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Velocity and Monetary Expansion in a Growing Economy with Interest-Rate Control

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Abstract

We analyze the income velocity of money in an endogenous growth model with an interest-rate control rule and a cash-in-advance (CIA) constraint. We show that the long-term relationship between the income velocity of money and the nominal growth rate of money supply depends not only on the form of the CIA constraint but also on the central bank's stance of interest-rate control rule.

Keywords and Phrases: velocity, an interest-rate control, endogenous growth, cash-in-advance constraint

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1 Introduction

This paper examines the long-run effects of an endogenous monetary expansion via the interest-rate control on income growth and on the velocity of money in the context of an AK growth model with a cash-in-advance (CIA) constraint.

In the existing literature, Suen and Yip (2005) and Chen and Guo (2008a) also study the AK growth model with a CIA constraint. The main finding of those studies can be summarized as follows. First, if the interetemporal elasticity of substitution in consumption is less than one, the balanced growth path is unique and determinate, while there may exist dual balanced growth paths if the interetermporal elasticity of substitution in consumption is higher than one. In the latter case, the balance-growth path (BGP) with a higher growth rate is locally indeterminate and there is a positive relation between the growth rate of nominal money supply and the velocity of money. Such a positive relationship is, however, empirically implausible. Chen and Guo (2008b) overcome this problem by assuming that the CIA constraint is more effective on investment than on consumption. They justify this assumption based on the recent increases in the consumer credit and in the cash holdings of firms.

These foregoing studies mentioned above assume that the central bank keeps the growth rate of nominal money supply constant. However, many central banks have shifted their policy stance from the base-money targeting to the interest-rate control, we re-examine the long-run relation between monetary growth and velocity of money under interest-rate control rules. Except for the monetary policy rule, we employ the same analytical framework as Chen and Guo (2008b) use.

The present paper reveals that the relation between money growth and velocity of money around the BGP depends on the stance of the monetary policy as well as on the form of CIA constraint. When the nominal interest rate responds to the current rate of inflation alone, the velocity of money is negatively related to the growth rate of nominal money supply around the unique BGP, if the CIA constraint is more binding for investment than consumption. However, if the interest rate responds to the growth rate of income as well and it responds to inflation more than one for one, then a lower velocity of money may be associated with a higher money growth under the normal CIA constraint which is more effective for consumption spending than for investment expenditure.

2 The Model

The representative household's problem is to maximize a discounted stream of utilities

$$\int_0^\infty \frac{c^{1-\sigma}}{1-\sigma} e^{-\rho t} dt, \quad \sigma > 0, \quad \rho > 0, \tag{1}$$

subject to

$$\dot{m} = y - \pi m - c - \nu + \tau, \tag{2}$$

$$\dot{k} = \nu - \delta k, \quad 0 \leq \delta \leq 1, \quad k_0 : \text{given}, \tag{3}$$

$$\psi_c c + \psi_\nu \nu \leq m, \quad 0 < \psi_c \leq 1, \quad 0 \leq \psi_\nu \leq 1, \tag{4}$$

where c is consumption, ρ denotes the time discount rate, k is the household's capital stock, δ denotes the capital depreciation rate and σ is the inverse of the intertemporal elasticity of substitution in consumption. Moreover, ν is gross investment, y is output, $\pi \equiv \dfrac{\dot{P}}{P}$ is the rate of inflation, P the price level, and m denotes the real money balances that equal the nominal money supply M divided by P. The household follows the budget constraint (2) and the dynamics of capital stock (3).

The generalized CIA constraint (4) means that parts of consumption and gross investment must be financed by the household's real money balances. The seigniorage is returned to households from the government as a lump-sum transfer so that the government's budget constraint is $\tau = \dot{m} + \pi m$.

The production function is given by

$$y = Ak, \quad A > 0, \tag{5}$$

and thus the market equilibrium condition for commodity, $y = \dot{k} + \delta k + c$, yields

$$\frac{\dot{k}}{k} = A - \delta - z, \tag{6}$$

where $z \equiv \dfrac{c}{k}$.

Following Taylor (1993), we assume that the monetary authority controls the nominal interest rate by observing the real income as well as inflation. Since we deal with a growing economy in which real income continuously expands, we assume that the monetary authority changes the nominal interest rate in response not to the level of income but to the growth rate of income. Although this formulation is not exactly the same as in Taylor (1993), it is common that interest rate should be raised in order to suppress the overheated economy. Specifically, we assume the following control rule:

$$R = R(\pi, g) = \pi^* \left(\frac{\pi}{\pi^*}\right)^{\phi} + (A - \delta)\left(\frac{g}{g^*}\right)^{\eta}, \quad \phi \geq 0, \quad \phi \neq 1, \quad \eta \geq 0, \quad A > \delta, \tag{7}$$

where R is the nominal interest rate, and g denotes the growth rate of real income given by

$$g = \frac{\dot{y}}{y} = \frac{\dot{k}}{k} = A - \delta - z. \tag{8}$$

In addition, $g^* > 0$ represents the balanced-growth rate of income and $\pi^* > 0$ denotes the target rate of inflation. If $\phi > 1$, the nominal interest rate rises more than one for one in response to a change in the rate of inflation. In this case, the interest control rule is said to be active as to inflation. Conversely, the rule (7) with $\phi < 1$ is defined as a passive monetary policy.

We focus on the economy's BGP on which income, capital, consumption and real money balances grow at a common rate. Combining (7) and (8) with the Fisher equation,

$$R - \pi = A - \delta, \tag{9}$$

which implies that the real interest rate equals to the real rate of return to capital, we obtain

$$A - \delta + \pi = \pi^* \left(\frac{\pi}{\pi^*} \right)^{\phi} + (A - \delta) \left(\frac{A - \delta - z}{g^*} \right)^{\eta}. \tag{10}$$

Therefore, we see that the equilibrium rate of inflation depends on z, that is, $\pi = \pi(z)$.

This type of monetary policy rule is also used in Fujisaki and Mino (2007), in which money is introduced as the form of money-in-the-utility into an AK growth model. Fujisaki and Mino (2007) show that the response of the nominal rate of interest to the growth rate of income affects macroeconomic stability.

3 Dynamic System and Balanced Growth Path

We consider the case in which ψ_ν is non-zero [1]. Denoting λ_m and λ_k as the shadow prices of real money balances and capital, we define $p \equiv \dfrac{\lambda_k}{\lambda_m}$. As shown in Appendix,

[1] Since money is superneutral when $\psi_\nu = 0$, interesting result cannot be obtained even if $\psi_c > 0$. Technically, $p = 1$ for all t in such a case so that the dynamic system is consisted by z alone and it does not depend on ψ_c.

we obtain the following dynamic system:

$$\dot{z} = \left\{ \frac{1}{\sigma} \left[l(p)\frac{\dot{p}}{p} + \frac{A}{p} - \rho - \delta \right] - A + \delta \right\} z + z^2, \tag{11}$$

$$\dot{p} = \frac{p^2}{\psi_\nu} + \left(\delta - \pi(z) - \frac{1}{\psi_\nu} \right) p - A. \tag{12}$$

where $l(p) = -\dfrac{\psi_c - \psi_\nu}{\psi_c p - (\psi_c - \psi_\nu)}$. The dynamic equation of p (12) seems to be much simple than that in Chen and Guo (2008a, 2008b), but this is due to the specification of monetary policy. Therefore, the dynamic system in this paper is essentially the same as in Chen and Guo (2008a, 2008b). Note that $\text{sign}[l(p)] = -\text{sign}[\psi_c - \psi_\nu]$, since $p > 1$ from the assumption that the CIA constraint is strictly binding in equilibrium.

In the BGP where $\dot{z} = \dot{p} = 0$ in (11) and (12), we obtain

$$g^* = \frac{1}{\sigma} \left[\frac{A}{p^*} - \rho - \delta \right] = A - \delta - z^* > 0, \tag{13}$$

$$\frac{p^*}{\psi_\nu} + \delta - \pi^* - \frac{1}{\psi_\nu} - \frac{A}{p^*} = 0. \tag{14}$$

As above conditions show, the BGP is uniquely determined [2], regardless of the magnitude of σ, which is in a marked contrast to the model with a fixed growth rate of nominal money supply.

[2] From (12), we have two p^*s, but one of them is negative, while another is positive and satisfies $p^* > 1$ since $\dot{p}(p = 1; z = z^*) = -(A - \delta) - \pi^* < 0$. Therefore, the number of plausible p^* is only one. Under the unique p^*, we can give the nontrivial unique z^*. Determinacy of this unique BGP is detailed in Appendix.

4 Velocity of Money

From now on, we investigate the relation between the velocity and the monetary expansion rate. From (5) and $\nu = y - c$, (4) is rewritten such that

$$m = (\psi_c - \psi_\nu)c + \psi_\nu Ak. \tag{15}$$

Differentiating this equation and defining μ as the nominal money growth rate, we obtain

$$\frac{\dot{m}}{m} = \mu - \pi = \frac{(\psi_c - \psi_\nu)\dot{c} + \psi_\nu A\dot{k}}{(\psi_c - \psi_\nu)c + \psi_\nu Ak}. \tag{16}$$

Substituting $\frac{\dot{z}}{z} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k}$ into (15), the endogenous nominal rate of monetary expansion is:

$$\mu = \mu(z, p) = q(z)\frac{\dot{z}}{z} + A - \delta - z + \pi(z), \tag{17}$$

where $q(z) \equiv \frac{(\psi_c - \psi_\nu)z}{(\psi_c - \psi_\nu)z + \psi_\nu A}$ is positive (resp. negative) if $\psi_c > \psi_\nu$ (resp. $\psi_c < \psi_\nu$). Under the interest-control rule, the nominal growth rate of money supply is endogenously determined.

Since the growth rate of real money supply equals that of real income around the BGP, that is, $\frac{\dot{m}}{m} = g^*$, we derive

$$\mu(z^*) = A - \delta - z^* + \pi(z^*). \tag{18}$$

From (10),

$$\pi'(z^*) = \frac{A - \delta}{A - \delta - z^*}\frac{\eta}{\phi - 1}, \tag{19}$$

and therefore sign$[\pi'(z^*)]$=sign$\left[\frac{\eta}{\phi - 1}\right]$ is satisfied around the BGP. As a result, we can obtain

$$\mu'(z^*) = \pi'(z^*) - 1 = \frac{A - \delta}{A - \delta - z^*}\frac{\eta}{\phi - 1} - 1, \tag{20}$$

Table 1: The Relation between Velocity and Money Supply

	$\psi_c > \psi_\nu$	$\psi_c < \psi_\nu$	$\psi_c = \psi_\nu$
$\eta > 0$, $\phi > 1$, high $\frac{A-\delta}{g^*}$	$-$	$+$	0
$\eta > 0$, $\phi > 1$, low $\frac{A-\delta}{g^*}$	$+$	$-$	0
$\eta > 0$, $0 < \phi < 1$	$+$	$-$	0
$\eta = 0$	$+$	$-$	0

and thus

$$\text{sign}\ [\mu'(z^*)] = \text{sign}\ \left(\frac{\eta}{\phi - 1} - \frac{A - \delta - z^*}{A - \delta} \right). \tag{21}$$

Note that $g^* = A - \delta - z^*$. The income velocity of money around the BGP is represented by

$$V^*(z^*) = \frac{y^*}{m^*} = \frac{A}{(\psi_c - \psi_\nu) z^* + \psi_\nu A}, \tag{22}$$

implying that

$$\text{sign}[V^{*\prime}(z^*)] = -\text{sign}[\psi_c - \psi_\nu]. \tag{23}$$

Combining (20) and (22), we can describe the effect of money supply on velocity around the BGP as in Table 1. In this table, + means a positive relation, − means a negative relation and 0 indicates that there is no relation.

5 Discussion

We consider intuitive implication of the results.

From (19), two effects of economic growth on the nominal expansion rate of money supply can be seen. First, a decrease z^* yields a higher balanced-growth rate g^*. The second is the change of the inflation rate via the interest-control rule. When the growth rate of income increases, the central bank should raise the nominal interest rate to stabilize economy. However, since the net real rate of return to capital is constant due to the assumption of AK technology, the real interest rate also should be kept constant by adjusting the rate of inflation to satisfy the Fisher equation [3]. This is achieved by depressing inflation if the monetary policy is generalized and active, because the decline in nominal interest rate is larger than that in the inflation rate. Therefore, the expansion rate of nominal money supply may fall. Otherwise, the rise of economic growth does not generate the fall of the inflation rate so that the nominal growth rate of money supply increases.

We consider the result in (22). In light of AK technology, decreasing z^* generates the same magnitude of positive movement in the investment-capital ratio $\frac{\nu^*}{k^*}$. Therefore, if $\psi_c > \psi_\nu$, velocity of money which means the ratio of capital to real money balances becomes larger. Conversely, when $\psi_c < \psi_\nu$, the negative relation between the velocity and the growth rate of income is produced.

Taylor (1993) suggests that the observable policy stance of the Federal Reserve may be described by setting $\phi = 1.5$ and $\eta = 0.5$. Moreover, it is plausible to consider that real investment expenditures for machines, factories and housing are less constrained by cash holdings than consumption spending. Therefore, we focus on the case under which $\eta > 0$, $\phi > 1$, and $\psi_c > \psi_\nu$. When the ratio of the net real rate of return on capital $A - \delta$ to the balanced-growth rate $g^* = A - \delta - z^*$ is higher enough to satisfy $0 < \frac{\phi - 1}{\eta} < \frac{A - \delta}{g^*}$, the negative relation between the nominal money expansion and the income velocity of money holds, regardless of whether or not the economy displays sunspot-driven fluctuations around the BGP. Since $A - \delta > g^*$, the condition $0 < 1 = \frac{\phi - 1}{\eta} < \frac{A - \delta}{g^*}$ holds for $\phi = 1.5$ and $\eta = 0.5$. This result is different from Chen and Guo (2008b), who show that the negative

[3] When the interest rate is controlled by the rate of inflation alone, this channel is not effective.

relation emerges only on the determinate BGP if $\psi_c > \psi_\nu$. Consequently, we can conclude that both the form of the CIA constraint and the central bank's policy stance are important for the relation between and velocity and money expansion.

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Appendix

The first-order conditions of the household's maximization problem are

$$c^{-\sigma} = \lambda_m + \psi_c \zeta, \tag{24}$$

$$\lambda_k - \lambda_m = \psi_\nu \zeta, \tag{25}$$

$$\dot{\lambda_k} = (\rho + \delta)\lambda_k - A\lambda_m, \tag{26}$$

$$\dot{\lambda_m} = (\rho + \pi)\lambda_m - \zeta, \tag{27}$$

$$\zeta(m - \psi_c c - \psi_\nu \nu) = 0, \quad \zeta \geq 0, \quad m \geq \psi_c c + \psi_\nu \nu, \tag{28}$$

together with the transversality conditions $\lim_{t\to\infty} e^{-\rho t}\lambda_{mt} m_t = 0$ and $\lim_{t\to\infty} e^{-\rho t}\lambda_{kt} k_t = 0$, where λ_m and λ_k are the shadow prices of real money balances and capital, respectively, and ζ represents the Lagrange multiplier for the CIA constraint (4). In the following, we assume that the CIA constraint (4) is strictly binding in equilibrium, and thus $\zeta > 0$ for all t. Using (23) through (26), we obtain the following dynamic equations:

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left[l(p)\frac{\dot{p}}{p} + \frac{A}{p} - \rho - \delta\right],$$

$$\frac{\dot{\lambda_k}}{\lambda_k} = \rho + \delta - \frac{A}{p},$$

$$\frac{\dot{\lambda_m}}{\lambda_m} = \left(\rho + \pi(z) + \frac{1}{\psi_\nu}\right) - \frac{p}{\psi_\nu}.$$

From these equations and (6), the dynamics equations (11) and (12) are derived.

We linearize the dynamic system (11) and (12) around the BGP to obtain:

$$\begin{bmatrix} \dot{z} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \dot{z}_z & \dot{z}_p \\ \dot{p}_z & \dot{p}_p \end{bmatrix} \begin{bmatrix} \hat{z} \\ \hat{p} \end{bmatrix} = J \begin{bmatrix} \hat{z} \\ \hat{p} \end{bmatrix}, \tag{29}$$

where $\hat{z} \equiv z - z^*$ and $\hat{p} \equiv p - p^*$. The elements of matrix J (28) are the following:

$$\dot{z}_z = \left.\frac{\partial \dot{z}}{\partial z}\right|_{BGP} = \left[\frac{l(p^*)\dot{p}_z}{\sigma p^*} + 1\right] z^* = \left(-\frac{\pi'(z^*)l(p^*)}{\sigma} + 1\right) z^*,$$

$$\dot{z}_p = \frac{\partial \dot{z}}{\partial p}\bigg|_{BGP} = \frac{z^*}{\sigma(p^*)^2}[l(p^*)\dot{p}_p p^* - A] = -\frac{z^*}{\sigma p^*}\frac{p^*(\psi_c - \psi_\nu) + A\psi_c\psi_\nu}{\psi_\nu[\psi_c p^* - (\psi_c - \psi_\nu)]},$$

$$\dot{p}_z = \frac{\partial \dot{p}}{\partial z}\bigg|_{BGP} = -\pi'(z^*)p^*,$$

$$\dot{p}_p = \frac{\partial \dot{p}}{\partial p}\bigg|_{BGP} = \frac{A}{p^*} + \frac{p^*}{\psi_\nu} > 0.$$

The trace and determinant of J are respectively given by:

$$\mathrm{tr}J = \dot{z}_z + \dot{p}_p = \left(-\frac{\pi'(z^*)l(p^*)}{\sigma} + 1\right)z^* + \frac{A}{p^*} + \frac{p^*}{\psi_\nu}, \tag{30}$$

$$\mathrm{det}J = \dot{z}_z\dot{p}_p - \dot{z}_p\dot{p}_z = \frac{z^*}{p^*}\left[\frac{A\psi_\nu + (p^*)^2}{\psi_\nu} - \frac{A\pi'(z^*)}{\sigma}\right]. \tag{31}$$

Since z and p are jump variables, if $\mathrm{tr}J > 0$ and $\mathrm{det}J > 0$, then the BGP is totally unstable so that the economy always stays on the BGP, that is, the equilibrium path is determinate. Otherwise, the equilibrium path is indeterminate. That is, the economy always stays on the BGP, or endogenous income fluctuations driven by sunspots are generated. Inspecting (29) and (30), we can find the following proposition, which shows that the generalization of the Taylor rule and the most generalized CIA constraint play a significant role in macroeconomic stability.

Proposition 1 *In the case of* $0 < \psi_c, \psi_\nu \leq 1$*, equilibrium determinacy holds either if (i) monetary policy rule responds only to the rate of inflation. or if (ii)* $\psi_c \leq \psi_\nu$ *and monetary policy is passive. Otherwise, BGP could be locally indeterminate.*