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CONSTRAINED EFFICIENCY AND STRATEGY-PROOFNESS IN PACKAGE ASSIGNMENT PROBLEMS WITH MONEY

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Constrained efficiency and strategy-proofness in package assignment problems with money*

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Abstract

We consider a package assignment problem with money, in which a finite set M of objects is allocated to agents. Each agent receives a package of objects and makes a payment, and has preferences over pairs consisting of a package and a payment. These preferences are not necessarily quasi-linear. The *admissible* set of object allocations is chosen by the planner to pursue specific objectives in conjunction with the rule. A rule satisfies *constrained efficiency* if no allocation—whose object allocation is admissible under the rule—Pareto dominates the allocation selected by the rule. We study the compatibility between constraints on admissible object allocations and desirable properties of rules, and characterize the rules that satisfy both. We establish that: *A rule satisfies constrained efficiency, no wastage, equal treatment of equals, strategy-proofness, individual rationality, and no subsidy if and only if its admissible set of object allocations is bundling unit-demand for some partition of M, satisfies no wastage and anonymity, and the rule is a bundling unit-demand minimum price Walrasian rule.*

JEL Classification Numbers. D44, D47, D71, D82

Keywords. Constrained Efficiency, Strategy-proofness, Equal treatment of equals, Non-quasi-linear preferences, Bundling unit-demand minimum price Warlasian rule, Package auctions

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1 Introduction

1.1 Constraints in auctions

Since the 1990s, governments in many countries have used auctions to allocate frequency licenses to cellphone carriers. These auctions not only generate substantial revenue but also have significant economic implications. In such auctions, it is common to impose constraints on license allocations. For example, to prevent monopolistic power or to ensure broader participation among carriers, a single carrier is often limited in the number of licenses it can acquire.[1] In some cases, certain licenses are set aside for new entrants to promote their participation in the cellphone market.[2]

Because electromagnetic frequencies are physically continuous, the allocation of frequency licenses is inherently flexible. However, licenses are bundled into specific frequency bands before the auction, and each carrier is allowed to obtain only one or a few such bundled bands. This bundling introduces a constraint on the allocation. Similar constraints frequently arise in other public auctions—for example, those involving land, housing, or other public assets distributed to citizens.

1.2 Compatibility between constraints and desirable properties

Although constraints are introduced to promote desirable allocations, their effects are not necessarily compatible with fundamental properties of rules—such as efficiency, fairness, and incentive compatibility. For example, by limiting the flexibility of allocations, constraints may compromise efficiency. Therefore, it is essential to examine how such constraints interact with these properties. This paper investigates the compatibility between constraints on object allocations and desirable rule properties, and characterizes the rules that satisfy both.

1.3 Main results

1.3.1 Model description

We consider a model with a set N of agents and a set M of objects, where objects are allocated to agents. Each agent receives a package of objects and makes a payment for it. Agents have preferences over pairs consisting of a package and a payment. These preferences are not necessarily quasi-linear and may reflect income effects and financial constraints, which

[1] For example, frequency license auctions in the USA (2020), UK (2018, 2021), France (2020), Italy (2018), Australia (2021), Korea (2018), and Spain (2018, 2022), among others.

[2] For example, frequency license auctions in Canada (2019, 2021), Korea (2024), and Belgium (2022), among others.

are particularly relevant in large-scale auctions, such as frequency license auctions and other public resource allocations.

A (*feasible*) *object allocation* specifies how objects are assigned to agents. An *allocation* includes both an object allocation and the agents' payments. An *allocation rule*, or a *rule* for short, is a function from a set of preference profiles to the set of admissible allocations.

We distinguish between *feasibility* and *admissibility* of object allocations. The *feasible set* is determined by technological constraints outside the planner's control, whereas the *admissible set* is a subset of the feasible set, chosen by the planner to achieve specific objectives in conjunction with the rule. Importantly, the admissible set is not exogenous but is defined as part of the rule itself: an *admissible object allocation* is one that can arise for some preference profile under the rule. In other words, the admissible set corresponds to the range of object allocations under the rule.

1.3.2 Desirable properties of rules

We assume that the planner is concerned only with her total revenue from an allocation. An allocation *Pareto dominates* another if each agent and the planner weakly prefer the former to the latter, and at least one agent or the planner strictly prefers it. A rule is *Pareto efficient* if it always selects an allocation that is not dominated by any other allocation whose object allocation is feasible. It is *constrained efficient* if it always selects an allocation that is not dominated by any other allocation whose object allocation is admissible under the rule. Since the admissible set of object allocations is a subset of the feasible set, *constrained efficiency* is generally weaker than *Pareto efficiency*, unless the two sets coincide. Moreover, the smaller the admissible set, the weaker the requirement of *constrained efficiency*. Thus, there is a trade-off: the stronger the requirement of *constrained efficiency*, the more diverse the set of admissible object allocations—though satisfying the stronger requirement becomes more demanding. A rule satisfies *no wastage* if all objects are always allocated to agents. Note that *no wastage* is also an efficiency requirement, though weaker than *Pareto efficiency*.

Strategy-proofness is a dominant strategy incentive compatibility condition, requiring that no agent ever benefit from misrepresenting his preferences. *Individual rationality* is a voluntary participation condition, requiring that each agent find his assigned pair (a package of objects and a payment) at least as desirable as receiving no object and paying nothing. *No subsidy* is a condition that prevents disinterested agents from participating in the rule solely to obtain subsidies, requiring that each agent's payment always be non-negative. *Equal treatment of equals* is a fundamental fairness condition, requiring that whenever two agents have identical preferences, they receive the same level of welfare.

1.3.3 Constraints compatible with desirable properties

We first investigate which admissible sets of object allocations are compatible with the desirable properties of *constrained efficiency*, *no wastage*, *equal treatment of equals*, *strategy-proofness*, *individual rationality*, and *no subsidy*.

A *constraint* is a subset of the set of feasible object allocations. Note that the admissible set of object allocations under a rule is itself a constraint. A constraint satisfies *anonymity* if for each object allocation in the constraint, any permutation of agents also yields an object allocation that belongs to the constraint. A constraint satisfies *no wastage* if, in every object allocation that belongs to the constraint, all objects are assigned to agents.

Given a partition $\mathcal{B}$ of M, a constraint is called $\mathcal{B}$-*bundling* if, in each object allocation in the constraint, each agent is allowed to receive a subset of $\mathcal{B}$ or nothing. A $\mathcal{B}$-bundling constraint is said to be of *unit-demand* if, in each object allocation in the constraint, each agent is allowed to receive exactly one element of $\mathcal{B}$ or nothing. Thus, under a $\mathcal{B}$-bundling unit-demand constraint, each agent may receive one package from $\mathcal{B}$ or nothing. If each package in $\mathcal{B}$ is interpreted as a single "object," then the $\mathcal{B}$-bundling unit-demand constraint becomes essentially equivalent to the *unit-demand model*, in which each agent is allowed to receive at most one object.

We establish that: *if there exists a rule satisfying constrained efficiency, no wastage, equal treatment of equals, strategy-proofness, individual rationality, and no subsidy, then the set of admissible object allocations must be $\mathcal{B}$-bundling unit-demand for some partition $\mathcal{B}$ of M with at most $|N|$ elements, and must satisfy no wastage and anonymity* (Proposition).

Proposition thus shows that compatibility with these desirable properties imposes strong restrictions on the admissible set: it must be much smaller than the feasible set of object allocations, thereby necessarily sacrificing *Pareto efficiency*.

1.4 Rules satisfying desirable properties

Next, we investigate rules that satisfy the desirable properties—namely, *constrained efficiency*, *no wastage*, *equal treatment of equals*, *strategy-proofness*, *individual rationality*, and *no subsidy*.

As noted earlier, when the admissible set of object allocations is $\mathcal{B}$-bundling unit-demand and satisfies both no wastage and anonymity—as in the conclusion of Proposition—the mathematical structure of our model becomes equivalent to the unit-demand model à la Demange and Gale (1985), in which each agent may receive at most one object. In that setting, the minimum price Walrasian (MPW) rules not only exist but are also the only rules that satisfy *Pareto efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy* (Demange and

Gale, 1985; Morimoto and Serizawa, 2015; Wakabayashi et al., 2025). For any $\mathcal{B}$-bundling unit-demand constraint satisfying no wastage and anonymity, we define the counterpart of an MPW rule in our model, referred to as a *bundling minimum price Walrasian* (*MPW*) *rule*.

By applying the above characterization results in the unit-demand model (Demange and Gale, 1985; Morimoto and Serizawa, 2015; Wakabayashi et al., 2025), we establish the following: *A rule satisfies constrained efficiency, no wastage, equal treatment of equals, strategy-proofness, individual rationality, and no subsidy if and only if the set of admissible object allocations is $\mathcal{B}$-bundling unit-demand for some partition $\mathcal{B}$ of M with at most $|N|$ elements, satisfies no wastage and anonymity, and the rule is a $\mathcal{B}$-bundling unit-demand MPW rule.* (Theorem)

1.5 Implications of results

Our results (Proposition and Theorem) suggest that, unless the set of admissible object allocations is carefully selected, it is impossible to design a rule that satisfies the desirable properties. Moreover, they have a practical implication: if the planner aims to satisfy the desirable properties—*constrained efficiency*, *no wastage*, *equal treatment of equals*, *strategy-proofness*, *individual rationality*, and *no subsidy*—then she must adopt a bundling unit-demand constraint satisfying *no wastage* and *anonymity*, and must employ the associated bundling unit-demand MPW rule.

It is worth noting that the constraints used in public auctions in many countries can be classified as bundling unit-demand constraints that satisfy both no wastage and anonymity.[3] While a primary motivation for adopting such constraints has been to reduce the risk of collusion among bidders (Binmore and Klemperer, 2002), our results provide a novel rationale for this auction design: bundling unit-demand constraints are the only ones that permit the existence of a rule satisfying *(constrained) efficiency*, *fairness*, and *strategy-proofness*.

1.6 Related literature

1.6.1 Quasi-linear preferences

The literature on object allocation problems with money is extensive. A common assumption in this literature is that agents have quasi-linear preferences. This assumption is particularly

[3]For example, the European 3G frequency license auctions in the U.K., the Netherlands, Italy, and Denmark, as well as recent auctions in Korea (2024), Hong Kong (2019), Finland (2018, 2020), and Poland (2023). Additionally, auctions allocating public housing to citizens inherently follow bundling unit-demand constraints.

useful because it renders the problem of efficient object allocation equivalent to simply maximizing the sum of valuations.

One of the most celebrated results in the literature is that if a class of preferences consists solely of quasi-linear preferences and is sufficiently rich, then the Vickrey rules (Vickrey, 1961) are the only rules that satisfy *Pareto efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy* (see, e.g., Holmström, 1979; Chew and Serizawa, 2007). Notably, Holmström's (1979) characterization continues to hold when *Pareto efficiency* is replaced by *constrained efficiency* and the Vickrey rules are replaced by the constrained Vickrey rules—that is, the constrained Vickrey rules are the only rules that satisfy *constrained efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy*.[4] This paper contributes to the literature by extending Holmström's result to settings with non-quasi-linear preferences, while also incorporating two additional but relatively weak requirements: a fairness condition—*equal treatment of equals*—and an efficiency condition—*no wastage*.

1.6.2 Non-quasi-linear preferences

Although the assumption of quasi-linear preferences is analytically convenient, it limits the applicability of results to situations where payments are small relative to agents' incomes or budgets, such that income effects and budget constraints can be ignored. However, in many important applications of object allocation problems with money—such as frequency license auctions—payments are typically large, making income effects and budget constraints non-negligible. Motivated by this limitation, a small but growing literature has begun to examine object allocation problems with money under non-quasi-linear preferences.

Some studies in this literature assume that agents have unit-demand preferences. As discussed in Section 1.4, when agents have unit-demand and non-quasi-linear preferences, the MPW rules are the only rules that satisfy *Pareto efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy* (see, e.g., Demange and Gale, 1985; Morimoto and Serizawa, 2015; Wakabayashi et al., 2025).

In contrast, other studies assume that agents have multi-demand preferences, as in this paper. A series of results has shown that when agents have multi-demand and non-quasi-linear preferences, no rule satisfies *Pareto efficiency*, *fairness*, and *strategy-proofness* simultaneously (see, e.g., Kazumura and Serizawa, 2016; Baisa, 2020; Malik and Mishra, 2021; Kazumura, 2022; Shinozaki et al., 2025). These impossibility results imply that at least one of these three properties—*Pareto efficiency*, *fairness*, or *strategy-proofness*—must be relaxed

[4]A *constrained Vickrey rule* modifies the standard Vickrey rule by restricting attention to admissible object allocations when determining the object allocation and the payments. See Definition 5 in Section 3.4.2 for the formal definition.

or abandoned in this setting. This paper contributes to the literature by relaxing *Pareto efficiency* to *constrained efficiency* and characterizing rules that satisfy the remaining desirable properties. Moreover, to the best of our knowledge, this is the first paper to explicitly study *constrained efficiency* and *strategy-proofness* in package assignment problems with money under non-quasi-linear preferences.

The aforementioned studies assume that the set of admissible object allocations is exogenously given, whereas our framework allows the planner to endogenously determine this set as part of the rule design. In this respect, the models considered in those studies can be seen as special cases of our more general framework.[5][6]

1.6.3 Walrasian equilibrium allocation under constraints

Recently, several papers have investigated the existence and structure of (standard, non-bundling) Walrasian equilibrium allocations in package assignment problems with money under (non-)quasi-linear preferences (see, e.g., Fleiner et al., 2019; Kojima et al., 2020; Schlegel, 2022; Baldwin et al., 2023; Nguyen and Vohra, 2024). While Walrasian rules —that is, rules that select a Walrasian equilibrium allocation for each preference profile —satisfy *Pareto efficiency* (or *constrained efficiency* in constrained settings) and *fairness* (under anonymous constraints), they generally fail to satisfy *strategy-proofness*. This paper complements these studies by focusing on rules that satisfy *strategy-proofness*.

Bando et al. (2025) analyze a two-sided many-to-one matching model with money, under individual constraints and quasi-linear preferences. They show that the gross substitutes condition (Kelso and Crawford, 1982) on the preferences of a multi-demand agent (i.e., a firm) is both necessary and sufficient for the existence of a *strategy-proof* Walrasian rule on the unit-demand side of the market (i.e., the workers). Since their study concerns a two-sided matching model with money and assumes quasi-linear preferences, their result is logically independent of ours.

1.7 Structure of the paper

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 presents the main results. Section 4 concludes. All formal proofs are provided in the Appendix.

[5] Two notable exceptions are Baisa (2020) and Shinozaki et al. (2025), which examine models with identical objects. Since the package assignment model studied in this paper accommodates heterogeneous objects, their models do not constitute special cases of the present framework.

[6] To be precise, our results do not imply theirs (and vice versa), as the previous studies do not explicitly assume fairness properties such as *equal treatment of equals*.

2 Model

There are $n \geq 2$ agents and $m \geq 1$ objects. The set of agents is denoted by $N \equiv \{1, \ldots, n\}$. Our generic notations for agents are i, j, k, etc. The set of objects is denoted by M with $|M| = m$. Our generic notations for objects are a, b, c, etc. Let $\mathcal{M} \equiv 2^M$.[7] A subset of M, i.e., an element of $\mathcal{M}$, is referred to as a **package**. Each agent $i \in N$ receives a package $A_i \in \mathcal{M}$ and pays $t_i \in \mathbb{R}$. A (consumption) pair consisting of a package and a payment of agent i is denoted by $z_i \equiv (A_i, t_i) \in \mathcal{M} \times \mathbb{R}$. Let $\mathbf{0} \equiv (\emptyset, 0) \in \mathcal{M} \times \mathbb{R}$ denote the pair where an agent $i \in N$ receives no object and makes no payment.

2.1 Preferences

Each agent $i \in N$ has a complete and transitive preference R_i over $\mathcal{M} \times \mathbb{R}$. The strict and indifference relations associated with R_i are denoted by P_i and I_i, respectively. We assume that each preference R_i satisfies the following properties.

Money monotonicity. For each $A_i \in \mathcal{M}$ and each t_i,$t'_i \in \mathbb{R}$ with $t_i < t'_i$, we have $(A_i, t_i)\ P_i\ (A_i, t'_i)$.

Object monotonicity. For each $A_i, A'_i \in \mathcal{M}$ with $A'_i \subsetneq A_i$ and each $t_i \in \mathbb{R}$, we have $(A_i, t_i)\ P_i\ (A'_i, t_i)$.

Possibility of compensation. For each $z_i \in \mathcal{M} \times \mathbb{R}$ and each $A_i \in \mathcal{M}$, there exist two payments $t_i, t'_i \in \mathbb{R}$ such that $(A_i, t_i)\ R_i\ z_i$ and $z_i\ R_i\ (A_i, t'_i)$.

Continuity. For each $z_i \in \mathcal{M} \times \mathbb{R}$, the upper contour set at z_i, $\{z'_i \in \mathcal{M} \times \mathbb{R} : z'_i\ R_i\ z_i\}$, and the lower contour set at z_i, $\{z'_i \in \mathcal{M} \times \mathbb{R} : z_i\ R_i\ z'_i\}$, are both closed.

Our generic notation for a class of preferences satisfying the above four properties is $\mathcal{R}$, which we refer to as a **domain**.[8] Let $\overline{\mathcal{R}}$ denote the class of all preferences satisfying the above four properties.

Given $R_i \in \mathcal{R}$, $A_i \in \mathcal{M}$, and $z_i \in \mathcal{M} \times \mathbb{R}$, possibility of compensation and continuity together imply that there exists a payment $V(A_i, z_i; R_i) \in \mathbb{R}$ such that $\big(A_i, V(A_i, z_i; R_i)\big)\ I_i\ z_i$.[9] By money monotonicity, such a payment $V(A_i, z_i; R_i)$ is unique. We call the payment

[7] Given a set G, 2^G denotes the power set of G, i.e., $2^G = \{G' : G' \subseteq G\}$.

[8] Note that, due to object monotonicity, any domain we consider in this paper does not include any unit-demand preference, where a preference R_i is said to exhibit *unit demand* if for each $A_i \in \mathcal{M}\backslash\{\emptyset\}$ and each $t_i \in \mathbb{R}$, there exists $a \in A_i$ such that $\big(\{a\}, t_i\big)\ R_i\ (A_i, t_i)$.

[9] For a formal proof of the existence of such a payment, see Lemma 1 of Kazumura and Serizawa (2016).

$V(A_i, z_i; R_i)$ the **valuation of A_i at z_i for R_i**. It represents the amount of payment that makes receiving package A_i together with the payment is indifferent to z_i according to preference R_i.

Given $R_i \in \mathcal{R}$, $A_i \in \mathcal{M}$, and $t_i \in \mathbb{R}$, let $w(A_i, t_i; R_i) \equiv V\big(A_i, (\emptyset, t_i); R_i\big) - t_i$. We call $w(A_i, t_i; R_i)$ the **willingness to pay of A_i at t_i for R_i**. It represents the maximal amount of money that an agent is willing to pay for the package A_i when he currently owns no object and has made a payment of t_i. By object monotonicity, for each $A_i, A'_i \in \mathcal{M}$ with $A'_i \subsetneq A_i$ and each $t_i \in \mathbb{R}$, it holds that $w(A_i, t_i; R) > w(A'_i, t_i; R_i)$.

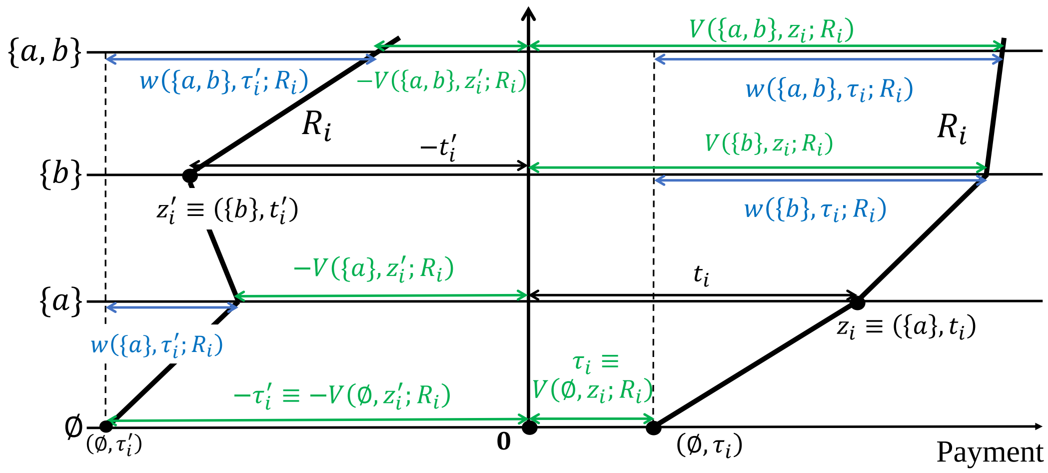

Figure 1: An illustration of a preference.

Figure 1 provides an illustration of a preference R_i. Each horizontal line represents a package, and each point on a line indicates a payment level. Thus, each point corresponds to a pair consisting of a package and a payment. The vertical line represents the set of pairs with zero payment. The solid lines represent the indifference curves associated with the preference R_i. Note that the valuations and willingness to pay for R_i are also depicted in Figure 1.

We introduce two special classes of preferences that are of particular importance.

First, the following class consists of preferences without income effects, which has been studied extensively in the literature (e.g., Holmström, 1979).

Definition 1. A preference $R_i \in \mathcal{R}$ is **quasi-linear** if for each $(A_i, t_i), (A'_i, t'_i) \in \mathcal{M} \times \mathbb{R}$ and each $\delta \in \mathbb{R}$, $(A_i, t_i)\ I_i\ (A'_i, t'_i)$ implies $(A_i, t_i + \delta)\ I_i\ (A'_i, t'_i + \delta)$.

Let $\mathcal{R}^Q$ denote the class of all quasi-linear preferences.

Figure 2 illustrates a quasi-linear preference. As shown in the figure, under a quasi-linear preference, the willingness to pay of each package is independent of the payment. Thus, for each $A_i \in \mathcal{M}$ and each $t_i, t'_i \in \mathbb{R}$, we have $w(A_i, t_i; R_i) = w(A_i, t'_i; R_i) \equiv w(A_i; R_i)$. Moreover, if a preference R_i is quasi-linear, it can be represented by a utility function $u_i : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$

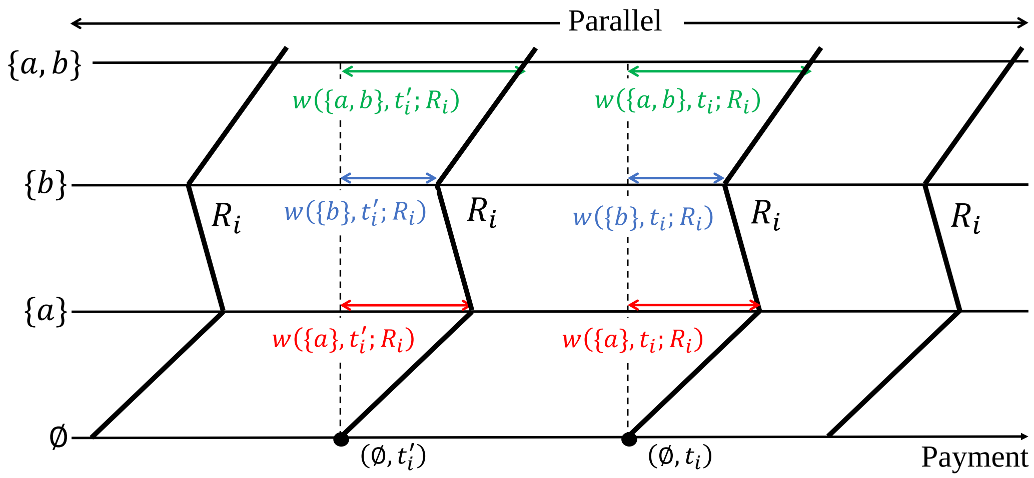

Figure 2: An illustration of a quasi-linear preference.

such that for each $z_i \equiv (A_i, t_i) \in \mathcal{M} \times \mathbb{R}$, $u_i(z_i) = w(A_i; R_i) - t_i$. Thus, our concept of willingness to pay aligns with the quasi-linear valuation under quasi-linear preferences.

Second, the following class consists of preferences with additive willingness to pay over packages.

Definition 2. A preference R_i is **additive** if for each $A_i, A_i' \in \mathcal{M}$ with $A_i \cap A_i' = \emptyset$ and each $t_i \in \mathbb{R}$, we have

$$w(A_i \cup A_i', t_i; R_i) = w(A_i, t_i; R_i) + w(A_i', t_i; R_i).$$

Let $\mathcal{R}^{Add}$ denote the class of all additive preferences. Note that $\mathcal{R}^{Add}$ includes preferences that are additive but not necessarily quasi-linear, i.e., $\mathcal{R}^{Add} \not\subseteq \mathcal{R}^{Q}$.

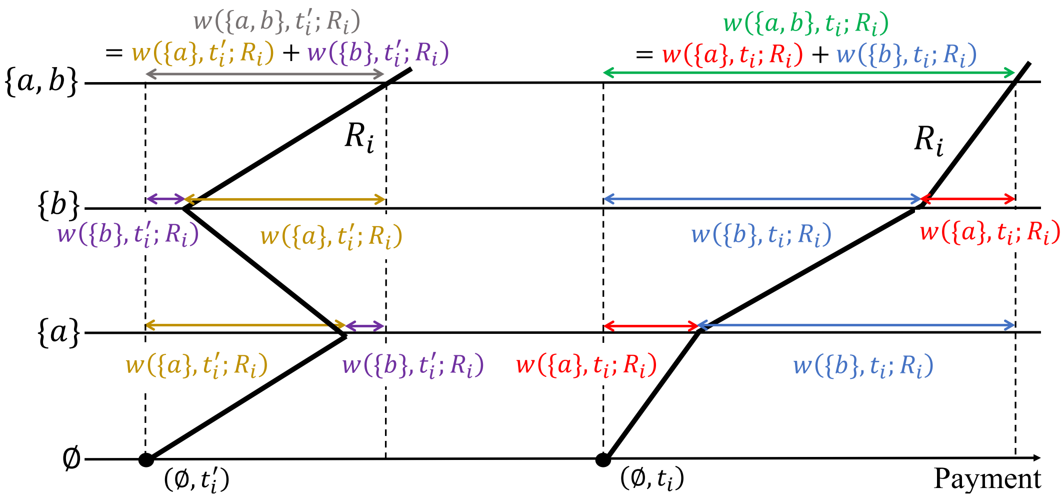

Figure 3: An illustration of an additive preference.

Figure 3 is an illustration of an addtive preference R_i. As shown in the figure, the willingness to pay of a package at each payment is equivalent to the sum of the willingness to pay of the individual objects contained in the package at that payment. Note that when a preference is additive but not quasi-linear, as depicted in Figure 3, the shapes of the indifference curves can vary depending on the payment, but each indifference curve consistently reflects the additive willingness to pay.

A domain $\mathcal{R}$ is said to be **rich** if it includes all additive preferences, i.e., $\mathcal{R} \supseteq \mathcal{R}^{Add}$. Many non-quasi-linear domains studied in the literature are rich. Examples include the *net substitutes domain* (Kelso and Crawford, 1982; Baldwin et al., 2023), the *net complements domain* (Rostek and Yoder, 2020; Baldwin et al., 2023), the *net substitutes and complements domain* (Sun and Yang, 2006; Baldwin et al., 2023), the *single improvement domain* (Gul and Stacchetti, 1999; Nguyen and Vohra, 2024), and the *no complementarities domain* (Gul and Stacchetti, 1999), among others. The conditions defining these domains are based on the Hicksian demand, which represents the demand for a package along the locus of its valuation. The additivity of preferences ensures that this locus satisfies the conditions characterizing these domains.[10]

2.2 Allocations

An **object allocation** is an n-tuple $A \equiv (A_i)_{i\in N} \in \mathcal{M}^n$ such that for each distinct $i, j \in N$, $A_i \cap A_j = \emptyset$. Let $\mathcal{A}$ denote the set of all object allocations. Given $A \in \mathcal{A}$ and $N' \subseteq N$, let $A_{-N'} \equiv (R_i)_{i\in N\setminus N'}$. In particular, for given $A \in \mathcal{A}$ and two distinct agents $i, j \in N$, let $A_{-i} \equiv (A_k)_{k\in N\setminus\{i\}}$ and $A_{-i,j} \equiv (A_k)_{k\in N\setminus\{i,j\}}$.

An **allocation** is an n-tuple $z \equiv (A_i, t_i)_{i\in N} \in (\mathcal{M} \times \mathbb{R})^n$ such that $(A_i)_{i\in N} \in \mathcal{A}$. Let Z denote the set of allocations. Given $z \equiv (A_i, t_i)_{i\in N} \in Z$, its associated object allocation and payment profile are denoted by $A \equiv (A_i)_{i\in N}$ and $t \equiv (t_i)_{i\in N}$ respectively. When convenient, we write $z \equiv (A, t) \in Z$.

2.3 Rules

A **preference profile** is an n-tuple $R \equiv (R_i)_{i\in N} \in \mathcal{R}^n$. Given $R \in \mathcal{R}^n$ and $N' \subseteq N$, let $R_{N'} \equiv (R_i)_{i\in N'}$ and $R_{-N'} \equiv (R_i)_{i\in N\setminus N'}$. In particular, for given $R \in \mathcal{R}^n$ and two distinct agents $i, j \in N$, let $R_{i,j} \equiv R_{\{i,j\}}$, $R_{-i} \equiv R_{-\{i\}}$, and $R_{-i,j} \equiv R_{-\{i,j\}}$.

An (**allocation**) **rule on $\mathcal{R}^n$** is a mapping $f : \mathcal{R}^n \to Z$. With a slight abuse of notation, we may write $f \equiv (A, t)$, where $A : \mathcal{R}^n \to \mathcal{A}$ and $t : \mathcal{R}^n \to \mathbb{R}^n$ are the object allocation and the payment rules associated with f, respectively. The package that agent i receives and his payment under a rule f at a preference profile R are denoted by $A_i(R)$ and $t_i(R)$, respectively, so that $f_i(R) = \big(A_i(R), t_i(R)\big)$.

[10] For the formal definitions of the domains listed above, see Example 11 in Appendix D.

2.4 Admissible object allocations and constraints

We refer to a subset of $\mathcal{A}$ as a **constraint**. Our generic notation for a constraint is $\mathcal{C} \subseteq \mathcal{A}$.

Given a rule f on $\mathcal{R}^n$, let $\mathcal{A}^f \equiv \{A \in \mathcal{A} : \exists R \in \mathcal{R}^n \text{ such that } A(R) = A\}$ denote the range of object allocations under a rule $f \equiv (A, t)$. Note that $\mathcal{A}^f \subseteq \mathcal{A}$ itself is a constraint.

We distinguish between *feasibility* and *admissibility* of object allocations.

The *feasible set* is determined by technology, which is beyond the control of the (social) planner. We assume that all object allocations in $\mathcal{A}$ are feasible.[11] Thus, $\mathcal{A}$ represents the *set of feasible object allocations*.

In contrast, an *admissible set* is a subset $\mathcal{C} \subseteq \mathcal{A}$ of the feasible set (i.e., a constraint), selected by the planner for policy purposes, together with a rule f—as illustrated in Example 1 below. Unlike the feasible set, an admissible set is part of the planner's policy choice.

We emphasize that the planner selects an admissible set $\mathcal{C}$ simultaneously with a rule f; that is, $\mathcal{C}$ is not exogenously fixed but is jointly determined with the rule. Therefore, the planner must choose a rule f such that every outcome object allocation $A \in \mathcal{A}^f$ is admissible under the chosen constraint (i.e., $\mathcal{A}^f \subseteq \mathcal{C}$). Conversely, the admissible set $\mathcal{C}$ must also be consistent with the rule f (i.e., $\mathcal{C} \subseteq \mathcal{A}^f$). Thus, given a rule f, we identify $\mathcal{A}^f$ with the admissible set $\mathcal{C}$ (i.e., $\mathcal{A}^f = \mathcal{C}$), and refer to it as the set of *admissible object allocations* (*under* f), or simply the *admissible set* (*under* f).

The following example illustrates the above point: an admissible set is chosen by the planner simultaneously with a rule.

Example 1 (Admissible object allocations). (i) Suppose that, in order to control the market power of agents after the allocation of objects, the planner restricts each agent from receiving more than three objects. Then, the planner selects a rule f such that $\mathcal{A}^f = \{A \in \mathcal{A} : \forall i \in N,\ |A_i| \leq 3\}$.
(ii) Suppose that agents 1 and 2 are newcomers, and to ensure their continued participation in the market, the planner sets aside some objects—say, a and b—specifically for them. Then, the planner selects a rule f such that $\mathcal{A}^f = \{A \in \mathcal{A} : \{a, b\} \subseteq A_1 \cup A_2\}$. □

The planner aims to design rules that satisfy certain desirable properties. However, such rules may fail to exist under some $\mathcal{A}^f$. Therefore, the choice of $\mathcal{A}^f$ must ensure compatibility with these properties. In Section 3, we examine the compatibility between $\mathcal{A}^f$ and the desirable properties. In the next subsection, we introduce these desirable properties of rules.

[11] All our results remain valid even if the feasible set is an arbitrary non-empty subset of $\mathcal{A}$.

2.5 Properties of rules

2.5.1 Efficiency properties

Given $R \in \mathcal{R}^n$, an allocation $z \equiv (A, t) \in Z$ is said to (**Pareto**) **dominate** another allocation $z' \equiv (A', t') \in Z$ **for** R if the following three conditions hold: (i) for each $i \in N$, $z_i \; R_i \; z'_i$; (ii) $\sum_{i \in N} t_i \geq \sum_{i \in N} t'_i$; and (iii) for some $j \in N$, $z_j \; P_j \; z'_j$, or $\sum_{i \in N} t_i > \sum_{i \in N} t'_i$. This notion of domination takes into account not only the preferences of the agents but also the planner's preference.[12] We assume that the planner is concerned only with her total revenue.[13]

The next property requires that a rule select an efficient allocation that is not dominated by any other allocation.

Pareto efficiency. For each $R \in \mathcal{R}^n$, there exists no $z \equiv (A, t) \in Z$ that dominates $f(R)$ for R.

The next property requires that a rule select an allocation that is efficient over the set of admissible object allocations $\mathcal{A}^f$. Note that we impose constraints only on object allocations, with no restrictions on payments.

Constrained efficiency. For each $R \in \mathcal{R}^n$, there exists no $z \equiv (A, t) \in Z$ with $A \in \mathcal{A}^f$ that dominates $f(R)$ for R.

If $\mathcal{A}^f = \mathcal{A}$, then *constrained efficiency* coincides with standard *Pareto efficiency*. If $\mathcal{A}^f \subsetneq \mathcal{A}$, then *constrained efficiency* is strictly weaker than *Pareto efficiency*, and the smaller $\mathcal{A}^f$ is, the weaker the condition becomes. Thus, if $\mathcal{A}^f$ includes a sufficiently rich variety of admissible object allocations, the planner can implement a wider range of outcomes, although it becomes more difficult to ensure *constrained efficiency*. Conversely, if $\mathcal{A}^f$ includes only a narrow set of admissible object allocations, *constrained efficiency* becomes easier to satisfy, but the planner can implement only a limited range of outcomes. This illustrates a fundamental trade-off between the ease of achieving *constrained efficiency* and the variety of admissible object allocations.

Remark 1 (Constrained efficiency). A rule $f \equiv (A, t)$ on $\mathcal{R}^n$ satisfies *constrained effi-*

[12] Without incorporating the planner's preference, any allocation $z \equiv (A, t) \in Z$ would be dominated by another allocation $z' \equiv (A', t') \in Z$ such that $A' = A$ and $t'_i < t_i$ for each $i \in N$.

[13] Accordingly, the planner has a (quasi-linear) preference R_0 over Z such that, for any two allocations $(A, t), (A', t') \in Z$, it holds that $(A, t) \; R_0 \; (A', t')$ if and only if $\sum_{i \in N} t_i \geq \sum_{i \in N} t'_i$.

ciency if and only if for each $R \in \mathcal{R}^n$, we have

$$A(R) \in \underset{A \in \mathcal{A}^f}{\arg\max} \sum_{i \in N} V\big(A_i, f_i(R); R_i\big).$$

If $R \in (\mathcal{R}^Q)^n$, then this condition is equivalent to

$$A(R) \in \underset{A \in \mathcal{A}^f}{\arg\max} \sum_{i \in N} w(A_i; R_i).$$

The next property requires that all objects be allocated to agents for each preference profile.

No wastage. For each $R \in \mathcal{R}^n$, $\bigcup_{i \in N} A_i(R) = M$.

Note that *Pareto efficiency* implies *no wastage*. Thus, one justification for *no wastage* is that it represents a mild form of efficiency. Another justification is based on practical observations: in real-life auctions, even if some objects are not sold in a one-shot auction, they are typically sold in subsequent auctions. As a result, all objects are eventually sold, and *no wastage* is effectively satisfied in the long run (Kazumura et al., 2020b).

2.5.2 Incentive properties

The following property is a dominant strategy incentive compatibility, which requires that no agent can ever benefit from misrepresenting his preferences.

Strategy-proofness. For each $R \in \mathcal{R}^n$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $f_i(R)\ R_i\ f_i(R'_i, R_{-i})$.

The next property requires that each agent have an incentive to participate in the rule voluntarily—that is, each agent find his outcome pair (a package and a payment) under the rule at least as desirable as receiving no object and paying nothing.

Individual rationality. For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R)\ R_i\ \mathbf{0}$.

The next property requires that each agent's payment always be non-negative.

No subsidy. For each $R \in \mathcal{R}^n$ and each $i \in N$, $t_i(R) \geq 0$.

If agents receive subsidies, those with no interest in the objects may participate in the rule solely to obtain them. *No subsidy* eliminates this incentive for disinterested agents. Moreover, when the objects are initially public assets, providing subsidies alongside the allocation may invite public criticism. *No subsidy* helps to avoid such criticism.

2.5.3 Fairness properties

Next, we introduce three fairness properties. The first requires that any two agents with identical preferences receive the same welfare level under the rule.

Equal treatment of equals. For each $R \in \mathcal{R}^n$ and each $i, j \in N$, if $R_i = R_j$, then $f_i(R)\ I_i\ f_j(R)$.

The second fairness property requires that if agents' preferences are permuted, then their welfare levels be permuted accordingly. Given a preference profile $R \in \mathcal{R}^n$ and a permutation $\pi : N \to N$ on N,[14] let $R^\pi \in \mathcal{R}^n$ denote the permuted preference profile according to π, such that for each $i \in N$, $R^\pi_i = R_{\pi(i)}$.

Anonymity. For each $R \in \mathcal{R}^n$, each permutation $\pi : N \to N$ on N, and each $i \in N$, $f_i(R)\ I_i\ f_{\pi(i)}(R^\pi)$.[15]

The third fairness property requires that no agent prefer any other agent's outcome pair (a package and a payment) to his own under the rule.

No envy. For each $R \in \mathcal{R}^n$ and each $i, j \in N$, $f_i(R)\ R_i\ f_j(R)$.

The following remark clarifies the relationships among the fairness properties and shows that *equal treatment of equals* is the weakest of them.

Remark 2 (Relationships between fairness properties). Let $\mathcal{R}$ be a domain, and let f be a rule on $\mathcal{R}^n$.

(i) If f satisfies *anonymity*, then it also satisfies *equal treatment of equals*, but the converse does not necessarily hold.

(ii) If f satisfies *no envy*, then it also satisfies *equal treatment of equals*, but again, the

[14] A *permutation* on a set G is a bijection from G to itself.

[15] This definition is equivalent to the following condition: for each $R \in \mathcal{R}^n$, each $i, j \in N$, and each $R'_i, R'_j \in \mathcal{R}$ with $R'_i = R_j$ and $R'_j = R_i$, we have $f_i(R)\ I_i\ f_j(R'_{i,j}, R_{-i,j})$ and $f_j(R)\ I_j\ f_i(R'_{i,j}, R_{-i,j})$. Note that *anonymity* is often referred to as *anonymity in welfare* in the literature.

converse does not necessarily hold.
(iii) In general, *anonymity* and *no envy* are independent—that is, neither property implies the other.

We assume that the planner aims to design a rule that satisfies the following desirable properties: *constrained efficiency*, *no wastage*, *equal treatment of equals*, *strategy-proofness*, *individual rationality*, and *no subsidy*.

3 Main results

In this section, we study rules satisfying the desirable properties.

3.1 Bundling unit-demand constraints

First, we examine the conditions that the desirable properties impose on the set of admissible object allocations $\mathcal{A}^f$.

A constraint $\mathcal{C} \subseteq \mathcal{A}$ satisfies **no wastage** if all objects are allocated to agents—that is, if $\bigcup_{i \in N} A_i = M$. As the name suggests, this condition is essential for a rule to satisfy *no wastage*.

A constraint $\mathcal{C}$ satisfies **anonymity** if it is a symmetric set—that is, for each $A \in \mathcal{C}$ and each permutation π of N, we have $A^\pi \in \mathcal{C}$. This condition is essential for a rule to satisfy fairness properties such as *equal treatment of equals*, *anonymity*, and *no envy*.

We introduce bundling constraints, under which objects are bundled into several packages, and each agent receives a collection of these packages instead of individual objects. Given a partition $\mathcal{B} \equiv \{B_1, \ldots, B_K\}$ of M,[16] a constraint $\mathcal{C}$ is said to be **$\mathcal{B}$-bundling** if for each $i \in N$, $A_i = \emptyset$ or there is $L \subseteq \{1, \ldots, K\}$ such that $A_i = \bigcup_{l \in L} B_l$. A special case of bundling is $\overline{\mathcal{B}} \equiv \big\{\{a\} : a \in M\big\}$, where no objects are bundled—that is, each package consists of a single object.

Example 2 (No wastage, anonymity, bundling constraints). Let $n = 2$ and $m = 5$. Let $M = \{a, b, c, d, e\}$. Let $\mathcal{B}$ be a partition of M such that $\mathcal{B} \equiv \{B_1, B_2\}$, where $B_1 \equiv \{a, b, c\}$ and $B_2 \equiv \{d, e\}$. $\mathcal{B}$-bundling constraint requires that each agent receive one of $\emptyset$, B_1, B_2 or M, but not any other package—such as $\{a\}$, $\{e\}$, $\{a, b\}$, $\{a, e\}$, $\{c, d\}$, $\{a, b, c, e\}$,

[16] A set $\mathcal{B} \equiv \{B_1, \ldots, B_K\}$ is a *partition* of M if (i) for each k, $B_k \subseteq M$ and $B_k \neq \emptyset$, (ii) for each distinct $k, k' \in \{1, \ldots, K\}$, $B_k \cap B_{k'} = \emptyset$, and (iii) $\bigcup_{k=1}^{K} B_k = M$.

etc. Let

$$\begin{aligned}
\mathcal{C}^1 &\equiv \big\{(\emptyset, B_1 \cup B_2), (\emptyset, B_2), (B_1, B_2), (B_1, \emptyset)\big\}, \\
\mathcal{C}^2 &\equiv \big\{(\emptyset, B_1 \cup B_2), (B_1, B_2), (B_2, B_1)\big\}, \\
\mathcal{C}^3 &\equiv \big\{(\emptyset, B_1), (B_1, \emptyset), (B_1 \cup B_2, \emptyset), (\emptyset, B_1 \cup B_2)\big\}.
\end{aligned}$$

Then, $\mathcal{C}^1$ is $\mathcal{B}$-bundling but satisfies neither no wastage nor anonymity. $\mathcal{C}^2$ is $\mathcal{B}$-bundling and satisfies no wastage but not anonymity. $\mathcal{C}^3$ is $\mathcal{B}$-bundling and satisfies anonymity but not no wastage. □

A $\mathcal{B}$-bundling constraint $\mathcal{C}$ is said to be **unit-demand** if each agent receives at most one package in $\mathcal{B}$—that is, for each $A \in \mathcal{C}$ and each $i \in N$, we have $A_i \in \mathcal{B} \cup \big\{\emptyset\big\}$. A constraint $\mathcal{C}$ is referred to as **bundling unit-demand** if it is a $\mathcal{B}$-bundling unit-demand constraint for some partition $\mathcal{B}$. Note that a $\mathcal{B}$-bundling unit-demand constraint can satisfy no wastage only if $|\mathcal{B}| \leq n$.

Example 3 (Bundling unit-demand constraint). Let $n = 3$ and $m = 5$. Let $M = \{a, b, c, d, e\}$. Let $\mathcal{B}$ be a partition of M such that $\mathcal{B} \equiv \{B_1, B_2\}$, where $B_1 \equiv \{a, b, c\}$ and $B_2 \equiv \{d, e\}$. Let $\mathcal{C}$ be a constraint such that

$$\mathcal{C} \equiv \big\{(\emptyset, B_1, B_2), (\emptyset, B_2, B_1), (B_1, \emptyset, B_2), (B_1, B_2, \emptyset), (B_2, \emptyset, B_1), (B_2, B_1, \emptyset)\big\}.$$

Then, $\mathcal{C}$ is a $\mathcal{B}$-bundling unit-demand constraint satisfying no wastage and anonymity. Recall that $\overline{\mathcal{B}} \equiv \big\{\{a'\} : a' \in M\big\}$. Since $\big|\overline{\mathcal{B}}\big| = m > n$, no $\overline{\mathcal{B}}$-bundling unit demand constraint satisfies no wastage. □

Note that, given a partition $\mathcal{B}$ of M with $|\mathcal{B}| \leq n$, the $\mathcal{B}$-bundling unit-demand constraint $\mathcal{C}$ that satisfies no wastage and anonymity is unique and can be identified with $\mathcal{B}$. Given a partition $\mathcal{B} \equiv \{B_1, \ldots, B_K\}$ of M with $|\mathcal{B}| \leq n$, let $\mathcal{C}^*(\mathcal{B})$ denote the **$\mathcal{B}$-bundling unit-demand constraint satisfying no wastage and anonymity**. That is,

$$\mathcal{C}^*(\mathcal{B}) \equiv \left\{ A \in \mathcal{A} : \forall i \in N,\ A_i \in \mathcal{B} \cup \{\varnothing\} \text{ and } \bigcup_{i \in N} A_i = M \right\}.$$

As mentioned above, no wastage and anonymity of $\mathcal{A}^f$ are essential for the associated rule f to satisfy *no wastage* and the fairness properties. In addition, the properties of *constrained efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy* impose further restrictions on $\mathcal{A}^f$. The following proposition shows that, in order for a rule f to satisfy all these properties, $\mathcal{A}^f$ must also be a bundling unit-demand constraint.

Proposition. *Let $\mathcal{R}$ be a rich domain. Let f be a rule on $\mathcal{R}^n$ satisfying constrained efficiency, no wastage, equal treatment of equals, strategy-proofness, individual rationality, and no subsidy. Then, $\mathcal{A}^f$ is $\mathcal{B}$-bundling unit-demand for some partition $\mathcal{B}$ of M with $|\mathcal{B}| \leq n$, and satisfies no wastage and anonymity—that is, $\mathcal{A}^f = \mathcal{C}^*(\mathcal{B})$.*

Although $\mathcal{A}^f$ is selected by the planner, it cannot be chosen arbitrarily. The planner must ensure that $\mathcal{A}^f$ is compatible with the desirable properties of rules. If a rule fails to satisfy some of the desirable properties under $\mathcal{A}^f$, then the planner must forgo either $\mathcal{A}^f$ or some of the desirable properties. Proposition shows that if the planner aims to design a rule f satisfying *constrained efficiency*, *no wastage*, *equal treatment of equals*, *strategy-proofness*, *individual rationality*, and *no subsidy*, then she must choose $\mathcal{A}^f = \mathcal{C}^*(\mathcal{B})$ for some partition $\mathcal{B}$ of M with $|\mathcal{B}| \leq n$.

Recall that the strength of *constrained efficiency* depends on $\mathcal{A}^f$: the larger $\mathcal{A}^f$ is, the more demanding the requirement becomes. Therefore, when designing a rule that satisfies the above properties—including *constrained efficiency*—the planner may need to ensure that $\mathcal{A}^f$ is sufficiently small. For example, in Example 2, $\mathcal{C}^*(\mathcal{B})$ contains only 6 object allocations. In general, the maximum number of object allocations that $\mathcal{C}^*(\mathcal{B})$ can contain is $n!$,[17] which is much smaller than the number of feasible object allocations. For instance, when $n = 3$ and $m = 5$, as in Example 2, the total number of feasible object allocations is: $1 + 15 + 90 + 270 + 405 + 243 = 1024$.[18] Thus, in Proposition, the set of admissible object allocations $\mathcal{A}^f$ under a rule f satisfying the desirable properties is significantly smaller than the set of feasible ones $\mathcal{A}$. This reduction represents the cost of satisfying those properties.

3.2 Bundling unit-demand MPW rule

Before turning to our main theorem, we briefly explain its underlying idea. Demange and Gale (1985), Morimoto and Serizawa (2015) (hereafter M&S), and Wakabayashi et al. (2025) (hereafter WSS) study the *unit-demand model*—that is, a model with heterogeneous objects where each agent receives at most one object—and show that, in this setting, the *minimum price Walrasian (MPW) rules* are the only rules satisfying *Pareto efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy*.

Given a $\mathcal{B}$-bundling unit-demand constraint for some partition $\mathcal{B}$ of M, we can reinterpret each package in $\mathcal{B}$ as a single "object." Under this interpretation, our model becomes mathematically equivalent to the unit-demand model. Accordingly, we can define a counterpart

[17] This maximum is attained when $|\mathcal{B}| = n$.

[18] To see this, note that the number of feasible object allocations in which $k = 0, 1, 2, 3, 4, 5$ objects are allocated to three agents is given by $3^k \times {}_5C_k$. The total number is obtained by summing these values over $k = 0$ to 5.

of the MPW rules in our setting, which we call the *bundling unit-demand minimum price Walrasian (MPW) rules*.

Our rich domain induces the same preference domain over packages and payments as those considered in M&S and WSS. As a result, the bundling unit-demand MPW rules inherit the key properties of the MPW rules: they satisfy *strategy-proofness*, *no wastage*, *equal treatment of equals*, *individual rationality*, and *no subsidy*. Moreover, since *constrained efficiency* corresponds to *Pareto efficiency* over admissible object allocations, the bundling unit-demand MPW rules also satisfy *constrained efficiency*. Finally, the results of M&S and WSS imply that the bundling unit-demand MPW rules are the only rules satisfying all these desirable properties.

Now, we move on to the formal discussion. Given a partition $\mathcal{B}$ of M, let $\mathcal{B}_0 \equiv \mathcal{B} \cup \{\emptyset\}$. Note that $\mathcal{B}_0 \subseteq \mathcal{M}$. Given a preference $R_i \in \mathcal{R}$, let $R_i|_{\mathcal{B}_0}$ denote the restriction of R_i to $\mathcal{B}_0 \times \mathbb{R}$. That is, for each $(B_i, t_i), (B'_i, t'_i) \in \mathcal{B}_0 \times \mathbb{R}$, $(B_i, t_i)\ R_i|_{\mathcal{B}_0}\ (B'_i, t'_i)$ if and only if $(B_i, t_i)\ R_i\ (B'_i, t'_i)$. Given a domain $\mathcal{R}$, let $\mathcal{R}|_{\mathcal{B}_0} \equiv \{R_i|_{\mathcal{B}_0} : R_i \in \mathcal{R}\}$. For a given preference profile $R \in \mathcal{R}^n$, let $R|_{\mathcal{B}_0} \equiv (R_i|_{\mathcal{B}_0})_{i \in N}$.

Given a partition $\mathcal{B}$ of M, a **$\mathcal{B}$-bundling price vector** is a vector $p \equiv (p_B)_{B \in \mathcal{B}_0} \in \mathbb{R}_+^{|\mathcal{B}_0|}$ such that $p_\emptyset = 0$. Note that a $\mathcal{B}$-bundling price vector assigns a price to each package $B_i \in \mathcal{B}$, not to each individual object. Given a preference $R_i \in \mathcal{R}$ and a $\mathcal{B}$-bundling price vector $p \in \mathbb{R}_+^{|\mathcal{B}_0|}$, the **$\mathcal{B}$-bundling unit-demand set for R_i at p** is defined as

$$D(R_i, p, \mathcal{B}) \equiv \left\{B_i \in \mathcal{B}_0 : \forall B'_i \in \mathcal{B}_0,\ (B_i, p_{B_i})\ R_i|_{\mathcal{B}_0}\ (B'_i, p_{B'_i})\right\}.$$

We introduce a bundling unit-demand Walrasian equilibrium, which serves as the counterpart to a Walrasian equilibrium in the unit-demand model, adapted to our setting with a bundling unit-demand constraint.[19]

Definition 3. Given a partition $\mathcal{B}$ of M and $R \in \mathcal{R}^n$, a pair $(z, p) \equiv \big((A, t), p\big) \in Z \times \mathbb{R}_+^{|\mathcal{B}_0|}$ of an allocation $z \equiv (A, t)$ with $A \in \mathcal{C}^*(\mathcal{B})$ and a $\mathcal{B}$-bundling price vector p, is a **$\mathcal{B}$-bundling unit-demand Walrasian equilibrium for R** if the following two conditions hold:
(i) For each $i \in N$, $A_i \in D(R_i, p, \mathcal{B})$ and $t_i = p_{B_i}$.
(ii) For each $B \in \mathcal{B}$, if there exists no $i \in N$ such that $A_i = B$, then $p_B = 0$.

Condition (i) states that each agent receives his most preferred package at given prices and pays the price of the package he receives. Condition (ii) states that the price of any package not allocated to any agent is zero.

Given a partition $\mathcal{B}$ of M and a preference profile $R \in \mathcal{R}^n$, if (z, p) is a $\mathcal{B}$-bundling

[19] For the definition of a Walrasian equilibrium in the unit-demand model, see, for example, M&S and WSS.

unit-demand Walrasian equilibrium for R, then z is called a **$\mathcal{B}$-bundling unit-demand Walrasian equilibrium allocation for** R, and p is called a **$\mathcal{B}$-bundling unit-demand Walrasian equilibrium price vector for** R. Let $P(R,\mathcal{B})$ denote the set of such price vectors.

It is known that in the unit-demand model, the set of Walrasian equilibrium price vectors forms a non-empty complete lattice, and hence there exists a unique minimum Walrasian equilibrium price vector with respect to the vector inequality (Demange and Gale, 1985; Alkan and Gale, 1990). The following fact states that an analogous result holds in our setting when a bundling unit-demand constraint is imposed and each package in the given partition is treated as an "object."

Fact 1 (Demange and Gale, 1985; Alkan and Gale, 1990). *Let $\mathcal{B}$ be a partition of M and $R \in \mathcal{R}^n$. Then, $P(R,\mathcal{B})$ forms a non-empty complete lattice, and has a (unique) minimum element $p \in P(R,\mathcal{B})$ such that for each $p' \in P(R,\mathcal{B})$, $p \leq p'$.*[20]

Given a partition $\mathcal{B}$ of M and a preference profile $R \in \mathcal{R}^n$, let $p^{\min}(R,\mathcal{B})$ denote the minimum element of $P(R,\mathcal{B})$, whose existence and uniqueness are guaranteed by Fact 1. Let $Z^{\min}(R,\mathcal{B})$ denote the set of $\mathcal{B}$-bundling unit-demand Walrasian equilibrium allocations supported by $p^{\min}(R,\mathcal{B})$—that is,

$$\begin{aligned}&Z^{\min}(R,\mathcal{B})\\ \equiv &\Big\{z \in Z : \big(z, p^{\min}(R,\mathcal{B})\big) \text{ is a } \mathcal{B}\text{-bundling unit-demand Walrasian equilibrium for } R\Big\}.\end{aligned}$$

Now, we are ready to define a bundling unit-demand MPW rule in our model, which serves as the counterpart to an MPW rule in the unit-demand model.

Definition 4. Given a partition $\mathcal{B}$ of M, a rule f on $\mathcal{R}^n$ with $\mathcal{A}^f = \mathcal{C}^*(\mathcal{B})$ is a **$\mathcal{B}$-bundling unit-demand minimum price Walrasian (MPW) rule** if for each $R \in \mathcal{R}^n$, $f(R) \in Z^{\min}(R,\mathcal{B})$.

Given a partition $\mathcal{B}$ of M and $R \in \mathcal{R}^n$, $Z^{\min}(R,\mathcal{B})$ is essentially unique.[21] Thus, for a given partition $\mathcal{B}$ of M, the $\mathcal{B}$-bundling unit-demand MPW rules are essentially unique. However, since there are many possible partitions of M, each partition gives rise to a distinct bundling unit-demand MPW rule. Therefore, there exists a large class of such rules, each corresponding to a different partition of M.

It is well known that, in the unit-demand model, the MPW rules satisfy several desirable properties: *Pareto efficiency*, *no wastage*, *equal treatment of equals*, *anonymity*, *no envy*,

[20] Given $p, p' \in \mathbb{R}^K$, $p \leq p'$ if and only if $p_k \leq p'_k$ for each $k = 1, \ldots, K$.

[21] To be precise, $Z^{\min}(R,\mathcal{B})$ is unique up to ties, and all allocations in $Z^{\min}(R,\mathcal{B})$ are welfare-equivalent in the sense that for each $z, z' \in Z^{\min}(R,\mathcal{B})$ and each $i \in N$, $z_i \; I_i \; z'_i$.

strategy-proofness, *individual rationality*, and *no subsidy* (Demange and Gale, 1985). Moreover, on a sufficiently rich domain—known as the *classical domain*—these rules are the only ones that satisfy *Pareto efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy* (M&S; WSS). Since a bundling unit-demand constraint renders the mathematical structure of our model equivalent to that of the unit-demand model, and the classical domain corresponds to $\overline{\mathcal{R}}|_{\mathcal{B}_0}$ in our setting, these results carry over to our model.

Fact 2 (Demange and Gale, 1985; Morimoto and Serizawa, 2015; Wakabayashi et al., 2025). *Let $\mathcal{B}$ be a partition of M such that $|\mathcal{B}| \leq n$. Let $\mathcal{R}$ be a domain such that $\mathcal{R}|_{\mathcal{B}_0} = \overline{\mathcal{R}}|_{\mathcal{B}_0}$.*
(i) A $\mathcal{B}$-bundling unit-demand MPW rule satisfies $\mathcal{A}^f = \mathcal{C}^*(\mathcal{B})$, and satisfies the following properties: constrained efficiency, no wastage, equal treatment of equals, anonymity, no envy, strategy-proofness, individual rationality, and no subsidy.
(ii) A rule on $\mathcal{R}^n$ with $\mathcal{A}^f = \mathcal{C}^*(\mathcal{B})$ satisfies constrained efficiency, strategy-proofness, individual rationality, and no subsidy if and only if it is a $\mathcal{B}$-bundling unit-demand MPW rule.

Note that Fact 2 focuses on rules f for which $\mathcal{A}^f = \mathcal{C}^*(\mathcal{B})$ for some partition $\mathcal{B}$ of M.

3.3 Main theorem

The following theorem presents the main result of this paper. It establishes that, on any rich domain, the bundling unit-demand MPW rules are the only rules that satisfy *constrained efficiency*, *no wastage*, *equal treatment of equals*, *strategy-proofness*, *individual rationality*, and *no subsidy*.

Theorem. *Let $\mathcal{R}$ be a rich domain. A rule f on $\mathcal{R}^n$ satisfies constrained efficiency, no wastage, equal treatment of equals, strategy-proofness, individual rationality, and no subsidy if and only if there exists a partition $\mathcal{B}$ of M such that: (i) $|\mathcal{B}| \leq n$, (ii) $\mathcal{A}^f = \mathcal{C}^*(\mathcal{B})$, and (iii) f is a $\mathcal{B}$-bundling unit-demand MPW rule.*

Recall that, $\mathcal{A}^f$ is not a fixed set but a variable chosen jointly with a rule f. Thus, the strength of the requirement for *constrained efficiency* depends on the choice of $\mathcal{A}^f$. As shown in Proposition, in order to satisfy *constrained efficiency*, *no wastage*, *equal treatment of equals*, *strategy-proofness*, *individual rationality*, and *no subsidy*, $\mathcal{A}^f$ must be selected from among the bundling unit-demand constraints.

Note that this proposition does not specify how the rule should select an outcome allocation from a given bundling unit-demand constraint for each preference profile. Theorem addresses this point: it states that, to satisfy the above properties, the rule must select an allocation according to a bundling unit-demand MPW rule.

Note that Theorem holds for any rich domain. In particular, it applies to cases where the objects are substitutes, complements, or both—namely, the net substitutes domain, the net complements domain, and the net substitutes and complements domain.

We have employed *equal treatment of equals* as a fairness property. Although it is one of the central fairness concepts in the literature, other important notions include *anonymity* and *no envy*. We now discuss how Theorem would change if we replace *equal treatment of equals* with *anonymity* or *no envy*.

First, recall that any bundling unit-demand MPW rule satisfies both *anonymity* and *no envy* (Fact 2 (i)). Thus, the "if" part of Theorem still holds even when *equal treatment of equals* is replaced with either *anonymity* or *no envy*.

Second, recall that both *anonymity* and *no envy* are stronger than *equal treatment of equals* (see Remark 2 (i) and (ii)). Thus, the "only if" part of Theorem also remains valid even if we replace *equal treatment of equals* with either *anonymity* or *no envy*.

Thus, the conclusion of Theorem remains unchanged if we replace *equal treatment of equals* with either *anonymity* or *no envy*.

Corollary 1. *Let $\mathcal{R}$ be a rich domain.*
(i) A rule f on $\mathcal{R}^n$ satisfies constrained efficiency, no wastage, anonymity, strategy-proofness, individual rationality, and no subsidy if and only if there exists a partition $\mathcal{B}$ of M such that: (i-i) $|\mathcal{B}| \leq n$, (i-ii) $\mathcal{A}^f = \mathcal{C}^*(\mathcal{B})$, and (i-iii) f is a $\mathcal{B}$-bundling unit-demand MPW rule.
(ii) A rule f on $\mathcal{R}^n$ satisfies constrained efficiency, no wastage, no envy, strategy-proofness, individual rationality, and no subsidy if and only if there exists a partition $\mathcal{B}$ of M such that: (ii-i) $|\mathcal{B}| \leq n$, (ii-ii) $\mathcal{A}^f = \mathcal{C}^*(\mathcal{B})$, and (ii-iii) f is a $\mathcal{B}$-bundling unit-demand MPW rule.

3.4 Outline of the proof

Given Proposition and Fact 2, the remaining step in proving Theorem is to verify that any rich domain $\mathcal{R}$ satisfies the domain condition stated in Fact 2—that is, $\mathcal{R}|_{\mathcal{B}_0} = \overline{\mathcal{R}}|_{\mathcal{B}_0}$ for some partition $\mathcal{B}$ of M. This step is relatively straightforward. Hence, the main difficulty in proving Theorem lies in establishing Proposition.

We further observe that:
(i) no wastage of $\mathcal{A}^f$ follows directly from *no wastage* of a rule f, and
(ii) once we show that $\mathcal{A}^f$ is a $\mathcal{B}$-bundling unit-demand for some partition $\mathcal{B}$ of M, *no wastage* of $\mathcal{A}^f$ implies that $|\mathcal{B}| \leq n$.

Therefore, the main challenge in proving Proposition is to establish that $\mathcal{A}^f$ is a $\mathcal{B}$-bundling unit-demand constraint for some partition $\mathcal{B}$ of M, and that $\mathcal{A}^f$ satisfies anonymity.

In this subsection, we outline the proofs of these two key properties.

3.4.1 Strategy-proofness

In the proof, the properties of *strategy-proof* rules play a crucial role. We therefore begin by reviewing these properties.

We introduce some notations. Given a rule $f \equiv (A, t)$ on $\mathcal{R}^n$ and an agent $i \in N$, let $\mathcal{M}_i$ denote the set of packages that may be assigned to agent i under f:

$$\mathcal{M}_i \equiv \big\{A_i \in \mathcal{M} : \exists R \in \mathcal{R}^n \text{ such that } A_i(R) = A_i\big\}.$$

Furthermore, given $R_{-i} \in \mathcal{R}^{n-1}$, agent i's **package option set for R_{-i} under f** is defined as the set of packages that agent i can possibly receive given R_{-i} under f:

$$\mathcal{M}_i(R_{-i}) \equiv \big\{A_i \in \mathcal{M} : \exists R_i \in \mathcal{R} \text{ such that } A_i(R_i, R_{-i}) = A_i\big\}.$$

For a given rule f on $\mathcal{R}^n$, it follows that for each $i \in N$ and each $R_{-i} \in \mathcal{R}^{n-1}$, we have $\mathcal{M}_i(R_{-i}) \subseteq \mathcal{M}_i \subseteq \mathcal{M}$.

It is well known that under a *strategy-proof* rule, each agent's payment depends only on the package he receives and the preferences of the other agents: once an agent's package is fixed, his own preference does not affect his payment. Formally, if a rule f on $\mathcal{R}^n$ satisfies *strategy-proofness*, then for each $i \in N$, each $R_{-i} \in \mathcal{R}^{n-1}$, and each $A_i \in \mathcal{M}_i(R_{-i})$, there exists a unique payment $t_i(R_{-i}; A_i) \in \mathbb{R}$ such that for some $R_i \in \mathcal{R}$, we have $f_i(R_i, R_{-i}) = \big(A_i, t_i(R_{-i}; A_i)\big)$.[22] Given $A_i \in \mathcal{M}_i(R_{-i})$, let $z_i(R_{-i}; A_i) \equiv \big(A_i, t_i(R_{-i}; A_i)\big)$.

Another well-known property of *strategy-proof* rules is that, for each $R \in \mathcal{R}^n$ and each $i \in N$, agent i receives the most preferred outcome pair from the set $\big\{z_i(R_{-i}; A_i) : A_i \in \mathcal{M}_i(R_{-i})\big\}$.[23] The following remark formalizes this observation.

Remark 3 (Strategy-proofness). Let $\mathcal{R}$ be a domain. Let f be a rule on $\mathcal{R}^n$ satisfying *strategy-proofness*. Let $R \in \mathcal{R}^n$ and $i \in N$. For each $A_i \in \mathcal{M}_i(R_{-i})$, $f_i(R)\ R_i\ z_i(R_{-i}; A_i)$.

[22] To see that there exists at most one such a payment, suppose for contradiction that there exist distinct $t_i, t'_i \in \mathbb{R}$ and $R_i, R'_i \in \mathcal{R}$ such that $f_i(R_i, R_{-i}) = (A_i, t_i)$ and $f_i(R'_i, R_{-i}) = (A_i, t'_i)$. Without loss of generality, suppose $t'_i < t_i$. Then, $f_i(R'_i, R_{-i}) = (A_i, t'_i)\ P_i\ (A_i, t_i) = f_i(R_i, R_{-i})$, which contradicts *strategy-proofness*.

[23] To see this, suppose for contradiction that $z_i(R_{-i}; A'_i)\ P_i\ z_i(R_{-i}; A_i) = f_i(R)$ for some $A'_i \in \mathcal{M}_i(R_{-i})$. Since $A'_i \in \mathcal{M}_i(R_{-i})$, there exists $R'_i \in \mathcal{R}$ such that $A_i(R'_i, R_{-i}) = A'_i$. Thus, $f_i(R'_i, R_{-i}) = z_i(R_{-i}; A'_i)\ P_i\ f_i(R)$, which contradicts *strategy-proofness*.

Given a non-empty set $\mathcal{M}' \subseteq \mathcal{M}$ of packages, a payment vector $\tau \in \mathbb{R}^{|\mathcal{M}'|}$ on $\mathcal{M}'$, and a package $A_i \in \mathcal{M}'$, we say that a preference $R_i \in \mathcal{R}$ **demands** A_i **at** τ **(on** $\mathcal{M}'$**)** if for each $A_i' \in \mathcal{M}' \backslash \{A_i\}$, it holds that $(A_i, \tau_{A_i})\ P_i\ (A_i', \tau_{A_i'})$. Note that under a *strategy-proof* rule f on $\mathcal{R}^n$, given $R \in \mathcal{R}^n$ and $i \in N$, if R_i demands $A_i \in \mathcal{M}_i(R_{-i})$ at $\tau \equiv \big(t_i(R_{-i}; A_i')\big)_{A_i' \in \mathcal{M}_i(R_{-i})}$ on $\mathcal{M}_i(R_{-i})$, then *strategy-proofness* implies $A_i(R) = A_i$ (see Remark 3).

Finally, a rule $f \equiv (A, t)$ on $\mathcal{R}^n$ satisfies **monotonicity** if for each $i \in N$, each $R_i, R_i' \in \mathcal{R}$, and each $R_{-i} \in \mathcal{R}^{n-1}$, the following inequality holds:

$$V\big(A_i(R), f_i(R); R_i\big) - V\big(A_i(R_i', R_{-i}), f_i(R); R_i\big) \geq V\big(A_i(R), f_i(R); R_i'\big) - V\big(A_i(R_i', R_{-i}), f_i(R); R_i'\big).$$

Note that if $R_i, R_i' \in \mathcal{R}^Q$, then this condition is equivalent to:

$$w\big(A_i(R); R_i\big) - w\big(A_i(R_i', R_{-i}); R_i\big) \geq w\big(A_i(R); R_i'\big) - w\big(A_i(R_i', R_{-i}); R_i'\big).$$

It is well established that monotonicity is a necessary condition for *strategy-proofness* (Bikhchandani et al., 2006; Kazumura et al., 2020a).

Fact 3 (Monotonicity). *Let $\mathcal{R}$ be a domain. Let f be a rule on $\mathcal{R}^n$ satisfying strategy-proofness. Then, f satisfies monotonicity.*

3.4.2 Characterization of the constrained Vickrey rules

The characterization of the (constrained) Vickrey rules (Vickrey, 1961) by Holmström (1979) also plays an important role in the proof.

Definition 5 (Vickrey, 1961). Given $\mathcal{R} \subseteq \mathcal{R}^Q$, a rule $f \equiv (A, t)$ on $\mathcal{R}^n$ is a **constrained Vickrey rule** if for each $R \in \mathcal{R}^n$, the following two conditions hold:
(i) We have

$$A(R) \in \operatorname*{arg\,max}_{A \in \mathcal{A}^f} \sum_{i \in N} w(A_i; R_i).$$

(ii) For each $i \in N$,

$$t_i(R) = \max_{A \in \mathcal{A}^f} \sum_{j \in N \backslash \{i\}} w(A_j; R_j) - \sum_{j \in N \backslash \{i\}} w\big(A_j(R); R_j\big).$$

Holmström (1979) considers a general model that includes the package assignment problems with money as a special case and characterizes the class of rules satisfying both (*constrained*) *efficiency* and *strategy-proofness* on sufficiently rich quasi-linear domains. His characterization result implies that the constrained Vickrey rules are the only rules satisfying *constrained efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy* on $(\mathcal{R}^{Add} \cap \mathcal{R}^{Q})^n$.[24]

Fact 4 (Holmström, 1979). *A rule f on $(\mathcal{R}^{Add} \cap \mathcal{R}^{Q})^n$ satisfies constrained efficiency, strategy-proofness, individual rationality, and no subsidy if and only if it is a constrained Vickrey rule.*

3.4.3 Proof of bundling unit-demand constraint: An outline

We now outline the proof that if a rule f on a rich domain $\mathcal{R}^n$ satisfies *constrained efficiency*, *no wastage*, *equal treatment of equals*, *strategy-proofness*, *individual rationality*, and *no subsidy*, then $\mathcal{A}^f$ must be a $\mathcal{B}$-bundling unit-demand constraint for some partition $\mathcal{B}$ of M. Let $\mathcal{R}$ be a rich domain, and let $f \equiv (A, t)$ be a rule on $\mathcal{R}^n$ that satisfies the above properties.

The proof proceeds in three steps:

(i) In the first step (Step 1), we establish that for each $i \in N$ and each $A_i, A'_i \in \mathcal{M}_i \backslash \{\emptyset\}$, it holds that $A_i \cap A'_i = \emptyset$. Thus, the set of packages available to agent i under the rule f, $\mathcal{M}_i$, consists of mutually disjoint packages, and agent i can receive at most one package from $\mathcal{M}_i$. Note, however, that $\mathcal{M}_i \backslash \{\emptyset\}$ may not form a partition of M, since we do not necessarily have $\bigcup \left(\mathcal{M}_i \backslash \{\emptyset\}\right) = M$.

(ii) In the second step (Step 2), we show that for each $i, j \in N$, $\mathcal{M}_i = \mathcal{M}_j$. In proving this result, the fairness property *equal treatment of equals* plays a key role.

(iii) In the third step (Step 3), we complete the proof by showing that $\mathcal{A}^f$ is a $\mathcal{B}$-bundling unit-demand for $\mathcal{B} \equiv \bigcup_{i \in N} \left(\mathcal{M}_i \setminus \{\emptyset\}\right)$. Given the results from Steps 1 and 2, no wastage of $\mathcal{A}^f$ (which follows from *no waastage* of f) ensures that $\mathcal{B}$ is indeed a partition of M.

Given Step 1, Steps 2 and 3 follow relatively easily. Thus, Step 1 is the crucial part of the proof, and we illustrate it for the simplest case where $n = m = 2$ and $\mathcal{R} = \overline{\mathcal{R}}$. Let $M = \{a, b\}$.

We establish that for each $i \in N$ and each distinct $A_i, A'_i \in \mathcal{M}_i \backslash \{\emptyset\}$, we have $A_i \cap A'_i = \emptyset$. Suppose for contradiction that there exist $i \in N$ and distinct $A_i, A'_i \in \mathcal{M}_i \backslash \{\emptyset\}$ such that $A_i \cap A'_i \neq \emptyset$. Without loss of generality, suppose $i = 1$. Since $A_1, A'_1 \neq \emptyset$, $A_1 \cap A'_1 \neq \emptyset$, and $M = \{a, b\}$, we may assume without loss of generality that $A_1 = \{a\}$ and $A'_1 = \{a, b\}$.

The proof consists of four claims.

[24] To be precise, Holmström (1979) studies a public goods model and establishes that, on a smoothly connected quasi-linear domain, the *Groves rules* (Groves, 1973) are the only rules satisfying *Pareto efficiency* and *strategy-proofness*. Note that $\mathcal{R}^{Add} \cap \mathcal{R}^{Q}$ is a smoothly connected quasi-linear domain, so his result applies to this domain. Moreover, if we interpret $\mathcal{A}^f$ as the set of public goods, his characterization implies that the (constrained) Groves rules are the only rules satisfying *constrained efficiency* and *strategy-proofness* on $(\mathcal{R}^{Add} \cap \mathcal{R}^{Q})^n$. Then, by incorporating *individual rationality* and *no subsidy*, we can further conclude that the constrained Vickrey rules are the only (constrained) Groves rules satisfying these additional properties on $(\mathcal{R}^{Add} \cap \mathcal{R}^{Q})^n$. Thus, we obtain Fact 4.

First, note that when agent 2 has an additive quasi-linear preference $R_2 \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ *constrained efficiency* implies that agent 1 can obtain any package $A_1'' \in \mathcal{M}_1$ by declaring a suitable preference $R_1 \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ for which the willingness to pay of A_1'' is sufficiently high and that of the objects not in A_1'' is sufficiently low.

Claim 1. *For each $A_1'' \in \mathcal{M}_1$ and each $R_2 \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$, there exists $R_1 \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ such that $A_1(R_1, R_2) = A_1''$, and so $A_1'' \in \mathcal{M}_1(R_2)$.*

Note that Claim 1 implies that for each $R_2 \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$, we have $\mathcal{M}_1(R_2) = \mathcal{M}_1$. In particular, since $A_1, A_1' \in \mathcal{M}_1$, it follows that for each $R_2 \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$, we have $A_1, A_1' \in \mathcal{M}_1(R_2)$.

Recall that, as discussed in Section 3.4.1, under a *strategy-proof* rule, an agent's payment depends solely on the package he receives and the preferences of the other agents. Also recall from Section 3.4.2 that the constrained Vickrey rules are the only rules satisfying *constrained efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy* on the domain $(\mathcal{R}^{Add} \cap \mathcal{R}^Q)^n$ (see Fact 4).

Combining these two facts, we conclude that when agent 2 has an additive quasi-linear preference, agent 1's payment must coincide with that under the constrained Vickrey rule.

Claim 2. *For each $R_2 \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ and each $A_1'' \in \mathcal{M}_1(R_2)$, we have*

$$t_1(R_2; A_1'') = w(M; R_2) - w(M \backslash A_1''; R_2) = w(A_1''; R_2).$$

Let $R_2 \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that $w(\{a\}; R_2) = 1$, $w(\{b\}; R_2) = 1$, and $w(\{a, b\}; R_2) = w(\{a\}; R_2) + w(\{b\}; R_2) = 2$. Let $\tau \in \mathbb{R}^{|\mathcal{M}|}$ be a payment vector on $\mathcal{M}$ such that for each $A_1'' \in \mathcal{M}_1(R_2)$, $\tau_{A_1} = t_1(R_2; A_1)$. Recall that $A_1, A_1' \in \mathcal{M}_1(R_2)$ (see Claim 1). According to Claim 2, we compute τ_{A_1} and $\tau_{A_1'}$ as follows:

$$\tau_{A_1} = t_1(R_2; A_1) = w(A_1; R_2) = w(\{a\}; R_2) = 1,$$
$$\tau_{A_1'} = t_1(R_2; A_1') = w(A_1'; R_2) = w(\{a, b\}; R_2) = 2.$$

Similarly, let $R_2' \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that $w(\{a\}; R_2') = 2$, $w(\{b\}; R_2') = 2$, and $w(\{a, b\}; R_2') = w(\{a\}; R_2') + w(\{b\}; R_2') = 4$. Let $\tau' \in \mathbb{R}^{|\mathcal{M}|}$ be a payment vector on $\mathcal{M}$ such that for each $A_1'' \in \mathcal{M}_1(R_2')$, $\tau'_{A_1} = t_1(R_2'; A_1)$. Since $A_1, A_1' \in \mathcal{M}_1(R_2')$ (see Claim 1), we apply Claim 2 to compute τ'_{A_1} and $\tau'_{A_1'}$ as follows:

$$\tau'_{A_1} = t_1(R_2'; A_1) = w(A_1; R_2') = w(\{a\}; R_2') = 2,$$
$$\tau'_{A_1'} = t_1(R_2'; A_1') = w(A_1'; R_2') = w(\{a, b\}; R_2') = 4.$$

Note that

$$\tau'_{A_1} = 2 > 1 = \tau_{A_1}. \tag{1}$$

This discrepancy in payments allows us to construct a (non-quasi-linear and not necessarily additive) preference $R_1 \in \overline{\mathcal{R}}$ such that R_1 demands $A_1 = \{a\}$ at τ and $A'_1 = \{a, b\}$ at τ'.

Claim 3 (Figure 4). *There exists $R_1 \in \overline{\mathcal{R}}$ that demands A_1 at τ and A'_1 at τ'.*

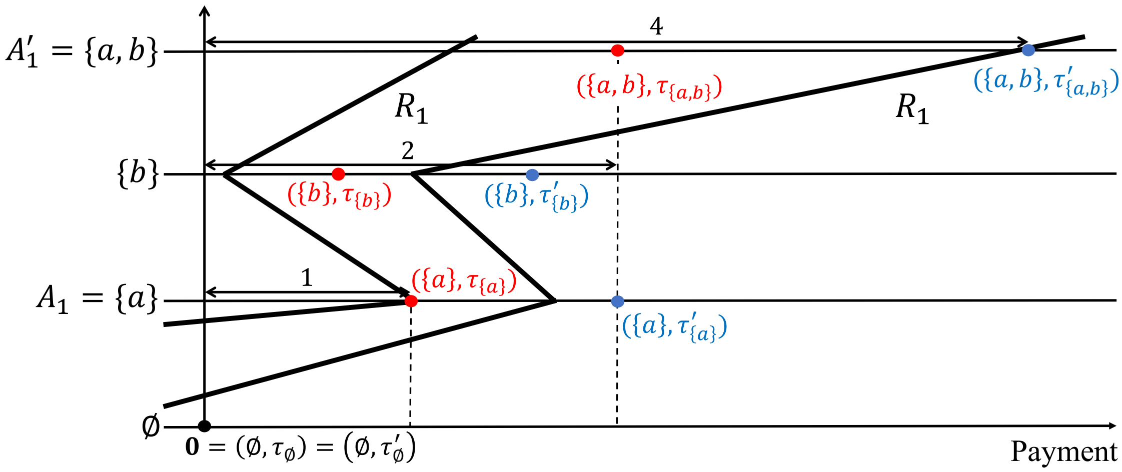

Figure 4: An illustration of Claim 3.

Figure 4 illustrates Claim 3.

Given Claim 3, *strategy-proofness* implies that $A_1(R) = A_1 = \{a\}$ and $A_1(R'_2, R_{-2}) = A'_1 = \{a, b\}$ (see Remark 3). Since there are only two agents, *no wastage* of f implies that agent 2 receives the remaining objects. Therefore, we obtain the following:

Claim 4. *We have $A_2(R) = M \backslash A_1(R) = \{b\}$ and $A_2(R_1, R'_2) = M \backslash A_1(R_1, R'_2) = \emptyset$.*

Claim 4 crucially relies on the fact that there are only two agents. Indeed, if there are three or more agents, *no wastage* does not necessarily imply that agent 2 receives the remaining objects.

Now, we are in a position to derive a contradiction. Observe that

$$\begin{aligned} &w\big(A_2(R); R'_2\big) - w\big(A_2(R_1, R'_2); R'_2\big) = w(\{b\}; R'_2) \\ &= 2 > 1 = w\big(\{b\}; R_2\big) = w\big(A_2(R); R_2\big) - w\big(A_2(R_1, R'_2); R_2\big), \end{aligned} \tag{2}$$

where the first and last equalities follow from Claim 4. However, this contradicts *monotonicity* of f (see Fact 3).

3.4.4 Challenges arising from many agents and domain restrictions

The outline of the proof presented in Section 3.4.3 assumes that there are two agents (i.e., $n = 2$) and that the domain is unrestricted (i.e., $\mathcal{R} = \overline{\mathcal{R}}$). The two-agent assumption simplifies the argument, as the *no wastage* property ensures that once the package assigned to one agent is determined, the other agent must receive the remaining objects.

In contrast, the full proof is substantially more complex, as it must account for both the allocation of objects among more than two agents and the restriction on the domain. In what follows, we describe how these features complicate the argument and explain how we address them.

Many agents. First, we explain how the existence of three or more agents complicates the proof. The outline in Section 3.4.3 crucially relies on two types of tractability regarding admissible object allocations: *intrapersonal tractability* and *interpersonal tractability*. Intrapersonal tractability refers to the tractability of the packages available to a single agent (see Claim 1), while interpersonal tractability refers to the tractability of the packages available to two agents (see Claim 4). In the outlined proof, both types of tractability are guaranteed by the two-agent assumption.

In the full proof, where there may be three or more agents, both types of tractability may fail under *constrained efficiency*. Indeed, recall that when there are only two agents, an agent can receive any package available under the rule by reporting certain preferences, regardless of the other agent's preferences (thus ensuring interpersonal tractability; see Claim 1). However, when there are three or more agents, the packages available to a given agent may depend on the preferences of the others—that is, intrapersonal tractability fails. The following example demonstrates this issue.

Example 4. Let $n = 3$ and $m = 3$, and let $M = \{a, b, c\}$. Let $\mathcal{R}$ be a rich domain. Let f be a rule on $\mathcal{R}^n$ satisfying *constrained efficiency* (and *no wastage*), such that

$$\mathcal{A}^f = \Big\{ \big(\{a\}, \{b\}, \{c\}\big), \big(\{a, b\}, \{c\}, \emptyset\big) \Big\}.$$

Let $R_{-1} \in (\mathcal{R}^{Add} \cap \mathcal{R}^Q)^2$ be such that $w\big(\{b\}; R_2\big) = 1$, $w\big(\{c\}; R_2\big) = 5$, and $w\big(\{c\}; R_3\big) = 1$. Since $w\big(\{b\}; R_2\big) + w\big(\{c\}; R_3\big) = 1 + 1 = 2 < 5 = w\big(\{c\}; R_2\big)$, *constrained efficiency* implies that for each $R_1 \in \mathcal{R}$, $A_1(R_1, R_{-1}) = \{a, b\}$. Thus, there exists no $R_1 \in \mathcal{R}$ such that $A_1(R_1, R_{-1}) = \{a\}$. Thus, $\mathcal{M}_1(R_{-1}) = \big\{\{a, b\}\big\}$. However, under other preferences of the remaining agents—for example, when all other agents have the same additive quasi-linear preferences—agent 1 can obtain either $\{a\}$ or $\{a, b\}$, depending on the preference he reports.

Thus, for some $R'_{-1} \in (\mathcal{R}^{Add} \cap \mathcal{R}^{Q})^2$, $\mathcal{M}_1(R'_{-1}) = \big\{\{a\},\{a,b\}\big\} = \mathcal{M}_1$. $\square$

To address this form of intractability, we proceed as follows: given a package $A_i \in \mathcal{M}_i$ that is available to agent i under the rule, we identify a preference profile $R_{-i} \in (\mathcal{R}^{Add} \cap \mathcal{R}^{Q})^{n-1}$ of the other agents such that $A_i \in \mathcal{M}_i(R_{-i})$ (see Lemma 2 in Appendix A.1.2). Note that this form of tractability is weaker than that established in Claim 1, where $\mathcal{M}_i(R_{-i}) = \mathcal{M}_i$ for each $R_{-i} \in (\mathcal{R}^{Add} \cap \mathcal{R}^{Q})^{n-1}$. Nevertheless, this partial tractability still provides a useful foundation for our analysis.

Furthermore, when there are three or more agents, determining how the remaining packages are allocated to the other agents once a package is assigned to one agent becomes a non-trivial problem—that is, interpersonal tractability fails. To recover interpersonal tractability, we exploit the implications of *equal treatment of equals*, along with the other desirable properties, to identify which packages may be available to two agents (see Lemma 5, Lemma 6, and Lemma 7 in Appendix A.1.3). Note that in the full proof of this part, *equal treatment of equals* plays a central role in restoring interpersonal tractability. By contrast, in the outline of the proof for the two-agent case, it plays no role, as full interpersonal tractability is already guaranteed by the *no wastage* property.

Note that non-quasi-linear preferences do not play an essential role in the above discussion on the tractability of admissible object allocations. Nevertheless, this perspective is novel, as prior studies have not explicitly examined tractability under *constrained efficiency* and *equal treatment of equals*.

Restricted domains. In addition to the challenges posed by allocation constraints, we must also address those arising from restricted domains. Since our richness condition requires that the domain include all (possibly non-quasi-linear) additive preferences, we are free to select only additive preferences in the proof. Claim 3 in the outline shows that there exists some preference $R_1 \in \overline{\mathcal{R}}$ that demands a package $A_1 = \{a\}$ at τ and another package $A'_1 = \{a,b\}$ at τ'. If we were allowed to choose non-additive, non-quasi-linear preferences from the domain, then constructing such a preference would be relatively straightforward (see Figure 4). The challenge, however, is to construct an additive non-quasi-linear preference that satisfies a property analogous to Claim 4. Although such an additive non-quasi-linear preference cannot be constructed for certain packages A_1, A'_1 and payment vectors τ, τ' under some constraints,[25] we identify conditions on the packages, payment vectors, and the constraint that ensure the

[25]For example, let $M = \{a,b\}$, and let $\tau, \tau' \in \mathbb{R}^{|\mathcal{M}|}$ be two object monotonic payment vectors such that $\tau_{\emptyset} = 0$, $\tau_{\{a\}} = \tau_{\{b\}} = 1$, $\tau_{\{a,b\}} = 2$, $\tau'_{\emptyset} = 0$, $\tau'_{\{a\}} = \tau'_{\{b\}} = 2$, and $\tau'_{\{a,b\}} = 4$. Then, no additive preference R_i demands $A_i = \emptyset$ at τ on $\mathcal{M}$ and $A'_i = \{a,b\}$ at τ' on $\mathcal{M}$. To see this, let $R_i \in \mathcal{R}^{Add}$ be an additive preference that demands $A'_i = \{a,b\}$ at τ' on $\mathcal{M}$. Then, since $\big(\{a,b\}, \tau'_{\{a,b\}}\big)\ P_i\ \big(\{b\}, \tau'_{\{b\}}\big)$,

existence of such a preference (see Lemma 8 and Lemma 9 in Appendix A.1.4).

3.4.5 Proof of anonymity: An outline

Next, we prove that $\mathcal{A}^f$ satisfies anonymity, assuming—as established in Section 3.4.3—that it is a $\mathcal{B}$-bundling unit-demand constraint for some partition $\mathcal{B}$ of M. As we shall see, the fairness condition of *equal treatment of equals*, together with the characterization of constrained Vickrey rules (Fact 4), plays a crucial role.

Let $\mathcal{R}$ be a rich domain, and let $f \equiv (A, t)$ be a rule on $\mathcal{R}^n$ that satisfies *constrained efficiency*, *no wastage*, *equal treatment of equals*, *strategy-proofness*, *individual rationality*, and *no subsidy*. Suppose for contradiction that $\mathcal{A}^f$ fails to satisfy anonymity. Assume $\mathcal{A}^f$ is a $\mathcal{B}$-bundling unit-demand constraint for some partition $\mathcal{B}$ of M.

To illustrate the essence of the argument, we focus on the simplest case where $n = m = 2$, as in Section 3.4.3, and let $M = \{a, b\}$. Since $m = 2$, there are only two possible partitions of M, which we consider in turn.

Case 1. $\mathcal{B} = \{M\}$.

Given that $\mathcal{A}^f$ is a $\mathcal{B}$-bundling unit-demand constraint, satisfies no wastage (which follows from *no wastage* of f), but violates anonymity, we must have either $\mathcal{A}^f = \big\{(M, \emptyset)\big\}$ or $\mathcal{A}^f = \big\{(\emptyset, M)\big\}$. Without loss of generality, let $\mathcal{A}^f = \big\{(M, \emptyset)\big\}$.

For each $i \in N$, let $R_i \in \mathcal{R}^{Add} \cap \mathcal{R}^{Q}$ be such that $w\big(\{a\}; R_i\big) = w\big(\{b\}; R_i\big) = 3$ and $w(M; R_i) = w\big(\{a\}; R_i\big) + w\big(\{b\}; R_i\big) = 6$. Note that $R_1 = R_2$. Note also that by richness of $\mathcal{R}$, $R \in \mathcal{R}^n$. Since f satisfies *constrained efficiency*, *stratey-proofness*, *individual rationality*, and *no subsidy*, its restriction to $(\mathcal{R}^{Add} \cap \mathcal{R}^{Q})^n$ satisfies the four properties as well. Thus,

---

$V\Big(\{b\}, (\{a,b\}, \tau'_{\{a,b\}}); R_i\Big) < \tau'_{\{b\}}$, which implies

$$V\Big(\{a,b\}, (\{a,b\}, \tau'_{\{a,b\}}); R_i\Big) - V\Big(\{b\}, (\{a,b\}, \tau'_{\{a,b\}}); R_i\Big) = \tau'_{\{a,b\}} - V\Big(\{b\}, (\{a,b\}, \tau'_{\{a,b\}})\Big) > \tau'_{\{a,b\}} - \tau'_{\{b\}} = 4 - 2 = 2.$$

Similarly, since $(\{a,b\}, \tau'_{\{a,b\}})\ P_i\ (\{a\}, \tau'_{\{a\}})$, $\tau'_{\{a\}} > V\Big(\{a\}, (\{a,b\}, \tau'_{\{a,b\}}); R_i\Big)$. Then,

$$V\Big(\{a\}, (\{a,b\}, \tau'_{\{a,b\}}); R_i\Big) - V\Big(\emptyset, (\{a,b\}, \tau'_{\{a,b\}}); R_i\Big) = V\Big(\{a,b\}, (\{a,b\}, \tau'_{\{a,b\}}); R_i\Big) - V\Big(\{b\}, (\{a,b\}, \tau'_{\{a,b\}}); R_i\Big) > 2,$$

where the first equality follows from additivity of R_i. Thus,

$$\tau'_{\{a\}} - V\Big(\emptyset, (\{a,b\}, \tau'_{\{a,b\}}); R_i\Big) > V\Big(\{a\}, (\{a,b\}, \tau'_{\{a,b\}}); R_i\Big) - V\Big(\emptyset, (\{a,b\}, \tau'_{\{a,b\}}); R_i\Big) > 2 = \tau'_{\{a\}},$$

where the first inequality follows from $\tau'_{\{a\}} > V\Big(\{a\}, (\{a,b\}, \tau'_{\{a,b\}}); R_i\Big)$. Thus, $V\Big(\emptyset, (\{a,b\}, \tau'_{\{a,b\}}); R_i\Big) < 0$, which implies $(\{a,b\}, \tau'_{\{a,b\}})\ P_i\ \mathbf{0}$. Thus, by $\tau_{\{a,b\}} = 2 < 4 = \tau'_{\{a,b\}}$, $(\{a,b\}, \tau_{\{a,b\}})\ P_i\ (\{a,b\}, \tau'_{\{a,b\}})\ P_i\ \mathbf{0}$. Thus, R_i does not demand $A_i = \emptyset$ at τ on $\mathcal{M}$.

it follows from Fact 4 that f coincides with a constrained Vickrey rule on $(\mathcal{R}^{Add} \cap \mathcal{R}^Q)^n$. Thus, since $R \in (\mathcal{R}^{Add} \cap \mathcal{R}^Q)^n$, $f(R)$ is an outcome of a constrained Vickrey rule for R. Thus, since $\mathcal{A}^f = \{(M, \emptyset)\}$, $A(R) = (M, \emptyset)$, $t_1(R) = w\big(\emptyset; R_2\big) - w\big(\emptyset; R_2\big) = 0$, and $t_2(R) = w(M; R_1) - w(M; R_1) = 0$. Thus,

$$w\big(A_1(R); R_1\big) - t_1(R) = w(M; R_1) = 6 \neq 0 = w(\emptyset; R_2) = w\big(A_2(R); R_2\big) - t_2(R).$$

However, since $R_1 = R_2$, this contradicts *equal treatment of equals*.

CASE 2. $\mathcal{B} = \big\{\{a\}, \{b\}\big\}$.

As in Case 1, $\mathcal{A}^f$ must be either $\Big\{\big(\{a\}, \{b\}\big)\Big\}$ or $\Big\{\big(\{b\}, \{a\}\big)\Big\}$. Without loss of generality, let $\mathcal{A}^f = \Big\{\big(\{a\}, \{b\}\big)\Big\}$.

For each $i \in N$, let $R_i \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that $w\big(\{a\}; R_i\big) = 3$ and $w\big(\{b\}; R_i\big) = 1$. Note that $R_1 = R_2$. It follows from richness of $\mathcal{R}$ that $R \in \mathcal{R}^n$. As in Case 1, since $R \in (\mathcal{R}^{Add} \cap \mathcal{R}^Q)^n$, $f(R)$ is an outcome of a constrained Vickrey rule for R (see Fact 4). Thus, since $\mathcal{A}^f = \Big\{\big(\{a\}, \{b\}\big)\Big\}$, $A(R) = \big(\{a\}, \{b\}\big)$, $t_1(R) = w\big(\{b\}; R_2\big) - w\big(\{b\}; R_2\big) = 0$, and $t_2(R) = w\big(\{a\}; R_1\big) - w\big(\{a\}; R_1\big) = 0$. Thus,

$$w\big(A_1(R); R_1\big) - t_1(R) = w\big(\{a\}; R_1\big) = 3 \neq 1 = w(\{b\}; R_2) = w\big(A_2(R); R_2\big) - t_2(R).$$

However, since $R_1 = R_2$, this contradicts *equal treatment of equals*.

3.5 Independence of the properties

All the properties in Theorem are indispensable. The following examples demonstrate that if any one of these properties is dropped, then there exists a rule that (i) differs from any bundling unit-demand MPW rule and (ii) satisfies all the remaining properties. Example 5 further shows that even when *equal treatment of equals* is replaced by stronger properties such as *anonymity* or *no envy*, *constrained efficiency* remains indispensable for Theorem. In all the following examples, let $\mathcal{R}$ be a rich domain.

Example 5 (Dropping constrained efficiency). Let $n = 4$ and $M = \{a, b\}$.[26] Let $\mathcal{B} = \big\{\{a\}, \{b\}\big\}$. Let g be a $\mathcal{B}$-bundling unit-demand MPW rule, and let $R_0 \in \mathcal{R}$. We define

[26]It is straightforward to extend the discussion here to the case where $n \geq 4$ and $m \geq 2$. Whether *constrained efficiency* is indispensable for Theorem when $n = 2$ remains an open question. When $m = 1$, the class of rules on a rich domain that satisfy *no wastage*, *equal treatment of equals*, *strategy-proofness*, *individual rationality*, and *no subsidy* coincides with the class of bundling unit-demand MPW rules (Sakai, 2013). Thus, in that case, *constrained efficiency* can be dispensed with in Theorem.

a rule f based on g as follows.

Informally, if all agents except one—say, agent i—have the same preference R_0, then agent i is allowed to choose between the outcome under g and the outcome under a $\{M\}$-bundling unit-demand MPW rule (which coincides with the bundling second-price rule for the grand bundle M). If there is no such agent for which all other agents have preference R_0, then f coincides with g.

The formal definition of f is as follows:

(i) If $|\{i \in N : R_i = R_0\}| = 3$, and for the unique $i \in N$ such that $R_i \neq R_0$, we have $\big(M, V(M, \mathbf{0}; R_0)\big) \; P_i \; g_i(R)$, then define $f_i(R) = \big(M, V(M, \mathbf{0}; R_0)\big)$ and $f_j(R) = \mathbf{0}$ for each $j \in N \backslash \{i\}$.

(ii) If $|\{i \in N : R_i = R_0\}| \neq 3$, or if $|\{i \in N : R_i = R_0\}| = 3$ and for the unique $i \in N$ such that $R_i \neq R_0$, $g_i(R) \; R_i \; \big(M, V(M, \mathbf{0}; R_0)\big)$, then let $f(R) = g(R)$.

Note that $\mathcal{A}^f = \big\{A \in \mathcal{A} : \forall i \in N,\ A_i = \emptyset, \{a\}, \{b\}, \text{ or } M,\ \bigcup_{i=1}^{4} A_i = M\big\}$. Note also that f is not a bundling unit-demand MPW rule with any partition of M.

Claim 5. *The rule f satisfies no wastage, equal treatment of equals, anonymity, no envy, strategy-proofness, individual rationality, and no subsidy, but it violates constrained efficiency.*

We defer the proof of Claim 5 to Appendix C. □

Example 6 (Dropping no wastage). Let f be the *no trade rule* on $\mathcal{R}^n$, that is, the rule such that for each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R) = \mathbf{0}$. Note that $\mathcal{A}^f = \big\{(\emptyset, \ldots, \emptyset)\big\}$. Then:

(i) f is not a bundling unit-demand MPW rule with any partition of M, and

(ii) it satisfies *constrained efficiency*, *equal treatment of equals*, *strategy-proofness*, *individual rationality*, and *no subsidy*, but violates *no wastage*. □

Example 7 (Dropping equal treatment of equals). Let $i \in N$, and let $A \in \mathcal{A}$ be such that $A_i = M$ and $A_j = \emptyset$ for each $j \in N \backslash \{i\}$. Let f be a rule on $\mathcal{R}^n$ such that for each $R \in \mathcal{R}^n$, we have $f_i(R) = (M, 0)$ and $f_j(R) = \mathbf{0}$ for each $j \in N \backslash \{i\}$—that is, agent i is the dictator under f. Note that $\mathcal{A}^f = \{A\}$. Then:

(i) f is not a bundling unit-demand MPW rule with any partition of M, and

(ii) it satisfies *constrained efficiency*, *no wastage*, *strategy-proofness individual rationality*, and *no subsidy*, but violates *equal treatment of equals*. □

Example 8 (Dropping strategy-proofness). Let f be a *generalized pay-as-bid rule* on $\mathcal{R}^n$, that is, a rule such that for each $R \in \mathcal{R}^n$, $A(R) \in \underset{A \in \mathcal{A}}{\arg\max} \sum_{i \in N} V(A_i, \mathbf{0}; R_i)$, and for each $i \in N$, $t_i(R) = V\big(A_i(R), \mathbf{0}; R_i\big)$. Note that $\mathcal{A}^f = \big\{A \in \mathcal{A} : \bigcup_{i \in N} A_i = M\big\}$. Then:

(i) f is not a bundling unit-demand MPW rule with any partition of M, and

(ii) it satisfies *constrained efficiency*, *no wastage*, *equal treatment of equals*, *individual rationality*, and *no subsidy*, but violates *strategy-proofness*. □

Example 9 (Dropping individual rationality). Let $\mathcal{B}$ be a partition of M. Let f be a $\mathcal{B}$-bundling unit-demand MPW rule with a (common and fixed) participation fee $e > 0$. Note that $\mathcal{A}^f = \mathcal{C}^*(\mathcal{B})$. Then:

(i) f is not a bundling unit-demand MPW rule with any partition of M, and

(ii) it satisfies *constrained efficiency*, *no wastage*, *equal treatment of equals*, *strategy-proofness*, and *no subsidy*, but violates *individual rationality*. □

Example 10 (Dropping no subsidy). Let $\mathcal{B}$ be a partition of M. Let f be a $\mathcal{B}$-bundling unit-demand MPW rule associated with a (common and fixed) participation subsidy $s < 0$. Note that $\mathcal{A}^f = \mathcal{C}^*(\mathcal{B})$. Then:

(i) f is not a bundling unit-demand MPW rule with any partition of M, and

(ii) it satisfies *constrained efficiency*, *no wastage*, *equal treatment of equals*, *strategy-proofness*, and *individual rationality*, but violates *no subsidy*. □

4 Conclusion

We have studied the package assignment problem with money, in which the set of admissible object allocations is selected by the planner. We have shown that the only admissible sets of object allocations that ensure the existence of a rule satisfying a set of desirable properties —namely, *constrained efficiency*, *no wastage*, *equal treatment of equals*, *strategy-proofness*, *individual rationality*, and *no subsidy*—are the bundling unit-demand constraints (Proposition). Furthermore, we have shown that the only rules satisfying these properties are the bundling unit-demand MPW rules (Theorem).

In practice, certain technological characteristics suggest that some objects are complements, while others are substitutes for most agents. In such cases, policymakers often aim to bundle complementary objects within the same packages and to separate substitutes into different ones in order to achieve more efficient allocations—an approach that corresponds to a special case of our bundling unit-demand constraints. Importantly, our domain richness condition is sufficiently weak to ensure that the results apply to such environments.

Bundling unit-demand constraints are commonly adopted in practical auction designs, including the European 3G frequency license auctions and recent 5G license auctions in several countries. Our results provide a novel theoretical justification for this widely used design principle.

Appendix

A Proof of Proposition

In this section, we provide the proof of Proposition.

A.1 Preliminaties

In this subsection, we present the lemmas that will be used in the proof of Proposition.

A.1.1 Strategy-proofness

We begin by presenting a lemma related to *strategy-proofness*.

The following lemma states that, under a *strategy-proof* rule, each agent who receives more objects must pay a higher amount. This result follows directly from object monotonicity and *strategy-proofness* (in particular, Remark 3), and the proof is therefore omitted.

Lemma 1 (Object monotonic payments). *Let $\mathcal{R}$ be a domain. Let f be a rule on $\mathcal{R}^n$ satisfying strategy-proofness. Let $i \in N$ and $R_{-i} \in \mathcal{R}^{n-1}$. For each $A_i, A_i' \in \mathcal{M}_i(R_{-i})$ with $A_i \supsetneq A_i'$, we have $t_i(R_{-i}; A_i) > t_i(R_{-i}; A_i')$.*

A.1.2 Intrapersonal tractability of object allocations

We now turn to the issue of intrapersonal tractability of admissible object allocations, corresponding to Claim 1 in the proof outline presented in Section 3.4.3. As discussed in Section 3.4.4, a key challenge posed by *constrained efficiency* is the potential loss of intrapersonal tractability—that is, the difficulty of identifying the set of packages available to an individual agent. Under *constrained efficiency*, this set may depend on the preferences of the other agents, which significantly complicates the analysis. To circumvent this difficulty, the following lemma provides a sufficient condition on the preference profiles of the other agents that ensures agent i can obtain a specified package A_i.

This condition consists of two components (see Figure 5):

(i) all other agents have the same willingness to pay of each object, except for those in A_i and one object a not in A_i;

(ii) there exists an agent $j \neq i$ who, conditional on agent i receiving A_i, is assigned object a and has a strictly higher willingness to pay of a than the other agents.

Note that if there are only two agents, this condition always holds, and the following lemma reduces to Claim 1 in Section 3.4.3.

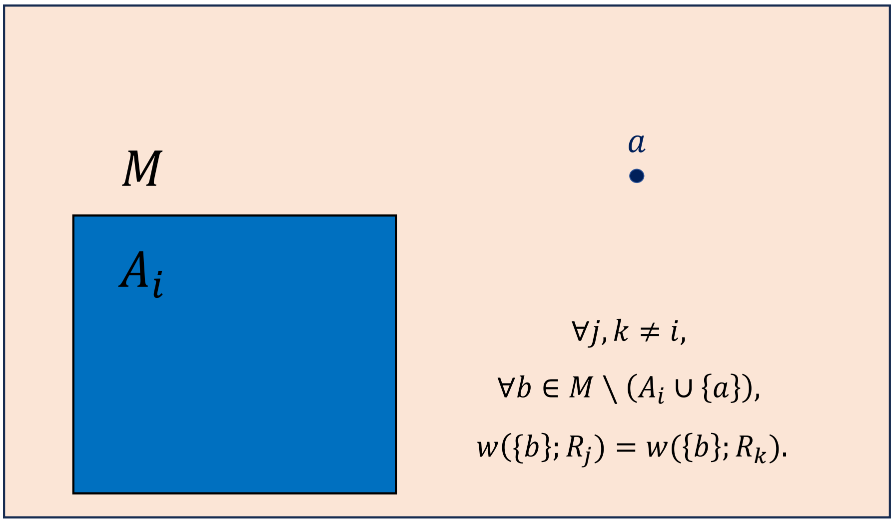

Figure 5: An illustration of the package A_i and the object a in Lemma 2.

Lemma 2 (Intrapersonal tractability). *Let $\mathcal{R}$ be a rich domain, and let f be a rule on $\mathcal{R}^n$ satisfying constrained efficiency and no wastage. Let $i \in N$ and $A_i \in \mathcal{M}_i$. Let $j \in N \setminus \{i\}$ and $a \in M \setminus A_i$ be such that for some $A_{-i} \in \mathcal{M}^{n-1}$, $(A_i, A_{-i}) \in \mathcal{A}^f$ and $a \in A_j$. Let $R_{-i} \in (\mathcal{R}^{Add} \cap \mathcal{R}^Q)^{n-1}$ be such that the following two conditions hold:*
(i) For each $k \in M \setminus \{i\}$ and each $b \in M \setminus \big(A_i \cup \{a\}\big)$, $w\big(\{b\}; R_k\big) = w\big(\{b\}; R_j\big)$.
(ii) For each $k \in N \setminus \{i\}$, $w\big(\{a\}; R_k\big) \leq w\big(\{a\}; R_j\big)$.
Then, there exists $R_i \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ such that $A_i(R_i, R_{-i}) = A_i$, and so $A_i \in \mathcal{M}_i(R_{-i})$.

Proof. Let $R_i \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be an additive quasi-linear preference such that the willingness to pay of the objects in A_i is sufficiently large, while the willingness to pay of the objects not in A_i is sufficiently small. Specifically, let $R_i \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ satisfy the following two conditions:

- For each $A_i' \in \mathcal{M}$ with $A_i' \supsetneq A_i$,

$$w(A_i' \setminus A_i; R_i) < w\big(A_i' \setminus A_i; R_j\big), \tag{3}$$

- For each $b \in A_i$,

$$w\big(\{b\}; R_i\big) > w(M \setminus A_i; R_i) + \sum_{k \in N \setminus \{i\}} w(M; R_k). \tag{4}$$

Note that, by object monotonicity of R_j, the right-hand side of (3) is positive, so we can indeed choose such a preference R_i satisfying (3). Moreover, by richness of $\mathcal{R}$, $R \in \mathcal{R}^n$.

We show that $A_i(R) = A_i$. Suppose for contradiction that $A_i(R) \neq A_i$. By our assumption, there exists $A_{-i} \in \mathcal{M}^{n-1}$ such that $(A_i, A_{-i}) \in \mathcal{A}^f$ and $a \in A_j$. We consider two cases.

CASE 1. $A_i(R) \supsetneq A_i$.

By no wastage of $\mathcal{A}^f$ (which follows from *no wastage* of f), $\bigcup_{k\in N} A_k(R) = \bigcup_{k\in N} A_k = M$. We have

$$\begin{aligned}\sum_{k\in N\setminus\{i\}} w(A_k;R_k) &= w\big(\{a\};R_j\big) + w\big(A_j\setminus\{a\};R_j\big) + \sum_{k\in N\setminus\{i,j\}} w(A_k;R_k)\\ &= w\big(\{a\};R_j\big) + w\left(\left(\bigcup_{k\in N\setminus\{i\}} A_k\right)\setminus\{a\};R_j\right)\\ &= w(M\setminus A_i;R_j), \end{aligned} \tag{5}$$

where the first equality follows from $a \in A_j$ and additivity of R_j; the second from additivity of R_{-i} and the assumption that $w\big(\{b\};R_k\big) = w\big(\{b\};R_j\big)$ for each $k \in N\setminus\{i\}$ and each $b \in M\setminus\big(A_i \cup \{a\}\big)$ (and hence giving objects of agents in $N\setminus\{i,j\}$ to agent j does not change the total willingness to pay among agents in $N\setminus\{i\}$); and the last from $\bigcup_{k\in N} A_k = M$ and additivity of R_j. Since $A_i(R) \supsetneq A_i$, we have $A_k(R) \subseteq M\setminus A_i$ for each $k \in N\setminus\{i\}$. Then,

$$\begin{aligned}\sum_{k\in N\setminus\{i\}} w\big(A_k(R);R_k\big) &\le w\big(\{a\};R_j\big) + w\big(A_j(R)\setminus\{a\};R_j\big) + \sum_{k\in N\setminus\{i,j\}} w\big(A_k(R)\setminus\{a\};R_k\big)\\ &= w\big(\{a\};R_j\big) + w\left(\left(\bigcup_{k\in N\setminus\{i\}} A_k(R)\right)\setminus\{a\};R_j\right)\\ &= w\big(M\setminus A_i(R);R_j\big), \end{aligned} \tag{6}$$

where the inequality follows from additivity of R_{-i} and the assumption that $w\big(\{a\};R_k\big) \le w\big(\{a\};R_j\big)$ for each $k \in N\setminus\{i\}$ (and hence assigning object a to agent j (weakly) increases the total willingness to pay among agents in $N\setminus\{i\}$); the first equality uses additivity of R_{-i} and that $A_k(R) \subseteq M\setminus A_i$ for each $k \in N\setminus\{i\}$, along with $w\big(\{b\};R_k\big) = w\big(\{b\};R_j\big)$ for each $b \in M\setminus\big(A_i \cup \{a\}\big)$ (and hence giving objects of agents in $N\setminus\{i,j\}$, except for object a, to agent j does not change the total willingness to pay); the last equality follows from $\bigcup_{k\in N\setminus\{i\}} A_k(R) = M\setminus A_i(R)$ (which follows from $\bigcup_{k\in N} A_k(R) = M$) and additivity of R_j.

We have

$$
\begin{aligned}
&\sum_{k\in N} w(A_k;R_k) - \sum_{k\in N} w\big(A_k(R);R_k\big) \\
&= \sum_{k\in N\setminus\{i\}} \Big(w(A_k;R_k) - w\big(A_k(R);R_k\big)\Big) - w\big(A_i(R)\setminus A_i;R_i\big) \\
&\geq w(M\setminus A_i;R_j) - w\big(A_i\setminus A_i(R);R_j\big) - w\big(A_i(R)\setminus A_i;R_i\big) \\
&= w\big(A_i(R)\setminus A_i;R_j\big) - w\big(A_i(R)\setminus A_i;R_i\big) > 0,
\end{aligned}
$$

where the first equality uses additivity of R_i and $A_i(R) \supsetneq A_i$; the first inequality follows from (5) and (6); the second equality uses additivity of R_j and $A_i(R) \supsetneq A_i$; and the last inequality follows from $A_i(R) \supsetneq A_i$ and (3). Since $A \in \mathcal{A}^f$, this contradicts *constrained efficiency*.

Case 2. $A_i(R) \not\supseteq A_i$.

By $A_i(R) \not\supseteq A_i$, $A_i\setminus A_i(R) \neq \emptyset$, so we can choose some $b \in A_i\setminus A_i(R)$. Then,

$$
\begin{aligned}
&\sum_{k\in N} w(A_k;R_k) - \sum_{k\in N} w\big(A_k(R);R_k\big) \\
&= w\big(A_i\setminus A_i(R);R_i\big) - w\big(A_i(R)\setminus A_i;R_i\big) - \sum_{k\in N\setminus\{i\}} \Big(w\big(A_k(R);R_k\big) - w(A_k;R_k)\Big) \\
&\geq w\big(\{b\};R_i\big) - w(M\setminus A_i;R_i) - \sum_{k\in N\setminus\{i\}} w(M;R_k) \\
&> 0,
\end{aligned}
$$

where the first equality uses additivity of R_i; the first inequality follows from $b \in A_i\setminus A_i(R)$ and object monotonicity; and the last inequality from (4). Since $A \in \mathcal{A}^f$, this contradicts *constrained efficiency*. □

A.1.3 Interpersonal tractability of object allocations

We now examine the interpersonal tractability of admissible object allocations, corresponding to Claim 4 in the proof outline presented in Section 3.4.3. As discussed in Section 3.4.4, *constrained efficiency* may compromise interpersonal tractability by making it difficult to determine which combinations of packages are available to multiple agents. To address this issue, we draw on the implications of *equal treatment of equals*, together with the other desirable properties, to identify which combinations of packages may be available to any two agents.

Throughout this subsection, let $\mathcal{R}$ be a rich domain, and let $f \equiv (A, t)$ be a rule on $\mathcal{R}^n$ satisfying *constrained efficiency*, *no wastage*, *equal treatment of equals*, *strategy-proofness*, *individual rationality*, and *no subsidy*.

The next two lemmas form the foundation of our analysis of interpersonal tractability. Note that by richness, $\mathcal{R}^{Add} \cap \mathcal{R}^{Q} \subseteq \mathcal{R}^{Add} \subseteq \mathcal{R}$.

Lemma 3. *Let $g \equiv (A^g, t^g)$ be the restriction of f to $(\mathcal{R}^{Add} \cap \mathcal{R}^Q)^n$. Then, $\mathcal{A}^f = \mathcal{A}^g$.*

Proof. Since g is the restriction of f to $(\mathcal{R}^{Add} \cap \mathcal{R}^Q)^n$, we have $\mathcal{A}^g \subseteq \mathcal{A}^f$. To show $\mathcal{A}^f \subseteq \mathcal{A}^g$, let $A \in \mathcal{A}^f$. For each $i \in N$, let $R_i \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that for each $a \in A_i$, $w(\{a\}; R_i) = m + 1$, and for each $a \in M \backslash A_i$, $w(\{a\}; R_i) = 1$. By richness, $R \in \mathcal{R}^n$. By the definition of R, $\{A\} = \arg\max_{A' \in \mathcal{A}} \sum_{i \in N} w(A'_i; R_i)$. Moreover, since $A \in \mathcal{A}^f$, we have $\{A\} = \arg\max_{A' \in \mathcal{A}^f} \sum_{i \in N} w(A'_i; R_i)$. Thus, by *constrained efficiency* of f, we have $A(R) = A$. Since g is the restriction of f to $(\mathcal{R}^{Add} \cap \mathcal{R}^Q)^n$ and $R \in (\mathcal{R}^{Add} \cap \mathcal{R}^Q)^n$, we also have $A^g(R) = A(R) = A$. Thus, $A \in \mathcal{A}^g$. □

Lemma 4. *Let $R \in (\mathcal{R}^{\text{Add}} \cap \mathcal{R}^Q)^n$, and let $i, j \in N$ satisfy $R_i = R_j$. Then,*

$$\max_{A \in \mathcal{A}^f} \sum_{k \in N \backslash \{i\}} w(A_k; R_k) = \max_{A \in \mathcal{A}^f} \sum_{k \in N \backslash \{j\}} w(A_k; R_k).$$

Proof. Let $g \equiv (A^g, t^g)$ be the restriction of f to $(\mathcal{R}^{Add} \cap \mathcal{R}^Q)^n$. Since f satisfies *constrained efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy*, so does g. Therefore, by Fact 4, g is a constrained Vickrey rule on $(\mathcal{R}^{Add} \cap \mathcal{R}^Q)^n$. Thus, for each $k \in N$, we have

$$\begin{aligned} w\big(A^g_k(R); R_k\big) - t^g_k(R) &= w\big(A^g_k(R); R_k\big) - \left(\max_{A \in \mathcal{A}^g} \sum_{l \in N \backslash \{k\}} w(A_l; R_l) - \sum_{l \in N \backslash \{k\}} w\big(A^g_l(R); R_l\big) \right) \\ &= \sum_{l \in N} w\big(A^g_l(R); R_l\big) - \max_{A \in \mathcal{A}^g} \sum_{l \in N \backslash \{k\}} w(A_l; R_l). \end{aligned} \quad (7)$$

Since $R_i = R_j$, *equal treatment of equals* of g implies that $g_i(R)\ I_i\ g_j(R)$, so $w\big(A^g_i(R); R_i\big) - t^g_i(R) = w\big(A^g_j(R); R_j\big) - t^g_j(R)$. Moreover, $\mathcal{A}^f = \mathcal{A}^g$ (see Lemma 3). Thus, by (7),

$$\sum_{k \in N} w\big(A^g_k(R); R_k\big) - \max_{A \in \mathcal{A}^f} \sum_{k \in N \backslash \{i\}} w(A_k; R_k) = \sum_{k \in N} w\big(A^g_k(R); R_k\big) - \max_{A \in \mathcal{A}^f} \sum_{k \in N \backslash \{j\}} w(A_k; R_k).$$

Canceling $\sum_{k \in N} w\big(A^g_k(R); R_k\big)$ from both sides yields the desired equality. □

The following lemma establishes a basic form of interpersonal tractability: if a package is available to some agent under an admissible object allocation, then there exists another admissible object allocation in which a different agent receives a package that includes all objects in the original package.

Lemma 5 (Interpersonal tractability (i)). *Let $A \in \mathcal{A}^f$, and let $i, j \in N$ be two distinct agents. Then, there exists $A' \in \mathcal{A}^f$ such that $A'_j \supseteq A_i$.*

Proof. Suppose for contradiction that for each $A' \in \mathcal{A}^f$, we have $A'_j \not\supseteq A_i$. Note that $A_i \neq \emptyset$. Let $R_i \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that for each $a \in A_i$, $w(\{a\}; R_i) = m + 1$, and for each $a \in M \backslash A_i$, $w(\{a\}; R_i) = 1$. Let $R_j = R_i$, and for each $k \in N \backslash \{i, j\}$, let $R_k \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that for each $a \in M$, $w(\{a\}; R_k) = 1$. Since $A \in \mathcal{A}^f$ and $i \in N \backslash \{j\}$, we have

$$\max_{A' \in \mathcal{A}^f} \sum_{k \in N \backslash \{j\}} w(A'_k; R_k) \geq w(A_i; R_i) = (m+1)|A_i|.$$

In contrast, for each $A' \in \mathcal{A}^f$, we have $A'_j \not\supseteq A_i$, and hence $|A'_j \cap A_i| < |A_i|$. Therefore, by the construction of R_{-i}, the total willingness to pay among agents in $N \backslash \{i\}$ is at most the amount obtained when agent j receives $|A_i| - 1$ objects from A_i, and the remaining objects are allocated to agents in $N \backslash \{i\}$ (that is, agent j cannot receive all the objects in A_i). Thus,

$$\max_{A' \in \mathcal{A}^f} \sum_{k \in N \backslash \{i\}} w(A'_k; R_k) \leq (m+1)\big(|A_i| - 1\big) + |M \backslash A_i| + 1 < (m+1)|A_i|,$$

where the last inequality uses $|M \backslash A_i| < m$ (since $A_i \neq \emptyset$). Combining these inequalities, we obtain

$$\max_{A' \in \mathcal{A}^f} \sum_{k \in N \backslash \{j\}} w(A'_k; R_k) \geq (m+1)|A_i| > \max_{A' \in \mathcal{A}^f} \sum_{k \in N \backslash \{i\}} w(A'_k; R_k),$$

which contradicts Lemma 4. □

Given a non-empty set $\mathcal{M}' \subseteq \mathcal{M}$ of packages, a package $A_i \in \mathcal{M}'$ is said to be **maximal (in $\mathcal{M}'$)** if there does not exist any $A'_i \in \mathcal{M}'$ such that $A_i \subsetneq A'_i$.

The following lemma shows that if a package is maximal in those available to a given agent under the rule, then it is also available to any other agent and remains maximal in that agent's available set under the rule.

Lemma 6 (Interpersonal tractability (ii)). *Let $i, j \in N$ be two distinct agents, and let $A_i \in \mathcal{M}_i$ be a package that is maximal in $\mathcal{M}_i$. Then, (i) $A_i \in \mathcal{M}_j$, and (ii) A_i is maximal in $\mathcal{M}_j$.*

Proof. By Lemma 5, there exists $A_j \in \mathcal{M}_j$ such that $A_j \supseteq A_i$. To prove (i) and (ii), it is sufficient to show that for each $A_j \in \mathcal{M}_j$ with $A_j \supseteq A_i$, we have $A_j = A_i$. Suppose for contradiction that there exists $A_j \in \mathcal{M}_j$ such that $A_j \supsetneq A_i$. Then, again by Lemma 5, there exists $A'_i \in \mathcal{M}_i$ such that $A'_i \supseteq A_j$. Since $A_j \supsetneq A_i$, we have $A'_i \supsetneq A_i$, contradicting the assumption that A_i is maximal in $\mathcal{M}_i$. □

The following lemma states that for any two agents, if one agent has a maximal package and there exists an object not included in that package, then there exists an admissible object allocation in which the agent receives the maximal package while the other agent receives that object.

Lemma 7 (Interpersonal tractability (iii)). *Let $i, j \in N$ be two distinct agents, and let $A_i \in \mathcal{M}_i \backslash \{\emptyset\}$ be a package that is maximal in $\mathcal{M}_i$. Let $a \in M \backslash A_i$. Then, there exists $A' \in \mathcal{A}^f$ such that $A'_i = A_i$ and $a \in A'_j$.*

Proof. By $A_i \in \mathcal{M}_i$, there exists $A_{-i} \in \mathcal{M}^{n-1}$ such that $A \equiv (A_i, A_{-i}) \in \mathcal{A}^f$. Since $\mathcal{A}^f$ satisfies no wastage (which follows from *no wastage* of f) and $a \in M \backslash A_i$, there exists $k \in N \backslash \{i\}$ such that $a \in A_k$. If $k = j$, then $A \in \mathcal{A}^f$ satisfies the desired properties: agent i receives A_i and $a \in A_j$. Suppose instead that $k \neq j$.

We now establish that there exists $A' \in \mathcal{A}^f$ such that $A'_i = A_i$ and $A'_j \supseteq A_k$. Note that for such $A' \in \mathcal{A}^f$, since $a \in A_k \subseteq A'_j$, A' satisfies the desired properties.

We proceed by contradiction. Suppose that for each $A' \in \mathcal{A}^f$ with $A'_i = A_i$, it holds that $A'_j \not\supseteq A_k$. Note that $A_k \neq \emptyset$. The proof now proceeds in three steps.

Step 1. We begin by constructing a preference profile. Let $R_i \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that for each $a \in A_i$, $w\big(\{a\}; R_i\big) = (m+1)^2$, and for each $a \in M \backslash A_i$, $w\big(\{a\}; R_i\big) = 1$. For each $l \in \{j, k\}$, let $R_l \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that for each $a \in A_k$, $w\big(\{a\}; R_l\big) = m + 1$, and for each $a \in M \backslash A_k$, $w\big(\{a\}; R_l\big) = 1$. Note that $R_j = R_k$. For each $l \in N \backslash \{i, j, k\}$, let $R_l \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that for each $a \in M$, $w\big(\{a\}; R_l\big) = 1$.

Step 2. Next, we show that for each $A' \in \underset{A'' \in \mathcal{A}^f}{\arg\max} \sum_{l \in N \backslash \{k\}} w(A''_l; R_l)$, we have $A'_i = A_i$. Let $A' \in \underset{A'' \in \mathcal{A}^f}{\arg\max} \sum_{l \in N \backslash \{k\}} w(A''_l; R_l)$. We claim that $A'_i \supseteq A_i$. Suppose for contradiction that $A'_i \not\supseteq A_i$. Then, $A'_i \cap A_i \subsetneq A_i$, and hence $|A'_i \cap A_i| < |A_i|$. Therefore,

$$\begin{aligned}\sum_{l \in N\setminus\{k\}} w(A'_l; R_l) &\leq (m+1)^2|A'_i \cap A_i| + |A'_j \cap A_k|(m+1) + m - \big(|A'_i \cap A_i| + |A'_j \cap A_k|\big) \\ &\leq (m+1)^2|A'_i \cap A_i| + m(m+1) + m \\ &< (m+1)^2|A_i| \\ &= w(A_i; R_i) \\ &\leq \sum_{l \in N\setminus\{k\}} w(A_l; R_l),\end{aligned}$$

where the right-hand side of the first inequality corresponds to the case in which all objects are allocated at A'; and the third inequality uses the facts that $|A'_i \cap A_i| < |A_i|$ and $(m+1)^2 > m(m+1)+m$. However, since $A \in \mathcal{A}^f$, this contradicts $A' \in \underset{A'' \in \mathcal{A}^f}{\arg\max} \sum_{l \in N\setminus\{k\}} w(A''_l; R_l)$. Thus, we must have $A'_i \supseteq A_i$, and since A_i is maximal in $\mathcal{M}_i$, it follows that $A'_i = A_i$.

Step 3. Finally, we derive a contradiction. Let $A' \in \underset{A'' \in \mathcal{A}^f}{\arg\max} \sum_{l \in N\setminus\{k\}} w(A''_l; R_l)$. By Step 2, $A'_i = A_i$. Thus, by our assumption, $A'_j \not\supseteq A_k$. Thus, $A'_j \cap A_k \subsetneq A_k$, and hence $|A'_j \cap A_k| < |A_k|$. Then, we have

$$\begin{aligned}\max_{A'' \in \mathcal{A}^f} \sum_{i \in N\setminus\{k\}} w(A''_l; R_l) &= \sum_{l \in N\setminus\{k\}} w(A'_l; R_l) \\ &\leq (m+1)^2|A'_i \cap A_i| + (m+1)|A'_j \cap A_k| + m - |A'_i \cap A_i| - |A'_j \cap A_k| \\ &\leq (m+1)^2|A_i| + (m+1)|A'_j \cap A_k| + m \\ &< (m+1)^2|A_i| + (m+1)|A_k| \\ &= w(A_i; R_i) + w(A_k; R_k) \\ &\leq \sum_{l \in N\setminus\{j\}} w(A_l; R_l) \\ &\leq \max_{A'' \in \mathcal{A}^f} \sum_{l \in N\setminus\{j\}} w(A''_l; R_l),\end{aligned}$$

where the right-hand side of the first inequality corresponds to the case in which all objects are allocated at A'; the second inequality uses $A'_i = A_i$; the third inequality uses $|A'_j \cap A_k| < |A_k|$ and $m+1 > m$; and the last follows from $A \in \mathcal{A}^f$. However, this contradicts Lemma 4, completing the proof. □

A.1.4 Existence of preferences in a rich domain

We examine the existence of an additive preference that demands a package A_i at a payment vector τ and another package A'_i at another payment vector τ'. The lemmas presented here correspond to Claim 3 in the proof outline provided in Section 3.4.3.

As discussed in Section 3.4.4, if we were allowed to choose arbitrary non-additive, non-quasi-linear preferences from the domain, then constructing such a preference would be relatively straightforward. The main challenge, however, lies in constructing an *additive*, non-quasi-linear preference.

We begin with the following remark, which describes how to construct a non-quasi-linear preference by interpolating between two given quasi-linear preferences.

Remark 4 (Figure 6). Let $A_i, A'_i \in \mathcal{M}$, and let $\tau_{A_i}, \tau'_{A'_i} \in \mathbb{R}$. Let $R_i, R'_i \in \mathcal{R}^Q$ be two quasi-linear preferences satisfying that for each $A''_i \in \mathcal{M}$, $w(A''_i; R_i) - w(A_i; R_i) + \tau_{A_i} < w(A''_i; R'_i) - w(A'_i; R'_i) + \tau'_{A'_i}$. Let $\tau'_{A_i} \equiv w(A_i, R'_i) - w(A'_i; R'_i) + \tau'_{A'_i}$, so $\tau_{A_i} < \tau'_{A_i}$. Let $\alpha : [\tau_{A_i}, \tau'_{A_i}] \to [0, 1]$ be such that for each $t_i \in [\tau_{A_i}, \tau'_{A_i}]$, $\alpha(t_i) = \frac{t_i - \tau_{A_i}}{\tau'_{A_i} - \tau_{A_i}}$. We now define a preference R''_i as follows: for each $A''_i \in \mathcal{M}$ and each $t_i \in \mathbb{R}$,

$$
\begin{aligned}
&t_i - V\big(A''_i, (A_i, t_i); R''_i\big) \\
&= \begin{cases} w(A_i; R_i) - w(A''_i; R_i) & \text{if } t_i \le \tau_{A_i}, \\ \big(1 - \alpha(t_i)\big)\big(w(A_i; R_i) - w(A''_i; R_i)\big) + \alpha(t_i)\big(w(A_i; R'_i) - w(A''_i; R'_i)\big) & \text{if } \tau_{A_i} \le t_i \le \tau'_{A_i}, \\ w(A_i; R'_i) - w(A''_i; R'_i) & \text{if } t_i \ge \tau'_{A_i}. \end{cases}
\end{aligned}
$$

Note that R''_i satisfies the following properties: for each $A''_i \in \mathcal{M}$,

$$
\begin{aligned}
\tau_{A_i} - V\big(A''_i, (A_i, \tau_{A_i}); R''_i\big) &= w(A_i; R_i) - w(A''_i; R_i), \\
\tau'_{A'_i} - V\big(A''_i, (A'_i, \tau'_{A'_i}); R''_i\big) &= w(A'_i; R'_i) - w(A''_i; R'_i).
\end{aligned}
$$

Figure 6 is an illustration of Remark 4. Remark 4 states that given two quasi-linear preferences R_i, R'_i such that for each $A''_i \in \mathcal{M}$, $w(A''_i; R_i) - w(A_i; R_i) + \tau_{A_i} < w(A''_i; R'_i) - w(A'_i; R'_i) + \tau'_{A'_i}$—that is, the indifference curve of R_i through (A_i, τ_{A_i}) lies entirely to the left of that of R'_i through $(A'_i, \tau'_{A'_i})$—we can construct a preference R''_i with the following properties:

(i) Each indifference curve to the left of (A_i, τ_{A_i}) is parallel to that of R_i.

(ii) Each indifference curve to the right of $(A'_i, \tau'_{A'_i})$ is parallel to that of R'_i.

(iii) Each indifference curve in between is formed by interpolating between the two, that is, by taking a convex combination of the corresponding willingness to pay for R_i and R'_i.

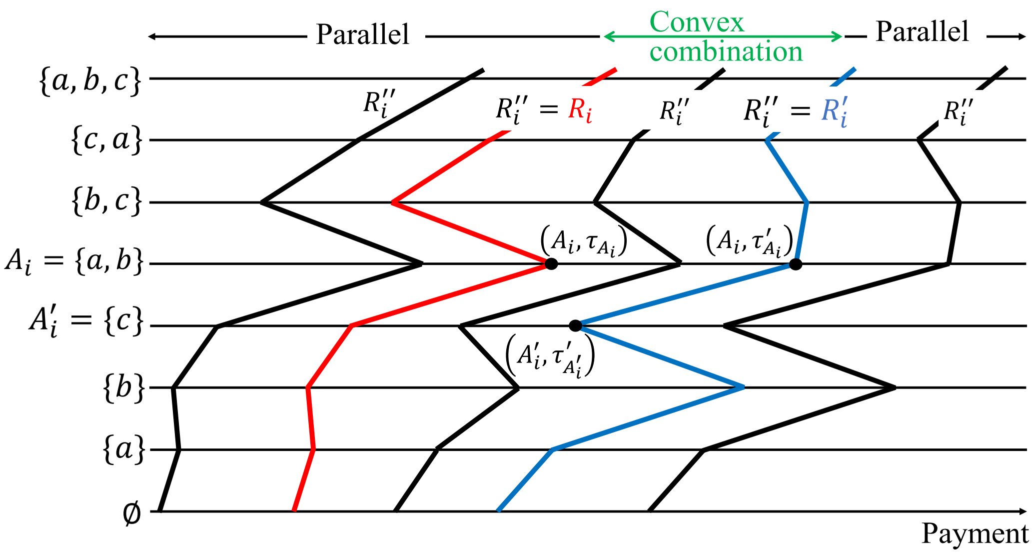

Figure 6: An illustration of Remark 4.

Given a non-empty set $\mathcal{M}' \subseteq \mathcal{M}$ of packages, a payment vector $\tau \in \mathbb{R}^{|\mathcal{M}'|}$ is said to be **object monotonic** (**on** $\mathcal{M}'$) if for each $A_i, A_i' \in \mathcal{M}'$ with $A_i' \supsetneq A_i$, it holds that $\tau_{A_i'} > \tau_{A_i}$.

Two distinct packages $A_i, A_i' \in \mathcal{M}'$ are said to be **adjacent** (**in** $\mathcal{M}'$) if one of the following holds:

(i) $A_i \subsetneq A_i'$, and there is no $A_i'' \in \mathcal{M}'$ such that $A_i \subsetneq A_i'' \subsetneq A_i'$; or

(ii) $A_i' \subsetneq A_i$, and there is no $A_i'' \in \mathcal{M}'$ such that $A_i' \subsetneq A_i'' \subsetneq A_i$.

The following lemma provides a sufficient condition for the existence of an additive non-quasi-linear preference that demands package A_i at a given payment vector τ on $\mathcal{M}'$, and demands another package A_i'—which is adjacent to A_i—at another payment vector τ' on $\mathcal{M}'$. Figure 7 illustrates the lemma. In the figure, a solid horizontal line indicates that the corresponding package is included in $\mathcal{M}'$, while a dotted horizontal line indicates that it is not. Accordingly, in Figure 7, $\mathcal{M}' = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}\}$.

Lemma 8 (Figure 7). *Let $\mathcal{M}' \subseteq \mathcal{M}$ be a non-empty set of packages. Let $A_i, A_i' \in \mathcal{M}'$ be two adjacent packages in $\mathcal{M}'$, satisfying the following conditions: (i) $A_i \neq \emptyset$, (ii) $A_i \subsetneq A_i'$, and (iii) for each $A_i'' \in \mathcal{M}'$ with $A_i'' \cap A_i' \neq \emptyset$, $A_i'' \subseteq A_i'$. Let $\tau, \tau' \in \mathbb{R}^{|\mathcal{M}'|}$ be two payment vectors on $\mathcal{M}'$ such that: (iv) $\tau_{A_i} < \tau'_{A_i}$, and (v) τ, τ' are object monotonic on $\mathcal{M}'$. Then, there exists an additive preference that demands A_i at τ on $\mathcal{M}'$, and A_i' at τ' on $\mathcal{M}'$.*

Proof. The proof proceeds in four steps.

Step 1. We begin by constructing two additive quasi-linear preferences. Let $K_i, L_i, \varepsilon_i \in \mathbb{R}_{++}$

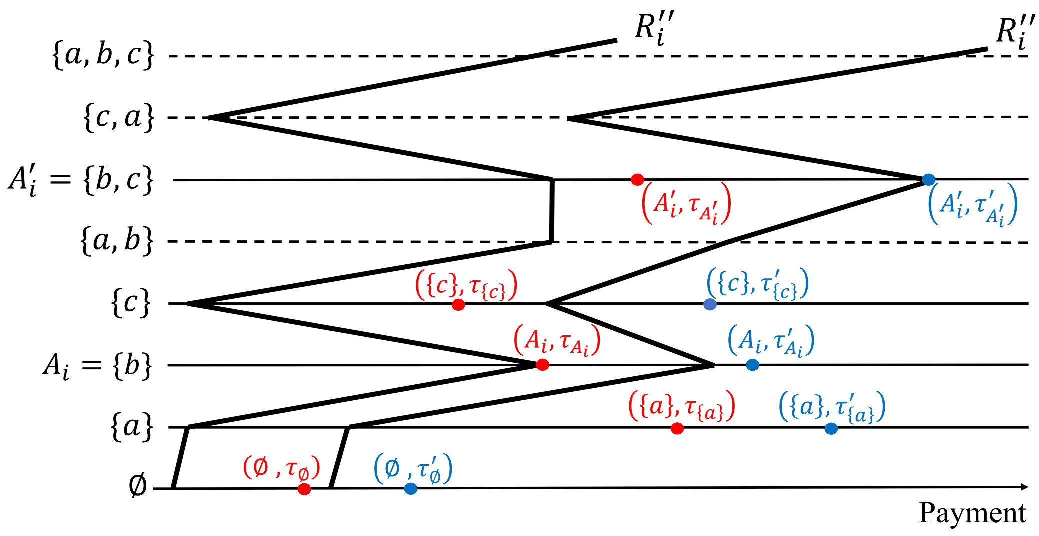

Figure 7: An illustration of Lemma 8.

be a triple of positive constants, where K_i is sufficiently large and ε_i is sufficiently small, such that the following conditions are satisfied:

- For each $A_i'' \in \mathcal{M}'$,

$$K_i > \varepsilon_i m + \max\left\{\tau_{A_i} - \tau_{A_i''}, \tau'_{A_i'} - \tau'_{A_i''}\right\}. \tag{8}$$

- For each $A_i'' \in \mathcal{M}'$ with $A_i'' \supsetneq A_i$,

$$\varepsilon_i|A_i''\backslash A_i| < \tau_{A_i''} - \tau_{A_i}, \tag{9}$$

$$\tau'_{A_i'} - \tau'_{A_i} < L_i|A_i'\backslash A_i| < \tau'_{A_i'} - \tau_{A_i}. \tag{10}$$

Since τ and τ' are object monotonic on $\mathcal{M}'$, we can always choose a sufficiently small $\varepsilon_i > 0$ satisfying (9). Furthermore, given object monotonicity of τ', the assumption that $A_i \subsetneq A_i'$, and that $\tau_{A_i} < \tau'_{A_i}$, we can select $L_i > 0$ satisfying (10).

Let $R_i \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that for each $a \in A_i$, $w(\{a\}; R_i) = K_i$, and each $a \in M\backslash A_i$, $w(\{a\}; R_i) = \varepsilon_i$. Similarly, let $R_i' \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that for each $a \in A_i$, $w(\{a\}; R_i') = K_i$, for each $a \in A_i'\backslash A_i$, $w(\{a\}; R_i') = L_i$, and for each $a \in M\backslash A_i'$, $w(\{a\}; R_i') = \varepsilon_i$.

Step 2. Next, we construct an additive non-quasi-linear preference based on R_i and R_i'. For each $A''_i \in \mathcal{M}$, observe that

$$\begin{aligned} w(A''_i; R_i) - w(A_i; R_i) + \tau_{A_i} &= \varepsilon_i |A''_i \setminus A_i| - K_i |A_i \setminus A''_i| + \tau_{A_i} \\ &< \varepsilon_i |A''_i \setminus A'_i| - K_i |A_i \setminus A''_i| - L_i |A'_i \setminus A_i| + \tau'_{A'_i} \\ &\leq \varepsilon_i |A''_i \setminus A'_i| - K_i |A_i \setminus A''_i| - L_i |A'_i \setminus (A_i \cup A''_i)| + \tau'_{A'_i} \\ &= w(A''_i; R'_i) - w(A'_i; R'_i) + \tau'_{A'_i}, \end{aligned}$$

where the first inequality follows from (10); and the second inequality uses $A'_i \setminus (A_i \cup A''_i) \subseteq A'_i \setminus A_i$ (and hence $|A'_i \setminus (A_i \cup A''_i)| \leq |A'_i \setminus A_i|$). This confirms that the indifference curve of R_i through (A_i, τ_{A_i}) lies entirely to the left of that of R'_i through $(A'_i, \tau'_{A'_i})$. Hence, by Remark 4, we can define a preference $R''_i \in \mathcal{R}$ such that for each $A''_i \in \mathcal{M}$,

$$\tau_{A_i} - V\big(A''_i, (A_i, \tau_{A_i}); R''_i\big) = w(A_i; R_i) - w(A''_i; R_i), \tag{11}$$

$$\tau'_{A'_i} - V\big(A''_i, (A'_i, \tau'_{A'_i}); R''_i\big) = w(A'_i; R'_i) - w(A''_i; R'_i). \tag{12}$$

By additivity of R_i, R'_i, we can ensure that $R''_i \in \mathcal{R}^{Add}$ (see Remark 4).

Step 3. We now show that R''_i demands A_i at τ on $\mathcal{M}'$. Let $A''_i \in \mathcal{M}' \setminus \{A_i\}$. If $A''_i \not\supseteq A_i$, then $A_i \setminus A''_i \neq \emptyset$, and we have

$$\tau_{A_i} - V\big(A''_i, (A_i, \tau_{A_i}); R''_i\big) = w(A_i; R_i) - w(A''_i; R_i) = K_i |A_i \setminus A''_i| - \varepsilon_i |A''_i \setminus A_i| > \tau_{A_i} - \tau_{A''_i},$$

where the first equality follows from (11); and the inequality from (8) and $A_i \setminus A''_i \neq \emptyset$. Instead, if $A''_i \supsetneq A_i$, then

$$\tau_{A_i} - V\big(A''_i, (A_i, \tau_{A_i}); R''_i\big) = w(A_i; R_i) - w(A''_i; R_i) = -\varepsilon_i |A''_i \setminus A_i| > \tau_{A_i} - \tau_{A''_i},$$

where the first equality follows from (11); the second equality from $A''_i \supsetneq A_i$; and the inequality from (9) and $A''_i \supsetneq A_i$. In both cases, we obtain $\tau_{A_i} - V\big(A''_i, (A_i, \tau_{A_i}); R''_i\big) > \tau_{A_i} - \tau_{A''_i}$, or equivalently, $V\big(A''_i, (A_i, \tau_{A_i}); R''_i\big) < \tau_{A''_i}$. This implies that $(A_i, \tau_{A_i})\ P''_i\ (A''_i, \tau_{A''_i})$. Since $A''_i \in \mathcal{M}' \setminus \{A_i\}$ was arbitrary, we conclude that R''_i demands A_i at τ on $\mathcal{M}'$.

Step 4. Finally, we show that R'_i demands A'_i at τ' on $\mathcal{M}'$. Let $A''_i \in \mathcal{M}' \setminus \{A'_i\}$. We consider two cases.

Case 1. $A''_i \cap A'_i = \emptyset$.

Since $A_i \subsetneq A'_i$, $A_i \neq \emptyset$ and $A''_i \cap A'_i = \emptyset$, it follows that $A_i \backslash A''_i \neq \emptyset$. Then,

$$\tau'_{A'_i} - V\big(A''_i, (A'_i, \tau'_{A'_i}); R''_i\big) = w(A'_i; R'_i) - w(A''_i; R'_i)$$
$$= |A_i \backslash A''_i| K_i + \big|A'_i \backslash (A_i \cup A''_i)\big| L_i - |A''_i \backslash A'_i| \varepsilon_i > \tau'_{A'_i} - \tau'_{A''_i},$$

where the first equality follows from (12), and the inequality from (8) and $A_i \backslash A''_i \neq \emptyset$.

Case 2. $A''_i \cap A'_i \neq \emptyset$.

By assumption, $A''_i \subseteq A'_i$. Thus, by $A''_i \neq A'_i$, we have $A''_i \subsetneq A'_i$. Since $A_i \subsetneq A'_i$, and A_i and A'_i are adjacent in $\mathcal{M}'$, we have either $A''_i = A_i$ or $A_i \not\subseteq A''_i$. If $A''_i = A_i$, then

$$\tau'_{A'_i} - V\big(A''_i, (A'_i, \tau'_{A'_i}); R''_i\big) = w(A'_i; R'_i) - w(A_i; R'_i) = |A'_i \backslash A_i| L_i > \tau'_{A'_i} - \tau'_{A''_i},$$

where the first equality follows from (12); the second equality uses $A_i \subsetneq A'_i$; and the inequality from (10). Instead, if $A_i \not\subseteq A''_i$, then $A_i \backslash A''_i \neq \emptyset$, so we have

$$\tau'_{A'_i} - V\big(A''_i, (A'_i, \tau'_{A'_i}); R''_i\big) = w(A'_i; R'_i) - w(A''_i; R'_i)$$
$$= |A_i \backslash A''_i| K_i + \big|A'_i \backslash (A_i \cup A''_i)\big| L_i - |A''_i \backslash A'_i| \varepsilon_i > \tau'_{A'_i} - \tau'_{A''_i},$$

where the first equality follows from (12), and the inequality uses (8) and $A_i \backslash A''_i \neq \emptyset$.

In all cases, we have $\tau'_{A'_i} - V\big(A''_i, (A'_i, \tau'_{A'_i}); R''_i\big) > \tau'_{A'_i} - \tau''_{A''_i}$, or equivalently, $V\big(A''_i, (A'_i, \tau'_{A'_i}); R''_i\big) < \tau'_{A''_i}$. Therefore, $(A'_i, \tau'_{A'_i})\ P''_i\ (A''_i, \tau'_{A''_i})$. Since $A''_i \in \mathcal{M}' \backslash \{A'_i\}$ was arbitrary, we conclude that R''_i demands A'_i at τ' on $\mathcal{M}'$. □

Furthermore, the next lemma provides a sufficient condition for the existence of an additive non-quasi-linear preference that demands a package A_i at a given payment vector τ on $\mathcal{M}'$, and a maximal package A'_i satisfying $A'_i \cap A_i = \emptyset$ at another given payment vector τ' on $\mathcal{M}'$. Figure 8 illustrates this result, where $\mathcal{M}' = \big\{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}\big\}$.

Lemma 9 (Figure 8). *Let $\mathcal{M}' \subseteq \mathcal{M}$ be a non-empty set of packages. Let $A_i, A'_i \in \mathcal{M}'$ be two packages satisfying the following conditions: (i) $A'_i \neq \emptyset$, (ii) $A_i \cap A'_i = \emptyset$, and (iii) A'_i is maximal in $\mathcal{M}'$. Let $\tau, \tau' \in \mathbb{R}^{|\mathcal{M}'|}$ be two payment vectors on $\mathcal{M}'$ such that: (iv) $\tau_{A_i} < \tau'_{A_i}$, and (v) τ, τ' are object monotonic on $\mathcal{M}'$. Then, there exists an additive preference that demands A_i at τ on $\mathcal{M}'$ and A'_i at τ' on $\mathcal{M}'$.*

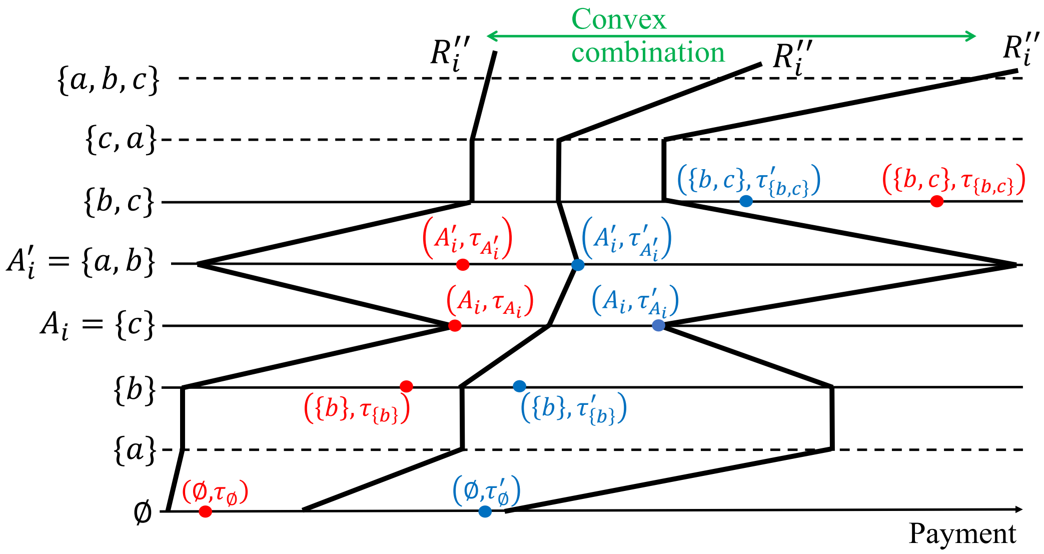

Figure 8: An illustration of Lemma 9.

Proof. We proceed with the proof in four steps.

STEP 1. We begin by constructing two additive quasi-linear preferences. Let $K_i, L_i, \varepsilon_i \in \mathbb{R}_{++}$ be a triple of positive constants, where K_i and L_i are sufficiently large and ε_i is sufficiently small, satisfying the following conditions:

- $L_i \geq K_i$.
- For each $A_i'' \in \mathcal{M}'$,

$$K_i > \varepsilon_i m + \max\left\{\tau_{A_i} - \tau_{A_i''}, \tau_{A_i} - \tau'_{A_i''}, \tau'_{A_i'} - \tau'_{A_i}\right\}. \tag{13}$$

- For each $A_i'' \in \mathcal{M}'$ with $A_i' \not\subseteq A_i''$,

$$\begin{aligned}&\frac{\tau'_{A_i'} - \tau_{A_i} + K_i|A_i| - \varepsilon_i|A_i'|}{L_i|A_i'| - \varepsilon_i|A_i| + K_i|A_i| - \varepsilon_i|A_i'|}\\ &< \frac{\tau'_{A_i''} - \tau'_{A_i} + K_i|A_i\backslash A_i''| - \varepsilon_i|A_i''\backslash A_i|}{L_i|A_i' \cap A_i''| + \varepsilon_i|A_i''\backslash A_i'| - \varepsilon_i|A_i| + K_i|A_i\backslash A_i''| - \varepsilon_i|A_i''\backslash A_i|}.\end{aligned} \tag{14}$$

- For each $A_i'' \in \mathcal{M}'$ with $A_i'' \supsetneq A_i$,

$$\varepsilon_i|A_i''\backslash A_i| < \min\left\{\tau_{A_i''} - \tau_{A_i}, \tau'_{A_i''} - \tau_{A_i}\right\}. \tag{15}$$

- $\varepsilon_i|A_i| < \tau'_{A_i} - \tau_{A_i}$.

For each $A_i'' \in \mathcal{M}'$ with $A_i' \not\subseteq A_i''$, $A_i' \backslash A_i'' \neq \emptyset$, so $A_i' \cap A_i'' \subsetneq A_i'$. Thus, we can choose L_i satisfying (14). Furthermore, since τ and τ' are object monotonic on $\mathcal{M}'$ and $\tau_{A_i} < \tau'_{A_i}$, we can choose $\varepsilon_i \in \mathbb{R}_{++}$ satisfying (15) and $\varepsilon_i |A_i| < \tau'_{A_i} - \tau_{A_i}$.

Let $R_i \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that for each $a \in A_i$, $w(\{a\}; R_i) = K_i$, and for each $a \in M \backslash A_i$, $w(\{a\}; R_i) = \varepsilon_i$. Similarly, let $R_i' \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that for each $a \in A_i'$, $w(\{a\}; R_i') = L_i$, and for each $a \in M \backslash A_i'$, $w(\{a\}; R_i') = \varepsilon_i$. Note that since $A_i \cap A_i' = \emptyset$, we have $w(A_i; R_i') = \varepsilon_i |A_i|$ and $w(A_i'; R_i') = L_i |A_i'|$.

Step 2. Next, we construct an additive non-quasi-linear preference based on the two quasi-linear preferences R_i and R_i' constructed in Step 1. For each $A_i'' \in \mathcal{M}$, we have

$$\begin{aligned} w(A_i''; R_i) - w(A_i; R_i) + \tau_{A_i} &= \varepsilon_i |A_i'' \backslash A_i| - K_i |A_i \backslash A_i''| + \tau_{A_i} \\ &\leq L_i |A_i'' \cap A_i'| + \varepsilon_i |A_i'' \backslash A_i| + \tau_{A_i} \\ &< L_i |A_i'' \cap A_i'| + \varepsilon_i |A_i'' \backslash A_i'| - \varepsilon_i |A_i| + \tau'_{A_i} \\ &= w(A_i''; R_i') - w(A_i; R_i') + \tau'_{A_i}, \end{aligned}$$

where the second inequality follows from $\varepsilon_i |A_i| < \tau'_{A_i} - \tau_{A_i}$. This implies that the indifference curve of R_i through (A_i, τ_{A_i}) lies entirely to the left of that of R_i' through (A_i, τ'_{A_i}). Thus, by Remark 4, the following preference $R_i'' \in \mathcal{R}$ is well-defined: for each $A_i'' \in \mathcal{M}$ and each $t_i \in \mathbb{R}$,

$$\begin{aligned} &t_i - V\big(A_i'', (A_i, t_i); R_i''\big) \\ &= \begin{cases} w(A_i; R_i) - w(A_i''; R_i) & \text{if } t_i \leq \tau_{A_i}, \\ \big(1 - \alpha(t_i)\big)\big(w(A_i; R_i) - w(A_i''; R_i)\big) + \alpha(t_i)\big(w(A_i; R_i') - w(A_i''; R_i')\big) & \text{if } \tau_{A_i} \leq t_i \leq \tau'_{A_i}, \\ w(A_i; R_i') - w(A_i''; R_i') & \text{if } t_i \geq \tau'_{A_i}, \end{cases} \end{aligned}$$

where $\alpha : [\tau_{A_i}, \tau'_{A_i}] \to [0, 1]$ is a function such that for each $t_i \in [\tau_{A_i}, \tau'_{A_i}]$, $\alpha(t_i) = \frac{t_i - \tau_{A_i}}{\tau'_{A_i} - \tau_{A_i}}$. By additivity of R_i, R_i', we have $R_i'' \in \mathcal{R}^{Add}$. By Remark 4, we also have that for each $A_i'' \in \mathcal{M}$,

$$\tau_{A_i} - V\big(A_i'', (A_i, \tau_{A_i}); R_i''\big) = w(A_i; R_i) - w(A_i''; R_i), \tag{16}$$

$$\tau'_{A_i} - V\big(A_i'', (A_i, \tau'_{A_i}); R_i''\big) = w(A_i; R_i') - w(A_i''; R_i'). \tag{17}$$

Using (13), (15), and (16), we can show that R_i'' demands A_i at τ on $\mathcal{M}'$ as in Step 3 of Lemma 8. In the next two steps, we will demonstrate that R_i'' demands A_i' at τ' on $\mathcal{M}'$.

Step 3. We now claim that for each $A_i'' \in \mathcal{M}'\backslash\{A_i\}$, it holds that $(A_i, \tau_{A_i})\ P_i''\ (A_i', \tau'_{A_i'})$. This can be established in the same manner as in Step 3 of Lemma 8, using (13), (15), and (16). We therefore omit the details.

Step 4. Finally, we show that R_i'' demands A_i' at τ' on $\mathcal{M}'$.

We first establish that $(A_i', \tau'_{A_i'})\ P_i''\ (A_i, \tau'_{A_i})$. Observe that

$$\tau'_{A_i} - V\big(A_i', (A_i, \tau'_{A_i}); R_i''\big) = w(A_i; R_i') - w(A_i'; R_i') = \varepsilon_i|A_i| - L_i|A_i'| \leq \varepsilon_i|A_i| - K_i|A_i'| < \tau'_{A_i} - \tau'_{A_i'},$$

where the first equality follows from (17); the first inequality from $L_i \geq K_i$; and the second inequality uses (13) and $A_i' \neq \emptyset$. Therefore, we have $V\big(A_i', (A_i, \tau'_{A_i}); R_i''\big) > \tau'_{A_i'}$, which implies that $(A_i', \tau'_{A_i'})\ P_i''\ (A_i, \tau'_{A_i})$.

Let $A_i'' \in \mathcal{M}'\backslash\{A_i, A_i'\}$. We consider the following two cases.

Case 1. $w(A_i; R_i') - \tau'_{A_i} \geq w(A_i''; R_i') - \tau'_{A_i''}$.

Then,

$$\tau'_{A_i} - V\big(A_i'', (A_i, \tau'_{A_i}); R_i''\big) = w(A_i; R_i') - w(A_i''; R_i') \geq \tau'_{A_i} - \tau'_{A_i''},$$

where the equality follows from (17), and the inequality uses the assumption that $w(A_i; R_i') - \tau'_{A_i} \geq w(A_i''; R_i') - \tau'_{A_i''}$. Thus, $V\big(A_i'', (A_i, \tau'_{A_i}); R_i''\big) \leq \tau'_{A_i''}$, so $(A_i, \tau'_{A_i})\ R_i''\ (A_i'', \tau'_{A_i''})$. This, together with $(A_i', \tau'_{A_i'})\ P_i''\ (A_i, \tau'_{A_i})$, implies that $(A_i', \tau'_{A_i'})\ P_i''\ (A_i'', \tau'_{A_i''})$.

Case 2. $w(A_i''; R_i') - \tau'_{A_i''} \geq w(A_i; R_i') - \tau'_{A_i}$.

Let $t_{A_i''} \equiv V\big(A_i, (A_i'', \tau'_{A_i''}); R_i''\big)$. Then, in a similar way to Case 1, we can show that $(A_i'', \tau'_{A_i''})\ R_i''\ (A_i, \tau'_{A_i})$, Thus, $(A_i, t_{A_i''})\ I_i''\ (A_i'', \tau'_{A_i''})\ R_i''\ (A_i, \tau'_{A_i})$, which implies that $t_{A_i''} \leq \tau'_{A_i}$. By Step 3, $(A_i, \tau_{A_i})\ P_i''\ (A_i'', \tau'_{A_i''})$. Thus, $(A_i, \tau_{A_i})\ P_i''\ (A_i'', \tau'_{A_i''})\ I_i''\ (A_i, t_{A_i''})$, which implies $t_{A_i''} > \tau_{A_i}$. Thus, $t_{A_i''} \in [\tau_{A_i}, \tau'_{A_i}]$. Thus,

$$\begin{aligned} t_{A_i''} - \tau'_{A_i''} &= t_{A_i''} - V\big(A_i'', (A_i, t_{A_i''}); R_i''\big) \\ &= \big(1 - \alpha(t_{A_i''})\big)\big(w(A_i; R_i) - w(A_i''; R_i)\big) + \alpha(t_{A_i''})\big(w(A_i; R_i') - w(A_i''; R_i')\big), \end{aligned}$$

where the first equality follows from $\tau'_{A_i''} = V\big(A_i'', (A_i, t_{A_i''}); R_i''\big)$, which in turn follows from $(A_i, t_{A_i''})\ I_i''\ (A_i'', \tau'_{A_i''})$; and the second equality follows from the definition of R_i''. Rearranging this, we obtain

$$\alpha(t_{A_i''}) = \frac{\tau'_{A_i''} - t_{A_i''} + \big(w(A_i;R_i) - w(A_i'';R_i)\big)}{\big(w(A_i'';R_i') - w(A_i;R_i')\big) + \big(w(A_i;R_i) - w(A_i'';R_i)\big)}$$
$$= \frac{\tau'_{A_i''} - t_{A_i''} + K_i|A_i\backslash A_i''| - \varepsilon_i|A_i''\backslash A_i|}{L_i|A_i''\cap A_i'| + \varepsilon_i|A_i''\backslash A_i'| - \varepsilon_i|A_i| + K_i|A_i\backslash A_i''| - \varepsilon_i|A_i''\backslash A_i|}.$$

Let $t_{A_i'} \equiv V\big(A_i,(A_i',\tau'_{A_i'});R_i''\big)$. By $(A_i,t_{A_i'})\ I_i''\ (A_i',\tau'_{A_i'})\ P_i''\ (A_i,\tau'_{A_i})$, we have $t_{A_i'} < \tau'_{A_i}$. By Step 3, $(A_i,\tau_{A_i})\ P_i''\ (A_i',\tau'_{A_i'})\ I_i''\ (A_i,t_{A_i'})$, which implies $t_{A_i'} > \tau_{A_i}$. Thus, $t_{A_i'} \in [\tau_{A_i},\tau'_{A_i}]$. Thus, in the same way as above, we can show that

$$\alpha(t_{A_i'}) = \frac{\tau'_{A_i'} - t_{A_i'} + \big(w(A_i;R_i) - w(A_i';R_i)\big)}{\big(w(A_i';R_i') - w(A_i;R_i')\big) + \big(w(A_i;R_i) - w(A_i';R_i)\big)}$$
$$= \frac{\tau'_{A_i'} - t_{A_i'} + K_i|A_i| - \varepsilon_i|A_i|}{L_i|A_i'| - \varepsilon_i|A_i| + K_i|A_i| - \varepsilon_i|A_i'|}.$$

Since A_i' is maximal in $\mathcal{M}'$, $A_i' \not\subseteq A_i''$. Then,

$$\begin{aligned}
\alpha(t_{A_i'}) &= \frac{\tau'_{A_i'} - t_{A_i'} + K_i|A_i| - \varepsilon_i|A_i|}{L_i|A_i'| - \varepsilon_i|A_i| + K_i|A_i| - \varepsilon_i|A_i'|}\\
&\leq \frac{\tau'_{A_i'} - \tau_{A_i} + K_i|A_i| - \varepsilon_i|A_i|}{L_i|A_i'| - \varepsilon_i|A_i| + K_i|A_i| - \varepsilon_i|A_i'|}\\
&< \frac{\tau'_{A_i''} - \tau'_{A_i} + K_i|A_i\backslash A_i''| - \varepsilon_i|A_i''\backslash A_i|}{L_i|A_i''\cap A_i'| + \varepsilon_i|A_i''\backslash A_i'| - \varepsilon_i|A_i| + K_i|A_i\backslash A_i''| - \varepsilon_i|A_i''\backslash A_i|}\\
&\leq \frac{\tau'_{A_i''} - t_{A_i''} + K_i|A_i\backslash A_i''| - \varepsilon_i|A_i''\backslash A_i|}{L_i|A_i''\cap A_i'| + \varepsilon_i|A_i''\backslash A_i'| - \varepsilon_i|A_i| + K_i|A_i\backslash A_i''| - \varepsilon_i|A_i''\backslash A_i|}\\
&= \alpha(t_{A_i''}),
\end{aligned}$$

where the first inequality follows from $t_{A_i'} \geq \tau_{A_i}$; the second inequality uses (14) and $A_i' \not\subseteq A_i''$; and the third inequality uses $t_{A_i''} \leq \tau'_{A_i}$. Since $\alpha(t_i) = \frac{t_i - \tau_{A_i}}{\tau'_{A_i} - \tau_{Ai}}$ is an increasing function in t_i, $\alpha(t_{A_i'}) < \alpha(t_{A_i''})$ implies that $t_{A_i'} < t_{A_i''}$. Thus, $V\big(A_i,(A_i',\tau'_{A_i'});R_i''\big) = t_{A_i'} < t_{A_i''} = V\big(A_i,(A_i'',\tau'_{A_i''});R_i''\big)$, which implies that $(A_i',\tau'_{A_i'})\ P_i''\ (A_i'',\tau'_{A_i''})$.

Thus, for each $A_i'' \in \mathcal{M}'\backslash\{A_i'\}$, $(A_i',\tau'_{A_i'})\ P_i''\ (A_i'',\tau'_{A_i''})$, so R_i'' demands A_i' at τ' on $\mathcal{M}'$. This completes the proof. □

A.2 Proof of Proposition

We now proceed to the proof of Proposition. Let $f \equiv (A,t)$ be a rule on $\mathcal{R}^n$ that satisfies *constrained efficiency*, *no wastage*, *equal treatment of equals*, *strategy-proofness*, *individual* *rationality*, and *no subsidy*.

We begin with a brief outline of the proof. To show $\mathcal{A}^f = \mathcal{C}^*(\mathcal{B})$ for some partition $\mathcal{B}$ of M with $|\mathcal{B}| \leq n$, it suffices to show the following four properties:

(i) $\mathcal{A}^f$ is a $\mathcal{B}$-bundling unit-demand constraint for some partition $\mathcal{B}$ of M.

(ii) $|\mathcal{B}| \leq n$.

(iii) $\mathcal{A}^f$ satisfies no wastage.

(iv) $\mathcal{A}^f$ satisfies anonymity.

Among these properties, (iii) follows directly from *no wastage* of f. Moreover, once (i) is established, (ii) follows as a consequence of *no wastage*. Thus, it remains to prove that $\mathcal{A}^f$ satisfies properties (i) and (iv). In the first part (Appendix A.2.1), we prove property (i). This part, outlined in Section 3.4.3, is the most technically involved in the paper and repeatedly relies on no wastage of $\mathcal{A}^f$ (i.e., property (iii)). In the second part (Appendix A.2.2), we prove property (iv). A simple outline of the proof of this part in the two-agent, two-object case is provided in Section 3.4.5. This part builds on the result that $\mathcal{A}^f$ is a bundling unit-demand constraint (i.e., property (i)) and again relies on no wastage of $\mathcal{A}^f$.

A.2.1 Bundling unit-demand constraint

In this subsection, we show that there exists a partition $\mathcal{B}$ of M such that $\mathcal{A}^f$ is a $\mathcal{B}$-bundling unit-demand constraint and $|\mathcal{B}| \leq n$. The proof proceeds in three steps.

Step 1. We first show that for each $i \in N$ and each distinct $A_i, A'_i \in \mathcal{M}_i \backslash \{\emptyset\}$, we have $A_i \cap A'_i = \emptyset$. Suppose for contradiction that there exist $i \in N$ and distinct $A_i, A'_i \in \mathcal{M}_i \backslash \{\emptyset\}$ such that $A_i \cap A'_i \neq \emptyset$. The proof basically follows the outline provided in Section 3.4.3.

If A'_i is not maximal in $\mathcal{M}_i$, then there exists $A''_i \in \mathcal{M}_i$ such that $A''_i \supsetneq A'_i$. Since $A_i \cap A'_i \neq \emptyset$, it follows that $A_i \cap A''_i \neq \emptyset$. Therefore, without loss of generality, we may assume that A'_i is maximal in $\mathcal{M}_i$.

Since $\mathcal{R}$ is rich, we have $\mathcal{R}^{Add} \cap \mathcal{R}^Q \subseteq \mathcal{R}^{Add} \subseteq \mathcal{R}$. Let g be the restriction of f on $\mathcal{R}^n$ to $(\mathcal{R}^{Add} \cap \mathcal{R}^Q)^n$. Since f satisfies *constrained efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy*, its restriction g also satisfies these properties. Thus, by Fact 3, g is a constrained Vickrey rule. Thus, for each $R \in (\mathcal{R}^{Add} \cap \mathcal{R}^Q)^n$, $f(R) = g(R)$ is an outcome of a constrained Vickrey rule, and hence

$$\begin{aligned} t_i(R) = t_i\big(R_{-i}; A_i(R)\big) &= \max_{A'' \in \mathcal{A}^g} \sum_{j \in N\setminus\{i\}} w(A''_j; R_j) - \sum_{j \in N\setminus\{i\}} w(A_j(R); R_j) \\ &= \max_{A'' \in \mathcal{A}^f} \sum_{j \in N\setminus\{i\}} w(A''_j; R_j) - \sum_{j \in N\setminus\{i\}} w(A_j(R); R_j), \end{aligned} \tag{18}$$

where the last equality uses $\mathcal{A}^f = \mathcal{A}^g$ (see Lemma 3). Note that (18) corresponds to Claim 2 in the outline of the proof presented in Section 3.4.3.

Let $R_0 \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that for each $a \in M$, $w\big(\{a\}; R_0\big) = 1$.

We consider two cases.

Case 1. For each $A''_i \in \mathcal{M}_i$ with $A''_i \cap A'_i \neq \emptyset$, it holds that $A''_i \subseteq A'_i$.

Since $A_i \cap A'_i \neq \emptyset$, we have $A_i \subseteq A'_i$. Given that $A_i \neq A'_i$, we have $A_i \subsetneq A'_i$. Without loss of generality, assume that A_i and A'_i are adjacent in $\mathcal{M}_i$. Since $A_i \neq \emptyset$, we can choose $a \in A_i$. Since $A_i \subsetneq A'_i$, we have $A'_i\setminus A_i \neq \emptyset$, so we can choose some $b \in A'_i\setminus A_i$. Note that $a \neq b$. Figure 9 illustrates the packages A_i and A'_i.

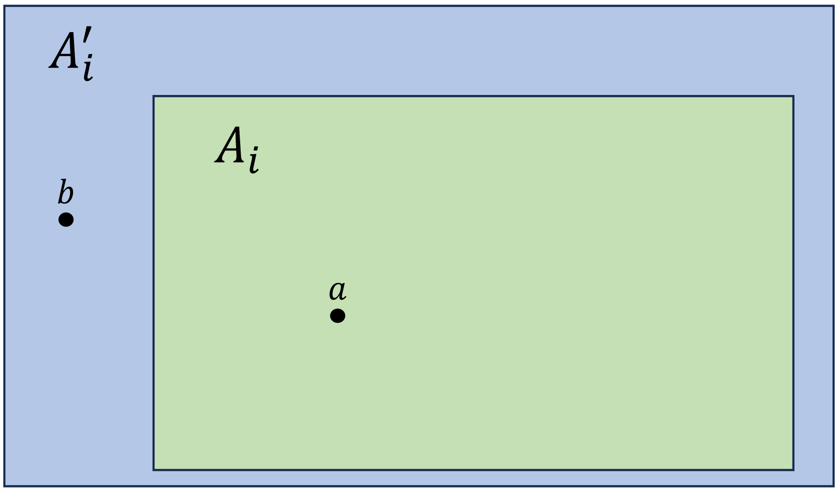

Figure 9: An illustration of the packages in Case 1.

By $A_i \in \mathcal{M}_i$, there exists $A_{-i} \in \mathcal{M}^{n-1}$ such that $A \equiv (A_i, A_{-i}) \in \mathcal{A}^f$. By no wastage of $\mathcal{A}^f$ (which follows from *no wastage* of f) and $b \notin A_i$, there exists $j \in N\setminus\{i\}$ such that $b \in A_j$. Let $R_j \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that $w\big(\{b\}; R_j\big) = m + 1$, and for each $c \in M\setminus\{b\}$, $w\big(\{c\}; R_j\big) = 1$. Let $R'_j \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that $w\big(\{a\}; R'_j\big) = 3m$, $w\big(\{b\}; R'_j\big) = m + 2$, and for each $c \in M\setminus\{a, b\}$, $w\big(\{c\}; R'_j\big) = 1$. For each $k \in N\setminus\{i, j\}$, let $R_k = R_0$.

Let $\tau, \tau' \in \mathbb{R}^{|\mathcal{M}_i|}$ be two object monotonic payment vectors on $\mathcal{M}_i$ such that for each

$A_i'' \in \mathcal{M}_i(R_{-i})$, $\tau_{A_i''} = t_i(R_{-i}; A_i'')$, and for each $A_i'' \in \mathcal{M}_i(R_j', R_{-i,j})$, $\tau'_{A_i''} = t_i(R_j', R_{-i,j}; A_i'')$. Note that by Lemma 1, we can choose such τ, τ' satisfying object monotonicity on $\mathcal{M}_i$.[27]

Recall that $A \in \mathcal{A}^f$ and $b \in A_j$. Thus, since $w(\{b\}; R_j) \geq w(\{b\}; R_k)$ and $w(\{c\}; R_j) = w(\{c\}; R_k)$ for each $k \in N \backslash \{i, j\}$ and each $c \in M \backslash (A_i \cup \{b\})$, Lemma 2 implies that there exists $R_i \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ such that $A_i(R_i, R_{-i}) = A_i$, and thus $A_i \in \mathcal{M}_i(R_{-i})$. This corresponds to Claim 1 in the outline of the proof presented in Section 3.4.3. Then, we have

$$\begin{aligned}
\tau_{A_i} &= t_i(R_{-i}; A_i) = t_i(R) \\
&= \max_{A'' \in \mathcal{A}^f} \sum_{k \in N \backslash \{i\}} w(A_k''; R_k) - \sum_{k \in N \backslash \{i\}} w(A_k(R); R_k) \\
&\leq \max_{A'' \in \mathcal{A}^f} \sum_{k \in N \backslash \{i\}} w(A_k''; R_k) \\
&\leq w(\{b\}; R_j) + w(M \backslash \{b\}; R_0) \\
&= 2m, \qquad (19)
\end{aligned}$$

where the first equality uses $A_i \in \mathcal{M}_i(R_{-i})$; the second equality follows from $A_i(R) = A_i$; the third equality uses that $R \in (\mathcal{R}^{Add} \cap \mathcal{R}^Q)^n$ and (18); and the second inequality follows from the fact that for each $c \in M \backslash \{b\}$, $w(\{b\}; R_j) \geq w(\{c\}; R_j) = w(\{c\}; R_0)$, so that assigning object b to agent j maximizes the total willingness to pay among agents in $N \backslash \{i\}$.

By the same argument as above, we can invoke Lemma 2 to claim that there exists $R_i' \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ such that $A_i(R_{i,j}', R_{-i,j}) = A_i$.[28] Thus, $A_i \in \mathcal{M}_i(R_j', R_{-i,j})$. Since A_i' is maximal in $\mathcal{M}_i$, $A_i' \in \mathcal{M}_j$ (see Lemma 6). Since $a \in A_i = A_i(R_{i,j}', R_{-i,j})$, $a \notin A_j(R_{i,j}', R_{-i,j})$. Given $A \in \mathcal{A}^f$, $a \notin A_j(R_{i,j}', R_{-i,j})$, and $b \in A_j$, *constrained efficiency* implies that $b \in A_j(R_{i,j}', R_{-i,j})$, because—conditional on agent j not receiving object a (i.e., $a \notin A_j(R_{i,j}', R_{-i,j})$)—assigning

[27] For each $A_i'' \in \mathcal{M}_i \backslash \mathcal{M}_i(R_{-i})$, define $\tau_{A_i''} \in \mathbb{R}$ as follows:

(i) If there is no $\tilde{A}_i \in \mathcal{M}_i$ such that $\tilde{A}_i \supsetneq A_i''$, then $\tau_{A_i''} = \max_{\tilde{A}_i \in \mathcal{M}_i(R_{-i})} \tau_{\tilde{A}_i} + 1$.

(ii) If there is no $\tilde{A}_i \in \mathcal{M}_i$ such that $\tilde{A}_i \subsetneq A_i''$, then $\tau_{A_i''} = \min_{\tilde{A}_i \in \mathcal{M}_i(R_{-i})} \tau_{\tilde{A}_i} - 1$.

(iii) If there are $\tilde{A}_i, \overline{A}_i \in \mathcal{M}_i$ such that $\tilde{A}_i \subsetneq A_i'' \subsetneq \overline{A}_i$, then

$$\max_{\tilde{A}_i \in \mathcal{M}_i(R_{-i}) : \tilde{A}_i \subsetneq A_i''} \tau_{\tilde{A}_i} < \tau_{A_i''} < \min_{\tilde{A}_i \in \mathcal{M}_i(R_{-i}) : \tilde{A}_i \supsetneq A_i''} \tau_{\tilde{A}_i}.$$

Note that in case (iii), we can choose such $\tau_{A_i''}$ due to Lemma 1. Then, using Lemma 1, it is straightforward to show that τ is an object monotonic payment vector on $\mathcal{M}_i$. An object monotonic payment vector τ' on $\mathcal{M}_i$ can be constructed in the same manner.

[28] This also corresponds to Claim 1 in the outline of the proof presented in Section 3.4.3.

object b to agent j maximizes the total willingness to pay. Then,

$$
\begin{aligned}
\tau'_{A_i} &= t_i(R'_j, R_{-i,j}; A_i) = t_i(R'_{i,j}, R_{-i,j}) \\
&= \max_{A'' \in \mathcal{A}^f} \left(w(A''_j; R'_j) + \sum_{k \in N \setminus \{i,j\}} w(A''_k; R_k) \right) \\
&\quad - \left(w\big(A_j(R'_{i,j}, R_{-i,j}); R'_j\big) + \sum_{k \in N \setminus \{i,j\}} w\big(A_k(R'_{i,j}, R_{-i,j}); R_k\big) \right) \\
&\geq w(A'_i; R_j) - \left(w\big(A_j(R'_{i,j}, R_{-i,j}); R'_j\big) + \sum_{k \in N \setminus \{i,j\}} w\big(A_k(R'_{i,j}, R_{-i,j}); R_k\big) \right) \\
&\geq w\big(\{a\}; R'_j\big) + w\big(\{b\}; R'_j\big) - \left(w\big(A_j(R'_{i,j}, R_{-i,j}); R'_j\big) + \sum_{k \in N \setminus \{i,j\}} w\big(A_k(R'_{i,j}, R_{-i,j}); R_k\big) \right) \\
&\geq w\big(\{a\}; R'_j\big) + w\big(\{b\}; R'_j\big) - \Big(w\big(\{b\}; R'_j\big) + w\big(M \setminus \{b\}; R_0\big) \Big) \\
&= w\big(\{a\}; R'_j\big) - w\big(M \setminus \{b\}; R_0\big) \\
&= 2m + 1,
\end{aligned} \tag{20}
$$

where the first equality uses that $A_i \in \mathcal{M}_i(R'_j, R_{-i,j})$; the second equality follows from $A_i(R'_{i,j}, R_{-i,j}) = A_i$; the third equality uses that $(R'_{i,j}, R_{-i,j}) \in (\mathcal{R}^{Add} \cap \mathcal{R}^Q)^n$ and (18); the first inequality uses $A'_i \in \mathcal{M}_j$; the second inequality follows from $a, b \in A'_i$ and additivity of R'_j; and the third inequality follows from $b \in A_j(R'_{i,j}, R_{-i,j})$, $a \notin A_j(R'_{i,j}, R_{-i,j})$, and from the fact that for each $c \in M \setminus \{a, b\}$, $w\big(\{b\}; R'_j\big) \geq w\big(\{c\}; R'_j\big) = w\big(\{c\}; R_0\big)$, so that assigning object b to agent j maximizes the total willingness to pay among agents in $N \setminus \{i\}$.

Combining (19) and (20), we obtain

$$\tau'_{A_i} \geq 2m + 1 > 2m \geq \tau_{A_i}.$$

This inequality corresponds to inequality (1) in the outline of the proof presented in Section 3.4.3. Recall that τ and τ' are both object monotonic on $\mathcal{M}_i$. Note that $\emptyset \neq A_i \subsetneq A'_i$, that A_i and A'_i are adjacent in $\mathcal{M}_i$, and that, by the assumption of Case 1, for each $A''_i \in \mathcal{M}_i$ with $A'_i \cap A''_i \neq \emptyset$, it holds that $A''_i \subseteq A'_i$. Thus, the assumptions of Lemma 8 are satisfied, and so the lemma implies that there exists $R''_i \in \mathcal{R}^{Add}$ that demands A_i at τ and A'_i at τ' on $\mathcal{M}_i$. Note that this corresponds to Claim 3 in the proof outline provided in Section 3.4.3.

Recall that $A_i \in \mathcal{M}_i(R_{-i})$, and note that $\mathcal{M}_i(R_{-i}) \subseteq \mathcal{M}_i$. Thus, since R''_i demands A_i at τ on $\mathcal{M}_i$, it follows that for each $A''_i \in \mathcal{M}_i(R_{-i}) \setminus \{A_i\}$, $z_i(R_{-i}; A_i) = (A_i, \tau_{A_i})\ P''_i\ (A''_i, \tau_{A''_i}) = z_i(R_{-i}; A''_i)$. Therefore, by *strategy-proofness*, $A_i(R''_i, R_{-i}) = A_i$ (see Remark 3). Since

$a \in A_i$, we have $a \in A_i(R''_i, R_{-i})$, and hence $a \notin A_j(R''_i, R_{-i})$. Also, since $A \in \mathcal{A}^f$, $a \notin A_j(R''_i, R_{-i})$, and $b \in A_j$, it follows from *constrained efficiency* that $b \in A_j(R''_i, R_{-i})$, as assigning object b to agent j, conditional on agent j not receiving object a (i.e., $a \notin A_j(R''_i, R_{-i})$), maximizes the total willingness to pay.

Since $w(\{c\}; R'_j) = w(\{c\}; R_0) = w(\{c\}; R_k)$ for each $k \in N \backslash \{i, j\}$ and each $c \in M \backslash A'_i$, Lemma 2 implies that $A'_i \in \mathcal{M}_i(R'_j, R_{-i,j})$. Thus, since $\mathcal{M}_i(R'_j, R_{-i,j}) \subseteq \mathcal{M}_i$ and R''_i demands A'_i at τ' on $\mathcal{M}_i$, it follows that for each $A''_i \in \mathcal{M}_i(R'_j, R_{-i,j}) \backslash \{A'_i\}$, $z_i(R'_j, R_{-i,j}; A'_i) = (A'_i, \tau'_{A'_i}) \; P''_i \; (A''_i, \tau'_{A''_i}) = z_i(R'_j, R_{-i,j}; A''_i)$. Therefore, by *strategy-proofness*, $A_i(R''_i, R'_j, R_{-i,j}) = A'_i$ (see Remark 3). Since $a, b \in A'_i = A_i(R''_i, R'_j, R_{-i,j})$, it follows that $a, b \notin A_j(R''_i, R'_j, R_{-i,j})$.

To sum up, we have

$$a \notin A_j(R''_i, R_{-i}), \quad b \in A_j(R''_i, R_{-i}), \quad a, b \notin A_j(R''_i, R'_j, R_{-i,j}). \tag{21}$$

This corresponds to Claim 4 in the outline of the proof presented in Section 3.4.3. Then,

$$\begin{aligned}
&w\big(A_j(R''_i, R_{-i}); R'_j\big) - w\big(A_j(R''_i, R'_j, R_{-i,j}); R'_j\big) \\
&= \big(|A_j(R''_i, R_{-i}) \backslash \{a\}| + (m+2)\big) - |A_j(R''_i, R'_j, R_{-i,j})| \\
&> \big(|A_j(R''_i, R_{-i}) \backslash \{a\}| + (m+1)\big) - |A_j(R''_i, R'_j, R_{-i,j})| \\
&= w\big(A_j(R''_i, R_{-i}); R_j\big) - w\big(A_j(R''_i, R'_j, R_{-i,j}); R_j\big),
\end{aligned}$$

where the first and the second equalities follow from (21) and additivity of R_j, R'_j. This inequality corresponds to inequality (2) in the outline of the proof presented in Section 3.4.3. However, it contradicts *monotonicity* of f (see Fact 3).

Case 2. There exists $A''_i \in \mathcal{M}_i$ such that $A'_i \cap A''_i \neq \emptyset$ and $A''_i \not\subseteq A'_i$.

By $A_i \cap A'_i \neq \emptyset$, we may assume without loss of generality that $A''_i = A_i$. Since $A_i \cap A'_i \neq \emptyset$, we can choose some $a \in A_i \cap A'_i$. Since $A_i \not\subseteq A'_i$, we also have $A_i \backslash A'_i \neq \emptyset$, so we can choose some $b \in A_i \backslash A'_i$. Note that $a \neq b$. If there exists $A''_i \in \mathcal{M}_i$ such that $A''_i \supsetneq A_i$, then $a \in A'_i \cap A''_i$ and $b \in A''_i \backslash A'_i$. Therefore, without loss of generality, we may assume that A_i is maximal in $\mathcal{M}_i$.

Let $j \in N \backslash \{i\}$. Since A'_i is maximal in $\mathcal{M}_i$, $A'_i \in \mathcal{M}_j$, and it is also maximal in $\mathcal{M}_j$ (see Lemma 6). Since A'_i is maximal in $\mathcal{M}_j$ and $b \notin A'_i$, there exists $A'' \in \mathcal{A}^f$ such that $A''_j = A'_i$ and $b \in A''_i$ (see Lemma 7). Since $A''_i \cap A''_j = \emptyset$ and $A''_j = A'_i$, we have $A''_i \cap A'_i = \emptyset$. Therefore, as $a \in A'_i$, we have $a \notin A''_i$.

In summary, we obtain

$$a, b \in A_i, \quad a \in A'_i, \quad b \notin A'_i, \quad b \in A''_i, \quad a \notin A''_i.$$

Figure 10 illustrates the packages A_i, A'_i, and A''_i.

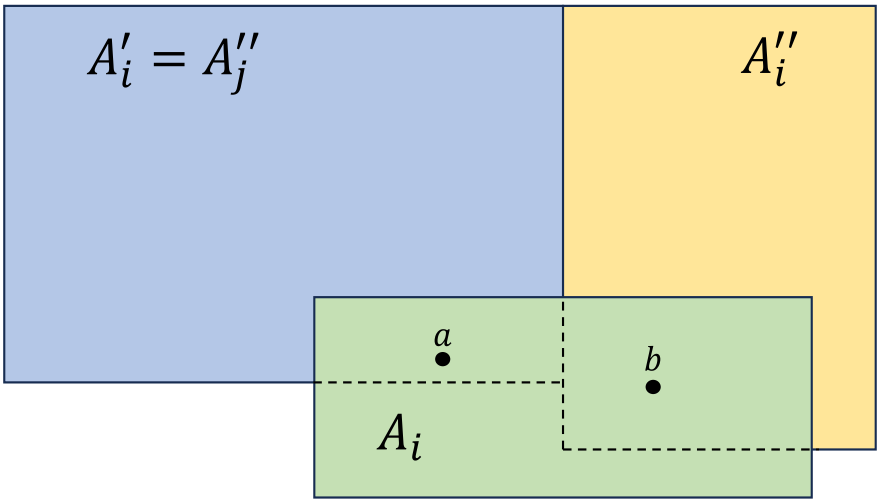

Figure 10: An illustration of the packages in Case 2.

Let $R_j \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that $w\big(\{a\}; R_j\big) = m+1$, and for each $c \in M\backslash\{a\}$, $w\big(\{c\}; R_j\big) = 1$. Let $R'_j \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that $w\big(\{a\}; R'_j\big) = 4m+1$, $w\big(\{b\}; R'_j\big) = 3m$, and for each $c \in M\backslash\{a, b\}$, $w\big(\{c\}; R'_j\big) = w\big(\{c\}; R_0\big)$. For each $k \in N\backslash\{i, j\}$, let $R_k = R_0$.

Let $\tau, \tau' \in \mathbb{R}^{|\mathcal{M}_i|}$ be two object monotonic payment vectors on $\mathcal{M}_i$ such that for each $A'''_i \in \mathcal{M}_i(R_{-i})$, we have $\tau_{A'''_i} = t_i(R_{-i}; A'''_i)$, and for each $A'''_i \in \mathcal{M}_i(R'_j, R_{-i,j})$, we have $\tau'_{A'''_i} = t_i(R'_j, R_{-i,j}; A'''_i)$. By Lemma 1, such τ, τ' can be chosen to satisfy object monotonicity on $\mathcal{M}_i$.[29]

Recall that $A'' \in \mathcal{A}^f$ and $a \in A''_j$. Thus, since $w\big(\{a\}; R_j\big) \geq w\big(\{a\}; R_k\big)$ and $w\big(\{c\}; R_j\big) = w\big(\{c\}; R_k\big)$ for each $k \in N\backslash\{i, j\}$ and each $c \in M\backslash\big(A''_i \cup \{a\}\big)$, Lemma 2 implies that there exists $R_i \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ such that $A_i(R_i, R_{-i}) = A''_i$. Thus, $A''_i \in \mathcal{M}_i(R_{-i})$. This corresponds

[29]For a detailed discussion, see footnote 25.

to Claim 1 in the outline of the proof presented in Section 3.4.3. Then,

$$\begin{aligned}
\tau_{A_i''} &= t_i(R_{-i}; A_i'') = t_i(R) \\
&= \max_{A''' \in \mathcal{A}^f} \sum_{k \in N \setminus \{i\}} w(A_k'''; R_k) - \sum_{k \in N \setminus \{i\}} w\big(A_k(R); R_k\big) \\
&\leq \max_{A''' \in \mathcal{A}^f} \sum_{k \in N \setminus \{i\}} w(A_k'''; R_k) \\
&\leq w\big(\{a\}; R_j\big) + w\big(M \setminus \{a\}; R_0\big) \\
&= 2m,
\end{aligned} \tag{22}$$

where the first equality follows from $A_i'' \in \mathcal{M}_i(R_{-i})$; the second equality from $A_i(R) = A_i''$; the third equality uses $R \in (\mathcal{R}^{Add} \cap \mathcal{R}^Q)^n$ and (18); and the second inequality follows from the fact that, for each $c \in M \setminus \{a\}$, we have $w\big(\{a\}; R_j\big) \geq w\big(\{c\}; R_j\big) = w\big(\{c\}; R_0\big)$, so assigning object a to agent j maximizes the total willingness to pay among agents in $N \setminus \{i\}$.

By the same argument as above, Lemma 2 implies that there exists $R_i' \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ such that $A_i(R_i', R_j', R_{-i,j}) = A_i''$.[30] Thus, $A_i'' \in \mathcal{M}_i(R_j', R_{-i,j})$. Since A_i' is maximal in $\mathcal{M}_i$, $A_i' \in \mathcal{M}_j$ (see Lemma 6). By $b \in A_i''$ and $A_i(R_{i,j}', R_{-i,j}) = A_i''$, we have $b \notin A_j(R_{i,j}', R_{-i,j})$. Thus, given $A'' \in \mathcal{A}^f$ and $a \in A_j''$, it follows from *constrained efficiency* that $a \in A_j(R_{i,j}', R_{-i,j})$, because—conditional on agent j not receiving object b (i.e., $b \notin A_j(R_{i,j}', R_{-i,j})$)—assigning object a to agent j maximizes the total willingness to pay. Then,

$$\begin{aligned}
&\tau_{A_i''}' = t_i(R_j', R_{-i,j}; A_i'') = t_i(R_{i,j}', R_{-i,j}) \\
&= \max_{A''' \in \mathcal{A}^f} \left(w(A_j'''; R_j') + \sum_{k \in N \setminus \{i,j\}} w(A_k'''; R_k) \right) \\
&\quad - \left(w\big(A_j(R_{i,j}', R_{-i,j}); R_j'\big) + \sum_{k \in N \setminus \{i,j\}} w\big(A_k(R_{i,j}', R_{-i,j}); R_k\big) \right) \\
&\geq w(A_i; R_j') - \left(w\big(A_j(R_{i,j}', R_{-i,j}); R_j'\big) + \sum_{k \in N \setminus \{i,j\}} w\big(A_k(R_{i,j}', R_{-i,j}); R_k\big) \right) \\
&\geq w\big(\{a\}; R_j'\big) + w\big(\{b\}; R_j'\big) - \left(w\big(A_j(R_{i,j}', R_{-i,j}); R_j'\big) + \sum_{k \in N \setminus \{i,j\}} w\big(A_k(R_{i,j}', R_{-i,j}); R_k\big) \right) \\
&\geq w\big(\{a\}; R_j'\big) + w\big(\{b\}; R_j'\big) - \Big(w\big(\{a\}; R_j'\big) + w\big(M \setminus \{a\}; R_0\big) \Big) \\
&= w\big(\{b\}; R_j'\big) - w\big(M \setminus \{a\}; R_0\big) \\
&= 2m + 1,
\end{aligned} \tag{23}$$

[30]Note that this also corresponds to Claim 1 in the outline of the proof presented in Section 3.4.3.

where the first equality follows from $A''_i \in \mathcal{M}_i(R'_j, R_{-i,j})$; the second equality from $A_i(R'_{i,j}, R_{-i,j}) = A''_i$; the third equality uses $(R'_{i,j}, R_{-i,j}) \in (\mathcal{R}^{Add} \cap \mathcal{R}^Q)^n$ and (18); the first inequality follows from $A_i \in \mathcal{M}_j$; the second inequality from $a, b \in A_i$ and additivity of R'_j; and the third inequality from the facts that $a \in A_j(R'_{i,j}, R_{-i,j})$, $b \notin A_j(R'_{i,j}, R_{-i,j})$, and for each $c \in M \backslash \{a, b\}$, $w\big(\{b\}; R'_j\big) \geq w\big(\{c\}; R'_j\big) = w\big(\{c\}; R_0\big)$, so that assigning object a to agent j maximizes the total willingness to pay among agents in $N \backslash \{i\}$.

By (22) and (23),

$$\tau'_{A''_i} \geq 2m + 1 > 2m \geq \tau_{A''_i}.$$

This inequality corresponds to inequality (1) in the outline of the proof presented in Section 3.4.3. Recall that τ and τ' are both object monotonic on $\mathcal{M}_i$. Thus, since $A'_i \cap A''_i = \emptyset$, $A'_i \neq \emptyset$, and A'_i is maximal in $\mathcal{M}_i$, we can invoke Lemma 9 to conclude that there exists $R''_i \in \mathcal{R}^{Add}$ that demands A''_i at τ and A'_i at τ' on $\mathcal{M}_i$. Note that this corresponds to Claim 3 in the proof outline provided in Section 3.4.3.

Recall that $A''_i \in \mathcal{M}_i(R_{-i})$. Thus, since $\mathcal{M}_i(R_{-i}) \subseteq \mathcal{M}_i$ and R''_i demands A''_i at τ on $\mathcal{M}_i$, it follows that for each $A'''_i \in \mathcal{M}_i(R_{-i}) \backslash \{A''_i\}$, $z_i(R_{-i}; A''_i) = (A''_i, \tau_{A''_i})\ P''_i\ (A'''_i, \tau_{A'''_i}) = z_i(R_{-i}; A'''_i)$. Therefore, by *strategy-proofness*, $A_i(R''_i, R_{-i}) = A''_i$ (see Remark 3). Since $b \in A''_i = A_i(R''_i, R_{-i})$, it follows that $b \notin A_j(R''_i, R_{-i})$. On the other hand, since $A'' \in \mathcal{A}^f$ and $a \in A''_j$, *constrained efficiency* implies that $a \in A_j(R''_i, R_{-i})$, as, conditional on agent j not receiving object b (i.e., $b \notin A_j(R''_i, R_{-i})$), assigning object a to agent j maximizes the total willingness to pay.

Recall that $A'_i \neq \emptyset$ is maximal in $\mathcal{M}_i$. Thus, since $b \notin A'_i$, there exists $A''' \in \mathcal{A}^f$ such that $A'''_i = A'_i$ and $b \in A'''_j$ (see Lemma 7). Moreover, since $w\big(\{b\}; R'_j\big) \geq w\big(\{b\}; R_k\big)$ and $w\big(\{c\}; R'_j\big) = w\big(\{c\}; R_k\big)$ for each $k \in N \backslash \{i, j\}$ and each $c \in M \backslash \big(A'_i \cup \{b\}\big)$, Lemma 3 implies that $A'_i \in \mathcal{M}_i(R'_j, R_{-i,j})$. Since $\mathcal{M}_i(R'_j, R_{-i,j}) \subseteq \mathcal{M}_i$ and R''_i demands A'_i at τ' on $\mathcal{M}_i$, it follows that for each $\tilde{A}_i \in \mathcal{M}_i(R'_j, R_{-i,j}) \backslash \{A'_i\}$, $z_i(R'_j, R_{-i,j}; A'_i) = (A'_i, \tau'_{A'_i})\ P''_i\ (\tilde{A}_i, \tau'_{\tilde{A}_i}) = z_i(R'_j, R_{-i,j}; \tilde{A}_i)$. Hence, by *strategy-proofness*, $A_i(R''_i, R'_j, R_{-i,j}) = A'_i$ (see Remark 3). Since $a \in A'_i = A_i(R''_i, R'_j, R_{-i,j})$, it follows that $a \notin A_j(R''_i, R'_j, R_{-i,j})$. Furthermore, since $A''' \in \mathcal{A}^f$ and $b \in A'''_j$, *constrained efficiency* implies that $b \in A_j(R''_i, R'_j, R_{-i,j})$, as, conditional on agent j not receiving object a (i.e., $a \notin A_j(R''_i, R'_j, R_{-i,j})$), assigning object b to agent j maximizes the total willingness to pay.

To sum up, we have

$$a \in A_j(R''_i, R_{-i}), \quad b \notin A_j(R''_i, R_{-i}), \quad a \notin A_j(R''_i, R'_j, R_{-i,j}), \quad b \in A_j(R''_i, R'_j, R_{-i,j}). \tag{24}$$

This corresponds to Claim 4 in the outline of the proof presented in Section 3.4.3. Then,

$$
\begin{aligned}
& w\big(A_j(R_i'', R_{-i}); R_j'\big) - w\big(A_j(R_i'', R_j', R_{-i,j}); R_j'\big) \\
&= \big(|A_j(R_i'', R_{-i})\backslash\{a\}| + (4m+1)\big) - \big(|A_j(R_i'', R_j', R_{-i,j})\backslash\{b\}| + 3m\big) \\
&> \big(|A_j(R_i'', R_{-i})\backslash\{a\}| + (m+1)\big) - \big(|A_j(R_i'', R_j', R_{-i,j})\backslash\{b\}| + 1\big) \\
&= w\big(A_j(R_i'', R_{-i}); R_j\big) - w\big(A_j(R_i'', R_j', R_{-i,j}); R_j\big),
\end{aligned}
$$

where the first and the second equalities follow from (24) and additivity of R_j, R_j'. This inequality corresponds to inequality (2) in the outline of the proof presented in Section 3.4.3. However, this contradicts *monotonicity* of f (see Fact 3).

STEP 2. In this step, we show that for each $i, j \in N$, $\mathcal{M}_i = \mathcal{M}_j$. Let $i, j \in N$ be two distinct agents. We show that $\mathcal{M}_i \subseteq \mathcal{M}_j$; the reverse inclusion follows symmetrically. Let $A_i \in \mathcal{M}_i$. If $A_i \neq \emptyset$, then by Step 1, A_i is maximal in $\mathcal{M}_i$. Hence, $A_i \in \mathcal{M}_j$ (see Lemma 6). Now suppose $A_i = \emptyset$. To show that $A_i \in \mathcal{M}_j$, suppose for contradiction that $A_i \notin \mathcal{M}_j$. For each $k \in N$, let $R_k \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that for each $a \in M$, $w\big(\{a\}; R_k\big) = 1$. Note that $R_i = R_j$.

By $A_i \in \mathcal{M}_i$, there exists $A_{-i} \in \mathcal{M}^{n-1}$ such that $A \equiv (A_i, A_{-i}) \in \mathcal{A}^f$. Given that $A_i = \emptyset$ and using no wastage of $\mathcal{A}^f$ (which follows from *no wastage* of f), we have $\bigcup_{k \in N\backslash\{i\}} A_k = M$. Therefore,

$$\max_{A' \in \mathcal{A}^f} \sum_{k \in N\backslash\{i\}} w(A_k'; R_k) \geq \sum_{k \in N\backslash\{i\}} w(A_k; R_k) = m, \tag{25}$$

where the inequality uses $A \in \mathcal{A}^f$.

Let $A' \in \mathcal{A}^f$. Since $A_i = \emptyset \notin \mathcal{M}_j$, we have $A_j' \neq \emptyset$. By no wastage of $\mathcal{A}^f$, it follows that $\bigcup_{k \in N\backslash\{j\}} A_k' = M\backslash A_j'$. Therefore,

$$\sum_{k \in N\backslash\{j\}} w(A_k'; R_k) = m - |A_j'| < m,$$

where the inequality follows from $A_j' \neq \emptyset$. Since $A' \in \mathcal{A}^f$ was arbitrary, we obtain

$$\max_{A' \in \mathcal{A}^f} \sum_{k \in N\backslash\{j\}} w(A_k'; R_k) < m. \tag{26}$$

By (25) and (26),

$$\max_{A' \in \mathcal{A}^f} \sum_{k \in N\backslash\{i\}} w(A_k'; R_k) \geq m > \max_{A' \in \mathcal{A}^f} \sum_{k \in N\backslash\{j\}} w(A_k'; R_k),$$

which contradicts Lemma 4.

STEP 3. We now complete the proof. Let $\mathcal{B} \equiv \bigcup_{i \in N} (\mathcal{M}_i \backslash \{\emptyset\})$. By no wastage of $\mathcal{A}^f$, we have $\bigcup \mathcal{B} = M$. By Step 2, for each $i \in N$, we have $\mathcal{B} = \mathcal{M}_i \backslash \{\emptyset\}$. Then, by Step 1, for each distinct $A_i, A'_i \in \mathcal{B}$, $A_i \cap A'_i = \emptyset$. Thus, $\mathcal{B}$ is a partition of M. Furthermore, for each $A \in \mathcal{A}^f$ and each $i \in N$, since $A_i \in \mathcal{M}_i$ and $\mathcal{B} = \mathcal{M}_i \backslash \{\emptyset\}$, we have $A_i \in \mathcal{B} \cup \{\emptyset\}$. Hence, $\mathcal{A}^f$ is a $\mathcal{B}$-bundling unit-demand constraint. Finally, by no wastage of $\mathcal{A}^f$, we have $|\mathcal{B}| \leq n$, since otherwise more than n non-empty packages would be assigned to distinct agents, contradicting feasibility. ■

A.2.2 Anonymity

In this subsection, we prove that $\mathcal{A}^f$ satisfies anonymity. The argument in this subsection corresponds to the outline given in Section 3.4.5.

A permutation $\pi : N \to N$ is said to be a **transposition on** N if there exist $i, j \in N$ such that $\pi(i) = j$, $\pi(j) = i$, and for each $k \in N \backslash \{i, j\}$, $\pi(k) = k$. Note that any permutation on N can be written as a product of transpositions. Hence, to prove that $\mathcal{A}^f$ satisfies anonymity, it suffices to show that for each $A \in \mathcal{A}^f$ and each transposition π on N, $A^\pi \in \mathcal{A}^f$. The proof proceeds in two steps.

STEP 1. We show that for each $A \in \mathcal{A}^f$ and each transposition π on N such that for some $i, j \in N$, $\pi(i) = j$, $\pi(j) = i$, and $A_j = \emptyset$, we have $A^\pi \in \mathcal{A}^f$. The argument in Step 1 corresponds to Case 1 in the outline presented in Section 3.4.5. Let $A \in \mathcal{A}^f$. Let π be a transposition on N such that for some distinct $i, j \in N$, $\pi(i) = j$, $\pi(j) = i$, and $A_j = \emptyset$. If $A_i = \emptyset$, then $A^\pi = A \in \mathcal{A}^f$. Suppose that $A_i \neq \emptyset$. To show $A^\pi \in \mathcal{A}^f$, suppose for contradiction that $A^\pi \notin \mathcal{A}^f$.

For each $k \in \{i, j\}$, let $R_k \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that for each $a \in A_i$, $w\big(\{a\}; R_k\big) = m+1$, and for each $a \in M \backslash A_i$, $w\big(\{a\}; R_k\big) = 1$. Note that $R_i = R_j$. For each $k \in N \backslash \{i, j\}$, let $R_k \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that for each $a \in A_k$, $w\big(\{a\}; R_k\big) = m+1$, and for each $a \in M \backslash A_k$, $w\big(\{a\}; R_k\big) = 1$.

By $A \in \mathcal{A}^f$,

$$\max_{A' \in \mathcal{A}^f} \sum_{k \in N \backslash \{j\}} w(A'_k; R_k) \geq \sum_{k \in N \backslash \{j\}} w(A_k; R_k) = (m+1) \sum_{k \in N \backslash \{j\}} |A_k|. \tag{27}$$

Let $A' \in \mathcal{A}^f$. We claim $A'_j \not\supseteq A_i$, or there exists $k \in N \backslash \{i, j\}$ such that $A'_k \not\supseteq A_k$. For contradiction, suppose that $A'_j \supseteq A_i$, and for each $k \in N \backslash \{i, j\}$, $A'_k \supseteq A_k$. Then, we have

$\bigcup_{k\in N\setminus\{i\}} A'_k \supseteq \bigcup_{k\in N\setminus\{j\}} A_k$. By no wastage of $\mathcal{A}^f$ (which follows from *no wastage* of f), $A \in \mathcal{A}^f$ implies that $\bigcup_{k\in N\setminus\{j\}} A_k = M\setminus A_j = M$, where the last equality follows from $A_j = \emptyset$. Also, by no wastage of $\mathcal{A}^f$ and $A' \in \mathcal{A}^f$, $\bigcup_{k\in N\setminus\{i\}} A'_k = M\setminus A'_i$. Thus, $M\setminus A'_i \supseteq M$, and hence $A'_i = \emptyset = A_j$. Since $\mathcal{A}^f$ is a bundling unit-demand constraint (see Appendix A.2.1) and $A_i \neq \emptyset$, A_i is maximal in $\mathcal{M}_i$. Thus, since $\mathcal{M}_i = \mathcal{M}_j$ (see Step 2 of Appendix A.2.1) A_i is also maximal in $\mathcal{M}_j$. Thus, by $A'_j \supseteq A_i$, $A'_j = A_i$. Let $N^+(A) \equiv \{k \in N : A_k \neq \emptyset\}$. Since $\mathcal{A}^f$ is a bundling unit-demand constraint, for each $k \in N^+(A)\setminus\{i,j\}$ such that $A_k \neq \emptyset$, A_k is maximal in $\mathcal{M}_k$, so $A'_k \supseteq A_k$ implies $A'_k = A_k$. Thus,

$$A'_j \cup \left(\bigcup_{k\in N^+(A)\setminus\{i,j\}} A'_k \right) = A_i \cup \left(\bigcup_{k\in N^+(A)\setminus\{i,j\}} A_k \right) = \bigcup_{k\in N} A_k = M,$$

where the second equality follows from $A_j = \emptyset$, and the last one from no wastage of $\mathcal{A}^f$. This implies that for each $k \in N\setminus\big(\{i,j\} \cup N^+(A)\big)$, $A'_k = \emptyset = A_k$. Therefore, we conclude that $A' = A^\pi$. However, this contradicts that $A' \in \mathcal{A}^f$ and $A^\pi \notin \mathcal{A}^f$.

Thus, we have $A'_j \not\supseteq A_i$, or there exists $k \in N\setminus\{i,j\}$ such that $A'_k \not\supseteq A_k$. Thus, we have $A'_j \cap A_i \subsetneq A_i$, or there exists $k \in N\setminus\{i,j\}$ such that $A'_k \cap A_k \subsetneq A_k$. Thus,

$$\begin{aligned}
\sum_{k\in N\setminus\{i\}} w(A'_k; R_k) &= (m+1)\left(|A'_j \cap A_i| + \sum_{k\in N\setminus\{i,j\}} |A'_k \cap A_k| \right) + |A'_j\setminus A_i| + \sum_{k\in N\setminus\{i,j\}} |A'_k\setminus A_k| \\
&\leq (m+1)\left(|A'_j \cap A_i| + \sum_{k\in N\setminus\{i,j\}} |A'_k \cap A_k| \right) + m \\
&< (m+1) \sum_{k\in N\setminus\{j\}} |A_k|,
\end{aligned}$$

where the second inequality uses the fact that $A'_j \cap A_i \subsetneq A_i$, or there exists $k \in N\setminus\{i,j\}$ such that $A'_k \cap A_k \subsetneq A_k$ (and hence $|A'_j \cap A_i| + \sum_{k\in N\setminus\{i,j\}} |A'_k \cap A_k| < \sum_{k\in N\setminus\{j\}} |A_k|$). Since $A' \in \mathcal{A}^f$ was arbitrary, we have

$$\max_{A''\in\mathcal{A}^f} \sum_{k\in N\setminus\{i\}} w(A''_k; R_k) < (m+1) \sum_{k\in N\setminus\{j\}} |A_k|. \tag{28}$$

By (27) and (28),

$$\max_{A''\in\mathcal{A}^f} \sum_{k\in N\setminus\{j\}} w(A''_k; R_k) \geq (m+1) \sum_{k\in N\setminus\{j\}} |A_k| > \max_{A''\in\mathcal{A}^f} \sum_{k\in N\setminus\{i\}} w(A''_k; R_k),$$

which contradicts Lemma 4.

Step 2. Next, we show that for each $A \in \mathcal{A}^f$ and each transposition π on N such that for some $i, j \in N$, $\pi(i) = j$, $\pi(j) = i$, and $A_i, A_j \neq \emptyset$, we have $A^\pi \in \mathcal{A}^f$. The argument in Step 2 corresponds to Case 2 in the outline presented in Section 3.4.5. Let $A \in \mathcal{A}^f$. Let π be a transposition on N such that for some distinct $i, j \in N$, $\pi(i) = j$, $\pi(j) = i$, and $A_i, A_j \neq \emptyset$. We show that $A^\pi \in \mathcal{A}^f$. There are two cases.

Case 1. There exists $k \in N \backslash \{i, j\}$ such that $A_k = \emptyset$.

agents

$i \quad j \quad k \quad N \setminus \{i,j,k\}$

$$\begin{aligned} A &= (A_i, A_j, A_k, A_{-\{i,j,k\}}) \\ A^{\pi^1} &= (A_k, A_j, A_i, A_{-\{i,j,k\}}) \\ A^{\pi^2\pi^1} &= (A_j, A_k, A_i, A_{-\{i,j,k\}}) \\ A^\pi = A^{\pi^3\pi^2\pi^1} &= (A_j, A_i, A_k, A_{-\{i,j,k\}}) \end{aligned}$$

Figure 11: An illustration of the packages A, A^{π^1}, $A^{\pi^2\pi^1}$, and $A^{\pi^3\pi^2\pi^1}$ in Case 1 of Step 2

We introduce a notation: given two transpositions π^1 and π^2 on N, we define $A^{\pi^2\pi^1} \equiv (A^{\pi^1})^{\pi^2}$. Let π^1 be a transposition on N such that $\pi^1(i) = k$ and $\pi^1(k) = i$. By $A_k = \emptyset$, Step 1 implies that $A^{\pi^1} \in \mathcal{A}^f$. Let π^2 be a transposition on N such that $\pi^2(i) = j$ and $\pi^2(j) = i$, and π^3 a transposition on N such that $\pi^3(j) = k$ and $\pi^3(k) = j$. Then, by successively applying Step 1 and using $A_k = \emptyset$, we obtain $A^{\pi^3\pi^2\pi^1} \in \mathcal{A}^f$. Thus, since $A^\pi = A^{\pi^3\pi^2\pi^1}$ (see Figure 11), it follows that $A^\pi \in \mathcal{A}^f$.

Case 2. For each $k \in N \backslash \{i, j\}$, $A_k \neq \emptyset$.

To show $A^\pi \in \mathcal{A}^f$, suppose for contradiction that $A^\pi \notin \mathcal{A}^f$.

We first claim that for each $A' \in \mathcal{A}^f$, $A'_j \not\supseteq A_i$, or there exists $k \in N \backslash \{i, j\}$ such that $A'_k \not\supseteq A_k$. Suppose for contradiction that there exists $A' \in \mathcal{A}^f$ such that $A'_j \supseteq A_i$, and for each $k \in N \backslash \{i, j\}$, $A'_k \supseteq A_k$. Since $\mathcal{A}^f$ is a bundling unit-demand constraint (see Appendix A.2.1) and $A_i \neq \emptyset$, it follows as in Step 1 that $A'_j = A_i$. Moreover, for each $k \in N \backslash \{i, j\}$, since $A_k \neq \emptyset$, A_k is maximal in $\mathcal{M}_k$. Hence, $A'_k \supseteq A_k$ implies that $A'_k = A_k$. It then follows that $\bigcup_{k \in N \backslash \{i\}} A'_k = \bigcup_{k \in N \backslash \{j\}} A_k$, and using no wastage of $\mathcal{A}^f$, we obtain $A'_i = A_j$.

Therefore, $A^\pi = A'$, contradicting the assumption that $A' \in \mathcal{A}^f$ and $A^\pi \notin \mathcal{A}^f$.

It follows that for each $A' \in \mathcal{A}^f$, $A'_j \not\supseteq A_i$, or there exists $k \in N \backslash \{i, j\}$ such that $A'_k \not\supseteq A_k$. The remaining part of the proof, including the construction of a preference profile, follows the same argument as in Step 1. Thus, we omit the details. ■

B Proof of Theorem

In this section, we present the proof of Theorem.

We show that for each rich domain $\mathcal{R}$ and each partition $\mathcal{B}$ of M, we have $\mathcal{R}|_{\mathcal{B}_0} = \overline{\mathcal{R}}|_{\mathcal{B}_0}$. Let $\mathcal{R}$ be a rich domain, and let $\mathcal{B}$ be a partition of M. Since $\mathcal{R} \subseteq \overline{\mathcal{R}}$, it follows that $\mathcal{R}|_{\mathcal{B}_0} \subseteq \overline{\mathcal{R}}|_{\mathcal{B}_0}$.

We now show the reverse inclusion, i.e., $\overline{\mathcal{R}}|_{\mathcal{B}_0} \subseteq \mathcal{R}|_{\mathcal{B}_0}$. Let $R_i|_{\mathcal{B}_0} \in \overline{\mathcal{R}}|_{\mathcal{B}_0}$, where $R_i \in \overline{\mathcal{R}}$. Let $R'_i \in \mathcal{R}^{Add}$ be such that for each $B_i \in \mathcal{B}$ and each $t_i \in \mathbb{R}$, $w(B_i, t_i; R'_i) = w(B_i, t_i; R_i)$. Note that such $R'_i \in \mathcal{R}^{Add}$ can be defined because $\mathcal{B}$ is a partition of M. By richness of $\mathcal{R}$, it follows that $R'_i \in \mathcal{R}$. Hence, $R'_i|_{\mathcal{B}_0} \in \mathcal{R}|_{\mathcal{B}_0}$. By the definition of R'_i, for each $B_i, B'_i \in \mathcal{B}_0$ and each $t_i, t'_i \in \mathbb{R}$, we have $(B_i, t_i)\ R'_i\ (B'_i, t_i)$ if and only if $(B_i, t_i)\ R_i\ (B'_i, t'_i)$. This implies that $R'_i|_{\mathcal{B}_0} = R|_{\mathcal{B}_0}$. Thus, since $R'_i|_{\mathcal{B}_0} \in \mathcal{R}|_{\mathcal{B}_0}$, we conclude that $R_i|_{\mathcal{B}_0} \in \mathcal{R}|_{\mathcal{B}_0}$.

We now complete the proof of Theorem. We have already shown $\mathcal{R}|_{\mathcal{B}_0} = \overline{\mathcal{R}}|_{\mathcal{B}_0}$. Thus, the "if" part of Theorem follows from Fact 2 (i). We next prove the "only if" part of Theorem. Suppose that a rule f on $\mathcal{R}^n$ satisfies *constrained efficiency*, *no wastage*, *equal treatment of equals*, *strategy-proofness*, *individual rationality*, and *no subsidy*. By Proposition, there exists a partition $\mathcal{B}$ of M such that $\mathcal{A}^f = \mathcal{C}^*(\mathcal{B})$ and $|\mathcal{B}| \leq n$. Since $\mathcal{R}|_{\mathcal{B}_0} = \overline{\mathcal{R}}|_{\mathcal{B}_0}$, it then follows from Fact 2 (ii) that f is a $\mathcal{B}$-bundling MPW rule. ■

C Proof of Claim 5

In this section, we provide the proof of Claim 5. For each $R \in \mathcal{R}^n$ such that $f_i(R) = \big(M, V(M, \mathbf{0}; R_0)\big)$ for some $i \in N$, we have $f(R) \neq g(R)$. Note that, by richness of $\mathcal{R}$, there exists such $R \in \mathcal{R}^n$. Thus, f is different from any bundling MPW rule.

We show that f satisfies the properties except for *constrained efficiency*. Since g satisfies *no wastage*, *equal treatment of equals*, *anonymity*, *no envy*, *individual rationality*, and *no subsidy*, f inherits these properties.

We now show that f satisfies *strategy-proofness*.[31] Let $R \in \mathcal{R}^4$. We show that agent 1

[31] If $n = 3$ and the domain includes non-additive preferences, the same type of rule f as in Example 5 is not *strategy-proof*. In the case $n = 3$, the rule is defined as follows. Let $R_0 \in \mathcal{R}$.
(i) If $|\{i \in N : R_i = R_0\}| = 2$, and for the unique $i \in N$ with $R_i \neq R_0$, it holds that $\big(M, V(M, \mathbf{0}; R_0)\big)\ P_i\ g_i(R)$, then define $f_i(R) = \big(M, V(M, \mathbf{0}; R_0)\big)$ and $f_j(R) = \mathbf{0}$ for each $j \in N \backslash \{i\}$.

cannot benefit from misrepresenting his preferences; the same argument applies to any $i \neq 1$.

There are three cases.

Case 1. $\left|\{i \in N \setminus \{1\} : R_i = R_0\}\right| \leq 1$.

In this case, for each $R'_1 \in \mathcal{R}$, we have $|\{i \in N : R'_i = R_0| \leq 2$, where $R'_i = R_i$ for each $i \in N \setminus \{1\}$. Hence, by the definition of the rule f, it follows that $f_1(R'_1, R_{-1}) = g_1(R'_1, R_{-1})$ for each $R'_1 \in \mathcal{R}$. Since g is *strategy-proof* (see Fact 2 (i)), agent 1 cannot benefit from misrepresenting his preferences.

Case 2. $\left|\{i \in N \setminus \{1\} : R_i = R_0\}\right| = 2$.

In this case, for each $R'_1 \in \mathcal{R}$, we have either $f_1(R'_1, R_{-1}) = g_1(R'_1, R_{-1})$ or $f_1(R'_1, R_{-1}) = \mathbf{0}$. We claim that $f_1(R)\ R_1\ g_1(R)$. Note that either (i) $f_1(R) = g_1(R)$ or (ii) $f_1(R) \neq g_1(R)$ and $f_1(R) = \mathbf{0}$.

In case (i), the claim holds trivially. In case (ii), since exactly two agents other than agent 1 have preference R_0, the rule assigns $f_1(R) = \mathbf{0}$ only when $R_1 = R_0$. In this case, we can compute $p^{\min}(R, \mathcal{B}) \equiv \left(p_a^{\min}(R, \mathcal{B}), p_b^{\min}(R, \mathcal{B})\right) = \Big(V\big(\{a\}, \mathbf{0}; R_0\big), V\big(\{b\}, \mathbf{0}; R_0\big)\Big)$.[32] Thus, since $R_1 = R_0$, we have $f_1(R) = \mathbf{0}\ I_1\ g_1(R)$.

(ii) If $|\{i \in N : R_i = R_0\}| \neq 2$, or if $|\{i \in N : R_i = R_0\}| = 2$ but the above condition $\big(M, V(M, \mathbf{0}; R_0)\big)\ P_i\ g_i(R)$ fails, then let $f(R) = g(R)$.

This rule f is not *strategy-proof*. To see this, let $R_0 \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that $w\big(\{a\}; R_0\big) = w\big(\{b\}; R_0\big) = 2$ and $w(M; R_0) = 4$. Let $R_1 \in \mathcal{R} \cap \mathcal{R}^Q$ be such that $w\big(\{a\}; R_1\big) = w\big(\{b\}; R_1\big) = 1$ and $w(M; R_1) = 5$, and let $R_2 = R_3 = R_0$. Note that agent 1 has a non-additive preference R_1.

Then, $g_2(R) = \big(\{a\}, 1\big)$ and $f_2(R) = \mathbf{0}$. Let $R'_2 \in \mathcal{R}^{Add} \cap \mathcal{R}^Q$ be such that $w\big(\{a\}; R'_2\big) = w\big(\{b\}; R'_2\big) = 3$ and $w(M; R'_2) = 6$. Then, $f_2(R'_2, R_{-2}) = g_2(R'_2, R_{-2}) = \big(\{a\}, 1\big)$. Hence, $f(R'_2, R_{-2})\ P_2\ f(R)$, and f fails *strategy-proofness*.

In this example, the non-additivity of R_1 is crucial for agent 2 to benefit from misreporting R'_2. Even when $n = 3$, if the domain includes only additive preferences, the same type of rule f as in Example 5 is *strategy-proof*. Therefore, *constrained efficiency* is indispensable for the conclusion of Theorem.

[32] To see this, let $z \in Z^{\min}(R, \mathcal{B})$ be a $\mathcal{B}$-bundling unit-demand Walrasian equilibrium allocation supported by $p^{\min}(\mathcal{B}, R)$. Given $R_1 = R_0$ and $\left|\{i \in N \setminus \{1\} : R_i = R_0\}\right| = 2$, exactly three agents have the same preference R_0. Since $\mathcal{B}$ contains only two packages, at least one of these agents—say agent j—must receive $\mathbf{0}$ under z. If $p_a^{\min}(R, \mathcal{B}) < V\big(\{a\}, \mathbf{0}; R_0\big)$, then agent j would strictly prefer package $\{a\}$ to his assigned package $A_j = \emptyset$ at price $p^{\min}(R, \mathcal{B})$, contradicting the definition of a $\mathcal{B}$-bundling unit-demand Walrasian equilibrium. Thus, $p_a^{\min}(R, \mathcal{B}) \geq V\big(\{a\}, \mathbf{0}; R_0\big)$. Similarly, we have $p_b^{\min}(R, \mathcal{B}) \geq V(\{b\}, \mathbf{0}; R_0)$.

Let $p' \equiv \Big(V\big(\{a\}, \mathbf{0}; R_0\big), V\big(\{b\}, \mathbf{0}; R_0\big)\Big)$ be a $\mathcal{B}$-bundling price vector. Let $k \in N$ be the unique agent with $R_k \neq R_0$ (such k exists since exactly three agents have preference R_0).

Consider an allocation $z' \equiv (A', t')$ such that: (i) $A' \in \mathcal{C}^*(\mathcal{B})$; (ii) agent k receives his most preferred package in $\mathcal{B}$ at price p'; (iii) the remaining packages in $\mathcal{B}$ are allocated arbitrarily among the other agents; and (iv) for each $l \in N$, $t'_l = p'_{A'_l}$.

Then, (z', p') is a $\mathcal{B}$-bundling unit-demand Walrasian equilibrium for R, so $p' \in P(\mathcal{B}, R)$. We already showed $p' \leq p^{\min}(R, \mathcal{B})$. Since $p^{\min}(R, \mathcal{B})$ is the minimum element of $P(\mathcal{B}, R)$ and $p' \in P(\mathcal{B}, R)$, we have $p^{\min}(R, \mathcal{B}) \leq p'$. Combining these inequalities yields $p^{\min}(R, \mathcal{B}) = p' = \Big(V\big(\{a\}, \mathbf{0}; R_0\big), V\big(\{b\}, \mathbf{0}; R_0\big)\Big)$.

Therefore, we have $f_1(R)\ R_1\ g_1(R)$. Given that $\big|\{i \in N\backslash\{1\} : R_i = R_0\}\big| = 2$, we have, for each $R_1' \in \mathcal{R}$, either $f_1(R_1', R_{-1}) = g_1(R_1', R_{-1})$ or $f_1(R_1', R_{-1}) = \mathbf{0}$. By *strategy-proofness* of g (see Fact 2 (i)), we have $f_1(R)\ R_1\ g_1(R)\ R_1\ g_1(R_1', R_{-1})$ for each $R_1' \in \mathcal{R}$, and by *individual rationality* of g (Fact 2 (i)), $f_1(R)\ R_1\ g_1(R)\ R_1\ \mathbf{0}$. Thus, in either case, agent 1 cannot benefit by misrepresenting his preferences.

Case 3. $\big|\{i \in N \setminus \{1\} : R_i = R_0\}\big| = 3$.

We show that $f_1(R)\ R_1\ g_1(R)$ and $f_1(R)\ R_1\ (M, V(M, \mathbf{0}; R_0))$. There are three cases.

First, suppose $R_1 = R_0$. Then, since more than three agents have preference R_0, $f_1(R) = g_1(R)$. It follows from *individual rationality* of g (see Fact 2 (i)) that $f_1(R) = g_1(R)\ R_1\ \mathbf{0}\ I_1\ \big(M, V(M, \mathbf{0}; R_1)\big) = \big(M, V(M, \mathbf{0}; R_0)\big)$.

Second, suppose $R_1 \neq R_0$ and $g_1(R)\ R_1\ \big(M, V(M, \mathbf{0}; R_0)\big)$. Then, since three agents other than agent 1 have preference R_0, we have $f_1(R) = g_1(R)$, and hence $f_1(R) = g_1(R)\ R_1\ \big(M, V(M, \mathbf{0}; R_0)\big)$.

Finally, if $R_1 \neq R_0$ and $\big(M, V(M, \mathbf{0}; R_0)\big)\ P_1\ g_1(R)$, then, again by the fact that three agents other than agent 1 have preference R_0, $f_1(R) = \big(M, V(M, \mathbf{0}; R_0)\big)$. Thus, $f_1(R) = \big(M, V(M, \mathbf{0}; R_0)\big)\ P_1\ g_1(R)$.

In all three cases, we conclude that $f_1(R)\ R_1\ g_1(R)$ and $f_1(R)\ R_1\ \big(M, V(M, \mathbf{0}; R_0)\big)$. Moreover, since $\big|\{i \in N \setminus \{1\} : R_i = R_0\}\big| = 3$, it follows from the definition of the rule f that for each $R_1' \in \mathcal{R}$, $f_1(R_1', R_{-1}) \in \Big\{g_1(R_1', R_{-1}), \big(M, V(M, \mathbf{0}; R_0)\big)\Big\}$. Then, by *strategy-proofness* of g (see Fact 2 (i)), we have $f_1(R)\ R_1\ g_1(R)\ R_1\ g_1(R_1', R_{-1})$. Thus, agent 1 cannot benefit from misrepresenting his preferences.

Finally, we show that f violates *constrained efficiency*. Let $R \in (\mathcal{R}^{Add} \cap \mathcal{R}^{Q})^4$ be such that: (i) $w\big(\{a\}; R_1\big) = 2$, $w\big(\{b\}; R_1\big) = 2$, and $w(M; R_1) = 4$; (ii) $R_2 = R_1$; (iii) $w\big(\{a\}; R_3\big) = 1$, $w\big(\{b\}; R_3\big) = 3$, and $w(M; R_3) = 4$; and (iv) $w\big(\{a\}; R_4\big) = 5$, $w\big(\{b\}; R_4\big) = 5$, and $w(M; R_4) = 10$. Then, $f(R) = g(R) = \Big(\mathbf{0}, \mathbf{0}, \big(\{b\}, 2\big), \big(\{a\}, 2\big)\Big)$. Let $z \equiv \big(\mathbf{0}, \mathbf{0}, (\emptyset, -1), (M, 7)\big)$. Note that $A \equiv (\emptyset, \emptyset, \emptyset, M) \in \mathcal{A}^f$, where A is the object allocation associated with z. Moreover, z Pareto dominates $f(R)$ for R. Therefore, f violates *constrained efficiency*. ■

D Rich domains

In this section, we present examples of rich domains to which Theorem applies.

A **price vector** is a vector $p \equiv (p_a)_{a \in M} \in \mathbb{R}^m_+$. Note that a price vector differs from a bundling price vector in that a price vector specifies the price of each *object*, whereas a bundling price vector specifies the price of each *package*. Thus, a price vector corresponds to a $\overline{\mathcal{B}}$-bundling price vector.

Given a preference $R_i \in \mathcal{R}$ and a price vector $p \in \mathbb{R}^m_+$, the (**Walrasian**) **demand set for R_i at p** is defined as

$$D(R_i, p) \equiv \left\{ A_i \in \mathcal{M} : \forall A'_i \in \mathcal{M}, \ \left(A_i, \sum_{a \in A_i} p_a \right) \ R_i \ \left(A'_i, \sum_{a \in A'_i} p_a \right) \right\},$$

where we set $\sum_{a \in \emptyset} p_a \equiv 0$. In words, the demand set for R_i at p is the set of most preferred packages at the given price vector p.

Given a preference $R_i \in \mathcal{R}$, a price vector $p \in \mathbb{R}^m_+$, and $z_i \in \mathcal{M} \times \mathbb{R}$, the **Hicksian demand set for R_i at p and z_i** is defined as

$$D_H(R_i, p, z_i) \equiv \left\{ A_i \in \mathcal{M} : \exists t_i \in \mathbb{R} \text{ such that } (A_i, t_i) \in \underset{(A'_i, t'_i) \in \mathcal{M} \times \mathbb{R} : (A'_i, t'_i) \, R_i \, z_i}{\arg\min} \left(\sum_{a \in A'_i} p_a - t'_i \right) \right\}.$$

Here, a payment is considered a good, with its price normalized to -1. Thus, the expenditure for (A_i, t_i) at price vector p is equal to $\sum_{a \in A_i} p_a - t_i$. The Hicksian demand set at p and z_i is the set of expenditure-minimizing packages that yield at least as high a welfare level as z_i.

The following are examples of rich domains. These domains have recently attracted attention because they ensure the existence of a Walrasian equilibrium without requiring quasi-linearity.

Example 11 (Rich domains). The following are all rich domains.

- A preference R_i satisfies the **net substitutes condition** (Kelso and Crawford, 1982; Baldwin et al., 2023) if for each price vector $p \in \mathbb{R}^m_+$, each $z_i \in \mathcal{M} \times \mathbb{R}$, each $a \in M$, each $\delta \in \mathbb{R}_{++}$, and each $A_i \in D_H(R_i, p, z_i)$, there exists $A'_i \in D_H(R_i, p + \delta \mathbf{e}_a, z_i)$ such that $A_i \backslash \{a\} \subseteq A'_i$. That is, a preference satisfies the net substitutes condition if, whenever the price of an object increases, the Hicksian demand for the other objects does not decrease. This condition reflects substitutability among all objects and guarantees the existence of a Walrasian equilibrium (Kelso and Crawford, 1982; Baldwin et al., 2023). Let $\mathcal{R}^{NS}$ denote the class of all preferences that satisfy the net substitutes condition. Then, $\mathcal{R}^{NS}$ is a rich domain (Kelso and Crawford, 1982; Baldwin et al., 2023).[33]

[33]To be more specific, the richness of $\mathcal{R}^{NS}$ follows from two established results:

– Baldwin et al. (2023) show that a preference R_i satisfies the net substitutes condition, provided that for

- A preference R_i satisfies the **net complements condition** (Rostek and Yoder, 2020; Baldwin et al., 2023) if for each $p, p' \in \mathbb{R}^m_+$ with $p \geq p'$, each $z_i \in \mathcal{M} \times \mathbb{R}$, each $A_i \in D_H(R_i, p, z_i)$, and each $A'_i \in D_H(R_i, p', z_i)$, we have that $A_i \cap A'_i \in D_H(R_i, p, z_i)$ and $A_i \cup A'_i \in D_H(R_i, p', z_i)$. Roughly speaking, a preference satisfies the net complements condition if, as prices increase, the Hicksian demand for the objects does not increase. The net complements condition reflects complementarity among all objects and ensures the existence of a Walrasian equilibrium (Rostek and Yoder, 2020; Baldwin et al., 2023). Let $\mathcal{R}^{NC}$ denote the class of all preferences that satisfy the net complements condition. Then, $\mathcal{R}^{NC}$ is a rich domain (Rostek and Yoder, 2020; Baldwin et al., 2023).

- Let $\mathcal{B}$ be a partition of M such that $|\mathcal{B}| = 2$. Thus, M is partitioned into two sets, M_1 and M_2. A preference R_i satisfies the **net substitutes and complements condition (with respect to $\mathcal{B}$)** (Sun and Yang, 2006; Baldwin et al., 2023) if for each price vector $p \in \mathbb{R}^m_+$, each $z_i \in X \times \mathbb{R}$, each distinct $j, k \in \{1, 2\}$, each $a \in M_j$, each $\delta \in \mathbb{R}_{++}$, and each $A_i \in D_H(R_i, p, z_i)$, there exists $A'_i \in D_H(R_i, p + \delta \mathbf{e}_a, z_i)$ such that $(A_i \cap M_j) \backslash \{a\} \subseteq A'_i \cap M_j$ and $A'_i \cap M_k \subseteq A_i \cap M_k$. That is, the net substitutes and complements condition requires that when the price of an object in M_j increases, the Hicksian demand for other objects in the same set does not decrease, while the demand for objects in the other set does not increase. This condition reflects substitutability within each set and complementarity across sets. It guarantees the existence of a Walrasian equilibrium (Sun and Yang, 2006; Baldwin et al., 2023). Given a partition $\mathcal{B}$ of M with $|\mathcal{B}| = 2$, let $\mathcal{R}^{NSC}(\mathcal{B})$ denote the class of all preferences that satisfy the net substitutes and complements condition with respect to $\mathcal{B}$. Then, $\mathcal{R}^{NSC}(\mathcal{B})$ is a rich domain (Sun and Yang, 2006; Baldwin et al., 2023).

- A preference R_i satisfies the **single improvement condition** (Gul and Stacchetti, 1999; Nguyen and Vohra, 2024) if for each price vector $p \in \mathbb{R}^m_+$ and each $A_i \notin D(R_i, p)$, there exists $A'_i \in \mathcal{M}$ such that $\left(A'_i, \sum_{a \in A'_i} p_a\right)$ P_i $\left(A_i, \sum_{a \in A_i} p_a\right)$, $|A_i \backslash A'_i| \leq 1$, and $|A'_i \backslash A_i| \leq 1$. That is, a preference satisfies the single improvement condition if any suboptimal bundle (in terms of Walrasian demand) can be improved by removing, adding, or swapping a single object. The single improvement condition is equivalent to the net substitutes condition under quasi-linear preferences (Gul and Stacchetti, 1999),

each payment $t_i \in \mathbb{R}$, the quasi-linear preference $R'_i \in \mathcal{R}^Q$ with willingness to pay $w(\cdot; R'_i) = w(\cdot, t_i; R_i)$ satisfies the net substitutes condition.

– Kelso and Crawford (1982) establish that all additive quasi-linear preferences satisfy the net substitutes condition.

By the same argument, one can also show that the net complements domain and the net substitutes and complements domain introduced below are both rich.

and has played a central role in the design of dynamic auctions that converge to a Walrasian equilibrium under quasi-linear preferences (Gul and Stacchetti, 2000).[34] It also guarantees the existence of a Walrasian equilibrium even without assuming quasi-linearity (Nguyen and Vohra, 2024). Let $\mathcal{R}^{SI}$ denote the class of all preferences that satisfy the single improvement condition. Then, $\mathcal{R}^{SI}$ is a rich domain.[35]

- Gul and Stacchetti (1999) also introduce another condition that is equivalent to both the net substitutes condition and the single improvement condition under quasi-linear preferences. A preference R_i satisfies the **no complementarities condition** (Gul and Stacchetti, 1999) if for each price vector $p \in \mathbb{R}^m_+$, each $A_i, A'_i \in D(R_i, p)$, and each $A''_i \in \mathcal{M}$ with $A''_i \subseteq A_i$, there exists $A'''_i \in \mathcal{M}$ such that $A'''_i \subseteq A'_i$ and $A_i \setminus (A''_i \cup A'''_i) \in D(R_i, p)$. This condition guarantees the existence of a Walrasian equilibrium without assuming quasi-linearity (Nguyen and Vohra, 2024). Let $\mathcal{R}^{NoC}$ denote the class of all preferences that satisfy the no complementarities condition. Then, $\mathcal{R}^{NoC}$ is a rich domain. □

We can also define the gross substitutes condition (Kelso and Crawford, 1982), the gross complements condition (Rostek and Yoder, 2020), and the gross substitutes and complements condition (Sun and Yang, 2006) in terms of the Walrasian demand set instead of the Hicksian demand set. However, none of these domains is rich.

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[34] Without quasi-linearity, the equivalence between the single improvement condition and the net substitutes condition does not necessarily hold. Similarly, under non-quasi-linear preferences, the equivalence between the no complementarities condition—introduced below—and the net substitutes condition does not necessarily hold.

[35] The proof sketch for the richness of $\mathcal{R}^{SI}$ is as follows. Since the single improvement condition concerns the Walrasian demand set at a given price vector, a preference R_i satisfies the condition if for each $t_i \in \mathbb{R}$, the quasi-linear preference $R'_i \in \mathcal{R}^Q$ with $w(\cdot; R'_i) = w(\cdot, t_i; R_i)$ satisfies the single improvement condition. Because every additive quasi-linear preference satisfies the single improvement condition (Gul and Stacchetti, 1999), it follows that every additive preference also satisfies the condition. Therefore, $\mathcal{R}^{SI}$ is rich. Similarly, $\mathcal{R}^{NoC}$, introduced below, is also rich.

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