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EQUITY IN AUCTION DESIGN WITH UNIT-DEMAND AGENTS AND NON-QUASILINEAR PREFERENCES

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Equity in auction design with unit-demand agents and non-quasilinear preferences [*]

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Abstract

We study a model of auction design where a seller is selling a set of objects to a set of agents who can be assigned no more than one object. Each agent's preference over (object, payment) pair need not be quasilinear. If the domain contains all classical preferences, we show that there is a unique mechanism, the minimum Walrasian equilibrium price (MWEP) mechanism, which is strategy-proof, individually rational, and satisfies equal treatment of equals, no-wastage (every object is allocated to some agent), and no-subsidy (no agent is subsidized). This provides an equity-based characterization of the MEWP mechanism, and complements the efficiency-based characterization of the MWEP mechanism known in the literature.

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1 Introduction

Many auctions in practice are conducted by public bodies such as governments, local authorities, and public institutions. Prominent examples include spectrum auctions and other public allocation problems involving exclusive rights, such as the allocation of ownership or development rights. In these contexts, favoring particular firms or individuals is neither permissible nor desirable, and is likely to attract public criticism.[1] Thus, in addition to standard incentive and participation constraints, fairness can be viewed as an additional constraint in such design problems.

We consider an environment in which there are multiple heterogeneous objects with unit-demand agents. Preferences of agents may not be quasilinear, and thus, may exhibit income effects. There are various notions of fairness, and the appropriate fairness notion may depend on the application. To cover as wide a range of applications as possible, we employ one of the weakest notions of fairness - *equal treatment of equals* (ETE). This criterion requires that if two agents have identical preferences, then the welfare levels they obtain from the mechanism are identical. This principle reflects Aristotle's conception of justice; he writes:[2]

> Justice is considered to mean equality. It does not mean equality - but equality for those who are equal, and not for all.

ETE can also be justified as a necessary condition for standard fairness desiderata such as no envy and anonymity.

We impose *strategy-proofness* (dominant strategy incentive compatibility) and *individual rationality* as incentive constraints. In addition, we consider two minor properties, no subsidy and no wastage. *No subsidy* requires agents' payments to be nonnegative, which is a standard assumption in auction design. *No wastage* requires that every object be assigned to some agent.[3]

In our model, the minimum Walrasian equilibrium price (MWEP) mechanism satisfies equal treatment of equals, strategy-proofness, individual rationality, no subsidy, and no

[1] For instance, in the context of spectrum auctions, fairness is recognized as an important criterion for choosing auction formats (McMillan, 1995; Kwerel and Strack, 2001).

[2] Aristotle, " Politics," Book III: The theory of citizenship and constitutions, C: The principle of oligarchy and democracy and the nature of distributional justice, Chapter 9.

[3] In our model, we assume that the number of agents is larger than that of objects.

wastage (Demange and Gale, 1985).[4] Equal treatment of equals is satisfied by a large class of mechanisms, and there also exist many mechanisms that satisfy strategy-proofness, individual rationality, no subsidy, and no wastage. Nevertheless, we show that if the domain of preferences is sufficiently rich (i.e., contains all *classical* preferences), the MWEP mechanism is the unique mechanism that satisfies equal treatment of equals, strategy-proofness, individual rationality, no subsidy, and no wastage (Theorem 1).

An important implication of our uniqueness result concerns efficiency. The MWEP mechanism is efficient. As a consequence, our main result implies that any mechanism that satisfies equal treatment of equals, strategy-proofness, individual rationality, no subsidy, and no wastage must be efficient (Corollary 1). Fairness and efficiency are often viewed as competing objectives in economics. In this environment, however, it is known that the strong fairness requirement of no envy, together with no wastage, implies efficiency (Svensson, 1983). Our result strengthens this insight: when combined with strategy-proofness, individual rationality, and no subsidy, equal treatment of equals—which is substantially weaker than no envy—together with no wastage implies efficiency.

While strategy-proofness, individual rationality and ETE are standard axioms in mechanism design, we briefly discuss our remaining axioms. [5] No subsidy is satisfied by all standard auction formats. It can also be justified as a requirement that prevents participation by "fake" bidders—agents who have (or report) very low valuations for the objects but seek to derive utility from subsidies. The no wastage axiom can be motivated as a very weak form of efficiency. It excludes auctions where reserve prices are kept. In practice, we see many auctions without a reserve price. Though reserve prices are used in many auctions too, often sellers, such as governments, fail to commit to these reserve prices by reselling the unsold objects (spectrum licenses, for example). Hence, no-wastage is a reasonable axiom in our model.

Our result relies heavily on the fact that the auction designer believes that preferences of agents may exhibit income effect, i.e., non-quasilinear. While quasilinearity is a reasonable assumption when payments of agents are small, ignoring income effects in auctions involving large payments (for instance, in selling high-worth natural resources like mines, spectrum

[4]We focus on deterministic mechanisms. Deterministic auction mechanisms are simple and transparent, and are often preferred by sellers over auctions that involve randomization.

[5]We do not consider interim individual rationality and Bayesian incentive compatibility. This allows us to work in a prior-free model.

etc.) makes economics models unrealistic. Bulow et al. (2017) mention budget-constraint among bidders (which induces non-quasilinear preferences) as one of the two main reasons why spectrum auctions are complex. When preferences are quasilinear, there are many mechanisms other than the MWEP mechanism that satisfy our five properties (Tierney, 2016). Moreover, our result does not hold even when agents have specific non-quasilinear preferences. Kazumura et al. (2020) show that the MWEP mechanism is not the only mechanism that satisfies our properties if objects are normal goods. Thus, a key assumption for our result is that the domain contains various preferences with income effects.[6] While such domain requirement seems to be demanding, this is reasonable for a seller who has no information about the extent and nature of income effect in agents' preferences over transfers, wants to be robust about this aspect of the model.

We study a model with unit-demand agents, i.e., agents can be assigned at most one out of many objects. Though restrictive, this model appears in practice in many settings. The unit-demand assumption can be justified based on institutional restrictions. For instance, while selling team franchises in professional sports leagues, it is common to restrict a buyer to buy at most one franchise; the first spectrum auction in UK restricted each bidder to buy at most one spectrum (Binmore and Klemperer, 2002). The unit-demand assumption can naturally arise from preferences of agents also. For instance, buyers in public housing markets are usually interested to buy at most one house (Andersson et al., 2016).

2 Related literature

There is a growing literature on fair mechanisms in auction models. While many papers impose fairness requirements stronger than ETE, the class of mechanisms that satisfy ETE and incentive constraints is not well understood.

When agents are unit-demand bidders, the only mechanism that satisfies no-envy, strategy-proofness, individual rationality, no subsidy, and no wastage is the MWEP mechanism (Svensson, 1983; Sakai, 2013; Morimoto and Serizawa, 2015). If objects are identical and preferences are quasilinear, the MWEP mechanism is characterized by anonymity, strategy-

[6]Another important assumption for our result is heterogeneity of objects. If objects are identical, the MWEP mechanism is not the only mechanism that satisfies the five properties. This is true both when preferences are quasilinear and when they can be non-quasilinear (Adachi, 2014).

proofness, individual rationality, no subsidy, and no wastage (Ashlagi and Serizawa, 2012). However, when objects are heterogeneous and preferences are quasilinear, the MWEP mechanism is not the unique mechanism that satisfies anonymity and these properties (Tierney, 2016). This is also true when objects are identical and preferences are allowed to be non-quasilinear (Adachi, 2014). Since ETE is weaker than anonymity, there are multiple mechanisms that satisfy ETE, strategy-proofness, individual rationality, no subsidy, and no wastage in these environments.[7] Moreover, even if objects are identical and preferences are quasilinear, there are many mechanisms that satisfy these properties. Against this background, our result is remarkable in that the MWEP mechanism is uniquely pinned down as by these properties.

Several recent papers, like ours, impose fairness conditions as a constraint in addition to incentive constraints. But they pursue different objectives. Kazumura et al. (2020) and Sakai and Serizawa (2021) show that the MWEP mechanism is ex-post revenue maximizing among mechanisms that satisfy ETE, strategy-proofness, individual rationality, no subsidy, and no wastage. Chen and Knyazev (2025) consider a single object model, and show that a second price auction with flexible reserve prices is expected revenue maximizing among mechanisms that satisfy anonymity and Bayesian incentive compatibility. While these papers adopt a similar approach to ours, they require revenue maximization as an additional axiom. In contrast, our result shows that the MWEP mechanism is uniquely characterized without specifying any particular objective, once fairness is imposed as a constraint.

The Vickrey-Clarke-Groves (VCG) mechanism plays a central role when preferences are quasilinear. It is a unique mechanism that satisfies efficiency, strategy-proofness, individual rationality, and no subsidy (Holmström, 1979; Chew and Serizawa, 2007). When preferences may not be quasilinear, however, the VCG is no longer efficient or strategy-proof. In such settings with unit-demand agents, the MWEP mechanism is characterized using efficiency, strategy-proofness, individual rationality, and no subsidy (Saitoh and Serizawa, 2008; Sakai, 2008; Morimoto and Serizawa, 2015; Zhou and Serizawa, 2018; Wakabayashi et al., 2025).

Our result parallels the characterization of the MWEP mechanism in Morimoto and Serizawa (2015). They consider the same model and domain and characterize the MWEP using efficiency. Since no-wastage is a significant weakening of Pareto efficiency, our result

[7] Tierney (2016) and Adachi (2014) characterize the MWEP mechanism using a continuity axiom in addition to anonymity and other properties.

highlights that equity can almost substitute efficiency objectives of an auction designer. We believe that such an alternate foundation is useful because fairness is an important desideratum in auction design and its relation to efficiency is not well understood in auction models. Besides, equal treatment of equals is a more *testable* axiom in practice than Pareto efficiency. For instance, there are legal implications of violating equal treatment of equals in auctions – Deb and Pai (2016) contain examples of some prominent lawsuits in the United States where auctioneers have been dragged to court for designing auction mechanisms that discriminate among bidders. On the other hand, violating Pareto efficiency usually does not have any legal implications.

In models with multi-demand agents, a Walrasian equilibrium may not exist.[8] Even when a Walrasian equilibrium exists, the MWEP mechanism may not be strategy-proof. Moreover, when preferences may not be quasilinear, an efficient and strategy-proof mechanism does not exist (Kazumura and Serizawa, 2016; Baisa, 2020; Malik and Mishra, 2021; Shinozaki and Serizawa, 2025b). Thus, Shinozaki and Serizawa (2025a) pursue constrained efficiency, and introduce the notion of the bundling unit-demand minimum price Walrasian mechanism. This mechanism partitions objects into several bundles and selects a minimum price Walrasian equilibrium when the bundles are regarded as objects. They show that this is the unique mechanism that satisfies constrained efficiency, ETE, strategy-proofness, and no subsidy.

3 Model and definitions

There are $n \geq 2$ agents and $m \geq 2$ objects with $n > m$. We denote the set of agents by $N \equiv \{1, \ldots, n\}$ and the set of objects by $M \equiv \{1, \ldots, m\}$. Let $L \equiv M \cup \{0\}$, where assigning 0 means not assigning any (real) object from M. We call 0 the null object, and unlike a real object in M, it can be assigned to any number of agents. Each agent receives at most one object and pays some amount of money. Thus, agents' common **consumption set** is $L \times \mathbb{R}$, and a generic **(consumption) bundle** for any agent $i \in N$ is a pair $z_i = (a, t) \in L \times \mathbb{R}$.

[8] There is a large literature on the existence of a Walrasian equilibrium; see, for example, Kelso and Crawford (1982), Sun and Yang (2006), Teytelboym (2014), and Baldwin and Klemperer (2019). The existence with non-quasilinear preferences is studied by Baldwin et al. (2023) and Nguyen and Vohra (2024).

3.1 Classical preferences

Each agent $i \in N$ has a complete and transitive preference R_i over $L \times \mathbb{R}$. Let P_i and I_i be the strict and indifference relations associated with R_i.

Definition 1 *A preference R_i is* **classical** *if it satisfies the following four conditions:*

1. **Money monotonicity.** *For every $t > t'$ and for every $a \in L$, we have $(a, t')\ P_i\ (a, t)$.*
2. **Desirability of objects.** *For every t and for every $a \in M$, we have $(a, t)\ P_i\ (0, t)$.*
3. **Continuity.** *For every $z \in L \times \mathbb{R}$, the sets $\{z' : z'\ R_i\ z\}$ and $\{z' : z\ R_i\ z'\}$ are closed.*
4. **Possibility of compensation.** *For every $z \in L \times \mathbb{R}$ and for every $a \in L$, there exists t and t' such that $z\ R_i\ (a, t)$ and $(a, t')\ R_i\ z$.*

Let $\mathcal{R}^C$ be the set of all classical preferences. We call $(\mathcal{R}^C)^n$ the **classical domain**. Throughout this paper, we assume the preferences to be classical. We use $\mathcal{R} \subseteq \mathcal{R}^C$ to denote an arbitrary domain of preferences.

A preference R_i is **quasilinear** if there exists a **valuation function** $v : L \to \mathbb{R}_+$ with $v(0) = 0$ such that for all $a, b \in L$ and for all $t, t' \in \mathbb{R}$, we have $(a, t)\ R_i\ (b, t')$ if and only if $v(a) - t \geq v(b) - t'$. We denote the set of all quasilinear preferences as $\mathcal{R}^Q$.

The existence of a valuation function $v : L \to \mathbb{R}$ is restricted to quasilinear preferences. However, the notion of valuation can be extended to classical preferences.

Definition 2 *The* **valuation** *of agent i with preference R_i for object $a \in L$ at consumption bundle z is defined as $V^{R_i}(a, z)$, which uniquely solves $(a, V^{R_i}(a, z))\ I_i\ z$.*

In other words, $V^{R_i}(a, z)$ is the unique amount of transfer that makes agent i indifferent between consumption bundles z and $(a, V^{R_i}(a, z))$. The existence of $V^{R_i}(a, z)$ and its uniqueness are guaranteed by the assumptions of classical preferences.[9] Note that if R_i is quasilinear, then for each $a \in L$, $V^{R_i}(a, (0, 0)) = v_i(a)$.

[9] See Kazumura and Serizawa (2016) for the formal proof.

3.2 Mechanisms

An **object allocation** is an n-tuple $(a_1, \ldots, a_n) \in L^n$ such that for each pair $i, j \in N$ with $i \neq j$, $a_i = a_j$ implies $a_i = a_j = 0$. We denote the set of object allocations by A. A **(feasible) allocation** is an n-tuple $z \equiv (z_1, \ldots, z_n) \equiv ((a_1, t_1), \ldots, (a_n, t_n)) \in (L \times \mathbb{R})^n$ such that $(a_1, \ldots, a_n) \in A$. We denote the set of feasible allocations by Z.

Fix a domain $\mathcal{R}^n \subseteq (\mathcal{R}^C)^n$. A **preference profile** is an n-tuple $R \equiv (R_1, \ldots, R_n) \in \mathcal{R}^n$. Given $R \in \mathcal{R}^n$ and $i \in N$, let $R_{-i} \equiv (R_j)_{j \neq i}$. Given $R \in \mathcal{R}^n$ and $N' \subseteq N$, let $R_{N'} = (R_i)_{i \in N'}$ and $R_{-N} \equiv (R_i)_{i \in N \setminus N'}$.

A **mechanism** on $\mathcal{R}^n$ is a function $f : \mathcal{R}^n \to Z$. Given a mechanism f and $R \in \mathcal{R}^n$, we denote the bundle assigned to agent i by $f_i(R)$ and we write $f_i(R) = (a_i(R), t_i(R))$, where $a_i(R)$ is the object assigned to agent i and $t_i(R)$ is his payment. We require a mechanism to satisfy the following properties.

Definition **3** *Let $f : \mathcal{R}^n \to Z$ be a mechanism defined on domain $\mathcal{R}$.*

1. *f is* **strategy-proof** *if for every $i \in N$, for every $R_{-i} \in \mathcal{R}^{n-1}$, and for every $R_i, R'_i \in \mathcal{R}$, we have*
$$f_i(R_i, R_{-i}) \ R_i \ f_i(R'_i, R_{-i}).$$

2. *f is* **(ex-post) individual rationality (IR)** *if for every $i \in N$, for every $R \in \mathcal{R}^n$, we have $f_i(R) \ R_i \ (0, 0)$.*

3. *f satisfies* **equal treatment of equals (ETE)** *if for every $i, j \in N$, for every $R \in \mathcal{R}^n$ with $R_i = R_j$, we have $f_i(R) \ I_i \ f_j(R)$.*

4. *f satisfies* **no wastage (NW)** *if for every $R \in \mathcal{R}^n$ and for every $a \in M$, there exists some $i \in N$ such that $a_i(R) = a$.*

5. *f satisfies* **no subsidy (NS)** *if for each $R \in \mathcal{R}^n$ and each $i \in N$, we have $t_i(R) \geq 0$.*

Earlier in Section 1, we discussed the above properties in detail. They can either be motivated by weak fairness (ETE) or weak efficiency (NW) or some practical concerns (NS). Next, we introduce the MWEP mechanism.

4 A characterization of the MWEP mechanism

In this section, we define the minimum Walrasian equilibrium price mechanism. A price vector $p \in \mathbb{R}_+^{|L|}$ defines a price for every object with $p_0 = 0$. At any price vector p, let $D(R_i, p) \equiv \{a \in L : (a, p_a) \; R_i \; (b, p_b) \; \forall \; b \in L\}$ denote the **demand set** of agent i with preference R_i at price vector p.

DEFINITION **4** *An object allocation $(a_1, \ldots, a_n) \in A$ and a price vector p is a* **Walrasian equilibrium** *at a preference profile $R \in (\mathcal{R}^C)^n$ if*

1. *$a_i \in D(R_i, p)$ for all $i \in N$ and*

2. *for all $a \in M$ with $a \neq a_i$ for all $i \in N$, we have $p_a = 0$.*

We refer to p and $\{z_i \equiv (a_i, p_{a_i})\}_{i \in N}$ defined above as a **Walrasian equilibrium price vector** *and a* **Walrasian equilibrium allocation** *at R respectively.*

It is known that a Walrasian equilibrium exists at each $R \in (\mathcal{R}^C)^n$ (Demange and Gale (1985); Alkan and Gale (1990)). A Walrasian equilibrium price vector p is a **minimum Walrasian equilibrium price vector** at preference profile R if for every Walrasian equilibrium price vector p' at R, we have $p_a \leq p'_a$ for all $a \in L$. Demange and Gale (1985) prove that if R is a profile of classical preferences, then the set of Walrasian equilibrium price vectors at R forms a lattice with a unique minimum and a unique maximum. We denote the minimum Walrasian equilibrium price vector at R as $p^{\min}(R)$. Note that for each $R \in \mathcal{R}^n$, even though $p^{\min}(R)$ is a unique price vector, there may be many object allocations that can support the Walrasian equilibrium. Let $Z^{\min}(R)$ denote the set of all allocations at a minimum Walrasian equilibrium at preference profile R. Note that if $((a_i, t_i))_{i \in N} \in Z^{\min}(R)$ then $t_i = p^{\min}_{a_i}(R)$, i.e., the transfer associated with an agent is the price of the object assigned to him in the Walrasian equilibrium.

DEFINITION **5** *A mechanism $f : \mathcal{R}^n \to Z$ is a* **minimum Walrasian equilibrium (MWEP) mechanism** *if*

$$f(R) \in Z^{\min}(R) \; \forall \; R \in \mathcal{R}^n.$$

Let $f^{\min} : \mathcal{R}^n \to Z$ denote an MWEP mechanism. For every $R \in \mathcal{R}$, an MWEP mechanism picks *any* one allocation in $Z^{\min}(R)$. It can be easily shown that each agent must be indifferent between all its allocations in $Z^{\min}(R)$. Hence, whenever we say *the* MWEP mechanism, we mean *any* MWEP mechanism.

Demange and Gale (1985) showed that the MWEP mechanism is strategy-proof and IR. Clearly, it also satisfies ETE, NW, and NS. Our main result is that on the classical domain, only the MWEP mechanism satisfies these properties.

THEOREM 1 *Let $f : (\mathcal{R}^C)^n \to Z$ be a mechanism. Then, f is strategy-proof, individually rational, and satisfies equal treatment of equals, no wastage, and no subsidy if and only if it is the MWEP mechanism.*

We postpone the proof of Theorem 1 to the appendix. We argue that the domain richness in Theorem 1 is somewhat necessary. Tierney (2016) contains an example which shows that there are strategy-proof and IR mechanisms satisfying ETE, NW, and NS that are not MWEP mechanisms if the domain of preferences is domain of quasilinear preferences.

Similarly, the axioms in Theorem 1 are necessary. Below, we provide some examples to establish this claim.

- STRATEGY-PROOFNESS. Consider a mechanism that chooses the maximum Walrasian equilibrium allocation at every profile. Such a mechanism will satisfy all the properties except strategy-proofness (Miyake, 1998).

- INDIVIDUAL RATIONALITY. Consider the MWEP mechanism supplemented by a constant participation fee (which is independent of the preferences of the agents and equal across all the agents). Such a mechanism will satisfy all the properties except IR.

- EQUAL TREATMENT OF EQUALS. Consider the following mechanism, where we treat some agent, say agent 1, differently. At every preference profile, agent 1 is asked to pick her best bundle in $Z_0 \equiv \{(a, 0) : a \in M\}$, i.e., she picks her best object at zero payment. Then, for the remaining agents and remaining objects, we use the MWEP mechanism for the subeconomy. Such a mechanism is clearly strategy-proof, individually rational, and satisfies no wastage and no subsidy. However, it fails equal treatment of equals.

- NO WASTAGE. Consider the mechanism which never sells any of the objects and does not ask agents to pay anything. This mechanism satisfies all the properties except no wastage.

- No subsidy. Consider the MWEP mechanism supplemented by a constant participation subsidy (which is independent of the preferences of the agents and equal across all the agents). Such a mechanism satisfies all the properties except no subsidy.

5 Connection to earlier characterizations

We describe the connection of our characterization to other characterizations of the MWEP mechanism in the literature. Most of these characterizations involve Pareto efficiency.

Definition 6 *A mechanism $f : \mathcal{R}^n \to Z$ is* **Pareto efficient** *if at every preference profile $R \in \mathcal{R}^n$, there exists no allocation $((\hat{a}_1, \hat{t}_1), \ldots, (\hat{a}_n, \hat{t}_n)) \in Z$ such that*

$$(\hat{a}_i, \hat{t}_i) \ R_i \ f_i(R) \qquad \forall \ i \in N,$$
$$\sum_{i \in N} \hat{t}_i \geq \sum_{i \in N} t_i(R),$$

with either the second inequality holding strictly or some agent i strictly preferring $(\hat{a}_i, \hat{t}_i)$ to $f_i(R)$.

The above definition is the appropriate notion of Pareto efficiency in this setting. Notice that by distributing some money among all the agents, we can always make each agent better off than the allocation in any mechanism. Hence, the above definition requires that there should not exist another allocation where the sum of transfers is not less and every agent is weakly better off.

The MWEP mechanism is Pareto efficient. The following theorem characterizes the MWEP mechanism using Pareto efficiency. We remind that $\mathcal{R}^Q$ denotes the set of quasilinear preferences.

Theorem 2 **(Holmström (1979); Morimoto and Serizawa (2015))** *Suppose $\mathcal{R} \in \{\mathcal{R}^Q, \mathcal{R}^C\}$ and let $f : \mathcal{R}^n \to Z$ be a mechanism on this domain. Then, f is strategy-proof, Pareto efficient, individually rational, and satisfies no subsidy if and only if it is an MWEP mechanism.*

Theorem 2 was proved for $\mathcal{R} = \mathcal{R}^Q$ by Holmström (1979) and for $\mathcal{R} = \mathcal{R}^C$ by Morimoto and Serizawa (2015). [10]

[10] Zhou and Serizawa (2018) have shown that Theorem 2 continues to hold in smaller non-quasilinear preference domains.

The efficiency characterization of Theorem 2 holds for both the quasilinear domain and the classical domain. This is not true for our characterization in Theorem 1. As discussed later, our characterization holds for the classical domain but breaks down for the quasilinear domain. We can use both the characterizations to point out an interesting connection between Pareto efficiency and ETE with no wastage in the classical domain.

Corollary **1** *Suppose $f : (\mathcal{R}^C)^n \to Z$ is a strategy-proof and individually rational mechanism satisfying no subsidy. Then, the following are equivalent.*

1. *f is Pareto efficient.*

2. *f satisfies equal treatment of equals and no wastage.*

Corollary 1 is true because in the classical domains, the only mechanism satisfying these axioms is the MWEP mechanism, which is not the case in the quasilinear domain. The connection between fairness and efficiency has been known in the literature. Svensson (1983) shows that no envy and no wastage imply Pareto efficiency.[11] Corollary 1 shows that equal treatment of equal, which is substantially weaker than no envy, and no wastage imply Pareto efficiency under additional auxiliary axioms such as strategy-proofness and individual rationality.

The other result which is worth explaining is Kazumura et al. (2020). They consider a class of domains that they call *rich* domains. Formally, it is defined as follows.

Definition **7** *A domain of preference $\mathcal{R}$ is* **rich** *if for all $a \in M$ and for all $\hat{p}$ with $\hat{p}_a > 0$ and $\hat{p}_b = 0$ for all $b \neq a$ and for every p with $p_x > \hat{p}_x$ for all $x \in M$, there exists a preference $R_i \in \mathcal{R}$ such that $D(R_i, \hat{p}) = \{a\}$ and $D(R_i, p) = \{0\}$.*

It is not difficult to see that many domains of preferences can be rich. Kazumura et al. (2020) show that the domain of quasilinear preferences ($\mathcal{R}^Q$), the domain of classical preferences ($\mathcal{R}^C$), and domains containing all *positive income effect* preferences satisfy richness. If a domain is rich, then the following result holds. Denote the *revenue* at a preference profile R from a mechanism f as $\text{Rev}^f(R) \equiv \sum_{i \in N} t_i(R)$, i.e., the sum of payments of all the agents at preference profile R.

[11] A mechanism $f : \mathcal{R}^n \to Z$ satisfies *no envy* if for each $R \in \mathcal{R}^n$ and each pair $i, j \in N$, $f_i(R)\ R_i\ f_j(R)$. It is clear that no envy implies equal treatment of equals.

Definition 8 *A mechanism $f : \mathcal{R}^n \to Z$ is* **ex-post revenue optimal** *among a class of mechanisms if for every mechanism $g : \mathcal{R}^n \to Z$ in this class,*

$$\text{Rev}^f(R) \geq \text{Rev}^g(R) \qquad \forall\ R \in \mathcal{R}^n.$$

Theorem 3 (**Kazumura et al. (2020)**) *Suppose $\mathcal{R}$ is a rich domain. Then, every MWEP mechanism is ex-post revenue optimal among the class of strategy-proof and IR mechanisms satisfying equal treatment of equals, no wastage, and no subsidy.*

Since the classical domain is rich, it follows that Theorem 3 is an easy corollary of our main result Theorem 1 when applied to the classical domain. However, such a conclusion cannot be drawn for other rich domains. For instance, the quasilinear domain of preferences is rich, and we know that Theorem 3 holds but Theorem 1 does not hold in that domain.

Two other characterizations are worth pointing. Sakai (2013) shows that if there is a *single* object, then the MWEP mechanism is the unique strategy-proof and individually rational mechanism satisfying ETE, NW, and NS in the classical domain. Hence, we generalize the result of Sakai (2013) to the case of multiple heterogeneous objects. Ashlagi and Serizawa (2012) characterizes the MWEP mechanism for the *multiple identical* objects case in the *quasilinear* domain, when agents can be assigned at most one object, using strategy-proofness, individual rationality, no subsidy, no wastage, and *anonymity* (which is stronger than equal treatment of equals). As we have discussed, our characterization does not hold in the quasilinear domain. Further, the result of Ashlagi and Serizawa (2012) does not extend to the case where objects are *heterogenous* as Tierney (2016) shows that there are strategy-proof and individually rational mechanisms satisfying no subsidy, no wastage, and anonymity which is not the MWEP mechanism. Also, the result of Ashlagi and Serizawa (2012) does not extend to the case where objects are identical but preferences are classical (Adachi, 2014).

6 Outline of the proof of Theorem 1

In this section, we first introduce a useful representation of preferences, which we often use in the proof of Theorem 1. We then explain the sketch of the proof using a simple example.

6.1 Indifference vectors

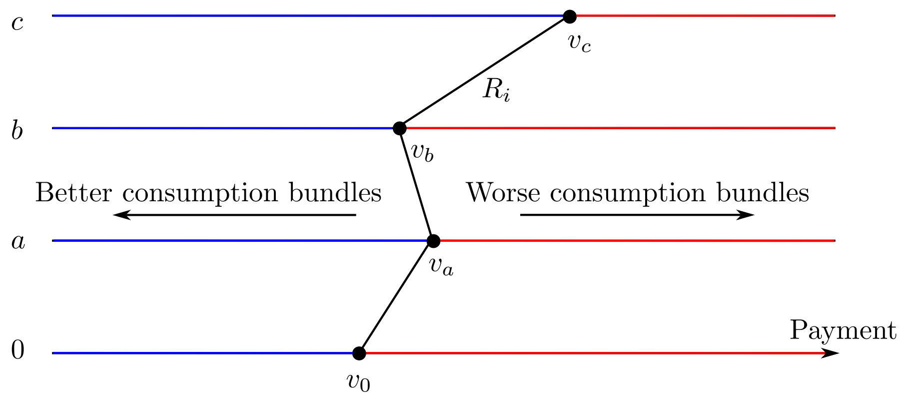

Figure 1: An indifference vector

A vector $v \in \mathbb{R}^{m+1}$ is an **indifference vector** of classical preference R_i if for all $a, b \in L$, we have $(a, v_a)\ I_i\ (b, v_b)$. Denote the set of all indifference vectors of R_i as $\mathcal{I}(R_i)$. A typical indifference vector v of a preference R_i can be represented by a diagram shown in Figure 1 for three objects (and the null object). Each horizontal line in the figure corresponds to a unique object. Each point on each of the horizontal lines corresponds to a payment level. So, the set of all consumption bundles are the four horizontal lines in Figure 1. As we go right along the horizontal lines, the payment of the agent increases. Hence, consumption bundles to the right (left) of the indifference vector v shown in Figure 1 are worse (respectively, better) than the four consumption bundles corresponding to v.

An equivalent way to think of a preference R_i is through its indifference vectors $\mathcal{I}(R_i)$, which is an infinite set. Hence, a preference consists of an infinite collection of such vectors: an illustration is shown in Figure 2. Note that for every classical preference R_i and for every distinct $v, v' \in \mathcal{I}(R_i)$, we have either $v > v'$ or $v' > v$, i.e., v and v' do not intersect.

6.2 Outline of the proof

We now provide an outline of the proof of Theorem 1. For brevity of exposition, we refer to any strategy-proof and individually rational mechanism satisfying equal treatment of equals, no wastage and no subsidy as a **desirable mechanism**.

Since the MWEP mechanism is desirable, what we need to show is that there is no other mechanism that is desirable. Kazumura et al. (2020) shows that if a mechanism is

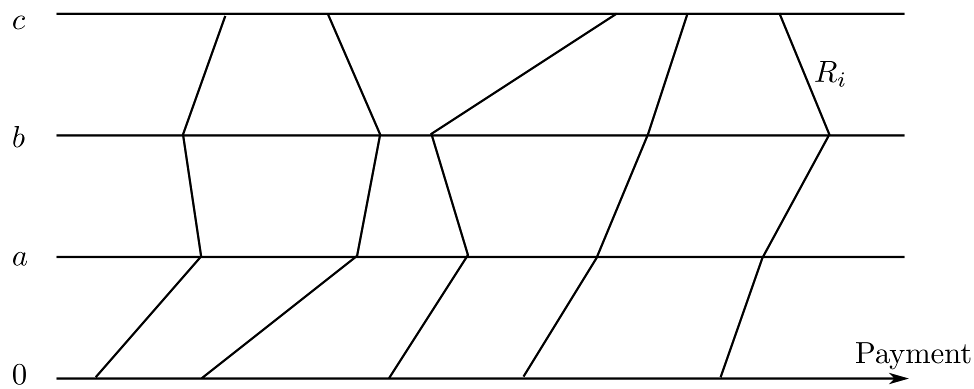

Figure 2: A preference and its indifference vectors

desirable, then for each preference profile, the mechanism assigns an allocation where each agent receives a bundle that is at least as desirable as bundles that he receives at minimum price Walrasian equilibrium allocations.

Fact **1** (**Kazumura et al.** (**2020**)) *Let f be a desirable mechanism on $\mathcal{R}^n$. For each $R \in \mathcal{R}^n$, each $((a_i, t_i))_{i\in N} \in Z^{\min}(R)$, and each $i \in N$, $f_i(R)\ R_i\ (a_i, t_i)$.*

Let $\mathcal{R} = \mathcal{R}^C$ and $f : \mathcal{R}^n \to Z$ be a desirable mechanism. By Fact 1, to complete the proof, it is sufficient to show that for each $R \in \mathcal{R}^n$ and each $i \in N$, $t_i(R) = p^{\min}_{a_{i(R)}}(R)$.[12] Fact 1 also implies that for each $R \in \mathcal{R}^n$ and each $i \in N$, $t_i(R) \leq p^{\min}_{a_{i(R)}}(R)$.[13] Therefore, all we need to show is that for each $R \in \mathcal{R}^n$ and each $i \in N$, $t_i(R) \geq p^{\min}_{a_{i(R)}}(R)$.

$$t_i(R) \geq p^{\min}_{a_i(R)}(R) \text{ for each } R \in \mathcal{R}^n \text{ and each } i \in N.$$

In this section, we overview the proof by focusing on a simple example. Suppose there are three agents and two objects. For convenience, denote $M = \{x_1, x_2\}$. Let $v \in \mathbb{R}_{++}$. We focus on a preference profile $R \in \mathcal{R}^3$ such that for each $i \in N$,

$$V^{R_i}(x_1, (0, 0)) = V^{R_i}(x_2, (0, 0)) = v.$$

[12]Let $R \in \mathcal{R}^n$ and suppose that for each $i \in N$, $t_i(R) = p^{\min}_{a_i(R)}(R)$. Let $((a_j, t_j))_{j\in N} \in Z^{\min}(R)$ and $i \in N$. By Fact 1, $(x_i(R), p^{\min}_{a_i(R)}(R)) = f_i(R)\ R_i\ (a_i, t_i)$. Thus, by $a_i \in D(R_i, p^{\min}(R))$ and thus, $x_i(R) \in D(R_i, p^{\min}(R))$. Hence $f(R) \in Z^{\min}(R)$.

[13]To see this, let $R \in \mathcal{R}^n$, $((a_i, t_i))_{i\in N} \in Z^{\min}(R)$, and $i \in N$. By Fact 1 and $a_i \in D(R_i, p)$, $f_i(R)\ R_i\ (a_i, t_i)\ R_i\ (x_i(R), p^{\min}_{a_i(R)}(R))$. This implies $t_i(R) \leq p^{\min}_{a_{i(R)}}(R)$.

Note that if we assume preferences to be quasilinear, this condition implies preferences in R are identical. In this case, since at least one agent receives $(0,0)$, *equal treatment of equals* immediately implies that for each $i \in N$, $f_i(R)\ I_i\ (0,0)\ I_i\ (x_1, p^{\min}_{x_1}(R))\ I_i\ (x_2, p^{\min}_{x_2}(R))$, completing the proof. However, we cannot make this assumption, and some of the preferences in R may not be identical. Here, we focus on the case where preferences in R are all distinct.

Suppose for contradiction that there is an agent, say agent 1, such that $t_1(R) < p^{\min}_{a_1(R)}(R)$. By no subsidy, $a_1(R) \neq 0$. Without loss of generality, assume $a_1(R) = x_1$. It is easy to see that $p^{\min}_{x_1}(R) = p^{\min}_{x_2}(R) = v$. Thus, $t_1(R) < p^{\min}_{x_1}(R) = v$.

The main difficulty of our proof is that *equal treatment of equals* can be used only when some agents have the same preference. Indeed, since preferences are all distinct at R, *equal treatment of equals* has no implication for the allocation at R. Therefore, by changing preferences from R, we have to construct a preference profile where some agents have identical preferences and *equal treatment of equals* induces a contradiction. A significant part of the proof is dedicated to construct such a preference profile. We give a sense of this construction below.

We first change agent 1's preference. Let $R'_1 \in \mathcal{R}$ satisfy the following properties:

1. $V^{R'_1}(x_2, f_1(R)) < 0$.

2. For each $x \in M$,
$$V^{R'_1}(x,(0,0)) \begin{cases} < t_1(R) + \epsilon & \text{if } x = x_1, \\ > v & \text{if } x = x_2, \end{cases}$$
where $\epsilon > 0$ is sufficiently close to zero so that $V^{R'_1}(x_1,(0,0)) < v$.

Figure 3 is an illustration of R'_1. The interpretation of R'_1 is as follows: The first condition means that, given that agent 1 receives $f_1(R)$, x_1 is more preferred to x_2 in the sense that agent 1 would not give up x_1 in exchange for x_2 unless she receives a positive amount of money. In this sense, agent 1 "favors" $f_1(R)$ – in the proof, we call such a preference a $f_1(R)$-favoring preference (see Appendix A.1). The second condition means that at $(0,0)$, x_2 is (much) more preferred and x_1 is (slightly) less preferred under R'_1 than under R_1. This condition also implies that at $(0,0)$, x_2 is preferred to x_1 in the sense that the valuation for x_2 at $(0,0)$ is higher than that for x_1 at $(0,0)$. Since x_1 is preferred to x_2 at $f_1(R)$, R'_1

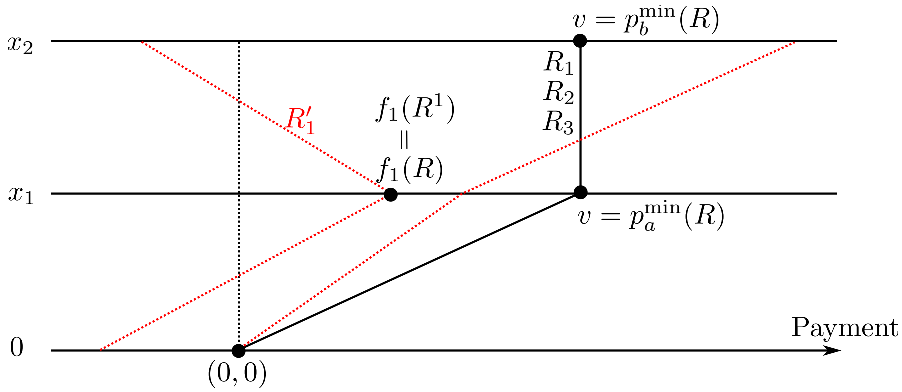

Figure 3: An illustration of R'_1.

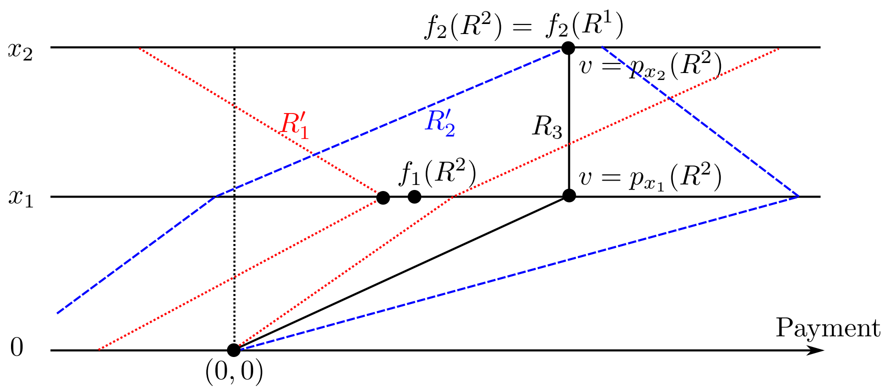

Figure 4: An illustration of R'_2.

exhibits income effects. In the formal proof, we need to define R'_1 to satisfy more delicate conditions to incorporate issues that arise in a general setting.

The implication of this delicate construction is the following. Denote $R^1 := (R'_1, R_2, R_3)$. First, we see that $f_1(R^1) = f_1(R)$. By strategy-proofness, $f_1(R^1)\ R'_1\ f_1(R)$. Thus, if $a_1(R^1) \neq x_1$, then by the construction of R'_1, $t_1(R^1) < 0$. This contradicts *no subsidy*. Thus, $a_1(R^1) = x_1$, and then, strategy-proofness immediately implies $f_1(R^1) = f_1(R)$. Next, by no wastage, there is an agent who receives x_2 at R^1. As we have seen, this is not agent 1. Without loss of generality, assume that agent 2 receives x_2.

Now, we change agent 2's preference. The idea is similar to R'_1, but the roles of x_1 and x_2 are interchanged. Let $R'_2 \in \mathcal{R}$ satisfy the following properties:

1. $V^{R'_2}(x_1, f_2(R^1)) < 0$.

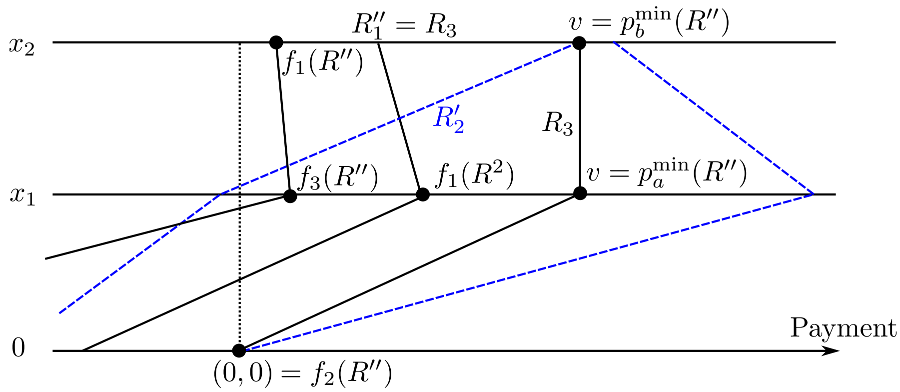

Figure 5: An illustration of R''.

2. For each $x \in M$,

$$V^{R_2'}(x,(0,0)) \begin{cases} > v & \text{if } x = x_1, \\ < t_2(R^1) + \epsilon & \text{if } x = x_2, \end{cases}$$

where $\epsilon > 0$ is sufficiently close to zero so that if $t_2(R^1) < v$, then $V^{R_2'}(x_2,(0,0)) < v$.

Again, here are the implications of the delicate construction of R_2'. See Figure 4 for an illustration of R_2'. Denote $R^2 = (R_1', R_2', R_3)$. Note that as in the case of R^1, strategy-proofness implies that $f_2(R^2) = f_2(R^1)$. Then, $a_1(R^2) = x_1$ or 0. Suppose $a_1(R^2) = 0$. Individual rationality and no subsidy imply $t_1(R^2) = 0$. It is easy to show that $p_{x_1}^{\min}(R^2) = p_{x_2}^{\min}(R^2) = v$ (see Figure 4). By the construction of R_1', we have $(x_2, p_{x_2}^{\min}(R^2))\ P_1'\ (0,0) = f_1(R^2)$. This contradicts Fact 1. Hence, $a_1(R^2) = x_1$. By $V^{R_1'}(x_1,(0,0)) < v$ and individual rationality, $t_1(R^2) < v$.

Note that by $t_1(R^2) < v$, $f_1(R^2)\ P_3\ (0,0)$. This is, in fact, a key condition to derive a contradiction. Now, let $R_1'' \in \mathcal{R}$ be such that $R_1'' = R_3$, and denote $R'' := (R_1'', R_2', R_3)$. For an illustration of R'', see Figure 5. By strategy-proofness, $f_1(R'')\ R_1''\ f_1(R^2)$. Thus, by $f_1(R^2)\ P_3\ (0,0)$ and $R_1'' = R_3$, $f_1(R'')\ P_1''\ (0,0)$. By no subsidy, this implies $a_1(R'') \neq 0$. Since agents 1 and 3 have the same preference, $f_3(R'')\ I_3\ f_1(R'')\ P_3\ (0,0)$. Hence, $a_3(R'') \neq 0$.

Since there are only two objects, agent 2 receives 0. By individual rationality and no subsidy, $f_2(R'') = (0,0)$. It is also easy to show that $p_{x_1}^{\min}(R'') = p_{x_2}^{\min}(R^*) = v$. However, by the construction of R_2', we have $(x_1, p_{x_1}^{\min}(R''))\ P_2'\ (0,0) = f_2(R'')$. This contradicts Fact 1 and we complete the proof.

In the formal proof, we need to change preferences of m agents, where m is the number of objects, and construct m preference profiles:

- $R^1 := (R'_1, R_2, \ldots, R_n)$,
- $R^2 := (R'_1, R'_2, R_3, \ldots, R_n)$,

  $\vdots$
- $R^m := (R'_1, \ldots, R'_m, R_{m+1}, \ldots, R_n)$.

Then, we show that, at R^m, there are $i \in \{1, \ldots, m\}$ and $j \in N \setminus \{1, \ldots, m\}$ such that $f_i(R^m)\ P_j\ (0, 0)$. In the above example, it was easy to find such a pair of agents, as the number of agents who prefer $f_1(R)$ to $(0, 0)$ exceeds the number of objects. However, this may not be the case in general. In fact, the proof for the existence of such a pair is the most difficult part in our proof. Out of eleven steps in our proof, ten steps are dedicated to show the existence of such a pair of agents.

7 Conclusion

The proof of Theorem 1 is tedious and long, and unfortunately, there is no intuition to explain the result. The proof involves carefully constructing classical preferences and putting together the implications of our axioms at these preferences. While we wished a simpler proof was available for our theorem, it seems unlikely. At the same time, we believe that our result is useful. It provides an equity foundation (using equal treatment of equals) of the MWEP mechanism instead of the efficiency foundation (using Pareto efficiency) that is well-known. It answers the following question for the classical domain of preferences: *Which strategy-proof and individually rational mechanisms satisfy no wastage, no subsidy, and equal treatment of equals?* Our result contributes to the literature on auctions with income effect. In future, we plan to explore an answer to this question in restricted domains of preferences like the quasilinear domain or the domain of positive income effect preferences.

A Proof of Theorem 1

A.1 Preliminaries

First, we state three useful facts. The first fact states that if $n > m$, then the minimum Walrasian equilibrium price vector is always positive.

Fact **2** *Suppose $n > m$. For each $R \in \mathcal{R}^n$ and each $a \in M$, $p_a^{\min}(R) > 0$.*

The following notions play an important role in the proof.

Definition **9** *Let $R \in \mathcal{R}^n$ and $p \in \mathbb{R}_+^{|L|}$ be a price vector. A set of real objects $M' \subseteq M$ is* **overdemanded at** p **for** R *if*

$$|\{i \in N : D(R_i, p) \subseteq M'\}| > |M'|.$$

A set of real objects $M' \subseteq M$ is **underdemanded at** p **for** R *if $p_a > 0$ for all $a \in M'$ and*

$$|\{i \in N : D(R_i, p) \cap M' \neq \emptyset\}| < |M'|,$$

and **weakly underdemanded at** p **for** R *if the above inequality holds weakly.*

The following fact is a characterization of the minimum Walrasian equilibrium price vector by means of overdemanded and weakly underdemanded sets.

Fact **3 (Mishra and Talman (2010); Morimoto and Serizawa (2015))** *Let $R \in \mathcal{R}^n$ and $p \in \mathbb{R}_+^{|L|}$ be a price vector. Then, p is a minimum Walrasian equilibrium price vector at R if and only if no set of real objects is overdemanded and no set of real objects is weakly underdemanded at p for R.*

Fact 3 implies the following fact.

Fact **4 (Demand connected sequence)** *Let $n > m$, $R \in \mathcal{R}^n$, and $((a_i, t_i))_{i \in N} \in Z^{\min}(R)$. Then, for every agent $i^* \in N$, there is a sequence of K distinct agents $\{i_k\}_{k=1}^K$ such that*[14]
(1) $i_1 = i^*$,
(2) $a_{i_K} = 0$ and for each $k \in \{1, \ldots, K-1\}$, $a_{i_k} \neq 0$, and
(3) for each $k \in \{2, \ldots, K\}$, $\{a_{i_{k-1}}, a_{i_k}\} \subseteq D(R_{i_k}, p^{\min}(R))$.

[14]If $a_{i^*} = 0$, then the sequence is $\{i_k\}_{k=1}^K = \{i^*\}$ and thus the latter part of Condition (2) and Condition (3) vacuously hold.

The formal proof of Fact 3 is given in Morimoto and Serizawa (2015).

Now we state two lemmas. The following lemma states that under a mechanism satisfying *individual rationality* and *no subsidy*, the payment of an agent who does not receive any real object is zero.

LEMMA **1** *Let f be a mechanism on $\mathcal{R}^n$ satisfying individual rationality and no subsidy. For each $R \in \mathcal{R}^n$ and each $i \in N$, if $a_i(R) = 0$, then $t_i(R) = 0$.*

We omit the proof because it is straightforward from individual rationality and no subsidy. Next, we introduce a notion of preferences called the "(a, t)-favoring" preferences.

DEFINITION **10** *Let $(a, t) \in M \times \mathbb{R}_+$. A preference relation $R_i \in \mathcal{R}$ is (a, t)***-favoring** *if for price vector p with $p_a = t$ and $p_b = 0$ for each $b \in L \setminus \{a\}$, $D(R_i, p) = \{a\}$.*

LEMMA **2** *Let f be a mechanism on $\mathcal{R}^n$ satisfying strategy-proofness and no subsidy. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $a_i(R) \neq 0$. Let $R'_i \in \mathcal{R}$ be $f_i(R)$-favoring. Then, $f_i(R'_i, R_{-i}) = f_i(R)$.*

For the formal proof of Lemma 2, see Morimoto and Serizawa (2015).

A.2 Proof of Theorem 1

Let f be a desirable mechanism on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$. By Fact 1, for each $i \in N$, $t_i(R) \leq p^{\min}_{a_i(R)}(R)$. Hence, the proof of our theorem is completed by establishing that for each $i \in N$, $t_i(R) \geq p^{\min}_{a_i(R)}(R)$.

Let $\overline{V} \in \mathbb{R}$ be such that

$$\overline{V} > \max_{i \in N} \max_{a \in M} V^{R_i}(a, (0, 0)).$$

Note that the right hand side of the inequality is well-defined by finiteness of N and M. Note also that $\overline{V} > 0$ by desirability of objects.

Next, we introduce the notion of *individually rational indifference-connected sequence*, which plays an important role in the proof.

DEFINITION **11** *Given $(a, t) \in M \times \mathbb{R}$, a pair $S := (\{i_j\}_{j=1}^k, \{a^j\}_{j=1}^k)$ of sequences of $k \leq m$ distinct agents and k distinct real objects is* **individually rational indifference-connected (IRIC) sequence from** *(a, t) if there is a sequence of consumption bundles $\mathbf{z} \equiv \{z^j\}_{j=1}^k \equiv \{(a^j, t^j)\}_{j=1}^k$ such that*

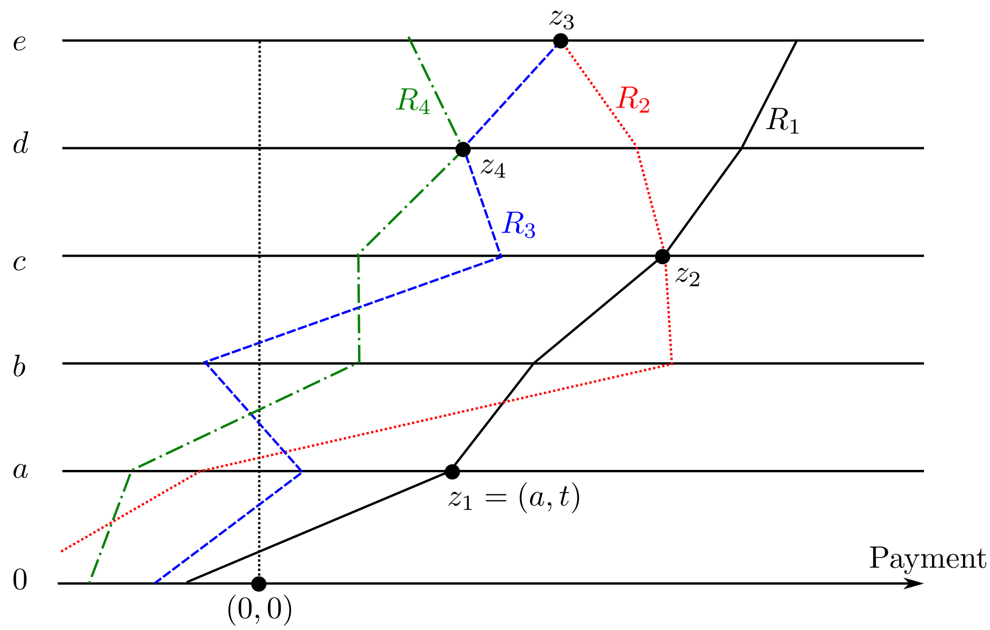

Figure 6: Illustration of an IRIC from (a, t).

1. $z^1 = (a, t)$ *i.e.,* $a^1 = a$ *and* $t^1 = t$

2. $z^j \; P_{i_j} \; (0, 0) \; \forall \; j \in \{1, \dots, k\}$

3. $z^j \; I_{i_j} \; z^{j+1} \; \forall \; j \in \{1, \dots, k-1\}$.

Figure 6 illustrates an IRIC sequence $S = (\{1, 2, 3, 4\}, \{a, c, e, d\})$ at (a, t). Note that the sequence of consumption bundles $\{z^1, z^2, z^3, z^4\}$ contains distinct agents and distinct real objects.

Let $\mathcal{S}(a, t)$ be the set of all IRIC sequences from (a, t). Note that, by continuity, for each $(a, t) \in M \times \mathbb{R}$ and each $S \in \mathcal{S}(a, t)$, there is $d > t$ such that for each $t' \in \mathbb{R}$ with $t' \leq d$, S is also an IRIC from (a, t') for K. Pick such a number for each $z \in M \times \mathbb{R}$ and each $S \in \mathcal{S}(z)$, and denote it by $d(z, S)$. Given $z \in M \times \mathbb{R}$, let

$$d(z) \equiv \begin{cases} \min\{d(z, S) : S \in \mathcal{S}(z)\} & \text{if } \mathcal{S}(z) \neq \emptyset, \\ \overline{V} & \text{otherwise.} \end{cases}$$

Note that $d(z)$ is well-defined since $\mathcal{S}(z)$ is finite. Note also that for each $z \in M \times \mathbb{R}$, $d(z) \leq \overline{V}$. [15]

[15]To see this, let $z \equiv (a, t) \in M \times \mathbb{R}$. If $\mathcal{S}(z) = \emptyset$, then $d(z) = \overline{V}$. Suppose $\mathcal{S}(z) \neq \emptyset$. Let $S =$

Now, assume for contradiction that there is an agent $i^* \in N$ such that

$$t_{i^*}(R) < p^{\min}_{a_{i^*}(R)}(R).$$

By *no subsidy*, $a_{i^*}(R) \neq 0$.

Denote the set of objects as $M \equiv \{x_1, \ldots, x_m\}$ instead of $\{1, \ldots, m\}$ for convenience. For convenience, we abuse notation and denote $x_{m+1} \equiv x_1$. For simplicity of notation, for each $R' \in \mathcal{R}^n$ and each $k \in \{1, \ldots, m+1\}$, we write $p^{\min}_k(R')$ instead of $p^{\min}_{x_k}(R')$. Without loss of generality, assume $a_{i^*}(R) = x_1$. Then, using the new notation, we have $t_{i^*}(R) < p^{\min}_1(R)$.

Step **1** *There exists a sequence $\{i_1, \ldots, i_m\}$ of m distinct agents and a preference profile of these agents $R'_{\{i_1,\ldots,i_m\}} \in \mathcal{R}^m$ such that for the sequence of $(m+1)$ preference profiles*

$$\begin{aligned}
R^0 &= R\\
R^1 &= (R'_{i_1}, R_{-i_1})\\
R^2 &= (R'_{\{i_1,i_2\}}, R_{-\{i_1,i_2\}})\\
\ldots &= \ldots\\
R^k &= (R'_{\{i_1,\ldots,i_k\}}, R_{-\{i_1,\ldots,i_k\}})\\
\ldots &= \ldots\\
R^m &= (R'_{\{i_1,\ldots,i_m\}}, R_{-\{i_1,\ldots,i_m\}})
\end{aligned}$$

the followings hold. For each $k \in \{1, \ldots, m\}$, $a_{i_k}(R^{k-1}) = x_k$, and R'_{i_k} satisfies the conditions below:[16]

(k-i) R'_{i_k} is $f_{i_k}(R^{k-1})$-favoring,

$(\{i_j\}^k_{j=1}, \{a^j\}^k_{j=1}) \in \mathcal{S}(z)$. By the definition of $d(z,S)$, S is also an IRIC sequence from $(a, d(z,S))$. Thus, by the definition of IRIC sequence, we have $(a, d(z,S))\ P_{i_1}\ (0,0)$. This implies $d(z,S) < V^{R_{i_1}}(a,(0,0))$. By $V^{R_{i_1}}(a,(0,0)) < \overline{V}$ and $d(z) \leq d(z,S)$, we obtain $d(z) \leq \overline{V}$.

[16] In the example in Section 6.2, the first condition of R'_1 corresponds to Condition (1-i) and the second one corresponds to Condition (1-ii).

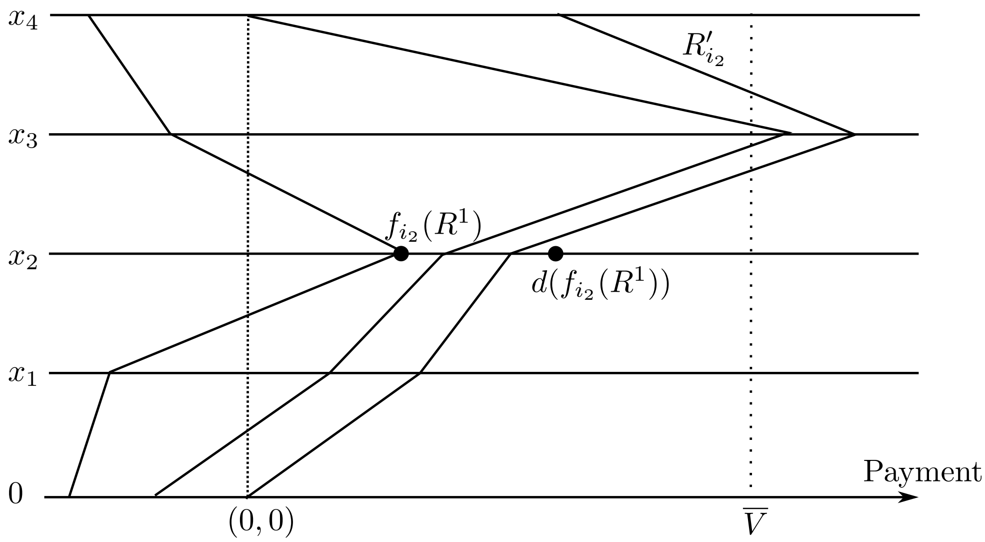

Figure 7: Illustration of R'_{i_2}.

(k-ii) For each $a \in M$,

$$V^{R'_{i_k}}(a,(0,0)) \begin{cases} < d(f_{i_k}(R^{k-1})) & \text{if } a = x_k, \\ > \overline{V} & \text{if } a = x_{k+1}, \\ < \overline{V} & \text{if } a \in M \setminus \{x_k, x_{k+1}\}, \end{cases}$$

(k-iii) for each $a \in M \setminus \{x_k, x_{k+1}\}$, $V^{R'_{i_k}}(x_{k+1},(a,0)) > \overline{V}$. [17]

Figure 7 illustrates R'_{i_k} for $k = 2$.

Proof of Step 1: We inductively construct $\{i_1, \ldots, i_m\}$ and $R'_{\{i_1,\ldots,i_m\}}$.

Induction base. Let $i_1 = i^*$ (the agent for which we have $t_{i^*}(R) < p_1^{\min}(R)$). By $R^0 = R$, we have $a_{i_1}(R^0) = a_{i^*}(R) = x_1$. Note that a preference relation satisfying (1-i), (1-ii), and (1-iii) exists if $t_{i_1}(R) < \overline{V}$. But this immediately follows because, by individual rationality, $f_{i_1}(R)\ R_{i_1}\ (0,0)$, and by the definition of $\overline{V}$, $t_{i_1}(R) \leq V^{R_{i_1}}(x_1,(0,0)) < \overline{V}$.

Induction argument. Let $k \in \{1, \ldots, m-1\}$. Assume that there exist $\{i_1, \ldots, i_k\}$ of k distinct agents and $R'_{\{i_1,\ldots,i_k\}} \in \mathcal{R}^k$ such that for each $\ell \in \{1, \ldots, k\}$, $a_{i_\ell}(R^{\ell-1}) = a_\ell$, and R'_{i_ℓ}

[17] In case of $m = 2$, some of the conditions are redundant. In this case, they vacuously hold, and cause no problem in the rest of the proof.

satisfies (ℓ-i), (ℓ-ii), and (ℓ-iii). By *no wastage*, there is $i \in N$ such that

$$a_i(R^k) = x_{k+1}. \tag{1}$$

CLAIM: $i \notin \{i_1, \ldots, i_k\}$.

Proof: By contradiction, suppose $i \in \{i_1, \ldots, i_k\}$. By $a_{i_k}(R^{k-1}) = x_k$, (k-i), and Lemma 2, we have $f_{i_k}(R^k) = f_{i_k}(R^{k-1})$. Thus, $i \neq i_k$. Thus, there is $i_\ell \in \{i_1, \ldots, i_{k-1}\}$ such that $i = i_\ell$. By $\ell \leq k-1 < k$, $x_{k+1} \notin \{x_\ell, x_{\ell+1}\}$. Thus,

$$\begin{aligned} V^{R'_i}(x_{\ell+1}, f_i(R^k)) &\geq V^{R'_i}(x_{\ell+1}, (x_{k+1}, 0)) && \text{by (1) and \textit{no subsidy}} \\ &= V^{R'_{i_\ell}}(x_{\ell+1}, (x_{k+1}, 0)) && \text{by } i = i_\ell \\ &> \overline{V}. && \text{by (ℓ-iii) and } x_{k+1} \notin \{x_\ell, x_{\ell+1}\} \end{aligned}$$

By Fact 1, $f_i(R^k)\ R'_i\ (x_{\ell+1}, p^{\min}_{\ell+1}(R^k))$. This implies $V^{R'_i}(x_{\ell+1}, f_i(R^k)) \leq p^{\min}_{\ell+1}(R^k)$. Thus, by $V^{R'_i}(x_{\ell+1}, f_i(R^k)) > \overline{V}$,

$$p^{\min}_{\ell+1}(R^k) > \overline{V}. \tag{2}$$

By the definition of $\overline{V}$ and (2), for each $j \in N \setminus \{i_1, \ldots, i_k\}$, $V^{R_j}(x_{\ell+1}, (0,0)) < \overline{V} < p^{\min}_{\ell+1}(R^k)$, which implies $x_{\ell+1} \notin D(R_j, p^{\min}(R^k))$. For each $i_{k'} \in \{i_1, \ldots, i_k\} \setminus \{i_\ell, i_{\ell+1}\}$

$$V^{R'_{i_{k'}}}(x_{\ell+1}, (0,0)) < \overline{V} < p^{\min}_{\ell+1}(R^k),$$

where the first inequality follows from $x_{\ell+1} \notin \{x_{k'}, x_{k'+1}\}$ and (k'-ii), and the last inequality follows from (2). Thus, for each $i_{k'} \in \{i_1, \ldots, i_k\} \setminus \{i_\ell, i_{\ell+1}\}$, $x_{\ell+1} \notin D(R'_{i_{k'}}, p^{\min}(R^k))$. Moreover, by ($\ell+1$-ii) and (2), we have $V^{R'_{i_{\ell+1}}}(x_{\ell+1}, (0,0)) < \overline{V} < p^{\min}_{\ell+1}(R^k)$, implying $x_{\ell+1} \notin D(R'_{i_{\ell+1}}, p^{\min}(R^k))$. Therefore,

$$|\{j \in N : D(R^k_j, p^{\min}(R^k)) \cap \{x_{\ell+1}\} \neq \emptyset\}| \leq 1.$$

This and Fact 2 imply that $\{x_{\ell+1}\}$ is weakly underdemanded at $p^{\min}(R^k)$ for R^k, contradicting Fact 3. $\square$

Let $i_{k+1} = i$. By Claim and induction hypothesis, the agents in $\{i_1, \ldots, i_k, i_{k+1}\}$ are distinct. By (1), $a_{i_{k+1}}(R^k) = x_{k+1}$. Note that a preference relation satisfying ($k+1$-i), ($k+1$-ii), and ($k+1$-iii) exists if $t_{i_{k+1}}(R^k) < \overline{V}$. But this immediately follows because,

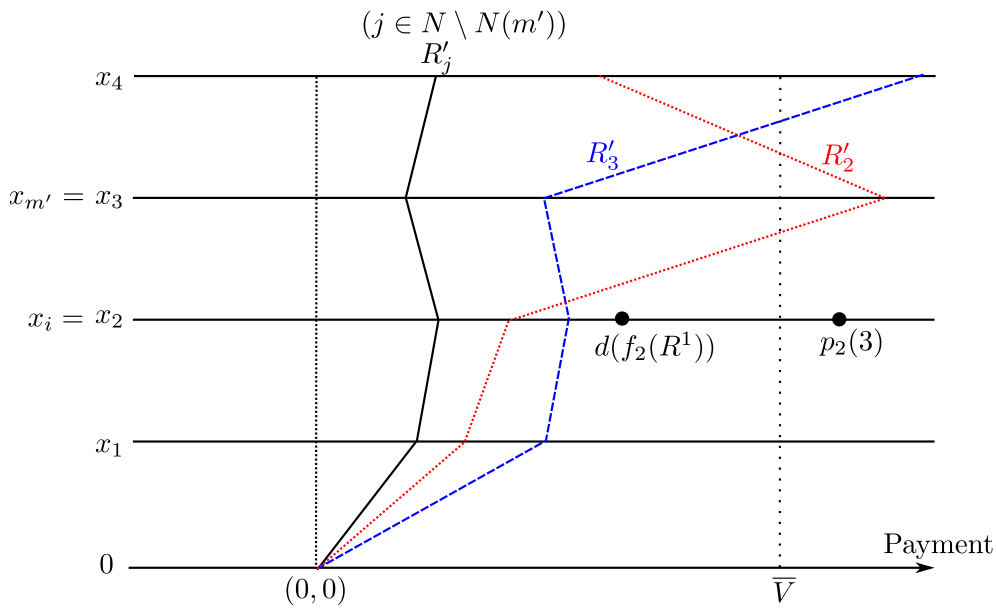

Figure 8: Illustration of the proof of Step 2 for $m' = 3$ and $i = 2$.

by individual rationality, $f_{i_{k+1}}(R^k)\ R_{i_{k+1}}\ (0,0)$, and by the definition of $\overline{V}$, $t_{i_{k+1}}(R^k) \leq V^{R_{i_{k+1}}}(x_{k+1},(0,0)) < \overline{V}$. This completes the proof of Step 1. ■

Without loss of generality, assume $\{i_1,\ldots,i_m\} = \{1,\ldots,m\}$. For each $m' \in \{1,\ldots,m\}$, let $N(m') \equiv \{1,\ldots,m'\}$ and $M(m') \equiv \{x_1,\ldots,x_{m'}\}$. For each $m' \in \{0,1,\ldots,m\}$, let $p(m') \equiv p^{\min}(R^{m'})$.

Step **2** *Let $m' \in \{1,\ldots,m\}$ and $x_i \in M$. Then, $p_i(m') < \overline{V}$.*

Proof of Step 2: (See Figure 8 for illustration.) Suppose by contradiction that $p_i(m') \geq \overline{V}$. By the definition of $\overline{V}$, for each $j \in N \setminus N(m')$, $V^{R_j}(x_i,(0,0)) < \overline{V} \leq p_i(m')$, which implies $(0,0)\ P_j\ (x_i,p_i(m'))$. Thus,

$$x_i \notin D(R_j,p(m')) \text{ for each } j \in N \setminus N(m'). \tag{1}$$

For each $j \in N(m')\setminus\{i-1,i\}$, by $x_i \in M\setminus\{x_j,x_{j+1}\}$ and (j-ii) in Step 1, $V^{R'_j}(x_i,(0,0)) < \overline{V} \leq p_i(m')$, which implies $(0,0)\ P'_j\ (x_i,p_i(m'))$. Thus,

$$x_i \notin D(R'_j,p(m')) \text{ for each } j \in N(m') \setminus \{i-1,i\}. \tag{2}$$

Suppose $i \in N(m')$. By (i-ii) in Step 1 and the fact that $d(f_i(R^{i-1})) \leq \overline{V}$, we get $V^{R'_i}(x_i, (0,0)) < d(f_i(R^{i-1})) \leq \overline{V} \leq p_i(m')$, which implies $(0,0)\, P'_i\, (x_i, p_i(m'))$. Thus,

$$x_i \notin D(R'_i, p(m')). \tag{3}$$

Thus, by (1), (2), and (3),

$$|\{j \in N : D(R_j^{m'}, p(m')) \cap \{x_i\} \neq \emptyset\}| \leq 1.$$

If $i \notin N(m')$, this inequality is immediately implied by (1) and (2).

Hence, in either case, $\{x_i\}$ is a weakly underdemanded set at $p(m')$ for $R^{m'}$. This contradicts Fact 3. This completes the proof of Step 2. ■

STEP **3** *Let $m' \in \{1, \ldots, m\}$, $((a_i, t_i))_{i \in N} \in Z^{\min}(R^{m'})$, and $i \in N(m')$. Then, the following properties hold.*

(i) (Twin demand property.) $D(R'_i, p(m')) \subseteq \{x_i, x_{i+1}\}$, and thus, $a_i \in \{x_i, x_{i+1}\}$.

(ii) (Unique demand property.) If $x_i \notin M_+(m')$, $D(R'_i, p(m')) = \{x_i\}$, and thus, $a_i = x_i$.

Proof of Step 3: We show the first property. By Step 2 and (i-ii) in Step 1, $p_{i+1}(m') < \overline{V} < V^{R'_i}(x_{i+1}, (0,0))$, which implies $(x_{i+1}, p_{i+1}(m'))\, P'_i\, (0,0)$. Thus, $0 \notin D(R'_i, p(m'))$.

Let $a \in M \setminus \{x_i, x_{i+1}\}$. By Step 2, ($i$-iii) in Step 1, and $p_a(m') \geq 0$, we have

$$p_{i+1}(m') < \overline{V} < V^{R'_i}(x_{i+1}, (a, 0)) \leq V^{R'_i}(x_{i+1}, (a, p_a(m'))),$$

which implies $(x_{i+1}, p_{i+1}(m'))\, P'_i\, (a, p_a(m'))$. Thus, $a \notin D(R'_i, p(m'))$. Hence, $D(R'_i, p(m')) \subseteq \{x_i, x_{i+1}\}$, and this immediately implies $a_i \in \{x_i, x_{i+1}\}$.

Next, we show the second property. Suppose $x_i \notin M_+(m')$. Then, $p_i(m') \leq t_i(R^{i-1})$. Since R'_i is $f_i(R^{i-1})$-favoring, for each $a \in L \setminus \{x_i\}$,

$$V^{R'_i}(a, (x_i, p_i(m'))) \leq V^{R'_i}(a, f_i(R^{i-1})) < 0 \leq p_a(m'),$$

implying that $(x_i, p_i(m'))\; P'_i\; (a, p_a(m'))$. Therefore, for each $a \in L \setminus \{x_i\}$, $a \notin D(R'_i, p(m'))$. Hence, $D(R'_i, p(m')) = \{x_i\}$, and this immediately implies $a_i = x_i$. ■

Now we introduce two notations. Given $m' \in \{1, \ldots, m\}$, let

$$M_+(m') \equiv \{x_k \in M(m') : p_k(m') > t_k(R^{k-1})\}, \text{ and}$$
$$M_{++}(m') \equiv \{x_k \in M(m') : p_k(m') > V^{R'_k}(x_k, (0,0))\}.$$

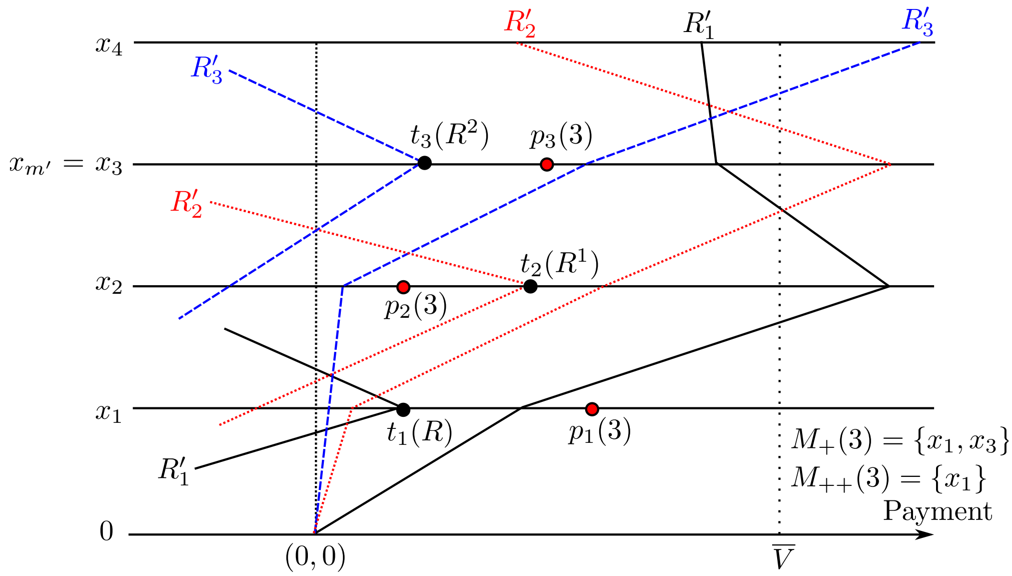

Figure 9: $M_+(3)$ and $M_{++}(3)$.

Figure 9 illustrates $M_+(m')$ and $M_{++}(m')$ for $m' = 3$. In this figure, $p_1(3) > V'_1(x_1, (0,0))$, $p_2(3) < t_2(R^1)$, and $t_3(R^2) < p_3(3) < V'_3(x_3, (0,0))$. Thus, $M_+(3) = \{x_1, x_3\}$ and $M_{++}(3) = \{x_1\}$.

Note that for each $m' \in \{1, \ldots, m\}$, $M_{++}(m') \subseteq M_+(m')$. To see this, let $m' \in \{1, \ldots, m\}$ and $x_k \in M_{++}(m')$. Then, $p_k(m') > V^{R'_k}(x_k, (0,0))$. By ($k$-i) in Step 1, R'_k is $f_k(R^{k-1})$-favoring. Thus, $t_k(R^{k-1}) < V^{R'_k}(x_k, (0,0)) < p_k(m')$. Thus, $x_k \in M_+(m')$.

Step 4 **(Outside demander)** *Let $m' \in \{1, \ldots, m\}$ be such that $M_+(m') \neq \emptyset$. Then, there is $i \in N \setminus N(m')$ such that $D(R_i, p(m')) \cap M_+(m') \neq \emptyset$.*

Proof of Step 4: Suppose for contradiction that

$$D(R_i, p(m')) \cap M_+(m') = \emptyset \text{ for each } i \in N \setminus N(m'). \tag{1}$$

By Step 3 (ii) (Unique demand property),

$$\{i \in N(m') : x_i \notin M_+(m')\} \subseteq \{i \in N(m') : D(R'_i, p(m')) \cap M_+(m') = \emptyset\}. \tag{2}$$

By (1) and (2),

$$\begin{aligned}
|\{i \in N :& D(R_i^{m'}, p(m')) \cap M_+(m') \neq \emptyset\}| \\
&= |\{i \in N(m') : D(R'_i, p(m')) \cap M_+(m') \neq \emptyset\}| && \text{by (1)} \\
&= |N(m')| - |\{i \in N(m') : D(R'_i, p(m')) \cap M_+(m') = \emptyset\}| \\
&\leq |N(m')| - |\{i \in N(m') : x_i \notin M_+(m')\}| && \text{by (2)} \\
&= |M(m')| - |M(m') \setminus M_+(m')| \\
&= |M_+(m')|. && \text{by } M_+(m') \subseteq M(m')
\end{aligned}$$

Thus, this inequality and Fact 2 imply that $M_+(m')$ is weakly underdemanded at $p(m')$ for $R^{m'}$. This contradicts Fact 3. ■

Step **5** *Let $m' \in \{1, \dots, m\}$. Let $\{i_1, \dots, i_K\} \subseteq N \setminus N(m')$ and $\{b_1, \dots, b_K\} \subseteq M$ be such that*
(a) $b_1 \in M_+(m') \setminus M_{++}(m')$,
(b) for each $k \in \{1, \dots, K-1\}$, $\{b_k, b_{k+1}\} \subseteq D(R_{i_k}, p(m'))$, and
(c) $b_K \in D(R_{i_K}, p(m'))$.
Then, $0 \notin D(R_{i_K}, p(m'))$.

Proof of Step 5: By (b), for each $k \in \{1, \dots, K-1\}$, $(b_k, p_{b_k}(m'))\, I_{i_k}\, (b_{k+1}, p_{b_{k+1}}(m'))$. Thus,

$$p_{b_{k+1}}(m') = V^{R_{i_k}}(b_{k+1}, (b_k, p_{b_k}(m'))) \text{ for each } k \in \{1, \dots, K-1\}. \tag{1}$$

We first show that $(\{i_1, \dots, i_K\}, \{b_1, \dots, b_K\})$ is an IRIC from $f_j(R^{j-1})$. By $b_1 \in M_+(m')$, there is $j \in \{1, \dots, m'\}$ such that $b_1 = x_j$. Let $r_1 \equiv t_j(R^{j-1})$, and for each $k \in \{2, \dots, K\}$, let $r_k \equiv V^{R_{i_{k-1}}}(b_k, (b_{k-1}, r_{k-1}))$. Then, $(b_1, r_1) = f_j(R^{j-1})$, and for each $k \in \{1, \dots, K-1\}$, $(b_k, r_k)\ I_{i_k} (b_{k+1}, r_{k+1})$.

Claim: *For each $k \in \{1, \dots, K\}$, $r_k < p_{b_k}(m')$.*

Proof: The proof is by induction.

Induction base. Let $k = 1$. By (a) and $b_1 = x_j$, $r_1 = t_j(R^{j-1}) < p_j(m')$.

Induction argument. Let $k \geq 1$ and assume that $r_k < p_{b_k}(m')$. Then,

$$\begin{aligned} p_{b_{k+1}}(m') &= V^{R_{i_k}}(b_{k+1}, (b_k, p_{b_k}(m'))) && \text{by (1)} \\ &> V^{R_{i_k}}(b_{k+1}, (b_k, r_k)) && \text{by } r_k < p_{b_k}(m') \\ &= r_{k+1} \end{aligned}$$

□

By (b) and (c), for each $k \in \{1, \ldots, K\}$, $(b_k, p_{b_k}(m'))\, R_{i_k}\, (0, 0)$. Thus, by Claim, for each $k \in \{1, \ldots, K\}$, $(b_k, r_k)\, P_{i_k}\, (b_k, p_{b_k}(m'))\, R_{i_k}\, (0, 0)$. Therefore, $(\{i_1, \ldots, i_K\}, \{b_1, \ldots, b_K\})$ is an IRIC from $f_j(R^{j-1})$.

By (a) and (j-ii) in Step 1,

$$p_{b_1}(m') \leq V^{R'_j}(x_j, (0, 0)) < d(f_j(R^{j-1})).$$

Therefore, by the definition of $d(f_j(R^{j-1}))$, $(\{i_1, \ldots, i_K\}, \{b_1, \ldots, b_K\})$ is an IRIC from $(x_j, p_j(m'))$. By (1), the corresponding sequence of bundles is $\{(b_k, p_{b_k}(m')\}_{k=1}^K$. Therefore,

$$(b_K, p_{b_K}(m'))\ P_{i_K}(0, 0).$$

Hence, $0 \notin D(R_{i_K}, p(m'))$, and this completes the proof of Step 5.

■

Step **6 (Outside receiver I)** *Let $m' \in \{1, \ldots, m-1\}$ be such that $M_+(m') \neq \emptyset$ and $M_{++}(m') = \emptyset$. Let $((a_i, t_i))_{i \in N} \in Z^{\min}(R^{m'})$. Then, there exists $i \in N \setminus N(m')$ such that $a_i \in M_+(m')$.*

Proof of Step 6: Suppose for contradiction that for each $i \in N \setminus N(m')$, $a_i \notin M_+(m')$. By Step 3 (ii) (Unique demand property), for each $x_j \in M(m') \setminus M_+(m')$, $x_j = a_j$. Thus,

$$\{i \in N \setminus N(m') : a_i \in M(m')\} = \emptyset. \tag{1}$$

By Step 3 (i) (Twin demand property), $m' < m$ and (1), [18]

$$a_i = x_i \text{ for each } i \in N(m'). \tag{2}$$

[18] The proof is as follows. By no wastage, there is $i \in N$ such that $a_i = x_1$. By (1), $i \in N(m')$. By Step 3 (i) (Twin demand property) and $m' < m$, $i \notin \{2, \ldots, m'\}$. Thus, $i = 1$. Then, by Step 3 (i) (Twin demand property), we can inductively show that $a_2 = x_2$, $a_3 = x_3$,..., $a_{m'} = x_{m'}$.

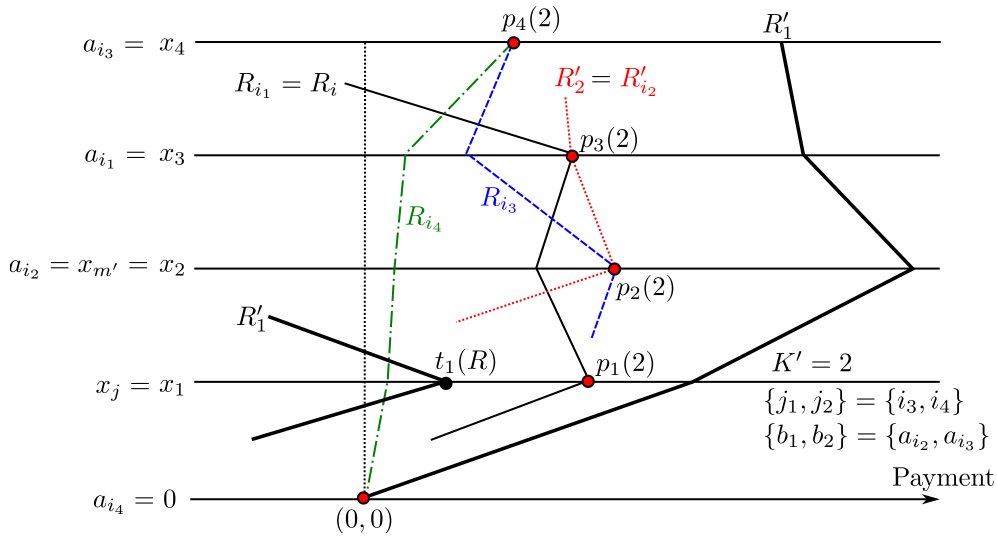

Figure 10: $\{i_k\}_{k=1}^K$ in the proof of Step 6.

By Step 4 (Outside demander) and $M_+(m') \neq \emptyset$, there exist $i \in N \setminus N(m')$ and $x_j \in M_+(m')$ such that $x_j \in D(R_i, p(m'))$. By Fact 4, there is a sequence $\{i_k\}_{k=1}^K$ of K distinct agents such that[19]

$$i_1 = i, \tag{3}$$

$$a_{i_K} = 0 \text{ and for each } k \in \{1, \ldots, K-1\},\ a_{i_k} \neq 0 \text{ and} \tag{4}$$

$$\text{for each } k \in \{2, \ldots, K\}, \{a_{i_{k-1}}, a_{i_k}\} \subseteq D(R_{i_k}^{m'}, p(m')). \tag{5}$$

Figure 10 illustrates the sequence $\{i_k\}_{k=1}^K$, for $m' = 2$, $K = 4$, and $x_j = x_1$.

Let $i_0 \equiv j$. By $x_j \in M(m')$, $i_0 \in N(m')$. By (2) and (4), $i_K \notin N(m')$. Thus, there is $K' \in \{0, 1, \ldots, K-1\}$ such that

$$i_{K'} \in N(m'), \text{ and} \tag{6}$$

$$\{i_{K'+1}, \ldots, i_K\} \cap N(m') = \emptyset. \tag{7}$$

Let $\{j_1, \ldots, j_{K-K'}\} \subseteq N$ and $\{b_1, \ldots, b_{K-K'}\} \subseteq M$ be such that for each $k \in \{1, \ldots, K-K'\}$,

$$j_k = i_{K'+k} \text{ and } b_k = a_{i_{K'+k-1}}.$$

[19]Note that it is possible that $a_i = 0$. In this case, the sequence $\{i_k\}_{k=1}^K$ consists only of agent i. That is, $K = 1$ and $\{i_k\}_{k=1}^K = \{i\}$.

In Figure 10, $K' = 2$ and $K - K' = 2$. Thus, $\{j_1, j_2\} = \{i_3, i_4\}$ and $\{b_1, b_2\} = \{a_{i_2}, a_{i_3}\} = \{x_2, x_4\}$.

Claim: *$\{j_1, \dots, j_{K-K'}\} \subseteq N \setminus N(m')$, $\{b_1, \dots, b_{K-K'}\} \subseteq M$, and the pair $(\{j_k\}_{k=1}^{K-K'}, \{b_k\}_{k=1}^{K-K'})$ satisfies the following conditions.*
(a) $b_1 \in M_+(m') \setminus M_{++}(m')$,
(b) for each $k \in \{1, \dots, K - K' - 1\}$, $\{b_k, b_{k+1}\} \subseteq D(R_{j_k}, p(m'))$, and
(c) $b_{K-K'} \in D(R_{j_{K-K'}}, p(m'))$.

Proof: By $\{j_1, \dots, j_{K-K'}\} = \{i_{K'+1}, \dots, i_K\}$ and (7), $\{j_1, \dots, j_{K-K'}\} \subseteq N \setminus N(m')$. By Step 3 (i) (Twin demand property) and $i_0 = j \in N(m')$, $a_{i_0} \neq 0$. By $\{b_1, \dots, b_{K-K'}\} = \{a_{i_{K'}}, \dots, a_{i_{K-1}}\}$, $a_{i_0} \neq 0$, and (5), $\{b_1, \dots, b_{K-K'}\} \subseteq M$.
Proof of (a): Note that by $M_{++}(m') = \emptyset$, (a) is equivalent to $b_1 \in M_+(m')$. If $K' = 0$, then by (2), $b_1 = a_{i_0} = a_j = x_j \in M_+(m')$.

Suppose $K' \geq 1$. By $i_1 = i \notin N(m')$ and (6), $K' > 1$. Then, by (5),

$$|D(R'_{i_{K'}}, p(m'))| \geq 2.$$

By (6) and (2),

$$b_1 = a_{i_{K'}} = x_{i_{K'}}.$$

Thus, Step 3 (ii) (Unique demand property) implies $b_1 \in M_+(m')$.
Proof of (b): Let $k \in \{1, \dots, K - K' - 1\}$. First, suppose $K' = 0$ and $k = 1$. Note that $\{b_1, b_2\} = \{a_{i_0}, a_{i_1}\}$. By $i_0 = j$, $j \in N(m')$, and (2), $a_{i_0} = x_j$. Thus, by (5),

$$\{b_1, b_2\} = \{x_j, a_i\} \subseteq D(R_i, p(m')) = D(R_{j_1}, p(m')).$$

Next, suppose either $K' \neq 0$ or $k \neq 1$. In both cases, we have $K' + k > 1$. Thus, $j_k = i_{K'+k} \neq i_1$. Therefore, by (5),

$$\{b_k, b_{k+1}\} = \{a_{i_{K'+k-1}}, a_{i_{K'+k}}\} \subseteq D(R_{i_{K'+k}}, p(m')) = D(R_{j_k}, p(m')).$$

Proof of (c): By $j_{K-K'} = i_K$, $b_{K-K'} = a_{i_{K-1}}$, and (5),

$$b_{K-K'} = a_{i_{K-1}} \in D(R_{i_K}, p(m')) = D(R_{j_{K-K'}}, p(m')).$$

□

By Step 5 and Claim, $0 \notin D(R_{j_{K-K'}}, p(m')) = D(R_{i_K}, p(m'))$. This contradicts (4). This completes the proof of Step 6. ■

STEP **7 (Outside receiver II)** *Let $m' \in \{1, \ldots, m-1\}$ be such that $M_{++}(m') \neq \emptyset$. Let $x_k \in M_{++}(m')$ and $((a_i, t_i))_{i \in N} \in Z^{\min}(R^{m'})$. Then, there exists $i \in N \setminus N(m')$ such that $a_i \in M(k) \cap M_+(m')$.*

Proof of Step 7: Suppose for contradiction that for each $i \in N \setminus N(m')$, $a_i \notin M(k) \cap M_+(m')$. By Step 3 (ii) (Unique demand property), for each $j \in N(k)$ with $x_j \notin M_+(m')$, $a_j = x_j$. Thus,

$$\{i \in N \setminus N(m') : a_i \in M(k)\} = \emptyset. \tag{1}$$

Note that by $k \geq 1$, $x_1 \in M(k)$. Thus by (1), for each $i \in N \setminus N(m')$, $a_i \neq x_1$. By $m' < m$ and Step 3 (i) (Twin demand property), for each $i \in N(m') \setminus \{1\}$, $a_i \neq x_1$. Thus, by *no wastage*, we conclude that $a_1 = x_1$. By using (1) and Step 3 (i) (Twin demand property) repeatedly, we obtain

$$a_2 = x_2, \; a_3 = x_3, \; \ldots, \; a_k = x_k.$$

By $x_k \in M_{++}(m')$, $p_k(m') > V^{R'_k}(x_k, (0,0))$. Thus,

$$(0,0) \; I'_k \; (x_k, V^{R'_k}(x_k, (0,0))) \; P'_k \; (x_k, p_k(m')).$$

This implies $x_k \notin D(R'_k, p(m'))$, contradicting $a_k = x_k$. ■

STEP **8** *Let $m' \in \{0, \ldots, m-1\}$ be such that $M_+(m') \neq \emptyset$. Let $((a_i, t_i))_{i \in N} \in Z^{\min}(R^{m'})$. Let $\{i_k\}_{k=1}^{K}$ be a sequence of K distinct agents such that*

$$i_1 = m' + 1, \tag{1}$$

$$a_{i_K} = 0 \textit{ and for each } k \in \{1, \ldots, K-1\}, \; a_{i_k} \neq 0 \textit{ and} \tag{2}$$

$$\textit{for each } k \in \{2, \ldots, K\}, \{a_{i_{k-1}}, a_{i_k}\} \subseteq D(R^{m'}_{i_k}, p(m')). \tag{3}$$

Suppose $m' \in \{i_1, \ldots, i_K\}$.[20] *Then $p_{m'+1}(m') > t_{m'+1}(R^{m'})$.*

[20]Note that this implies $K > 1$.

Proof of Step 8: This step will be proved using five claims, which we state and prove as we go along the proof of this step. Assume for contradiction that $p_{m'+1}(m') \leq t_{m'+1}(R^{m'})$. By $M_+(m') \neq \emptyset$, and Steps 6 (Outside receiver I) and 7 (Outside receiver II), there exist $i \in N \setminus N(m')$ and $x_\ell \in M_+(m')$ such that [21]

$$a_i = x_\ell. \tag{4}$$

Then, by using Step 3 (i) (Twin demand property) repeatedly, we obtain

$$a_\ell = x_{\ell+1},\ a_{\ell+1} = x_{\ell+2},\ \ldots,\ a_{m'} = x_{m'+1}. \tag{5}$$

Claim 1: *Let $k \in \{1, \ldots, K-1\}$ be such that $i_k \notin N(m')$ and $i_{k+1} \in N(m')$. Then $i_k = i$.*

Proof: By $i_{k+1} \in N(m')$, (3), and Step 3 (i) (Twin demand property), $a_{i_k} \in D(R'_{i_{k+1}}, p(m')) \subseteq M(m'+1)$. By Step 3 (i) (Twin demand property) and $a_i \in M_+(m')$, $\{a_j : j \in N(m') \cup \{i\}\} = M(m'+1)$. Thus, by $i_k \notin N(m')$, $i_k = i$. □

By $m' \in \{i_1, \ldots, i_K\}$, $\{i_1, \ldots, i_K\} \cap N(m') \neq \emptyset$. Thus, by (1), there is $K' \in \{1, \ldots, K-1\}$ such that

$$\{i_1, \ldots, i_{K'}\} \cap N(m') = \emptyset \text{ and } i_{K'+1} \in N(m').$$

By Claim 1,

$$i_{K'} = i. \tag{6}$$

By $m' \in \{i_1, \ldots, i_K\}$, there is $K'' \in \{1, \ldots, K\}$ such that

$$i_{K''} = m'.$$

By $\{i_1, \ldots, i_{K'}\} \cap N(m') = \emptyset$ and $i_{K''} = m' \in N(m')$, we have $K'' > K'$. Note that by $a_{i_{K''}} = x_{m'+1}$ and (2), $K'' < K$. Therefore,

$$K' < K'' < K.$$

Figure 11 is an illustration of the sequence $\{i_k\}_{k=1}^K$ for $m' = 2$ and $K = 5$. In this figure,

[21] If $M_{++}(m') = \emptyset$, then Step 6 (Outside receiver I) implies there is $i \in N \setminus N(m')$ such that $x_i \in M_+(m')$. If there is $x_k \in M$ such that $x_k \in M_{++}(m')$, then Step 7 (Outside receiver II) implies that there is $i \in N \setminus N(m')$ such that $x_i \in M(k) \cap M_+(m') \subseteq M_+(m')$.

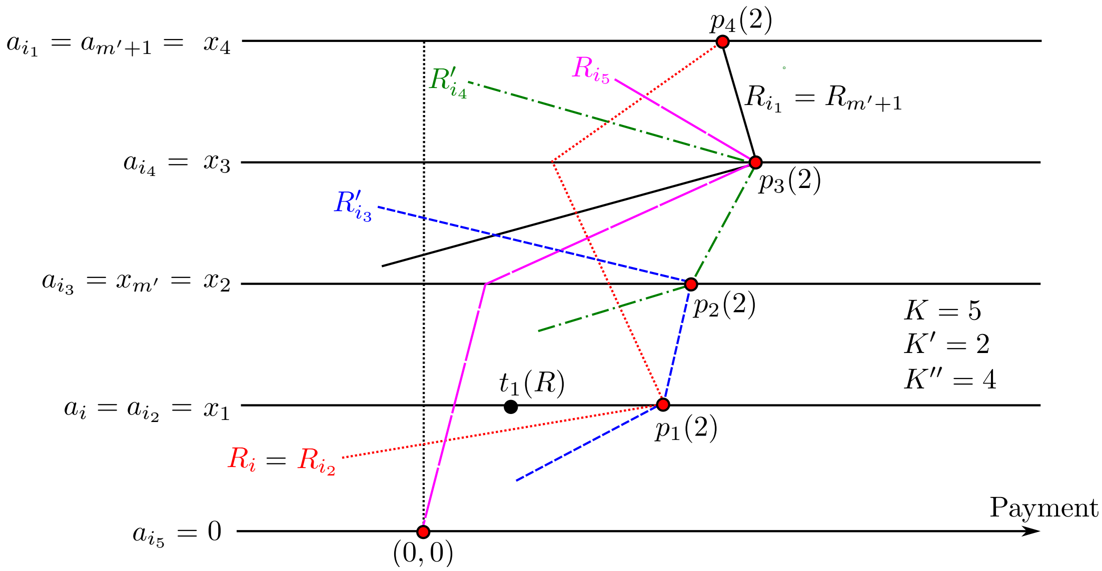

Figure 11: Illustration of $\{i_k\}_{k=1}^K$ for $m' = 2$ and $K = 5$.

$K' = 2$ and $K'' = 4$.

Claim 2: $\{i_{K''+1}, \ldots, i_K\} \cap N(m') = \emptyset$.

Proof: First, we show $i_{K''+1} \notin N(m')$. Suppose for contradiction $i_{K''+1} \in N(m')$. By $i_{K''} = m'$, $i_{K''+1} \in N(m'-1)$. Thus, by Step 3 (i) (Twin demand property), $x_{m'+1} \notin D(R'_{i_{K''+1}}, p(m'))$. However, by (5) and (3),

$$x_{m'+1} = a_{i_{K''}} \in D(R'_{i_{K''+1}}, p(m')),$$

a contradiction. Hence, $i_{K''+1} \notin N(m')$.

Now, suppose for contradiction that $\{i_{K''+1}, \ldots, i_K\} \cap N(m') \neq \emptyset$. Then, by $i_{K''+1} \notin N(m')$, there is $K^* \in \{K''+1, \ldots, K-1\}$ such that $i_{K^*} \notin N(m')$ and $i_{K^*+1} \in N(m')$. By Claim 1, we have $i_{K^*} = i$. By (6), $i_{K^*} = i_{K'}$. Since agents $\{i_1, \ldots, i_K\}$ are distinct, $K^* = K'$. However, this is a contradiction because $K' < K'' < K^*$. □

Claim 3: $\{\ell, \ldots, m'\} \subseteq \{i_{K'+1}, \ldots, i_{K''}\}$.

Proof: Suppose for contradiction that there is $k \in \{\ell, \ldots, m'\}$ such that $k \notin \{i_{K'+1}, \ldots, i_{K''}\}$. Without loss of generality, we can let k be the largest number in $\{\ell, \ldots, m'\}$ such that

$k \notin \{i_{K'+1}, \ldots, i_{K''}\}$. By $i_{K''} = m'$, $k < m'$. Thus, there is $K^* \in \{K'+1, \ldots, K''\}$ such that $i_{K^*} = k+1$.

By (3) and Step 3 (i) (Twin demand property),

$$\{a_{i_{K^*-1}}, a_{i_{K^*}}\} \subseteq D(R^{m'}_{i_{K^*}}, p(m')) = D(R'_{k+1}, p(m')) \subseteq \{x_{k+1}, x_{k+2}\}.$$

Further, by (5), $a_{i_{K^*}} = x_{k+2}$. Thus, $a_{i_{K^*-1}} = x_{k+1}$. Hence, by (5), $i_{K^*-1} = k$.

By $k \notin \{i_{K'+1}, \ldots, i_{K''}\}$ and $K^* \in \{K'+1, \ldots, K''\}$, $K^* - 1 = K'$. This and (6) imply that $i = k$. However, this is a contradiction since $i \notin N(m')$ and $k \in N(m')$. □

Figure 12 summarizes the properties of the sequences $\{i_k\}_{k=1}^K$ and $\{a_k\}_{k=1}^K$, which we have

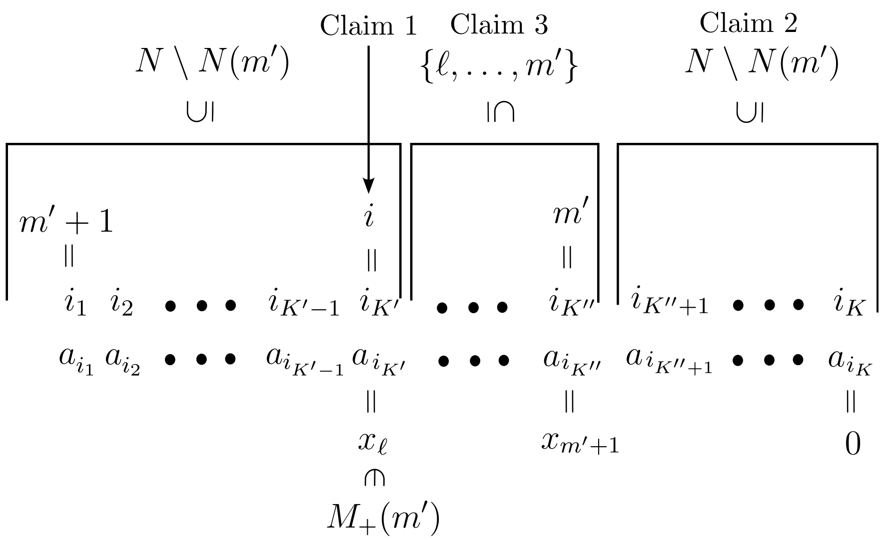

Figure 12: Properties of $\{i_k\}_{k=1}^K$ and $\{a_k\}_{k=1}^K$.

uncovered so far in the proof.

Claim 4: $M_{++}(m') = \emptyset$.

Proof: Suppose for contradiction that $M_{++}(m') \neq \emptyset$. Let $x_{k^*} \in M_{++}(m')$. Note that Step 3 (i) (Twin demand property), there is at most one agent $j \in N \setminus N(m')$ such that $a_j \in M(m')$. Thus, by $a_i \in M(m')$, $i \in N \setminus N(m')$, and Step 7 (Outside receiver II), $a_i \in M(k^*) \cap M_+(m')$. Therefore, by $a_i = x_\ell$, $\ell \leq k^*$.

By Claim 3, there is $K^* \in \{K'+1,\dots,K''\}$ such that $i_{K^*} = k^*$. By (3) and Step 3 (i) (Twin demand property),

$$\{a_{i_{K^*-1}}, a_{i_{K^*}}\} \subseteq D(R^{m'}_{i_{K^*}}, p(m')) = D(R'_{k^*}, p(m')) \subseteq \{x_{k^*}, x_{k^*+1}\},$$

which implies $\{a_{i_{K^*-1}}, a_{i_{K^*}}\} = \{x_{k^*}, x_{k^*+1}\}$. Thus, $x_{k^*} \in D(R'_{k^*}, p(m'))$. However, by $x_{k^*} \in M_{++}(m')$, $p_{k^*}(m') > V^{R'_k}(x_{k^*}, (0,0))$. This implies $(0,0)\ P'_{k^*}\ (x_{k^*}, p(m'))$, and thus, $x_{k^*} \notin D(R'_{k^*}, p(m'))$, a contradiction. □

We now construct two new sequences $\{j_k\}_{k=1}^{K'+K-K''}$ and $\{b_k\}_{k=1}^{K'+K-K''}$ using the sequences $\{i_k\}_{k=1}^{K}$ and $\{a_k\}_{k=1}^{K}$. Let $\{j_k\}_{k=1}^{K'+K-K''}$ and $\{b_k\}_{k=1}^{K'+K-K''}$ be such that for each $k \in \{1,\dots,K'+$

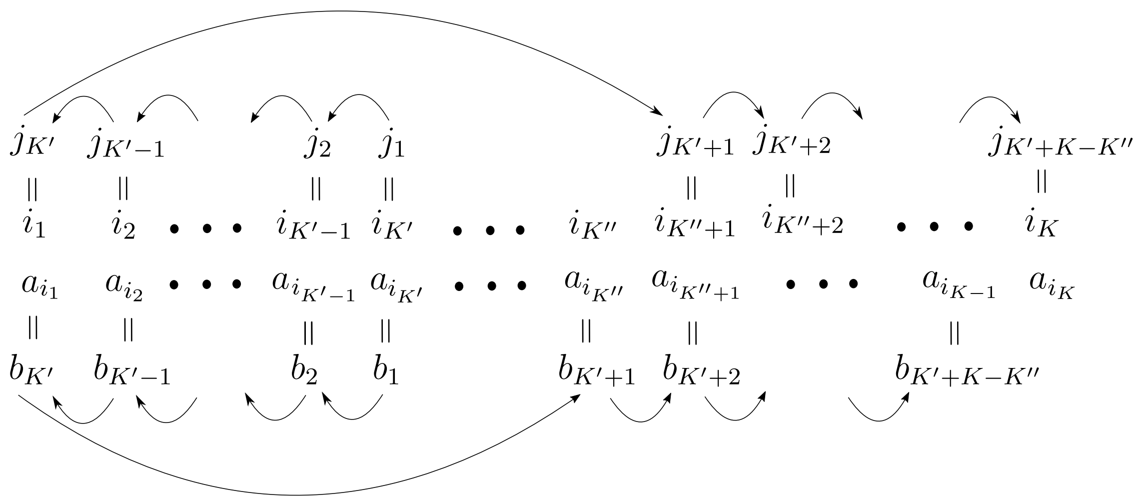

Figure 13: Illustration of $\{j_k\}_{k=1}^{K'+K-K''}$ and $\{b_k\}_{k=1}^{K'+K-K''}$.

$K-K''\}$,

$$j_k = \begin{cases} i_{K'+1-k} & \text{if } k \le K', \\ i_{K''+k-K'} & \text{if } k > K', \end{cases} \text{ and } b_k = \begin{cases} a_{i_{K'+1-k}} & \text{if } k \le K', \\ a_{i_{K''+k-K'-1}} & \text{if } k > K'. \end{cases}$$

Figure 13 is an illustration of $\{j_k\}_{k=1}^{K'+K-K''}$ and $\{b_k\}_{k=1}^{K'+K-K''}$.

Claim 5: *$\{j_1,\dots,j_{K'+K-K''}\} \subseteq N \setminus N(m')$ and $(\{j_k\}_{k=1}^{K'+K-K''}, \{b_k\}_{k=1}^{K'+K-K''})$ satisfies the following conditions.*
(a) $b_1 \in M_+(m') \setminus M_{++}(m')$,

(b) for each $k \in \{1, \ldots, K' + K - K'' - 1\}$, $\{b_k, b_{k+1}\} \subseteq D(R_{j_k}, p(m'))$, *and*

(c) $b_{K'+K-K''} \in D(R_{i_{K'+K-K''}}, p(m'))$.

Proof: First, we show $\{j_1, \ldots, j_{K'+K-K''}\} \subseteq N \setminus N(m')$. Let $k \in \{1, \ldots, K'\}$. By $K'+1-k \leq K'$, $j_k = i_{K'+1-k} \in \{i_1, \ldots, i_{K'}\}$. By the definition of K', $\{i_1, \ldots, i_{K'}\} \cap N(m') = \emptyset$. Thus, $j_k \notin N(m')$.

Next, let $k \in \{K' + 1, \ldots, K' + K - K''\}$. By $K'' + k - K' > K''$, $j_k = i_{K''+k-K'} \in \{i_{K''+1}, \ldots, i_K\}$. By Claim 2, $j_k \notin N(m')$. Hence, $\{j_1, \ldots, j_{K'+K-K''}\} \subseteq N \setminus N(m')$.

Proof of (a): By (6), $b_1 = a_{i_{K'}} = a_i$. Thus, by $a_i \in M_+(m')$ and Claim 4, $b_1 \in M_+(m') = M_+(m') \setminus M_{++}(m')$.

Proof of (b): Let $k \in \{1, \ldots, K' + K - K'' - 1\}$. There are three cases.

Case 1: $k \leq K' - 1$.

We have $j_k = i_{K'+1-k}$ and $\{b_k, b_{k+1}\} = \{a_{i_{K'+1-k}}, a_{i_{K'-k}}\}$. Thus, by (3),

$$\{b_k, b_{k+1}\} = \{a_{i_{K'+1-k}}, a_{i_{K'-k}}\} \subseteq D(R_{i_{K'+1-k}}, p(m')) = D(R_{j_k}, p(m')).$$

Case 2: $k = K'$.

By (1), $j_{K'} = i_1 = m' + 1$. Thus, $b_{K'} = a_{i_1} = a_{m'+1}$. By $a_{m'+1} \in D(R_{m'+1}, p(m'))$, $b_{K'} \in D(R_{j_{K'}}, p(m'))$. By $a_{m'+1}(R^{m'}) = x_{m'+1}$, $t_{m'+1}(R^{m'}) \geq p_{m'+1}(m')$, and Fact 1,

$$(x_{m'+1}, p_{m'+1}(m'))\, R_{m'+1}\, f_{m'+1}(R^{m'})\, R_{m'+1}\, (a_{m'+1}, p_{a_{m'+1}}(m')).$$

Thus, by $a_{m'+1} \in D(R_{m'+1}, p(m'))$, $x_{m'+1} \in D(R_{m'+1}, p(m'))$. By $b_{K'+1} = a_{i_{K''}} = a_{m'} = x_{m'+1}$, $b_{K'+1} \in D(R_{j_{K'}}, p(m'))$.

Case 3: $k \geq K' + 1$.

We have $j_k = i_{K''+k-K'}$ and $\{b_k, b_{k+1}\} = \{a_{i_{K''+k-K'-1}}, a_{i_{K''+k-K'}}\}$. Thus, by (3),

$$\{b_k, b_{k+1}\} = \{a_{i_{K''+k-K'-1}}, a_{i_{K''+k-K'}}\} \subseteq D(R_{i_{K''+k-K'}}, p(m')) = D(R_{j_k}, p(m')).$$

Proof of (c): Note that $j_{K'+K-K''} = i_K$ and $b_{K'+K-K''} = a_{i_{K-1}}$. Thus, by (3),

$$b_{K'+K-K''} = a_{i_{K-1}} \in D(R_{i_K}, p(m')) = D(R_{j_{K'+K-K''}}, p(m')).$$

□

By Claim 5 and Step 5, $0 \notin D(R_{j_{K'+K-K"}}, p(m'))$. However, by (2), $a_{j_{K'+K-K"}} = a_{i_K} = 0$. This contradicts $((a_j,t_j))_{j\in N} \in Z^{\min}(R^{m'})$, which completes the proof of step 8. ■

For the next step, it is convenient to introduce the following notations: Let $M(0) = M_+(0) = \emptyset$.

STEP **9** *Let $m' \in \{0,\ldots,m-1\}$. Suppose that $M_+(m') \neq \emptyset$ or $p_{m'+1}(m') > t_{m'+1}(R^{m'})$. Then, there is $i^* \in N \setminus N(m'+1)$ such that the following two conditions hold:*
(a) If $p_{m'+1}(m') > t_{m'+1}(R^{m'})$, then

$$D(R_{i^*}, p(m')) \cap (M_+(m') \cup \{x_{m'+1}\}) \neq \emptyset.$$

Else,

$$D(R_{i^*}, p(m')) \cap M_+(m') \neq \emptyset.$$

(b) If $m' \leq m-2$, then there is a set $\{j_{m'+2},\ldots,j_m\} \subseteq N\setminus(N(m'+1)\cup\{i^*\})$ of $m-(m'+1)$ distinct agents such that for each $k \in \{m'+2,\ldots,m\}$, $x_k \in D(R_{j_k}, p(m'))$.

Proof of Step 9: Let $((a_i,t_i))_{i\in N} \in Z^{\min}(R^{m'})$. For agent $m'+1$, by Fact 4, there is a sequence $\{i_k\}_{k=1}^{K}$ of K distinct agents such that[22]

$$i_1 = m'+1, \tag{1}$$
$$a_{i_K} = 0 \text{ and for each } k \in \{1,\ldots,K-1\},\ a_{i_k} \neq 0 \text{ and} \tag{2}$$
$$\text{for each } k \in \{2,\ldots,K\}, \{a_{i_{k-1}}, a_{i_k}\} \subseteq D(R^{m'}_{i_k}, p(m')). \tag{3}$$

CLAIM 1: *Let $x_j \in M(m') \cap \{a_{i_1},\ldots,a_{i_K}\}$. Then, $x_j \in M_+(m')$.*

Proof: Suppose for contradiction that $x_j \notin M_+(m')$. Then, by Step 3 (ii) (Unique demand property), $D(R'_j, p(m')) = \{x_j\}$. Thus, $a_j = x_j$. Therefore, by $x_j \in \{a_{i_1},\ldots,a_{i_K}\}$, $j \in \{i_1,\ldots,i_K\}$.

If $j = i_1$, then by (1), $j = m'+1$, which contradicts $j \in \{1,\ldots,m'\}$. Thus $j = i_k$ for some $k \in \{2,\ldots,K\}$. However, by (3),

$$\{a_{i_{k-1}}, a_{i_k}\} \subseteq D(R^{m'}_{i_k}, p(m')) = D(R'_j, p(m')) = \{x_j\},$$

[22]If $a_{m'+1} = 0$, then $\{i_k\}_{k=1}^{K} = \{m'+1\}$ and thus, the latter part of (2) and (3) vacuously hold.

a contradiction. $\square$

Note that by $|N(m')| < |M(m'+1)|$ and *no wastage*, there is $i \in N \setminus N(m')$ such that $a_i \in M(m'+1)$. By Step 3 (i) (Twin demand property),

$$\{a_1, \dots, a_{m'}, a_i\} = M(m'+1). \tag{4}$$

Note that by Step 3 (ii) (Unique demand property), for each $x_j \in M(m') \setminus M_+(m')$, $a_j = x_j$. Thus,

$$a_i \in M_+(m') \cup \{x_{m'+1}\}. \tag{5}$$

When $m' \leq m-2$, we define $\{j_{m'+2}, \dots, j_m\} \subseteq N$ as follows: First note that by $((a_i, t_i))_{i \in N} \in Z^{\min}(R^{m'})$, for each $k \in \{m'+2, \dots, m\}$, there is $i(k) \in N$ such that $a_{i(k)} = x_k$. For each $k \in \{m'+2, \dots, m\}$, let

$$j_k \equiv \begin{cases} i_{K'+1} & \text{if } i(k) = i_{K'} \text{ for some } K' \in \{1, \dots, K-1\}, \\ i(k) & \text{otherwise.} \end{cases}$$

Note that by $a_i \in M(m'+1)$

$$i \notin \{i(m'+2), \dots, i(m)\}. \tag{6}$$

In the following claims, we show that the agents $j_{m'+2}, \dots, j_m$ are distinct, $\{j_{m'+2}, \dots, j_m\} \subseteq N \setminus N(m'+1)$, and for each $k \in \{m'+2, \dots, m\}$, $x_k \in D(R_{j_k}, p(m'))$.

Claim 2. *Suppose $m' \leq m-2$. Let $k, \ell \in \{m'+2, \dots, m\}$ be such that $k \neq \ell$. Then, $j_k \neq j_\ell$.*

Proof: First suppose $i(k), i(\ell) \notin \{i_1, \dots, i_{K-1}\}$. Then, $j_k = i(k)$ and $j_\ell = i(\ell)$. By $i(k) \neq i(\ell)$, $j_k \neq j_\ell$.

Next, suppose $i(k) \notin \{i_1, \dots, i_{K-1}\}$ and $i(\ell) \in \{i_1, \dots, i_{K-1}\}$. Then, $j_k = i(k)$, and there is $K' \in \{1, \dots, K-1\}$ such that $i(\ell) = i_{K'}$ and $j_\ell = i_{K'+1}$. By $a_{i_K} = 0$ and $a_{i(k)} = x_{i(k)}$, $j_k = i(k) \neq i_K$, which implies $j_k \notin \{i_1, \dots, i_K\}$. By $j_\ell = i_{K'+1}$, $j_\ell \in \{i_1, \dots, i_K\}$, Thus, $j_k \neq j_\ell$. The same argument holds for the converse case.

Finally, suppose $i(k), i(\ell) \in \{i_1, \dots, i_{K-1}\}$. There are $K', K'' \in \{1, \dots, K-1\}$ such that $i_{K'} = i(k)$ and $j_k = i_{K'+1}$, and $i_{K''} = i(\ell)$ and $j_\ell = i_{K''+1}$. By $i_{K'} = i(k) \neq i(\ell) = i_{K''}$,

$K' \neq K''$. This implies $K'+1 \neq K''+1$. Thus, $j_k = i_{K'+1} \neq i_{K''+1} = j_\ell$. $\square$

Claim 3. *Suppose $m' \leq m-2$. Let $k \in \{m'+2, \ldots, m\}$. Then, $j_k \notin N(m'+1)$.*

Proof: First, suppose $i(k) \notin \{i_1, \ldots, i_{K-1}\}$. Then $j_k = i(k)$. By Step 3 (i) (Twin demand property) and $a_{i(k)} = x_k \notin M(m'+1)$, $j_k \notin N(m')$. In addition, by (1), $i(k) \neq i_1 = m'+1$.

Next, suppose $i(k) \in \{i_1, \ldots, i_{K-1}\}$. There is $K' \in \{1, \ldots, K-1\}$ such that $i(k) = i_{K'}$ and $j_k = i_{K'+1}$. By $i(k) = i_{K'}$, $a_{i_{K'}} = x_k$. By (3), $x_k = a_{i_{K'}} \in D(R^{m'}_{i_{K'+1}}, p(m'))$. Thus, by $x_k \notin M(m'+1)$ and Step 3 (i) (Twin demand property), $j_k = i_{K'+1} \notin N(m')$. In addition, by $K'+1 > 1$ and (1), $j_k \neq i_1 = m'+1$. $\square$

Claim 4. *Suppose $m' \leq m-2$. Let $k \in \{m'+2, \ldots, m\}$. Then, $x_k \in D(R_{j_k}, p(m'))$.*

Proof: First, suppose $i(k) \notin \{i_1, \ldots, i_{K-1}\}$. Then $j_k = i(k)$. By $a_{i(k)} = x_k$, $x_k \in D(R_{j_k}, p(m'))$.

Next, suppose $i(k) \in \{i_1, \ldots, i_{K-1}\}$. There is $K' \in \{1, \ldots, K-1\}$ such that $i(k) = i_{K'}$ and $j_k = i_{K'+1}$. By $i(k) = i_{K'}$, $a_{i_{K'}} = x_k$. By (3), $x_k = a_{i_{K'}} \in D(R^{m'}_{i_{K'+1}}, p(m')) = D(R_{j_k}, p(m'))$. $\square$

We now complete the proof of this step by considering three disjoint cases.

Case 1: $M(m'+1) \cap \{a_{i_1}, \ldots, a_{i_K}\} = \emptyset$.

Let

$$i^* \equiv i.$$

By (5) and $M(m'+1) \cap \{a_{i_1}, \ldots, a_{i_K}\} = \emptyset$,

$$i \notin \{i_1, \ldots, i_K\}. \tag{7}$$

Thus, by (1), $i \neq i_1 = m'+1$. Thus, by $i \notin N(m')$, $i^* \in N \setminus N(m'+1)$.

Proof of (a). By (5) and $i^* = i$, $D(R_{i^*}, p(m')) \cap (M_+(m') \cup \{x_{m'+1}\}) \neq \emptyset$. Thus, we are done when $p_{m'+1}(m') > t_{m'+1}(R^{m'})$. Now, suppose $p_{m'+1}(m') \leq t_{m'+1}(R^{m'})$. Since either $M_+(m') \neq \emptyset$ or $p_{m'+1}(m') > t_{m'+1}(R^{m'})$, we have $M_+(m') \neq \emptyset$. By Steps 6 (Outside receiver I) and 7 (Outside receiver II), there is $k \in N \setminus N(m')$ such that $a_k \in M_+(m')$. By (4), $k = i$.

Thus, $a_i \in M_+(m')$. Thus, by $i^* = i$, $D(R_{i^*}, p(m')) \cap M_+(m') \neq \emptyset$.

Proof of (b). Suppose $m' \leq m - 2$. Let $k \in \{m' + 2, \ldots, m\}$. By Claims 2, 3, and 4, we only need to show $j_k \neq i^*$. First, suppose $i(k) \notin \{i_1, \ldots, i_K\}$. Then, $j_k = i(k)$. Thus, by (6), $j_k = i(k) \neq i = i^*$.

Next, suppose $i(k) \in \{i_1, \ldots, i_K\}$. There is $K' \in \{1, \ldots, K-1\}$ such that $i(k) = i_{K'}$ and $j_k = i_{K'+1}$. By $j_k \in \{i_1, \ldots, i_K\}$ and (7), $j_k \neq i = i^*$.

Case 2: *$M(m') \cap \{a_{i_1}, \ldots, a_{i_K}\} \neq \emptyset$ and $x_{m'+1} \notin \{a_{i_1}, \ldots, a_{i_K}\}$.*

By $M(m') \cap \{a_{i_1}, \ldots, a_{i_K}\} \neq \emptyset$ and (2), there is $K' \in \{1, \ldots, K-1\}$ such that

$$a_{i_{K'}} \in M(m') \text{ and} \tag{8}$$

$$\{a_{i_{K'+1}}, \ldots, a_{i_K}\} \cap M(m') = \emptyset. \tag{9}$$

Let

$$i^* \equiv i_{K'+1}.$$

By $a_{i_{K'+1}} \notin M(m')$ and Step 3 (i) (Twin demand property), $i_{K'+1} \notin N(m'-1)$. If $i_{K'+1} = m'$, then by $a_{i_{K'+1}} \notin M(m')$ and Step 3 (i) (Twin demand property), $a_{i_{K'+1}} = x_{m'+1}$, which is impossible since $x_{m'+1} \notin \{a_{i_1}, \ldots, a_{i_K}\}$. Thus, $i_{K'+1} \neq m'$. Finally, by $K' + 1 > 1$ and (1), $i_{K'+1} \neq i_1 = m' + 1$. Hence, $i^* = i_{K'+1} \in N \setminus N(m' + 1)$.

Proof of (a). By (3) and $i^* = i_{K'+1}$, $a_{i_{K'}} \in D(R^{m'}_{i_{K'+1}}, p(m')) = D(R_{i^*}, p(m'))$. By Claim 1 and $a_{i_{K'}} \in M(m')$, $a_{i_{K'}} \in M_+(m')$. Thus, $D(R_{i^*}, p(m')) \cap M_+(m') \neq \emptyset$.

Proof of (b). Suppose $m' \leq m - 2$. Let $k \in \{m' + 2, \ldots, m\}$. By Claims 2, 3, and 4, we only need to show $j_k \neq i^*$. First, suppose $i(k) \notin \{i_1, \ldots, i_K\}$. Then, $j_k = i(k)$. By $i(k) \notin \{i_1, \ldots, i_K\}$ and $i^* = i_{K'+1} \in \{i_1, \ldots, i_K\}$, $j_k \neq i^*$.

Next, suppose $i(k) \in \{i_1, \ldots, i_K\}$. There is $K^* \in \{1, \ldots, K-1\}$ such that $i(k) = i_{K^*}$ and $j_k = i_{K^*+1}$. By $a_{i_{K'}} \in M(m')$ and $a_{i_{K^*}} = a_{i(k)} = x_k \notin M(m' + 1)$, $K' \neq K^*$. Thus, $j_k = i_{K^*+1} \neq i_{K'+1} = i^*$.

Case 3: *$x_{m'+1} \in \{a_{i_1}, \ldots, a_{i_K}\}$.*

Let $K' \in \{1, \ldots, K\}$ be such that

$$a_{i_{K'}} = x_{m'+1}.$$

By (2), $K' < K$. Thus, $i_{K'+1}$ exists. Let

$$i^* \equiv i_{K'+1}.$$

We show $i^* \in N \setminus N(m'+1)$. By $a_{i_{K'}} = x_{m'+1}$, (3), and Step 3 (i) (Twin demand property), $i_{K'+1} \notin N(m'-1)$. By $K'+1 > 1$ and (1), $i_{K'+1} \neq i_1 = m'+1$.

Note that by (4), (5), and Step 3 (i) (Twin demand property),

$$a_i = x_{m'+1} \text{ or } a_{m'} = x_{m'+1}.$$

Suppose $a_{m'} = x_{m'+1}$. Then $i_{K'} = m'$ and thus $i_{K'+1} \neq m'$. Suppose $a_i = x_{m'+1}$. This implies $i \neq i^*$. If $M_+(m') \neq \emptyset$, then Steps 6 (Outside receiver I) and 7 (Outside receiver II) imply that $a_j \in M_+(m')$ for some $j \in N \setminus N(m')$, and by (4), $j = i$, contradicting $a_i = x_{m'+1}$. Thus, $M_+(m') = \emptyset$. Then, by Claim 1, $a_{i_{K'+1}} \notin M(m')$. Moreover, by $a_{i_{K'}} = x_{m'+1}$, $a_{i_{K'+1}} \neq x_{m'+1}$. Thus, by Step 3 (i) (Twin demand property), $i_{K'+1} \neq m'$. Hence, $i^* = i_{K'+1} \in N \setminus N(m'+1)$.

Proof of (a). First, we show that $p_{m'+1}(m') > t_{m'+1}(R^{m'})$. Suppose $M_+(m') \neq \emptyset$. Then, as we have seen in the above paragraph, we can show that $a_i \neq x_{m'+1}$. Since either $a_i = x_{m'+1}$ or $a_{m'} = x_{m'+1}$, $a_{m'} = x_{m'+1}$. By $x_{m'+1} \in \{a_{i_1}, \ldots, a_{i_K}\}$, $m' \in \{i_1, \ldots, i_K\}$. Then, by Step 8, $p_{m'+1}(m') > t_{m'+1}(R^{m'})$. Since $M_+(m') \neq \emptyset$ or $p_{m'+1}(m') > t_{m'+1}(R^{m'})$, we can conclude that $p_{m'+1}(m') > t_{m'+1}(R^{m'})$.

By $a_{i_{K'}} = x_{m'+1}$ and (3), $x_{m'+1} \in D(R^{m'}_{i_{K'+1}}, p(m')) = D(R_{i^*}, p(m'))$. Thus, $D(R_{i^*}, p(m')) \cap (M_+(m') \cup \{x_{m'+1}\}) \neq \emptyset$.

Proof of (b). Suppose $m' \leq m-2$. Let $k \in \{m'+2, \ldots, m\}$. By Claims 2, 3, and 4, we only need to show $j_k \neq i^*$. First, suppose $i(k) \notin \{i_1, \ldots, i_K\}$. Then, $j_k = i(k)$. By $i(k) \notin \{i_1, \ldots, i_K\}$ and $i^* = i_{K'+1} \in \{i_1, \ldots, i_K\}$, $j_k \neq i^*$.

Next, suppose $i(k) \in \{i_1, \ldots, i_K\}$. There is $K^* \in \{1, \ldots, K-1\}$ such that $i(k) = i_{K^*}$ and $j_k = i_{K^*+1}$. By $a_{i_{K^*}} = a_{i(k)} = x_k \neq x_{m'+1} = a_{i_{K'}}$, $K^* \neq K'$. Thus, $j_k = i_{K^*+1} \neq i_{K'+1} = i^*$. ■

Step **10** *There are $i \in N \setminus N(m)$ and $j \in \{1, \ldots, m\}$ such that $f_j(R^{j-1})\, P_i\, (0,0)$.*

Proof of Step 10: The proof consists of two substeps.

Substep 10-1 *Let* $m' \in \{0, 1, \ldots, m-2\}$ *be such that* $M_+(m') \neq \emptyset$ *or* $p_{m'+1}(m') > t_{m'+1}(R^{m'})$. *Then,* $M_+(m'+1) \neq \emptyset$.

Proof: Suppose for contradiction that $M_+(m'+1) = \emptyset$. By $M_+(m') \neq \emptyset$ or $p_{m'+1}(m') > t_{m'+1}(R^{m'})$, Step 9 (a) implies that there is $i^* \in N \setminus N(m'+1)$ such that if $p_{m'+1}(m') > t_{m'+1}(R^{m'})$, then

$$D(R_{i^*}, p(m')) \cap (M_+(m') \cup \{x_{m'+1}\}) \neq \emptyset, \tag{1}$$

else,

$$D(R_{i^*}, p(m')) \cap M_+(m') \neq \emptyset. \tag{2}$$

Moreover, by Step 9 (b) and $m' \leq m-2$, there is a set $\{j_{m'+2}, \ldots, j_m\} \subseteq N \setminus (N(m'+1) \cup \{i^*\})$ of $m-(m'+1)$ distinct agents such that for each $k \in \{m'+2, \ldots, m\}$, $x_k \in D(R_{j_k}, p(m'))$.

Let $((a_i, t_i))_{i \in N} \in Z^{\min}(R^{m'+1})$. Note that by $M_+(m'+1) = \emptyset$ and Step 3 (ii) (Unique demand property),

$$a_i = x_i \text{ for each } i \in N(m'+1). \tag{3}$$

Claim 1: *Let* $i \in N \setminus N(m'+1)$. *Suppose that there is* $x_k \in M$ *such that* $x_k \in D(R_i, p(m'))$ *and* $p_k(m'+1) < p_k(m')$. *Then,* $p_{a_i}(m'+1) < p_{a_i}(m')$.

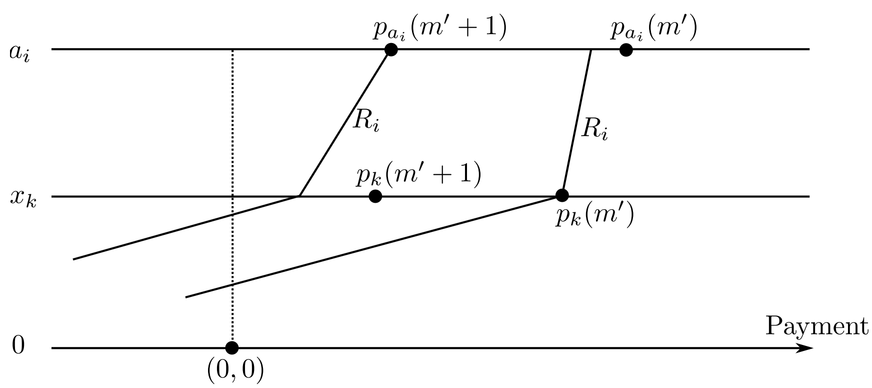

Figure 14: Illustration of the proof of Claim 1.

Proof: (See Figure 14 for illustration.) Note that

$$(a_i, p_{a_i}(m'+1)) \; R_i \; (x_k, p_k(m'+1)) \; P_i \; (x_k, p_k(m')) \; R_i \; (a_i, p_{a_i}(m')),$$

where the first relation follows from $a_i \in D(R_i, p(m'+1))$, the second from $p_k(m'+1) < p_k(m')$, and the last from $x_k \in D(R_i, p(m'))$. This implies $p_{a_i}(m'+1) < p_{a_i}(m')$. □

Claim 2: *For each $K \in \mathbb{N}$, there exists a set $N_K \equiv \{i_1, \dots, i_K\} \subseteq N \setminus N(m'+1)$ of K distinct agents such that $i^* \in N_K$ and for each $a \in \{a_{i_1}, \dots, a_{i_K}\}$, $p_a(m'+1) < p_a(m')$.*

Proof: The proof is by induction.

Induction base. Let $K = 1$ and $N_1 = \{i^*\}$. By $i^* \in N \setminus N(m'+1)$, $N_1 \subseteq N \setminus N(m'+1)$. Note that if we show that there is $a \in D(R_{i^*}, p(m'))$ such that $p_a(m'+1) < p_a(m')$, then Claim 1 implies that $p_{a_{i^*}}(m'+1) < p_{a_{i^*}}(m')$.

First, suppose that there is $x_k \in M_+(m')$ such that $x_k \in D(R_{i^*}, p(m'))$. By $M_+(m'+1) = \emptyset$, $p_k(m'+1) \leq t_k(R^{k-1})$. By $x_k \in M_+(m')$, $p_k(m') > t_k(R^{k-1})$. Thus, $p_k(m'+1) < p_k(m')$.

Next, suppose that $D(R_{i^*}, p(m')) \cap M_+(m') = \emptyset$. By (2) we have $p_{m'+1}(m') > t_{m'+1}(R^{m'})$. Then, by (1), $x_{m'+1} \in D(R_{i^*}, p(m'))$. By $M_+(m'+1) = \emptyset$, $p_{m'+1}(m'+1) \leq t_{m'+1}(R^{m'})$. Thus, $p_{m'+1}(m'+1) < p_{m'+1}(m')$.

Induction argument. Let $K \geq 1$ and assume that there is a set $N_K = \{i_1, \dots, i_K\} \subseteq N \setminus N(m'+1)$ of K distinct agents such that

$$i^* \in N_K, \text{ and} \tag{4}$$

$$\text{for each } a \in \{a_{i_1}, \dots, a_{i_K}\},\ p_a(m'+1) < p_a(m'). \tag{5}$$

By (3) and $N_K \subseteq N \setminus N(m'+1)$, $\{a_{i_1}, \dots, a_{i_K}\} \subseteq M \setminus M(m'+1)$. Without loss of generality, assume that $\{a_{i_1}, \dots, a_{i_K}\} = \{x_{m'+2}, \dots, x_{m'+K+1}\}$. By $i^* \in N_K$ and $i^* \notin \{j_{m'+2}, \dots, j_{m'+K+1}\}$, there is $j_k \in \{j_{m'+2}, \dots, j_{m'+K+1}\}$ such that $j_k \notin N_K$.

Let $N_{K+1} \equiv N_K \cup \{j_k\}$. By $N_K \subseteq N \setminus N(m'+1)$ and $j_k \notin N(m'+1)$, $N_{K+1} \subseteq N \setminus N(m'+1)$. By (4), $i^* \in N_K \subseteq N_{K+1}$. Note that by $j_k \in \{j_{m'+2}, \dots, j_{m'+K+1}\}$, $x_k \in \{x_{m'+2}, \dots x_{m'+K+1}\} = \{a_{i_1}, \dots, a_{i_K}\}$. Thus, by (5), $p_k(m'+1) < p_k(m')$. Thus, by $x_k \in D(R_{j_k}, p(m'))$, Claim 1 implies that $p_{a_{j_k}}(m'+1) < p_{a_{j_k}}(m')$. This completes the proof of Claim 2. □

Claim 2 completes the proof of Substep 10-1 because N is finite. ■

Substep 10-2 *Completing the proof of Step 10.*

Remember that $p_1(0) > t_1(R^0)$. Thus, by using Substep 10-1 repeatedly, we have $M_+(m-1) \neq \emptyset$. Then, by Step 9 (a), there is $i \in N \setminus N(m)$ such that if $p_m(m-1) > t_m(R^{m-1})$, then

$$D(R_i, p(m-1)) \cap (M_+(m-1) \cup \{x_m\}) \neq \emptyset, \tag{6}$$

else,

$$D(R_i, p(m-1)) \cap M_+(m-1) \neq \emptyset. \tag{7}$$

First, suppose that there is $x_j \in M_+(m-1)$ such that $x_j \in D(R_i, p(m-1))$. By $x_j \in M_+(m-1)$, $p_j(m-1) > t_j(R^{j-1})$. Thus, by $x_j \in D(R_i, p(m-1))$, $f_j(R^{j-1})\, P_i\, (x_j, p_j(m-1))\ R_i\ (0,0)$.

Next, suppose that $D(R_i, p(m-1)) \cap M_+(m-1) = \emptyset$. By (6) and (7), $p_m(m-1) > t_m(R^{m-1})$ and $x_m \in D(R_i, p(m-1))$. Thus, $f_m(R^{m-1})\, P_i\, (x_m, p_m(m-1))\ R_i\ (0,0)$. ■

STEP **11** *Completing the proof.*

First, we show the following.

CLAIM: *Let $i \in N(m)$. Then, $a_i(R^m) \in \{x_i, x_{i+1}\}$.*

Proof: Suppose for contradiction that $a_i(R^m) \notin \{x_i, x_{i+1}\}$. Then,

$$\begin{aligned} V^{R'_i}(x_{i+1}, f_i(R^m)) &\geq V^{R'_i}(x_{i+1}, (a_i(R^m), 0)) && \text{by } \textit{no subsidy} \\ &> \overline{V} && \text{by } (i\text{-iii}) \text{ in Step 1} \\ &> p_{i+1}(m). && \text{by Step 2} \end{aligned}$$

This implies $(x_{i+1}, p_{i+1}(m))\, P'_i\, f_i(R^m)$, contradicting Fact 1. □

Since R'_m is $f_m(R^{m-1})$-favoring, Lemma 2 implies $f_m(R^m) = f_m(R^{m-1})$. Thus, by $a_m(R^{m-1}) = x_m$, $a_m(R^m) = x_m$. Therefore, by applying Claim repeatedly, we obtain

$$a_i(R^m) = x_i \text{ for each } i \in N(m).$$

Thus, by *individual rationality*,

$$t_i(R^m) \leq V^{R'_i}(x_i, (0,0)) \text{ for each } i \in N(m). \tag{1}$$

By Step 10, there are $i \in N \setminus N(m)$ and $j \in \{1, \ldots, m\}$ such that $f_j(R^{j-1})\, P_i\, (0,0)$. Note that this implies that $(\{i\}, \{x_j\})$ is an IRIC from $f_j(R^{j-1})$. By (1) and (j-ii) in Step 1, $t_j(R^m) \leq V^{R'_j}(x_j, (0,0)) < d(f_j(R^{j-1}))$. Therefore, $(\{i\}, \{x_j\})$ is an IRIC from $f_j(R^m)$, and thus, $f_j(R^m)\, P_i\, (0,0)$

Let $R''_j \equiv R_i$. By *strategy-proofness* and $f_j(R^m)\, P_i\, (0,0)$, $f_j(R''_j, R^m_{-j})\, R''_j\, f_j(R^m)\, P''_j\, (0,0)$. By *equal treatment of equals* and $R_i = R''_j$, $f_i(R''_j, R^m_{-j})\, I_i\, f_j(R''_j, R^m_{-j})\, P_i\, (0,0)$. Thus, by Lemma 1,

$$a_j(R''_j, R^m_{-j}) \neq 0 \text{ and } a_i(R''_j, R^m_{-j}) \neq 0.$$

By $a_i(R''_j, R^m_{-j}) \neq 0$ and $|N(m)| = m$, there is $k \in N(m)$ such that $a_k(R''_j, R^m_{-j}) = 0$. By $a_j(R''_j, R^m_{-j}) \neq 0$, $k \neq j$. By Lemma 1, $t_k(R''_j, R^m_{-j}) = 0$. Let $p \equiv p^{\min}(R''_j, R^m_{-j})$. Similarly to Step 2, we can show $p_{x_{k+1}} < \bar{V}$. By (k-ii), $p_{x_{k+1}} < \bar{V} < V^{R'_k}(x_{k+1}, (0,0))$. Thus, we have $(x_{k+1}, p_{x_{k+1}})\ P'_k\ (0,0) = f_k(R''_j, R^m_{-j})$. This contradicts Fact 1 ■

References

Adachi, T. (2014): "Equity and the Vickrey allocation rule on general preference domains," *Social Choice and Welfare*, 42, 813–830.

Alkan, A. and D. Gale (1990): "The core of the matching game," *Games and economic behavior*, 2, 203–212.

Andersson, T., L. Ehlers, and L.-G. Svensson (2016): "Transferring ownership of public housing to existing tenants: A market design approach," *Journal of Economic Theory*, 165, 643–671.

Ashlagi, I. and S. Serizawa (2012): "Characterizing Vickrey Allocation Rule by Anonymity," *Social Choice and Welfare*, 38, 531–542.

Baisa, B. (2020): "Efficient multi-unit auctions for normal goods," *Theoretical Economics*, 15, 361–413.

Baldwin, E., R. Jagadeesan, P. Klemperer, and A. Teytelboym (2023): "The equilibrium existence duality," *Journal of Political Economy*, 131, 1440—1476.

Baldwin, E. and P. Klemperer (2019): “Understanding preferences:“demand types”, and the existence of equilibrium with indivisibilities,” *Econometrica*, 87, 867–932.

Binmore, K. and P. Klemperer (2002): “The biggest auction ever: the sale of the British 3G telecom licences,” *The Economic Journal*, 112, C74–C76.

Bulow, J., J. Levin, and P. Milgrom (2017): *Winning Play in Spectrum Auctions*, Cambridge University Press, chap. Handbook of Spectrum Auction Design.

Chen, B. and D. Knyazev (2025): “Optimal discrimination-free auctions,” *Management Science, forthcoming.*

Chew, S. H. and S. Serizawa (2007): “Characterizing the Vickrey combinatorial auction by induction,” *Economic Theory*, 33, 393–406.

Deb, R. and M. Pai (2016): “Discrimination via symmetric auctions,” *American Economic Journal: Microeconomics*, 9, 275–314.

Demange, G. and D. Gale (1985): “The strategy structure of two-sided matching markets,” *Econometrica*, 53, 873–888.

Holmström, B. (1979): “Groves’ scheme on restricted domains,” *Econometrica*, 1137–1144.

Kazumura, T., D. Mishra, and S. Serizawa (2020): “Strategy-proof multi-object mechanism design: Ex-post revenue maximization with no wastage,” 188, 105036, journal of Economic Theory.

Kazumura, T. and S. Serizawa (2016): “Efficiency and strategy-proofness in object assignment problems with multi-demand preferences,” *Social Choice and Welfare*, 47, 633–663.

Kelso, A. S. and V. P. Crawford (1982): “Job matching, coalition formation, and gross substitutes,” *Econometrica: Journal of the Econometric Society*, 1483–1504.

Kwerel, E. R. and W. Strack (2001): *Auctioning spectrum rights*, US Federal Communications Commission.

Malik, K. and D. Mishra (2021): "Pareto efficient combinatorial auctions: dichotomous preferences without quasilinearity," *Journal of Economic Theory*, 191, 105128.

McMillan, J. (1995): "Why auction the spectrum?" *Telecommunications policy*, 19, 191–199.

Mishra, D. and D. Talman (2010): "Characterization of the Walrasian equilibria of the assignment model," *Journal of Mathematical Economics*, 46, 6–20.

Miyake, M. (1998): "On the incentive properties of multi-item auctions," *International Journal of Game Theory*, 27, 1–19.

Morimoto, S. and S. Serizawa (2015): "Strategy-proofness and efficiency with non-quasi-linear preferences: A characterization of minimum price Walrasian rule," *Theoretical Economics*, 10, 445–487.

Nguyen, T. and R. Vohra (2024): "(Near) Substitute Preferences and Equilibria with Indivisibilities," *Journal of Political Economy*, 132, 4122–4154.

Saitoh, H. and S. Serizawa (2008): "Vickrey allocation rule with income effect," *Economic Theory*, 35, 391–401.

Sakai, R. and S. Serizawa (2021): "Strategy-proof mechanism design with non-quasi-linear preferences: Ex-post revenue maximization for an arbitrary number of objects," *Social Choice and Welfare*, 60, 103–120.

Sakai, T. (2008): "Second price auctions on general preference domains: two characterizations," *Economic Theory*, 37, 347–356.

——— (2013): "An equity characterization of second price auctions when preferences may not be quasilinear," *Review of Economic Design*, 17, 17–26.

Shinozaki, H. and S. Serizawa (2025a): "Constrained efficiency and strategy-proofness in package assignment problems with money," ISER Discussion paper No. 1292.

——— (2025b): "fficient and strategy-proof multi-unit object allocation problems with money: decreasing incremental valuations with income effects," *Economic Theory, forthcoming.*

Sun, N. and Z. Yang (2006): "Equilibria and indivisibilities: Gross substitutes and complements," *Econometrica*, 74, 1385–1402.

Svensson, L.-G. (1983): "Large indivisibles: An analysis with respect to price equilibrium and fairness," *Econometrica*, 51, 939–954.

Teytelboym, A. (2014): "Gross substitutes and complements: A simple generalization," *Economic Letters*, 123, 135–138.

Tierney, R. (2016): "Incentives in a Job-market Clearing house," Mimeo.

Wakabayashi, Y., R. Sakai, and S. Serizawa (2025): "A general characterization of the minimum price Walrasian rule with reserve prices," *Games and Economic Behavior*, 150, 1–26.

Zhou, Y. and S. Serizawa (2018): "Strategy-proofness and efficiency for non-quasi-linear and common-tiered-object preferences: Characterization of minimum price rule," *Games and Economic Behavior*, 109, 327–363.