}\left(\frac{\partial x_1}{\partial m_1}+\frac{\partial x_1}{\partial \hat{m}_2}\right) & -E_3-E_4-\frac{\partial Q}{\partial \theta}
\end{vmatrix} \\
&= \left(\frac{\partial Q}{\partial \theta}\right)^2 \begin{vmatrix}
-X-\frac{\partial Q}{\partial \theta} & X \\
X & -X-\frac{\partial Q}{\partial \theta}
\end{vmatrix} \\
&= \left(\frac{\partial Q}{\partial \theta}\right)^2 \left\{ \left(\frac{\partial Q}{\partial \theta}\right)^2 + 2X\frac{\partial Q}{\partial \theta} \right\} \\
&= \left(\frac{\partial Q}{\partial \theta}\right)^3 \left(\frac{\partial Q}{\partial \theta} + 2X\right) \neq 0.
\end{aligned}
$$

Here X appearing in the third to the last line is defined as $X = 0.5\frac{\partial Q}{\partial \bar{x}}(\frac{\partial x_2}{\partial m_2}+\frac{\partial x_2}{\partial \hat{m}_1})$, and to obtain this equality, we use the fact that $X = 0.5\frac{\partial Q}{\partial \bar{x}}(\frac{\partial x_1}{\partial m_1}+\frac{\partial x_1}{\partial \hat{m}_2}) = 0.5\frac{\partial Q}{\partial \bar{x}}(\frac{\partial \hat{x}_2}{\partial m_1}+\frac{\partial \hat{x}_2}{\partial \hat{m}_2}) = 0.5\frac{\partial Q}{\partial \bar{x}}(\frac{\partial \hat{x}_1}{\partial m_2}+\frac{\partial \hat{x}_1}{\partial \hat{m}_1})$. The last inequality follows from (64) and $\frac{\partial Q}{\partial \theta} = R(x_1(\theta^*), x_2(\theta^*), a) \neq 0$.

The rest of the proof is exactly the same as that of Proposition 2, and hence omitted.

D.2 Proof of Proposition 9

In the proof of Proposition 3, we have seen that the evolution of $(m^t, \xi^t,)$ is described by the difference equations (32) through (35), and its asymptotic motion is approximated by the ODE (40) through (43). It is straightforward to see that the same result holds even in the model of the false-consensus effect, provided that θ_i, I_i, $\hat{\theta}_i$, and $\hat{I}_i$ appearing these equations are replaced with the ones defined in this appendix, and that $R(x_i, \hat{x}_{-i}, A_i)$ and $R(x_i, \hat{x}_{-i}, \hat{A}_{-i})$ appearing in (32) and

(33) are replaced with with $R((0.5+\gamma)x_i+(0.5-\gamma)\hat{x}_{-i},A_i)$ and $R((0.5+\gamma)x_i+(0.5-\gamma)\hat{x}_{-i},A_{-i})$, respectively.

Also, Lemma 6 remains true in the model of the self-consensus effect. Indeed, just as in the proof of the original lemma, we can show that the Jacobian of the ODE has eigenvalues $\lambda = -1$ (multiplicity 4) and $\lambda = -\xi^*$ (multiplicity 2), and that the remaining eigenvalues solve the quadratic equation (45). Now, since $\hat{\theta}_{-i}$ solves

$$Q(0.5(x_i(m_i,\hat{m}_{-i})+x_{-i}(m_{-i},\hat{m}_i)),a,\theta^*)-Q((0.5+\gamma)x_i(m_i,\hat{m}_{-i})+(0.5-\gamma)\hat{x}_{-i}(m_i,\hat{m}_{-i}),a,\theta)=0,$$

by the implicit function theorem, we have

$$\begin{aligned}
&\frac{\partial\hat{\theta}_{-i}}{\partial m_{-i}}+\frac{\partial\hat{\theta}_{-i}}{\partial\hat{m}_i}=0.5X,\\
&\frac{\partial\hat{\theta}_{-i}}{\partial m_i}+\frac{\partial\hat{\theta}_{-i}}{\partial\hat{m}_{-i}}=-\gamma X-(0.5-\gamma)X=-0.5X
\end{aligned}$$

where

$$X=\frac{\frac{\partial Q}{\partial\bar{x}}\left(\frac{\partial x_1}{\partial m_1}+\frac{\partial x_1}{\partial\hat{m}_2}\right)}{\frac{\partial Q}{\partial\theta}}=\frac{\frac{\partial Q}{\partial\bar{x}}\left(\frac{\partial x_2}{\partial m_2}+\frac{\partial x_2}{\partial\hat{m}_1}\right)}{\frac{\partial Q}{\partial\theta}}=\frac{\frac{\partial Q}{\partial\bar{x}}\left(\frac{\partial\hat{x}_1}{\partial m_2}+\frac{\partial\hat{x}_1}{\partial\hat{m}_1}\right)}{\frac{\partial Q}{\partial\theta}}=\frac{\frac{\partial Q}{\partial\bar{x}}\left(\frac{\partial\hat{x}_2}{\partial m_1}+\frac{\partial\hat{x}_2}{\partial\hat{m}_2}\right)}{\frac{\partial Q}{\partial\theta}}.$$

Hence we have

$$\left(\frac{\partial\hat{\theta}_1}{\partial m_1}+\frac{\partial\hat{\theta}_1}{\partial\hat{m}_2}\right)\left(\frac{\partial\hat{\theta}_2}{\partial m_2}+\frac{\partial\hat{\theta}_2}{\partial\hat{m}_1}\right)=\left(\frac{\partial\hat{\theta}_1}{\partial m_2}+\frac{\partial\hat{\theta}_1}{\partial\hat{m}_1}\right)\left(\frac{\partial\hat{\theta}_2}{\partial m_1}+\frac{\partial\hat{\theta}_2}{\partial\hat{m}_2}\right).$$

Using this, (45) reduces to (47). Then the argument similar to the one in the proof of the original Lemma 6 shows that the same result holds even in this case.

The rest of the proof is essentially the same as that of the proof of Proposition 3; we only need to replace $R(x_i,\hat{x}_{-i},A_i)$ and $R(x_i,\hat{x}_{-i},\hat{A}_{-i})$ with $R((0.5+\gamma)x_i+(0.5-\gamma)\hat{x}_{-i},A_i)$ and $R((0.5+\gamma)x_i+(0.5-\gamma)\hat{x}_{-i},A_{-i})$, respectively.